Entropies for Negatively Curved Manifolds

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This is a survey of several notions of entropy related to a compact manifold of negative curvature and of some relations between them. Namely, let (M, g) be a C^∞ compact boundaryless Riemannian connected manifold with negative curvature. After recalling the basic definitions, we will define and state the first properties of

(1) the volume entropy V,
(2) the dynamical entropies of the geodesic flow, in particular the entropy H of the Liouville measure and the topological entropy (which coincides with V),
(3) the stochastic entropy h_ρ of a family of (biased) diffusions related to the stable foliation of the geodesic flow,
(4) the relative dynamical entropy of natural stochastic flows representing the (biased) diffusions.

Most of the material in this survey are not new, some are classical, and we apologize in advance for any inaccuracy in the attributions. New observations are Theorems 2.5 and 4.9, but the main goal of this survey is to present together related notions that are spread out in the literature. In particular, we are interested in the different so-called rigidity results and problems that (aim to) characterize locally

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symmetric spaces among negatively curved manifolds by equalities in general entropy inequalities.

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1 Local Symmetry and Volume Growth

Let (M, g) be a C^∞ compact boundaryless connected d-dimensional Riemannian manifold and for u, v vector fields on M we denote $\nabla_u v$ the covariant derivative of v in the direction of u. Given $u, v \in T_x M$, the *curvature tensor* R associates with a vector $w \in T_x M$ the vector $R(u, v)w$ given by

$$R(u, v)w \;=\; \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w.$$

The space (M, g) is called *locally symmetric* if $\nabla R = 0$.

Consider the case (M, g) has *negative sectional curvature*; i.e., for all non-colinear $u, v \in T_x M$, $x \in \widetilde{M}$, the *sectional curvature* $K(u, v) := \dfrac{< R(u, v)v, u >}{|u \wedge v|^2}$ is negative. Simply connected locally symmetric spaces of negative sectional curvature are non-compact. They have been classified and are one of the hyperbolic spaces $\mathbb{H}^n_{\mathbb{R}}, \mathbb{H}^n_{\mathbb{C}}, \mathbb{H}^n_{\mathbb{H}}, \mathbb{H}^2_{\mathbb{O}}$, respectively of dimension respectively $n, 2n, 4n, 16$. Hyperbolic spaces are obtained as quotients of semisimple Lie groups of real rank one (respectively $SO(n, 1), SU(n, 1), Sp(n, 1), F_{4(-20)}$), endowed with the metrics coming from the Killing forms, by maximal compact subgroups. By general results of Borel [6] and Selberg [51], these spaces admit compact boundaryless quotient manifolds and those locally symmetric (M, g_0) are the basic examples of our objects of study. Clearly, C^2 small C^∞ perturbations of g_0 on the same space M yield other examples of compact negatively curved manifolds. Different examples of non-locally symmetric, compact, negatively curved manifolds have been constructed (see [16, 18, 22, 45]). They are supposed to be abundant, even if constructing explicit ones is often delicate.

It is natural to ask if we can recognize locally symmetric spaces through global properties or quantities. One supportive example is the volume entropy. Let $\widetilde{M}$ be the universal cover space of M such that $M = \widetilde{M}/\Gamma$, where $\Gamma := \Pi_1(M)$ is the fundamental group of M, and endow $\widetilde{M}$ with metric $\widetilde{g}$, which is the Γ-invariant extension of g. The volumes on (M, g) and $(\widetilde{M}, \widetilde{g})$ are denoted Vol_g and $\mathrm{Vol}_{\widetilde{g}}$, respectively. (We will fix a connected fundamental domain M_0 for the action of Γ on $\widetilde{M}$. The restriction of $\mathrm{Vol}_{\widetilde{g}}$ on M_0 is also denoted Vol_g.) For $x \in \widetilde{M}$, let

$B_{\widetilde{M}}(x,r), r > 0$, denote the ball centered at x with radius r. The following limit exists (independent of $x \in \widetilde{M}$) and defines the *volume entropy* (Manning, [43]):

$$V(g) := \lim_{r\to\infty} \frac{1}{r} \log \mathrm{Vol}_{\widetilde{g}} B_{\widetilde{M}}(x,r).$$

Since (M, g) is negatively curved, by Bishop comparison theorem, $V(g) > 0$. The following rigidity result is shown by Besson–Courtois–Gallot [5]:

Theorem 1.1 ([5]) *Let* (M, g_0) *be closed locally symmetric space of negative curvature, and consider another metric* g *on* M *with negative curvature and such that* $\mathrm{Vol}_g(M) = \mathrm{Vol}_{g_0}(M)$. *Then,*

$$V(g) \ \geq V(g_0).$$

If $d = dim(M) > 2$*, one has equality only if* (M, g) *is isometric to* (M, g_0).

If $d = 2$, equality holds if, and only if, the curvature is constant (Katok, [30]). In the case $d > 2$, Katok [30] proved Theorem 1.1 under the hypothesis that g is conformally equivalent to g_0.

Remark 1.2 The theorem holds even if g' is a metric on another manifold M', homotopically equivalent to M.

The locally symmetric property can also be interpreted as geodesic symmetry. A *geodesic* in M is a curve $t \mapsto \gamma(t), t \in \mathbb{R}$, such that, if $\dot\gamma(t) := \frac{d}{dt}\gamma(s)\big|_{s=t}$, satisfies $\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0$ for all t. For all $v \in TM$, there is a unique geodesic γ_v such that $\dot\gamma_v(0) = v$. The *exponential map* $\exp_x : T_xM \to M$ is given by $\exp_x v = \gamma_v(1)$. By compactness, there exists $\iota > 0$ such that, for all $x \in M$, $\exp_x$ is a diffeomorphism between the ball of radius ι in (T_xM, g_x) and the ball of radius ι about x in M. The Cartan–Ambrose–Hicks Theorem implies that the space is locally symmetric if, and only if, for any $x \in M$, the geodesic symmetry about x defined by $y \mapsto \exp_x(-\exp_x^{-1} y)$ is a local isometry.

One natural dynamics related to geodesics is the geodesic flow. Let $SM := \{v, v \in TM : \|v\| = 1\}$ be the unit tangent bundle. The *geodesic flow* φ_t on SM is such that $\varphi_t(v) = \dot\gamma_v(t)$ for $t \in \mathbb{R}$. Denote $\overline{X}(v) \in T_vSM$ the vector field on SM generating the geodesic flow. The derivative $D_v\varphi_t$ is described using *Jacobi fields*. Let $s \mapsto v(s)$ be a curve in SM with $v(0) = v, \dot v(0) = w \in T_vSM$. Then, $s \mapsto \gamma_{v(s)}(t)$ is a curve with tangent vector $J(t)$ at $\gamma_v(t)$. $J(t)$ satisfies the *Jacobi equation:*

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma} J(t) + R(J(t), \dot\gamma(t))\dot\gamma(t) = 0. \tag{1.1}$$

Proof By definition,

$$R(J(t), \dot\gamma(t))\dot\gamma(t) = \nabla_{J(t)}\nabla_{\dot\gamma(t)}\dot\gamma(t) - \nabla_{\dot\gamma(t)}\nabla_{J(t)}\dot\gamma(t) - \nabla_{[J(t),\dot\gamma(t)]}\dot\gamma(t).$$

We have $\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0$ by definition, $[J(t), \dot\gamma(t)] = [\frac{\partial}{\partial s}, \frac{\partial}{\partial t}] = 0$ and so $\nabla_{J(t)}\dot\gamma(t) = \nabla_{\dot\gamma(t)}J(t)$ (we use the fact that $\nabla_u v - \nabla_v u = [u, v]$). □

We will consider C^∞ compact boundaryless connected Riemannian manifolds with negative sectional curvature. It follows from (1.1) that $t \mapsto \|J(t)\|^2$ is a strictly convex function (by a direct computation). In particular, $\exp_x$ is a diffeomorphism from T_xM to the universal cover $\widetilde M$. Two geodesic rays γ_1, γ_2 in $\widetilde M$ are said to be equivalent if $\sup_{t\geq 0} d(\gamma_1(t), \gamma_2(t)) < \infty$. The space of equivalence classes $\partial\widetilde M := \{[\gamma_v(t), t \geq 0], v \in TM\}$ is the *geometric boundary at infinity*. For $x \in \widetilde M$, $\pi_x : S_x\widetilde M \to \partial\widetilde M$, $\pi_x(v) = [\gamma_v(t), t \geq 0]$ is one-to-one (π_x is injective by convexity (of $t \mapsto d(\gamma_v(t), \gamma_w(t))$ for $w \in S_x\widetilde M$ with $w \neq v$) and for any geodesic ray γ, any $t > 0$, one can find $v_t \in S_x\widetilde M$ such that $\gamma(t) \in \gamma_{v_t}(s), s \geq 0$; any limit point v of $v_t, t \to +\infty$, is such that γ_v is equivalent to γ). Thus, the unit tangent bundle $S\widetilde M$ is identified with $\widetilde M \times \partial\widetilde M$. For any two points ξ, η in $\partial\widetilde M$, there is a unique geodesic $\gamma_{\eta,\xi}$ (up to time translation) such that $\gamma_{\eta,\xi}(+\infty) := \lim_{t\to+\infty}\gamma_{\eta,\xi}(t) = \xi$ and $\gamma_{\eta,\xi}(-\infty) := \lim_{t\to-\infty}\gamma_{\eta,\xi}(t) = \eta$. The topology on $\widetilde M \times \partial\widetilde M$ is such that two pairs (x, ξ) and (y, η) are close if x and y are close and the distance from x to the geodesic $\gamma_{\eta,\xi}$ is large. The group Γ acts discretely and cocompactly on $\widetilde M$. The action of Γ extends continuously to $\partial\widetilde M$ and the diagonal action of Γ on $\widetilde M \times \partial\widetilde M$ is again discrete and cocompact. The quotient $(\widetilde M \times \partial\widetilde M)/\Gamma = S\widetilde M/\Gamma$ is identified with SM.

We continue to use φ_t to denote the geodesic flow on $S\widetilde M$. It has the *Anosov property* [3]: each $\varphi_t, t \neq 0$, has no fixed point and there is a continuous decomposition $\{T_vS\widetilde M = E^{ss}(v) \oplus \overline X(v) \oplus E^{su}(v), v \in S\widetilde M\}$ with $\overline X(v)$ being the geodesic spray tangent to the flow direction and constants $C, C > 0$, $\lambda, \lambda \in (0, 1)$, such that, for $t > 0$,

$$\|D_v\varphi_t w_s\| \leq C\lambda^t\|w_s\|,\ \forall w_s \in E^{ss}(v),\ \|D_v\varphi_{-t}w_u\| \leq C\lambda^t\|w_u\|,\ \forall w_u \in E^{su}(v).$$

For $v = (x, \xi) \in S\widetilde M$, the *stable manifold at* v of the geodesic flow,

$$\widetilde W^s(v) := \left\{w : \sup_{t\geq 0} d(\varphi_t w, \varphi_t v) < +\infty\right\}$$

is tangent to $E^{ss}(v) \oplus \overline X(v)$. The $\widetilde W^s(v)$ can be identified with $\widetilde M \times \{\xi\}$ and hence is endowed naturally with the metric $\widetilde g$. The quotient $(\widetilde M \times \{\xi\})/\Gamma$ is the *stable manifold* $W^s(v)$. As ξ varies, they form a Hölder continuous lamination $\mathcal W^s$ of SM into C^∞ manifolds of dimension d which is called the *stable foliation*. Therefore, the metric on each individual stable manifold comes from the local identification with $\widetilde M$. The *strong stable manifold at* v,

$$\widetilde W^{ss}(v) := \left\{(y, \xi) : \lim_{t\to+\infty} d(\gamma_{x,\xi}(t), \gamma_{y,\xi}(t)) = 0\right\}$$

has tangent $E^{ss}(v)$. Let $\underline v$ be the projection of v on SM; then, $\widetilde W^{ss}(v)$ projects onto

$$W^{ss}(\underline{v}) := \left\{w \in SM : \lim_{t\to+\infty} d(\gamma_w(t), \gamma_{\underline{v}}(t)) = 0\right\}.$$

The collection of $\{W^{ss}(\underline{v}), \underline{v} \in SM\}$ forms a Hölder continuous lamination $\mathcal{W}^{ss}$ of SM into C^∞ manifolds of dimension $d-1$ which is called the *strong stable foliation*.

For $v = (x, \xi) \in S\widetilde{M}$, define the *Busemann function*

$$b_{x,\xi}(y) = b_{x,\xi}(y, \xi) \ := \ \lim_{z\to\xi} \left(d(y, z) - d(x, z)\right), \ \forall y \in \widetilde{M}.$$

The level set $\{(y, \xi) : b_{x,\xi}(y, \xi) = 0\}$ coincides with $\widetilde{W}^{ss}(x, \xi)$ and the set of its foot points is the horosphere of (x, ξ). Denote Div^s, ∇^s the divergence and gradient along $\widetilde{W}^s$ (and W^s) induced by the metric $\widetilde{g}$ on $\widetilde{M} \times \{\xi\}$, $\Delta^s = \mathrm{Div}^s\nabla^s$. Then,

$$\nabla_y b_{x,\xi}(y)|_{y=x} = -(x, \xi) \text{ or } \nabla^s_w b_v(w)|_{w=v} = -\overline{X}(v).$$

Set

$$B(x, \xi) := \Delta_y b_{x,\xi}(y)|_{y=x} = -\mathrm{Div}^s\overline{X}(v).$$

Geometrically, the $B(x, \xi)$ is the mean curvature at x of the horosphere of (x, ξ). The function B is a Γ-invariant function on $S\widetilde{M}$. We still denote B the function on the quotient SM. From the definition follows:

$$B(v) = -\frac{d}{dt} \log \mathrm{Det} D_v\varphi_t|_{W^{ss}(v)}\Big|_{t=0}. \tag{1.2}$$

So, dynamically, $-B$ tells the exponential growth rate of the volume on W^{ss} under the geodesic flow φ_t, $t > 0$. It follows from (1.2) that the function B is Hölder continuous on SM. The main property of the function B is the following, whose proof combines the works of Benoist–Foulon–Labourie [4], Foulon–Labourie [20], and Besson–Courtois–Gallot [5].

Theorem 1.3 ([4, 5, 20]) *The function B is constant if, and only if, the space (M, g) is locally symmetric.*

Remark 1.4 There is a positive operator U on the orthogonal space to v in T_xM satisfying the Riccati equation $\dot{U}+U^2+R(\cdot, \dot\gamma(t))\dot\gamma(t) = 0$ and such that $B = \mathrm{Tr}U$. If $d = 2$, the equation reduces to $\dot{B} + B^2 + K = 0$. Clearly, if B is constant, then the curvature K is the constant $-B^2$. If $d = 3$, one can also conclude from the Riccati equation and some matrix calculations that B is constant if, and only if, the sectional curvature is constant (see Knieper [33]).

2 Dynamical Entropy and an Application of Thermodynamical Formalism

More quantities related to V, B can be introduced through a dynamical point of view.

2.1 Dynamical Entropy

Let T be a continuous transformation of a compact metric space $\mathbf{X}$. For $x \in \mathbf{X}, \varepsilon > 0, n \in \mathbb{N}$, define the *Bowen ball* $B(x, \varepsilon, n)$

$$B(x, \varepsilon, n) := \{y \in \mathbf{X} : d(T^j y, T^j x) < \varepsilon \text{ for } 0 \leq j \leq n\}$$

and the *entropy* $h_m(T)$ of a T-invariant probability measure m

$$h_m(T) := \sup_{\varepsilon} \int \left(\limsup_n -\frac{1}{n} \log m(B(x, \varepsilon, n)) \right) dm(x).$$

It is easy to see that for $j \in \mathbb{Z}, h_m(T^j) = |j| h_m(T)$. A useful upper bound of $h_m(T)$ is given by Ruelle inequality [50] using the average maximal exponential growth rate of all the parallelograms under the iteration of the tangent map DT.

Theorem 2.1 (Ruelle, [50]) *Assume* $\mathbf{X}$ *is a compact manifold and* T *a* C^1 *mapping of* $\mathbf{X}$*. Then, for any* T*-invariant probability measure* m*,*

$$h_m(T) \ \leq \ \int \left(\sup_k \limsup_n \frac{1}{n} \log \| \wedge^k D_x T^n \| \right) dm(x),$$

where $\wedge^k D_x T^n$ *denotes the* k*-th exterior power of* $D_x T^n$*.*

Corollary 2.2 *If* $\mathbf{X} = SM$*, where* (M, g) *is a compact, boundaryless,* C^2 *Riemannian manifold with negative sectional curvature and dimension* d*,* m *a geodesic flow invariant probability measure, and* $t \in \mathbb{R}$*,*

$$h_m(\varphi_t) \ \leq \ |t| \int_{SM} B \, dm.$$

Proof For $v \in SM, t < 0, |t|$ large, the highest value of $\| \wedge^k D_v \varphi_t \|$ is obtained for $k = d - 1$ and is the Jacobian of $D_v \varphi_t$ restricted to $T_v W^{ss}(v)$. By (1.1), this is $e^{\int_t^0 B(\varphi_s v)\, ds}$. By the ergodic theorem,

$$\lim_{n \to +\infty} \frac{1}{n} \log \left\| \wedge^{d-1} D_v \varphi_{nt} |_{W^{ss}} \right\| = \lim_{n \to +\infty} \frac{1}{n} \int_{nt}^0 B(\varphi_s v) \, ds$$

exists and has integral $|t| \int B\,dm$. The conclusion follows by Ruelle inequality. □

Another general inequality is given by

Theorem 2.3 (Manning, [43]) *Let (M, g) be a compact, boundaryless, C^2 Riemannian manifold with negative sectional curvature and dimension d, m a geodesic flow invariant probability measure, and $t \in \mathbb{R}$,*

$$h_m(\varphi_t) \;\leq\; |t| V.$$

Remark 2.4 The proof of Theorem 2.3 is based on the following consequence of nonpositive curvature ([43], Lemma page 571). For any $v, w \in SM$, any $r \geq 1$,

$$\begin{aligned}\max\{\sup_{0\leq s\leq 1} d(\varphi_s v, \varphi_s w), \sup_{r-1\leq s\leq r} d(\varphi_s v, \varphi_s w)\} &\leq \sup_{0\leq s\leq r} d(\varphi_s v, \varphi_s w)\\ &\leq \sup_{0\leq s\leq 1} d(\varphi_s v, \varphi_s w)\\ &\quad + \sup_{r-1\leq s\leq r} d(\varphi_s v, \varphi_s w).\end{aligned}$$

This observation can also be used to give a direct proof of Corollary 2.2.

2.2 Thermodynamical Formalism

For simplicity, we introduce the notion of pressure by the classical variational principle. Let $(\mathbf{X}, T)$ be a continuous mapping of a compact metric space. The *Pressure* $P(F)$ of a continuous function $F : \mathbf{X} \to \mathbb{R}$ is defined by

$$P(F) \;:=\; \sup_m \left\{ h_m(T) + \int F\,dm \right\},$$

where m runs over all T-invariant probability measures. Let $\mathbf{X} = SM$, where M is closed negatively curved and $T = \varphi_1$. From Ruelle and Manning inequalities follow

$$P(-B) \leq 0 \quad \text{and} \quad P(0) \leq V.$$

We will construct later the *Liouville measure* m_L with the property (Theorem 2.6)

$$h_{m_L}(\varphi_1) = \int B\,dm_L =: H \tag{2.1}$$

and the *Bowen–Margulis measure* m_{BM} such that (Theorem 3.3)

$$h_{m_{BM}}(\varphi_1) = V. \tag{2.2}$$

This will show that $P(-B) = 0$ and $P(0) = V$. Using these properties, we can prove:

Theorem 2.5 *Let (SM, φ_t) be the geodesic flow on a closed manifold of negative curvature. Let $\mathcal{M}$ be the set of φ_t-invariant probability measures, H and V as defined above. Then,*

$$\inf_{m\in\mathcal{M}} \int B\,dm \le H \le V \le \sup_{m\in\mathcal{M}} \int B\,dm, \tag{2.3}$$

with equality in one of the inequalities if, and only if, $m_L = m_{BM}$. Moreover, in that case, $\int B\,dm = V$ for all $m \in \mathcal{M}$.

Proof Since the function B is Hölder continuous on SM, for each $s \in \mathbb{R}$, there exists a unique invariant probability measure m_s (equilibrium measure for sB) such that $\mathrm{P}(s) := P(sB) = h_{m_s}(\varphi_1) + s\int B\,dm_s$ [46, Proposition 4.10].[1] For example, by (2.1), (2.2), m_L, m_{BM} are equilibrium measures for $-B$ and 0, respectively. Together with Corollary 2.2, we obtain

$$\inf_{m\in\mathcal{M}} \int B\,dm \le \int B\,dm_L = H \le \sup_{m\in\mathcal{M}} \{h_m(\varphi_1)\} = V \le \int B\,dm_{BM} \le \sup_{m\in\mathcal{M}} \int B\,dm,$$

which gives (2.3).

Clearly, using the uniqueness of m_s, we have that $H = V$ if, and only if $m_L = m_{BM}$. To show any equality in the other inequalities of (2.3) holds if, and only if, $m_L = m_{BM}$, we use properties of the Pressure function, in particular of the convex function $s \mapsto \mathrm{P}(s)$. We already know that $\mathrm{P}(-1) = 0$ and that $\mathrm{P}(0) = V$. From the definition follows that $\inf_{m\in\mathcal{M}} \int B\,dm$ and $\sup_{m\in\mathcal{M}} \int B\,dm$ are the slopes of the asymptotes of the function $\mathrm{P}(s)$ as $s \to -\infty$ and $+\infty$, respectively. Since the function B is Hölder continuous on SM, the function $s \mapsto \mathrm{P}(s)$ is real analytic [46, Proposition 4.8]. Moreover, the slope at s is given by $\int B\,dm_s$ [46, Proposition 4.10]. Now, if $H = \inf_{m\in\mathcal{M}} \int B\,dm$, the function $s \mapsto \mathrm{P}(s)$ is affine on $[-\infty, -1]$ and thus everywhere. Since the slopes of $\mathrm{P}(s)$ at -1 and 0 are $\int B\,dm_L = H$ and $\int B\,dm_{BM}$, respectively, and $H \le V \le \int B\,dm_{BM}$, hence we must have $V = \int B\,dm_{BM}$, which implies that m_{BM} coincides with m_L and $V = H$. Finally, if $V = \sup_{m\in\mathcal{M}} \int B\,dm$, the measure m_{BM} is the equilibrium measure for $-B$, which must coincide with m_L.

Assume m_{BM} and m_L coincide, then by [46, Proposition 4.9], there exists a continuous function F on SM, C^1 along the trajectories of the geodesic flow, such that

[1] Chapter 4 in [46] is only concerned with subshifts of finite type. The extension of [46] Propositions 4.8, 4.9, 4.10 to suspended flows is direct (see [46], Chapter 6) and the application to geodesic flows on compact negatively curved manifolds is standard (cf. [46], Appendix 3).

$$-B = \mathrm{P}(-1) - \mathrm{P}(0) + \frac{\partial}{\partial t} F \circ \varphi_t\big|_{t=0}.$$

In particular, $\int B\, dm = P(0) = V$ for all $m \in \mathcal{M}$. □

2.3 Liouville Measure

For $x \in \widetilde{M}$, let λ_x denote the pull back measure on $\partial \widetilde{M}$ of the Lebesgue probability measure on $S_x \widetilde{M}$ through the mapping $\pi_x^{-1} : \partial \widetilde{M} \mapsto S_x \widetilde{M},\ \xi \mapsto (x, \xi)$. Define a measure $\widetilde{m}_L$ on $\widetilde{M} \times \partial \widetilde{M}$ by setting

$$\int F(x,\xi)\, d\widetilde{m}_L = \int_{\widetilde{M}} \left(\int_{\partial \widetilde{M}} F(x,\xi)\, d\lambda_x(\xi) \right) \frac{d\mathrm{Vol}_{\widetilde{g}}(x)}{\mathrm{Vol}_g(M)}.$$

It is clear from the definition that the measure $\widetilde{m}_L$ is Γ-invariant. There is a $D\varphi_t$-invariant 2-form on $\overline{X}^{\perp}$ in TSM defined by the Wronskian $\mathcal{W}$

$$\mathcal{W}\big((J_1, J_1'), (J_2, J_2')\big) \ := \ < J_1(t), J_2'(t) > - < J_1'(t), J_2(t) > .$$

Assume M is orientable. The $(2d-1)$-form $\wedge^{d-1}\mathcal{W} \wedge dt$ is nondegenerate and invariant. For $v \in SM$, take a positively oriented orthonormal basis $\{e_0, \cdots, e_{n-1}\}$ in $T_x M$ such that $e_0 = v$. By computing $\wedge^{d-1}\mathcal{W} \wedge dt$ on the $(2d-1)$-vector $\big((e_1, 0), (0, e_1), \cdots, (e_{n-1}, 0), (0, e_{n-1}), \overline{X}\big)$, one sees that the measure associated with this volume form is the one we defined. So the measure $\widetilde{m}_L$ is invariant under the geodesic flow. We do the same computation on a double cover of M if M is not orientable.

The measure m_L on SM that extends to $\widetilde{m}_L$ is a φ_t-invariant probability measure which is called the *Liouville probability measure*. It satisfies

Theorem 2.6 *For all $t \in \mathbb{R}$, $\ h_{m_L}(\varphi_t) = |t| \int B\, dm_L$.*

Proof (Sketch) It suffices to prove the theorem for $t = -1$. In the definition of entropy, we can use the flow Bowen balls $\mathbf{B}(v, \varepsilon, r)$, $\varepsilon, r > 0$,

$$\mathbf{B}(v, \varepsilon, r) := \left\{ w : \sup_{-r \le s \le 0} d(\varphi_s v, \varphi_s w) < \varepsilon \right\}.$$

By Remark 2.4,

$$\mathbf{B}(v, \varepsilon/2, 1) \cap \varphi_{r-1}\mathbf{B}(\varphi_{-r+1}v, \varepsilon/2, 1) \subset \mathbf{B}(v, \varepsilon, r) \subset \mathbf{B}(v, \varepsilon, 1) \cap \varphi_{r-1}\mathbf{B}(\varphi_{-r+1}v, \varepsilon, 1).$$

Estimating the Liouville measure of $\mathbf{B}(v, \varepsilon, 1) \cap \varphi_{r-1}\mathbf{B}(\varphi_{-r+1}v, \varepsilon, 1)$ reduces to estimating the d-dimensional measure of $B^s(v, \varepsilon) \cap \varphi_{r-1} B^s(\varphi_{-r+1}v, \varepsilon)$, where $B^s(v, a)$ is the ball of radius a and center v in $W^s(v)$. It follows from (1.2) that this measure is, up to error terms that depend on ε small enough, but not on r, equal to

$$\mathrm{Det} D_{\varphi_{-r+1}v}\varphi_r|_{W^s(\varphi_{-r+1}v)} = e^{-\int_{-r+1}^0 B(\varphi_s v)\,ds}.$$

It follows that, if one takes ε small enough,

$$h_{m_L}(\varphi_{-1}) = \lim_{r\to+\infty} \frac{1}{r} \int_{SM} \left(\int_{-r+1}^0 B(\varphi_s v)\,ds \right) dm_L(v) \;=\; \int_{SM} B\, dm_L.$$

□

Observe that, since m_L is a measure realizing the maximum in $P(-B)$, it is ergodic.

Remark 2.7 Basic facts about ergodic theory and thermodynamic formalism are in Bowen [8]; see also Parry–Pollicott [46]. The definition of the entropy given here is due to Brin–Katok [11]. The ergodicity of m_L with respect to the geodesic flow is a landmark result of Anosov [3].

3 Patterson–Sullivan, Bowen–Margulis, Burger–Roblin

In analogy to the construction of the measure m_L, one can obtain the Bowen–Margulis measure m_{BM} using a class of measures (Patterson–Sullivan measures) on the boundary at infinity.

3.1 Patterson–Sullivan

Theorem 3.1 *There exists a family of measures on $\partial\widetilde{M}$, $x \mapsto \nu_x$, $x \in \widetilde{M}$, such that*

$$\nu_{\beta x} = \beta_* \nu_x, \text{ for } \beta \in \Gamma, \text{ and } \frac{d\nu_y}{d\nu_x}(\xi) = e^{-V b_{x,\xi}(y)}. \tag{3.1}$$

The family is unique if normalized by $\int_M \nu_x(\partial\widetilde{M})\, d\mathrm{Vol}_g(x) \;=\; 1$. *Moreover, the measures ν_x are continuous.*

Proof We first show the existence of such a family. Fix $x_0 \in \widetilde{M}$. It suffices to construct the family $\nu_{\beta x_0}$, $\beta \in \Gamma$, such that

$$\text{for all } \beta \in \Gamma,\ \nu_{\beta x_0} = \beta_* \nu_{x_0} \text{ and } \frac{d\nu_{\beta x_0}}{d\nu_{x_0}}(\xi) = e^{-V b_{x_0,\xi}(\beta x_0)}. \tag{3.2}$$

Indeed, assume such a family $\nu_{\beta x_0}, \beta \in \Gamma$, is constructed, we then set $\nu_y := e^{-Vb_{x_0,\xi}(y)}\nu_{x_0}$ for all $y \in \widetilde{M}$. Using the cocycle property of the Busemann function:

$$b_{x,\xi}(\beta y) = b_{x,\beta^{-1}\xi}(y) + b_{x,\xi}(\beta x), \ \forall x, y \in \widetilde{M}, \xi \in \partial\widetilde{M},$$

one can easily check that the class of measures $\{\nu_y\}$ satisfies the requirement of (3.1).

Recall $V = \lim_{R\to+\infty} \frac{1}{R} \log \mathrm{Vol}_{\widetilde{g}} B_{\widetilde{M}}(x_0, R)$. Set, for $s > V$, a family $\nu^s_{\beta x_0}, \beta \in \Gamma$, with $d\nu^s_{\beta x_0}(y) := \dfrac{e^{-sd(\beta x_0,y)}\, d\mathrm{Vol}_{\widetilde{g}}(y)}{\int_{\widetilde{M}} e^{-sd(x_0,y)}\, d\mathrm{Vol}_{\widetilde{g}}(y)}$. We have

$$\begin{aligned}\beta_* d\nu^s_{x_0}(y) \ = \ d\nu^s_{x_0}(\beta^{-1}y) \ &= \ \frac{e^{-sd(x_0,\beta^{-1}y)}\, d\mathrm{Vol}_{\widetilde{g}}(y)}{\int_{\widetilde{M}} e^{-sd(x_0,y)}\, d\mathrm{Vol}_{\widetilde{g}}(y)} \\ &= \ \frac{e^{-sd(\beta x_0,y)}\, d\mathrm{Vol}_{\widetilde{g}}(y)}{\int_{\widetilde{M}} e^{-sd(x_0,y)}\, d\mathrm{Vol}_{\widetilde{g}}(y)} \ = \ d\nu^s_{\beta x_0}(y).\end{aligned}$$

Recall that $\widetilde{M} \cup \partial\widetilde{M}$ is compact and assume that $\int_{\widetilde{M}} e^{-sd(x_0,y)}\, d\mathrm{Vol}_{\widetilde{g}}(y) \to \infty$ as $s \searrow V$. Choose $s_n \searrow V$ such that $\nu^{s_n}_{x_0}$ weak* converge to ν_{x_0}. Then, ν_{x_0} is supported by $\partial\widetilde{M}$. Moreover, for any $\beta \in \Gamma$, $\nu^{s_n}_{\beta x_0}$ weak* converge as well and call $\nu_{\beta x_0} := \lim_{s_n\searrow V} \nu^{s_n}_{\beta x_0}$. The family $\nu_{\beta x_0}, \beta \in \Gamma$, satisfies (3.2). Indeed, $\nu_{\beta x_0} = \beta_*\nu_{x_0}$. Moreover, consider an open cone C based on x_0. We have, for any $\beta \in \Gamma$,

$$\nu_{\beta x_0}(C) = \lim_{s_n\searrow V} \nu^{s_n}_{\beta x_0}(C) = \lim_{s_n\searrow V} \int_C e^{-s_n(d(\beta x_0,y)-d(x_0,y))}\, d\nu^{s_n}_{x_0}(y).$$

As $s_n \searrow V$, most of the $\nu^{s_n}_{x_0}$ measure is supported by a neighborhood of $\partial\widetilde{M}$ and, for y close to $\xi \in \partial\widetilde{M}$, $d(\beta x_0, y) - d(x_0, y)$ is close to $b_{x_0,\xi}(\beta x_0)$. The density property follows.

If $\int_{\widetilde{M}} e^{-sd(x_0,y)}\, d\mathrm{Vol}_{\widetilde{g}}(y)$ is bounded, use Patterson's trick [47, Lemma 3.1]: one can find a real function L on $\mathbb{R}_+$ such that

$$\lim_{s\searrow V} \int_{\widetilde{M}} L(d(x_0, y))e^{-sd(x_0,y)}\, d\mathrm{Vol}_{\widetilde{g}}(y) = \infty \quad \text{and} \quad \forall a \in \mathbb{R}, \ \lim_{t\to+\infty} \frac{L(t+a)}{L(t)} = 1.$$

We can then replace the previous family $\nu^s_{\beta x_0}, \beta \in \Gamma$, by $\nu'^s_{\beta x_0}\beta \in \Gamma$, with $d\nu'^s_{\beta x_0}(y) := \frac{L(d(\beta x_0,y))e^{-sd(\beta x_0,y)}\, d\mathrm{Vol}_{\widetilde{g}}(y)}{\int_{\widetilde{M}} L(d(x_0,y))e^{-sd(x_0,y)}\, d\mathrm{Vol}_{\widetilde{g}}(y)}$.

The function $x \mapsto \nu_x(\partial\widetilde{M})$ is Γ-invariant and continuous; in particular, it is bounded. This implies that the measure ν_{x_0} is continuous since otherwise, there is $\xi \in \partial\widetilde{M}$ with $\nu_{x_0}(\{\xi\}) = a > 0$. When $\{y_n\}_{n\in\mathbb{N}} \in \widetilde{M}$ converge to ξ, $\nu_{y_n}(\{\xi\}) = e^{-Vb_{x_0,\xi}(y_n)}a \to +\infty$, a contradiction.

We will see later (Remark 3.5) that such a family is unique, up to multiplication by a constant factor. □

The family $\nu_x, x \in \widetilde{M}$, is called the family of *Patterson–Sullivan measures*.

3.2 Bowen–Margulis

Define, for $x \in \widetilde{M}, \xi, \eta \in \partial \widetilde{M}$, the *Gromov product*

$$(\xi, \eta)_x := \frac{1}{2} \lim_{y \to \xi, z \to \eta} (d(x, y) + d(x, z) - d(y, z)) .$$

The Gromov product is a nonnegative number (by the triangle inequality) and because of pinched negative curvature, the Gromov product is finite; actually it is (exercise) uniformly bounded away from the distance from x to the geodesic $\gamma_{\eta,\xi}$. Moreover, the Gromov product satisfies the cocycle relation

$$(\xi, \eta)_{x'} - (\xi, \eta)_x = \frac{1}{2}(b_{x,\xi}(x') + b_{x,\eta}(x')). \tag{3.3}$$

Let $\widetilde{M}^{(2)} := \{(\xi, \eta) \in \partial \widetilde{M} \times \partial \widetilde{M}, \xi \neq \eta\}$. Then, $S\widetilde{M}$ is identified with $\widetilde{M}^{(2)} \times \mathbb{R}$ by the *Hopf coordinates:*

$$v \mapsto (\gamma_v(+\infty), \gamma_v(-\infty), b_v(x_0)).$$

Proposition 3.2 *Let $\nu_x, x \in \widetilde{M}$, be the family of Patterson–Sullivan measures. The measure ν with $d\nu(\xi, \eta) := \frac{d\nu_x(\xi) \times d\nu_x(\eta)}{e^{-2V(\xi,\eta)_x}}$ does not depend on x. The measure $\nu \times dt$ on $\widetilde{M}^{(2)} \times \mathbb{R}$ is Γ-invariant and invariant by the geodesic flow.*

Proof The first affirmation follows directly from the cocycle relation (3.3). In particular, the measure ν is Γ-invariant on $\partial \widetilde{M} \times \partial \widetilde{M}$. The measure ν is supported by $\widetilde{M}^{(2)}$ because ν_x is continuous. The actions of Γ and of φ_s in Hopf coordinates are given by:

$$\beta(\xi, \eta, t) = (\beta\xi, \beta\eta, t + b_{x_0,\xi}(\beta^{-1}x_0)), \text{ for } \beta \in \Gamma,$$
$$\varphi_s(\xi, \eta, t) = (\xi, \eta, t + s).$$

The invariance of $\nu \times dt$ under the actions of Γ and of φ_s follows. □

We call *Bowen–Margulis measure* m_{BM} the unique probability measure on SM such that its Γ-invariant extension is proportional to $\nu \times dt$. It satisfies

Theorem 3.3 $h_{m_{BM}}(\varphi_t) = |t|V.$

Proof (Sketch) We follow the sketch of the proof of Theorem 2.6. We have to estimate m_{BM} $(\mathbf{B}(v, \varepsilon, 1) \cap \varphi_{r-1}\mathbf{B}(\varphi_{-r+1}v, \varepsilon, 1))$. Choose ε small enough that this set lifts to $S\widetilde{M}$ into a set of the same form. In Hopf coordinates, this is, up to some constant A, of the form:

$\mathbf{B}(v, \varepsilon, 1)$

$$\asymp \left\{(\xi, \eta, t) : \xi \in C(\varphi_{1/2}v, A^{\pm 1}\varepsilon), \eta \in C(-\varphi_{1/2}v, A^{\pm 1}\varepsilon), b_v(x_0) \le t \le b_v(x_0)+1\right\},$$

where, for $w \in S\widetilde{M}$ and $0 < \delta < \pi$, $C(w, \delta)$ is the cone of geodesics starting from w with an angle smaller than δ. Our set $\mathbf{B}(v, \varepsilon, 1) \cap \varphi_{r-1}\mathbf{B}(\varphi_{-r+1}v, \varepsilon, 1)$ is

$$\left\{(\xi, \eta, t) : \xi \in C(\varphi_{1/2}v, A^{\pm 1}\varepsilon), \eta \in C(-\varphi_{-r+3/2}v, A^{\pm 1}\varepsilon), b_v(x_0) \le t \le b_v(x_0)+1\right\}.$$

The $\nu \times dt$ measure of this set is within $A^{\pm 2}e^{(-r+3/2)V} m_{BM}(\mathbf{B}(v, \varepsilon, 1))$. □

Corollary 3.4 *$P(0) = V$ and m_{BM} is the measure of maximal entropy for the geodesic flow φ_t. In particular, m_{BM} is ergodic.*

Remark 3.5 It also follows from this construction that the Patterson–Sullivan family ν_x is unique. Indeed, let ν'_x be another Patterson–Sullivan family. One can construct as above a family ν', $d\nu'(\xi, \eta) := \frac{d\nu_x(\xi) \times d\nu'_x(\eta)}{e^{-2V(\xi,\eta)_x}}$. By the same reasoning, the measure $\nu' \times dt$ is proportional to an invariant probability measure with entropy V. It follows that ν' is proportional to ν; i.e., ν'_x is proportional to ν_x for all x.

3.3 Burger–Roblin

Define a measure $\widetilde{m}_{BR}$ on $\widetilde{M} \times \partial\widetilde{M}$ by setting, for all continuous function F with compact support on $S\widetilde{M}$,

$$\int F(x, \xi)\, d\widetilde{m}_{BR} = \int_{\widetilde{M}} \left(\int_{\partial\widetilde{M}} F(x, \xi)\, d\nu_x(\xi) \right) d\mathrm{Vol}_{\widetilde{g}}(x). \tag{3.4}$$

It follows from the definition that the measure $\widetilde{m}_{BR}$ is Γ-invariant. Call m_{BR} the induced measure on SM; by our normalization, we have $m_{BR}(SM) = 1$. The measure m_{BR} is called the *Burger–Roblin measure.* Many of its properties follow from

Theorem 3.6 *For any vector field Z on SM such that $Z(v)$ is tangent to $W^s(v)$ for all $v \in SM$, we have*

$$\int_{SM} \mathrm{Div}^s Z(v) + V < Z(v), \overline{X}(v) > dm_{BR}(v) \ = \ 0. \tag{3.5}$$

Proof Using a partition of unity, we may assume that Z has compact support inside a flow-box for the foliation. Choosing a reference point x_0, we can write $dm_{BR}(x, \xi) = e^{-Vb_{x_0,\xi}(y)} d\nu_{x_0}(y) d\mathrm{Vol}_{\widetilde{g}}(y)$. Since Z has compact support on each local stable leaf $W^s_{loc}(x, \xi)$, we have

$$\int_{W^s_{loc}(x,\xi)} \mathrm{Div}^s_y \left(e^{-Vb_{x_0,\xi}(y)} Z(y,\xi)\right)\Big|_{y=z} d\mathrm{Vol}_{\widetilde{g}}(z) = 0$$

for all $(x,\xi) \in S\widetilde{M}$. Then, (3.5) follows by developing

$$\begin{aligned}
&\mathrm{Div}^s_y \left(e^{-Vb_{x_0,\xi}(y)} Z(y,\xi)\right)\Big|_{y=z} \\
&= \left(\mathrm{Div}^s_y Z(y,\xi)\Big|_{y=z} + V < Z(z,\xi), \overline{X}(z,\xi) >\right) e^{-Vb_{x_0,\xi}(z)}.
\end{aligned}$$

□

Corollary 3.7 $\int B\, dm_{BR} = V$.

Proof Apply (3.5) to $Z = \overline{X}$. □

Corollary 3.8 *The operator $\Delta^s + V\overline{X}$ is symmetric for m_{BR}: for $F_1, F_2 \in C^\infty(SM)$, the set of smooth functions on SM,*

$$\int_{SM} F_1(\Delta^s + V\overline{X})F_2\, dm_{BR} = \int_{SM} F_2(\Delta^s + V\overline{X})F_1\, dm_{BR}.$$

Hence, m_{BR} is also stationary for the operator $\Delta^s + V\overline{X}$, i.e., for all $F \in C^\infty(SM)$, $\int_{SM}(\Delta^s + V\overline{X})F\, dm_{BR} = 0$.

Proof Apply (3.5) to $Z = F_1\nabla^s F_2$ to get

$$\int_{SM} F_1(\Delta^s + V\overline{X})F_2\, dm_{BR} = -\int_{SM} < \nabla^s F_1, \nabla^s F_2 > dm_{BR}.$$

The Right Hand Side is invariant when switching F_1 and F_2. □

Corollary 3.9 *The measure m_{BR} is symmetric for the Laplacian Δ^{ss} along the strong stable foliation $\mathcal{W}^{ss}$: for $F_1, F_2 \in C^\infty(SM)$,*

$$\int_{SM} F_1\Delta^{ss}F_2\, dm_{BR} = \int_{SM} F_2\Delta^{ss}F_1\, dm_{BR}.$$

So, m_{BR} is also stationary for the operator Δ^{ss}, i.e., for all $F \in C^\infty(SM)$, $\int_{SM} \Delta^{ss}F\, dm_{BR} = 0$.

Proof Apply (3.5) to $Z = F_1 \frac{d}{dt} F_2 \circ \varphi_t\big|_{t=0}\overline{X}$ to obtain that

$$\begin{aligned}
&\int_{SM} F_1 \left(\frac{d^2}{dt^2} F_2 \circ \varphi_t\Big|_{t=0} - B\frac{d}{dt} F_2 \circ \varphi_t\Big|_{t=0} + V\frac{d}{dt} F_2 \circ \varphi_t\Big|_{t=0}\right) dm_{BR} \\
&= -\int_{SM} \overline{X}F_1\overline{X}F_2\, dm_{BR}.
\end{aligned}$$

Recall that in horospherical coordinates, Δ^s can be written as

$$\Delta^s F = \frac{d^2}{dt^2} F \circ \varphi_t \big|_{t=0} - B \frac{d}{dt} F \circ \varphi_t \big|_{t=0} + \Delta^{ss} F.$$

Replacing in the formula above, we get that

$$- \int_{SM} F_1 \Delta^{ss} F_2 \, dm_{BR} + \int_{SM} F_1 (\Delta^s + V\overline{X}) F_2 \, dm_{BR} \ = \ - \int_{SM} \overline{X} F_1 \overline{X} F_2 \, dm_{BR}.$$

The conclusion follows from Corollary 3.8. □

Remark 3.10 We observe that m_{BR} is ergodic. Indeed, strong stable manifolds have polynomial volume growth[2] , so a symmetric measure for the Laplacian Δ^{ss} along the strong stable foliation $\mathcal{W}^{ss}$ is given locally by the product of the Lebesgue measure along the W^{ss} leaves and some family of measures on the transversals (Kaimanovich, [28]). This family has to be *invariant* under the holonomy map of the W^{ss} leaves. By Bowen–Marcus [9], there exists only one holonomy-invariant family on the transversals to the $\mathcal{W}^{ss}$ foliation, up to a multiplication by a constant factor.

Remark 3.11 The family of measures in this section has a long history. The invariant measures for the $\mathcal{W}^{ss}$ foliation were first constructed by Margulis [44] and used to construct the invariant measure m_{BM}. Margulis' construction (in the strong unstable case) amounts to taking the limit of the normalized Lebesgue measure on $\varphi_T S_x M$ (see also Knieper [33]). Margulis did not state that the measure m_{BM} has maximal entropy, and the measure of maximal entropy was constructed by Bowen (cf. Bowen [8], Bowen–Ruelle [10]) as the limit as $T \to +\infty$ of equidistributed measures on closed geodesics of length smaller than T. Bowen also showed that the measure of maximal entropy is unique, so that the two constructions give the same measure m_{BM}. Independently, Patterson [47] constructed the measures ν_x in the case of hyperbolic surfaces, not necessarily compact; Sullivan [52] extended the construction to a general hyperbolic space, observed that it is, up to normalization, the Hausdorff measure on the limit set of the discrete group in its Hausdorff dimension for the angle metric, that it is also the conformal measure for the action of the group on its limit set and moreover, the exit measure of the Brownian motion with suitable drift. He also made its connection with the measure of maximal entropy (in the constant curvature case). Hamenstädt [24] connected m_{BM} with the Patterson–Sullivan construction and then many authors extended the Patterson–Sullivan construction to many circumstances (see Paulin–Pollicott–Schapira [48] for a detailed recent survey). Again in the hyperbolic geometrically finite case, Burger [12] considered m_{BR} as the measure invariant by the horocycle action; finally,

[2]There are constants C, k such that the volume of the balls of radius r for the induced metric on strong stable manifolds is bounded by Cr^k.

Roblin [49] considered the general case of a group acting discretely on a $CAT(-1)$ space. What is remarkable is that in all these constructions, these measures were introduced as tools, and not, like here, as objects interesting in their own right. A posteriori, their interest comes from all these applications.

4 A Family of Stable Diffusions; Probabilistic Rigidity

Recall (Corollary 3.8) that the Burger–Roblin measure m_{BR} is a stationary measure for $\Delta^s + V\overline{X}$. In this section, we study the stationary measures for $\Delta^s + \rho\overline{X}$, $\rho < V$, characterize them in analogy to m_{BR}, and state a rigidity result concerning these measures.

4.1 Foliated Diffusions

A differential operator $\mathcal{L}$ on SM is called *subordinate to the stable foliation* $\mathcal{W}^s$ if, for any $F \in C^\infty(SM)$, $\mathcal{L}F(v)$ depends only on the values of F along $W^s(v)$. It is given by a Γ-equivariant family $\mathcal{L}_\xi$ on $\widetilde{M} \times \{\xi\}$. A probability measure m is called *stationary* for $\mathcal{L}$ (or $\mathcal{L}$*-stationary*, $\mathcal{L}$*-harmonic*) if, for all $F \in C^\infty(SM)$,

$$\int \mathcal{L}F(v)\, dm(v) \;=\; 0.$$

Theorem 4.1 (Garnett, [21]) *Assume $\mathcal{L}$-stationary is an operator which is subordinate to $\mathcal{W}^s$, has continuous coefficients, and is elliptic on W^s leaves. Then, the set of $\mathcal{L}$-stationary probability measures is a non-empty convex compact set. Extremal points are called ergodic.*

We will consider the operators $\mathcal{L}^\rho := \Delta^s + \rho\overline{X}$ for $\rho \in \mathbb{R}$. Clearly, each $\mathcal{L}^\rho$ is subordinate to $\mathcal{W}^s$ and for all $F \in C^\infty(SM)$,

$$\mathcal{L}^\rho_\xi F(x,\xi) = \Delta^s_y F(y,\xi)|_{y=x} + \rho < \overline{X}, \nabla^s_y F(y,\xi)|_{y=x} >_{x,\xi} .$$

For a fixed ξ, $\mathcal{L}^\rho_\xi$ is elliptic on $\widetilde{M}$ and Markovian ($\mathcal{L}^\rho_\xi 1 = 0$). Hence, by Theorem 4.1, there is always some $\mathcal{L}^\rho$-stationary measure. Let m_ρ be a $\mathcal{L}^\rho$-stationary measure. Then, locally [21], on a local flow-box of the lamination the measure m_ρ has conditional measures along the leaves that are absolutely continuous with respect to Lebesgue, and the density K^ρ satisfies $\mathcal{L}^{\rho *}K^\rho = 0$, where $\mathcal{L}^{\rho *}$ is the formal adjoint of $\mathcal{L}^\rho$ with respect to Lebesgue measure on the leaf, i.e.,

$$\mathcal{L}^{\rho *}F \;=\; \Delta^s F - \rho \mathrm{Div}^s(F\overline{X}). \tag{4.1}$$

Globally, there exists a Γ-equivariant family of measures ν_x^ρ such that the Γ-invariant extension $\widetilde{m}_\rho$ of m_ρ is given by a formula analogous to (3.4):

$$\int F(x,\xi)\, d\widetilde{m}_\rho \;=\; \int_{\widetilde{M}} \left(\int_{\partial \widetilde{M}} F(x,\xi)\, d\nu_x^\rho(\xi) \right) d\mathrm{Vol}_{\widetilde{g}}(x).$$

Indeed, choose a transversal to the foliation $\mathcal{W}^s$, say the sphere $S_{x_0}M$ and write SM as $M_0 \times S_{x_0}M$. A stationary measure m_ρ is given by an integral for some measure $d\nu(\xi)$ of measures of the form $\mathsf{K}^\rho(x,\xi)\, d\mathrm{Vol}_g(x)$, where Vol_g is the volume on M_0. We can arrange that $\mathsf{K}^\rho(x_0,\xi) = 1$, ν-a.e.. For a lift $\widetilde{x}_0 =: x$, set $\nu_x^\rho = (\pi_x)_*\nu$. The family $\nu_{\beta x}^\rho$, $\beta \in \Gamma$, is Γ-equivariant by construction. Starting from a different point $y_0 \in M_0$, the same construction gives a Γ-equivariant family $\nu_{\beta y}^\rho$, $\beta \in \Gamma$, for the lifts y of y_0. By construction also,

$$\frac{d\nu_y^\rho}{d\nu_x^\rho}(\xi) \;=\; \frac{\mathsf{K}^\rho(y,\xi)}{\mathsf{K}^\rho(x,\xi)}.$$

The same proof as for the relation (3.5) yields, for any vector field Z on SM such that $Z(v)$ is tangent to $W^s(v)$ for all $v \in SM$,

$$\int_{SM} \mathrm{Div}^s Z + < Z, \nabla_y^s \log \mathsf{K}^\rho(y,\xi)\big|_{y=x} > \, dm_\rho(v) = 0. \tag{4.2}$$

For each $\mathcal{L}^\rho$, there is a *diffusion*, i.e., a Γ-equivariant family of probability measures $\widetilde{\mathbb{P}}_{x,\xi}^\rho$ on $C(\mathbb{R}_+, S\widetilde{M})$ such that $t \mapsto \widetilde{\omega}(t)$ is a Markov process with generator $\mathcal{L}_\xi^\rho$, $\widetilde{\mathbb{P}}_{x,\xi}^\rho$-a.s. $\widetilde{\omega}(0) = (x,\xi)$ and $\widetilde{\omega}(t) \in \widetilde{M} \times \{\xi\}$, $\forall t > 0$. The distribution of $\widetilde{\omega}(t)$ under $\widetilde{\mathbb{P}}_{x,\xi}^\rho$ is $p_\xi^\rho(t,x,y) d\mathrm{Vol}_{\widetilde{g}}(y)\delta_\xi(\eta)$, where $p_\xi^\rho(t,x,y)$ is the fundamental solution of the equation $\frac{\partial F}{\partial t} = \mathcal{L}_\xi^\rho F$. The quotient $\mathbb{P}_v^\rho$ defines a Markov process on SM such that for all $t \geq 0$, $\omega(t) \in W^s(\omega(0))$. For any $\mathcal{L}^\rho$-stationary measure m_ρ, the probability measure $\mathbb{P}_{m_\rho}^\rho := \int \mathbb{P}_v^\rho \, dm_\rho(v)$ is invariant under the shift on $C(\mathbb{R}_+, SM)$ (cf. [21, 26]). If the measure m_ρ is an extremal point of the set of stationary measures for $\mathcal{L}^\rho$, then the probability measure $\mathbb{P}_{m_\rho}^\rho$ is invariant ergodic under the shift on $C(\mathbb{R}_+, SM)$.

Proposition 4.2 *Let m_ρ be a stationary ergodic measure for $\mathcal{L}^\rho$. Then, for $\mathbb{P}_{m_\rho}^\rho$ a.e. ω and any lift $\widetilde{\omega}$ of ω to $S\widetilde{M}$,*

$$\lim_{t \to +\infty} \frac{1}{t} b_{\widetilde{\omega}(0)}(\widetilde{\omega}(t)) \;=\; -\rho + \int B\, dm_\rho \;=:\; \ell_\rho(m_\rho). \tag{4.3}$$

In particular, for $\rho = V, m_\rho = m_{BR}$, *we have* $\ell_V(m_{BR}) = V - \int B\, dm_{BR} = 0$.

By Remark 3.10, the measure m_{BR} is ergodic.

Proof Let $\sigma_t, t \in \mathbb{R}_+$, be the shift transformation on $C(\mathbb{R}_+, S\widetilde{M})$. For any $\widetilde{\omega} \in C(\mathbb{R}_+, S\widetilde{M})$, $t, s \in \mathbb{R}_+$, $b_{\widetilde{\omega}(0)}(\widetilde{\omega}(t+s)) = b_{\widetilde{\omega}(0)}(\widetilde{\omega}(t)) + b_{\sigma_t\widetilde{\omega}(0)}(\sigma_t\widetilde{\omega}(s))$. By Γ-equivariance, $b_{\widetilde{\omega}(0)}(\widetilde{\omega}(t))$ takes the same value for all $\widetilde{\omega}$ with the same projection in $C(\mathbb{R}_+, SM)$ and defines an additive functional on $C(\mathbb{R}_+, SM)$. Moreover, $\sup_{0\le t\le 1} b_{\widetilde{\omega}(0)}(\widetilde{\omega}(t)) \le \sup_{0\le t\le 1} d(\widetilde{\omega}(0), \widetilde{\omega}(t))$, so that the convergence in (4.3) holds $\mathbb{P}^\rho_{m_\rho}$-a.e. and in $L^1(\mathbb{P}^\rho_{m_\rho})$. By ergodicity of the process and additivity of the functional $b_{\widetilde{\omega}(0)}(\widetilde{\omega}(t))$, the limit is $\frac{1}{t}\mathbb{E}^\rho_{m_\rho}\left(b_{\widetilde{\omega}(0)}(\widetilde{\omega}(t))\right)$, for all $t > 0$. In particular,

$$\begin{aligned}\lim_{t\to+\infty} \frac{1}{t} b_{\widetilde{\omega}(0)}(\widetilde{\omega}(t)) &= \lim_{t\to 0^+} \frac{1}{t}\mathbb{E}^\rho_{m_\rho}\left(b_{\widetilde{\omega}(0)}(\widetilde{\omega}(t))\right)\\ &= \int_{SM} \Delta^s_y b_{x,\xi}(y)\big|_{y=x} + \rho < \overline{X}, \nabla^s_y b_{x,\xi}(y)\big|_{y=x} >_{x,\xi}\ dm_\rho(x,\xi).\end{aligned}$$

Equation (4.3) follows. □

Following Ancona [1] and Hamenstädt [26], we call our operator $\mathcal{L}^\rho$ *weakly coercive* if there is some $\varepsilon > 0$ such that for all $\xi \in \partial\widetilde{M}$, there exists a positive superharmonic function for the operator $\mathcal{L}^\rho_\xi + \varepsilon$ (i.e., a positive F such that $\mathcal{L}^\rho_\xi F + \varepsilon F \le 0$). As a corollary of Proposition 4.2, we see that if m_ρ is a $\mathcal{L}^\rho$-stationary measure with $\ell_\rho(m_\rho) > 0$, then for $\widetilde{m}_\rho$ almost all $\widetilde{\omega}(0)$ and $\widetilde{\mathbb{P}}^\rho_{x,\xi}$ almost all $\widetilde{\omega}$, $\widetilde{\omega}(+\infty) = \lim_{t\to+\infty}\widetilde{\omega}(t) \in (\partial\widetilde{M} \setminus \{\xi\}) \times \{\xi\}$. This, together with the negative curvature and the cocompact assumption of the underlying space, implies that

Corollary 4.3 [26, Corollary 3.10] *Assume the operator* $\mathcal{L}^\rho$ *is such that there exists some* $\mathcal{L}^\rho$*-stationary ergodic measure* m_ρ *with* $\ell_\rho(m_\rho) > 0$. *Then,* $\mathcal{L}^\rho$ *is weakly coercive.*

4.2 Stable Diffusions

For a weakly coercive $\mathcal{L}^\rho$, we want to understand more about its diffusions. Hamenstädt developed in [26] many tools for the study of the foliated diffusions subordinate to the stable foliation $\mathcal{W}^s$, using dynamics and thermodynamical formalism. We review in this subsection her results when applied for our $\mathcal{L}^\rho$.

For each $\mathcal{L}^\rho$, $\rho \in \mathbb{R}$, recall that $p^\rho_\xi(t,x,y)$ is the fundamental solution of the equation $\frac{\partial F}{\partial t} = \mathcal{L}^\rho_\xi F$. We write $G^\rho_\xi(x,y)$ for the Green function of $\mathcal{L}^\rho$: for $x, y \in \widetilde{M}$,

$$G^\rho_\xi(x,y) := \int_0^\infty p^\rho_\xi(t,x,y)\, dt.$$

For weakly coercive operators on a pinched negatively curved simply connected manifold, Ancona's Martin boundary theory [1] shows the following

Theorem 4.4 ([1]) *Assume that the operator $\Delta^s + \rho \overline{X}$ is weakly coercive and recall that the sectional curvature of $\widetilde{M}$ is between two constants $-a^2$ and $-b^2$. There exists a constant C such that for any $\xi \in \partial \widetilde{M}$, any three points x, y, z in that order on the same geodesic in $\widetilde{M}$ and such that $d(x, y), d(y, z) \geq 1$, we have:*

$$C^{-1} G_\xi^\rho(x, y) G_\xi^\rho(y, z) \;\leq\; G_\xi^\rho(x, z) \;\leq\; C G_\xi^\rho(x, y) G_\xi^\rho(y, z). \tag{4.4}$$

(In particular, by Corollary 4.3, the inequality (4.4) holds for ρ such that there is an ergodic $\mathcal{L}^\rho$-stationary measure m_ρ with $\ell_\rho(m_\rho) > 0$.)

Ancona [1] deduced from (4.4) that the *Martin boundary* of each weakly coercive operator $\mathcal{L}_\xi^\rho$ is the geometric boundary $\partial \widetilde{M}$. Namely, for any $x, y \in \widetilde{M}$, $\xi, \eta \in \partial \widetilde{M}$, there exists a function $K_{\xi,\eta}^\rho(x, y)$ such that

$$\lim_{z \to \eta} \frac{G_\xi^\rho(y, z)}{G_\xi^\rho(x, z)} \;=\; K_{\xi,\eta}^\rho(x, y).$$

The function $K_{\xi,\eta}^\rho(x, y)$ is $\mathcal{L}_\xi^\rho$-harmonic and therefore smooth in x and y. Moreover, the functions $(x, \eta) \mapsto K_{\xi,\eta}^\rho(x, y)$, $(x, \eta) \mapsto \nabla_y K_{\xi,\eta}^\rho(x, y)\big|_{y=x}$ are Hölder continuous (cf. [26], Appendix B). By uniformity of the constant C in (4.4), the functions $(x, \xi) \mapsto K_{\xi,\eta}^\rho(x, y)$, $\xi \mapsto \nabla_y K_{\xi,\eta}^\rho(x, y)\big|_{y=x}$ are continuous into the space of Hölder continuous functions on SM (see e.g. [37], Proposition 3.9).

Let $\mathcal{L}^{\rho *}$ be the leafwise formal adjoint of $\mathcal{L}^\rho$ (see (4.1)). Then, $\mathcal{L}^{\rho *}$ is subordinate to $\mathcal{W}^s$ and the corresponding Green function $G_\xi^{\rho *}(x, y)$ is given by $G_\xi^{\rho *}(x, y) = G_\xi^\rho(y, x)$. In particular, the Green function $G_\xi^{\rho *}(x, y)$ satisfies (4.4) as well and we find, for $\xi, \eta \in \partial \widetilde{M}$, $x, y \in \widetilde{M}$, the Martin kernel $K_{\xi,\eta}^{\rho *}(x, y)$ given by:

$$K_{\xi,\eta}^{\rho *}(x, y) \;=\; \lim_{z \to \eta} \frac{G_\xi^{\rho *}(y, z)}{G_\xi^{\rho *}(x, z)} \;=\; \lim_{z \to \eta} \frac{G_\xi^\rho(z, y)}{G_\xi^\rho(z, x)}.$$

Again, the function $K_{\xi,\eta}^{\rho *}(x, y)$ is $\mathcal{L}_\xi^\rho$-harmonic and therefore smooth in x and y. Moreover, the functions $(x, \eta) \mapsto K_{\xi,\eta}^{\rho *}(x, y)$, $(x, \eta) \mapsto \nabla_y K_{\xi,\eta}^{\rho *}(x, y)\big|_{y=x}$ are Hölder continuous and the functions $(x, \xi) \mapsto K_{\xi,\eta}^{\rho *}(x, y)$, $\xi \mapsto \nabla_y K_{\xi,\eta}^{\rho *}(x, y)\big|_{y=x}$ are continuous into the space of Hölder continuous functions on SM. Observe also that the relation (4.4) is satisfied also by the resolvent $G_\xi^{\lambda, \rho *}(x, y) := \int_0^\infty e^{-\lambda t} p_\xi^\rho(t, y, x)\, dt$, uniformly for $\lambda > 0$ close to 0 and for $\xi \in \partial \widetilde{M}$, so that we also have:

$$K_{\xi,\eta}^{\rho*}(x,y) = \lim_{z\to\eta,\lambda\to 0^+} \frac{G_\xi^{\lambda,\rho*}(y,z)}{G_\xi^{\lambda,\rho*}(x,z)}. \tag{4.5}$$

We can use the function $K_{\xi,\eta}^{\rho,*}(x,y)$ to express the function K^ρ in (4.2).

Proposition 4.5 *Assume $\ell_\rho(m_\rho) > 0$ and m_ρ is ergodic. Then, the corresponding K^ρ in (4.2) is given by $\frac{\mathsf{K}^\rho(y,\xi)}{\mathsf{K}^\rho(x,\xi)} = K_{\xi,\xi}^{\rho*}(x,y)$.*

Proof Let ν_x^ρ be the family such that $d\widetilde{m}_\rho(x,\xi) = d\mathrm{Vol}_{\widetilde{g}}(x)d\nu_x^\rho(\xi)$. For $F \in C(SM)$, the set of continuous functions on SM, set $\widetilde{F}$ for the Γ-periodic function on $\widetilde{M}\times\partial\widetilde{M}$ extending F. Since m_ρ is ergodic, we have, for m_ρ-a.e. (x,ξ),

$$\int_{SM} F\,dm_\rho \;=\; \lim_{\lambda\to 0^+}\lambda\int_0^\infty e^{-\lambda t}\left(\int p_\xi^\rho(t,x,y)\widetilde{F}(y,\xi)\,d\mathrm{Vol}_{\widetilde{g}}(y)\right)dt.$$

The inner integral can be written

$$\sum_{\beta\in\Gamma}\int p_\xi^\rho(t,x,\beta y)\widetilde{F}(\beta y,\xi)\,d\mathrm{Vol}_g(y) = \sum_{\beta\in\Gamma}\int p_{\beta^{-1}\xi}^\rho(t,\beta^{-1}x,y)\widetilde{F}(y,\beta^{-1}\xi)\,d\mathrm{Vol}_g(y),$$

where Vol_g is the restriction of $\mathrm{Vol}_{\widetilde{g}}$ on the fundamental domain M_0, so that we have

$$\int_{SM} F\,dm_\rho = \lim_{\lambda\to 0^+}\sum_{\beta\in\Gamma}\lambda\int G_{\beta^{-1}\xi}^{\lambda,\rho*}(y,\beta^{-1}x)F(y,\beta^{-1}\xi)\,d\mathrm{Vol}_g(y).$$

By Harnack inequality, all ratios $\frac{G_{\beta^{-1}\xi}^{\lambda,\rho*}(y,\beta^{-1}x)}{G_{\beta^{-1}\xi}^{\lambda,\rho*}(z,\beta^{-1}x)}$ for $y,z\in M_0$ are of the same order as soon as $d(\beta^{-1}x, M_0)\geq 1$. Choose an open $A\subset\partial\widetilde{M}$ disjoint from $\{\xi\}$. If, for β large enough, $\beta^{-1}\xi\in A$, then $\beta^{-1}x$ is close to A. Then, by (4.5) and Harnack inequality, given $\varepsilon>0$, for all $x\in M_0$, $\xi\in\partial\widetilde{M}$, for all $\beta\in\Gamma$ so that $\beta^{-1}x$ is close enough to $\beta^{-1}\xi$, y' close enough to y, z' close enough to z,

$$\frac{G_{\beta^{-1}\xi}^{\lambda,\rho*}(y',\beta^{-1}x)}{G_{\beta^{-1}\xi}^{\lambda,\rho*}(z',\beta^{-1}x)} \sim^{1+\varepsilon} K_{\beta^{-1}\xi,\beta^{-1}\xi}^{\rho,*}(z,y),$$

where, for $a,b\in\mathbb{R}$, $a\sim^{1+\varepsilon} b$ means $(1+\varepsilon)^{-1}b\leq a\leq(1+\varepsilon)b$. Consider as functions F_y, F_z the indicator of $\mathcal{U}_y\times A, \mathcal{U}_z\times A$, where $\mathcal{U}_y,\mathcal{U}_z$ are respectively small neighborhoods of y, z. Then

$$\int_{SM} F_y\,dm_\rho = \int_{\mathcal{U}_y}\nu_{y'}^\rho(A)\,d\mathrm{Vol}_g(y') = \lim_{\lambda\to 0^+}\sum_{\beta\in\Gamma,\beta^{-1}\xi\in A}\lambda\int_{\mathcal{U}_y} G_{\beta^{-1}\xi}^{\lambda,\rho*}(y',\beta^{-1}x)\,d\mathrm{Vol}_g(y'),$$

$$\int_{SM} F_z \, dm_\rho = \int_{\mathcal{U}_z} \nu_{z'}^\rho(A) \, d\mathrm{Vol}_g(z') = \lim_{\lambda \to 0^+} \sum_{\beta \in \Gamma, \beta^{-1}\xi \in A} \lambda \int_{\mathcal{U}_z} G_{\beta^{-1}\xi}^{\lambda,\rho*}(z', \beta^{-1}x) \, d\mathrm{Vol}_g(z').$$

As $\lambda \to 0^+$, the β's involved in the sums are such that the distance $d(y, \beta^{-1}x), d(z, \beta^{-1}x)$ is larger and larger. It follows that, for ν_z^ρ-a.e. η,

$$\frac{d\nu_y^\rho}{d\nu_z^\rho}(\eta) = K_{\eta,\eta}^{\rho,*}(z, y).$$

□

Corollary 4.6 *Assume $\ell_\rho(m_\rho) > 0$ for some ergodic $\mathcal{L}^\rho$-stationary measure m_ρ. Then, m_ρ is the only $\mathcal{L}^\rho$-stationary probability measure.*

Proof By Proposition 4.5, any ergodic $\mathcal{L}^\rho$-stationary measure is described by a Γ-equivariant family of measures at the boundary ν_x that satisfies

$$\frac{d\nu_y}{d\nu_z}(\eta) = K_{\eta,\eta}^{\rho,*}(z, y).$$

Since the cocycle depends Hölder-continuously on η, there is a unique equivariant family with that property (see, e.g., [36, Théorème 1.d], [48, Corollary 5.12]). □

4.3 Stochastic Entropy and Rigidity

Let m_ρ be an ergodic $\mathcal{L}^\rho$-stationary measure, and assume that $\ell_\rho(m_\rho) > 0$. The following theorems are the counterpart of the more familiar random walks properties in our setting.

Theorem 4.7 (Kaimanovich, [27]) *Let m_ρ be an ergodic $\mathcal{L}^\rho$-stationary measure, and assume that $\ell_\rho(m_\rho) > 0$. For $\mathbb{P}_{m_\rho}$-a.e. $\omega \in C(\mathbb{R}_+, SM)$, the following limits exist*

$$\begin{aligned} h_\rho(m_\rho) &= \lim_{t \to +\infty} -\frac{1}{t} \log p_\xi^\rho(t, \widetilde{\omega}(0), \widetilde{\omega}(t)) \\ &= \lim_{t \to +\infty} -\frac{1}{t} \log G_\xi^\rho(\widetilde{\omega}(0), \widetilde{\omega}(t)), \end{aligned}$$

where $\widetilde{\omega}(t), t \geq 0$, is a lift of ω to $S\widetilde{M}$. Moreover,

$$h_\rho(m_\rho) = \int_{SM} \left(\|\nabla^s \log K^\rho(x, \xi)\|^2 - \rho B(x, \xi) \right) dm_\rho.$$

Proof The first part is proven in details in [38], Proposition 2.4. For the final formula, we follow [38], Erratum. Since the notations are not exactly the same, for the sake of clarity, we give the main ideas of the proof. We firstly claim is that, since $\ell_\rho(m_\rho) > 0$, for $\mathbb{P}_{m_\rho}$-a.e. $\omega \in C(\mathbb{R}_+, SM)$,

$$\limsup_{t\to+\infty} \left|\log G_\xi^\rho(\widetilde{\omega}(0), \widetilde{\omega}(t)) - \log K_{\xi,\xi}^{\rho *}(\widetilde{\omega}(0), \widetilde{\omega}(t))\right| < +\infty.$$

Indeed, let z_t be the point on the geodesic ray $\gamma_{\widetilde{\omega}(t),\xi}$ closest to x. Then, as $t \to +\infty$,

$$G_\xi^\rho(\widetilde{\omega}(0), \widetilde{\omega}(t)) \asymp G_\xi^\rho(z_t, \widetilde{\omega}(t)) \asymp \frac{G_\xi^\rho(y, \widetilde{\omega}(t))}{G_\xi^\rho(y, z_t)}$$

for all y on the geodesic going from $\widetilde{\omega}(t)$ to ξ with $d(y, \widetilde{\omega}(t)) \geq d(y, z_t) + 1$, where $\asymp$ means up to some multiplicative constant independent of t. The first $\asymp$ comes from Harnack inequality using the fact that $\sup_t d(x, z_t)$ is finite $\mathbb{P}_{m_\rho}$-almost everywhere. (Since $\ell_\rho(m_\rho) > 0$, for $\mathbb{P}_{m_\rho}$-a.e. $\omega \in C(\mathbb{R}_+, SM)$, $\eta = \lim_{t\to+\infty} \widetilde{\omega}(t)$ differs from ξ and $d(x, z_t)$, as $t \to +\infty$, converge to the distance between x and $\gamma_{\xi,\eta}$.) The second $\asymp$ comes from Ancona inequality (4.4). Replace $\frac{G_\xi^\rho(y,\widetilde{\omega}(t))}{G_\xi^\rho(y,z_t)}$ by its limit as $y \to \xi$, which is $K_{\xi,\xi}^{\rho *}(z_t, \widetilde{\omega}(t))$ by (4.5), which is itself $\asymp K_{\xi,\xi}^{\rho *}(\widetilde{\omega}(0), \widetilde{\omega}(t))$ by Harnack inequality again. It follows that, for $\mathbb{P}_{m_\rho}^\rho$-a.e. $\omega \in C(\mathbb{R}_+, SM)$,

$$h_\rho(m_\rho) = \lim_{t\to+\infty} -\frac{1}{t} \log K_{\xi,\xi}^{\rho *}(\widetilde{\omega}(0), \widetilde{\omega}(t)).$$

By Harnack inequality, there is a constant C such that $|\log K_{\xi,\xi}^{\rho *}(\widetilde{\omega}(0), \widetilde{\omega}(t))| \leq Cd(\widetilde{\omega}(0), \widetilde{\omega}(t))$. Since $\log K_{\xi,\xi}^{\rho *}(\widetilde{\omega}(0), \widetilde{\omega}(t))$ is additive along the trajectories, and $\mathbb{P}_{m_\rho}^\rho$ is shift ergodic, the limit reduces to

$$\begin{aligned} h_\rho(m_\rho) &= \lim_{t\to 0^+} -\frac{1}{t} \mathbb{E}_{m_\rho} \log K_{\xi,\xi}^{\rho *}(\widetilde{\omega}(0), \widetilde{\omega}(t)) \\ &= -\int_{SM} \left(\Delta_y^s \log K_{\xi,\xi}^{\rho *}(x, y)\big|_{y=x} + \rho < \overline{X}, \nabla_y^s \log K_{\xi,\xi}^{\rho *}(x, y)\big|_{y=x} >_{x,\xi}\right) dm_\rho(x, \xi) \\ &= -\int_{SM} \left(\Delta^s \log \mathsf{K}^\rho(x, \xi) + \rho < \overline{X}, \nabla^s \log \mathsf{K}^\rho > (x, \xi)\right) dm_\rho(x, \xi), \end{aligned}$$

where we used Proposition 4.5 to replace $\nabla_y^s \log K_{\xi,\xi}^{\rho *}(x, y)\big|_{y=x}$ by $\nabla^s \log \mathsf{K}^\rho(x, \xi)$. Finally, we use (4.2) applied to $Z = \nabla^s \log \mathsf{K}^\rho(x, \xi)$ to write

$$-\int_{SM} \Delta^s \log \mathsf{K}^\rho(x, \xi)\, dm_\rho(x, \xi) = \int_{SM} \|\nabla^s \log \mathsf{K}^\rho(x, \xi)\|^2\, dm_\rho(x, \xi)$$

and applied to $Z = \overline{X}$ to write

$$\int B\, dm_\rho = \int < \overline{X}, \nabla^s \log \mathsf{K}^\rho > dm_\rho. \tag{4.6}$$

The formula for the entropy follows. □

Theorem 4.8 (Guivarc'h, [23]) *Assume that* $\ell_\rho(m_\rho) > 0$. *Then,* $h_\rho(m_\rho) \leq \ell_\rho(m_\rho)V$.

Proof Fix $(x, \xi) \in S\widetilde{M}$ such that $\frac{1}{t} b_{x,\xi}(\widetilde{\omega}(t)) \to \ell_\rho(m_\rho)$ and $-\frac{1}{t} \log p^\rho_\xi(t, \widetilde{\omega}(0), \widetilde{\omega}(t)) \to h_\rho(m_\rho)$, $\widetilde{\mathbb{P}}^\rho_{x,\xi}$-a.e., as $t \to +\infty$. There is a constant $\widetilde{C}$ depending only on the curvature bounds such that one can find a partition $\mathcal{A} = \{A_k, k \in \mathbb{N}\}$ of $\widetilde{M}$ such that the sets A_k have diameter at most $\widetilde{C}$ and inner diameter at least 1. Set for $k \in \mathbb{N}, t > 0$, $q^\rho_k(t) := \widetilde{\mathbb{P}}^\rho_{x,\xi}(\{\widetilde{\omega} : \widetilde{\omega}(t) \in A_k\})$. The family $\{q^\rho_k(t), k \in \mathbb{N}\}$ is a probability on $\mathbb{N}$ with the property that, with high probability, $q^\rho_k(t) \leq e^{-t(h_\rho(m_\rho)-\varepsilon)}$ and $k \in N_t$, where $N_t := \{k : A_k \subset B(x, t(\ell_\rho(m_\rho) + \varepsilon))\}$. Then,

$$-\sum_{k \in N_t} q^\rho_k(t) \log q^\rho_k(t) \leq \sum_{k \in N_t} q^\rho_k(t) \times \log \# N_t.$$

Since $\# N_t \leq C e^{t(\ell_\rho(m_\rho)+\varepsilon)(V+\varepsilon)}$, for some constant C, Theorem 4.8 follows. □

Theorem 4.9 *Assume that* $\ell_\rho(m_\rho) > 0$. *Then,* $\int B\, dm_\rho \leq V$, *with equality in this inequality only when* (M, g) *is locally symmetric.*

Proof Recall Equation (4.6): $\int B\, dm_\rho = \int < \overline{X}, \nabla^s \log \mathsf{K}^\rho > dm_\rho$, so that, by Schwarz inequality,

$$\left(\int B\, dm_\rho \right)^2 \leq \int_{SM} \| \nabla^s \log \mathsf{K}^\rho_{x,\xi} \|^2 \, dm_\rho,$$

with equality only if $\nabla^s \log \mathsf{K}^\rho = \tau(\rho)\overline{X}$ for some real number $\tau(\rho)$. Abbreviate $h_\rho(m_\rho), \ell_\rho(m_\rho)$ as h_ρ, ℓ_ρ. We write

$$\begin{aligned} h_\rho &= \int_{SM} \left(\| \nabla^s \log \mathsf{K}^\rho_{x,\xi} \|^2 - \rho B(x, \xi) \right) dm_\rho \\ &\geq \left(\int B\, dm_\rho \right)^2 - \rho \int B\, dm_\rho = \ell_\rho \int B\, dm_\rho. \end{aligned}$$

We indeed have $\int B\, dm_\rho \leq V$, with equality only if $\nabla^s \log \mathsf{K}^\rho = \tau(\rho)\overline{X}$ for some real number $\tau(\rho)$. Then, Equation (3.5) holds with V replaced by $\tau(\rho)$. The proof of Corollary 3.9 applies and the operator Δ^{ss} is symmetric with respect to the measure m_ρ. By Remark 3.10, $m_\rho = m_{BR}$. Then, $\tau(\rho) = V$ and from $\int B\, dm_\rho = \int B\, dm_{BR} = V$ and $\ell_\rho(m_\rho) > 0$, we have $\rho \neq V$. We have

$0 = \mathcal{L}_y^{\rho *} e^{-Vb_{x,\xi}(y)}\big|_{y=x} = (V - B(x,\xi))(V - \rho)$. It follows that $B = V$ is constant. By Theorem 1.3, the space (M, g) is locally symmetric. □

The conclusion in Theorem 4.9 actually holds true for all $\rho < V$ due to the following.

Proposition 4.10 *Let $\rho \in \mathbb{R}$. There is some $\mathcal{L}^\rho$-stationary ergodic measure m_ρ such that $\ell_\rho(m_\rho) > 0$ if, and only if, $\rho < V$. Moreover, the measures m_ρ weak* converge to m_{BR} as $\rho \nearrow V$.*

Proof Let ρ_0 be such that there is some $\mathcal{L}^{\rho_0}$-stationary measure m_{ρ_0} with $\ell_{\rho_0}(m_{\rho_0}) \leq 0$, but such that there exist $\{\rho_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to+\infty} \rho_n = \rho_0$ and $\ell_{\rho_n}(m_{\rho_n}) > 0$ (we know that m_{ρ_n} is unique by Corollary 4.6). Observe that by Equation (4.3), $\ell_\rho > 0$ for ρ sufficiently close to $-\infty$. On the other hand, if $\ell_{\rho_n}(m_{\rho_n}) > 0$, we must have $\rho_n < V$ by Equation (4.3) and Theorem 4.9. Therefore one can choose ρ_0 and ρ_n with those properties. Let m be a weak* limit of the measures m_{ρ_n}. We are going to show that $m = m_{BR}$ and that $\rho_0 = V$.

Observe that $\ell_{\rho_0}(m) \leq 0$ since otherwise m is the only stationary measure and we cannot have $\ell_{\rho_0}(m_{\rho_0}) \leq 0$ for some other $\mathcal{L}^{\rho_0}$-stationary measure m_{ρ_0}. On the other hand, $\ell_{\rho_0}(m) \geq 0$ by continuity, so $\ell_{\rho_0}(m) = 0$ and $\lim_{n\to+\infty} \ell_{\rho_n}(m_{\rho_n}) = 0$. By Theorem 4.8, $\lim_{n\to+\infty} h_{\rho_n}(m_{\rho_n}) = 0$ as well. We have

$$0 = \lim_{n\to+\infty} h_{\rho_n}(m_{\rho_n}) = \lim_{n\to+\infty} \int_{SM} \left(\|\nabla^s \log \mathsf{K}^{\rho_n}(x,\xi)\|^2 - \rho_n B(x,\xi) \right) dm_{\rho_n}$$

$$= \lim_{n\to+\infty} \int_{SM} \left(\|\nabla^s \log \mathsf{K}^{\rho_n}(x,\xi)\|^2 - \rho_n < \overline{X}, \nabla^s \log \mathsf{K}^{\rho_n} > \right) dm_{\rho_n}.$$

Write $Z_n := \nabla^s \log \mathsf{K}^{\rho_n}(x,\xi) - \left(\int_{SM} < \overline{X}, \nabla^s \log \mathsf{K}^{\rho_n} > dm_{\rho_n}\right) \overline{X}$. We have

$$\lim_{n\to+\infty} \int_{SM} \|Z_n\|^2 \, dm_{\rho_n} = \lim_{n\to+\infty} \int_{SM} \left(\|\nabla^s \log \mathsf{K}^{\rho_n}(x,\xi)\|^2 \right) dm_{\rho_n} - \left(\int_{SM} B \, dm_{\rho_n} \right)^2$$

$$= \lim_{n\to+\infty} \left(h_{\rho_n}(m_{\rho_n}) - \ell_{\rho_n}(m_{\rho_n}) \int_{SM} B \, dm_{\rho_n} \right)$$

and so $\lim_{n\to+\infty} \int_{SM} \|Z_n\|^2 \, dm_{\rho_n} = 0$. In other words, Equation (3.5) holds with V replaced by $\int_{SM} B \, dm_{\rho_n}$ with an error $\int_{SM} < Z, Z_n > dm_{\rho_n}$. The proof of Corollary 3.9 applies and the operator Δ^{ss} is symmetric with respect to the measure m_{ρ_n}, up to an error which goes to 0 as $n \to +\infty$. It follows that the operator Δ^{ss} is symmetric with respect to the limit measure m. By Remark 3.10, $m = m_{BR}$. Since $\ell_{\rho_0}(m) = 0$, $\rho_0 = \int_{SM} B \, dm = \int_{SM} B \, dm_{BR} = V$. □

Remark 4.11 Anderson and Schoen [2] described the Martin boundary for the Laplacian on a simply connected manifold with pinched negative curvature. Regularity of the Martin kernel in the [2] proof yields, in the cocompact case, nice properties of the harmonic measure (i.e., the stationary measure for $\mathcal{L}^0 = \Delta^s$).

This was observed by [25, 29] and [34]. Ancona [1] extended [2]'s results to the general weakly coercive operator and proved the basic inequality (4.4). This allowed Hamenstädt to consider the general case that $\mathcal{L} = \Delta^s + Y$, with Y^*, the dual of Y in the cotangent bundle to the stable foliation over SM, satisfying $dY^* = 0$ leafwisely [26]. The criterion she obtained for the existence of a $\mathcal{L}$-stationary ergodic measure m with $\ell_{\mathcal{L}}(m) := \int_{M_0 \times \partial \widetilde{M}} \big(- < Y, \overline{X} > + B \big) \, dm > 0$ is $P\big(- < \overline{X}, Y > \big) > 0$. Our presentation follows [26], with a few simplifications when $Y = \rho \overline{X}$. Theorem 4.9 was shown by Kaimanovich [27] in the case $\rho = 0$. From [26], Theorem A (2), the measure m_{BR} is the only symmetric measure for $\mathcal{L}^V$. It is not known whether m_{BR} is the only stationary measure for $\mathcal{L}^V$. The second statement in Proposition 4.10 would also follow from such a uniqueness result.

5 Stochastic Flows of Diffeomorphisms and a Relative Entropy

In this section, we introduce a stochastic flow associated with $\mathcal{L}^\rho$. In the case of $\rho = 0$ our object has been considered as a *stochastic (analogue of) the geodesic flow* (cf. [14, 17]). It gives rise to a random walk on the space of homeomorphisms of a bigger compact manifold and the relative entropy of this random walk of homeomorphisms is our fourth entropy. The continuity of this entropy as $\rho \to -\infty$ will be used to prove that the measures m_ρ converge to m_L as $\rho \to -\infty$ (see Theorem 5.5 below).

5.1 Stochastic Flow Adapted to $\mathcal{L}^\rho$

Let $O\widetilde{M}$ be the orthonormal frame bundle (OFB) of $(\widetilde{M}, \widetilde{g})$:

$$O\widetilde{M} := \big\{ x \mapsto u(x) : \; u(x) = (u_1, \cdots, u_d) \in O(S_x \widetilde{M}) \big\}$$

and consider $O\widetilde{M} \times \{\xi\} =: O^s S\widetilde{M}$, the OFB in $T\widetilde{W}^s$ and $O^s SM := O^s S\widetilde{M}/\Gamma$, the OFB in TW^s. For $v \in S\widetilde{M}, u \in O^s_v S\widetilde{M}$, the *horizontal* subspace of $T_u O^s S\widetilde{M}$ is the space of directions w such that $\nabla_u w = 0$.

Denote $D^r(O^s S\widetilde{M})$ ($r \in \mathbb{N}$ or $r = \infty$) the space of homeomorphisms Φ such that

$$\Phi(x, u, \xi) := \big(\phi_\xi(x, u), \xi \big) \, ,$$

where ϕ_ξ is a C^r diffeomorphism of $O\widetilde{M}$, which depends continuously on ξ in $\partial \widetilde{M}$. We use stochastic flow theory to define a random walk on $D^\infty(O^s S\widetilde{M})$.

Theorem 5.1 ([17]) *Let $(\Omega, \mathbb{P})$ be a $\mathbb{R}^d$ Brownian motion (with covariance $2t\mathbb{I}$). For $\mathbb{P}$-a.e. $\omega \in \Omega$, all $t > 0$, there exists $\Phi_t^\rho = \left(\phi_{\xi,t}^\rho, \xi\right) \in D^\infty(O^s S\widetilde{M})$ such that for all $u \in O^s S\widetilde{M}$, $(\omega, t) \mapsto u_t = \phi_{\xi,t}^\rho(u)$ solves the Stratonovich Stochastic Differential Equation (SDE)*

$$du_t = \rho \widehat{X}(u_t) + \sum_{i=1}^{d} \widehat{H}(u_t^i) \circ dB_t^i, \tag{5.1}$$

where $\widehat{X}, \widehat{H}(u^i)$ are the horizontal lifts of $\overline{X}, u^i \in T_v \widetilde{W}^s(v)$ to $T_u O^s S\widetilde{M}$. Moreover,

1) *for $\mathbb{P}$-a.e. $\omega \in \Omega$, all $t, s > 0$, $\rho < V$, $\xi \in \partial \widetilde{M}$,*

$$\phi_{\xi,t+s}^\rho(\omega) = \phi_{\xi,t}^\rho(\sigma_s \omega) \circ \phi_{\xi,s}^\rho(\omega),$$

   where σ_s is the shift on Ω,
2) *for $\mathbb{P}$-a.e. $\omega \in \Omega$, for all $\beta \in \Gamma$, all $t > 0$, $D\beta \circ \phi_{\xi,t}^\rho(\omega) = \phi_{\xi,t}^\rho(\omega) \circ D\beta$, and*
3) *for $\mathbb{P}$-a.e. $\omega \in \Omega$, all $t > 0$, $\rho \mapsto \Phi_t^\rho(\omega)$ is continuous in $D^\infty(O^s S\widetilde{M})$ and the derivatives are solutions to the derivative SDE.*

Relation (5.1) implies that for all $(x, \xi, u), u \in OS_x\widetilde{M}$, the projection of $\phi_{\xi,t}^\rho(\omega)(u)$ on $S\widetilde{M}$ is a realization of the $\mathcal{L}^\rho$ diffusion starting from (x, ξ).

Property 1) and independence of the increments of the Brownian motion give that if $\kappa_{\rho,s}$ is the distribution of $\Phi_{\rho,s}(\omega)$ in $D^\infty(O^s S\widetilde{M})$, we can write

$$\kappa_{\rho,s+t} = \kappa_{\rho,t} * \kappa_{\rho,s},$$

where $*$ denotes the convolution in the group $D^\infty(O^s S\widetilde{M})$. So we have a *stochastic flow*. Property 2) yields a stochastic flow on $D^\infty(O^s SM)$. Property 3) will allow to control derivatives.

Fix $t > 0$. A probability measure $\overline{m}$ on $O^s SM$ is said to be *stationary* for $\kappa_{\rho,t}$, if for any $F \in C(O^s SM)$, the set of continuous functions on $O^s SM$,

$$\int_{O^s SM} F(u)\, d\overline{m}(u) = \int_{D^\infty(O^s SM)} \int_{O^s SM} F(\Phi u)\, d\overline{m}(u)\, d\kappa_{\rho,t}(\Phi).$$

Proposition 5.2 *Fix any $\rho < V, t > 0$. The probability measure $\overline{m}_\rho$ on $O^s SM$ that projects to m_ρ on SM and is the normalized Lebesgue measure on the fibers is stationary for $\kappa_{\rho,t}$. If we identify $O^s SM = \{(x, u, \xi) : x \in M_0, u \in O_x\widetilde{M}, \xi \in \partial\widetilde{M}\}$, then, up to a normalizing constant,*

$$d\overline{m}_\rho(x, u, \xi) = d\nu_x^\rho(\xi) d\mathrm{Vol}(x, u).$$

5.2 Entropy of a Random Transformation

There is a notion of entropy for random transformations with a stationary measure (see [31] for details).

Let $\mathbf{X}$ be a compact metric space and $D^0\mathbf{X}$ the group of homeomorphisms of $\mathbf{X}$. Let κ be a probability measure on $D^0\mathbf{X}$ and let $\overline{m}$ be a stationary measure for κ. Let σ be the shift on $(D^0\mathbf{X})^{\otimes\mathbb{N}}$, $\mathcal{K} = \kappa^{\otimes\mathbb{N}}$ the Bernoulli σ-invariant measure, $\overline{\sigma}$ the skew-product transformation on $(D^0\mathbf{X})^{\otimes\mathbb{N}} \times \mathbf{X}$

$$\overline{\sigma}(\underline{\phi}, x) := (\sigma\underline{\phi}, \phi_0 x), \ \forall \underline{\phi} = (\phi_0, \phi_1, \cdots) \in (D^0\mathbf{X})^{\otimes\mathbb{N}}.$$

Proposition 5.3 *Let $\overline{m}$ be a stationary measure for κ. Then, the measure $\mathcal{K} \times \overline{m}$ is $\overline{\sigma}$-invariant.*

For $\underline{\phi} \in (D^0\mathbf{X})^{\otimes\mathbb{N}}$, $x \in \mathbf{X}$, $\varepsilon > 0$, $n \in \mathbb{N}$, define a *random Bowen ball* by

$$\overline{B}(\underline{\phi}, x, \varepsilon, n) := \{y : \ y \in \mathbf{X}, d(\phi_k \circ \cdots \circ \phi_0 y, \phi_k \circ \cdots \circ \phi_0 x) < \varepsilon, \ \forall 0 \le k < n\}$$

and the *relative entropy* $h_{\overline{m}}(\mathcal{K})$ as the $\mathcal{K}$-a.e. value of

$$\sup_{\varepsilon} \int_{\mathbf{X}} \limsup_{n\to+\infty} -\frac{1}{n} \log \overline{m}(\overline{B}(\underline{\phi}, x, \varepsilon, n))\, d\overline{m}(x).$$

With the preceding notations, take $\mathbf{X} = O^sSM$, $\kappa = \kappa_{\rho,t}$ for some (ρ, t), $\rho < V, t > 0$, and the stationary measure $\overline{m}_\rho$. We want to estimate the relative entropy $h_{\overline{m}_\rho}(\mathcal{K}_{\rho,t})$.

Proposition 5.4 ([39]) *We have*

$$h_{\overline{m}_\rho}(\mathcal{K}_{\rho,t}) \ge \int \log \left|\mathrm{Det} D_u \Phi\big|_{T_u O^s S\widetilde{M}}\right| \, d\kappa_{\rho,t}(\Phi)\, d\overline{m}_\rho(u).$$

Recall that $\overline{m}_\rho$ has absolutely continuous conditional measures on the foliation $\overline{\mathcal{W}}^s$ defined by $(O\widetilde{M} \times \{\xi\})/\Gamma$. The proof uses ingredients from the proof of Pesin formula in the non-uniformly hyperbolic case (cf. [42]) and the non-invertible case [40, 41]. Observe that, even if $\Phi_{-1}\big|_{\overline{\mathcal{W}}^s}$ has only nonnegative exponents, there might be negative exponents for the random walk, and the inequality in Proposition 5.4 might be strict.

5.3 Continuity of the Relative Entropy

We now indicate the main ideas of the proof of the following theorem

Theorem 5.5 **([39])** *For $\rho < V$, let m_ρ be the stationary measure for the diffusion on SM with generator $\mathcal{L}^\rho = \Delta^s + \rho\overline{X}$. Then, as $\rho \to -\infty$, m_ρ weak* converge to the Liouville measure m_L.*

Corollary 5.6 $\lim_{\rho\to-\infty} \int B\,dm_\rho \ = \ \int B\,dm_L \ = \ H.$

Proof Set $\kappa_\rho = \kappa_{\rho,\frac{-1}{\rho}}$. We first observe that as $\rho \to -\infty$, κ_ρ weak* converge on $D^\infty(O^sSM)$ to the Dirac measure on the reverse frame flow Φ_{-1}. Moreover, for any $r \in \mathbb{N}$, $r \geq 1$,

$$\limsup_{\rho\to-\infty} C_r(\rho) < +\infty, \text{ where } C_r(\rho) := \int \|\Phi\|_{D^r(O^sSM)}\,d\kappa_\rho(\Phi),$$

where $\|\cdot\|_{D^r(O^sSM)}$ is the supremum of leafwise C^r norm. Indeed, by definition, κ_ρ is the distribution of the time one of the stochastic flow associated with the Stratonovich SDE

$$du_t = -\widehat{X}(u_t) + \frac{-1}{\rho}\sum_{i=1}^d \widehat{H}(u_t^i)\circ dB_t^i.$$

When $\rho \to -\infty$, the SDE converge to the ODE on O^sSM, $du_t = -\widehat{X}(u_t)$. The convergence, and the control on C_r, follow by continuity of the solutions in $D^\infty(O^sSM)$.

Let then m be a weak* limit of the measures m_ρ as $\rho \to -\infty$, $\overline{m}$ its extension to O^sSM by the Lebesgue measure on the fibers. The measure m is φ_{-1} invariant, $\overline{m}$ is the weak* limit of the measures $\overline{m}_\rho$, and $\overline{m}$ is Φ_{-1} invariant. Moreover, $h_m(\varphi_{-1}) = h_{\overline{m}}(\Phi_{-1})$ (this is a compact isometric extension) and

$$\int \log\left|\mathrm{Det} D_v\varphi_{-1}|_{T_vW^s(v)}\right|\,dm(v) = \int \log\left|\mathrm{Det} D_u\Phi_{-1}|_{T_uO^s\widetilde{SM}}\right|\,d\overline{m}(u)$$
$$= \lim_{\rho\to-\infty}\int \log\left|\mathrm{Det} D_u\Phi|_{T_uO^s\widetilde{SM}}\right|\,d\overline{m}_\rho(u)\,d\kappa_\rho(\Phi).$$

By [10], the Liouville measure is the only φ_{-1} invariant measure with

$$h_m(\varphi_{-1}) = \int \log\left|\mathrm{Det} D_v\varphi_{-1}|_{T_vW^s(v)}\right|\,dm(v).$$

To conclude the theorem, using Proposition 5.4, it suffices to show

$$h_{\overline{m}}(\Phi_{-1}) \geq \limsup_{\rho\to-\infty} h_{\overline{m}_\rho}(\mathcal{K}_\rho).$$

This will follow from the properties of the *topological relative conditional entropy* in the next subsection. ☐

5.4 Topological Relative Conditional Entropy

The following definition extends the definition of Bowen [7] to the random case, following Kifer–Yomdin [32] and Cowieson–Young [15].

For $\varepsilon > 0$ and $\underline{\phi} \in (D^0 X)^{\otimes \mathbb{N}}$, $x \in \mathbf{X}$, $\tau > 0^+$, $n \in \mathbb{N}$, set $r(\varepsilon, \underline{\phi}, x, \tau, n)$ for the smallest number of random $\overline{B}(\underline{\phi}, y, \tau, n)$ balls needed to cover $\overline{B}(\underline{\phi}, x, \varepsilon, n)$ and

$$h_{loc}(\varepsilon, \underline{\phi}) := \sup_x \lim_{\tau \to 0^+} \limsup_{n \to +\infty} \frac{1}{n} \log r(\varepsilon, \underline{\phi}, x, \tau, n).$$

The function $\underline{\phi} \mapsto h_{loc}(\varepsilon, \underline{\phi})$ is σ-invariant. For $\mathbf{X} = O^s SM$, write $h_{\rho,loc}(\varepsilon)$ for the $\mathcal{K}_\rho$-essential value of $h_{loc}(\varepsilon, \underline{\phi})$. The conclusion follows from the two following facts (cf. [39], Section 4).

Proposition 5.7 *For all $\varepsilon > 0$,*

$$h_{\overline{m}}(\Phi_{-1}) \geq \limsup_{\rho \to -\infty} h_{\overline{m}_\rho}(\mathcal{K}_\rho) - \limsup_{\rho \to -\infty} h_{\rho,loc}(\varepsilon).$$

Proposition 5.8 *There is a constant C such that, for all $r \in \mathbb{N}$, $r \geq 1$, there is ρ_r such that, for $\rho < \rho_r$,*

$$\lim_{\varepsilon \to 0^+} \sup_{\rho < \rho_r} h_{\rho,loc}(\varepsilon) \leq \frac{C}{r} C_1,$$

where $C_1 = \sup_{\rho < \rho_1} \int \|\Phi\|_{D^1(O^s SM)} \, d\kappa_\rho(\Phi)$.

Proposition 5.7 in the deterministic case is due to Bowen [7]. Proposition 5.8 in the deterministic case is a famous result of Yomdin [53, 54] and Buzzi [13]. By Proposition 5.8, since r is arbitrary, $\lim_{\varepsilon \to 0^+} \limsup_{\rho \to -\infty} h_{\rho,loc}(\varepsilon) = 0$. Proposition 5.7 then yields the claimed inequality.

5.5 Conclusion. Katok's Conjecture

Let (M, g) be a C^∞ d-dimensional Riemannian manifold with negative curvature. We introduced in Sections 1 and 2 the numbers H, the entropy of the Liouville measure for the geodesic flow, V, the topological entropy of the geodesic flow, and the function B on SM. The function B is constant if, and only if (M, g) is a locally symmetric space (Theorem 1.3). Using thermodynamical formalism, $H \leq V$ and if $H = V$, there exists a continuous function F on SM, C^1 along the trajectories of the flow, such that $B = V - \frac{\partial}{\partial t} F \circ \varphi_t \big|_{t=0}$ (see Theorem 2.5). Katok's conjecture (see [35] and [55] for some history of this topic) is that this can only happen when (M, g) is a locally symmetric space, that is, when B is constant on SM. This was proven by Katok [30] in dimension 2 and more generally if g is conformally equivalent to a locally symmetric g_0. It was also proven by Flaminio [19] in a C^2 neighborhood of a constant curvature metric g_0. Here, we introduced a family of measures m_ρ, $\rho \leq V$, such that $\int B\, dm_V = V$ and for $\rho < V$, $\int B\, dm_\rho \leq V$ with equality only in the case of locally symmetric spaces (Theorem 4.9). Finally, in the C^∞ case, we also show that $\lim_{\rho \to -\infty} \int B\, dm_\rho = H$ (Corollary 5.6).

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