Complexity and Integrability in 4D Bi-rational Maps with Two Invariants

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Abstract In this letter we give fourth-order autonomous recurrence relations with two invariants, whose degree growth is cubic or exponential. These examples contradict the common belief that maps with sufficiently many invariants can have at most quadratic growth. Cubic growth may reflect the existence of non-elliptic fibrations of invariants, whereas we conjecture that the exponentially growing cases lack the necessary conditions for the applicability of the discrete Liouville theorem.

Keywords Integrability · Complexity · Algebraic entropy · Growth of bi-rational maps · Higher-order difference equation

1 Introduction

Bi-rational maps in two dimensions have played a crucial role in the study of integrable discrete dynamical systems since the seminal paper of Penrose and Smith (1981) and the introduction of the QRT mappings in Quispel et al. (1988, 1989). Elliptic curves and rational elliptic surfaces proved to be one of the main tools in understanding the geometry behind this kind of integrability, see Sakai (2001), Duistermaat (2011), Tsuda (2004). In this letter we give examples of higher-order maps

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whose properties go beyond those of the two-dimensional maps, and show that the geometry of elliptic fibrations is no longer sufficient to explain their behaviour.

Up to now the QRT mappings appear to describe almost the totality of the known integrable examples in dimension two with some notable exceptions (Viallet et al. 2004; Duistermaat 2011). However, no general framework exists for higher order maps. A generalization of the QRT scheme (Quispel et al. 1988, 1989) in dimension four was given in Capel and Sahadevan (2001). Certain maps obtained in Capel and Sahadevan (2001) were shown in Hay (2007) to be autonomous reductions of members of q-Painlevé hierarchies (*multiplicative equations* in Sakai's scheme Sakai 2001). Since hierarchies are known also for the *additive* discrete Painlevé equations (Cresswell and Joshi 1999), it is clear that the cases considered in Capel and Sahadevan (2001) cannot exhaust all the possible integrable autonomous maps in four dimensions, as already shown in Joshi and Viallet (2018). It is important to mention that there are also other examples of discrete mappings of higher orders produced either by periodic or symmetry reductions of integrable partial difference equations (Papageorgiou et al. 1990; Quispel et al. 1991; van der Kamp and Quispel 2010; Levi and Winternitz 2006) or as Kahan-Hirota-Kimura discretization (Kahan 1993; Kimura and Hirota 2000) of continuous integrable systems (Petrera and Suris 2010; Celledoni et al. 2013, 2014; Petrera et al. 2009).

In this letter, we focus on the study of integrability properties of autonomous recurrence relations. Here an autonomous recurrence relation is given by a bi-rational map of the complex projective space into itself:

$$\varphi\colon [\mathbf{x}] \in \mathbb{CP}^n \to [\mathbf{x}'] \in \mathbb{CP}^n, \tag{1}$$

where $n > 1$.[1] We take $[\mathbf{x}] = \left[x_1 : x_2 : \cdots : x_{n+1}\right]$ and $[\mathbf{x}'] = \left[x_1' : x_2' : \cdots : x_{n+1}'\right]$ to be homogeneous coordinates on $\mathbb{CP}^n$. Moreover we recall that a bi-rational map is a rational map $\varphi\colon V \to W$ of algebraic varieties V and W such that there exists a map $\psi\colon W \to V$, which is the *inverse* of φ in the dense subset where both maps are defined (Shafarevich 1994).

Integrability for autonomous recurrence relations (discrete equations) can be characterized in different ways. In the continuous case, for finite dimensional systems, integrability is usually understood as the existence of a "sufficiently" high number of *first integrals*, i.e. of *non-trivial* functions constant along the solution of the differential system. In the Hamiltonian setting a characterization of integrability was given by Liouville (1855). In the case of map (1) the analogue of first integrals are the *invariants*. To be more precise we state the following:

Definition 1 An invariant of a bi-rational map $\varphi\colon \mathbb{CP}^n \to \mathbb{CP}^n$ is a homogeneous function $I\colon \mathbb{CP}^n \to \mathbb{C}$ such that it is left unaltered by action of the map, i.e.

$$\varphi^*(I) = I, \tag{2}$$

where $\varphi^*(I)$ means the pullback of I through the map φ, i.e. $\varphi^*(I) = I(\varphi([\mathbf{x}]))$. For $n > 1$, an invariant is said to be *non-degenerate* if:

[1] Bi-rational maps in $\mathbb{CP}^1$ are just Möbius transformations so everything is trivial.

$$\frac{\partial I}{\partial x_1}\frac{\partial I}{\partial x_n} \neq 0. \tag{3}$$

Otherwise an invariant is said to be *degenerate*.

In what follows we will concentrate on a particular class of invariants:

Definition 2 An invariant I is said to be *polynomial*, if in the affine chart $[x_1 : \cdots : x_n : 1]$ the function I is a polynomial function.

A polynomial invariant in the sense of Definition 2 written in homogeneous variables is always a homogeneous rational function of degree 0. The form of the polynomial invariant in homogeneous coordinates is then given by:

$$I([\mathbf{x}]) = \frac{I'([\mathbf{x}])}{t^d}, \quad d = \deg I'([\mathbf{x}]), \tag{4}$$

where deg is the total degree.

To better characterize the properties of these invariants we introduce the following:

Definition 3 Given a polynomial function $F\colon \mathbb{CP}^n \to V$, where V can be either $\mathbb{CP}^n$ or $\mathbb{C}$, we define the *degree pattern* of F to be:

$$\operatorname{dp} F = \left(\deg_{x_1} F, \deg_{x_2} F, \ldots, \deg_{x_n} F\right). \tag{5}$$

Remark 1 The degree pattern of a polynomial function F is not invariant under general bi-rational transformations. However, the degree pattern of a polynomial function F is invariant under scaling and translations, which are transformations of the form:

$$\chi\colon [\mathbf{x}] \to [a\mathbf{x} + b], \quad a \in \mathbb{C} \setminus \{0\}, b \in \mathbb{CP}^n. \tag{6}$$

Example 1 Consider the following map in $\mathbb{CP}^2$:

$$\varphi\colon [x : y : t] \mapsto [-y(x^2 - t^2) + 2axt^2 : x(x^2 - t^2) : t(x^2 - t^2)]. \tag{7}$$

This map is known as the *McMillan map* (McMillan 1971) and possesses the following invariant:

$$t^4 I_{\text{McM}} = x^2y^2 + (x^2 + y^2 - 2axy)t^2. \tag{8}$$

We have $\operatorname{dp} I_{\text{McM}} = (2, 2)$, i.e. it is a *bi-quadratic* polynomial. We also note that the invariant of a QRT map (Quispel et al. 1988, 1989), I_{QRT}, which is a generalization of the McMillan map (7), is the ratio of two *bi-quadratics* in the dynamical variables of $\mathbb{CP}^2$. Hence QRT mappings leave invariant a pencil of curves of degree pattern $(2, 2)$.

Example 2 The invariants of the maps presented in Capel and Sahadevan (2001), I_{CS}, are ratios of *bi-quadratics* in all the four dynamical variables of $\mathbb{CP}^4$, i.e. ratios of polynomial of degree pattern (2, 2, 2, 2). In this sense the classification of Capel and Sahadevan (2001) is an extension of the one in Quispel et al. (1988, 1989).

Finally we will consider invariants that are not of the most general kind, but satisfy the following condition.

Definition 4 We say that a invariant $I : \mathbb{CP}^n \to \mathbb{C}$ is *symmetric* if it is left unaltered by the following involution:

$$\iota\colon \left[x_1 : x_2 : \cdots : x_n : x_{n+1}\right] \to \left[x_n : x_{n-1} : \cdots : x_1 : x_{n+1}\right], \tag{9}$$

i.e. $\iota^*(I) = I$.

We then have the following characterization of integrability for autonomous recurrence relations:

(i) **Existence of invariants** An n-dimensional map is (super) integrable if *there exist $n-1$ invariants.*

(ii) **Liouville integrability** (Veselov 1991; Maeda 1987; Bruschi et al. 1991) An n-dimensional map (in affine coordinates) is integrable if *it preserves a Poisson structure of rank $2r$ and $r+n-2r = n-r$ functionally independent invariants in involution with respect to this Poisson structure*. In affine coordinates $\mathbf{w} = (w_{n-1}, \ldots, w_0) = \left[w_{n-1} : \cdots : w_0 : 1\right]$ we say that a map $\varphi\colon \mathbf{w} \mapsto \mathbf{w}'$ is called a Poisson map of rank $2r \leq n$ if there is a skew-symmetric matrix $J(\mathbf{w})$ of rank $2r$ satisfies the Jacobian identity

$$\sum_{l=1}^{n} \left(J_{li} \frac{\partial J_{jk}}{\partial w_{l-1}} + J_{lj} \frac{\partial J_{ki}}{\partial w_{l-1}} + J_{lk} \frac{\partial J_{ij}}{\partial w_{l-1}} \right) = 0, \quad \forall i, j, k, \tag{10}$$

and

$$\mathrm{d}\,\varphi J(\mathbf{w})\,\mathrm{d}\,\varphi^T = J(\mathbf{w}^{'}), \tag{11}$$

where $\mathrm{d}\,\varphi$ is the Jacobian matrix of the map φ, see Capel and Sahadevan (2001), Olver (1986). The Poisson bracket of two smooth functions f and g is defined as

$$\{f, g\} = \nabla f \; J \; (\nabla g)^T, \tag{12}$$

where ∇f is the gradient of f. We can easily see that $\{w_{i-1}, w_{j-1}\} = J_{ij}$. We note that in the case where the Poisson structure has full rank, i.e. $n = 2r$, we only need $n/2$ invariants which are in involution. In this case the Poisson matrix is invertible, and its inverse is called a *symplectic matrix*. A symplectic matrix give raise to a *symplectic structure*.

(iii) **Existence of a Lax pair** (Lax 1968) An n-dimensional map is integrable if it *arises as compatibility condition of an overdetermined linear system*. We emphasize the fact that the Lax pair needs to provide us some integrability aspects of the maps such as invariants or solutions of the non-linear system. It is known in the literature that not all the Lax pairs satisfy such conditions (Calogero and Nucci 1991; Hay and Butler 2012, 2015; Gubbiotti et al. 2016). Lax pairs that do not satisfy such conditions are called *fake Lax pairs* and their existence cannot be used to prove integrability of a given system.

(iv) **Low growth condition** (Veselov 1992; Falqui and Viallet 1993; Bellon and Viallet 1999) An n-dimensional *bi-rational* map is integrable if *the degree of growth of the iterated map* φ^k *is polynomial with respect to the initial conditions* $[\mathbf{x_0}]$. Integrability is then equivalent to the vanishing of the *algebraic entropy*:

$$\varepsilon = \lim_{k\to\infty} \frac{1}{k} \log \deg_{[\mathbf{x_0}]} \varphi^k. \tag{13}$$

Algebraic entropy is a measure of the *complexity* of a map, analogous to the one introduced by Arnol'd (1990) for diffeomorphisms. In this sense growth is given by computing the number of intersections of the successive images of a straight line with a generic hyperplane in complex projective space (Veselov 1992).

We emphasize the fact that the above list is not completely exhaustive of all the possible definitions of integrability. Since we are focused on autonomous recurrence relations we choose to cover only the most used definition for these ones.

Remark 2 Here, we collect some observations about algebraic entropy and how to evaluate it.

(i) Algebraic entropy is invariant under bi-rational maps (Bellon and Viallet 1999).
(ii) In principle, the definition of algebraic entropy in Eq. (13) requires us to compute all the iterates of a bi-rational map φ to obtain the sequence $\left\{d_k = \deg_{[\mathbf{x_0}]} \varphi^k\right\}_{k=0}^{\infty}$. Fortunately, for the majority of applications the form of the sequence can be inferred by using generating functions (Lando 2003):

$$g(z) = \sum_{n=0}^{\infty} d_k z^k. \tag{14}$$

(iii) In almost all cases, the generating function turns out to be a rational function, which can be inferred from a finite number of iterates of the dynamical system. It then becomes a predictive tool, which can be tested using further terms of the sequence of degrees. In this paper, we find inferred generating functions for 4 cases given in Eqs. (23), (28), (47), and (53). In each case, the type of argument required to show that the given generating function is indeed the correct one may be found in Viallet (2015).

(iv) When a generating function is available, the algebraic entropy is then given by the logarithm of the smallest pole of the generating function, see Gubbiotti (2016), Grammaticos et al. (2009).

Remark 3 The condition of Liouville integrability (Maeda 1987; Veselov 1991; Bruschi et al. 1991) is stronger than the existence of invariants. Indeed, for a map, being measure preserving and preserving a Poisson/symplectic structure are very strong conditions. However, they lead to a great drop in the number of invariants needed for integrability. The same can be said for the existence of a Lax pair, since it is well known that a well-posed Lax pair gives all the invariants of the system through the spectral relations. Finally, the low growth condition means that the complexity of the map is very low, and it is known that invariants help in reducing the complexity of a map. Indeed the growth of a map possessing invariants cannot be generic since the motion is constrained to take place on the intersection of hypersurfaces defined by the invariants. For maps in $\mathbb{CP}^2$, it was proved in Diller and Favre (2001) that the growth can be only bounded, linear, quadratic or exponential. Linear cases are trivially integrable in the sense of invariants. We note that for polynomial maps, it was already known from Veselov (1992) that the growth can be only linear or exponential. It is known that QRT mappings and other maps with invariants in $\mathbb{CP}^2$ possess quadratic growth (Duistermaat 2011), so the two notions are actually equivalent for a large class of integrable systems.

Now we discuss briefly the concept of *duality* for rational maps, which was introduced in Quispel et al. (2005). Let us assume that our map φ possesses L independent invariants, i.e. I_j for $j \in \{1, \dots, L\}$. Then we can form the linear combination:

$$H = \alpha_1 I_1 + \cdots + \alpha_L I_L. \tag{15}$$

For an unspecified autonomous recurrence relation

$$\left[x_1 : x_2 : \cdots : x_{n+1}\right] \mapsto \left[x_1' : x_1 : \cdots : x_n\right], \tag{16}$$

we can write down the invariant condition for H (15):

$$\widehat{H}(x_1', [\mathbf{x}]) = H\left([\mathbf{x}']\right) - H\left([\mathbf{x}]\right) = 0. \tag{17}$$

Since we know that $[\mathbf{x}'] = \varphi\left([\mathbf{x}]\right)$ is a solution of (17) we have the following factorization:

$$\widehat{H}(x_1', [\mathbf{x}]) = A\left(x_1', [\mathbf{x}]\right) B\left(x_1', [\mathbf{x}]\right). \tag{18}$$

We can assume without loss of generality that the map φ corresponds to the annihilation of A in (18). Now since $\deg_{x_1'} \widehat{H} = \deg_{x_1} H$ and $\deg_{x_n} \widehat{H} = \deg_{x_n} H$ we have that if $\deg_{x_1} H, \deg_{x_n} H > 1$ the factor B in (18) is non constant.[2] In general, since

[2] We remark that this assertion is possible because we are assuming that all the invariants are non-degenerate. It is easy to see that degenerate invariants can violate this property.

the map φ is bi-rational, we have the following equalities:

$$\deg B_{x_1'} = \deg_{x_1'} \widehat{H} - \deg_{x_1'} A = \deg_{x_1} H - 1, \tag{19a}$$
$$\deg B_{x_n} = \deg_{x_n} \widehat{H} - \deg_{x_n} A = \deg_{x_n} H - 1. \tag{19b}$$

Therefore we have that if $\deg_{x_1} H, \deg_{x_n} H > 2$, the annihilation of B does not define a bi-rational map in general, but may define an algebraic one. However when $\deg_{x_1} H, \deg_{x_n} H = 2$ the annihilation of B defines a bi-rational projective map. We call this map the *dual map* and we denote it by $\varphi^\vee$.

Remark 4 We note that in principle for $\deg_{x_1} H = \deg_{x_n} H = d > 2$, more general factorizations can be considered:

$$\widehat{H}\left(x_1', [\mathbf{x}]\right) = \prod_{i=1}^{d} A_i\left(x_1', [\mathbf{x}]\right), \tag{20}$$

but we will not consider this case here.

Now assume that the invariants (and hence the map φ) depend on some *arbitrary constants* $I_i = I_i([\mathbf{x}]; a_i)$, for $i = 1, \ldots, M$. Choosing some of the a_i in such a way that there remains M arbitrary constants and such that for a subset a_{i_k} we can write Eq. (15) in the following way:

$$H = a_{i_1} J_1 + a_{i_2} J_2 + \cdots + a_{i_K} J_{a_{i_K}}, \tag{21}$$

where $J_i = J_i([\mathbf{x}])$, $i = 1, 2, \ldots, K$ are new functions. The parameters a_{i_k} do not appear in the dual maps in the same way as the parameters α_i do not appear in the main maps. Therefore, using the factorization (18) the J_i functions are invariants for the dual maps.

Remark 5 In fact, one can consider more general combinations than linear combinations given in (15) and (21). However, we only consider those linear combinations given (15) and (21) in this paper.

It is clear from Eq. (21) that even though the dual map is naturally equipped with some invariants, it is not *necessarily* equipped with a sufficient number of invariants to claim integrability. In fact there exist examples of dual maps with any possible behaviour, integrable, superintegrable and non-integrable (Joshi and Viallet 2018; Gubbiotti et al. 2020).

In a recent paper (Joshi and Viallet 2018), the authors considered the *autonomous limit* of the second member of the $\mathrm{d}P_{\mathrm{I}}$ and $\mathrm{d}P_{\mathrm{II}}$ hierarchies (Cresswell and Joshi 1999). We will denote these equations as $\mathrm{d}P_{\mathrm{I}}^{(2)}$ and $\mathrm{d}P_{\mathrm{II}}^{(2)}$ equations. These $\mathrm{d}P_{\mathrm{I}}^{(2)}$ and $\mathrm{d}P_{\mathrm{II}}^{(2)}$ equations are given by autonomous recurrence relations of order four, and showed to be integrable according to the algebraic entropy approach. They showed that both maps possess two invariants, one of degree pattern $(1, 3, 3, 1)$ and one of degree pattern $(2, 4, 4, 2)$. Using these invariants, they showed that the dual maps of the $\mathrm{d}P_{\mathrm{I}}^{(2)}$ and $\mathrm{d}P_{\mathrm{II}}^{(2)}$ equations are integrable according to the algebraic entropy test and moreover, produced some invariants, showing that these dual maps were actually superintegrable. Finally they gave a scheme to construct autonomous recurrence relations with the assigned degree pattern $(1, 3, 3, 1)$ associated with I_{low} and $(2, 4, 4, 2)$ associated with I_{high} and they provided some new examples out of this construction.

In (Gubbiotti et al. 2020) we consider the problem of finding all fourth order birational maps $\varphi\colon [x : y : z : u : t] \mapsto [x' : y' : z' : u' : t']$ possessing a polynomial symmetric invariant I_{low} such that $\mathrm{dp}\, I_{\mathrm{low}} = (1, 3, 3, 1)$ *where the only non-zero coefficients are those appearing in the* $(1, 3, 3, 1)$ *invariant of both the* $\mathrm{d}P_{\mathrm{I}}^{(2)}$ and $\mathrm{d}P_{\mathrm{II}}^{(2)}$ *equation*, and such that φ possesses a polynomial symmetric invariant I_{high} such that $\mathrm{dp}\, I_{\mathrm{high}} = (2, 4, 4, 2)$. The two invariants I_{low} and I_{high} are assumed to be functionally independent and non-degenerate. Within this class we have found the known $\mathrm{d}P_{\mathrm{I}}^{(2)}$ and $\mathrm{d}P_{\mathrm{II}}^{(2)}$ equations as well as new examples of maps with these properties.

In this letter we will present in detail four particular examples of this class. In Sect. 2, we will discuss two pairs of main-dual maps. We will discuss the integrability property of these maps in light of their invariants and of their growth. We will present maps possessing two invariants and integrable according to the algebraic entropy test with *cubic growth*. This implies that another rational invariant cannot exist. Indeed, the orbits of superintegrable maps with rational invariant are confined to *elliptic curves* and the growth is at most *quadratic* (Bellon 1999; Gizatullin 1980). From this general statement follows that a four-dimensional map with cubic growth can possess at most two rational invariants. We note that some examples of cubic growth were already presented in Joshi and Viallet (2018). However, it was pointed out that these examples can be deflated to lower dimensional maps with quadratic growth. This also holds for our maps, i.e. we can deflate them to integrable maps in lower dimension. Furthermore, we will present a map with two invariants and *exponential growth*, that is non-integrable according to the algebraic entropy test. We discuss some possible reasons why this map is non-integrable even though it possesses two invariants. In the final Section, we will give some conclusions and an outlook on the future perspectives of this approach.

2 Notable Examples

In this section we discuss two pairs of maps, which arise as part of a systematic classification to be presented in Gubbiotti et al. (2020). The interest in these particular maps arises since the relation between their invariants and growth properties is non trivial. In both cases the main maps possess two functionally independent invariants, but they behave differently. One map has *cubic* degree growth, while the other one has *exponential* degree growth. Therefore, even though these two maps have the same number of invariants with the same degree patterns, one map is integrable and the other one is non-integrable. In addition, in both cases the degree growth property of the dual maps reflect the growth of the main map. However, we note that the degree growth of the dual map does not always reflect that of the main map (Gubbiotti et al. 2020).

2.1 (P.i) and Its Dual Map (Q.i)

Consider the map $[\mathbf{x}] \mapsto \varphi_i([\mathbf{x}]) = [\mathbf{x}']$ given as follows:

$$\begin{aligned} x' &= \{[\nu t^2(x+z) + uz^2]y + t^2\mu uz + (x+z)^2 y^2\}d - at^4, \\ y' &= x^2 d(t^2\mu + xy), \quad z' = yxd(t^2\mu + xy), \\ u' &= zxd(t^2\mu + xy), \quad t' = txd(t^2\mu + xy). \end{aligned} \tag{P.i}$$

This map depends on four parameters a, d and μ, ν.

From the construction in Gubbiotti et al. (2020) we know that the map (P.i) possesses the following invariants:

$$t^6 I_{\text{low}}^{\text{P.i}} = at^4 yz + d\left[\nu y^2 z^2 - yz(ux - uz - xy)\mu\right] t^2 - y^2 z^2 d(ux - xy - yz - uz), \tag{22a}$$

$$\begin{aligned} t^8 I_{\text{high}}^{\text{P.i}} = {} & [(uz + xy - yz)\mu - \nu yz]\, at^6 \\ & + \left[yz(xy + yz + uz)a + d\mu^2(uz + xy - yz)^2 \right.\\ & \left. + 2d\mu\nu yz(ux - yz) - d\nu^2 y^2 z^2\right] t^4 \\ & + \left[2dzy(uz + xy - yz)(xy + yz + uz)\mu + 2dy^2 z^2 \nu ux\right] t^2 \\ & + dy^2 z^2 (xy + yz + uz)^2. \end{aligned} \tag{22b}$$

Moreover, the map (P.i) has the following degrees of iterates:

$$\begin{aligned} \{d_n\}_{\text{P.i}} = {} & 1, 4, 12, 28, 52, 86, 130, 188, 260, 348, 452, \\ & 576, 720, 886, 1074, 1288, 1528, 1796, 2092 \ldots \end{aligned} \tag{23}$$

The sequence (23) is fitted by:

$$g_{\text{P.i}}(s) = \frac{s^7 - 3s^6 + s^5 - s^4 + 3s^3 + 3s^2 + s + 1}{(s+1)(s^2+1)(s-1)^4}. \tag{24}$$

This generating function can be found by using the first 15 iterates, and its validity can be confirmed by using further iterates. See Remark 2 for the justification.

Due to the presence of $(s-1)^4$ in the denominator we have that the growth of the map (P.i) is fitted by a *cubic polynomial*. As discussed in the Introduction this means

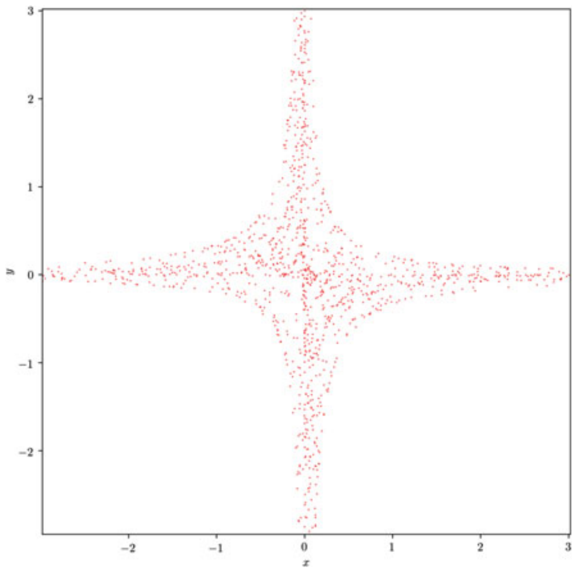


Fig. 1 Affine orbit of equation (P.i) with parameters $a = 6$, $\mu = 3$, $\nu = 4$ and $d = 6$ and initial conditions $(x, y, z, u) = (0.02, 0.05, 0.06, 0.07)$

at once that the map is integrable according to the algebraic entropy test and that another rational invariant cannot exist. This suggests that the geometry of the orbits of the map (P.i) is nontrivial, and goes beyond the existence of *elliptic fibrations*.

Explicit numerical calculations and drawings suggest that in the case of map (P.i), no additional invariant exists. Indeed, if an additional third invariant, even algebraic, existed then all the orbits of equation (P.i) would lie on a curve. On the other hand referring to Fig. 1 we see that a generic orbit of equation (P.i) does not lie on a curve. This implies that no such invariant exists.

The dual map $[\mathbf{x}] \mapsto \varphi_i^{\vee}([\mathbf{x}]) = [\mathbf{x}']$ of (P.i) is given by:

$$\begin{aligned} x' &= [\beta(2xy - 2yz + uz)\mu + (\beta\nu - \alpha)y(x - z)]\, t^2 \\ &\quad + \beta y(z^2 y - x^2 y + uz^2), \\ y' &= x^2 \beta(t^2\mu + xy), \quad z' = yx\beta(t^2\mu + xy), \\ u' &= zx\beta(t^2\mu + xy), \quad t' = tx\beta(t^2\mu + xy). \end{aligned} \tag{Q.i}$$

This map depends on four parameters α, β, and μ, ν. The parameters μ and ν are shared with the main map (P.i).

The main map (P.i) possesses two invariants and depends on a and d whereas the dual map (Q.i) does not depend on them. Then according to (21) we can write down the invariants for the dual map (Q.i) as:

$$\alpha I_{\text{low}}^{\text{P.i}} + \beta I_{\text{high}}^{\text{P.i}} = a I_{\text{low}}^{\text{Q.i}} + d I_{\text{high}}^{\text{Q.i}}. \tag{25}$$

Therefore, we obtain the following expressions:

$$
\begin{aligned}
t^4 I_{\text{low}}^{\text{Q.i}} &= [yz\alpha + (\mu xy - yz\mu - y\nu z + \mu uz)\beta]t^2 \\
&\quad + \beta yz(xy + yz + uz),
\end{aligned} \tag{26a}
$$

$$
\begin{aligned}
t^8 I_{\text{high}}^{\text{Q.i}} &= \{[y^2z^2\nu - yz(ux - uz - xy)\mu]\alpha \\
&\quad + [(uz + xy - yz)^2\mu^2 + 2yz(ux - yz)\nu\mu - \nu^2y^2z^2]\beta\}t^4 \\
&\quad + \{z^2y^2(xy + yz - ux + uz)\alpha \\
&\quad + [2yz(uz + xy - yz)(xy + yz + uz)\mu + 2y^2z^2\nu ux]\beta\}t^2 \\
&\quad + z^2y^2(xy + yz + uz)^2\beta.
\end{aligned} \tag{26b}
$$

We remark that the invariant (26a) has degree pattern (1, 2, 2, 1) which differs from dp $I_{\text{low}}^{\text{P.i}}$.

The map (Q.i) has the following degrees of iterates:

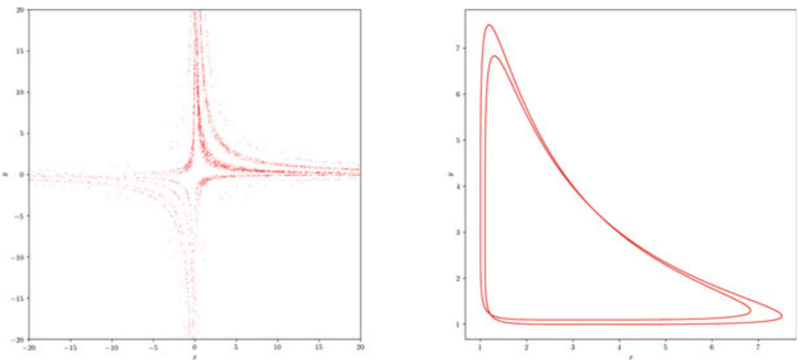
$$
\{d_n\}_{\text{Q.i}} = 1, 4, 12, 26, 48, 78, 118, 170, 234, 312, 406, 516, 644, 792\ldots \tag{27}
$$

fitted by the generating function:

$$
g_{\text{Q.i}}(s) = \frac{(s^3 - 2s^2 - 1)(s^3 - s^2 - s - 1)}{(s^2 + s + 1)(s - 1)^4}. \tag{28}
$$

This means that the dual map is integrable according to the algebraic entropy test with *cubic* growth, just like the main map. See Remark 2 for the justification.

Explicit numerical calculations and drawings suggest that also in the case of map (Q.i), no additional invariant exists. Indeed, if an additional third invariant, even algebraic, existed then all the orbits of equation (Q.i) would lie on a curve. In this case we are actually able to find some orbits lying on a curve, see Fig. 2b. However,



(a) Parameters $\alpha = 3$, $\mu = 3$, $\nu = 7$ and $\beta = 3$.

(b) Parameters $\alpha = 3$, $\mu = 6$, $\nu = 8$ and $\beta = 9$.

Fig. 2 Affine orbit of equation (Q.i) with different parameters but the same initial conditions $(x, y, z, u) = (3, 4, 1, 3)$

it is possible to find orbits of equation (Q.i) that do not lie on a curve. An example of such orbit is shown in Fig. 2a. Therefore, we can conclude that a *globally defined* third invariant does not exist. The existence of some closed orbits like in Fig. 2b suggests the existence of a non-analytic invariant existing only in some regions of the space.

Therefore, the pair of main-dual maps (P.i) and (Q.i) consists of two integrable equations with non-standard degree of growth. However, as remarked above the degree pattern of the invariants of the maps (P.i) and (Q.i) differ slightly.

We now consider the maps (P.i) and (Q.i) in affine coordinates, which are given by

$$\varphi : (w_3, w_2, w_1, w_0) \mapsto (w_4, w_3, w_2, w_1), \tag{29}$$

where

$$w_4 = \frac{N_1}{d w_3 \left(w_2 w_3 + \mu\right)}, \tag{AP.i}$$

$$w_4 = \frac{N_2}{\beta\, w_3 \left(w_2 w_3 + \mu\right)}, \tag{AQ.i}$$

with

$$\begin{aligned} N_1 = {} & -d\big(w_0 w_1^2 w_2 + w_1^2 w_2^2 + 2 w_1 w_2^2 w_3 + w_2^2 w_3^2 + \mu w_0 w_1 \\ & + \nu w_1 w_2 + \nu w_2 w_3\big) - a, \end{aligned} \tag{30}$$

$$\begin{aligned} N_2 = {} & \beta w_0 w_1^2 w_2 + \beta \mu w_0 w_1 + \beta w_1^2 w_2^2 + (\alpha - 2\beta\mu - \beta\nu)\, w_1 w_2 \\ & - \beta w_3^2 w_2^2 + (2\beta\mu + \beta\nu - \alpha)\, w_2 w_3. \end{aligned} \tag{31}$$

Invariants for these maps are obtained from I_{low} and I_{high} respectively by taking $t = 1$, $u = w_0$, $z = w_1$, $y = w_2$, and $x = w_3$.

We note that when a Poisson structure has the full rank, using Eq. (11), one gets

$$\left[\det(\mathrm{d}\,\varphi)\right]^2 = \frac{\det\left(J(\mathbf{w}')\right)}{\det\left(J(\mathbf{w})\right)}. \tag{32}$$

This implies that the map φ is either volume or anti-volume preserving.

We recall that a map φ is called (anti) volume preserving if there is a function $\Omega(\mathbf{w})$ such that the following volume form is preserved

$$\Omega(\mathbf{w})\,\mathrm{d}\,w_0 \wedge \mathrm{d}\,w_1 \wedge \cdots \wedge \mathrm{d}\,w_{n-1} = \pm\Omega(\mathbf{w}')\,\mathrm{d}\,w_0' \wedge \mathrm{d}\,w_1' \wedge \cdots \wedge \mathrm{d}\,w_{n-1}'. \tag{33}$$

Thus, we can write

$$\frac{\partial\left(w_0', w_1', \ldots, w_{n-1}'\right)}{\partial\left(w_0, w_1, \ldots, w_{n-1}\right)} = \pm\frac{\Omega(\mathbf{w})}{\Omega(\mathbf{w}')}, \tag{34}$$

where the left hand side is the determinant of the Jacobian matrix of the map φ. In Byrnes et al. (1999) it was proved that if a map in n dimension is (anti) volume preserving and possesses $n-2$ invariant, then we can construct an (anti) Poisson structure of rank 2 from these invariants. However, these invariants turn out to be Casimirs (functions that Poisson commute with all other functions) of this Poisson bracket. Therefore, in order to have Liouville integrability, we need an extra invariant in addition to the known $n-2$ invariants if we want to use Poisson structures constructed this way. In other words, the map is super integrable. Thus, to discuss Liouville integrability of the maps (AP.i) and (AQ.i), we need to find a Poisson bracket of rank 4 as we already predicted that the third invariant does not exist. We do not have that information for these maps but we can show they reduce to three dimensional Liouville integrable maps via a process called deflation (Joshi and Viallet 2018). *Mutatis mutandis*, this process will preserve the invariants, and in dimension three two invariants are sufficient to claim integrability in the general sense as discussed in the Introduction.

It is easy to check that the maps (AP.i) and (AQ.i) are volume and anti-volume preserving, respectively, with respect to the same volume form:

$$\Omega = w_1 w_2 (w_1 w_2 + \mu). \tag{35}$$

We now construct the (anti) Poisson structures for these two maps following (Byrnes et al. 1999). We consider the dual multi-vector of the volume form

$$\tau = m \frac{\partial}{w_0} \wedge \frac{\partial}{w_1} \wedge \frac{\partial}{w_2} \wedge \frac{\partial}{w_3}, \tag{36}$$

where $m = 1/\Omega$. A degenerate Poisson structure for the map (AP.i) and a degenerate anti-Poisson structure for the map (AQ.i) are given by the following contraction

$$J = \tau \rfloor \mathrm{d} I_{\text{low}} \rfloor \mathrm{d} I_{\text{high}}, \tag{37}$$

where I_{low} and I_{high} are invariants for these maps in affine coordinates. Since these (anti) Poisson structures are quite big, we do not present them here.

Remark 6 The Poisson structures which can be constructed using the method of Byrnes et al. (1999) are degenerate and cannot be used to explain the integrability of the two maps (AP.i) and (AQ.i).

We also note that the maps (AP.i) and (AQ.i) can be reduced to three dimensional maps using a deflation $v_i = w_i w_{i+1}$. The recurrences for these maps are denoted by (DP.i) and (DQ.i) and are given as follows

$$d\mu\,(v_0 + v_3) + d\nu\,(v_1 + v_2) + d\left(v_0 v_1 + v_1^2 + 2 v_1 v_2 + v_2^2 + v_2 v_3\right) + a = 0, \tag{DP.i}$$

$$\beta\mu\left(-v_0+2\,\beta v_1-2\,\beta v_2+v_3\right)+\left(\beta\nu-\alpha\right)\left(v_1-v_2\right) \qquad \text{(DQ.i)}$$
$$+\beta\left(-v_0v_1-v_1^2+v_2^2+v_2v_3\right)=0.$$

Each of the maps (DP.i) and (DQ.i) has two functionally independent invariants which can be obtained directly from I_{low} and I_{high} even though they live in a different space. One can check that the map (DP.i) and (DQ.i) are anti-volume preserving and volume preserving with $\Omega = v_1 + \mu$. Therefore, we can construct their (anti) Poisson structure using the three dimensional version of (37). Using the following invariant from I_{low} for (DP.i)

$$I_1^{\mathrm{P.i}} = d\mu v_0v_1 - d\mu v_0v_2 + d\mu v_1v_2 + d\nu {v_1}^2 + dv_0{v_1}^2 - dv_0v_1v_2 + d{v_1}^3 + d{v_1}^2v_2 + av_1, \qquad (38)$$

we have found that the map (dP.i) has an anti-Poisson structure given by

$$J_{12}^{\mathrm{P.i}} = d(v_1 - v_0),\;\; J_{2,3}^{\mathrm{P.i}} = d(v_1 - v_2),$$
$$J_{13}^{\mathrm{P.i}} = \frac{-d\mu v_0 - d\mu v_2 - 2d\nu v_1 - 2dv_0v_1 + dv_0v_2 - 3d{v_1}^2 - 2dv_1v_2 - a}{\mu + v_1}.$$

Similarly, for the map (DQ.i) we obtain the invariant

$$I_1^{\mathrm{DQ.i}} = \beta\mu v_0 - \beta\mu v_1 + \beta\mu v_2 - \nu\beta v_1 + \beta v_0v_1 + \beta {v_1}^2 + \beta v_1v_2 + \alpha v_1, \qquad (39)$$

and the corresponding Poisson structure

$$J^{\mathrm{Q.i}} = \begin{bmatrix} 0 & \beta & \dfrac{\beta\left(\mu+\nu-v_0-2v_1-v_2\right)-\alpha}{\mu+v_1} \\ -\beta & 0 & \beta \\ -\dfrac{\beta\left(\mu+\nu-v_0-2v_1-v_2\right)-\alpha}{\mu+v_1} & -\beta & 0 \end{bmatrix}. \qquad (40)$$

For these constructions, $I_1^{\mathrm{P.i}}$ and $I_1^{\mathrm{Q.i}}$ are Casimirs for their associated (anti) Poisson structures. Their second (anti) Poisson structures can be obtained from the invariant I_{high} but we do not present here as they are quite big.

It is important to note that the (anti) Poisson structures of (AP.i) and (AQ.i) under inflation give us the trivial Poisson structures for (DP.i) and (DQ.i), i.e. $J = \mathbf{0}$, where $\mathbf{0}$ is the zero matrix. On the other hand, from the common factor that appears in the Poisson structure of (AP.i), we have found that there exists an anti-invariant $K^{\mathrm{P.i}}$ for this map, i.e. $K^{\mathrm{P.i}}(\mathbf{w}) = -K^{\mathrm{P.i}}(\mathbf{w}')$where

$$K^{\mathrm{P.i}} = 2d\left(w_2w_1^2w_0 + w_2^2w_1^2 + w_1w_2^2w_3 + \mu w_0w_1 - \mu w_1w_2 + \mu w_2w_3 + \nu w_1w_2\right) + a. \qquad (41)$$

However, $K^{\text{P.i}}$ is not independent of $I_{low}^{\text{P.i}}$ and $I_{high}^{\text{P.i}}$ since we have

$$\left(K^{\text{P.i}}\right)^2 - 4d\ I_{\text{high}}^{\text{P.i}} - 8d\ \nu I_{\text{low}}^{\text{P.i}} = a^2. \tag{42}$$

Using this anti-invariant, we obtain the following anti-invariant for the map (DP.i)

$$K^{\text{DP.i}} = 2d\mu v_0 - 2d\mu v_1 + 2d\mu v_2 + 2d\nu v_1 + 2dv_0 v_1 + 2d{v_1}^2 + 2dv_1 v_2 + a. \tag{43}$$

Therefore, using this anti-invariant, we get a Poisson structure for (DP.i) as follows (after factoring out a constant term)

$$J_2^{\text{P.i}} = \begin{bmatrix} 0 & 1 & \dfrac{\mu - \nu - v_0 - 2v_1 - v_2}{\mu + v_1} \\ -1 & 0 & 1 \\ -\dfrac{\mu - \nu - v_0 - 2v_1 - v_2}{\mu + v_1} & -1 & 0 \end{bmatrix}. \tag{44}$$

We can check directly that the invariants inherited from the affine map (AP.i) are in involution with respect to the Poisson structure (44). In the sense of the definition given in the Introduction, this means that the reduced maps (DP.i) and (DQ.i) are Liouville integrable.

Remark 7 We notice that we can always use the invariants (38) and (39) to reduce the three dimensional maps (DP.i) and (DQ.i) to two dimensional maps and relate them to QRT maps. To be more specific we have that the reduced map of (DQ.i) preserves a bi-quadratic curve so that it is of the QRT type. On the other hand, using the anti-invariant, the reduced map of (DP.i) sends a bi-quadratic to another bi-quadratic and fits in the framework of Roberts and Jogia (2015).

2.2 (P.ii) and Its Dual Map (Q.ii)

Consider the map $[\mathbf{x}] \mapsto \varphi_{\text{ii}}([\mathbf{x}]) = [\mathbf{x}']$ given as follows:

$$\begin{aligned} x' &= \left[(x^2 + z^2)y - uz^2\right]\mu - t^2(u - 2y), \\ y' &= x(t^2 + \mu x^2), \quad z' = y(t^2 + \mu x^2), \\ u' &= z(t^2 + \mu x^2), \quad t' = t(t^2 + \mu x^2). \end{aligned} \tag{P.ii}$$

This map only depends on the parameter μ.

From the construction in Gubbiotti et al. (2020) we know that the map (P.ii) has the following invariants:

$$t^5 I_{\text{low}}^{\text{P.ii}} = (x-z)(u-y)\left(t^2+z^2\mu\right)\left(\mu y^2+t^2\right), \tag{45a}$$

$$\begin{aligned} t^6 I_{\text{high}}^{\text{P.ii}} &= \left[(x-z)^2 y^4 + y^2 z^4 - 2yz^4 u + u^2 z^4\right]\mu^2 \\ &\quad + 2t^2\left[\left(x^2-2xz+2z^2\right)y^2 - 2yz^2 u + u^2 z^2\right]\mu \\ &\quad + t^4\left(z^2+u^2+x^2+y^2-2uy-2xz\right). \end{aligned} \tag{45b}$$

Moreover, the map (P.ii) has the following degrees of iterates:

$$\{d_n\}_{\text{P.ii}} = 1, 3, 9, 21, 45, 93, 189, 381, 765, 1533\ldots \tag{46}$$

fitted by the generating function:

$$g_{\text{P.ii}}(s) = \frac{1+2s^2}{(2s-1)(s-1)}. \tag{47}$$

This means that despite the existence of the two invariants (45) the map (P.ii) is non-integrable according to the algebraic entropy test: its entropy is positive and given by $\varepsilon = \log 2$. See Remark 2 for the justification.

Therefore we have that the map (P.ii) is an example of non-integrable admitting two invariants.

Again following (Byrnes et al. 1999) we can produce a Poisson structure of rank 2 for (P.ii) as the affine version of (P.ii) is volume preserving with $\Omega = (1+\mu w_1^2)(1+\mu w_2^2)$, where we have taken $t=1, u=w_0, z=w_1, y=w_2$, and $x=w_3$. By the construction, the two invariants (45) become Casimir functions for it, so again the existence of such Poisson structure does not imply any form of Liouville integrability. However, we notice that there are common factors appear at every non-zero entry of this structure. Thus, we have found the following anti-invariant for the map (P.ii) using these common factors

$$\begin{aligned} K^{\text{P.ii}} &= \left[\mu\left(w_0 w_1^2 - {w_1}^2 w_2 - w_1 w_2^2 + w_2^2 w_3\right) + w_0 - w_1 - w_2 + w_3\right] \times \\ &\qquad \left[\mu\left(w_0 w_1^2 - w_1^2 w_2 + w_1 w_2^2 - w_2^2 w_3\right) + w_0 + w_1 - w_2 - w_3\right] \\ &= F_1 F_2. \end{aligned} \tag{48}$$

This suggests that we should check each factor of $K^{P.ii}$ to see whether they are (anti) invariants of (P.ii). By direct calculation we can see that the first factor F_1 is an anti-invariant and F_2 is an invariant for (P.ii), but they are not functionally independent of I_{low} and I_{high}. In fact, their relations are

$$I_{\text{high}}^{\text{P.ii}} - F_1^2 + 2I_{\text{low}}^{\text{P.ii}} = 0, \text{ and } I_{\text{high}}^{\text{P.ii}} - F_2^2 - 2I_{\text{low}}^{\text{P.ii}} = 0. \tag{49}$$

Therefore, the map (P.ii) actually has two invariants of degrees $(1, 2, 2, 1)$ and $(1, 3, 3, 1)$. Nevertheless, despite the existence of such invariants the map (P.ii) is non-integrable in the sense of the algebraic entropy.

Remark 8 We can use F_1 and F_2 to construct an anti-Poisson structure for (P.ii) using the formula (37):

$$\begin{gathered}J_{1,2} = -1,\ J_{2,3} = 1,\ J_{3,4} = -1,\\ J_{1,3} = \frac{2\mu w_1 (w_2 - w_0)}{\mu w_1^2 + 1},\ J_{2,4} = -\frac{2\mu w_2 (w_3 - w_1)}{\mu w_2^2 + 1},\\ J_{1,4} = -\frac{\mu^2 w_1 w_2 [4 (w_0 w_1 - w_0 w_3 + w_2 w_3) - 3 w_1 w_2] + \mu \left(w_1^2 + w_2^2\right) + 1}{\left(\mu w_1^2 + 1\right)\left(\mu w_2^2 + 1\right)}.\end{gathered} \tag{50}$$

We have checked that F_2 and $I_{\text{low}}^{P.ii}$ are in involution with respect to this anti-Poisson structure. A Poisson structure can be obtained by multiplying this anti-Poisson structure with the anti-invariant F_1.

The dual map $[\mathbf{x}] \mapsto \varphi_{ii}^{\vee}([\mathbf{x}]) = [\mathbf{x}']$ of (P.ii) is given as follows:

$$\begin{aligned}x' &= \alpha \left[\left(x^2 - z^2\right) y + u z^2\right] \mu + t^2 \alpha u + \beta y^2 (x - z) \mu\\ &\quad + t^2 \beta (x - z),\\ y' &= \alpha x \left(t^2 + \mu x^2\right), \quad z' = \alpha y \left(t^2 + \mu x^2\right),\\ u' &= \alpha z \left(t^2 + \mu x^2\right), \quad t' = \alpha t \left(t^2 + \mu x^2\right).\end{aligned} \tag{Q.ii}$$

This map depends on three parameters α, β and μ. The parameter μ is shared with the main map (P.ii).

Since the main map (P.ii) possesses two invariants depending only on one parameter μ then according to (21) we can write down only a single invariant for the dual map (Q.ii):

$$I^{\text{Q.ii}} = \alpha I_{\text{high}}^{\text{P.ii}} + \beta I_{\text{low}}^{\text{P.ii}}. \tag{51}$$

The invariant (51) has degree pattern (2, 4, 4, 2).

We have then that the dual map (Q.ii) has the following fast-growing degrees of iterates:

$$\{d_n\}_{\text{Q.ii}} = 1, 3, 9, 21, 45, 93, 189, 381, 765, 1533, 3069 \ldots. \tag{52}$$

The growth of degrees evident in (52) is clearly exponential and is fitted by the generating function

$$g_{\text{Q.ii}}(s) = \frac{1 + 2s^2}{(2s - 1)(s - 1)}. \tag{53}$$

This confirms that the algebraic entropy is positive and equal to $\varepsilon = \log 2$. See Remark 2 for the justification.

This means that the dual map is non-integrable with same rate of growth as the main map. In this case we can show that the map is anti-volume preserving with the same measure as the main map (P.ii). Moreover, we proved that the map (Q.ii) do not posses any addition invariant up to order 14. Therefore at the present stage we cannot construct a Poisson structure using the method of Byrnes et al. (1999).

3 Conclusions and Outlook

In this letter, we gave some examples of fourth order bi-rational maps with two invariants possessing interesting degree growth properties. These examples come from our classification of all the fourth-order autonomous recurrence relations possessing two invariants in a given class of degree patterns (Gubbiotti et al. 2020).

The first pair of bi-rational maps is given by the map (P.i) and its dual (Q.i) and consists of integrable maps with cubic growth. The interest in maps with cubic growth arises from geometrical considerations: maps with polynomial but higher than quadratic growth, can arise only in dimension greater than two (Diller and Favre 2001) and prove, in the case of superintegrable maps, the existence of non-elliptic fibrations of invariant varieties (Bellon and Viallet 1999). The interest in maps with this type of growth arose recently following the examples given in Joshi and Viallet (2018) and we expect them to lead to many new and interesting geometric structures.

The second pair of fourth order bi-rational maps, given by the map (P.ii) and its dual (Q.ii), consists of non-integrable maps with exponential growth. There are various possible reasons why the map (P.ii) is non-integrable despite possessing two invariants. To claim integrability with two invariants according to the discrete Liouville theorem (Maeda 1987; Bruschi et al. 1991; Veselov 1991) we need to prove that the map has a symplectic structure and that the two invariants commute with respect to this symplectic structure. Hence, either the map (P.ii) does not admit *any* symplectic structure, or the map (P.ii) admits only symplectic structures such that the two invariants (45) do not commute. Since, usually, from a set of non-commuting invariants it is possible to find a set of functionally independent commuting invariants we conjecture that Eq. (P.ii) is devoid of a non-degenerate Poisson structure.

Work is in progress to characterize the surfaces generated by the invariants in both integrable and non-integrable cases. We expect this to give new results in the geometric theory of integrable systems.

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