Trivariate Interpolated Galerkin Finite Elements for the Poisson Equation

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Abstract When applying finite element method to the Poisson equation on a domain in $\mathbb{R}^3$, we replace some Lagrange nodal basis functions by bubble functions whose dual functionals are the values of the Laplacian. To compute the coefficients of these Laplacian basis functions instead of solving a large linear system, we interpolate the right hand side function in the Poisson equation. The finite element solution is then the Galerkin projection on a smaller vector space. We construct a qudratic and a cubic nonconforming interpolated finite elements, and quartic and higher degree conforming interpolated finite elements on arbitrary tetrahedral partitions. The main advantage of our method is that the number of unknowns that require solving a large system of equations on each element is reduced. We show that the interpolated Galerkin finite element method retains the optimal order of convergence. Numerical results confirming the theory are provided as well as comparisons with the standard finite elements.

Keywords Conforming finite element · Conforming finite element · Interpolated finite element · Tetrahedral grid · Poisson equation

1 Introduction

When solving partial differential equations using finite element method, the full space P_k of polynomials of degree $\leq k$ on each element is typically used in order to achieve the optimal order of approximation. Occasionally, the P_k polynomial space may be enriched by the so-called bubble functions. This is done for stability or continuity, while the order of approximation is not increased, cf. [2–4, 6–10, 15–

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18]. The only exception when a proper subspace of P_k is used while retaining the optimal order of convergence ($O(h^k)$ in the H^1-norm) can be found in [12, 13]. In these papers, we constructed a harmonic finite element method for solving the Laplace equation (1.1),

$$\begin{aligned} -\Delta u &= 0, \quad \text{in } \Omega, \\ u &= f, \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded polygonal domain in $\mathbb{R}^2$. In this method only harmonic polynomials are used in constructing the finite element space because the exact solution is harmonic. However, the harmonic finite element method of [12, 13] cannot be applied (directly) to the Poisson equation,

$$\begin{aligned} -\Delta u &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

where Ω is a bounded polyhedral domain in $\mathbb{R}^3$.

Let $\mathcal{T}_h$ be a tetrahedral grid of size h on a polyhedral domain Ω in $\mathbb{R}^3$. Let $\partial\mathcal{K} = \cup_{K\in\mathcal{T}_h}\partial K$. When using the standard Lagrange finite elements to solve (1.2), the solution is given by

$$u_h = \sum_{\mathbf{x}_i\in\partial\mathcal{K}\setminus\partial\Omega} u_i\phi_i + \sum_{\mathbf{x}_i\in\Omega\setminus\partial\mathcal{K}} u_j\phi_j + \sum_{\mathbf{x}_k\in\partial\Omega} c_k\phi_k, \tag{1.3}$$

where $\{\phi_i, \phi_j, \phi_k\}$ are the nodal basis functions at element-boundary, element-interior and domain boundary, respectively, c_k are interpolated values on the boundary, and both u_i and u_j are obtained from the Galerkin projection by solving of a linear system of equations.

In [14] an interpolated Galerkin finite element method is proposed for the 2D Poisson equation. In this paper, we extend this idea to the trivariate setting. The main idea can be described as follows. We add non-harmonic polynomial basis functions to the harmonic finite element solution of [12, 13] to obtain a solution to (1.2). That is, the solution is obtained as

$$u_h = \sum_{\mathbf{x}_i\in\partial\mathcal{K}\setminus\partial\Omega} u_i\phi_i + \sum_{\mathbf{x}_i\in\Omega\setminus\partial\mathcal{K}} c_j\phi_j + \sum_{\mathbf{x}_k\in\partial\Omega} c_k\phi_k, \tag{1.4}$$

where both c_j and c_k are interpolated values (of the right hand side function f, or of the boundary condition), and only u_i are obtained from the Galerkin projection. In these constructions, the linear system of Galerkin projection equations involves only the unknowns on $\partial\mathcal{K} \setminus \partial\Omega$. The number of unknowns on each element is reduced from $\binom{k+3}{3}$ to $2k^2+2$, i.e., from $O(k^3)$ to $O(k^2)$. Compared to the standard finite element, the new linear system is smaller (good for a direct solver) and has a better condition number (by numerical examples in this paper.)

This method is similar to, but different from, the standard Lagrange finite element method with static condensation. In the latter, internal degrees of freedom on each element remain unknowns and are represented by the element-boundary unknowns. For example, the Jacobi iterative solutions of condensed equations are identical to those of original equations with a proper unknown ordering (internal unknowns first). That is, the static condensation is a method for solving linear systems of equations arising from the high order finite element discretization, which does not define a different system. In the new method the coefficients of some degrees of freedom are no longer unknowns but given directly by the data. For ease of analysis we use a local integral of the right hand side function f to determine these coefficients. We can simply use the pointwise values of f instead. The new method is like the standard Lagrange finite element method when some "boundary values" are given on every element.

The paper is organized as follows. In Sects. 2 and 3, for arbitrary tetrahedral partitions, we construct a P_2 and a P_3 nonconforming interpolated Galerkin finite elements with one internal Laplacian basis function for each tetrahedron. In Sect. 4, for arbitrary tetrahedral partitions, we construct quartic and higher degree conforming interpolated Galerkin finite elements with $\binom{k-1}{3}$ internal Laplacian basis functions for each tetrahedron. In Sect. 5, we show that the interpolated Galerkin finite element solution converges at the optimal order. In Sect. 6, numerical tests are provided to compare the interpolated Galerkin finite elements (P_2 to P_6) with the standard ones.

2 The P_2 Nonconforming Interpolated Galerkin Finite Element

Let $\mathcal{T}_h$ be a quasi-uniform tetrahedral grid of size h on a polyhedral domain Ω in $\mathbb{R}^3$. On all interior tetrahedra, a P_2 nonconforming finite element function must have continuous moments of degree one. Let $K := [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4]$ be a tetrahedron in $\mathcal{T}_h$ with vertices v_i, and let $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ be the barycentric coordinates associated with K. That is, λ_i is a linear function on K assuming value 1 at $\mathbf{x}_i$, and vanishing on the face F_i opposite vertex $\mathbf{x}_i$. From [1, 5], we know that there is only one nonconforming quadratic bubble function per K:

$$\phi_0 = c_0(2 - 4\sum_{i=1}^{4} \lambda_i^2), \tag{2.1}$$

where the constant c_0 is determined by (2.2) below, satisfying three vanishing 1-moment conditions on every face F_i of K, and one 0-moment of Laplacian condition on K:

$$\int_{F_i} \phi_0 \lambda_j^\alpha \lambda_k^\beta \lambda_l^\gamma = 0, \quad i \neq j \neq k \neq l, \ \alpha + \beta + \gamma = 1,$$

$$\int_K \Delta\phi_0 \phi_0 \, d\mathbf{x} = 1. \tag{2.2}$$

For a P_2 element on K, there are ten domain points located at the vertices and mid-edges of K. Let $\{\psi_i\}_{i=1}^{10}$ be the Lagrange basis functions of a conforming P_2 element on K, i.e., each ψ_i assumes value 1 at one domain point and vanishes at the remaining nine. We define the interpolated Galerkin finite element basis as follows:

$$\phi_i = \psi_i - \phi_0 \int_K \Delta\psi_i \phi_0 d\mathbf{x},$$

$$\int_K \Delta\phi_i \, d\mathbf{x} = 0, \quad i = 1, \ldots, 10.$$

We define the P_2 nonconforming interpolated Galerkin finite element space by

$$\begin{aligned} V_h = \{v_h \mid v_h &\text{ has continuous 1-moments on face triangle,} \\ &v_h \text{ has vanishing 1-moments on boundary triangle,} \\ &v_h|_K = \sum_{i=1}^{10} c_i \phi_i + u_0 \phi_0 \ \text{ on each } K \in \mathcal{T}_h\}. \end{aligned} \tag{2.3}$$

The interpolated Galerkin finite element solution for the Poisson equation (1.2) is defined by

$$u_h = \sum_{K \in \mathcal{T}_h} \Big(\sum_{i=1}^{10} c_i \phi_i - \phi_0 \int_K f(\mathbf{x}) \phi_0 d\mathbf{x} \Big) \in V_h \tag{2.4}$$

such that

$$(\nabla_h u_h, \nabla_h v_h) = (f, v_h) \quad \forall v_h = \sum_{K \in \mathcal{T}_h} (\sum_{i=1}^{10} v_i \phi_i) \in V_h, \tag{2.5}$$

where ∇_h denotes a picewise defined gradient, and the dependency of ϕ_i on K is omitted for brevity of notation.

3 The P_3 Nonconforming Interpolated Galerkin Finite Element

Let K be a tetrahedron with the associated barycentric coordinates $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, as defined in Sect. 2. Then there is precisely one nonconforming cubic bubble function on K,

$$\phi_0 = c_0\Big(\sum_{i=1}^{4}(5\lambda_i^3 + 90\frac{\lambda_1\lambda_2\lambda_3\lambda_4}{\lambda_i}) - 3\Big), \tag{3.1}$$

satisfying vanishing 1-moment conditions on every face F_i of K, and one 0-moment of Laplacian condition on K:

$$\int_{F_i} \phi_0\lambda_j^\alpha\lambda_k^\beta\lambda_l^\gamma = 0, \quad i \neq j \neq k \neq l,\ \alpha+\beta+\gamma = 2,$$
$$\int_K \Delta\phi_0\phi_0\, d\mathbf{x} = 1.$$

For cubic finite elements, there are twenty domain points in each K, and twenty Lagrange basis functions $\{\psi_i\}_{i=1}^{20}$. We define the interpolated Galerkin finite element basis as follows

$$\phi_i = \psi_i - \phi_0 \int_K \Delta\psi_i\phi_0 d\mathbf{x}, \quad i = 1, .., 20.$$

The P_3 nonconforming interpolated Galerkin finite element space is defined by

$$\begin{aligned} V_h = \{v_h \mid\ & v_h \text{ has continuous 2-moments on face triangle,} \\ & v_h \text{ has vanishing 2-moments on boundary triangle,} \\ & v_h|_K = \sum_{i=1}^{20} c_i\phi_i + u_0\phi_0 \text{ on each } K \in \mathcal{T}_h\}. \end{aligned} \tag{3.2}$$

The P_3 interpolated Galerkin finite element solution for the Poisson equation (1.2) is defined by

$$u_h = \sum_{K\in\mathcal{T}_h}\Big(\sum_{i=1}^{20} c_i\phi_i - \phi_0 \int_K f(\mathbf{x})\phi_0 d\mathbf{x}\Big) \in V_h \tag{3.3}$$

such that

$$(\nabla_h u_h, \nabla_h v_h) = (f, v_h) \quad \forall v_h = \sum_{K \in \mathcal{T}_h} (\sum_{i=1}^{20} v_i \phi_i) \in V_h. \tag{3.4}$$

4 The P_k, $k \geq 4$, Conforming Interpolated Galerkin Finite Element

Let K be a tetrahedron with the associated barycentric coordinates $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, as defined in Sect. 2. For $k \geq 4$, there are $\binom{k-1}{3}$ domain points strictly interior to K. We shall refer to them as internal degrees of freedom. In this section, we define a P_k interpolated Galerkin conforming finite element on general tetrahedral grids, where the internal degrees of freedom are determined by interpolating the values of the function f on the right hand side of (1.2).

We first deal with $\binom{k+3}{3} - \binom{k-1}{3} = 2k^2 + 2$ domain points on the boundary of K:

$$\mathcal{D} := \Big\{ (i_1 \mathbf{x}_1 + i_2 \mathbf{x}_2 + i_3 \mathbf{x}_3 + i_4 + \mathbf{x}_4)/k \mid 0 \leq i_j \leq k, \sum_{j=1}^{4} i_j = k, \prod_{j=1}^{4} i_j = 0 \Big\}. \tag{4.1}$$

The first $(2k^2 + 2)$ linear functionals $F_l := u(\xi_l)$, $\xi_l \in \mathcal{D}$, $l = 1, \ldots, 2k^2 + 2$, (the dual basis of the finite element basis) are nodal values at these face Lagrange nodes. The remaining $\binom{k-1}{3}$ linear functionals are the weighted Laplacian $(k-4)$-moments corresponding to the strictly interior domain points. Let $\mathcal{B}$ be a basis for P_{k-4}, and let

$$\Big\{ F_j(\Delta u) = \int_K p_j \prod_{i=1}^{4} \lambda_i \Delta u \, d\mathbf{x} \mid p_j \in \mathcal{B}, \quad j = 2k^2 + 3, \ldots, \binom{k+3}{3} \Big\}. \tag{4.2}$$

Lemma 1 *The set of linear functionals in* (4.1) *and* (4.2) *uniquely determines a polynomial of degree* $\leq k$.

Proof We have a square linear system of equations. Thus, we only need to show the uniqueness of the solution. Let u_h have zero values for all these linear functionals. Therefore, u_h is identically zero on the boundary of K. Then

$$u_h = u_4 \prod_{i=1}^{4} \lambda_i \qquad \text{for some } u_4 \in P_{k-4}.$$

Letting $p = u_4$ in (4.2), we obtain

$$0 = \int_K u_4 \prod_{i=1}^{4} \lambda_i \Delta u \, d\mathbf{x} = -\int_K \nabla u_h \cdot \nabla u_h d\mathbf{x}$$

and, consequently, $\nabla u_h = 0$ on K. Thus, u_h is a constant on K. As $u_h = 0$ on ∂K, $u_h = 0$.

Let $\{\phi_i\}_{i=1}^{\dim P_k}$ be the basis of P_k dual to the set of linear functions defined by (4.1) and (4.2). In particular, the first $2k^2 + 2$ functions ϕ_i are dual to (4.1), and the remaining ones are dual to (4.2) Then, the P_k $(k \geq 4)$ interpolated Galerkin finite element space is defined as follows:

$$V_h = \{v_h \in H_0^1(\Omega) \ : v_h|_K = \sum_{i=1}^{2k^2+2} c_i \phi_i + \sum_{j=2k^2+3}^{\dim P_k} v_j \phi_j \ \text{ on each } K \in \mathcal{T}_h\}, \tag{4.3}$$

where each ϕ_i and ϕ_j depend on K. The interpolated Galerkin finite element solution for the Poisson equation (1.2) is defined by

$$u_h = \sum_{K \in \mathcal{T}_h} \Big(\sum_{i=1}^{2k^2+2} c_i \phi_i - \sum_{j=2k^2+3}^{\dim P_k} F_j(f) \phi_j \Big) \in V_h \tag{4.4}$$

such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h = \sum_{K \in \mathcal{T}_h} (\sum_{i=1}^{2k^2+2} v_i \phi_i) \in V_h. \tag{4.5}$$

5 Convergence Theory

We prove convergence for conforming and nonconforming interpolated Galerkin finite elements separately. The conforming case is considered first.

Theorem 1 *Let u and u_h be the exact solution of* (1.2) *and the finite element solution of* (4.5)*, respectively. Then*

$$\|u - u_h\|_1 \leq C h^k \|u\|_{k+1}, \tag{5.1}$$

where $\|\cdot\|_i$ is the standard Sobolev $H^i(\Omega)$ norm

Proof Testing (1.2) by $v_h = \sum_{K \in \mathcal{T}_h} \sum_{i=1}^{2k^2+2} v_i \phi_i \in H_0^1(\Omega)$, we have

$$(\nabla u, \nabla v_h) = (f, v_h). \tag{5.2}$$

Subtracting (4.5) from (5.2),

$$(\nabla(u - u_h), \nabla v_h) = 0. \tag{5.3}$$

On one element K, testing (1.2) by $v_h = \phi_j \in H_0^1(K)$ for $j > 2k^2 + 2$, using (4.2) we obtain

$$\begin{aligned}(\nabla(u - u_h), \nabla \phi_j) &= -\int_K \Delta u \phi_j d\mathbf{x} + \int_K \Delta u_h \phi_j d\mathbf{x} \\ &= \int_K f \phi_j d\mathbf{x} - F_j(f) = 0.\end{aligned} \tag{5.4}$$

Combining (5.3) and (5.4) implies

$$\begin{aligned}|u - u_h|_1^2 &= (\nabla(u - u_h), \nabla(u - I_h u)) \\ &\leq |u - u_h|_1 |u - I_h u|_1,\end{aligned}$$

where I_h is the interpolation operator to V_h. The following inequalities complete the proof:

$$\|u - u_h\|_1 \leq C|u - u_h|_1 \leq C|u - I_h u|_1 \leq Ch^k \|u\|_{k+1}.$$

Next we consider the two nonconforming cases.

Theorem 2 *Let u and u_h be the exact solution of* (1.2) *and either the finite element solution of* (2.5) *or of* (3.4)*, respectively. Then*

$$|u - u_h|_{1,h} \leq Ch^k \|u\|_{k+1}, \tag{5.5}$$

where $|\cdot|_{1,h}^2 = (\nabla_h \cdot, \nabla_h \cdot)$, $k = 2$ and 3 for (2.5) *and* (3.4)*, respectively, and $\|\cdot\|_{k+1}$ is the standard Sobolev $H^{k+1}(\Omega)$ norm.*

Proof We shall prove the case of $k = 2$. The proof of the cubic case is similar. Let $\tilde{u}_h = \sum_{K \in \mathcal{T}_h} \left(\sum_{i=1}^{10} \tilde{u}_i \phi_i + \tilde{u}_0 \phi_0 \right) \in V_h$ be the Galerkin finite element solution, i.e.,

$$(\nabla_h \tilde{u}_h, \nabla_h v_h) = (f, v_h) \quad \forall v_h \in V_h. \tag{5.6}$$

Testing (5.6) by $v_h = \phi_0$ on some $K \in \mathcal{T}_h$, we get

$$(\nabla_h \tilde{u}_h, \nabla_h \phi_0) = (f, \phi_0)_K = \int_K f(\mathbf{x}) \phi_0 \, d\mathbf{x},$$

$$(\nabla_h \tilde{u}_h, \nabla_h \phi_0) = -\int_K \Delta \tilde{u}_h \phi_0 \, d\mathbf{x}$$
$$= -\tilde{u}_0 \int_K \Delta \phi_0 \phi_0 \, d\mathbf{x} = -\tilde{u}_0,$$

where in the integration by parts, we use the fact $\nabla \tilde{u}_h \cdot \underline{\mathrm{n}}$ is a polynomial of a smaller (in fact, one less) degree on the boundary of K. That is, $u_h = \tilde{u}_h$, i.e., u_h satisfies (5.6).

Let $w_h \in V_h$. Then

$$|u - u_h|_{1,h} \leq |u - w_h|_{1,h} + |u_h - w_h|_{1,h} = |u - w_h|_{1,h}$$
$$+ \sup_{v_h \in V_h} \frac{(\nabla_h (u_h - w_h), \nabla_h v_h)}{|v_h|_{1,h}} \leq |u - w_h|_{1,h}$$
$$+ \sup_{v_h \in V_h} \frac{(\nabla_h (u - u_h), \nabla_h v_h)}{|v_h|_{1,h}} + \sup_{v_h \in V_h} \frac{(\nabla_h (u - w_h), \nabla_h v_h)}{|v_h|_{1,h}}$$
$$\leq 2|u - w_h|_{1,h} + \sup_{v_h \in V_h} \frac{(\nabla_h (u - u_h), \nabla_h v_h)}{|v_h|_{1,h}}.$$

The first term is the bounded by the interpolation error, i.e., the right hand side of (5.5). We estimate the second term. Let $[v_h]$ denote the jump on an (internal) triangle e of $\mathcal{T}_h$, after choosing an orientation for e. Then

$$(\nabla_h (u - u_h), \nabla_h v_h) = \sum_{K \in \mathcal{T}_h} \int_\partial K \frac{\partial u}{\partial \underline{\mathrm{n}}} v_h dS = \sum_{e \in \partial \mathcal{T}_h} \int_e \frac{\partial u}{\partial \underline{\mathrm{n}}} [v_h] dS$$
$$= \sum_{e \in \partial \mathcal{T}_h} \int_e \Big(\frac{\partial u}{\partial \underline{\mathrm{n}}} - \Pi_e \frac{\partial u}{\partial \underline{\mathrm{n}}} \Big) (v_h|_{e^+} - \Pi_e v_h|_{e^+} - v_h|_{e^-} + \Pi_e v_h|_{e^-}) dS$$
$$= \left(\sum_{e \in \partial \mathcal{T}_h} \int_e \Big(\frac{\partial u}{\partial \underline{\mathrm{n}}} - \Pi_e \frac{\partial u}{\partial \underline{\mathrm{n}}} \Big)^2 dS \right)^{1/2} \left(\sum_{e \in \partial \mathcal{T}_h} \int_{e^\pm} (v_h - \Pi_e v_h)^2 dS \right)^{1/2},$$

where Π_e is the L^2 projection onto the space of bivariate linear polynomials $P_1(e)$. By the trace inequality, we continue above estimation,

$$(\nabla_h (u - u_h), \nabla_h v_h) \leq C \left(\sum_{e \in \partial \mathcal{T}_h} \int_e \Big(\frac{\partial u}{\partial \underline{\mathrm{n}}} - \frac{\partial I_h u}{\partial \underline{\mathrm{n}}} \Big)^2 dS \right)^{1/2}$$
$$\cdot \left(\sum_{K \in \mathcal{T}_h} (\frac{1}{h} \| v_h - E_h v_h \|^2_{L^2(K)} + h \| \nabla (v_h - E_h v_h) \|^2_{L^2(K)}) \right)^{1/2}$$

$$\leq C\left(\sum_{K\in\mathcal{T}_h}(\frac{1}{h}\|\nabla(u-I_hu)\|^2_{L^2(K)}+h\|D^2(u-I_hu)\|^2_{L^2(K)})\right)^{1/2} h^{1/2}|v_h|_{1,h}$$

$$\leq C\left(\sum_{K\in\mathcal{T}_h}(h^{2k-1}|u|^2_{H^{k+1}(K)}\right)^{1/2} h^{1/2}|v_h|_{1,h}\leq Ch^k|u|_{k+1}|v_h|_{1,h},$$

where I_h is the standard interpolation operator to V_h, see [11], and $E_h v_h \in P_k(K)$ is a stable extension of (moments of) $\Pi_e v_h$ inside K. The proof is complete.

6 Numerical Tests

Let the domain of the boundary value problem (1.2) be $\Omega = [0, 1]^3$, and let $f(x) = 3\pi^2 \sin \pi x \sin \pi y \sin \pi z$. The exact solution is $u(x, y) = \sin \pi x \sin \pi y \sin \pi z$. In all numerical tests on P_k interpolated Galerkin finite element methods in this section, we choose a family of uniform grids shown in Fig. 1.

We solve problem (1.2) first by the P_2 interpolated Galerkin conforming finite element method defined in (2.3), and by the P_2 nonconforming finite element method, on same grids. The errors and the orders of convergence are listed in Table 1. We have one order of superconvergence for the interpolated Galerkin finite element method (2.5), in both H^1 semi-norm and L^2 norm. We note that the standard P_2 conforming finite element method has one order of superconvergence in both H^1 semi-norm and L^2 norm. But the nonconforming P_2 element has the optimal order of convergence only.

Next we solve the same problem by the interpolated Galerkin P_3 finite element method (3.4) and by the P_3 nonconforming finite element method. The errors and the orders of convergence are listed in Table 2. Both methods converge in the optimal order.

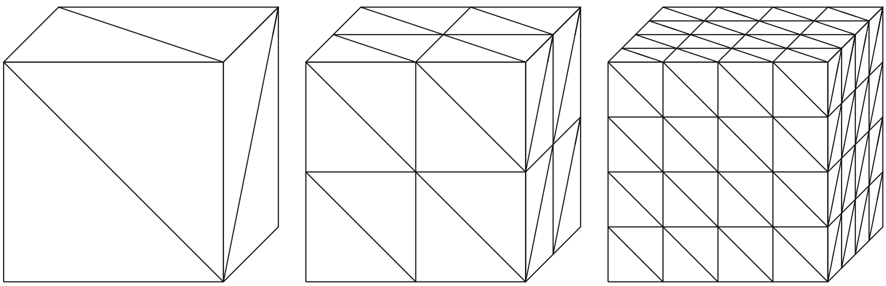

Fig. 1 The first three levels of grids

Table 1 The error $e_h = I_h u - u_h$ and the order of convergence, by the P_2 interpolated Galerkin finite element (2.3) and by the P_2 nonconforming finite element

	$\|e_h\|_0$	h^n	$\|e_h\|_1$	h^n	$\|e_h\|_0$	h^n	$\|e_h\|_1$	h^n
Grid	P_2 interpolated element				P_2 nonconforming element			
3	0.3298E−02	3.2	0.5367E−01	2.3	0.3769E−02	2.9	0.1079E+00	1.8
4	0.2538E−03	3.7	0.8380E−02	2.7	0.4439E−03	3.1	0.2751E−01	2.0
5	0.1704E−04	3.9	0.1150E−02	2.9	0.5315E−04	3.1	0.6861E−02	2.0
6	0.1089E−05	4.0	0.1493E−03	2.9	0.6542E−05	3.0	0.1712E−02	2.0
7	0.6859E−07	4.0	0.1898E−04	3.0	0.8141E−06	3.0	0.4276E−03	2.0

Table 2 The error $e_h = I_h u - u_h$ and the order of convergence, by the P_3 interpolated Galerkin finite element (3.2) and by the P_3 nonconforming finite element

	$\|e_h\|_0$	h^n	$\|e_h\|_1$	h^n	$\|e_h\|_0$	h^n	$\|e_h\|_1$	h^n
Grid	P_3 interpolated element				P_3 nonconforming element			
3	0.3943E−03	4.0	0.1453E−01	2.7	0.3957E−03	4.0	0.1464E−01	2.7
4	0.2320E−04	4.1	0.1995E−02	2.9	0.2341E−04	4.1	0.2004E−02	2.9
5	0.1423E−05	4.0	0.2589E−03	2.9	0.1439E−05	4.0	0.2598E−03	2.9
6	0.8854E−07	4.0	0.3283E−04	3.0	0.8964E−07	4.0	0.3293E−04	3.0

Table 3 Comparison of P_4 interpolated Galerkin and conforming Lagrange finite elements

	P_4 interpolated element				P_4 Lagrange element			
Grid	$\|e_h\|_0$	h^n	$\|e_h\|_1$	h^n	$\|e_h\|_0$	h^n	$\|e_h\|_1$	h^n
3	0.5624E−04	4.8	0.2450E−02	3.8	0.5577E−04	4.8	0.2471E−02	3.8
4	0.1794E−05	5.0	0.1587E−03	3.9	0.1789E−05	5.0	0.1592E−03	4.0
5	0.5592E−07	5.0	0.1000E−04	4.0	0.5587E−07	5.0	0.1002E−04	4.0
# unknowns	225471				250047			
# iterations	927				3050			
CPU	95.5				308.6			

Finally, we solve the problem by the interpolated Galerkin P_4, P_5, and P_6 finite element methods, (4.3) with $k = 4, 5, 6$, and by the P_4, P_5, and P_6 conforming finite element methods. The errors and the orders of convergence are listed in Tables 3, 4, and 5. The optimal order of convergence is achieved in every case. Also in the table, we list the number of unknowns, the number of conjugate iterations used in solving the resulting linear system of equations, and the computing time, on the last level computation. The number of unknowns for the P_6 element is only about 2/3 of that of the P_6 Lagrange element. The number of iterations for the P_6 interpolated element is less than 1/16 of that of the Lagrange element. The conditioning of the system of the new element is much better while giving also a slightly better solution. For the P_6 elements, the new method uses less than 1/10 of the computer time than that of the standard finite element.

Table 4 Comparison of P_5 interpolated Galerkin and conforming Lagrange finite elements

Grid	P_5 interpolated element				P_5 Lagrange element			
	$\|e_h\|_0$	h^n	$\|e_h\|_1$	h^n	$\|e_h\|_0$	h^n	$\|e_h\|_1$	h^n
2	0.3343E−03	5.7	0.9230E−02	4.7	0.3402E−03	5.7	0.9354E−02	4.7
3	0.5719E−05	5.9	0.3282E−03	4.8	0.5739E−05	5.9	0.3295E−03	4.8
4	0.8999E−07	6.0	0.1065E−04	4.9	0.8967E−07	6.0	0.1068E−04	4.9
# unknowns	47031				59319			
# iterations	877				7080			
CPU	31.1				267.0			

Table 5 Comparison of P_6 interpolated Galerkin and conforming Lagrange finite elements

Grid	P_6 interpolated element				P_6 Lagrange element			
	$\|e_h\|_0$	h^n	$\|e_h\|_1$	h^n	$\|e_h\|_0$	h^n	$\|e_h\|_1$	h^n
1	0.2234E−02	0.0	0.5041E−01		0.3098E−02	0.0	0.6026E−01	
2	0.5798E−04	5.3	0.2134E−02	4.6	0.5866E−04	5.7	0.2153E−02	4.8
3	0.5037E−06	6.8	0.3700E−04	5.8	0.5046E−06	6.9	0.3713E−04	5.9
# unknowns	8327				12167			
# iterations	876				14335			
CPU	18.0				181.5			

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