The Replicator Dynamics for Games in Metric Spaces: Finite Approximations

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1 Introduction

In this paper, we are interested in evolutionary games, in which the interaction of strategies is studied as a dynamical system. We are interested in the special case in which the strategies' interactions follow a specific dynamical system known as the replicator dynamics.

An evolutionary game is said to be *symmetric* if there are two players only and, furthermore, they have the same strategy sets and the same payoff functions. This type of game models interactions of the strategies of a single population. In contrast, an *asymmetric* evolutionary game, also known as *multipopulation games*, is a game with a finite set of players (or populations) each of which has a different set of strategies and different payoff functions.

In our model, the pure strategies set of each player (or population) is a metric space and consequently the replicator dynamics lives in a Banach space (a space of finite signed measures). In particular, if we have n players each of which has m_i strategies, for $i = 1, \ldots, n$, then the replicator dynamics is in $\mathbb{R}^m$, where $m = \sum_{i=1}^{n} m_i$.

The main goal of this paper is to establish conditions under which a finite-dimensional dynamical system approximates the replicator dynamics for games with strategies in metric spaces. In this manner, we can use numerical analysis techniques for finite-dimensional differential equations to approximate a solution to the replicator dynamics, which lives in an infinite-dimensional Banach space. This is important because it will allow us to study games with pure strategies in metric spaces such as models in oligopoly theory, international trade theory, war of attrition, and public goods, among others. To achieve this goal, we first present a finite-dimensional

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approximation technique for games in metric spaces and we give a proposal of a finite-dimensional dynamical system to approximate evolutionary dynamics in a Banach space, see Sect. 4. After, in Sects. 5 and 6, we establish general approximation theorems for the replicator dynamics in metric spaces and use these results for a finite-dimensional approximation given in Sect. 4, see Notes 1 and 3.

Oechssler and Riedel [24] propose two approximation theorems for symmetric games. The first theorem establishes the proximity in the strong topology of two paths generated by two dynamical systems (the original model and a discrete approximation of the model) with the same initial conditions. The second theorem establishes the proximity in the weak topology of two paths with different initial conditions, and these paths satisfy the same differential equation.

We propose here two approximation results with hypotheses less restrictive than those by Oechssler and Riedel [24]. Our approximation theorems extend the results in [24]. In our case, the approximation theorems are for symmetric and asymmetric games. Also, we establish the proximity of two paths generated by two different dynamical systems (the original model and a discrete approximation model) with different initial conditions. In addition, our approximation results are studied in the strong topology using the norm of total variation, and also in the weak topology using the Kantorovich–Rubinstein metric. This last point is important because the initial conditions and the paths (by consequence) of the original dynamics model and the finite-dimensional dynamic approximation may be very far between them (both initial conditions and paths) in terms of the strong topology, but very close between them in terms of the weak topology.

These approximations require different hypotheses. The first approximation theorem, Theorem 1, requires a proximity in the strong topology of the two initial conditions, and it only requires that the payoff functions for the original model be bounded. The second approximation result, Theorem 2, weakens the hypothesis of proximity of the two initial conditions (it only imposes a condition of proximity in the weak topology), but it requires that the payoff functions for the original model be Lipschitz continuous.

There are several publications on the replicator dynamics in games with strategies in metric spaces. For instance, conditions for the existence of solutions, as in Bomze [4], Oechssler and Riedel [23], Cleveland and Ackleh [7], Mendoza-Palacios and Hernández-Lerma [21] (for asymmetric games). Similarly, conditions for dynamic stability, as in Bomze [3], Oechssler and Riedel [23, 24], Eshel and Sansone [9], Veelen and Spreij [30], Cressman and Hofbauer [8], Mendoza-Palacios and Hernández-Lerma [21, 22].

The paper is organized as follows. Section 2 presents notation and technical requirements. Section 3 describes the replicator dynamics and its relation to evolutionary games. Some important technical issues are also summarized. Section 4 introduces a finite-dimensional game to approximate evolutionary games in a Banach space. Section 5 establishes an approximation theorem for the replicator dynamics on measure spaces by means of dynamical systems in finite-dimensional spaces. The distance for this first approximation is the total variation norm. Section 6 establishes an approximation theorem using the Kantorovich–Rubinstein metric. Section 7 proposes an example to illustrate our results. We conclude in Sect. 8 with some general comments on possible extensions. An appendix contains results of some technical facts.

2 Technical Preliminaries

2.1 Spaces of Signed Measures

Consider a separable metric space (A, ϑ) and its Borel σ-algebra $\mathcal{B}(A)$. Let $\mathbb{M}(A)$ be the Banach space of finite signed measures μ on $\mathcal{B}(A)$ endowed with the total variation norm

$$\|\mu\| := \sup_{\|f\|\leq 1} \left| \int_A f(a)\mu(da) \right| = |\mu|(A). \tag{1}$$

The supremum in (1) is taken over functions in the Banach space $\mathbb{B}(A)$ of real-valued bounded measurable functions on A, endowed with the supremum norm

$$\|f\| := \sup_{a\in A} |f(a)|. \tag{2}$$

Consider the subset $\mathbb{C}(A) \subset \mathbb{B}(A)$ of all real-valued continuous and bounded functions on A. Consider the dual pair $(\mathbb{C}(A), \mathbb{M}(A))$ given by the bilinear form $\langle \cdot, \cdot \rangle : \mathbb{C}(A) \times \mathbb{M}(A) \to \mathbb{R}$

$$\langle g, \mu \rangle = \int_A g(a)\mu(da). \tag{3}$$

We consider the *weak topology* on $\mathbb{M}(A)$ (induced by $\mathbb{C}(A)$), i.e., the topology under which all elements of $\mathbb{C}(A)$ when regarded as linear functionals $\langle g, \cdot \rangle$ on $\mathbb{M}(A)$ are continuous.

2.2 The Kantorovich–Rubinstein Metric

There are many metrics that metrize the weak topology on $\mathbb{P}(A)$. Here we use the Kantorovich–Rubinstein metric. Let (A, ϑ) be a separable metric space, and $\mathbb{P}(A)$ the set of probability measure on A. For any $\mu, \nu \in \mathbb{P}(A)$ we define the **the Kantorovich–Rubinstein metric** r_{kr} as

$$r_{kr}(\mu, \nu) := \sup_{f\in\mathbb{L}(A)} \left\{ \int_A f(a)\mu(da) - \int_A f(a)\nu(da) \ : \ \|f\|_L \leq 1 \right\}, \tag{4}$$

where $(\mathbb{L}(A), \|\cdot\|_L)$ is the space of continuous real-valued functions on A that satisfy the Lipschitz condition

$$\|f\|_L := \sup\left\{|f(a) - f(b)|/\vartheta(a,b),\ \ a, b \in A,\ a \neq b\right\} < \infty. \tag{5}$$

Let a_0 be a fixed point in A, and

$$\mathbb{M}_K(A) := \left\{\mu \in \mathbb{M}(A) :\ \sup_{f\in\mathbb{L}(A)} \int_A |f(a)|\mu(da) < \infty\right\}.$$

The Kantorovich–Rubinstein metric r_{kr} can be extended as a norm on $\mathbb{M}_K(A)$ defined as

$$\|\mu\|_{kr} := |\mu(A)| + \sup_{f\in\mathbb{L}(A)} \left\{\int_A f(a)\mu(da)\ :\ \|f\|_L \leq 1,\ f(a_0) = 0\right\} \tag{6}$$

for any μ in $\mathbb{M}_K(A)$ (see Bogachev [2], Chap. 8).

Remark 1 Note that for any $\mu, \nu \in \mathbb{P}(A)$, $r_{kr}(\mu, \nu) = \|\mu - \nu\|_{kr}$. Indeed if $\mu, \nu \in \mathbb{P}(A)$, then

$$\sup_{f\in\mathbb{L}(A)} \left\{\int_A f(a)\mu(da) - \int_A f(a)\nu(da)\ :\ \|f\|_L \leq 1\right\}$$

$$= \sup_{f\in\mathbb{L}(A)} \left\{\int_A [f(a) - f(a_0)]\mu(da) - \int_A [f(a) - f(a_0)]\nu(da)\ :\ \|f\|_L \leq 1\right\}$$

$$= \sup_{g\in\mathbb{L}(A)} \left\{\int_A g(a)\mu(da) - \int_A g(a)\nu(da)\ :\ \|g\|_L \leq 1,\ g(a_0) = 0\right\}.$$

2.3 Product Spaces

Consider two separable metric spaces X and Y with their Borel σ-algebras $\mathcal{B}(X)$ and $\mathcal{B}(Y)$. We denote by $\mathcal{B}(X) \times \mathcal{B}(Y)$ the product σ-algebra on $X \times Y$. For $\mu \in \mathbb{M}(X)$ and $\nu \in \mathbb{M}(Y)$, we denote their product by $\mu \times \nu$ and it holds that

$$\|\mu \times \nu\| \leq \|\mu\|\|\nu\|. \tag{7}$$

As a consequence, $\mu \times \nu$ is in $\mathbb{M}(X \times Y)$ (see by example Heidergott and Leahu [11], Lemma 4.2.).

Now consider a finite family of metric spaces $\{X_i\}_{i=1}^n$ and their σ-algebras $\mathcal{B}(X_i)$, as well as the Banach spaces $(\mathbb{M}(X_i), \|\cdot\|)$ and $(\mathbb{M}_K(X_i), \|\cdot\|_{kr})$. For $i = 1, \ldots, n$, let $\mu_i \in \mathbb{M}(X_i)$ and consider the elements $\mu = (\mu_1, ..., \mu_n)$ in the product space $\mathbb{M}(X_1) \times ... \times \mathbb{M}(X_n)$ with the norm

$$\|\mu\|_\infty := \max_{1\le i\le n} \|\mu_i\| < \infty. \tag{8}$$

These elements form the Banach space $(\mathbb{M}(X_1) \times ... \times \mathbb{M}(X_n), \|\cdot\|_\infty)$. We can similarly define the Banach space $(\mathbb{M}_K(X_1) \times ... \times \mathbb{M}_K(X_n), \|\cdot\|_\infty^{kr})$, where

$$\|\mu\|_\infty^{kr} := \max_{1\le i\le n} \|\mu_i\|_{kr} < \infty. \tag{9}$$

2.4 Differentiability

Definition 1 Let A be a separable metric space. We say that a mapping $\mu : [0, \infty) \to \mathbb{M}(A)$ is strongly differentiable if there exists $\mu'(t) \in \mathbb{M}(A)$ such that, for every $t > 0$,

$$\lim_{\epsilon\to 0} \left\| \frac{\mu(t+\epsilon) - \mu(t)}{\epsilon} - \mu'(t) \right\| = 0. \tag{10}$$

Note that, by (1), the left-hand side in (10) can be expressed more explicitly as

$$\lim_{\epsilon\to 0} \sup_{\|g\|\le 1} \left| \frac{1}{\epsilon} \left[\int_A g(a)\mu(t+\epsilon, da) - \int_A g(a)\mu(t, da) \right] - \int_A g(a)\mu'(t, da) \right|.$$

The signed measure μ' in (10) is called the strong derivative of μ.

For weak differentiability, see Remark 3.

3 The Replicator Dynamics and Evolutionary Games

3.1 Asymmetric Evolutionary Games

Let $I := \{1, 2, \ldots, n\}$ be the set of different species (or players). Each individual of the species $i \in I$ can choose a single element a_i in a set of characteristics (strategies or actions) A_i, which is a separable metric space. For every $i \in I$ and every vector $a := (a_1, ..., a_n)$ in the Cartesian product $A := A_1 \times ... \times A_n$, we write a as (a_i, a_{-i}) where $a_{-i} := (a_1, ..., a_{i-1}, a_{i+1}, ..., a_n)$ is in

$$A_{-i} := A_1 \times ... \times A_{i-1} \times A_{i+1} \times ... \times A_n.$$

For each $i \in I$, let $\mathcal{B}(A_i)$ be the Borel σ-algebra of A_i and $\mathbb{P}(A_i)$ the set of probability measures on A_i, also known as the set of *mixed strategies*. A probability measure $\mu_i \in \mathbb{P}(A_i)$ assigns a population distribution over the action set A_i of the species i.

Finally, for each species i we assign a payoff function $J_i : \mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n) \to \mathbb{R}$ that explains the interrelation with the population of other species, and which is defined as

$$J_i(\mu_1, ..., \mu_n) := \int_{A_1} ... \int_{A_n} U_i(a_1, ..., a_n)\mu_n(da_n)...\mu_1(da_1), \tag{11}$$

where $U_i : A_1 \times ... \times A_n \to \mathbb{R}$ is a given measurable function.

For every $i \in I$ and every vector $\mu := (\mu_1, ..., \mu_n)$ in $\mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n)$, we sometimes write μ as (μ_i, μ_{-i}), where $\mu_{-i} := (\mu_1, ..., \mu_{i-1}, \mu_{i+1}, ..., \mu_n)$ is in $\mathbb{P}(A_1) \times ... \times \mathbb{P}(A_{i-1}) \times \mathbb{P}(A_{i+1}) \times ... \times \mathbb{P}(A_n)$. If $\delta_{\{a_i\}}$ is a probability measure concentrated at $a_i \in A_i$, the vector $(\delta_{\{a_i\}}, \mu_{-i})$ is written as (a_i, μ_{-i}), and so

$$J_i(\delta_{\{a_i\}}, \mu_{-i}) = J_i(a_i, \mu_{-i}). \tag{12}$$

In particular, (11) yields

$$J_i(\mu_i, \mu_{-i}) := \int_{A_i} J_i(a_i, \mu_{-i})\mu_i(da_i). \tag{13}$$

In an evolutionary game, the dynamics of the strategies is determined by a system of differential equations of the form

$$\mu_i'(t) = F_i(\mu_1(t), ..., \mu_n(t)) \quad \forall\ i \in I, \quad t \geq 0, \tag{14}$$

with some initial condition $\mu_i(0) = \mu_{i,0}$ for each $i \in I$. The notation $\mu_i'(t)$ represents the strong derivative of $\mu_i(t)$ in the Banach space $\mathbb{M}(A_i)$ (see Definition 1). For each $i \in I$, $F_i(\cdot)$ is a mapping

$$F_i : \mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n) \to \mathbb{M}(A_i).$$

Let

$$F : \mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n) \to \mathbb{M}(A_1) \times ... \times \mathbb{M}(A_n)$$

be such that $F(\mu) := (F_1(\mu), ..., F_n(\mu))$, and consider the vector

$$\mu'(t) := (\mu_1'(t), ..., \mu_n'(t)).$$

Hence, the system (14) can be expressed as

$$\mu'(t) = F(\mu(t)), \tag{15}$$

and we can see that the system lives in the Cartesian product of signed measures

$$\mathbb{M}(A_1) \times ... \times \mathbb{M}(A_n),$$

which is a Banach space with norm as in (8).

More explicitly, we may write (14) as

$$\mu_i'(t, E_i) = F_i(\mu(t), E_i) \quad \forall\, i \in I,\ E_i \in \mathcal{B}(A_i),\ t \geq 0, \tag{16}$$

where $\mu_i'(t, E_i)$ and $F_i(\mu(t), E_i)$ denote the signed measures $\mu_i'(t)$ and $F_i(\mu(t))$ valued at $E_i \in \mathcal{B}(A_i)$.

We shall be working with a special class of asymmetric evolutionary games which can be described as

$$\Big[I, \Big\{\mathbb{P}(A_i)\Big\}_{i\in I}, \Big\{J_i(\cdot)\Big\}_{i\in I}, \Big\{\mu_i'(t) = F_i(\mu(t))\Big\}_{i\in I}\Big], \tag{17}$$

where

(i) $I = \{1, ..., n\}$ is the finite set of players;
(ii) for each player $i \in I$ we have a set $\mathbb{P}(A_i)$ of mixed actions and a payoff function $J_i : \mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n) \to \mathbb{R}$ (as in (12)); and
(iii) the replicator function $F_i(\mu(t))$, where

$$F_i(\mu(t), E_i) := \int_{E_i} \Big[J_i(a_i, \mu_{-i}(t)) - J_i(\mu_i(t), \mu_{-i}(t))\Big]\mu_i(t, da_i). \tag{18}$$

Conditions for the existence of solutions and dynamic stability for asymmetric games are given, for instance, by Mendoza-Palacios and Hernández-Lerma [21], Theorems 4.3 and 4.5.

3.2 The Symmetric Case

We can obtain from (17) a *symmetric* evolutionary game when $I := \{1, 2\}$ and the sets of actions and payoff functions are the same for both players, i.e., $A = A_1 = A_2$ and $U(a, b) = U_1(a, b) = U_2(b, a)$, for all $a, b \in A$. As a consequence, the sets of mixed actions and the expected payoff functions are the same for both players, that is, $\mathbb{P}(A) = \mathbb{P}(A_1) = \mathbb{P}(A_2)$ and $J(\mu, \nu) = J_1(\mu, \nu) = J_2(\nu, \mu)$ for all $\mu, \nu \in \mathbb{P}(A)$. This kind of model determines the dynamic interaction of strategies of a unique species through the replicator dynamics $\mu'(t) = F(\mu(t))$, where $F : \mathbb{P}(A) \to \mathbb{M}(A)$ is given by

$$F(\mu(t), E) := \int_E \Big[J(a, \mu(t)) - J(\mu(t), \mu(t))\Big]\mu(t, da) \ \ \forall E \in \mathcal{B}(A). \tag{19}$$

As in (17), we can describe a symmetric evolutionary game in a compact form as

$$\left[I = \{1, 2\},\ \ \mathbb{P}(A),\ \ J(\cdot),\ \ \mu'(t) = F(\mu(t))\right]. \tag{20}$$

There are several papers on the replicator dynamics in symmetric games with strategies in metric spaces. In particular, for conditions on the existence of solutions, see, for instance, Bomze [4], Oechssler and Riedel [23], Cleveland and Ackleh [7]. Similarly, conditions for dynamic stability are given by Bomze [3], Oechssler and Riedel [23, 24], Eshel and Sansone [9], Veelen and Spreij [30], Cressman and Hofbauer [8], Mendoza-Palacios and Hernández-Lerma [22], among others.

4 Discrete Approximations to the Replicator Dynamics

To obtain a finite-dimensional approximation of the replicator dynamics (15) (with $F_i(\cdot)$ in (18)), for an asymmetric (17) (or symmetric (20)) model, we can apply the following Theorems 1 and 2 to a discrete approximation of the payoff functions U_i and the initial probability measures $\mu_{i,0}$, for i in I. For some approximation techniques for the payoff function in games, see Bishop and Cannings [1], Simon [29].

4.1 Games with Strategies in an Real Interval

Oechssler and Riedel [24] propose a finite approximation for a symmetric game. Following [24], consider an asymmetric game (17) where, for every i in I, $A_i = [c_{i,1}, c_{i,2}]$ (for some real numbers with $c_{i,1} < c_{i,2}$) and U_i is a real-valued bounded function. For every i in I, consider the partition $P_{k_i} := \{\xi_{m_i}\}_{m_i=0}^{2^{k_i}-1}$ over A_i, where

$$\xi_{m_i} := [a_{m_i}, a_{m_i+1}),\ \ a_{m_i} = c_{i,1} + \frac{m_i[c_{i,2} - c_{i,1}]}{2^{k_i}},$$

for $m_i = 0, 1, ..., 2^{k_i} - 1$ and $\xi_{2^{k_i}-1} := [a_{2^{k_i}-1}, c_{i,2}]$. For every i in I, the discrete approximation to U_i is given by the function

$$U_{k_i}(x_1, ..., x_i, ..., x_n) := U_i(a_{m_1}, ..a_{m_i}, ..., a_{m_n}),$$

if $(x_1, ..., x_i, ..., x_n)$ is in $\xi_{m_1} \times \cdots \times \xi_{m_i} \times \cdots \times \xi_{m_n}$. Also, for each i in I we approximate a probability measure $\mu_i \in \mathbb{P}(A_i)$ by a discrete probability distribution μ_{k_i} on the partition set P_{k_i}. Then we can write the approximation to the payoff function (11) as

$$J_{k_i}(\mu_{k_1}, ..., \mu_{k_n}) := \sum_{\xi_{m_1} \in P_{k_1}} ... \sum_{\xi_{m_n} \in P_{k_n}} U_i(a_{m_1}, ..., a_{m_n})\mu_{k_n}(\xi_{m_n}) \cdots \mu_{k_1}(\xi_{m_1}). \quad (21)$$

For every $i \in I$ and every vector $\mu_k := (\mu_{k_1}, ..., \mu_{k_n})$ in $\mathbb{P}(P_{k_1}) \times ... \times \mathbb{P}(P_{k_n})$, we write μ_k as (μ_{k_i}, μ_{-k_i}), where $\mu_{k_{-i}} := (\mu_{k_1}, ..., \mu_{k_{i-1}}, \mu_{k_{i+1}}, ..., \mu_{k_n})$ is in $\mathbb{P}(P_{k_1}) \times ... \times \mathbb{P}(P_{k_{i-1}}) \times \mathbb{P}(P_{k_{i+1}}) \times ... \times \mathbb{P}(P_{k_n})$. If $\delta_{\{\xi_{m_i}\}}$ is a probability measure concentrated at $\xi_{m_i} \in P_{k_i}$, the vector $(\delta_{\{\xi_{m_i}\}}, \mu_{-i})$ is written as (a_{m_i}, μ_{-i}), and so

$$J_{k_i}(\delta_{\{\xi_{m_i}\}}, \mu_{k_{-i}}) = J_{k_i}(a_{m_i}, \mu_{k_{-i}}). \quad (22)$$

In particular, (21) yields

$$J_{k_i}(\mu_{k_i}, \mu_{k_{-i}}) := \sum_{\xi_{m_i} \in P_{k_i}} J_{k_i}(a_{m_i}, \mu_{k_{-i}})\mu_{k_i}(\xi_{m_i}). \quad (23)$$

Note that $\mu_k := (\mu_{k_1}, ..., \mu_{k_n})$ in $\mathbb{P}(P_{k_1}) \times ... \times \mathbb{P}(P_{k_n})$ is a vector of measures in $\mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n)$. Then for any $i \in I$ and $E_i \in \mathcal{B}(A_i) \cap P_{k_i}$, the replicator induced by $\{U_{k_i}\}_{i \in I}$ has the form,

$$\mu'_{k_i}(t, E_i) = \sum_{\xi_{m_i} \in E_i \cap P_{k_i}} \Big[J_{k_i}(a_{m_{k_i}}, \mu_{k_{-i}}(t)) - J_{k_i}(\mu_{k_i}(t), \mu_{k_{-i}}(t))\Big]\mu_{k_i}(t, \xi_{m_i}), \quad (24)$$

which is equivalent to the system of differential equations in $\mathbb{R}^{2^{k_1}+...+2^{k_n}}$ of the form:

$$\mu'_{k_i}(t, \xi_{m_i}) = \Big[J_{k_i}(a_{m_i}, \mu_{k_{-i}}(t)) - J_{k_i}(\mu_{k_i}(t), \mu_{k_{-i}}(t))\Big]\mu_{k_i}(t, \xi_{m_i}), \quad (25)$$

for $i = 1, 2, \dots, n$ and $m_i = 0, 1, \dots, 2^{ki} - 1$, with initial condition $\{\mu_{k_i,0}(\xi_{m_i})\}_{m_i=0}^{2^{ki-1}}$.

Hence, using Theorem 1 or Theorem 2, we can approximate (14), (15) (with $F_i(\cdot)$ as (18)) by a system of differential equations in $\mathbb{R}^{2^{k_1}+...+2^{k_n}}$ of the form (25).

4.2 Games with Strategies in Compact Metric Spaces

Similarly as in Sect. 4.1, consider an asymmetric game (17) where, for every i in I, A_i is a compact metric space and U_i is a real-valued bounded function. For every i in I, consider the partition $P_{k_i} := \{A_{m_i}\}_{m_i=0}^{2^{k_i}-1}$ over A_i. For every i in I and a fixed profile $(a_{m_1}, ..a_{m_i}, ..., a_{m_n}) \in A_{m_1} \times \cdots \times A_{m_i} \times \cdots \times A_{m_n}$, the discrete approximation to U_i is given by the function

$$U_{k_i}(x_1, ..., x_i, ..., x_n) := U_i(a_{m_1}, ..a_{m_i}, ..., a_{m_n}),$$

if $(x_1, ..., x_i, ..., x_n)$ is in $A_{m_1} \times \cdots \times A_{m_i} \times \cdots \times A_{m_n}$. If for each i in I we can approximate any probability measure $\mu_i \in \mathbb{P}(A_i)$ by a discrete probability distribution μ_{k_i} on the partition set P_{k_i}, then we can write the approximation to the payoff function (11) as

$$J_{k_i}(\mu_{k_1}, ..., \mu_{k_n}) := \sum_{A_{m_1} \in P_{k_1}} ... \sum_{A_{m_n} \in P_{k_n}} U_i(a_{m_1}, ..., a_{m_n}) \mu_{k_n}(A_{m_n}) \cdots \mu_{k_1}(A_{m_1}). \quad (26)$$

For every $i \in I$ and every vector $\mu_k := (\mu_{k_1}, ..., \mu_{k_n})$ in $\mathbb{P}(P_{k_1}) \times ... \times \mathbb{P}(P_{k_n})$, we write μ_k as (μ_{k_i}, μ_{-k_i}), where $\mu_{k_{-i}} := (\mu_{k_1}, ..., \mu_{k_{i-1}}, \mu_{k_{i+1}}, ..., \mu_{k_n})$ is in $\mathbb{P}(P_{k_1}) \times ... \times \mathbb{P}(P_{k_{i-1}}) \times \mathbb{P}(P_{k_{i+1}}) \times ... \times \mathbb{P}(P_{k_n})$. Note that $\mu_k := (\mu_{k_1}, ..., \mu_{k_n})$ in $\mathbb{P}(P_{k_1}) \times ... \times \mathbb{P}(P_{k_n})$ is a vector of measures in $\mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n)$. Then for any $i \in I$ and $E_i \in \mathcal{B}(A_i) \cap P_{k_i}$, the replicator induced by $\{U_{k_i}\}_{i \in I}$ has the following form:

$$\mu'_{k_i}(t, E_i) = \sum_{A_{m_i} \in E_i \cap P_{k_i}} \Big[J_{k_i}(a_{m_{k_i}}, \mu_{k_{-i}}(t)) - J_{k_i}(\mu_{k_i}(t), \mu_{k_{-i}}(t)) \Big] \mu_{k_i}(t, A_{m_i}), \quad (27)$$

which is equivalent to the system of differential equations in $\mathbb{R}^{2^{k_1}+...+2^{k_n}}$ of the form:

$$\mu'_{k_i}(t, A_{m_i}) = \Big[J_{k_i}(a_{m_i}, \mu_{k_{-i}}(t)) - J_{k_i}(\mu_{k_i}(t), \mu_{k_{-i}}(t)) \Big] \mu_{k_i}(t, A_{m_i}), \quad (28)$$

for $i = 1, 2, \ldots, n$ and $m_i = 0, 1, \ldots, 2^{k_i} - 1$, with initial condition $\{\mu_{k_i,0}(A_{m_i})\}_{m_i=0}^{2^{k_i}-1}$.

As in Sect. 4.1, using Theorem 1 or Theorem 2, we can approximate (14), (15) (with $F_i(\cdot)$ as (18)) by a system of differential equations in $\mathbb{R}^{2^{k_1}+...+2^{k_n}}$.

5 An Approximation Theorem in the Strong Form

In this section, we provide an approximation theorem that gives conditions under which we can approximate (14), (15) (with $F_i(\cdot)$ as in (18)) by a finite-dimensional dynamical system of the form (25) under the total variation norm (1).

The proof of this theorem uses the following two lemmas, which are proved in the appendix.

Lemma 1 *For each i in I, let A_i be a separable metric space. If each map μ_i : $[0, \infty) \to \mathbb{M}(A_i)$ is* strongly differentiable*, then*

$$\frac{d\|\mu(t)\|_\infty}{dt} \leq \|\mu'(t)\|_\infty.$$

Proof See Appendix.

Lemma 2 *For each i in I, let A_i be a separable metric space and let $F(\cdot)$ be as in (14), (15) (with F_i as in (18)). Suppose that for each i in I the payoff function $U_i(\cdot)$*

in (18) is bounded. Then

$$\|F(\nu) - F(\mu)\|_\infty \le Q\|\nu - \mu\|_\infty \quad \forall \mu, \nu \in \mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n), \tag{29}$$

where $Q := (2n+1)H$ *and* $H := \max_{i \in I} \|U_i\|$.

Proof See Appendix.

Theorem 1 *For each* i *in* I, *let* A_i *be a separable metric space and let* $U_i, U_i^\epsilon : A_1 \times ... \times A_n \to \mathbb{R}$ *be bounded functions such that* $\max_{i \in I} \|U_i - U_i^\epsilon\| < \epsilon$, *where* $\|\cdot\|$ *is the sup norm in (2). Consider the replicator dynamics induced by* $\{U_i\}_{i=1}^n$ *and* $\{U_i^\epsilon\}_{i=1}^n$, *i.e.,*

$$\mu_i'(t, E_i) = \int_{E_i} \Big[J_i(a_i, \mu_{-i}(t)) - J_i(\mu_i(t), \mu_{-i}(t))\Big]\mu_i(t, da_i), \tag{30}$$

$$\nu_i'(t, E_i) = \int_{E_i} \Big[J_i^\epsilon(a_i, \nu_{-i}(t)) - J_i^\epsilon(\nu_i(t), \nu_{-i}(t))\Big]\nu_i(t, da_i), \tag{31}$$

for each $i \in I$, $E \in \mathcal{B}(A_i)$, *and* $t \ge 0$. *If* $\mu(\cdot)$ *and* $\nu(\cdot)$ *are solutions of (30) and (31), respectively, with initial conditions* $\mu(0) = \mu_0$ *and* $\nu(0) = \nu_0$, *then for* $T < \infty$

$$\sup_{t \in [0,T]} \|\mu(t) - \nu(t)\|_\infty < \|\mu_0 - \nu_0\|_\infty e^{QT} + 2\epsilon\left(e^{QT} - \frac{1}{Q}\right). \tag{32}$$

where $Q := (2n+1)H$ *and* $H := \max_{i \in I} \|U_i\|$.

Proof For each i in I and $t \ge 0$, let

$$\beta_i(a_i|\mu) := J_i(a_i, \mu_{-i}) - J_i(\mu_i, \mu_{-i}), \quad \beta_i^\epsilon(a_i|\nu_i) := J_i^\epsilon(a_i, \nu_{-i}) - J_i^\epsilon(\nu_i, \nu_{-i}),$$

and

$$F_i(\mu, E_i) := \int_{E_i} \beta_i(a_i|\mu)\mu_i(da_i), \qquad F_i^\epsilon(\nu, E_i) := \int_{E_i} \beta_i^\epsilon(a_i|\nu)\nu_i(da_i).$$

Since U_i is bounded, by Lemma 2 there exists $Q > 0$ such that

$$\|F(\nu) - F(\mu)\|_\infty \le Q\|\nu - \mu\|_\infty \quad \forall \mu, \nu \in \mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n). \tag{33}$$

Actually, $Q := (2n+1)H$ and $H := \max_{i \in I} \|U_i\|$. We also have that, for all $i \in I$ and $\nu \in \mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n)$,

$$\|F_i(\nu) - F_i^\epsilon(\nu)\| \le \int_{A_i} |\beta_i(a_i|\nu) - \beta_i^\epsilon(a_i|\nu)|\nu_i(da_i) \le 2\|U_i - U_i^\epsilon\| \le 2\epsilon,$$

so

$$\|F(\nu) - F^{\epsilon}(\nu)\|_{\infty} \leq 2\epsilon. \tag{34}$$

By Lemma 1 and (33), (34), we have

$$\begin{aligned}
\frac{d\|\mu(t) - \nu(t)\|_{\infty}}{dt} &\leq \|\mu'(t) - \nu'(t)\|_{\infty} \\
&= \|F(\mu(t)) - F^{\epsilon}(\nu(t))\|_{\infty} \\
&\leq \|F(\mu(t)) - F(\nu(t))\|_{\infty} + \|F(\nu(t)) - F^{\epsilon}(\nu(t))\|_{\infty} \\
&\leq Q\|\mu(t) - \nu(t)\|_{\infty} + 2\epsilon.
\end{aligned}$$

Then

$$\frac{d\|\mu(t) - \nu(t)\|_{\infty}}{dt} - Q\|\mu(t) - \nu(t)\|_{\infty} \leq 2\epsilon.$$

Multiplying by e^{-Qt} we get

$$\frac{d\|\mu(t) - \nu(t)\|_{\infty} e^{-Qt}}{dt} - Q\|\mu(t) - \nu(t)\|_{\infty} e^{-Qt} \leq 2\epsilon e^{-Qt},$$

and integrating in the interval $[0, t]$, where $t \leq T$, we get

$$\|\mu(t) - \nu(t)\|_{\infty} e^{-Qt} - \|\mu_0 - \nu_0\|_{\infty} e^{-Q0} \leq 2\epsilon \left(\frac{1 - e^{-Qt}}{Q}\right).$$

Then for all $t \in [0, T]$

$$\begin{aligned}
\|\mu(t) - \nu(t)\|_{\infty} &= \|\mu_0 - \nu_0\|_{\infty} e^{Qt} + 2\epsilon \left(\frac{e^{Qt} - 1}{Q}\right) \\
&\leq \|\mu_0 - \nu_0\|_{\infty} e^{QT} + 2\epsilon \left(\frac{e^{QT} - 1}{Q}\right),
\end{aligned}$$

which yields (32). □

Remark 2 The last argument in the proof of Theorem 1 is a particular case of the well-known Gronwall–Bellman inequality: If $f(\cdot)$ is nonnegative and $f'(t) \leq Qf(t) + c$ for all $t \geq 0$, where Q and c are nonnegative constants, then

$$f(t) \leq f(0)e^{Qt} + cQ^{-1}(e^{Qt} - 1) \text{ for all } t \geq 0.$$

For the reader's convenience, we included the proof here. □

Note 1 As in Sects. 4.1 and 4.2, consider a game with strategies in compact metric spaces. For each player $i \in I$ consider a partition P_{k_i} of A_i and suppose that the initial condition $\mu_{i,0} \in \mathbb{P}(A_i)$ of (30) can be approximated in the variation norm by a discrete probability distribution $\mu_{k_i,0} \in \mathbb{P}(P_{k_i})$. Then for any $i \in I$ and $E_i \in \mathcal{B}(A_i) \cap P_{k_i}$, (31) can be written as (27) (or (24)), with U_i^{ϵ} as (26) (or (21)). So, in this particular case, (30) can be approximated by a system of differential equations in $\mathbb{R}^{2^{k_1}+\ldots+2^{k_n}}$ of the form (28).

Note 2 For the existence of the replicator dynamic, only the boundedness of the payoff functions is necessary (see Sect. 4 in [21]). So, the hypothesis of compactness on the set of strategies is not necessary in Theorem 1. Hence, the hypothesis of compactness on the set of strategies is also not necessary to approximate (30) by a finite-dimensional dynamical system. For example, it is sufficient that there exists a discrete probability distribution with finite values for any probability distribution over the set of strategies. For this last case, it is enough that for each $i \in I$, let A_i be a separable metric space, see Theorem 6.3, p. 44 in [26]. However, the compactness on the set of strategies ensures the existence of Nash equilibrium.

Corollary 1 *Let us assume the hypotheses of Theorem 1. Suppose that for each i in I, there exists a sequence of functions $\{U_i^{\epsilon_n}\}_{n=1}^{\infty}$ and probability measure vectors $\{\nu^n\}_{n=1}^{\infty}$ such that $\max_{i\in I} \|U_i - U_i^{\epsilon_n}\| \to 0$ and $\|\mu_0 - \nu_0^n\|_{\infty} \to 0$. If $\mu(\cdot)$ and $\nu^n(\cdot)$ are solutions of (30) and (31), respectively, with initial conditions $\mu(0) = \mu_0$ and $\nu^n(0) = \nu_0^n$, then for $T < \infty$,*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|\mu(t) - \nu_n(t)\|_{\infty} = 0.$$

6 An Approximation Theorem in the Weak Form

The next approximation result, Theorem 2, establishes the proximity of two paths generated by two different dynamical systems (the original model and a discrete approximating model) with different initial conditions, under the weak topology. To this end we use the Kantorovich–Rubinstein norm $\|\cdot\|_{kr}$ on $\mathbb{M}(A)$, which metrizes the weak topology.

Remark 3 Let A be a separable metric space. We say that a mapping $\mu : [0, \infty) \to \mathbb{M}(A)$ is weakly differentiable if there exists $\mu'(t) \in \mathbb{M}(A)$ such that for every $t > 0$ and $g \in \mathbb{C}(A)$

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[\int_A g(a)\mu(t+\epsilon, da) - \int_A g(a)\mu(t, da)\right] = \int_A g(a)\mu'(t, da). \tag{35}$$

If $\|\cdot\|_{k,r}$ is the Kantorovich–Rubinstein metric in (4), then (35) is equivalent to

$$\lim_{\epsilon\to 0} \left\| \frac{\mu(t+\epsilon) - \mu(t)}{\epsilon} - \mu'(t) \right\|_{kr} = 0. \tag{36}$$

Moreover if $\mu'(t)$ is the strong derivative of $\mu(t)$, then it is also the weak derivative of $\mu(t)$. Conversely, if $\mu'(t)$ is the weak derivative of $\mu(t)$ and $\mu(t)$ is continuous in t with the norm (1), then it is the strong derivative of $\mu(t)$. (See Heidergott, Hordijk, and Leahu [11].)

Lemma 3 *For each i in I, let A_i be a separable metric space. If each map $\mu_i : [0, \infty) \to \mathbb{M}(A_i)$ is* strongly differentiable*, then*

$$\frac{d\|\mu(t)\|_{\infty}^{kr}}{dt} \leq \|\mu'(t)\|_{\infty}^{kr}.$$

Proof The proof is similar to that of Lemma 1. □

Lemma 4 *For each i in I, consider a bounded separable metric space (A_i, ϑ_i) (with diameter $C_i > 0$) and the metric space $(A_1 \times ... \times A_n, \vartheta^*)$, where $\vartheta^*(a, b) = \max_{i \in I}\{\vartheta_i(a_i, b_i)\}$ for any a, b in $A_1 \times ... \times A_n$. Let $F(\cdot)$ be as in (14), (15) (with F_i as in (18)). For each i in I, suppose that the payoff function $U_i(\cdot)$ in (11) is bounded and satisfies that $\|U_i\|_L < \infty$. Then there exists $Q > 0$ such that*

$$\|F(\nu) - F(\mu)\|_{\infty}^{kr} \leq Q\|\nu - \mu\|_{\infty}^{kr} \tag{37}$$

for all $\mu, \nu \in \mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n) \cap \mathbb{M}_K(A_1) \times ... \times \mathbb{M}_K(A_n)$, where $Q := [2H + (2n - 1)CH_L]$, $H := \max_{i \in I} \|U_i\|$, $H_L := \max_{i \in I} \|U_i\|_L$, and $C := \max_{i \in I} C_i$.

Proof See Appendix.

Theorem 2 *For each i in I, let (A_i, ϑ_i) be a bounded separable metric space (with diameter $C_i > 0$), and $U_i, U_i^{\epsilon} : A_1 \times ... \times A_n \to \mathbb{R}$ be two bounded functions such that $\max_{i \in I} \|U_i - U_i^{\epsilon}\| < \epsilon$.. For each i in I, suppose that $\|U_i\|_L < \infty$ and consider the replicator dynamics induced by $\{U_i\}_{i=1}^n$ and $\{U_i^{\epsilon}\}_{i=1}^n$, as in (30) and (31). If $\mu(\cdot)$ and $\nu(\cdot)$ are solutions of (30) and (31), respectively, with initial conditions $\mu(0) = \mu_0$ and $\nu(0) = \nu_0$, then for $T < \infty$*

$$\sup_{t \in [0,T]} \|\mu(t) - \nu(t)\|_{\infty}^{kr} < \|\mu_0 - \nu_0\|_{\infty}^{kr} e^{QT} + 2\epsilon \left(e^{QT} - \frac{1}{Q}\right). \tag{38}$$

where $Q := [2H + (2n - 1)CH_L]$, $H := \max_{i \in I} \|U_i\|$, $H_L := \max_{i \in I} \|U_i\|_L$, and $C := \max_{i \in I} C_i$.

Proof For each i in I and $t \geq 0$, let

$$\beta_i(a_i|\mu) := J_i(a_i, \mu_i) - J_i(\mu_i, \mu_{-i}), \quad \beta_i^{\epsilon}(a_i|\nu_i) := J_i^{\epsilon}(a_i, \nu_{-i}) - J_i^{\epsilon}(\nu_i, \nu_{-i}),$$

and

$$F_i(\mu, E_i) := \int_{E_i} \beta_i(a_i|\mu)\mu_i(da_i), \qquad F_i^{\epsilon}(\nu, E_i) := \int_{E_i} \beta_i^{\epsilon}(a_i|\nu)\nu_i(da_i).$$

By Lemma 4 there exists $Q > 0$ such that

$$\|F(\nu) - F(\mu)\|_{\infty}^{kr} \leq Q\|\nu - \mu\|_{\infty}^{kr} \tag{39}$$

for all $\mu, \nu \in \mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n) \cap \mathbb{M}_K(A_1) \times ... \times \mathbb{M}_K(A_n)$. Actually, $Q := [2H + (2n-1)CH_L]$, $H := \max_{i \in I} \|U_i\|$, $H_L := \max_{i \in I} \|U_i\|_L$, and $C := \max_{i \in I} C_i$.

We also have that, for all i, in I and

$$\nu \in \mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n) \cap \mathbb{M}_K(A_1) \times ... \times \mathbb{M}_K(A_n),$$

$$\begin{aligned} \|F_i(\nu) - F_i^{\epsilon}(\nu)\|_{kr} &\leq \sup_{\substack{\|f\|_L \leq 1 \\ f(a_i^0)=0}} \int_{A_i} f(a_i)|\beta_i(a_i|\nu) - \beta_i^{\epsilon}(a_i|\nu)|\nu_i(da_i) \\ &\leq 2\|U_i - U_i^{\epsilon}\| \sup_{\substack{\|f\|_L \leq 1 \\ f(a_i^0)=0}} \int_{A_i} f(a_i)\nu_i(da_i) \\ &\leq 2C\epsilon. \end{aligned}$$

Then[1]

$$\|F(\nu) - F^{\epsilon}(\nu)\|_{\infty}^{kr} \leq 2C\epsilon. \tag{40}$$

By Lemma 3 and (39), (40) we have

$$\begin{aligned} \frac{d\|\mu(t) - \nu(t)\|_{\infty}^{kr}}{dt} &\leq \|\mu'(t) - \nu'(t)\|_{\infty}^{kr} \\ &= \|F(\mu(t)) - F^{\epsilon}(\nu(t))\|_{\infty}^{kr} \\ &\leq \|F(\mu(t)) - F(\nu(t))\|_{\infty}^{kr} + \|F(\nu(t)) - F^{\epsilon}(\nu(t))\|_{\infty}^{kr} \\ &\leq Q\|\mu(t) - \nu(t)\|_{\infty}^{kr} + 2C\epsilon. \end{aligned}$$

(See Remark 2 after Theorem 1.) The rest of the proof is similar to that done in Theorem 1. ☐

Note 3 As in Sects. 4.1 and 4.2, consider a game with strategies in compact metric spaces. For each player $i \in I$ let $\|U_i\|_L < \infty$ and consider a partition P_{k_i} of A_i.

[1]Note that if f satisfies that $\|f\|_L \leq 1$ and $f(a_i^0) = 0$, then $f(a_i) \leq \vartheta_i(a_i, a_i^0) \leq C_i$ for all $a_i \in A_i$. Therefore $\sup_{\substack{\|f\|_L \leq 1 \\ f(a_i^0)=0}} \int_{A_i} f(a_i)\nu_i(da_i) \leq C$.

Suppose that the initial condition $\mu_{i,0} \in \mathbb{P}(A_i)$ of (30) can be approximated in the weak form by a discrete probability distribution $\mu_{k_i,0} \in \mathbb{P}(P_{k_i})$, then for any $i \in I$ and $E_i \in \mathcal{B}(A_i) \cap P_{k_i}$, (31) can be written as (27) (or (24)), with U_i^{ϵ} as (26) (or (21)). So, in this particular case, (30) can be approximated by a system of differential equations in $\mathbb{R}^{2^{k_1}+\ldots+2^{k_n}}$ of the form (28).

Corollary 2 *Let us assume the hypotheses of Theorem 2. Suppose that for each i in I, there exist a sequences of functions $\{U_i^{\epsilon_n}\}_{n=1}^{\infty}$ and of vectors of probability measures $\{\nu^n\}_{n=1}^{\infty}$ such that $\max_{i\in I} \|U_i - U_i^{\epsilon_n}\| \to 0$ and $\|\mu_0 - \nu_0^n\|_{\infty}^{kr} \to 0$. If $\mu(\cdot)$ and $\nu^n(\cdot)$ are solutions of (30) and (31), respectively, with initial conditions $\mu(0) = \mu_0$ and $\nu^n(0) = \nu_0^n$, then, for $T < \infty$,*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|\mu(t) - \nu_n(t)\|_{\infty}^{kr} = 0.$$

7 Examples

7.1 A Linear-Quadratic Model: Symmetric Case

In this subsection, we consider a symmetric game in which we have two players with the following payoff function:

$$U(x, y) = -ax^2 - bxy + cx + dy, \tag{41}$$

with $a, b, c > 0$ and d any real number.

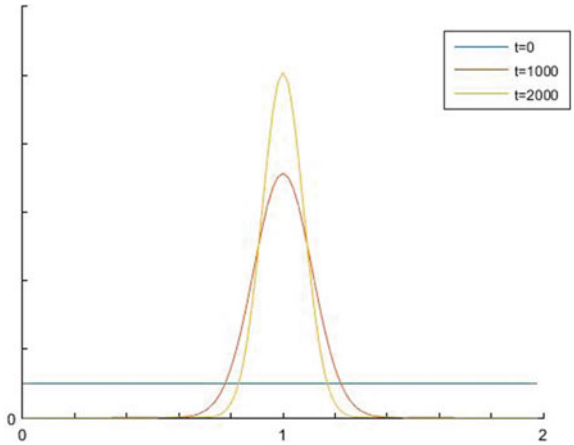
Let $A = [0, M]$, for $M > 0$, be the strategy set. If $2c(a - b) > 0$ and $4a^2 - b^2 > 0$, then we have an interior Nash equilibrium strategy (NES)

$$x^* = \frac{2c(a - b)}{4a^2 - b^2}.$$

Let $\mu(t)$ be the solution of the symmetric replicator dynamics induced by (41). Then if the initial condition is such that $\mu_0(x*) > 0$, we have that $\mu(t) \to \delta_{x^*}$ in distribution (see, [21–23]).

Consider a game where $a = 2,\ b = 1,\ c = 5,\ d = 1,\ M = 2$. For this game, the payoff function (41) is bounded Lipschitz and by Theorem 2 we can approximate the replicator dynamics by a finite-dimensional dynamical system of the form (25) under the Kantorovich–Rubinstein norm. Figure 1 shows a numerical approximation for this game where the Nash equilibrium is $x^* = 1$. For this numerical approximation, we consider a partition with 100 elements with the same size and use the forward Euler method for solving ordinary differential equations. We consider the uniform distribution as initial condition. We show the distribution for the times 0, 1000, and 2000.

Fig. 1 Linear Quadratic Model: Symmetric Case



Note that, under the strong norm, the Nash equilibrium $x^* = 1$ cannot be approximated by any probability measure with continuous density function.

7.2 Graduated Risk Game

The graduated risk game is a symmetric game (proposed by Maynard Smith and Parker [20]), where two players compete for a resource of value $v > 0$. Each player selects the "level of aggression" for the game. This "level of aggression" is captured by a probability distribution $x \in [0, 1]$, where x is the probability that neither player is injured, and $\frac{1}{2}(1 - x)$ is the probability that player one (or player two) is injured. If the player is injured its payoff is $v - c$ (with $c > 0$), and hence the expected payoff for the player is

$$U(x, y) = \begin{cases} vy + \frac{v-c}{2}(1 - y) & \text{if } y > x, \\ \frac{v-c}{2}(1 - x) & \text{if } y \leq x, \end{cases} \tag{42}$$

where x and y are the "levels of aggression" selected by the player and her opponent, respectively.

If $v < c$, this game has a Nash equilibrium strategy with the density function

$$\frac{d\mu^*(x)}{dx} = \frac{\alpha - 1}{2} x^{\frac{\alpha-3}{2}}, \tag{43}$$

where $\alpha = \frac{c}{v}$ (see Maynard Smith and Parker [20], and Bishop and Cannings [1]).

Let $\mu(t)$ be the solution of the symmetric replicator dynamics induced by (42). Then, for any initial condition μ_0 with support $[0, 1]$, we have that $\mu(t) \to \delta_{x^*}$ in distribution (see [22]).

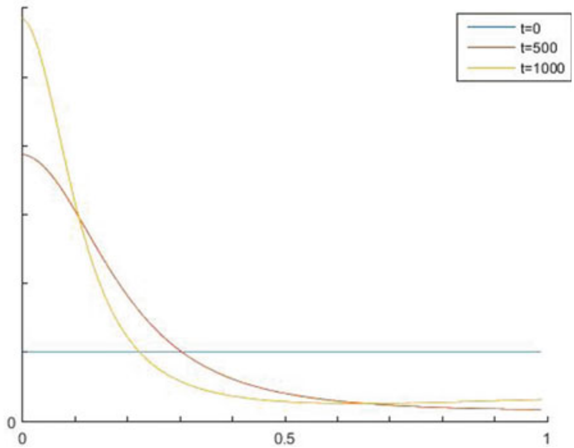


Fig. 2 Graduate Risk Game: Case $c = 10$; $v = 6 : 5$

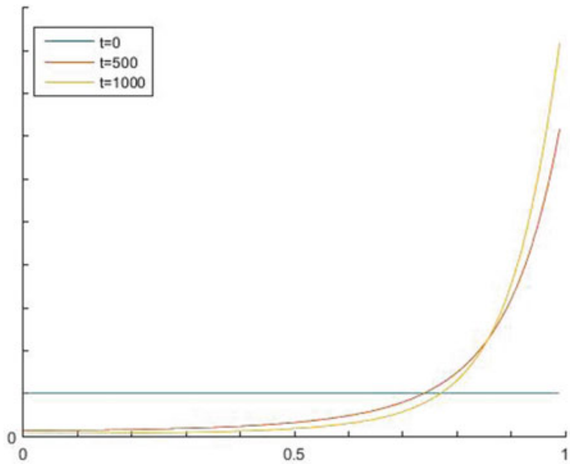


Fig. 3 Graduate Risk Game: Case $c = 10$; $v = 0 : 5$

Consider a game where $c = 10, \ v = 6.5$. For this game, the payoff function (42) is bounded, and by Theorem 1 we can approximate the replicator dynamics by a finite-dimensional dynamical system of the form (25) under the strong norm (1). Figure 2 shows a numerical approximation for this game. For this numerical approximation, we consider a partition with 100 elements with the same size, and use the forward Euler method for solving ordinary differential equations. We consider the uniform distribution as initial condition. We show the distribution for the times 0, 500, and 1000.

In the same way, Fig. 3 shows a numerical approximation for a game where $c = 10, \ v = 0.5$. For this numerical approximation, we consider a partition with 100 elements with the same size, and use the forward Euler method for solving ordinary differential equations. We consider the uniform distribution as initial condition. We show the distribution for the times 0, 500, and 1000.

8 Comments

In this paper, we introduced a model of asymmetric evolutionary games with strategies on measurable spaces. The model can be reduced, of course, to the symmetric case. We established conditions to approximate the replicator dynamics in a measure space by a sequence of dynamical systems on finite spaces. Finally, we presented two examples. The first one may be applicable to oligopoly models, theory of international trade, and public good models. The second example deals with a graduated risk game.

There are many questions, however, that remain open. For instance, the replicator dynamics has been studied in other general spaces without direct applications in game theory such as Kravvaritis et al. [15–18], and Papanicolaou and Smyrlis [25] studied conditions for stability and examples for these general cases. These extensions may be applicable in areas such as migration, regional sciences, and spatial economics (see Fujita et al. [10] Chaps. 5 and 6). An open question: can we establish conditions to approximate the replicator dynamics for general spaces by a sequence of dynamical systems on finite spaces?

In the theory of evolutionary games, there are several interesting dynamics, for instance, the imitation dynamics, the monotone-selection dynamics, the best-response dynamics, the Brown–von Neumann–Nash dynamics, and so forth (see, for instance, Hofbauer and Sigmund [13, 14], Sandholm [28]). Some of this evolutionary dynamics have been extended to games with strategies in a space of probability measures. For instance, Hofbauer et al. [12] extend the Brown–von Neumann–Nash dynamics; Lahkar and Riedel extend the logit dynamics [19]. These publications establish conditions for the existence of solutions and the stability of the corresponding dynamical systems. Cheung proposes a general theory for pairwise comparison dynamics [5] and for imitative dynamics [6]. Ruijgrok and Ruijgrok [27] extend the replicator dynamics with a mutation term. An open question: can we establish conditions to approximate other evolutionary dynamics for measurable spaces by a sequence of dynamical systems on finite spaces?

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Appendix: Proof of Lemmas

For the proof of Lemmas 2 and 4, it is convenient to rewrite (11) as

$$\mathcal{I}_{(\mu_1,\ldots,\mu_n)}U_i := \int_{A_1} \ldots \int_{A_n} U_i(a_1, \ldots, a_n)\mu_n(da_n)\ldots\mu_1(da_1). \tag{44}$$

Hence (12) becomes

$$J_i(a_i, \mu_{-i}) = \int_{A_{-i}} U_i(a_i, a_{-i})\mu_{-i}(da_{-i}) \\ = \mathcal{I}_{(\mu_1,\ldots,\mu_{i-1},\mu_{i+1},\ldots,\mu_n)} U_i(a_i). \tag{45}$$

Proof of Lemma 1

We have the following inequalities:

$$\begin{aligned}
\frac{d\|\mu(t)\|_\infty}{dt} &= \frac{d}{dt}\max_{i\in I}[\|\mu_i(t)\|] \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\max_{i\in I}[\|\mu_i(t+\epsilon)\|] - \max_{i\in I}[\|\mu_i(t)\|]\right] \\
&\leq \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\max_{i\in I}[\|\mu_i(t+\epsilon)\| - \|\mu_i(t)\|]\right] \\
&\leq \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\max_{i\in I}[\|\mu_i(t+\epsilon) - \mu_i(t)\|]\right] \\
&= \max_{i\in I}\left[\lim_{\epsilon\to 0}\left\|\frac{\mu_i(t+\epsilon)-\mu_i(t)}{\epsilon}\right\|\right] \\
&= \max_{i\in I}\left[\|\mu_i'(t)\|\right] \\
&= \|\mu'(t)\|.
\end{aligned}$$

□

Proof of Lemma 2

For any i in I and μ, ν in $\mathbb{P}(A_1) \times \ldots \times \mathbb{P}(A_1)$, using (44) we obtain

$$\begin{aligned}
&\left|\int_A U_i(a)\eta(da) - \int_A U_i(a)\nu(da)\right| \\
&\qquad \leq |\mathcal{I}_{(\eta_1,\eta_2,\ldots,\eta_n)}U_i - \mathcal{I}_{(\nu_1,\eta_2,\ldots,\eta_n)}U_i| \\
&\qquad\quad + |\mathcal{I}_{(\nu_1,\eta_2,\eta_3,\ldots,\eta_n)}U_i - \mathcal{I}_{(\nu_1,\nu_2,\eta_3,\ldots,\eta_n)}U_i| \\
&\qquad\quad + \ \ldots \\
&\qquad\quad + |\mathcal{I}_{(\nu_1,\ldots,\nu_{n-2},\eta_{n-1},\eta_n)}U_i - \mathcal{I}_{(\nu_1,\ldots,\nu_{n-2},\nu_{n-1},\eta_n)}U_i| \\
&\qquad\quad + |\mathcal{I}_{(\nu_1,\ldots,\nu_{n-1},\eta_n)}U_i - \mathcal{I}_{(\nu_1,\ldots,\nu_{n-1},\nu_n)}U_i| \\
&\qquad \leq \|U_i\|\|\eta_2 \times \ldots. \times \eta_n\|\|\eta_1 - \nu_1\| \\
&\qquad\quad + \|U_i\|\|\nu_1 \times \eta_3 \times \ldots \times \eta_n\|\|\eta_2 - \nu_2\| \\
&\qquad\quad + \ \ldots \\
&\qquad\quad + \|U_i\|\|\nu_1 \times \ldots \times \nu_{n-2} \times \eta_n\|\|\eta_{n-1} - \nu_{n-1}\|
\end{aligned}$$

$$
\begin{aligned}
&+ \|U_i\| \|\nu_1 \times .... \times \nu_{n-1}\| \|\eta_n - \nu_n\| \\
&\leq n\|U_i\| \max_{j \in I} \|\eta_j - \nu_j\|.
\end{aligned} \tag{46}
$$

Similarly, using (45),

$$
|J_i(a_i, \mu_{-i}) - J_i(a_i, \nu_{-i})| \leq (n-1)\|U_i\| \|\mu - \nu\|_\infty. \tag{47}
$$

Using (46) and (47), we have

$$
\begin{aligned}
\|F_i(\mu) - F_i(\nu)\|_\infty &= \sup_{\|f\| \leq 1} \int_{A_i} f(a_i)[F_i(\mu) - F_i(\nu)](da_i) \\
&\leq \sup_{\|f\| \leq 1} \int_{A_i} f(a_i)|J_i(a_i, \mu_{-i})|[\mu_i - \nu_i](da) \\
&\quad + \sup_{\|f\| \leq 1} \int_{A_i} f(a_i)|J_i(a_i, \mu_{-i}) - J_i(a_i, \nu_{-i})|\nu_i(da) \\
&\quad + \sup_{\|f\| \leq 1} \int_{A} f(a_i)|J_i(\mu_i, \mu_{-i})|[\nu_i - \mu_i](da) \\
&\quad + \sup_{\|f\| \leq 1} \int_{A} f(a_i)|J_i(\nu_i, \nu_{-i}) - J(\mu_i, \mu_{-i})|\nu_i(da) \\
&\leq \|U_i\| \|\mu_i - \nu_i\| + (n-1)\|U_i\| \|\mu - \nu\|_\infty \|\nu_i\| \\
&\quad + \|U_i\| \|\mu_i - \nu_i\| + n\|U_i\| \|\mu - \nu\|_\infty \|\nu_i\| \\
&\leq H\|\mu - \nu\|_\infty + (n-1)H\|\mu - \nu\|_\infty + H\|\mu - \nu\|_\infty + nH\|\mu - \nu\|_\infty \\
&= (2n+1)H\|\mu - \nu\|_\infty,
\end{aligned}
$$

where $H := \max_{i \in I} \|U_i\|$. □

Proof of Lemma 4

For any i and j in I and a_{-j} in A_{-j} let

$$
\|U_i(\cdot, a_{-j})\|_L := \sup_{a_j, b_j \in A_j} \frac{|U_i(a_j, a_{-j}) - U_i(b_j, a_{-j})|}{\vartheta^*((a_j, a_{-j}), (b_j, a_{-j}))} \leq \|U_i\|_L, \quad \text{and}
$$

$$
U_i^j := \frac{U_i(a_j, a_{-j})}{\|U_i(\cdot, a_{-j})\|_L}.
$$

Then for any i in I and μ, ν in $\mathbb{P}(A_1) \times ... \times \mathbb{P}(A_1)$, using (44) we see that

$$\left|\int_A U_i(a)\eta(da) - \int_A U_i(a)\nu(da)\right|$$

$$\begin{aligned}
&\leq \|U_i(\cdot, a_{-1})\|_L |\mathcal{I}_{(\eta_1,\eta_2,\ldots,\eta_n)} U_i^1 - \mathcal{I}_{(\nu_1,\eta_2,\ldots,\eta_n)} U_i^1| \\
&\quad + \|U_i(\cdot, a_{-2})\|_L |\mathcal{I}_{(\nu_1,\eta_2,\eta_3,\ldots,\eta_n)} U_i^2 - \mathcal{I}_{(\nu_1,\nu_2,\eta_3,\ldots,\eta_n)} U_i^2| \\
&\quad + \ \ldots \\
&\quad + \|U_i(\cdot, a_{-(n-1)})\|_L |\mathcal{I}_{(\nu_1,\ldots,\nu_{n-2},\eta_{n-1},\eta_n)} U_i^{n-1} - \mathcal{I}_{(\nu_1,\ldots,\nu_{n-2},\nu_{n-1},\eta_n)} U_i^{n-1}| \\
&\quad + \|U_i(\cdot, a_{-n})\|_L |\mathcal{I}_{(\nu_1,\ldots,\nu_{n-1},\eta_n)} U_i^n - \mathcal{I}_{(\nu_1,\ldots,\nu_{n-1},\nu_n)} U_i^n| \\
&\leq \|U_i\|_L \|\eta_2 \times \ldots . \times \eta_n\| \|\eta_1 - \nu_1\|_{kr} \\
&\quad + \|U_i\|_L \|\nu_1 \times \eta_3 \times \ldots \times \eta_n\| \|\eta_2 - \nu_2\|_{kr} \\
&\quad + \ \ldots \\
&\quad + \|U_i\|_L \|\nu_1 \times \ldots \times \nu_{n-2} \times \eta_n\| \|\eta_{n-1} - \nu_{n-1}\|_{kr} \\
&\quad + \|U_i\|_L \|\nu_1 \times \ldots . \times \nu_{n-1}\| \|\eta_n - \nu_n\|_{kr} \\
&\leq n\|U_i\|_L \|\eta_j - \nu_j\|_\infty^{kr}.
\end{aligned} \tag{48}$$

Similarly, using (45),

$$|J_i(a_i, \mu_{-i}) - J_i(a_i, \nu_{-i})| \leq (n-1)\|U_i\|_L \|\mu - \nu\|_\infty^{kr}. \tag{49}$$

Using (48) and (49) we have

$$\|F_i(\mu) - F_i(\nu)\|_{kr}$$

$$\begin{aligned}
&= \sup_{\substack{\|f\|_L \leq 1 \\ f(a_0)=0}} \int_{A_i} f(a_i)[F_i(\mu) - F_i(\nu)](da_i) \\
&\leq \sup_{\substack{\|f\|_L \leq 1 \\ f(a_0)=0}} \int_{A_i} f(a_i)|J_i(a_i, \mu_{-i})|[\mu_i - \nu_i](da) \\
&\quad + \sup_{\substack{\|f\|_L \leq 1 \\ f(a_0)=0}} \int_{A_i} f(a_i)|J_i(a_i, \mu_{-i}) - J_i(a_i, \nu_{-i})|\nu_i(da) \\
&\quad + \sup_{\substack{\|f\|_L \leq 1 \\ f(a_0)=0}} \int_{A} f(a_i)|J_i(\mu_i, \mu_{-i})|[\nu_i - \mu_i](da) \\
&\quad + \sup_{\substack{\|f\|_L \leq 1 \\ f(a_0)=0}} \int_{A} f(a_i)|J_i(\nu_i, \nu_{-i}) - J(\mu_i, \mu_{-i})|\nu_i(da) \\
&\leq \|U_i\| \|\mu_i - \nu_i\|_{kr} + (n-1)\|U_i\|_L \|\mu - \nu\|_\infty^{kr} \sup_{\substack{\|f\|_L \leq 1 \\ f(a_0)=0}} \int_{A_i} f(a_i)\nu_i(da_i)
\end{aligned}$$

$$+ \|U_i\| \|\mu_i - \nu_i\|_{kr} + n\|U_i\|_L \|\mu - \nu\|_\infty^{kr} \sup_{\substack{\|f\|_L \le 1 \\ f(a_0)=0}} \int_{A_i} f(a_i)\nu_i(da_i)$$

$$\le 2H\|\mu - \nu\|_\infty^{kr} + (2n-1)H_L\|\mu - \nu\|_\infty^{kr} C_i$$

$$= [2H + (2n-1)CH_L]\|\mu - \nu\|_\infty,$$

where $H := \max_{i\in I} \|U_i\|$, $H_L := \max_{i\in I} \|U_i\|_L$, and $C := \max_{i\in I} C_i$. □

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