Nonsingular Rational Solutions to Integrable Models

Gegenhasi, Xing-Biao Hu, Shi-Hao Li, and Bao Wang

Abstract In the literature, there have been considerable interests in the study of nonsingular rational solutions for nonlinear integrable models. These nonsingular rational solutions have appeared with different names in a variety of nonlinear systems, say, algebraic solitons, algebraic solitrary waves and lump solutions etc. More importantly, these nonsingular rational solutions have played a key role in the study of rogue waves. In the paper, we will develop a new procedure to generate lump solutions via Bäcklund transformations and nonlinear superposition formulae for some integrable models. It is shown that our procedure can be utilized to some well-studied equations such as KPI equation, elliptic Toda equation and BKP equation, but also to comparatively less-studied DJKM equation, Novikov-Veselov equation and negative flow of the BKP equation.

Keywords Lump solution · Bäcklund transformation · Nonlinear superposition formulae

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1 Introduction

The theory of modern integrable systems originated from the work on the celebrated Korteweg-de Vries (KdV) equation. It is a prototype water wave model involving a broad variety of mathematical methods. This theory allows one to study a wide range of phenomena and problems arising from physics, biology, and pure and applied mathematics. The special significance of integrable systems is that they combine tractability with nonlinearity. Hence, these systems enable one to explore nonlinear phenomena while working with explicit solutions. One of the interesting explicit solutions in nonlinear dynamics is that of solitons. Kruskal and Zabusky first discovered solitons in the mid-1960s when they worked on the KdV equation. A soliton is essentially a localized object that may be found in diverse areas of physics, such as gravitation and field theory, plasma and solid state physics, and hydrodynamics. The importance of solitons stems from the exhibition of particle-type interactions and the characterization of the long time asymptotic behavior of the solution.

There are some other types of explicit solutions available in the literature. One of them is so-called rational solutions, which is important to be found for integrable equations. It provides us a criterion for integrability as the existence of an infinite sequence of rational solutions appears to be equivalent to the Painlevé property (Newell 1987), and the rational solutions are of, at least, potential value in physical applications. In this regard, of particularly interesting are an important class of what we called nonsingular rational solutions. To the best of our knowledge, the study of nonsingular rational solutions to integrable equations can be traced back to Ames (1967) where N.J. Zabusky found simplest nonsingular rational solution $u = -\frac{4q}{1+4q^2x^2}$ to the Gardner equation

$$u_t + 12quu_x + 6u^2u_x + u_{xxx} = 0.$$

In the literature, there are three types of nonsingular rational solutions: (1) Algebraic solitons; (2) Lump solutions; (3) Rogue wave solutions. There are some examples which exhibit nonsingular rational solutions. In the case of algebraic solitons, a typical example is the Benjamin-Ono (BO) equation

$$u_t + 4uu_x + Hu_{xx} = 0, \quad Hu(x,t) \equiv \frac{1}{\pi}\mathrm{PV}\int_{-\infty}^{\infty}\frac{u(y,t)}{y-x}dy. \tag{1}$$

In Ono (1975), Ono obtained 1-soliton solution $u = \frac{a}{a^2(x-at-x_0)^2+1}$. Some further results about the algebraic solitons of the BO equation could be found in Matsuno (1982a, b), Case (1979). The second example of algebraic solitons is the mKdV equation $v_t + 6v^2v_x + v_{xxx} = 0$, whose simplest algebraic solution was also given by Ono (1976) $v = v_0 - \frac{4v_0}{4v_0^2(x-6v_0^2t)^2+1}$. Furthermore, N-algebraic solitons were found in Ablowitz and Satsuma (1978). As for lump solutions, the result can be traced to Manakov et al. (1977) where Manakov et al. gave lump solutions to the KPI equation. In particular, in Ablowitz and Satsuma (1978); Satsuma and Ablowitz (1979), Ablowitz and Satsuma developed a new method to seek lump solutions to the KPI equation and DSI equation by taking the "long-wave" limit of the soliton solutions and there have been many results about this topic; please see Feng et al. (1999), Grammaticos et al. (2007), Ablowitz et al. (2000), Villarroel and Ablowitz (1999), Ma (2015), Villarroel and Ablowitz (1994), Gilson and Nimmo (1990), Hu and Willox (1996). The third line of research about nonsingular rational solutions is rogue wave solutions, which is of physical significance. As is known, the NLS equation $iu_t + u_{xx} + 2|u|^2u = 0$ admits the following rogue wave solution $u = \left(1 - \frac{4(1+4it)}{1+4x^2+16t^2}\right) e^{2it}$. Obviously, by taking $u \longrightarrow ue^{-2it}$, we may get a nonsingular rational solution of the equation

$$iu_t + u_{xx} + 2(|u|^2 - 1)u = 0.$$

For more examples, please see, e.g., Kharif et al. (2009), Solli et al. (2007), Peregrine (1983), Dubard et al. (2010), Dubard and Matveev (2011), Gaillard (2011), Guo et al. (2012), Ohta and Yang (2012), Li et al. (2013), Ohta and Yang (2012, 2013) and references therein.

The purpose of this paper is to develop a new procedure to generate lump solutions to several integrable models. Different from those by Ablowitz and Satsuma by taking the "long-wave" limit of the soliton solutions obtained and those by Ablowitz and Villarroel based on inverse scattering transform, the technique we develop here is via Bäcklund transformations and nonlinear superposition formulae in Hirota's bilinear formalism (Hirota and Satsuma 1978). We will apply our procedure to the some known examples such as KPI equation, two-dimensional Toda equation, BKP equation to show how it works and further to the DJKM equation, Novikov-Veselov equation and negative flow of BKP equation to show its effectiveness.

2 The Lump Solutions of KP Equation

The KP equation takes the form

$$(u_t + 6uu_x + u_{xxx})_x + \alpha u_{yy} = 0. \tag{2}$$

Traditionally, the Eq. (2) with $\alpha = -1$ is called KPI, and the one for $\alpha = 1$ is KPII. The KPI equation does not have stable soliton solutions but has localized solutions that decay algebraically as $x^2 + y^2 \to \infty$ and are called lumps. The lump solutions of KPI have been first obtained by Manakov et al. (1977) and also by Ablowitz and Satsuma (1978). Subsequently, Ablowitz and Satsuma derived the determinant form of the N-lump solution for the KPI equation by taking limits of the corresponding soliton solutions in Satsuma and Ablowitz (1979). In the following, we will use the bilinear Bäcklund transformation and the nonlinear superposition formula to rederive the N-lump solutions of KPI equation.

Through the dependent variable transformation $u = 2(\ln f)_{xx}$, the Eq. (2) can be written in bilinear form

$$(D_x D_t + D_x^4 + \alpha D_y^2) f \cdot f = 0, \tag{3}$$

where the bilinear operator $D_x^m D_t^k$ is defined by Hirota (2004)

$$D_x^m D_t^k a \cdot b \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k a(x,t)b(x',t')\bigg|_{x'=x,t'=t}.$$

A bilinear Bäcklund transformation for Eq. (3) is given by Nakamura (1981), Hu (1997)

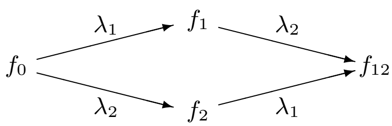
$$(aD_y + D_x^2 + \lambda D_x) f \cdot f' = 0, \tag{4}$$
$$(D_t + D_x^3 - 3a\lambda D_y - 3aD_x D_y) f \cdot f' = 0, \tag{5}$$

where $a^2 = \frac{1}{3}\alpha$ and λ is an arbitrary constant. We represent (4)–(5) symbolically by $f \xrightarrow{\lambda} f'$. The associated nonlinear superposition formula for the Eq. (3) is stated in the following proposition (Nakamura 1981; Hu 1997).

Proposition 1 *Let f_0 be a nonzero solution of (3) and suppose that f_1 and f_2 are solutions of (3) such that $f_0 \xrightarrow{\lambda_i} f_i$ $(i = 1, 2)$. Then f_{12} defined by*

$$f_0 f_{12} = c[D_x + \frac{1}{2}(\lambda_2 - \lambda_1)] f_1 \cdot f_2, \quad c \text{ is a nonzero real constant} \tag{6}$$

is a new solution to (3) which is related to f_1 and f_2 under bilinear BT (4)–(5) with parameters λ_2 and λ_1 respectively, i.e.



In Hu (1997), it has been shown if we choose $\theta_i = x + p_i y - \alpha p_i^2 t$, then the Bäcklund transformation tells us $1 \xrightarrow{\lambda_i = -ap_i} f_i = \theta_i + \beta_i$ (where β_i is a constant). By using proposition 1, we can obtain the following solution to the KP equation

$$f_{12} = \frac{2}{a(p_1 - p_2)}[f_{1x} f_2 - f_1 f_{2x} + \frac{1}{2}(\lambda_2 - \lambda_1) f_1 f_2] =$$
$$\theta_1\theta_2 + (\beta_1 + \frac{2}{a(p_1 - p_2)})\theta_2 + (\beta_2 - \frac{2}{a(p_1 - p_2)})\theta_1 + \beta_1\beta_2 + \frac{2(\beta_2 - \beta_1)}{a(p_1 - p_2)}, \tag{7}$$

by taking $c=\frac{2}{a(p_1-p_2)}$ in (6). If $\alpha=-1,\ p_2=p_1^*,\ \beta_1=-\frac{2}{a(p_1-p_2)},\ \beta_2=\frac{2}{a(p_1-p_2)}$ in (7), then we obtain the 1-lump solution

$$f_{12}=\theta_1\theta_1^*-\frac{12}{(p_1-p_1^*)^2}>0.$$

Furthermore, we can obtain an N-lump solution of the KP equation by using the nonlinear superposition formula repeatedly. For this purpose, we have the following proposition.

Proposition 2

$$F_N=c_N\begin{vmatrix} f_1 & f_2 & \cdots & f_N \\ (-\partial_x+\frac{\lambda_1}{2})f_1 & (-\partial_x+\frac{\lambda_2}{2})f_2 & \cdots & (-\partial_x+\frac{\lambda_N}{2})f_N \\ \vdots & \vdots & & \vdots \\ (-\partial_x+\frac{\lambda_1}{2})^{N-1}f_1 & (-\partial_x+\frac{\lambda_2}{2})^{N-1}f_2 & \cdots & (-\partial_x+\frac{\lambda_N}{2})^{N-1}f_N \end{vmatrix} \tag{8}$$

is a determinant solution to the KP equation (3), where $f_i(i=1,2,\ldots,N)$ is obtained from the seed solution f_0 by using Bäcklund transformation (4) *and* (5)*, i.e.* $f_0\xrightarrow{\lambda_i}f_i$.

In order to obtain the N-lump solution, we take $f_i=\theta_i+\beta_i,\theta_i=x+p_iy-\alpha p_i^2t,\lambda_i=-ap_i,\beta_i=\sum\limits_{j\neq i}\frac{2}{\lambda_i-\lambda_j}$ for $i=1,2,\ldots,N$ and $c_N=\prod\limits_{1\le i<j\le N}\frac{2}{\lambda_j-\lambda_i}$. In this case, from (8), we have

$$F_N=c_N\begin{vmatrix} \theta_1+\beta_1 & \cdots & \theta_N+\beta_N \\ -1+\frac{\lambda_1}{2}(\theta_1+\beta_1) & \cdots & -1+\frac{\lambda_N}{2}(\theta_N+\beta_N) \\ \vdots & & \vdots \\ (-N+1)(\frac{\lambda_1}{2})^{N-2}+(\frac{\lambda_1}{2})^{N-1}(\theta_1+\beta_1) & \cdots & (-N+1)(\frac{\lambda_N}{N})^{N-2}+(\frac{\lambda_N}{2})^{N-1}(\theta_N+\beta_N) \end{vmatrix}.$$

It can be verified that the above determinant can be written as the product of the determinants

$$\begin{vmatrix} 1 & \frac{\lambda_1}{2} & \cdots & (\frac{\lambda_1}{2})^{N-1} \\ 1 & \frac{\lambda_2}{2} & \cdots & (\frac{\lambda_2}{2})^{N-1} \\ \vdots & \vdots & & \vdots \\ 1 & \frac{\lambda_N}{2} & \cdots & (\frac{\lambda_N}{2})^{N-1} \end{vmatrix}\times\begin{vmatrix} \theta_1 & \frac{2}{\lambda_1-\lambda_2} & \cdots & \frac{2}{\lambda_1-\lambda_N} \\ -\frac{2}{\lambda_1-\lambda_2} & \theta_2 & \cdots & \frac{2}{\lambda_2-\lambda_N} \\ \vdots & \vdots & \vdots & \vdots \\ -\frac{2}{\lambda_1-\lambda_N} & -\frac{2}{\lambda_2-\lambda_N} & \cdots & \theta_N \end{vmatrix}.$$

By using the basic property of Vandermonde determinant, we know

$$F_N = \begin{vmatrix} \theta_1 & \frac{2}{\lambda_1-\lambda_2} & \cdots & \frac{2}{\lambda_1-\lambda_N} \\ -\frac{2}{\lambda_1-\lambda_2} & \theta_2 & \cdots & \frac{2}{\lambda_2-\lambda_N} \\ \vdots & \vdots & \vdots & \vdots \\ -\frac{2}{\lambda_1-\lambda_N} & -\frac{2}{\lambda_2-\lambda_N} & \cdots & \theta_N \end{vmatrix}. \tag{9}$$

If we choose $N = 2M,\ p_{M+i} = (p_i)^*(i = 1, 2, \ldots, M)$, then F_N gives the M-lump solutions of KPI equation which coincides with those obtained in Satsuma and Ablowitz (1979). The positivity of (9) could be found in Ohta and Yang (2013) for an affirmative answer.

3 The Lump Solutions of the DJKM Equation

The second equation of the KP hierarchy is the DJKM equation which is written as

$$w_{xxxxy} + 2w_{xxx}w_y + 4w_{xxy}w_x + 6w_{xy}w_{xx} - w_{yyy} - 2w_{xxt} = 0. \tag{10}$$

Through the dependent variable transformation $w = 2(\ln f)_x$, the Eq. (10) can be transformed into the multilinear form

$$D_x[(D_x^3 D_y - D_x D_t)f \cdot f] \cdot f^2 + \frac{1}{2}D_y[(D_x^4 - 3D_y^2)f \cdot f] \cdot f^2 = 0. \tag{11}$$

A bilinear Bäcklund transformation for Eq. (11) is given by

$$(D_x^2 + iD_y + \lambda + \mu D_x)f \cdot f' = 0, \tag{12a}$$

$$(iD_t + \frac{3}{2}\lambda D_y - \frac{1}{2}D_y^2 - \frac{i}{2}\mu^2 D_y - \frac{i}{2}\mu D_x D_y - \frac{i}{2}D_x^2 D_y)f \cdot f' = 0, \tag{12b}$$

where λ, μ are arbitrary constants. If we take $\lambda = 0$ for simplicity, then Bäcklund transformation (12a) and (12b) can be symbolically written as $f \xrightarrow{\mu} f'$. The associated nonlinear superposition formula for the Eq. (11) is stated in the following proposition.

Proposition 3 *Let f_0 be a nonzero solution of (11) and suppose that f_1 and f_2 are solutions of (11) such that $f_0 \xrightarrow{\mu_i} f_i$ $(i = 1, 2)$. Then f_{12} defined by*

$$f_0 f_{12} = c[D_x + \frac{1}{2}(\mu_2 - \mu_1)]f_1 \cdot f_2, \tag{13}$$

is a new solution to (11) which is related to f_1 and f_2 under bilinear BT with parameters μ_2 and μ_1 respectively. Here c is a nonzero real constant.

Similar with the KP case, we obtain the 1-lump solution to the DJKM equation by using the Bäcklund transformation and nonlinear superposition formula. By setting $\theta_i = x + p_i y - \frac{1}{2}p_i^3 t$, then from bilinear BT, one obtains $1 \overset{\mu_i=-ip_i}{\longrightarrow} f_i = \theta_i + \beta_i$ (where β_i is a constant). Now from the nonlinear superposition formula (13), we obtain the following solution of the DJKM equation

$$f_{12} = \frac{2}{\mu_2 - \mu_1}[f_{1x}f_2 - f_1 f_{2x} + \frac{1}{2}(\mu_2 - \mu_1)f_1 f_2] =$$
$$\theta_1\theta_2 + (\beta_1 + \frac{2}{\mu_2 - \mu_1})\theta_2 + (\beta_2 - \frac{2}{\mu_2 - \mu_1})\theta_1 + [\beta_1\beta_2 + \frac{2(\beta_2 - \beta_1)}{\mu_2 - \mu_1}] \quad (14)$$

by taking $c = \frac{2}{\mu_2 - \mu_1}$ in (13). If we choose $p_2 = p_1^*,\ \beta_1 = \frac{2}{\mu_1 - \mu_2},\ \beta_2 = \frac{2}{\mu_2 - \mu_1}$ in (14), then we obtain $\mu_2 = -\mu_1^*,\ \theta_2 = \theta_1^*$ and the 1-lump solution

$$f_{12} = \theta_1\theta_1^* + \frac{4}{(\mu_1 + \mu_1^*)^2} = |\theta_1|^2 + \frac{4}{(\mu_1 + \mu_1^*)^2} > 0. \quad (15)$$

The N-lump solution could be found by using the nonlinear superposition formula repeatedly.

Proposition 4

$$F_N = c_N \begin{vmatrix} f_1 & f_2 & \cdots & f_N \\ (-\partial_x + \frac{\mu_1}{2})f_1 & (-\partial_x + \frac{\mu_2}{2})f_2 & \cdots & (-\partial_x + \frac{\mu_N}{2})f_N \\ \vdots & \vdots & \vdots & \vdots \\ (-\partial_x + \frac{\mu_1}{2})^{N-1}f_1 & (-\partial_x + \frac{\mu_2}{2})^{N-1}f_2 & \cdots & (-\partial_x + \frac{\mu_N}{2})^{N-1}f_N \end{vmatrix}$$

is a determinant solution to the DJKM equation (11), where $f_i (i = 1, 2, \ldots, N)$ are obtained from seed solution f_0 by using BT (12a)–(12b) $f_0 \overset{\mu_i}{\longrightarrow} f_i$.

In order to obtain the multi-lump solution, we take $f_i = \theta_i + \beta_i,\ \theta_i = x + p_i y - \frac{1}{2}p_i^3 t,\ \mu_i = -ip_i,\ \beta_i = \sum_{j\neq i} \frac{2}{\mu_i - \mu_j}$ for $i = 1, 2, \ldots, N$. After the proper choices of parameters, the determinant F_N could be written as

$$F_N = c_N \begin{vmatrix} \theta_1 + \beta_1 & \cdots & \theta_N + \beta_N \\ \vdots & & \vdots \\ (-N+1)(\frac{\mu_1}{2})^{N-2} + (\frac{\mu_1}{2})^{N-1}(\theta_1 + \beta_1) & \cdots & (-N+1)(\frac{\mu_N}{N})^{N-2} + (\frac{\mu_N}{2})^{N-1}(\theta_N + \beta_N) \end{vmatrix}.$$

It can be verified that the above determinant is also a product of determinants

$$\begin{vmatrix} 1 & \frac{\mu_1}{2} & \cdots & (\frac{\mu_1}{2})^{N-1} \\ 1 & \frac{\mu_2}{2} & \cdots & (\frac{\mu_2}{2})^{N-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \frac{\mu_N}{2} & \cdots & (\frac{\mu_N}{2})^{N-1} \end{vmatrix} \times \begin{vmatrix} \theta_1 & \frac{2}{\mu_1 - \mu_2} & \cdots & \frac{2}{\mu_1 - \mu_N} \\ -\frac{2}{\mu_1 - \mu_2} & \theta_2 & \cdots & \frac{2}{\mu_2 - \mu_N} \\ \vdots & \vdots & \vdots & \vdots \\ -\frac{2}{\mu_1 - \mu_N} & -\frac{2}{\mu_2 - \mu_N} & \cdots & \theta_N \end{vmatrix}. \quad (16)$$

The choice of $c_N = \prod_{1\le i<j\le N} \frac{2}{\mu_j-\mu_i}$ gives us

$$F_N = \begin{vmatrix} \theta_1 & \frac{2}{\mu_1-\mu_2} & \cdots & \frac{2}{\mu_1-\mu_N} \\ -\frac{2}{\mu_1-\mu_2} & \theta_2 & \cdots & \frac{2}{\mu_2-\mu_N} \\ \vdots & \vdots & \vdots & \vdots \\ -\frac{2}{\mu_1-\mu_N} & -\frac{2}{\mu_2-\mu_N} & \cdots & \theta_N \end{vmatrix}. \tag{17}$$

For $N = 2M$ and $p_{M+i} = (p_i)^*$ $(i = 1, 2, \ldots, M)$, we could find that $\theta_{M+i} = (\theta_i)^*$, $\mu_{M+i} = -(\mu_i)^*$ and the positivity of F_N is the same with KP case. Therefore, in this case, F_N is the M-lump solution of the DJKM equation.

4 The Lump Solutions of the Elliptic Toda Equation

We now consider the so-called elliptic Toda equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\log u_n) = u_{n+1} - 2u_n + u_{n-1}.$$

This equation has been studied in Villarroel (1998); Villarroel and Ablowitz (1994), where the inverse scattering method was applied to obtain lump solutions. By the use of variable transformation $u_n = \frac{f_{n+1}f_{n-1}}{f_n^2}$, we can obtain the following bilinear form

$$(D_x^2 + D_y^2) f_n \cdot f_n = (2e^{D_n} - 2) f_n \cdot f_n \tag{18}$$

which admits a Bäcklund transformation as follows

$$(D_x + iD_y + \lambda^{-1}e^{-D_n} + \mu) f \cdot g = 0, \tag{19a}$$

$$((D_x - iD_y)e^{-\frac{1}{2}D_n} - \lambda e^{\frac{1}{2}D_n} + \gamma e^{-\frac{1}{2}D_n}) f \cdot g = 0. \tag{19b}$$

Furthermore, from the Bäcklund transformation, we may get the following superposition formula.

Proposition 5 *Let $f_0(n)$ be a nonzero solution of Eq. (18) and suppose that $f_1(n)$ and $f_2(n)$ are solutions of (18) such that $f_0(n) \xrightarrow{\lambda_i} f_i(n)$ $(i = 1, 2)$, then there exists the following nonlinear superposition formula*

$$e^{-\frac{1}{2}D_n} f_0 \cdot f_{12} = c(\lambda_1 e^{-\frac{1}{2}D_n} - \lambda_2 e^{\frac{1}{2}D_n}) f_1 \cdot f_2 \tag{20}$$

where f_{12} is a new solution of (18) related to f_1 and f_2 with parameters λ_2 and λ_1 respectively. Here c is a nonzero constant.

In order to get the lump solution, we choose $f_0 = 1$ and $f_i(i = 1, 2)$ as linear functions with respect to x, y and n, i.e. $1 \xrightarrow{\lambda_i} f_i = \theta_i + \beta_i = n + p_i x + q_i y + \beta_i$. Then from the Bäcklund transformation (19a) and (19b), we may get $\mu_i = -\lambda_i^{-1}$, $\gamma_i = \lambda_i$, $p_j = \frac{1}{2}(\lambda_j^{-1} + \lambda_j)$ and $q_j = \frac{1}{2i}(\lambda_j^{-1} - \lambda_j)$. Therefore, we get the seed function of the lump solutions

$$\theta_j = n + \frac{1}{2}(\lambda_j^{-1} + \lambda_j)x + \frac{1}{2i}(\lambda_j^{-1} - \lambda_j)y, \quad j = 1, 2.$$

Therefore the nonlinear superposition formula (20) becomes

$$f_{12}(n) = c(\lambda_1 f_1(n-1) f_2(n) - \lambda_2 f_1(n) f_2(n-1)). \tag{21}$$

In this case, if we take $c = \frac{1}{\lambda_1 - \lambda_2}$ and $f_i = \theta_i + \beta_i$, then (21) can be written as

$$\begin{aligned} f_{12}(n) = \theta_1\theta_2 + \frac{1}{\lambda_1 - \lambda_2}(\lambda_1\beta_2 - \lambda_2(\beta_2 - 1))\theta_1 \\ + \frac{1}{\lambda_1 - \lambda_2}(\lambda_1(\beta_1 - 1) - \lambda_2\beta_1)\theta_2 + \beta_1\beta_2 + \frac{1}{\lambda_1 - \lambda_2}(\lambda_2\beta_1 - \lambda_1\beta_2). \end{aligned} \tag{22}$$

Furthermore, if we take $\beta_1 = \frac{\lambda_1}{\lambda_1 - \lambda_2}$, $\beta_2 = -\frac{\lambda_2}{\lambda_1 - \lambda_2}$, then we have:

$$f_{12}(n) = \theta_1\theta_2 + A, \tag{23}$$

where $A = \frac{\lambda_1\lambda_2}{(\lambda_1 - \lambda_2)^2}$. Obviously, if we choose $\lambda_1 \neq \lambda_2$, then $\theta_1 = \theta_2^*$, $A > 0$, and therefore we get 1-lump solution of the elliptic Toda equation which is shown in Fig. 1.

Proposition 6 *The elliptic Toda equation admits the general nonlinear superposition formula*

$$e^{-\frac{1}{2}D_n} F_{N-1} \cdot F_{N+1} = c(\lambda_N e^{-\frac{1}{2}D_n} - \lambda_{N+1} e^{\frac{1}{2}D_n}) F_N \cdot \hat{F}_N, \tag{24}$$

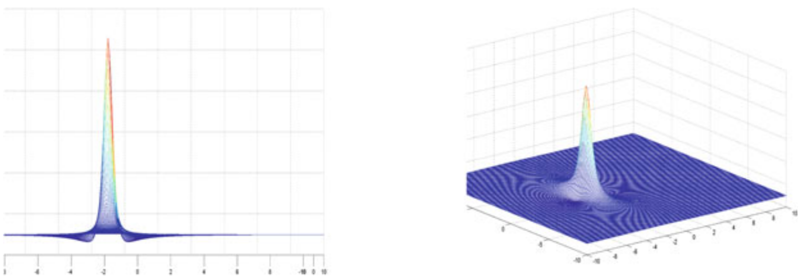


Fig. 1 1-lump solution of the elliptic Toda equation

where

$$F_N(n) = \begin{vmatrix} f_1(n) & \cdots & f_N(n) \\ \vdots & & \vdots \\ (-\lambda_1)^{N-1} f_1(n-N+1) & \cdots & (-\lambda_N)^{N-1} f_N(n-N+1) \end{vmatrix}$$
$$:= |1(n), \ldots, N(n)|.$$
$$\hat{F}_N(n) = |1(n), \ldots, N-1(n), N+1(n)|\,.$$

Here $\{f_j(n,x,y),\ j=1,2,\ldots,N+1\}$ *are the seed functions* $f_j(n,x,y) = n + \frac{1}{2}(\lambda_j^{-1}+\lambda_j)x + \frac{1}{2i}(\lambda_j^{-1}-\lambda_j)y + \beta_j$.

Proof It is noted that (24) can be alternatively written as:

$$F_{N-1}(n-1)F_{N+1}(n) = (\lambda_N F_N(n-1)\hat{F}_N(n) - \lambda_{N+1} F_N(n)\hat{F}_N(n-1)) \tag{25}$$

and

$$F_{N-1}(n-1) = |1(n-1), \ldots, N-1(n-1)| = \prod_{i=1}^{N-1}(-\lambda_i) D \begin{bmatrix} 1 \\ N \end{bmatrix}$$

where the determinant D means $F_N(n)$ and $D\begin{bmatrix} j \\ k \end{bmatrix}$ means the $(N-1)$-th minor of D whose j-th row and k-th column are deleted. By taking the explicit forms of F and $\hat{F}$ into the Eq. (25), we may see the nonlinear superposition formula is a Jacobi identity.

Inspired by the 1-lump solution, we now choose $f_j(n) = \theta_j(n) + \beta_j = n + \frac{1}{2}(\lambda_j^{-1}+\lambda_j)x + \frac{1}{2i}(\lambda_j^{-1}-\lambda_j)y + \beta_j$, and therefore the solution $F_N(n)$ can be written as

$$F_N(n) = \begin{vmatrix} \theta_1+\beta_1 & \cdots & \theta_N+\beta_N \\ -\lambda_1(\theta_1+\beta_1-1) & \cdots & -\lambda_N(\theta_N+\beta_N-1) \\ \vdots & & \vdots \\ (-\lambda_1)^{N-1}(\theta_1+\beta_1-N+1) & \cdots & (-\lambda_N)^{N-1}(\theta_N+\beta_N-N+1) \end{vmatrix},$$

from which we see that if and only if we take $\beta_i = \lambda_i \sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j}$, we can get F_{2M} without the odd term. On the other hand, from the determinant identity, we may get

$$F_N(n) = \begin{vmatrix} 1 & -\lambda_1 & \cdots & (-\lambda_1)^{N-1} \\ 1 & -\lambda_2 & \cdots & (-\lambda_2)^{N-1} \\ \vdots & \vdots & & \vdots \\ 1 & -\lambda_N & \cdots & (-\lambda_N)^{N-1} \end{vmatrix} \times \begin{vmatrix} \theta_1 & \frac{-\lambda_1}{\lambda_1-\lambda_2} & \cdots & \frac{-\lambda_1}{\lambda_1-\lambda_N} \\ \frac{-\lambda_2}{\lambda_2-\lambda_1} & \theta_2 & \cdots & \frac{-\lambda_2}{\lambda_2-\lambda_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-\lambda_N}{\lambda_N-\lambda_1} & \frac{-\lambda_N}{\lambda_N-\lambda_2} & \cdots & \theta_N \end{vmatrix}. \tag{26}$$

In this case, we have the following determinant solution

$$F_N(n) = \begin{vmatrix} \theta_1 & \frac{-\lambda_1}{\lambda_1-\lambda_2} & \cdots & \frac{-\lambda_1}{\lambda_1-\lambda_N} \\ \frac{-\lambda_2}{\lambda_2-\lambda_1} & \theta_2 & \cdots & \frac{-\lambda_2}{\lambda_2-\lambda_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-\lambda_N}{\lambda_N-\lambda_1} & \frac{-\lambda_N}{\lambda_N-\lambda_2} & \cdots & \theta_N \end{vmatrix}. \tag{27}$$

In the following, we want to construct lump solutions from (27). Here we just consider the case of $N = 4$, and set the parameters as $\lambda_3 = \frac{1}{\lambda_1^*}$, $\lambda_4 = \frac{1}{\lambda_2^*}$. In this case, we have

$$F_4 = \theta_1\theta_1^*\theta_2\theta_2^* + \frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_2)^2}\theta_1^*\theta_2^* + c.c + \frac{\lambda_1^*\lambda_2}{(\lambda_1^*\lambda_2-1)^2}\theta_1\theta_2^* + c.c$$
$$+ \frac{\lambda_1\lambda_1^*}{(\lambda_1\lambda_1^*-1)^2}\theta_2\theta_2^* + \frac{\lambda_2\lambda_2^*}{(\lambda_2\lambda_2^*-1)^2}\theta_1\theta_1^* + A,$$

where c.c means the complex conjugate and A is greater than zero. It means F_4 is 2-lump solution of the Toda equation and Fig. 2 shows 2-lump solution of the Toda equation.

In general, Villarroel has shown in Villarroel (1998) that the F_{2N} given by (27) is always greater than 0 if $\lambda_{N+i}\lambda_i^* = 1$ and $\{\lambda_i, 1 \leq i \leq 2N\}$ are off the unit circle.

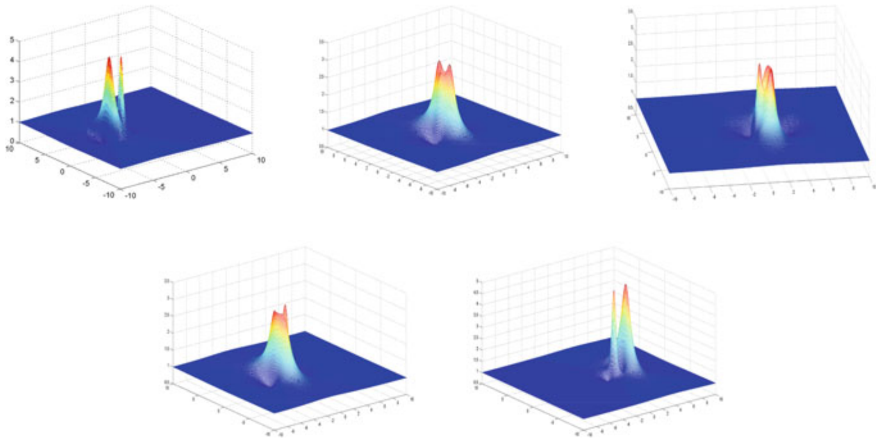


Fig. 2 The interaction of 2-lumps of the Toda equation

5 The Lump Solution of the BKP Equation

In Gilson and Nimmo (1990), the lump solution of the BKP equation has been considered by Claire Gilson and Jon Nimmo. In this part, we would like to show the Bäcklund transformation and nonlinear superposition formula can also provide us a Pfaffian form to the lump solution of BKP, which indicates this technique could also be used for the B_∞-type equations and Pfaffian forms.

Consider the BKP equation

$$(u_t + 15uu_{3x} + 15u_x^3 - 15u_xu_y + u_{5x})_x + 5u_{3x,y} - 5u_{yy} = 0.$$

Through the bilinear transformation $u = 2(\log f)_x$, we obtain the bilinear form for the BKP equation

$$(D_x^6 - 5D_x^3D_y - 5D_y^2 + 9D_xD_t)f \cdot f = 0, \tag{28}$$

whose Bäcklund transformation is indicated as follows (Hirota 2004)

$$(D_x^3 - D_y - 3kD_x^2 + 3k^2D_x)f \cdot g = 0, \tag{29a}$$

$$(-D_x^5 - 5D_x^2D_y + 5kD_x^4 + 5k^2D_x^3 - 10k^2D_y + 10kD_xD_y + 6D_t)f \cdot g = 0. \tag{29b}$$

Furthermore, we have the following nonlinear superposition formula.

Proposition 7 *Let f_0 be a nonzero solution of Eq. (28) and suppose that f_1 and f_2 are solutions such that $f_0 \xrightarrow{\lambda_i} f_i$ $(i = 1, 2)$, then there exists the following nonlinear superposition formula*

$$[D_x - (k_1 + k_2)]f_0 \cdot f_{12} = c[D_x + (k_1 - k_2)]f_1 \cdot f_2 \tag{30}$$

where f_{12} is a new solution related to f_1 and f_2 with parameters λ_2 and λ_1 respectively. Here c is a nonzero constant.

For the Bäcklund transformation (29a) and (29b), if we take $f_0 = 1$ and $f_i (i = 1, 2)$ as the linear functions, then $f_i = \theta_i + \beta_i = x + 3k_i^2y + 5k_i^4t + \beta_i$, $i = 1, 2$. In this case, the nonlinear superposition formula becomes

$$-\frac{d}{dx}f_{12} - (k_1 + k_2)f_{12} = f_2 - f_1 + (k_1 - k_2)f_1 \cdot f_2. \tag{31}$$

By solving this ordinary differential equation, we may obtain the solution of the BKP equation

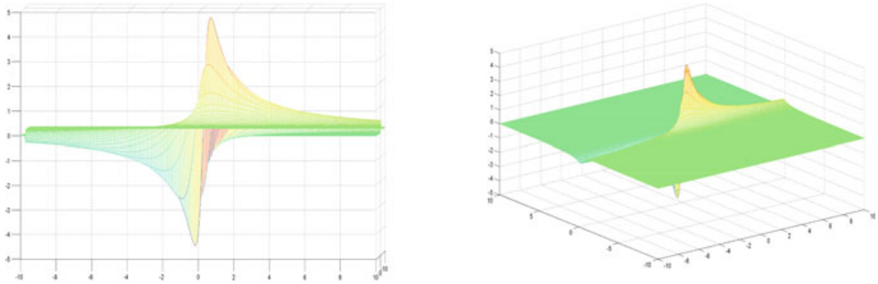


Fig. 3 The figure of 1-lump solution of the BKP equation

$$\begin{aligned} f_{12} &= \frac{k_2-k_1}{k_1+k_2} f_1 f_2 + \frac{2k_1}{(k_1+k_2)^2} f_1 - \frac{2k_2}{(k_1+k_2)^2} f_2 + 2\frac{k_2-k_1}{(k_1+k_2)^3} \\ &= \frac{k_2-k_1}{k_1+k_2}\theta_1\theta_2 + \frac{(k_2^2-k_1^2)\beta_2+2k_1}{(k_1+k_2)^2}\theta_1 + \frac{(k_2^2-k_1^2)\beta_1-2k_2}{(k_1+k_2)^2}\theta_2 + A, \end{aligned}$$

where $A = \frac{k_2-k_1}{k_1+k_2}\beta_1\beta_2 + \frac{\beta_1-\beta_2}{k_1+k_2} - \frac{k_2-k_1}{(k_1+k_2)^2}(\beta_1+\beta_2) + 2\frac{k_2-k_1}{(k_1+k_2)^3}$. It can be verified that if we take $\beta_1 = \frac{-2k_2}{k_1^2-k_2^2}$, $\beta_2 = \frac{2k_1}{k_1^2-k_2^2}$, $k_2 = k_1^*$ and $|Imk_1| > |Rek_1|$, then the 1-lump solution could be obtained.

Remark 1 Notice that the first order ordinary differential equation (31) may have a general solution, however, in the lump-solution case, we just consider the polynomial solution of f, hence this solution is unique in this sense.

In Fig. 3, the 1-lump solution of the BKP equation is drawn for a particular choice of the parameters.

Proposition 8 *BKP equation has a general nonlinear superposition formula as follows*

$$[D_x - (k_{2n+1}+k_{2n+2})]F_{2n}\cdot F_{2n+2} = [D_x + (k_{2n+1}-k_{2n+2})]\hat{F}_{2n+1}\cdot F_{2n+1}. \quad (32)$$

In particular, the solution F_{2n}, F_{2n+1} and $\hat{F}_{2n+1}$ have the Pfaffian forms

$$F_{2n} = (1,\ldots,2n),\ F_{2n+1} = (d_0,1,\ldots,2n+1),\ \hat{F}_{2n+1} = (d_0,1,\ldots,2n,2n+2), \quad (33)$$

in which the Pfaff element satisfies the following relationship

$$\begin{aligned} &(d_0,i) = f_i = \theta_i + \beta_i = x + 3k_i^2 y + 5k_i^4 t + \beta_i, \\ &(d_1,i) = \frac{d}{dx}(d_0,i) + k_i(d_0,i) = 1 + k_i f_i, \\ &\frac{d}{dx}(i,j) + (k_i+k_j)(i,j) = (d_0,d_1,i,j),\ (d_0,d_1) = 0. \end{aligned} \quad (34)$$

In order to prove the proposition, we need following lemmas.

Lemma 1 *Under the assumption of the Pfaff element (33), we have*

$$\frac{d}{dx}(1,\dots,2n)+(\sum_{i=1}^{2n}k_i)(1,\dots,2n)=(d_0,d_1,1,\dots,2n). \tag{35}$$

Proof We will prove this conclusion by induction. For n=1, it is just the assumption we set in (33). By assumption, it is known that

$$\frac{d}{dx}(2,\dots,\hat{j},\dots,2n+2)+\sum_{i=2,i\neq j}^{2n+2}k_i(2,\dots,\hat{j},\dots,2n+2)=(d_0,d_1,2,\dots,\hat{j},\dots,2n+2)$$

holds for Pfaffian of order n. Then for Pfaffian of order $n+1$, we have

$$\begin{aligned}
&(d_0,d_1,1,\dots,2n+2)\\
&=\sum_{j=2}^{2n+2}(-1)^j[(d_0,d_1,1,j)(2,\dots,\hat{j},\dots,2n+2)+(1,j)(d_0,d_1,2,\dots,\hat{j},\dots,2n+2)]\\
&=\sum_{j=2}^{2n+2}(-1)^j\{[\frac{d}{dx}(1,j)+(k_1+k_j)(1,j)](2,\dots,\hat{j},\dots,2n+2)\\
&\qquad+(1,j)[\frac{d}{dx}(2,\dots,\hat{j},\dots,2n+2)+\sum_{i\neq j}k_i(2,\dots,\hat{j},\dots,2n+2)]\}\\
&=\frac{d}{dx}(1,\dots,2n+2)+\sum_{i=1}^{2n+2}k_i(1,\dots,2n+2),
\end{aligned}$$

which completes the proof.

Lemma 2 *Under the assumption of the Pfaffian element (33), we also have*

$$(d_1,1,\dots,2n+1)=\frac{d}{dx}(d_0,1,\dots,2n+1)+(\sum_{i=1}^{2n+1}k_i)(d_0,1,\dots,2n+1), \tag{36a}$$

$$(d_1,1,\dots,2n,2n+2)=\frac{d}{dx}(d_0,1,\dots,2n,2n+2)+(\sum_{i\neq 2n+1}k_i)(d_0,1,\dots,2n,2n+2). \tag{36b}$$

Proof We just prove the first equation, and the second one can be verified in a similar way. By expansion of Pfaffian, one has

$$
\begin{aligned}
&(d_1, 1, \ldots, 2n+1) \\
&= \sum_{j=1}^{2n+1} (-1)^j (d_1, j)(1, \ldots, \hat{j}, 2n+1) \\
&= \sum_{j=1}^{2n+1} (-1)^{j-1} [\frac{d}{dx}(d_0, j) + k_j (d_0, j)](1, \ldots, \hat{j}, \ldots, 2n+1) \\
&= \frac{d}{dx}(d_0, 1, \ldots, 2n+1) + \sum_{j=1}^{2n+1} (-1)^{j-1} k_j (d_0, j)(1, \ldots, \hat{j}, \ldots, 2n+1) \\
&- \sum_{j=1}^{2n+1} (-1)^{j-1} (d_0, j)[(d_0, d_1, 1, \ldots, \hat{j}, \ldots, 2n+1) - (\sum_{i=1, i\neq j}^{2n+1} k_i)(1, \ldots, \hat{j}, \ldots, 2n+1)] \\
&= \frac{d}{dx}(d_0, 1, \ldots, 2n+1) + (\sum_{i=1}^{2n+1} k_i)(d_0, \ldots, 2n+1),
\end{aligned}
$$

and the equation is verified.

The Lemma 1 tells us the left side of the nonlinear superposition formula can be written as

$$(d_0, d_1, 1, \ldots, 2n)(1, \ldots, 2n+2) - (1, \ldots, 2n)(d_0, d_1, 1, \ldots, 2n+2), \tag{37}$$

while the Lemma 2 shows the right side can be written as

$$-(d_1, 1, \ldots, 2n+1)(d_0, 1, \ldots, 2n, 2n+2) + (d_0, 1, \ldots, 2n+1)(d_1, 1, \ldots, 2n, 2n+2). \tag{38}$$

Therefore, under these two lemmas, we find that the nonlinear superposition formula of BKP equation (30) can be written as

$$
\begin{aligned}
&(d_0, d_1, 1, \ldots, 2n)(1, \ldots, 2n+2) - (1, \ldots, 2n)(d_0, d_1, 1, \ldots, 2n+2) = \\
&\quad -(d_1, 1, \ldots, 2n+1)(d_0, 1, \ldots, 2n, 2n+2) + (d_0, 1, \ldots, 2n+1)(d_1, 1, \ldots, 2n, 2n+2),
\end{aligned}
$$

which is the Pfaffian identity (Hirota 2004).

And then we would like to prove the F_{2n} given in (33) is always positive or always negative under some constrains. Following the method mentioned in Gilson and Nimmo (1990), we first consider the determinant of $2n \times 2n$ skew-symmetric matrix $A = (a_{i,j})_{1\le i,j\le 2n}$ which can be represented as the square of Pfaffian given in (33):

$$F_{2n}^2 = (1, 2, \ldots, 2n)^2 = \det A. \tag{39}$$

Applying Eqs. (33) and (39), we can derive:

$$a_{i,j} = \frac{k_i - k_j}{k_i + k_j}[(f_i + \frac{2k_j}{k_i^2 - k_j^2})(f_j - \frac{2k_i}{k_i^2 - k_j^2}) + \frac{2(k_i^2 + k_j^2)}{(k_i^2 - k_j^2)^2}]. \tag{40}$$

If we set $k_i = k^*_{n+i}$, $\beta_i = \beta^*_{n+i}$ and $|\mathrm{Im}k_i| > |\mathrm{Re}k_i|$, then the determinant of A can be written as the following form:

$$\det A = \det \begin{vmatrix} \mathrm{C} & \mathrm{B} \\ -\mathrm{B}^* & \mathrm{C}^* \end{vmatrix}, \tag{41}$$

which is always positive. In Eq. (41), $B = (b_{i,j})_{1\le i,j\le n}$, $C = (c_{i,j})_{1\le i,j\le n}$ are two $n \times n$ matrices, whose element $b_{i,j}$, $c_{i,j}$ are given by:

$$b_{i,j} = \frac{k_i - k_j^\star}{k_i + k_j^\star}[(f_i + \frac{2k_j^\star}{k_i^2 - k_j^{\star 2}})(f_j^\star - \frac{2k_i}{k_i^2 - k_j^{\star 2}}) + \frac{2(k_i^2 + k_j^{\star 2})}{(k_i^2 - k_j^{\star 2})^2}],$$
$$c_{i,j} = \frac{k_i - k_j}{k_i + k_j}[(f_i + \frac{2k_j}{k_i^2 - k_j^2})(f_j - \frac{2k_i}{k_i^2 - k_j^2}) + \frac{2(k_i^2 + k_j^2)}{(k_i^2 - k_j^2)^2}].$$

Since $F_{2n}^2 > 0$ by taking $k_i = k^*_{n+i}$, $\beta_i = \beta^*_{n+i}$ and $|\mathrm{Im}k_i| > |\mathrm{Re}k_i|$ in F_{2n} , and the lump solution is a continuous function, so the F_{2n} is always positive or always negative. Therefore, the solution F_{2n} with $k_i = k^*_{n+i}$, $\beta_i = \beta^*_{n+i}$ and $|\mathrm{Im}k_i| > |\mathrm{Re}k_i|$ is the nonsingular rational solution of the BKP equation.

6 The Lump Solutions of the Novikov-Veselov Equation

In this part, we want to discuss the lump solution of the Novikov-Veselov equation

$$2u_t + u_{xxx} + u_{yyy} + 3(u\partial_y^{-1}u_x)_x + 3(u\partial_x^{-1}u_y)_y = 0, \tag{42}$$

which can be viewed as an extension the KdV equation in two spatial dimensions and one temporal dimension. Bäcklund Transformation and nonlinear superposition formula and 1,2-lump solutions have been studied in Hu and Willox (1996). Here we revisited some important facts.

Under the dependent variable transformation $u = u_0 + 2(\log f)_{xy}$ with u_0 a constant, the Eq. (42) can be transformed into the multilinear form and enjoys the following Bäcklund transformation

$$(D_xD_y - \mu D_x - \lambda D_y + \lambda\mu + u_0)f \cdot f' = 0, \tag{43a}$$
$$(2D_t + D_x^3 + D_y^3 + 3\lambda^2 D_x - 3\lambda D_x^2 + 3\mu^2 D_y - 3\mu D_y^2)f \cdot f' = 0, \tag{43b}$$

where λ and μ are arbitrary constants. The nonlinear superposition formula can be stated as follows.

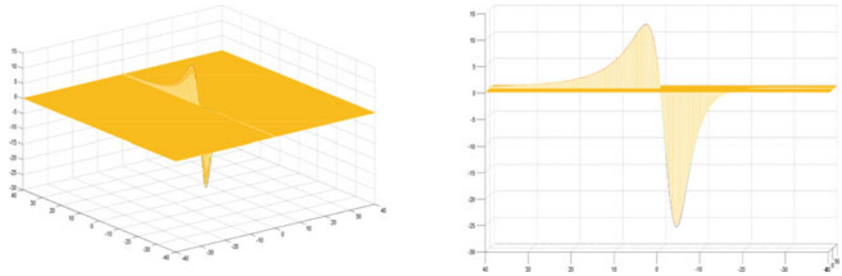


Fig. 4 1-lump solution of the Novikov-Veselov equation

Proposition 9 *Let* f_0 *be a nonzero solution of (42) and suppose that* f_1 *and* f_2 *are solutions of (42) such that* $f_0 \xrightarrow{\mu_i} f_i$ $(i = 1, 2)$. *Then* f_{12} *defined by*

$$[D_x - (k_1 + k_2)]f_0 \cdot f_{12} = c[D_x + (k_1 - k_2)]f_1 \cdot f_2 \tag{44}$$

is a new solution to (42) which is related to f_1 *and* f_2 *under bilinear BT (43a) and (43b) with parameters* k_2 *and* k_1 *respectively. Here c is a nonzero real constant.*

To obtain the lump solutions, we have to take $f_0 = 1$ and $f_i = \theta_i + \beta_i = k_i^2 x + u_0 y - \frac{3}{2(k_i^4 + u_0^3/k_i^2)} t + \beta_i$. For 1-lump solution, if we set $k_2 = k_1^*,\ \beta_1 = \beta_2^*$ and $\mathrm{Im}k_i > \mathrm{Re}k_i,\ (i = 1, 2)$, then

$$f_{12} = (\theta_1 + \frac{2k_1^2 k_2}{k_1^2 - k_2^2}) \times c.c. + 2\frac{k_1^2 k_2^2 (k_1^2 + k_2^2)}{(k_1^2 - k_2^2)^2},$$

where c.c. means the complex conjugate. Obviously, f_{12} is positive and it is a 1-lump solution. We depict the 1-lump solution of the Novikov-Veselov equation in Fig. 4.

Noticing that the nonlinear superposition formula of the Novikov-Veselov equation (44) is the same as the BKP equation (30), the Novikov-Veselov equation (44) possesses the same structure of solution as the BKP equation except the seed function. Hence we have the following proposition.

Proposition 10 *Novikov-Veselov equation owns a general nonlinear superposition formula*

$$[D_x - (k_{2n+1} + k_{2n+2})]F_{2n} \cdot F_{2n+2} = [D_x + (k_{2n+1} - k_{2n+2})]\hat{F}_{2n+1} \cdot F_{2n+1}, \tag{45}$$

where

$$F_{2n} = (1, \ldots, 2n),\ F_{2n+1} = (d_0, 1, \ldots, 2n+1),\ \hat{F}_{2n+1} = (d_0, 1, \ldots, 2n, 2n+2), \tag{46}$$

where the Pfaffian elements satisfy the following relationships

$$
\begin{aligned}
(d_0, i) &= f_i = \theta_i + \beta_i = k_i^2 x + u_0 y - \frac{3}{2(k_i^4 + u_0^3/k_i^2)} t + \beta_i, \\
(d_1, i) &= \frac{d}{dx}(d_0, i) + k_i (d_0, i), \\
\frac{d}{dx}(i, j) &+ (k_i + k_j)(i, j) = (d_0, d_1, i, j),\ (d_0, d_1) = 0.
\end{aligned} \tag{47}
$$

Since the proof of this proposition is similar to that of BKP equation, we omit it here. If we set $k_i = k_{n+i}^*$, $\beta_i = \beta_{n+i}^*$ and $|\mathrm{Im}k_i| > |\mathrm{Re}k_i|$, then we can show in a similar way in Sect. 5 that F_{2n} is always positive or always negative. Therefore, we get the N-lump solution of the Novikov-Veselov equation, which has the representation of (46) with $k_i = k_{n+i}^*$, $\beta_i = \beta_{n+i}^*$ and $|\mathrm{Im}k_i| > |\mathrm{Re}k_i|$.

7 The Lump Solutions for Negative Flow of BKP Equation

In Hirota (2004), Sect. 3.3, the author proposed another shallow wave equation, called the negative flow of BKP equation

$$
u_{yt} - u_{xxxy} - 3(u_x u_y)_x + 3u_{xx} = 0. \tag{48}
$$

By the dependent variable transformation $u = 2(\log f)_x$, it can be transformed into a bilinear form

$$
[(D_t - D_x^3)D_y + 3D_x^2] f \cdot f = 0,
$$

which possesses the following Bäcklund transformation

$$
(D_x D_y + \lambda^{-1} D_x + \lambda D_y) f \cdot f' = 0, \tag{49a}
$$

$$
(D_x^3 + 3\lambda D_x^2 + 3\lambda^2 D_x - D_t) f \cdot f' = 0. \tag{49b}
$$

Furthermore, we have the following result.

Proposition 11 *Let f_0 be a nonzero solution of Eq. (48) and suppose that f_1 and f_2 are solutions of (48) such that $f_0 \xrightarrow{\lambda_i} f_i$ $(i = 1, 2)$, then there exists a following nonlinear superposition formula*

$$
(D_x + (k_1 - k_2)) f_0 \cdot f_{12} = c(D_x - (k_1 - k_2)) f_1 \cdot f_2, \tag{50}
$$

where f_{12} is a new solution of (48) related to f_1 and f_2 under bilinear BT (49a) and (49b) with parameters k_2 and k_1 respectively. Here c is a nonzero constant.

A 1-lump solution of the negative flow for BKP equation is derived in the following. Starting with $f_0 = 1$, $f_i = x - k_i^2 y + 3k_i^2 t + \beta_i (i = 1, 2)$, we may obtain the following solution

$$f_{12} = \frac{k_2 - k_1}{k_1 + k_2} f_1 f_2 + \frac{2k_1}{(k_1 + k_2)^2} f_1 - \frac{2k_2}{(k_1 + k_2)^2} f_2 + 2\frac{k_2 - k_1}{(k_1 + k_2)^3}$$
$$= \frac{k_2 - k_1}{k_1 + k_2}\theta_1\theta_2 + \frac{(k_2^2 - k_1^2)\beta_2 + 2k_1}{(k_1 + k_2)^2}\theta_1 + \frac{(k_2^2 - k_1^2)\beta_1 - 2k_2}{(k_1 + k_2)^2}\theta_2 + A,$$

where $A = \frac{k_2-k_1}{k_1+k_2}\beta_1\beta_2 + \frac{\beta_1-\beta_2}{k_1+k_2} - \frac{k_2-k_1}{(k_1+k_2)^2}(\beta_1 + \beta_2) + 2\frac{k_2-k_1}{(k_1+k_2)^3}$. If we take $\beta_1 = \frac{-2k_2}{k_1^2-k_2^2}$, $\beta_2 = \frac{2k_1}{k_1^2-k_2^2}$, $k_2 = k_1^*$ and $|\mathrm{Im}k_1| > |\mathrm{Re}k_1|$, we get the 1-lump solution.

In order to obtain N-lump solutions, we need to establish a general nonlinear superposition formula for the negative flow BKP equation.

Proposition 12 *The negative flow BKP equation owns a general nonlinear superposition formula*

$$[D_x - (k_{2n+1} + k_{2n+2})]F_{2n} \cdot F_{2n+2} = [D_x + (k_{2n+1} - k_{2n+2})]\hat{F}_{2n+1} \cdot F_{2n+1} \quad (51)$$

and the solutions F_{2n}, F_{2n+1} and $\hat{F}_{2n+1}$ are expressed as Pfaffians

$$F_{2n} = (1, \ldots, 2n),\ F_{2n+1} = (d_0, 1, \ldots, 2n+1),\ \hat{F}_{2n+1} = (d_0, 1, \ldots, 2n, 2n+2), \quad (52)$$

where the Pfaff elements satisfy the following relations

$$(d_0, i) = f_i = \theta_i + \beta_i = x - k_i^2 y + 3k_i^2 t + \beta_i,$$
$$(d_1, i) = \frac{d}{dx}(d_0, i) + k_i(d_0, i),$$
$$\frac{d}{dx}(i, j) + (k_i + k_j)(i, j) = (d_0, d_1, i, j),\ (d_0, d_1) = 0. \quad (53)$$

The proof of Proposition 12 is similar to the case of BKP equation, so we omit it here. Furthermore, we can show in a similar way in Sect. 5 that F_{2n} in (52) with $k_i = k_{n+i}^*$, $\beta_i = \beta_{n+i}^*$ and $|\mathrm{Im}k_i| > |\mathrm{Re}k_i|$ gives the N-lump solution of the negative flow of BKP equation.

8 Conclusion

It is truly remarkable that the lump solutions of several integrable models could be obtained by Bäcklund transformations and nonlinear superposition formulae and the effectiveness presents itself in this paper. It is natural to expect that this technique can be applied to more equations in AKP and BKP type, also for CKP and DKP type equations. The lack of the bilinear Bäcklund transformation of CKP equation brings us essential difficulty to construct the nonlinear superposition formula, as well as the lump solution. In particular, we also expect to develop the similar technique to generate the lump solutions for the discrete integrable lattices.

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