On Eigenvalue Distribution of Varying Hankel and Toeplitz Matrices with Entries of Power Growth or Decay

Gidon Kowalsky and Doron S. Lubinsky

Abstract We study the distribution of eigenvalues of varying Toeplitz and Hankel matrices such as $\left[a_{n+k-j}\right]_{j,k}$ and $\left[a_{n+k+j}\right]_{j,k}$ where a_n behaves roughly like n^{β} for some non-0 complex number β, and $n \to \infty$. This complements earlier work on these matrices when the coefficients $\{a_n\}$ arise from entire functions.

Keywords Toeplitz matrices · Hankel matrices · Eigenvalue distribution

1 Introduction and Results

The distribution of eigenvalues of Toeplitz matrices $\left[c_{k-j}\right]_{1\leq j,k\leq n}$ is a much studied topic, especially when their entries are trigonometric moments [1, 2, 5, 7, 9, 18, 19, 26, 29, 30]. There is a classic paper of Widom [28] dealing with both finite and infinite Hankel matrices $\left[c_{j+k}\right]$. There is a large literature on random Hankel and Toeplitz matrices, see for example, [3, 10, 12, 13, 21, 22]. Generalizations of Toeplitz matrix sequences are considered and studied in [7].

Our interest arises from classical function theory and Padé approximation. There is a connection to complex function theory: Polya [20] proved that if $f(z) = \sum_{j=0}^{\infty} a_j/z^j$ can be analytically continued to a function analytic in the complex plane outside a set of logarithmic capacity $\tau \geq 0$, then

$$\limsup_{n\to\infty} \left|\det\left[a_{n-j+k}\right]_{1\leq j,k\leq n}\right|^{1/n^2} \leq \tau.$$

There are many extensions of this result [4, 16].

In the recent paper [16], we analyzed distribution of the eigenvalues of such matrices under appropriate hypotheses on

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G. E. Fasshauer et al. (eds.), *Approximation Theory XVI*, Springer Proceedings in Mathematics & Statistics 336, https://doi.org/10.1007/978-3-030-57464-2_8

$$q_j = \frac{a_{j-1}a_{j+1}}{a_j^2}.$$

The motivation comes from Padé approximation for functions such as

$$f(z) = \sum_{j=0}^{\infty} z^j / (j!)^{1/\alpha}, \ \alpha > 0, \tag{1.1}$$

for which (cf. [14, 15])

$$q_j = \exp\left(-\frac{1}{\alpha j} + O\left(\frac{1}{j^2}\right)\right).$$

More generally, we considered series

$$f(z) = \sum_{j=0}^{\infty} a_j z^j,$$

that satisfy

$$q_j = \frac{a_{j-1}a_{j+1}}{a_j^2} = \exp\left(-\frac{1}{\rho_j}\left(1 + o\left(\rho_j^{-1/2}\right)\right)\right),$$

with appropriate smoothly increasing or decreasing sequences $\{\rho_j\}$ of positive numbers. We proved, under mild conditions on $\{\rho_j\}$, the following assertions about the eigenvalues $\{\lambda_{nj}\}_{j=1}^{n}$ of the normalized matrix $\frac{1}{a_n}\left[a_{n+k-j}\right]_{1\le j,k\le n}$:

1. The eigenvalue of largest modulus satisfies

$$\max_{1\le j\le n} \left|\lambda_{nj}\right| = \sqrt{2\pi\rho_n}\,(1 + o(1)).$$

2. The set of all limit points of $\left\{\lambda_{nj}/\sqrt{2\pi\rho_n}\right\}_{1\le j\le n, n\ge 1}$ is [0, 1].
3. The scaled zero counting measures

$$\mu_n = \frac{1}{n}\sum_{j=1}^{n}\left(\operatorname{Re}\lambda_{nj}\right)\delta_{\lambda_{nj}/\sqrt{2\pi\rho_m}}$$

admit the weak convergence

$$d\mu_n \xrightarrow{*} |\pi \log t|^{-1/2}\, dt \tag{1.2}$$

in the sense that for each function f defined and continuous in an open subset of the plane containing [0, 1],

$$\lim_{n\to\infty}\int f\,d\mu_n = \int_0^1 f(t)\,|\pi\log t|^{-1/2}\,dt. \tag{1.3}$$

The hypotheses in [16] treat a broad array of entire functions of zero, finite positive, or infinite order, and also some power series of finite radius of convergence. However the hypotheses exclude the case where the coefficients have power growth or decay. It is the purpose of this paper to study that case. The general sequences of Toeplitz matrices in [7] differ from our situation in that our sequences of varying matrices require a different normalization as $n \to \infty$, and a different formulation for the eigenvalue counting measures. Moreover, in Widom's paper [28], the matrices treated have the form $\left[c_{j+k}\right]_{0\le j,k\le n}$, whereas in this paper the top left-hand corner element is a_m with m growing to ∞, so the results and methods are different. We consider the Hankel matrices

$$H_{mn} = \left[a_{m+k+j}\right]_{0\le j,k\le n-1}$$

and Toeplitz matrices

$$T_{mn} = \left[a_{m+k-j}\right]_{1\le j,k\le n}$$

where a_n behaves roughly like n^{β}.

Our approach is also quite different from that in [16], due to the different growth rates. There we used a similarity transformation on T_{mn} and showed that the eigenvalues of T_{mn}/a_m behaved like those of the matrix $E_{mn} = -\left[e^{-\frac{(j-k)^2}{2\rho_n}}\right]_{1\le j,k\le n}$. There roughly $O\left(\sqrt{n}\right)$ central bands of the matrix dominate and one can compute the asymptotics of the trace of E_{mn}^k for each fixed $k = 0, 1, 2, \ldots$. This approach fails for the sequences we consider here, as all bands contribute, and indeed we get a different weak limit from that above.

2 Hankel Matrices

In this section, we state our results for Hankel matrices $\left[a_{m+j+k}\right]_{0\le j,k\le n-1}$ where the a_j grow or decay like j^{β}. Of course if β is real, these matrices are real and symmetric, so have real eigenvalues. In the special case, where $\beta < 0$ and $a_j = j^{\beta}$, these matrices are actually positive definite, so have positive eigenvalues. Indeed this follows directly from the fact that for $\beta < 0$ and $j \ge 1$.

$$j^{\beta}=\frac{1}{\Gamma(-\beta)}\int_0^1 s^j\left(\log\frac{1}{s}\right)^{-\beta-1}s^{-1}ds.$$

This identity in turn follows from the standard integral for the gamma function

$$\Gamma(-\beta)=\int_0^{\infty}t^{-\beta-1}e^{-t}dt$$

by the substitution $s=e^{-t/j}$. Our first result allows possibly complex β. As above we let

$$H_{mn}=\left[a_{m+j+k}\right]_{0\le j,k\le n-1}. \tag{2.1}$$

We also let $\Lambda(H_{mn}/a_m)$ denote the collection of all eigenvalues of H_{mn}/a_m, and form the weighted counting measure

$$\mu_{mn}=\frac{1}{n^2}\sum_{\lambda\in\Lambda(H_{mn}/a_m)}\lambda^2\delta_{\lambda/n}. \tag{2.2}$$

Thus μ_{mn} places mass $\left(\frac{\lambda}{n}\right)^2$ at $\frac{1}{n}\lambda$ for each eigenvalue λ of H_{mn}/a_m. This is rather different from the usual eigenvalue counting measures, but is needed in our situation. The weighting reflects the fact that eigenvalues of H_{mn}/a_m tend to cluster around 0. For general sequences of Hankel and other matrices, this clustering effect has been extensively explored—see [6, 8, 23, 27].

Theorem 2.1 *Fix $k\ge 1$ and $R>0$. Assume $m=m(n)\to\infty$ in such a way that $m/n\to R$ as $n\to\infty$. Assume that $\beta\in\mathbb{C}$ and given $R>0$, we have as $n\to\infty$, uniformly for $0\le\ell\le Rm$,*

$$\frac{a_{m+\ell}}{a_m}=\left(1+\frac{\ell}{m}\right)^{\beta}(1+o(1)). \tag{2.3}$$

Then

(I)

$$\limsup_{n\to\infty}\frac{1}{n}\sup\{|\lambda|:\lambda\in\Lambda(H_{mn}/a_m)\}$$
$$\le\int_0^1\max_{0\le y\le 1}\left(1+\frac{x+y}{R}\right)^{\operatorname{Re}\beta}dx. \tag{2.4}$$

In particular, the supports of $\{\mu_{mn}\}_{n\ge 1}$ are contained in a compact set independent of n.

(II)

$$\limsup_{n\to\infty} \left|\mu_{mn}\right| (\mathbb{C}) \leq \int_0^1 \int_0^1 \left(1 + \frac{x+y}{R}\right)^{2\operatorname{Re}\beta} dx\, dy. \tag{2.5}$$

(III) For $k \geq 1$,

$$\lim_{n\to\infty} \frac{1}{n^k} Tr\left(\left[\frac{H_{mn}}{a_m}\right]^k\right) = c_k, \tag{2.6}$$

where

$$c_k = R^k \int_0^{1/R} \int_0^{1/R} \dots \int_0^{1/R} (1 + t_1 + t_2)^\beta \dots$$
$$(1 + t_{k-1} + t_k)^\beta (1 + t_k + t_1)^\beta dt_1 dt_2 \dots dt_k. \tag{2.7}$$

Consequently for $k \geq 0$,

$$\lim_{n\to\infty} \int \lambda^k d\mu_{mn}(\lambda) = c_{k+2}. \tag{2.8}$$

Corollary 2.2 *Assume that* β *is real and all* $\{a_j\}$ *are real. Then there is a finite positive measure* ω *with compact support on the real line such that for all functions* f *continuous on the real line with compact support,*

$$\lim_{n\to\infty} \int f(t)\, d\mu_{mn}(t) = \int f(t)\, d\omega(t). \tag{2.9}$$

The measure ω *is uniquely determined by the moment conditions*

$$\int t^k d\omega(t) = c_{k+2}, k \geq 0.$$

Remarks

(a) Note that (2.3) is satisfied if $a_n = n^\beta b_n$, where $\frac{b_{n+\ell}}{b_n} = 1 + o(1)$ for $0 \leq \ell \leq Rm$. For example this is true if $a_n = n^\beta (\log n)^\gamma (\log\log n)^\kappa$ for some γ, κ.
(b) If we do not assume that the $\{a_j\}$ are real, then we can only prove convergence for functions f analytic in a ball center 0 of large enough radius, as in Corollary 3.2 below.
(c) It is obviously of interest to find an explicit form for ω. There is a classic technique for simplices that provides an explicit value for similar Dirichlet-Liouville multiple integrals [11, 25], but it does not seem to work for cubes.
(d) Note that our eigenvalue counting measure μ_{mn} has a different normalization and scaling to standard ones, so we cannot apply standard results such as in [7].

We prove Theorem 2.1 and Corollary 2.2 in Sect. 4.

3 Toeplitz Matrices

As above, we let

$$T_{mn} = \left[a_{m+k-j}\right]_{1 \leq j,k \leq n}.$$

Here we set $a_j = 0$ if $j < 0$. We also let

$$\nu_{mn} = \frac{1}{n^2} \sum_{\lambda \in \Lambda(T_{mn}/a_m)} \lambda^2 \delta_{\lambda/n}. \tag{3.1}$$

We prove:

Theorem 3.1 *Let $R \geq 1$. Assume $m = m(n) \to \infty$ in such a way that $m/n \to R$ as $n \to \infty$. Let $\beta \in \mathbb{C}$. Assume that given $\varepsilon \in (0,1)$, we have as $n \to \infty$, uniformly for $-m(1-\varepsilon) \leq \ell \leq (R-1)m$,*

$$\frac{a_{m+\ell}}{a_m} = \left(1 + \frac{\ell}{m}\right)^{\beta} (1 + o(1)). \tag{3.2}$$

If $R = 1$, we assume in addition that $\operatorname{Re}\beta > -1$*and*

$$\lim_{\varepsilon \to 0+} \left(\limsup_{n \to \infty} \frac{1}{n|a_n|} \sum_{j=1}^{[\varepsilon n]} |a_j| \right) = 0. \tag{3.3}$$

Then

(I)

$$\limsup_{n \to \infty} \frac{1}{n} \sup\{|\lambda| : \lambda \in \Lambda(T_{mn}/a_m)\} \leq \int_0^1 \max_{0 \leq y \leq 1} \left(1 + \frac{x-y}{R}\right)^{\operatorname{Re}\beta} dx.$$

In particular, the supports of $\{\nu_{mn}\}_{n \geq 1}$ are contained in a compact set independent of n.

(II)

$$\limsup_{n \to \infty} |\nu_{mn}|(\mathbb{C}) \leq \int_0^1 \int_0^1 \left(1 + \frac{x-y}{R}\right)^{2\operatorname{Re}\beta} dx\,dy.$$

(III) For $k \geq 1$,

$$\lim_{n\to\infty}\frac{1}{n^k}Tr\left(\left[\frac{T_{mn}}{a_m}\right]^k\right)=d_k,$$

where

$$d_k=R^k\int_0^{1/R}\int_0^{1/R}\dots\int_0^{1/R}(1+t_1-t_2)^{\beta}\dots$$
$$(1+t_{k-1}-t_k)^{\beta}(1+t_k-t_1)^{\beta}\,dt_1dt_2\dots dt_k.$$

Consequently for $k\geq 0$*,*

$$\lim_{n\to\infty}\int\lambda^k d\nu_{mn}(\lambda)=d_{k+2}. \tag{3.4}$$

Corollary 3.2 *There is a finite complex measure* ω *with compact support in the plane such that for all functions* f *analytic in the ball center* 0*, radius* $\int_0^1\max_{0\leq y\leq 1}\left(1+\frac{x-y}{R}\right)^{\mathrm{Re}\,\beta}dx$*,*

$$\lim_{n\to\infty}\int f(t)\,d\nu_{mn}(t)=\int f(t)\,d\omega(t). \tag{3.5}$$

The measure ω *admits the moment conditions*

$$\int t^k d\omega(t)=d_{k+2},\ k\geq 0.$$

Here in the case $R=1$*, we assume* $\mathrm{Re}\,\beta>-1$.

We note that it is not clear if the complex valued measure ω is uniquely determined by the moment conditions, as it is supported in the complex plane. We prove the results of this section in Sect. 5.

4 Proof of Theorem 2.1 and Corollary 2.2

Proof of Theorem 2.1(I) It follows from Gershgorin's Theorem [17, p. 146] that every eigenvalue λ of H_{mn}/a_m satisfies

$$\frac{|\lambda|}{n}\leq\max_{0\leq j\leq n-1}\frac{1}{n}\sum_{k=0}^{n-1}\left|\frac{a_{m+k+j}}{a_m}\right|.$$

Our hypothesis (2.3) gives uniformly for $0\leq j,k\leq n-1$,

$$\left|\frac{a_{m+k+j}}{a_m}\right| = \left|\left(1+\frac{k+j}{m}\right)^{\beta}(1+o(1))\right|$$
$$= \left(1+\frac{k+j}{Rn(1+o(1))}\right)^{\operatorname{Re}\beta}(1+o(1))$$
$$\leq \max_{1\leq \ell\leq n}\left(1+\frac{k+\ell}{Rn}\right)^{\operatorname{Re}\beta}(1+o(1)),$$

so that

$$\frac{|\lambda|}{n} \leq \frac{1}{n}\sum_{k=0}^{n-1}\max_{0\leq y\leq 1}\left(1+\frac{k}{Rn}+\frac{y}{R}\right)^{\operatorname{Re}\beta}+o(1)$$
$$\to \int_0^1 \max_{0\leq y\leq 1}\left(1+\frac{x}{R}+\frac{y}{R}\right)^{\operatorname{Re}\beta}dx$$

as $n\to\infty$. ■

Proof of Theorem 2.1(II) By Schur's Inequality [17, p. 142],

$$\left|\mu_{mn}\right|(\mathbb{C}) = \frac{1}{n^2}\sum_{\lambda\in\Lambda(H_{mn}/a_m)}|\lambda|^2 \leq \frac{1}{n^2}\sum_{j,k=0}^{n-1}\left|\frac{a_{m+j+k}}{a_m}\right|^2$$
$$= \frac{1}{n^2}\sum_{j,k=0}^{n-1}\left|\left(1+\frac{j+k}{m}\right)^{\beta}(1+o(1))\right|^2$$
$$= \frac{1}{n^2}\sum_{j,k=0}^{n-1}\left(1+\frac{j+k}{Rn}\right)^{2\operatorname{Re}\beta}(1+o(1))$$
$$\to \int_0^1\int_0^1\left(1+\frac{x+y}{R}\right)^{2\operatorname{Re}\beta}dx\,dy$$

as $n\to\infty$. ■

Proof of Theorem 2.1(III) Now

$$\frac{1}{n^k}Tr\left(\left[\frac{H_{mn}}{a_m}\right]^k\right)$$
$$= \frac{1}{n^k}\sum_{j_1=0}^{n-1}\sum_{j_2=0}^{n-1}\cdots\sum_{j_k=0}^{n-1}\frac{a_{m+j_1+j_2}}{a_m}\frac{a_{m+j_2+j_3}}{a_m}\cdots\frac{a_{m+j_{k-1}+j_k}}{a_m}\frac{a_{m+j_k+j_1}}{a_m}$$

$$= \frac{1}{n^k} \sum_{j_1=0}^{n-1} \sum_{j_2=0}^{n-1} \cdots \sum_{j_k=0}^{n-1} \left(1 + \frac{j_1 + j_2}{m}\right)^\beta \left(1 + \frac{j_2 + j_3}{m}\right)^\beta \cdots$$

$$\left(1 + \frac{j_k + j_1}{m}\right)^\beta (1 + o(1))$$

$$= \frac{1}{n^k} \sum_{j_1=0}^{n-1} \sum_{j_2=0}^{n-1} \cdots \sum_{j_k=0}^{n-1} \left(1 + \frac{j_1 + j_2}{nR(1 + o(1))}\right)^\beta \left(1 + \frac{j_2 + j_3}{nR(1 + o(1))}\right)^\beta \cdots$$

$$\left(1 + \frac{j_k + j_1}{nR(1 + o(1))}\right)^\beta (1 + o(1))$$

$$= \frac{1}{n^k} \sum_{j_1=0}^{n-1} \sum_{j_2=0}^{n-1} \cdots \sum_{j_k=0}^{n-1} \left(1 + \frac{j_1 + j_2}{nR}\right)^\beta \left(1 + \frac{j_2 + j_3}{nR}\right)^\beta \cdots$$

$$\left(1 + \frac{j_k + j_1}{nR}\right)^\beta + o(1),$$

since each of the n^k terms are bounded independently of n and each index j_i, $1 \leq i \leq k$. The sum in the last line is a Riemann sum for the multiple integral

$$\int_0^1 \int_0^1 \cdots \int_0^1 \left(1 + \frac{x_1 + x_2}{R}\right)^\beta \cdots$$

$$\left(1 + \frac{x_{k-1} + x_k}{R}\right)^\beta \left(1 + \frac{x_k + x_1}{R}\right)^\beta dx_1 dx_2 \ldots dx_k$$

and so we obtain the result (2.7), after making the substitution $x_j = Rt_j$ for $1 \leq j \leq k$. Finally, from (2.2),

$$\int \lambda^j d\mu_{mn}(\lambda) = \frac{1}{n^{j+2}} Tr\left(\left[\frac{H_{mn}}{a_m}\right]^{j+2}\right).$$

Then (2.8) follows. ■

Proof of Corollary 2.2 Firstly as H_{mn}/a_m is real and symmetric, all its eigenvalues are real. It follows that μ_{mn} is a positive measure supported on the real line. Moreover, Theorem 2.1 shows that the supports of all μ_{mn} are contained in a bounded interval independent of n, while their total mass is bounded independent of n. By Helly's Theorem (or if you prefer the Banach-Alaoglu Theorem) every subsequence of $\{\mu_{mn}\}$ contains another subsequence converging weakly to some positive measure ω with compact support in the real line. It follows from Theorem 2.1(III) that for $j \geq 0$,

$$\int t^j d\omega(t) = c_{j+2}.$$

As the Hausdorff moment problem [24] (or moment problem for a bounded interval) has a unique solution, ω is independent of the subsequence. Then the full sequence $\{\mu_{mn}\}$ converges weakly to ω. ■

For the largest eigenvalue for this positive case, we prove:

Lemma 4.1 *Assume β is real and all $\{a_j\}$are real. Let $\lambda_{\max}$ denote the largest eigenvalue of H_{mn}/a_m. Then*

$$\liminf_{n\to\infty} \frac{1}{n}\lambda_{\max} \geq \int_0^1 \int_0^1 \left(1 + \frac{x+y}{R}\right)^\beta dx\, dy$$

and

$$\limsup_{n\to\infty} \frac{1}{n}\lambda_{\max} \leq \left(\int_0^1 \int_0^1 \left(1 + \frac{x+y}{R}\right)^{2\beta} dx\, dy\right)^{1/2}.$$

Proof As H_{mn}/a_m is real symmetric, its largest eigenvalue $\lambda_{\max}$ satisfies

$$\lambda_{\max} = \sup\left\{\sum_{j,k=0}^{n-1} \frac{a_{m+j+k}}{a_m} x_j x_k : \sum_{j=0}^{n-1} x_j^2 = 1\right\}.$$

Choosing all $x_j = \frac{1}{\sqrt{n}}$, we see much as above that

$$\liminf_{n\to\infty} \frac{1}{n}\lambda_{\max} \geq \lim_{n\to\infty} \frac{1}{n^2} \sum_{j,k=0}^{n-1} \left(1 + \frac{j+k}{Rn}\right)^\beta (1 + o(1))$$
$$= \int_0^1 \int_0^1 \left(1 + \frac{x+y}{R}\right)^\beta dx\, dy.$$

In the other direction, two applications of the Cauchy-Schwarz inequality give, if $\sum_{j=0}^{n-1} x_j^2 = 1$,

$$\left|\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \frac{a_{m+j+k}}{a_m} x_j x_k\right|$$
$$\leq \sum_{j=0}^{n-1} |x_j| \left(\sum_{k=0}^{n-1} \left(\frac{a_{m+j+k}}{a_m}\right)^2\right)^{1/2} \left(\sum_{k=0}^{n-1} x_k^2\right)^{1/2}$$

$$\leq \left(\sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \left(\frac{a_{m+j+k}}{a_m} \right)^2 \right)^{1/2} \left(\sum_{j=0}^{n-1} x_j^2 \right)^{1/2},$$

so much as above,

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{n} \lambda_{\max} \\ &\leq \lim_{n\to\infty} \left(\frac{1}{n^2} \sum_{j,k=0}^{n-1} \left(1 + \frac{j+k}{Rn} \right)^{2\beta} (1 + o(1)) \right)^{1/2} \\ &= \left(\int_0^1 \int_0^1 \left(1 + \frac{x+y}{R} \right)^{2\beta} dx\, dy \right)^{1/2}. \end{aligned}$$

■

5 Proof of Theorem 3.1 and Corollary 3.2

Toeplitz matrices are more delicate, as reflected both in the hypotheses and proofs. In the sequel, we let

$$\phi(\varepsilon) = \limsup_{n\to\infty} \frac{1}{n |a_n|} \sum_{j=1}^{[\varepsilon n]+1} |a_j|, \ \varepsilon \in [0, 1].$$

If $R = 1$, our hypothesis (3.3) is that $\phi(\varepsilon) \to 0$ as $\varepsilon \to 0+$.

Proof of Theorem 3.1(I) It follows from Gershgorin's Theorem that every eigenvalue λ of T_{mn}/a_m satisfies

$$\frac{|\lambda|}{n} \leq \max_{1\leq j\leq n} \frac{1}{n} \sum_{k=1}^{n} \left| \frac{a_{m+k-j}}{a_m} \right|. \tag{5.1}$$

Assume first $R > 1$. We can use our asymptotic (3.2) to deduce that

$$\begin{aligned} \frac{|\lambda|}{n} &\leq \max_{1\leq j\leq n} \frac{1}{n} \sum_{k=1}^{n} \left| \left(1 + \frac{k-j}{m} \right)^{\beta} (1 + o(1)) \right| \\ &\leq \max_{1\leq j\leq n} \frac{1}{n} \sum_{k=1}^{n} \left(1 + \frac{k-j}{m} \right)^{\operatorname{Re}\beta} + o(1) \end{aligned}$$

$$\leq \max_{1\leq j\leq n} \frac{1}{n} \sum_{k=1}^{n} \left(1 + \frac{k-j}{Rn\left(1+o\left(1\right)\right)}\right)^{\operatorname{Re}\beta} + o\left(1\right)$$

$$\leq \frac{1}{n} \sum_{k=1}^{n} \max_{0\leq y\leq 1} \left(1 + \frac{k}{Rn} - \frac{y}{R}\right)^{\operatorname{Re}\beta} + o\left(1\right)$$

$$\rightarrow \int_{0}^{1} \max_{0\leq y\leq 1} \left(1 + \frac{x-y}{R}\right)^{\operatorname{Re}\beta} dx.$$

Now suppose that $R = 1$. Choose a subsequence $\mathcal{S}$ of integers n and then for $n \in \mathcal{S}$, choose $j = j\left(n\right) \in [1, n]$, such that

$$\limsup_{n\rightarrow\infty} \left(\max_{1\leq j\leq n} \frac{1}{n} \sum_{k=1}^{n} \left|\frac{a_{m+k-j}}{a_m}\right|\right) = \lim_{n\rightarrow\infty, n\in\mathcal{S}} \frac{1}{n} \sum_{k=1}^{n} \left|\frac{a_{m+k-j(n)}}{a_m}\right|. \tag{5.2}$$

By choosing a further subsequence, which we also denote by $\mathcal{S}$, we may assume that for some $\alpha \in [0, 1]$,

$$\lim_{n\rightarrow\infty} \frac{j\left(n\right)}{n} = \alpha.$$

Fix $\varepsilon \in \left(0, \frac{1}{2}\right)$. Observe that if $k - j \geq -\left(1-\varepsilon\right) m$, we can apply (3.2). Here as $n \rightarrow \infty$, this inequality is asymptotically equivalent to $k \geq \left(\alpha + \varepsilon - 1\right) n \left(1 + o\left(1\right)\right)$. Then for $n \in \mathcal{S}$ and $j = j\left(n\right)$,

$$\frac{1}{n} \sum_{\substack{k:1\leq k\leq n \\ \text{and } k-j\geq-(1-\varepsilon)m}} \left|\frac{a_{m+k-j}}{a_m}\right|$$

$$\leq \frac{1}{n} \sum_{k\leq n:k\geq\max\{1,(\alpha+\varepsilon-1)n(1+o(1))\}} \left|\left(1 + \frac{k-j}{m}\right)^{\beta} \left(1 + o\left(1\right)\right)\right|$$

$$\leq \frac{1}{n} \sum_{k\leq n:k\geq\max\{1,(\alpha+\varepsilon-1)n(1+o(1))\}} \left(1 + \frac{k - \alpha n\left(1+o\left(1\right)\right)}{n\left(1+o\left(1\right)\right)}\right)^{\operatorname{Re}\beta} + o\left(1\right)$$

$$= \int_{\max\{0,\alpha+\varepsilon-1\}}^{1} \left(1 + x - \alpha\right)^{\operatorname{Re}\beta} dx + o\left(1\right).$$

Next, recall that $a_j = 0$ for $j < 0$. If $k - j \leq -\left(1-\varepsilon\right) m$, then $m + k - j \leq \varepsilon m$. Then as $m/n \rightarrow 1$ as $n \rightarrow \infty$, we have for large enough n and $j \geq 1$,

$$\frac{1}{n}\sum_{\substack{k:1\leq k\leq n\\ \text{and } k-j\leq -(1-\varepsilon)m}}\left|\frac{a_{m+k-j}}{a_m}\right|$$
$$\leq \frac{1+o(1)}{m|a_m|}\sum_{\ell=1}^{[\varepsilon m]+1}|a_\ell| \leq \phi(\varepsilon)+o(1).$$

Adding the two sums together, we obtain

$$\limsup_{n\to\infty}\left(\max_{1\leq j\leq n}\frac{1}{n}\sum_{k=1}^{n}\left|\frac{a_{m+k-j}}{a_m}\right|\right)$$
$$\leq \int_{\max\{0,\alpha+\varepsilon-1\}}^{1}(1+x-\alpha)^{\operatorname{Re}\beta}\,dx+\phi(\varepsilon).$$

Letting $\varepsilon\to 0+$, and using Dominated Convergence, we obtain

$$\limsup_{n\to\infty}\left(\max_{1\leq j\leq n}\frac{1}{n}\sum_{k=1}^{n}\left|\frac{a_{m+k-j}}{a_m}\right|\right)\leq \int_{\max\{0,\alpha-1\}}^{1}(1+x-\alpha)^{\operatorname{Re}\beta}\,dx$$
$$\leq \int_0^1\max_{0\leq y\leq 1}(1+x-y)^{\operatorname{Re}\beta}\,dx.$$

So we obtain the result for $R=1$. ■

Proof of Theorem 3.1(II) As in the proof of Theorem 2.1(II), Schur's inequality gives

$$|\nu_{mn}|(\mathbb{C})=\frac{1}{n^2}\sum_{\lambda\in\Lambda(T_{mn}/a_m)}|\lambda|^2\leq\frac{1}{n^2}\sum_{j,k=0}^{n-1}\left|\frac{a_{m+k-j}}{a_m}\right|^2.$$

Suppose first $R>1$. Then for large enough n, if $0\leq j,k\leq n-1$,

$$m+k-j\geq Rn(1+o(1))-n+1$$
$$\geq (R-1)n+o(n)$$
$$\geq \frac{R-1}{R}m+o(m),$$

so uniformly for such j,k, (3.2) gives

$$\frac{a_{m+k-j}}{a_m}=\left(1+\frac{k-j}{Rn}\right)^{\beta}(1+o(1)). \tag{5.3}$$

Then

$$\begin{aligned} |\nu_{mn}|(\mathbb{C}) &\leq \frac{1}{n^2}\sum_{j,k=0}^{n-1}\left|\frac{a_{m+k-j}}{a_m}\right|^2 \\ &\leq \frac{1}{n^2}\sum_{j,k=0}^{n-1}\left(1+\frac{k-j}{Rn}\right)^{2\operatorname{Re}\beta}(1+o(1)) \\ &\to \int_0^1\int_0^1\left(1+\frac{y-x}{R}\right)^{2\operatorname{Re}\beta}dx\,dy \end{aligned}$$

as $n\to\infty$. Next, let $R=1$. Much as above, we can see that given $\varepsilon\in(0,1)$,

$$\frac{1}{n^2}\sum_{0\leq j,k\leq n-1:k-j\geq-(1-\varepsilon)m}\left|\frac{a_{m+k-j}}{a_m}\right|^2$$

may be bounded above by a Riemann sum for the integral

$$\int\int_{\{(x,y):x,y\in[0,1]\text{ and }y-x\geq-(1-\varepsilon)\}}(1+y-x)^{2\operatorname{Re}\beta}\,dx\,dy$$

multiplied by $1+o(1)$. To deal with the tail sum, first observe that as $m=m(n)=m(1+o(1))$,

$$\begin{aligned} \frac{1}{n}\sum_{j=1}^{3n}\frac{|a_j|}{|a_m|} &\leq (1+o(1))\,\phi\left(\frac{1}{4}\right)+\frac{1}{n}\sum_{j=\left[\frac{1}{4}n\right]}^{3n}\left|\frac{a_j}{a_n}\right| \\ &\leq (1+o(1))\,\phi\left(\frac{1}{4}\right)+\frac{1+o(1)}{n}\sum_{\ell=\left[\frac{1}{4}n\right]-n}^{2n}\left|\frac{a_{n+\ell}}{a_n}\right| \\ &\leq (1+o(1))\,\phi\left(\frac{1}{4}\right)+\frac{1+o(1)}{n}\sum_{\ell=\left[\frac{1}{4}n\right]-n}^{2n}\left|1+\frac{\ell}{n}\right|^{\operatorname{Re}\beta}(1+o(1)) \\ &\leq (1+o(1))\,\phi\left(\frac{1}{4}\right)+(1+o(1))\int_{-3/4}^{2}|1+x|^{\operatorname{Re}\beta}\,dx. \end{aligned}$$

It follows that for some C independent of m,n,

$$\frac{1}{n}\sum_{j=1}^{3n}\frac{|a_j|}{|a_m|}\leq C. \tag{5.4}$$

Then

$$\begin{aligned}
&\frac{1}{n^2} \sum_{0\le j,k\le n-1:k-j\le -(1-\varepsilon)m} \left|\frac{a_{m+k-j}}{a_m}\right|^2 \\
&\le \left(\frac{1}{n}\sup_{1\le \ell\le 2m}\left|\frac{a_\ell}{a_m}\right|\right)\left(\frac{1}{n}\sum_{\ell=1}^{[\varepsilon m]}\left|\frac{a_\ell}{a_m}\right|\right) \\
&\le C\phi(\varepsilon),
\end{aligned}$$

in view of (5.4). This and the estimate above give

$$\begin{aligned}
&\limsup_{n\to\infty} |\nu_{mn}|(\mathbb{C}) \\
&= \limsup_{n\to\infty} \frac{1}{n^2} \sum_{\lambda\in\Lambda(T_{mn}/a_m)} |\lambda|^2 \\
&\le \int\int_{\{(x,y):x,y\in[0,1] \text{ and } y-x\ge -(1-\varepsilon)\}} (1+y-x)^{2\operatorname{Re}\beta}\,dx\,dy + C\phi(\varepsilon).
\end{aligned}$$

Letting $\varepsilon \to 0+$ and using our hypothesis (3.3) gives the result. ■

Proof of Theorem 3.1(III)

Step 1 Suppose first $R > 1$. Then for large enough n, we have (5.3) and also

$$\sup_{1\le j,\ell\le n}\left|\frac{a_{m+j-\ell}}{a_m}\right| = O(1). \tag{5.5}$$

Then

$$\begin{aligned}
&\frac{1}{n^k} Tr\left(\left[\frac{T_{mn}}{a_m}\right]^k\right) \\
&= \frac{1}{n^k}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\cdots\sum_{j_k=1}^{n} \frac{a_{m+j_2-j_1}}{a_m}\frac{a_{m+j_3-j_2}}{a_m}\cdots\frac{a_{m+j_k-j_{k-1}}}{a_m}\frac{a_{m+j_1-j_k}}{a_m} \\
&= \frac{1}{n^k}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\cdots\sum_{j_k=1}^{n}\left(1+\frac{j_2-j_1}{m}\right)^\beta\left(1+\frac{j_3-j_2}{m}\right)^\beta\cdots \\
&\quad \left(1+\frac{j_1-j_k}{m}\right)^\beta(1+o(1)) \\
&= \frac{1}{n^k}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\cdots\sum_{j_k=1}^{n}\left(1+\frac{j_2-j_1}{Rn(1+o(1))}\right)^\beta\left(1+\frac{j_3-j_2}{Rn(1+o(1))}\right)^\beta\cdots
\end{aligned}$$

$$\left(1+\frac{j_1-j_k}{Rn\,(1+o\,(1))}\right)^{\beta}(1+o\,(1))$$
$$=\frac{1}{n^k}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\cdots\sum_{j_k=1}^{n}\left(1+\frac{j_2-j_1}{Rn}\right)^{\beta}\left(1+\frac{j_3-j_2}{Rn}\right)^{\beta}\cdots$$
$$\left(1+\frac{j_1-j_k}{Rn}\right)^{\beta}+o\,(1)\,.$$

The sum in the last line is a Riemann sum for the multiple integral

$$\int_0^1\int_0^1\cdots\int_0^1\left(1+\frac{x_2-x_1}{R}\right)^{\beta}\left(1+\frac{x_3-x_2}{R}\right)^{\beta}\cdots$$
$$\left(1+\frac{x_1-x_k}{R}\right)^{\beta}dx_1dx_2\ldots dx_k$$

and so we obtain the result, after making the substitution $x_j=Rt_j$ for $1\le j\le k$.

Step 2 Now we turn to the more delicate case where $R=1$ and $\operatorname{Re}\beta>-1$. Fix $\varepsilon>0$. We observe that if $k-j\ge -m\,(1-\varepsilon)$, then we have (5.3). Then identifying $j_{k+1}=j_1$,

$$\frac{1}{n^k}\sum_{\substack{1\le j_1,j_2,\ldots,j_k\le n\\ \text{all } j_{i+1}-j_i\ge -m(1-\varepsilon)}}^{n}\frac{a_{m+j_2-j_1}}{a_m}\frac{a_{m+j_3-j_2}}{a_m}\cdots\frac{a_{m+j_k-j_{k-1}}}{a_m}\frac{a_{m+j_1-j_k}}{a_m}$$
$$=\frac{1}{n^k}\sum_{\substack{1\le j_1,j_2,\ldots,j_k\le n\\ \text{all } j_{i+1}-j_i\ge -m(1-\varepsilon)}}^{n}\left(1+\frac{j_2-j_1}{m}\right)^{\beta}\left(1+\frac{j_3-j_2}{m}\right)^{\beta}\cdots \tag{5.6}$$
$$\left(1+\frac{j_1-j_k}{m}\right)^{\beta}(1+o\,(1))$$
$$=\frac{1}{n^k}\sum_{\substack{1\le j_1,j_2,\ldots,j_k\le n\\ \text{all } j_{i+1}-j_i\ge -m(1-\varepsilon)}}^{n}\left(1+\frac{j_2-j_1}{n}\right)^{\beta}\left(1+\frac{j_3-j_2}{n}\right)^{\beta}\cdots$$
$$\left(1+\frac{j_1-j_k}{n}\right)^{\beta}+o\,(1)$$
$$=\int\cdots\int_{\mathcal{S}}(1+x_2-x_1)^{\beta}(1+x_3-x_2)^{\beta}\cdots$$
$$(1+x_1-x_k)^{\beta}dx_1dx_2\ldots dx_k+o\,(1)$$

where $\mathcal{S}=\left\{(x_1,x_2,\ldots,x_k)\in[0,1]^k : x_{j+1}-x_j\ge -(1-\varepsilon)\text{ for each } j\right\}$. Here we identify $x_{k+1}=x_1$. To treat the remaining terms in the sum where at least one $j_{i+1} - j_i \leq -m(1-\varepsilon)$, we proceed as follows: necessarily $j_{i+1} \leq n - m + \varepsilon m \leq 2\varepsilon m$, for large enough n, while $1 \leq m + j_{i+1} - j_i \leq \varepsilon m$, so

$$\frac{1}{n} \sum_{j_i : j_{i+1} - j_i \leq -m(1-\varepsilon)} \left| \frac{a_{m+j_{i+1}-j_i}}{a_m} \right| \leq \frac{1+o(1)}{n} \frac{1}{|a_m|} \sum_{\ell=1}^{[\varepsilon m]} |a_\ell| \leq (1+o(1)) \phi(\varepsilon).$$

Then

$$\frac{1}{n^k} \sum_{\substack{1 \leq j_1, j_2, \ldots, j_k \leq n \\ \text{for some } i,\ j_{i+1} - j_i \geq -m(1-\varepsilon)}}^{n} \left| \frac{a_{m+j_2-j_1}}{a_m} \frac{a_{m+j_3-j_2}}{a_m} \cdots \frac{a_{m+j_k-j_{k-1}}}{a_m} \frac{a_{m+j_1-j_k}}{a_m} \right|$$
$$\leq C^{k-1} (1+o(1)) \phi(\varepsilon),$$

recall (5.4). We now combine this with (5.6) and then let $\varepsilon \to 0+$ to get the result. Also (3.4) follows from (3.1). ■

Proof of Corollary 3.2 Since $\{\nu_{mn}\}$ have support in a compact set independent of n and total mass bounded independent of n, we can choose weakly convergent subsequences with limit ω. (One can think of applying Helly's Theorem to the decomposition of μ_{mn} into first real and imaginary parts and then positive and negative parts of each of those.) All weak limits of subsequences have the same moments $\left\{d_{j+2}\right\}_{j\geq 0}$. We have that if f is a polynomial,

$$\lim_{n\to\infty} \int P(t)\, d\nu_{mn}(t) = \int P(t)\, d\omega(t).$$

Note that the same limit holds for the full sequence of integers because all weak limits ω have the same power moments. As such polynomials are dense in the class of functions analytic in any ball, the result follows. ■

Acknowledgments Research supported by NSF grant DMS1800251 and Georgia Tech Mathematics REU Program NSF Grant DMS181843.

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