Linear and Nonlinear Harmonic Boundaries of Graphs; An Approach with ℓ^p-Cohomology in Degree One

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1 Introduction

Graphs are defined by their vertices (henceforth X) and their edges $E \subset X \times X$. In a sense understanding a graph means to understand how the vertices and edges work together. In a finite graph, it is common to reduce the whole graph to the incidence matrix.

In an oriented graph, the incidence matrix B has $|X|$ lines and $|E|$ columns. Each column contains a -1 and a $+1$ to indicate the source and target of every edge. This matrix not only encodes the whole graph, but also a very familiar operation: the vector space $\mathbb{R}^{|X|}$ is the space of functions on the vertices, $\mathbb{R}^{|E|}$ the space of functions on the edges, and the matrix B is the gradient. More precisely, given a function on the vertices f (that is an element of $\mathbb{R}^{|X|}$), Bf is a function on the edges and its value on the edge (x, y) from x to y is $f(y) - f(x)$.

For infinite graphs, the gradient encompasses also all the information of the graph. Most people would no longer refer to it as a matrix though, but rather as an operator. In short, ℓ^p-cohomology in degree one aims at understanding the image of this operator.

The history of the topic can be split in two "cases". The case $p = 2$ has been largely studied and offers even more connections to other fields of mathematics (see Lück [38] or Eckmann [14] among many references). The case $p \neq 2$ has been introduced through Zucker (see [63] and references therein) to study compactifications of manifolds and Gromov (see [28, §8]) as a large-scale invariant of groups. Since then, applications have been found to harmonic functions, many

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notions of boundaries, representation theory of groups, quasi-isometry and packing of graphs; see §2 for details.

The main aim of this paper is to present the connection between ℓ^p-cohomology in degree one and harmonic functions, *i.e.* to interpret it as a special subspace of the Poisson boundary. As such the presentation tries to streamline some results of [21, 22] and [24].

Here is a thinned out version of this result (the actual result applies to a larger class of graphs, but the statement becomes technical).

Theorem 1.1 *Let G be the Cayley graph of a group which is not virtually nilpotent. Fix some* $p \in]1, \infty[$(*and not* $p \in [1 \infty[$). *Then* (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (5) *where*

(1) *The reduced ℓ^p-cohomology in degree one vanishes.*
(2) *For any functions f with gradient in ℓ^p there is a $c \in \mathbb{R}$ so that* $\lim f(x_n) = c$ *for any sequence x_n going to infinity.*
(3) *There are no non-constant harmonic functions with gradient in ℓ^p.*
(4) *There are no non-constant bounded harmonic functions with gradient in ℓ^p.*
(5) *For any $q < p$, the reduced ℓ^q-cohomology in degree one vanishes.*

See §4 and Theorem 4.1 for details. Among others, this has applications to the question whether the Poisson boundary is invariant under quasi-isometries (see Corollary 4.16).

Organisation: §1.2 gives the definition of ℓ^p-cohomology in degree one. §1.3 follows with examples which are not too hard to grasp. §2 presents some applications of ℓ^p-cohomology in degree one to other problems and topics. §3 shows how ℓ^1-cohomology in degree one can be seen as a space of functions on the ends of the group, giving a first sign that ℓ^p-cohomology has to do with ideal boundaries of graphs. §4 tackles the connection between ℓ^p-cohomology in degree one and harmonic functions. Lastly, §5.1 tries to summarises some other results and §5.2 presents some questions. But first, let us start with some preliminaries.

1.1 Conventions and Preliminaries

The conventions are that a graph $\Gamma = (X, E)$ is defined by X, its set of vertices, and E, its set of edges. All graphs will be assumed to be of bounded valency and the set of vertices X will always be assumed to be countable. The set of edges will be thought of as a subset of $X \times X$. The set of edges will be assumed symmetric (*i.e.* $(x, y) \in E \implies (y, x) \in E$). Functions will take value in $\mathbb{R}$ (but we could easily work with $\mathbb{C}$ too). Functions on E will often be anti-symmetric (*i.e.* $f(x, y) = -f(y, x)$). This said $\ell^p(X)$ is the Banach space of functions on the vertices which are p-summable, while $\ell^p(E)$ will be the subspace of functions on the edges which are p-summable.

The gradient $\nabla : \mathbb{R}^X \to \mathbb{R}^E$ is defined by $\nabla g(x, y) = g(y) - g(x)$.

$c_0(X)$ denotes the space of functions f which tend to 0 at infinity. This can be defined as follows: $f \in c_0(X)$, if for any sequence of finite sets $A_n \subset X$ with $\cup A_n = X$ and $A_i \subset A_{i+1}$, $\sup_{x \notin A_n} f(x) \xrightarrow{n\to\infty} 0$. Another possible description is the closure of finitely supported functions in ℓ^∞-norm.

Lastly, p' will denote the Hölder conjugate exponent of p, *i.e.* $p' = p/(p-1)$ (with the usual convention that 1 and ∞ are conjugate).

1.2 ℓ^p-Cohomology in Degree One

So our lofty goal is to understand the gradient map from $\mathbb{R}^X$ to $\mathbb{R}^E$. The first thing is that $\mathbb{R}^E$ is way too big as a space, even if one restricts to anti-symmetric functions. Indeed, any function on the edge which does not sum to 0 along a 2-cycle cannot come from the gradient.

Hence we restrict to the image of the gradient (or the kernel of the second coboundary operator [from edges to cycles] if you are curious about the origins of the name "cohomology").

The next step is to bring some simple functional analysis by restricting to ℓ^p-spaces.

At last we have ℓ^p-cohomology in degree one: given that the gradient of some function is in $\ell^p(E)$, can this gradient be approximated by gradients of functions in $\ell^p(X)$?

More precisely, the ℓ^p-cohomology in degree one of the graph Γ is the quotient

$$\ell^p H^1(\Gamma) := (\ell^p(E) \cap \nabla\mathbb{R}^X)/\nabla\ell^p(X).$$

Unfortunately, the image of ∇ is not always closed. In order to avoid dealing with unseparated space (and space which trivially have lots of things in their ℓ^p cohomology), the focus is usually on the largest separated quotient, the reduced ℓ^p-cohomology:

$$\underline{\ell^p H}^1(\Gamma) := (\ell^p(E) \cap \nabla\mathbb{R}^X)/\overline{\nabla\ell^p(X)}^{\ell^p(E)}.$$

Now, if you are wondering when is the image of ∇ actually closed, then

Theorem 1.2 *Let $p \in [1,\infty[$. The image of $\nabla : \ell^p(X) \to \ell^p(E)$ is closed if and only if the graph is amenable (*i.e.* there is a sequence of finite sets F_n such that $\frac{|\partial F_n|}{|F_n|} \to 0$, where ∂F is the set of edges with only one extremity in F).*

One direction of the proof is straightforward:

Proof of the "easy" Part Assume there is a sequence of sets $F_n \subset X$ so that $\frac{|\partial F_n|}{|F_n|} \to 0$. Take $f_n = \frac{1}{|F_n|^{1/p}} \mathbb{1}_{F_n}$ where $\mathbb{1}_F$ is the characteristic function of the set F (the function which takes value 1 on F and 0 elsewhere). By construction $\|f_n\|_{\ell^p(X)} = 1$.

But ∇f_n takes value $\pm\frac{1}{|F_n|^{1/p}}$ on ∂F_n (the sign depends on whether the edge points towards or away from the set F_n) and 0 elsewhere. Hence (the upcoming factor of 2 comes from the two orientation of the edges)

$$\|\nabla f_n\|_{\ell^p(E)} = \frac{1}{|F_n|^{1/p}}\|\mathbb{1}_{\partial F_n}\|_{\ell^p(E)} = \frac{1}{|F_n|^{1/p}}(2|\partial F_n|)^{1/p} = 2^{1/p}\big(\tfrac{|\partial F_n|}{|F_n|}\big)^{1/p}.$$

By hypothesis, this sequence tends to 0. As a consequence of the closed image theorem (an operator has a closed image if and only if it has a bounded inverse), the image of ∇ is not closed. □

The other direction of the statement is a typical technical slicing argument (given a sequence of functions f_n with norm 1 whose gradient tends to 0, look at "well-chosen" level sets of these functions). As it is quite technical, the proof would bring us off-topic, so the reader is encouraged to look up surveys on amenability for all the details (a very nice book, which is not so easy to find was written by Greenleaf [26]; there are some surveys freely available in Internet).

Most of the times it is much more convenient to think only in term of functions. To this end, introduce the Banach space of p-Dirichlet functions as the space of functions f on X such that $\nabla f \in \ell^p(E)$. It will be denoted $\mathsf{D}^p(\Gamma)$.

In order to introduce the $\mathsf{D}^p(\Gamma)$-norm on $\mathbb{R}^X$, it is necessary to choose a vertex, denoted e_Γ. This said $\|f\|^p_{\mathsf{D}^p(\Gamma)} = \|\nabla f\|^p_{\ell^p(E)} + |f(e_\Gamma)|^p$.

By taking the primitive of these gradients, one may also prefer to think of reduced ℓ^p-cohomology in degree one as:

$$\underline{\ell^p H}^1(\Gamma) := \mathsf{D}^p(\Gamma)/\overline{\ell^p(X)+\mathbb{R}}^{\mathsf{D}^p}.$$

A common abuse of language and notation happens, as one says that the reduced cohomology is equal to the non-reduced one: this means that the "natural" quotient map $\ell^p H^1(\Gamma) \to \underline{\ell^p H}^1(\Gamma)$ is injective. By Theorem 1.2 above, this happens exactly when the graph is non-amenable.

1.3 Some Examples

Before moving on to general statements, the reader might want to look at some simple examples. Since most of our examples come from Cayley graphs, let us also shortly recall their construction.

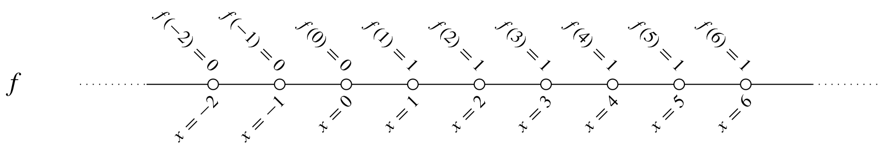
Given a finitely generated group G and a finite set S, the Cayley graph $\mathrm{Cay}(G, S)$ is the graph whose vertices are the element of G and $(\gamma, \gamma') \in E$ if $\exists s \in S$ such that $s^{-1}\gamma = \gamma'$. (This convention might be unusual from the point of view of random walks, but is much more convenient to write convolutions.) In order for the resulting graph to have a symmetric edge set, S is always going to be symmetric (*i.e.* $s \in$

$S \implies s^{-1} \in S$). Also, Cayley graphs are always going to be connected (*i.e.* S is generating).

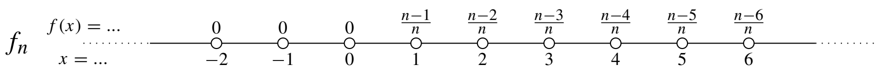
Example 1.3 The group $\mathbb{Z}$ with its most tempting generating set $\{\pm 1\}$ has the line as its Cayley graph.

Since there are no cycles, the question is: are all elements of $\ell^p(E)$ in the closure of $\nabla\ell^p(X)$? The simplest element of $\ell^p(E)$ is the "Dirac mass" (due to our convention that edges are oriented and function on edges are anti-symmetric, this is $\delta_{(1,0)} - \delta_{(0,1)}$), so that seems a nice place to start.

It is somehow easier to represent it as a function in $\mathsf{D}^p(X)$: namely $f(x) = 0$ for $x \leq 0$ and $f(x) = 1$ for $x > 1$.



This function looks hard to approximate: it is not even finitely supported. But remember, we are trying to approximate its gradient (not the values the function takes).



and f_n stays 0 once it reaches 0 (so for $x \geq n$). Now the important point is that we want $\nabla(f - f_n)$ to tend to 0 ($f - f_n$ obviously does not). A quick computation shows that $\nabla(f - f_n)$ takes on $2n$ edges (recall that (0, 1) and (1, 0) are both edges) the value $\pm\frac{1}{n}$. Hence

$$\|\nabla(f - f_n)\|_{\ell^p} = \big(2n\frac{1}{n^p}\big)^{1/p} = \frac{2^{1/p}}{n^{1-1/p}}.$$

This tends to 0 given that $p > 1$.

This shows that the basis of $\ell^p(E)$ is in $\overline{\nabla\ell^p(X)}$. Since this basis is dense in $\ell^p(E)$, we just showed that $\underline{\ell^p H}^1(\mathbb{Z})$ is trivial when $p > 1$.

And what about $p = 1$? Well, there is a trick (see Martin & Valette [40, Example 3 in §4] who mention hearing it from M. Bourdon). Let us quickly outline it here, it will be discussed at length in §3.

Note that any function on the line with gradient in ℓ^1 has a value as $x \to +\infty$ and $x \to -\infty$. For any $g \in \mathsf{D}^1(X)$, define $L(g) = \lim\limits_{x\to+\infty} g(x) - \lim\limits_{x\to-\infty} g(x)$. Then $L : \mathsf{D}^1(X) \to \mathbb{R}$ is a bounded operator. Its kernel contains $\ell^1(X)$ and so it will also contain its closure in the D^1-norm. Since our function f above has $L(f) = 1 \neq 0$, it lies outside of the closure $\ell^1(X)$.

As an upshot, $\underline{\ell^1 H}^1(\mathbb{Z})$ is not trivial (in fact, it is a one-dimensional real space). ♢

Example 1.4 Another simple example are the Cayley graphs of free groups on k generators F_k (resp. free products $C_2 * C_2 * \ldots * C_2$ where C_2 is the group with two elements). The Cayley graphs for the "standard" generating sets (*i.e.* the k free generators, resp. the generators of each C_2 factor) are $2k$-regular (resp. k-regular) trees.

Again, since trees have no cycles one gets that $\ell^p(E) = \ell^p(E) \cap \nabla\mathbb{R}^X$. If $k = 1$ (resp. $k = 2$), then we obtain the same graph as in the previous example. So we may assume that $k > 1$ (resp. $k > 2$). Now it comes in very handy to note that these graphs are not amenable. By Theorem 1.2, this means that $\nabla\ell^p(X)$ is closed or, if one thinks in terms of functions, that $\ell^p(X)$ is a closed subspace in the D^p-norm.

But functions in $\ell^p(X)$ also belong to $c_0(X)$ (the space of functions which tend to 0 at ∞, see §1.1). Hence, if we can find a function with gradient in ℓ^p which is not in $c_0(X)$, then we are done.

But this is fairly easy: (a) pick some edge, (b) removing it will disconnect the tree in two components, (c) set f to be identically 0 on one component and identically 1 on the other, (d) the gradient of this function is supported on one edge, so it lies definitively in D^p.

Consequently, $\underline{\ell^p H}^1$ and $\ell^p H^1$, are non-trivial for any p. ♢

It is straightforward to generalise this to any tree which is not amenable: one just has to make sure that the edge disconnects the tree in two infinite components. In fact, the argument applies to any non-amenable graph which can be disconnected into two infinite components by removing a finite number of edges.

These two examples are somehow extreme in the sense that $\underline{\ell^p H}^1$ is either trivial for all $p > 1$ or not. However, in the case of hyperbolic space, it turns out the p for which $\underline{\ell^p H}^1$ passes from trivial to non-trivial is a significant number (see §2.2.2 for details).

Also, the last example might make you think that almost all non-amenable graphs have a non-trivial $\underline{\ell^p H}^1$ (for some p). But it turns out it is often hard to construct an element of $\mathsf{D}^p(X) \setminus c_0(X)$. The following proposition can partially explain why (as well as generalising Example 1.3 and introducing some important proof technique).

Proposition 1.5 *Assume Γ is the Cayley graph of a group G whose centre $Z := Z(G)$ is infinite. Then $\underline{\ell^p H}^1(\Gamma) = \{0\}$ for all $p > 1$.*

Proof Needless to say, elements of the centre have the very nice property that, for any $g \in G$, $zg = gz$. This translates in a graph theoretical property. Indeed, the action of an element g of G on the right is a graph automorphism. The action on the left by the same element g means one follows the path labelled by the generators $s_i \in S$ so that $g = s_n s_{n-1} \ldots s_2 s_1$.

So being in the centre means that if you follow (starting at any vertex) a path labelled by $z = s_n s_{n-1} \ldots s_2 s_1$, then this is a graph automorphism.

Here is why elements of the centre are so special for this problem. Let $\rho_z f(g) := f(gz)$. Write $z = s_n s_{n-1} \dots s_2 s_1$ and let $t_i = s_i s_{i-1} \dots s_2 s_1$ (with $t_0 = e$ the identity element in G). Then

$$f(g) - \rho_z f(g) = f(g) - f(gz) = f(g) - f(zg) = \sum_{i=1}^{n} f(t_i g) - f(t_{i-1} g).$$

Note that this last expression is a sum of n values of the gradient of f. Hence, by the triangle inequality, $\|f - \rho_z f\|_{\ell^p(X)} \leq n \|\nabla f\|_{\ell^p(E)}$. This implies that f and $\rho_z f$ belong to the same equivalence class.

This can be used to bring the following plan into action. Given some function f with gradient in ℓ^p, consider $\rho_{z_n} f$ where z_n is some sequence of elements of the centre which goes to infinity. Since $\rho_{z_n} f$ are images under graph automorphisms of f, we are effectively translating the gradient of f to infinity.

Since $\ell^p(E) \subset c_0(E)$, this means that $\nabla \rho_{z_n} f$ tends point-wise to 0. Point-wise convergence is synonymous with weak* convergence. But weak* and weak convergence coincide in the reflexive case. And a classical consequence of the Hahn–Banach theorem is that weak and norm convergence to 0 also coincide.

So we found a way to build a sequence of elements which all belong to the equivalence class of f (in the quotient space $\nabla \mathbb{R}^X \cap \ell^p(E)/\overline{\ell^p(X)}$) and whose gradients tend (in norm) to 0. This shows that 0 is in the (closure) of the class too.

But we made no specific assumption on the function f, hence 0 is in the equivalence class of any function, and $\underline{\ell^p H}^1(\Gamma) = \{0\}$. □

The previous proposition can be found [often with weaker hypothesis] in Kappos [32, Theorem 6.4], Martin & Valette [40, Theorem 4.3], Puls [52, Theorem 5.3], Tessera [59, Proposition 3] or [20, Theorem 3.2].

There are many groups with an infinite centre many of them are not amenable. This hopefully contrasts with Example 1.4.

2 Applications

Before we move to our main focus (which has to do with harmonic functions), here is an overview of the different applications of ℓ^p-cohomology to themes.

2.1 Quasi-Isometries

One of the original motivation of ℓ^p-cohomology was to use it as an invariant of quasi-isometry, see Gromov [28, §8].

Let us briefly recall that a map $f : (X, d_X) \to (Y, d_Y)$ between two metric spaces is a quasi-isometry, if there is a constant $K > 1$ such that:

$$\frac{1}{K}d_X(x,x') - K \le d_Y\big(f(x), f(x')\big) \le K d_X(x,x') + K.$$

There are few important "exercises" on this concept, here are two: (1) "being quasi-isometric" is an equivalence relation; (2) a graph (with its combinatorial distance) can be quasi-isometric to a manifold (with its Riemannian metric).

In fact, Kanai has shown [31] that any Riemannian manifold with Ricci curvature and injectivity radius bounded from below is quasi-isometric to a graph (of bounded valency).

Theorem 2.1 (See Élek [15, §3] or Pansu [45]) *If two graphs of bounded valency Γ and Γ' are quasi-isometric, then they have the same ℓ^p-cohomology (in all degrees, reduced or not).*

The result is actually much more powerful, in the sense that it holds in a larger category (measure metric spaces; see above-mentioned references). For shorter proofs in more specific situations see Puls [55, Lemma 6.1] or Bourdon & Pajot [7, Théorème 1.1].

The previous theorem is sometimes very convenient, since it means that results can be transferred between graphs and Riemannian manifolds. This allows for a great flexibility in the methods that can be used to prove the results.

A consequence of 2.1 is that, if G is a finitely generated group, the ℓ^p-cohomology in degree one of any two Cayley graphs (for a finite generating set) is isomorphic. Indeed, the identity map on the vertices is a quasi-isometry between the Cayley graphs (hint: write the generators of one Cayley graph as words in the generators of the other Cayley graph). Thus, one may speak of the ℓ^p-cohomology of a group without making reference to a Cayley graph.

In [47] and [48], Pansu computed the ℓ^p cohomology (in degree 1 and above) of a variety of homogeneous spaces with pinched negative curvature. He then used the triviality or non-triviality of this cohomology to show that many of these spaces are not quasi-isometric, thus answering an old question of Berger.

The study of quasi-isometries also motivated some variants of ℓ^p-cohomology. First, by considering Orlicz spaces (instead of just ℓ^p spaces) Carrasco Piaggio [11] proved a fixed-point result for self-quasi-isometries of (many) Heintze groups.

Second, there is a body of work on the L_{pq}-cohomology (investigations of the quotients of the form $\mathrm{d}^p/(\mathbb{R} + \ell^q)$. The interested reader is encourage to look at Gol'shtein & Troyanov [19], Kopylov [36] and references therein.

2.2 Boundaries

S. Zucker was one of the first person to introduce ℓ^p-cohomology and use it to study manifolds with thin ends (see [63] and references therein). There are however many other applications to other ideal boundaries of spaces.

The ends are another typical "ideal boundary" for a space, and it turns out that the reduced ℓ^1-cohomology in degree one is isomorphic to the space of function on the ends modulo constant functions (see §3 for details).

2.2.1 Poisson Boundary

There is also a strong connection between ℓ^p-cohomology in degree one and harmonic functions. This particular topic will be explained in more details in §4.

The short version is that (if the isoperimetric dimension of the graph is large enough then) a function with a non-trivial cohomology class gives rise to a non-constant bounded harmonic function. This is easier to see in the case $p = 2$, but it extends to other $p \neq 2$ (if the isoperimetric dimension is large enough).

This is interesting since the Poisson boundary (which can be roughly thought of as the space of bounded harmonic function) is not an invariant of quasi-isometry (see, for example, T. Lyons' examples [39]). Namely, there are quasi-isometric graphs one of which has many non-constant bounded harmonic functions, while the other has none.

Theorem 2.1 can be invoked to show that the ℓ^p-cohomology in degree one gives rise to a part of the Poisson boundary which is invariant under quasi-isometries.

2.2.2 Boundary of Hyperbolic Spaces

There are also applications of ℓ^p-cohomology to the boundary of hyperbolic spaces, more precisely to problems which are related to the famous

Conjecture 2.2 (J. Cannon) *Let Γ be a hyperbolic group whose ideal boundary is a 2-sphere. Then Γ is virtually a cocompact lattice in* $\mathrm{PSL}(2, \mathbb{C})$.

Using a result of Keith & Laakso [34, Corollary 1.0.3], Bonk & Kleiner [2] were able to show that if Γ is a hyperbolic group whose ideal boundary is a 2-sphere and the conformal dimension is achieved by some metric, then Γ is virtually a cocompact lattice in $\mathrm{PSL}(2, \mathbb{C})$.

Further results by Bourdon & Pajot [7] show that, for hyperbolic spaces, one can define a L^p-dimension as the infimum over all p for which $\ell^p H^1$ is non-trivial. It turns out that the L^p-dimension coincides with the conformal dimension if there is a metric which achieves the conformal dimension.

Bourdon & Pajot [7] gave examples where these dimensions do not coincide, hence one cannot expect that the strategy from Bonk & Kleiner [2] works out of the box. On the positive side, there has been further work (using ℓ^p-cohomology) by Bourdon & Kleiner [4] which covers the case of Coxeter groups.

For a proof that any hyperbolic space has a non-trivial ℓ^p-cohomology in degree one starting at some p_0 see either Bourdon & Pajot [7], Élek [15] or Puls [54].

M. Bourdon pointed out to the author a very interesting point (see also [3, §2.4.1]). A result of Puls [54, Theorem 1.3] shows that if a group has a non-trivial

Floyd boundary for a Floyd function $\phi(g) = a^{-d(e,g)}$ (where $a > 1$), then its [reduced] ℓ^p-cohomology will be non-trivial for all p such that $\phi \in \ell^p(G)$. A careful reading of the construction of Gerasimov [18] shows that relatively hyperbolic groups will have non-trivial Floyd boundaries satisfying these conditions. Consequently, their reduced ℓ^p cohomology is non-trivial for all p larger than some p_0.

On the other hand, D. Osin pointed out to the author that some acylindrically hyperbolic groups have a trivial ℓ^p-cohomology for all $p \in [1, \infty[$ (these are right-angled Artin groups corresponding to the graph •–•–•–•–•).

Pansu [44] showed that among continuous Lie groups, having non-trivial reduced ℓ^p-cohomology is equivalent to hyperbolicity. This extends to algebraic groups over local fields of characteristic 0 by a result of de Cornulier & Tessera [13].

Lastly, Bourdon & Pajot [7, §3, Proposition 4.1 and the following Remarques] also showed that for p larger than the conformal dimension of the boundary, functions with ℓ^p-gradient, when extended to the boundary of [Gromov] hyperbolic spaces, can separate points of its boundary. In fact, they show that (non-trivial) Lipschitz functions on the boundary give rise to (non-trivial) classes in ℓ^p-cohomology. Bourdon & Kleiner [5, Theorem 3.8(1)] showed that if p is strictly smaller than the conformal boundary, then extensions of ℓ^p classes no longer separate points.

2.2.3 "Nonlinear" Boundaries

Reduced ℓ^p-cohomology (in degree one) is very strongly related to p-harmonic functions. When $p = 2$, this is the same as harmonic functions, but for $p \neq 2$ these are a nonlinear variation of the harmonic equation.

When p is an integer, p-harmonic functions come up naturally when studying a relaxation of conformal maps (called quasi-regular maps). Given two manifolds M and N of dimension p, a map $f : M \to N$ is called quasi-regular if there is a constant C so that $\|\mathrm{d}f\|^p \leq C|\det \mathrm{d}f|$.

When $g : N \to \mathbb{R}$ is a function, the p-Laplacian is $\Delta_p = \mathrm{div}(|\nabla g|^{p-2}\nabla g)$ and p-harmonic functions are functions whose p-Laplacian is 0. A quasi-regular map will allow to pull-back [non-constant] p-harmonic functions, so the existence or absence of [non-constant] p-harmonic functions can be used as obstruction to the existence of quasi-regular maps.

In addition to quasi-regular maps, there are also interesting limiting cases for the p-harmonic equation: when $p \to 1$ this is related to the mean curvature operator and when $p \to \infty$ to Lipschitz extensions.

In the setting of graphs there are two things which might be unclear:

(1) what is the divergence? see §4.
(2) what is a quasi-regular map? see either Benjamini, Schramm & Timàr [10, §1.1] or §2.4.

Furthermore, much like harmonic functions can be used to construct a Royden boundary and a harmonic boundary, p-harmonic functions can be used to construct a p-Royden boundary and a p-harmonic boundary. For the definitions see Puls [55, §2.1]. These boundaries are spaces constructed with the help of the Gelfand transform which can be associated with the classes of the reduced ℓ^p-cohomology in degree one. See paragraph after Lemma 4.7 for details.

The relation between reduced ℓ^p-cohomology in degree one and p-harmonic function is fairly straightforward (see Puls [53] or Martin & Valette [40] for details). Basically, given $f \in \mathsf{D}^p(\Gamma)$, one can try to search for the element which belongs to the same equivalence class as f but whose norm is minimal. For $p \in]1, \infty[$ such an element will exist by convexity of the norm. Furthermore, for all g of finite support on the edges $\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla f + t \nabla g\|^p_{\ell^p(E)}\big|_{t=0} = 0$ (by minimality of the norm of this element). Massaging this last equation (and the fact that g is an arbitrary function of finite support) will show that f is p-harmonic.

Other known consequences of the triviality of the reduced ℓ^p-cohomology in degree one include the triviality of the p-capacity between finite sets and ∞ (see Yamasaki [62] and Puls [55, Corollary 2.3]) and existence of continuous translation invariant linear functionals on $\mathsf{D}^p(\Gamma)/\mathbb{R}$ (see [55, §8]).

2.3 Representation Theory

For infinite groups it is often interesting to look at their representation on infinite dimensional space. For example, Property (T) is defined using the topology on the space of unitary representations in Hilbert spaces. It can also be expressed as a condition on the first cohomology of these representations.

It turns out that ℓ^p-cohomology in degree one (of some Cayley graph of a finitely generated group) is the same thing as the first cohomology of the regular representation (in ℓ^p), see Martin & Valette [40] or Puls [52]. There is also a nice text from Bourdon [3] on the topic (isometric actions on Banach space are equivalent to cohomology linear representations).

Though it might seem a very particular case, it turns out this has a direct and indirect application to Hilbertian representations. The direct application is that triviality of the reduced ℓ^p-cohomology in degree one implies that the reduced first cohomology of any unitary representation with coefficients in ℓ^p is trivial. (The coefficients of a unitary representation π are the functions $\kappa(\gamma) := \langle \pi_\gamma \xi \mid \xi' \rangle$ where ξ, ξ' are elements of the Hilbert space.)

The indirect application is that techniques that are useful to show the vanishing (or non-triviality) of ℓ^p-cohomology may also be applied for unitary representations. See [24] for more details.

2.4 Sphere Packings

A last nice application of reduced ℓ^p-cohomology in degree one is to sphere packings of graphs. Circle packings are a lovely topic which the reader should definitively try to read a survey about (for example, Stephenson [58] and Rohde [56]). The question of realising a graph as the contact graph of some spheres (of varying radius) is a natural generalisation of the circle case.

In fact, one can even relax the hypothesis significantly by requiring that the spheres be some (contractible) domains whose ratio $\frac{\text{outer radius}}{\text{inner radius}}$ is bounded by some constant. With this relaxation, every finite graph can be realised as a contact graph (although the bound on the ratio might get large). But is that true for infinite graphs?

Benjamini & Schramm explore this question in [9] and show that [non-constant] p-harmonic functions can be an obstruction to such packings. Since non-triviality of the reduced ℓ^p-cohomology in degree one is equivalent to the existence of non-constant p-harmonic functions, this gives yet another application of ℓ^p-cohomology.

This topic has been developed further by Pansu in [49].

3 ℓ^1-Cohomology and the Ends

One of the apparent features of Examples 1.3 and 1.4 is that cutting the graph in two infinite components by removing an edge helps a lot to find non-trivial elements of $\underline{\ell^p H}^1$.

This feature will be heavily supported in this section as we show that:

1. a function in $\mathsf{D}^1(\Gamma)$ can be assigned a value on each end of the graph (see below for the definition of the ends of a graph).
2. the function is trivial in reduced ℓ^1-cohomology in degree one if and only if it takes the same value on all the ends.

The ends of a graph are the infinite components of a group which cannot be separated by a finite (*i.e.* compact) set. More precisely, an end ξ is a function from finite sets to infinite connected components of their complement so that $\xi(F) \cap \xi(F') \neq \emptyset$ (for any F and F'). It may also be seen as an equivalence class of (infinite) rays who eventually leave any finite set. Two rays r and r' are equivalent if, for any finite set F, the infinite part of r and r' lie in the same (infinite) connected component.

Thanks to Stallings' theorem, groups with infinitely many ends contain an (non-trivial) amalgamated product or a (non-trivial) HNN extension. Being without ends is equivalent to being finite, and amenable groups may not have infinitely many ends. This may be seen using Stallings' theorem, see also Moon & Valette [41] for a direct proof.

Here is an idea of the proof. Assume there are 3 ends or more, that is upon removing the finite set F, there remains [at least] 3 infinite components, say K_1, K_2 and K_3. By vertex-transitivity, it may be assumed that the identity element belongs to F. Let $c = \max_{f \in F} d(e, f)$ where $d(e, \cdot)$ is the distance to the identity element. Pick elements $h_i \in K_i$ so that $d(e, k_i) > 2c$. Then it is not too hard to check that the set Fh_i (the groups acts on the right by graph automorphism) disconnects K_i in [at least] two infinite components. The technical part comes in when you need to show that Fh_ih_j (for $i \neq j$) further disconnects those components. It then follows that the subgroup generated by $\langle h_1, h_2, h_3 \rangle$ is isomorphic to a free product $H_1 * H_2 * H_3$ where $H_i = \langle h_i \rangle$ is cyclic (finite or infinite). This then implies the group contains a free subgroup and, hence, is not amenable.

Groups with two ends admit $\mathbb{Z}$ as a finite index subgroup. These groups are peculiar, as they have non-trivial reduced ℓ^1-cohomology in degree 1, even if their reduced ℓ^p-cohomology (in all degrees) vanishes for $1 < p < \infty$.

So outside virtually-$\mathbb{Z}$ groups, all infinite amenable groups have one end.

Before moving on, let me mention that the results of this section were first written up in [21, Appendix A]. This result was partially remarked by Pansu (essentially, case where there is one end). As mentioned in Example 1.3, the special case of the group $\mathbb{Z}$ was written down in Martin & Valette [40, Example 3 in §4], who learned it from M. Bourdon (like the author, a former student of Pansu). So the case with two ends was already known to Bourdon. There is only a small step to make to the general case, so that the author is uncertain if he deserves any credit there.

Proposition 3.1 *Let Γ be a connected graph, then $\underline{\ell^1 H}^1(\Gamma) = 0$ if and only if the number of ends of Γ is ≤ 1. More precisely, let $\mathbb{V} = \mathbb{R}^{\text{ends}(\Gamma)}/\mathbb{R}$ be the vector space of functions on ends modulo constants. There is a boundary value map $\beta : \mathsf{D}^1(\Gamma) \to \mathbb{V}$ such that $\beta(g) = \beta(h) \iff [g] = [h] \in \underline{\ell^1 H}^1(\Gamma)$.*

Note that the isomorphism β between $\underline{\ell^1 H}^1(\Gamma)$ and $\mathbb{R}^{\text{ends}(\Gamma)}/\mathbb{R}$ is in the category of vector spaces, not of normed vector spaces. In a few cases, the norm on $\mathbb{V}$ resembles the norm of the quotient $\ell^\infty(|\text{ends}|)/\mathbb{R}$ (see Question 5.1). The proof is barely different from the argument of M. Bourdon found in Martin & Valette [40, Example 3 in §4].

Proof Note that $\mathsf{D}^1(\Gamma) \subset \ell^\infty(X)$: if $g \in \mathsf{D}^1(\Gamma)$, then, for P a path from x to y,

$$|g(y)| = |g(x) + \sum_{e \in P} g(e)| \leq |g(x)| + \|\nabla g\|_{\ell^1(E)}.$$

In fact, $\|g\|_{\ell^\infty(X)} \leq \|g\|_{\mathsf{D}^1(\Gamma)} + \inf_{x \in X} |g(x)|$. Since functions in ℓ^1 decrease at ∞, if one removes a large enough finite set, the function g on the resulting graph is almost constant. In particular, it is possible to define a value of g on each end: let B_n be the ball of radius n at some fixed vertex (root) o, then

$$\beta g(\xi) := \lim_{n\to\infty} g(x_n) \text{ where } x_n \in \xi(B_n).$$

Alternatively, if $r : \mathbb{Z}_{\geq 0} \to X$ is a ray representing the end ξ, then the value at ξ can also be defined as $\lim\limits_{n\to\infty} g\big(r(n)\big)$. It is fairly straightforward to check these limits do not depend on the choice (of x_n and o or of the ray r).

Fix an end ξ_0. Then, define $\beta : \mathsf{D}^1(\Gamma) \to \mathbb{V}$ by changing with a constant the value of g to be 0 at ξ_0 and then looking at the values at the ends. This map is continuous and trivial on $\ell^1(X) + \mathbb{R}$ (since functions in $\ell^1(X)$ have trivial value at the ends). By continuity, $\overline{\ell^1(X) + \mathbb{R}}^{\mathsf{D}^1(\Gamma)} \subset \ker \beta$.

Assume, $\beta(f) = 0$, this means that, $\forall \epsilon > 0, \exists X_\epsilon \subset X$ a finite set such that $f(X_\epsilon^{\mathsf{c}}) \subset [-\epsilon, \epsilon]$. Set

$$f_\epsilon(\gamma) = \begin{cases} \epsilon f(\gamma)/|f(\gamma)| & \text{if } |f(\gamma)| > \epsilon, \\ f(\gamma) & \text{otherwise.} \end{cases}$$

Then $g_\epsilon := f - f_\epsilon$ is finitely supported, so in $\ell^1(X)$. Furthermore, $\|f - g_\epsilon\|_{\mathsf{D}^1(\Gamma)} = \|f_\epsilon\|_{\mathsf{D}^1(\Gamma)}$.

Let X_ϵ be as before, then

$$\nabla f_\epsilon \text{ is } \begin{cases} \text{equal to } \nabla f & \text{on } E \cap (X_\epsilon^{\mathsf{c}} \times X_\epsilon^{\mathsf{c}}), \\ \text{smaller in } |\cdot| \text{ than } \nabla f & \text{on } \partial X_\epsilon, \\ 0 & \text{on } E \cap (X_\epsilon \times X_\epsilon). \end{cases}$$

But $E \cap (X_\epsilon \times X_\epsilon)$ increases, as $\epsilon \to 0$, to the whole of E. More importantly, the ℓ^1-norm of ∇f outside this set tends to 0. Thus $\|f_\epsilon\|_{\mathsf{D}^1(\Gamma)} \to 0$ as $\epsilon \to 0$, and consequently $g_\epsilon \to f$ as $\epsilon \to 0$. Since g_ϵ are finitely supported, they belongs to $\ell^1(X)$. This shows that $f \in \overline{\ell^1(X)}^{\mathsf{D}^1(\Gamma)}$. □

Groups with two ends step strangely out of the crowd: although their reduced ℓ^p-cohomology is always trivial if $p > 1$, it is non-trivial for $p = 1$ (actually isomorphic to the base field). An amusing corollary is

Corollary 3.2 *Let G be a finitely generated group. G has infinitely many ends if and only if for some (and hence all) Cayley graph Γ, $\forall p \in [1, \infty[$, $\underline{\ell^p H}^1(\Gamma) \neq 0$. G has two ends if and only if for some (and hence all) Cayley graph Γ, $\forall p \in]1, \infty[$, $\underline{\ell^p H}^1(\Gamma) = 0$ but $\underline{\ell^1 H}^1(\Gamma) = \mathbb{R}$.*

Proof Use Proposition 3.1 for reduced ℓ^1-cohomology, use any vanishing theorem on groups of polynomial growth (such groups have an infinite centre, so see Proposition 1.5, Kappos [32] or Tessera [59]) to get the remaining values of p for groups with two ends.

Theorem 4.10 (which we have not discussed yet) will give the conclusion for groups with infinitely many ends (which are in particular non-amenable). □

It is worth noting that Bekka & Valette showed in [1, Lemma 2, p.316] that (for G discrete) the cohomology $H^1(G, \mathbb{C}G)$ is also isomorphic (as a vector space) to $\mathscr{A}\!\mathscr{V}$. Furthermore, by [1, Proposition 1], there is an embedding $H^1(G, \mathbb{C}G) \hookrightarrow \ell^1 H^1(G)$. A careful reading would probably reveal this remains injective in reduced cohomology (the only case to check is when G has two ends).

4 ℓ^p-Cohomology and Harmonic Functions

In §3, we dealt with one of the apparent features of Examples 1.3 and 1.4. Another feature which is present in those examples as well as the previous section is that it is very useful to think in terms of values at infinity.

However, for functions with gradient in ℓ^p with $p > 1$ this is somewhat counter intuitive. Indeed, the reader can quickly come up with a function on the graph of the line (a Cayley graph of $\mathbb{Z}$) which grows to ∞ even though its gradient is in ℓ^2. Nevertheless, this obstacle can be overcome.

The main motivation in this section is to show that the reduced ℓ^p-cohomology in degree one can be seen as a space of function on an ideal boundary, namely the Poisson boundary. The oldest result in this direction is a theorem of Lohoué [37] which says that in a non-amenable graph there is exactly one harmonic function in each equivalence class of $\underline{\ell^p H}^1(\Gamma)$. The results presented in this section come from [21], with some simplifications in the presentation coming mostly from [24].

In contrast to the result of Lohoué [37], the amenable case is trickier, so this result can only be generalised to some extent. To say how, some preliminary definitions are required.

Isoperimetric profiles. For $F \subset X$ a subset of the vertices, recall that ∂F is set of edges between F and F^{c}. Let $d \in \mathbb{R}_{\geq 1}$. Then, a graph Γ has

$$\begin{aligned} &\mathrm{IS}_d \text{ if there is a } \kappa > 0 \text{ such that for all finite } F \subset X,\ |F|^{(d-1)/d} \leq \kappa |\partial F|; \\ &\mathrm{IS}_\omega \text{ if there is a } \kappa > 0 \text{ such that for all finite } F \subset X,\ |F| \leq \kappa |\partial F|. \end{aligned}$$

Quasi-homogeneous graphs with a certain (uniformly bounded below) volume growth in n^d will satisfy these isoperimetric profiles, see Woess' book [61, (4.18) Theorem].

A Cayley graph will satisfy IS_d (for any $d \leq \delta$) if the growth of balls in this Cayley graph is bounded below by Kn^δ (for some $K > 0$). A Cayley graph will not satisfy IS_d (for any $d > \delta$) if the growth of balls in this Cayley graph is bounded above by $K'n^\delta$ (for some $K' > 0$).

Using Gromov's theorem on groups of polynomial growth [27], that the only groups which do not satisfy IS_d for all d are virtually nilpotent groups.

Cayley graphs of a group G does not satisfy IS_ω if and only if G is amenable. (There are many amenable groups which are not virtually nilpotent.) The upcoming result will apply best to groups which are not virtually nilpotent. See [61, §14] for more details.

Values at infinity. It is difficult to speak of a value at infinity, since it is not clear with what we can identify infinity (yet). However it is easy to say if a function is constant at infinity. This means that it belongs to $\mathbb{R} + c_0(X)$, *i.e.* a constant function plus an element of $c_0(X)$.

More precisely, let B_n be a sequence of balls in the graph with the same centre and B_n^{c} the sequence of their complement. On a connected graph, a function $f : X \to \mathbb{R}$ is constant at infinity if $\exists c \in \mathbb{R}$ so that $\forall \epsilon > 0$, $\exists n_\epsilon$ satisfying $f(B_{n_\epsilon}^{\mathsf{c}}) \subset [c - \epsilon, c + \epsilon]$.

Harmonic functions. A function $f : X \to \mathbb{R}$ is harmonic if it satisfies the mean-value property: for any vertex $x \in X$, $\sum\limits_{y \in N(x)} \big(f(y) - f(x)\big) = 0$ (where $N(x)$ denotes the neighbours of x).

Let us define the following spaces of harmonic functions:

- $\mathscr{H}(\Gamma)$ is the space of harmonic functions.
- $\mathscr{H}\mathsf{D}^p(\Gamma) = \mathscr{H}(\Gamma) \cap \mathsf{D}^p(\Gamma)$ is the space of harmonic functions whose gradient is in ℓ^p.
- $\mathsf{B}\mathscr{H}\mathsf{D}^p(\Gamma) = \ell^\infty(X) \cap \mathscr{H}(\Gamma) \cap \mathsf{D}^p(\Gamma)$ is the space of bounded harmonic functions whose gradient is in ℓ^p.

Divergence. There is another way to define harmonic functions by introducing the divergence. For two finitely supported function f and g on a countable set Y, define the pairing $\langle f \mid g \rangle_Y = \sum_{y \in Y} f(y)g(y)$. (The subscript Y will often be dropped.) This allows to define the adjoint of the gradient ∇, denoted ∇^* and called divergence, by $\langle f \mid \nabla g \rangle_E = \langle \nabla^* f \mid g \rangle_X$. More precisely, for $f : E \to \mathbb{R}$, one finds

$$\nabla^* f(x) = \sum_{y \in N(x)} f(y, x) - \sum_{y \in N(x)} f(x, y).$$

In particular

$$\nabla^* \nabla f(x) = 2 \sum_{y \in N(x)} \big(f(y) - f(x)\big).$$

Thus, harmonic functions are exactly the functions for which the divergence of the gradient is trivial.

Four conditions. Define for $p \geq 1$:

(1_p) The reduced ℓ^p-cohomology in degree one vanishes (for short, $\underline{\ell^p H}^1 = \{0\}$).
(2_p) All functions in $\mathsf{D}^p(G)$ take only one value at infinity.
(3_p) There are no non-constant functions in $\mathscr{H}\mathsf{D}^p(G)$.
(4_p) There are no non-constant functions in $\mathsf{B}\mathscr{H}\mathsf{D}^p(G)$.

For the record, note that $(1_1) \iff (2_1) \iff$ the number of ends is ≤ 1 (see Proposition 3.1 above).

Here is the best known to date extension of Lohoué's result [37].

Theorem 4.1 *Assume a graph Γ is of bounded valency and has* IS_d. *For* $1 < p < d/2$, $(1_p) \iff (2_p) \implies (3_p) \implies (4_p)$ *and, for* $q \geq \frac{dp}{d-2p}$, $(4_q) \implies (1_p)$.

If Γ has IS_d *for all d, then "$\forall p \in]1,\infty[$, (i_p) holds" where $i \in \{1,2,3,4\}$ are four equivalent conditions.*

The proof is split as follows: $(1_p) \iff (2_p)$ is the content of §4.2 (see Corollary 4.9). $(2_p) \implies (3_p)$ is a fairly easy consequence of the maximum principle (see Lemma 4.12). $(3_p) \implies (4_p)$ is obvious (since $\mathsf{B}\mathscr{H}\mathsf{D}^p(\Gamma) \subset \mathscr{H}\mathsf{D}^p(\Gamma)$). $(4_p) \implies (1_q)$ is the bulk of §4.3 (see Theorem 4.14).

4.1 Reduction to Bounded Functions

Now the first step in order to associate a value at infinity to any function in $\mathsf{D}^p(\Gamma)$ is to show that one can restrict to bounded functions.

This is basically the content of Lemma 4.4 from Holopainen & Soardi [30]. The Lemma is there stated in terms of p-harmonic functions, but its proof can be adapted without much difficulty.

We will use $[f] \in \underline{\ell^p H}^1(\Gamma)$ to denote the equivalence class of the function f, *i.e.* the closure of $f + \ell^p(X)$ in D^p-norm (or $\overline{f + \ell^p(X)}^{\mathsf{D}^p}$).

Lemma 4.2 (Holopainen & Soardi [30], 1994) *Let $g \in \mathsf{D}^p(\Gamma)$ be such that $g \notin [0] \in \underline{\ell^p H}^1(\Gamma)$. For $t \in \mathbb{R}_{>0}$, let g_t be defined as*

$$g_t(x) = \begin{cases} g(x) & \text{if } |g(x)| < t, \\ t\frac{g(x)}{|g(x)|} & \text{if } |g(x)| \geq t. \end{cases}$$

Then there exists t_0 such that $g_t \notin [0]$, for any $t > t_0$.

In particular, the reduced ℓ^p cohomology is trivial if and only if all bounded functions in $\mathsf{D}^p(\Gamma)$ have trivial classes.

Proof The proof goes essentially as in Proposition 3.1. Assume without loss of generality that $g(o) = 0$ for some preferred vertex (*i.e.* root) $o \in X$. Since $\|\nabla g\|_{\ell^\infty(E)} \leq \|\nabla g\|_{\ell^p(E)} =: K$, given $x \in X$ and P a path from o to x,

$$|g(x)| = |g(x) - g(o)| = \sum_{e \in P : o \to x} \nabla g(e) \leq d(o,x)\|\nabla g\|_{\ell^p(E)}.$$

In particular, g_t is identical to g on $B_{t/K}$. Hence $\|g - g_t\|_{D^p(\Gamma)} \leq \|\nabla g\|_{\ell^p(B^{\mathsf{c}}_{t/K})}$, where $\ell^p(B^{\mathsf{c}}_{t/K})$ denotes the ℓ^p-norm restricted to edges which are not inside $B_{t/K}$. Because $\nabla g \in \ell^p(E)$, $\|\nabla g\|_{\ell^p(B^{\mathsf{c}}_{t/K})}$ tends to 0, as t tends to ∞.

Now if there is an infinite sequence t_n such that g_{t_n} are in $[0]$ and $t_n \to \infty$, then g_{t_n} is a sequence of functions in $[0]$ which tends (in D^p-norm) to g. This implies $g \in [0]$, a contradiction. Hence, for some t_0, $g_t \notin [0]$ given that $t > t_0$. □

4.2 Values at Infinity

The aim of the current subsection is to show that (if the proper isoperimetric profile is present) functions in $\mathsf{D}^p(\Gamma)$ corresponding to the trivial class are exactly those which are constant at infinity. Some concepts from nonlinear potential theory will also come in handy.

Definition 4.3 Let (X, E) be an infinite connected graph. The **inverse** p**-capacity** of a vertex $x \in X$ is

$$\mathrm{icp}_p(x) := \big(\inf\{\|\nabla f\|_{\ell^p E} \mid f : X \to \mathbb{C} \text{ is finitely supported and } f(x) = 1\}\big)^{-1}.$$

The graph is called p**-parabolic** if $\mathrm{icp}_p(x) = +\infty$ for some $x \in X$. A graph is called p-**hyperbolic** if it is not p-parabolic.

One might also like to call the inverse p-capacity the "p-resistance to ∞". (When $p = 2$ capacity and resistance are strongly related.)

Recall (see Holopainen [29], Puls [55] or Yamasaki [62]) that if $\mathrm{icp}_p(x_0) = 0$ for some x_0, then $\mathrm{icp}_p(x) = 0$ for all $x \in X$. Recall also that 2-parabolicity is equivalent to recurrence.

Remark 4.4

1. If the graph Γ is vertex-transitive, $\mathrm{icp}_p(x) = \mathrm{icp}_p(y)$ for all $x, y \in X$. Let $\mathrm{icp}_p(\Gamma) := \mathrm{icp}_p(x)$ be this constant. It is also easy to see that if the automorphism group acts co-compactly on the graph, the inverse p-capacity is bounded from below.
2. Note that in the definition of p-capacity, one may also assume that the functions take value only in $\mathbb{R}_{\geq 0}$. Indeed, looking at $|f|$ instead of f reduces the norm of the gradient. Likewise, one can even assume f takes value only in $[0, 1]$ as truncating f at values larger than 1 will again reduce the norm of the gradient. ♢

The following proposition is an adaptation of a result of Keller, Lenz, Schmidt & Wojchiechowski [35, Theorem 2.1].

Proposition 4.5 *Assume Γ is vertex-transitive and has* IS_d. *Let $p < d$. If $f \in D^p(\Gamma)$ represents a trivial class in $\underline{\ell^p H}^1(\Gamma)$, then f is constant at infinity.*

Furthermore, $c_0(X) \subset D^p(\Gamma)$ and $\forall f \in c_0(X)$, $\|f\|_{\ell^\infty(X)} \leq \mathrm{icp}_p(\Gamma)\|\nabla f\|_{\ell^p(E)}$.

Proof A consequence of the Sobolev embedding corresponding to IS_d is that the graph is p-hyperbolic. See Troyanov [60, §7] as well as Woess' book [61, §4 and §14] and references therein for details.

As Γ is p-hyperbolic and by Remark 4.4.2, one has $\forall f$ of finite support $|f(x)| \leq \mathrm{icp}_p(x)\|\nabla f\|_p$. However, by Remark 4.4.1, there is no dependence on x on the right-hand side. So $\forall x \in G, \forall f$ of finite support $|f(x)| \leq \mathrm{icp}_p\|\nabla f\|_p$ where icp_p is $\mathrm{icp}_p(\Gamma)$. Trivially this implies

$$\forall f : G \to \mathbb{C} \text{ of finite support } \|f\|_\infty \leq \mathrm{icp}_p\|\nabla f\|_p.$$

As a first consequence, assume $f_n \overset{\mathsf{D}^p}{\to} f$ with f_n finitely supported. Then f_n also converge to f in $\ell^\infty(X)$. Since $c_0(X)$ is the closure of finitely supported functions in $\ell^\infty(X)$, this shows that $f \in \overline{\ell^p(X)}^{\mathsf{D}^p}$ implies $f \in c_0(X)$. In other words, if f represents a trivial class in reduced ℓ^p-cohomology, then f is constant at infinity.

As a second consequence, let us show the "Furthermore". Pick some $f \in c_0(X)$. Apply the inequality to $g_\epsilon = f - f_\epsilon$ where f_ϵ is the truncation of f:

$$f_\epsilon(x) = \begin{cases} \epsilon f(x)/|f(x)| & \text{if } |f(x)| > \epsilon \\ f(x) & \text{else.} \end{cases}$$

Indeed, g_ϵ is finitely supported so it satisfies $\|g_\epsilon\|_\infty \leq \text{icp}_p \|\nabla g_\epsilon\|_p$ (recall that $\text{icp}_p = \text{icp}_p(\Gamma)$). Also $\|\nabla g_\epsilon\|_p \leq \|\nabla f\|_p$ and $\|f\|_\infty \leq \epsilon + \|g_\epsilon\|_\infty$. Hence $\|f\|_\infty \leq \epsilon + \text{icp}_p \|\nabla f\|_p$ and the conclusion follows by letting $\epsilon \to 0$. □

The above proposition gives the following very nice characterisation of functions corresponding to the trivial class.

Corollary 4.6 *Assume Γ is vertex-transitive and has* IS_d. *Let $d > p$. $f \in \mathsf{D}^p(\Gamma)$ represents a trivial class in $\underline{\ell^p H}^1(\Gamma)$ if and only if f is constant at infinity.*

Proof Without loss of generality the constant at infinity is 0 (because one may add a constant function to f). Considering again $g_\epsilon = f - f_\epsilon$ (where f_ϵ is the truncation of f as in Lemma 4.2 and Proposition 4.5), one can check that, as $\epsilon \to 0$, $g_\epsilon \overset{\mathsf{D}^p}{\to} f$. Since g_ϵ is finitely supported, it is in $\ell^p(X)$ (and this concludes the proof). □

The above results are very nice, but they do require a fairly strong hypothesis, namely that the graph is vertex-transitive. If the isoperimetric profile is good enough, this can be remedied.

As in Keller, Lenz, Schmidt & Wojchiechowski [35], say that the graph Γ is **uniformly p-hyperbolic** if $\text{icp}_p(\Gamma) := \sup_{x \in X} \text{icp}_p(x)$ is finite. One can show:

Lemma 4.7 *If Γ is a graph of bounded valency with d-dimensional isoperimetry and $d > 2p$, then Γ is uniformly p-hyperbolic.*

Proof First, recall that d-dimensional isoperimetry implies that the Green's kernel ($k_o := \sum_{n \geq 0} P_o^n$ where P_o^n is the random walk distribution at times n starting at the vertex o) has an $\ell^q(X)$-norm (for some $q < p' = \frac{p}{p-1}$) which is bounded independently from o.

Indeed, d-dimensional isoperimetry implies that $\|P_o^n\|_\infty \leq \kappa n^{-d/2}$ (where $\kappa \in \mathbb{R}$ comes from the constant in the isoperimetric profile; see Woess' book [61, (14.5) Corollary] for details). From there, one gets that $\|P_o^n\|_q^q \leq \|P_o^n\|_q^{q-1} \|P_o^n\|_1 \leq \kappa^{q-1} n^{-d(q-1)/2}$. This implies that $\|k_o\|_q \leq \sum_{n \geq 0} \kappa^{1/q'} n^{-d/2q'}$ (a series which converges if $d > 2q'$).

Second, let f be a finitely supported function with $f(o) = 1$, then

$$\langle \nabla f \mid \nabla k_o \rangle = \langle f \mid \nabla^* \nabla k_o \rangle = \langle f \mid \delta_o \rangle = f(o) = 1.$$

Since $\|\nabla f\|_p \geq \|\nabla k_o\|_{p'}^{-1} \langle \nabla f \mid \nabla k_o \rangle$, $\|\nabla k_o\|_{p'} \leq 2\nu \|k_o\|_{p'} \leq 2\nu \|k_o\|_q = 2\nu\kappa_q^{-1}$ (where ν is the maximal valency of a vertex) and there is no dependence in o, this means that $\mathrm{icp}_p(\Gamma) \leq \kappa_q / 2\nu$.

Noting that, for the above, the conditions $q \leq p'$ and $2q' < d$ need to hold, one gets that the bound holds as long as $2p < d$. □

Remark 4.8 Pansu pointed out the following shortcut. The Sobolev (or Nash) inequality corresponding to IS_d and the exponent p is actually: for any finitely supported function f and all $p < d$, $\|f\|_{\frac{pd}{d-p}} \leq K \|\nabla f\|_p$ (where K depends only on the constant in the isoperimetric profile and p). Consequently, $\inf\{\|\nabla f\|_{\ell^p E} \mid f : X \to \mathbb{C}$ is finitely supported and $f(x) = 1\} \geq \frac{1}{K}$. Hence $\mathrm{icp}_p(x) \leq K$ for any x. So the graph is uniformly hyperbolic for any $p < d$. ♢

An amusing corollary is that, most of the time (*i.e.* if the isoperimetric dimension of the graph is large enough), the p-Royden and p-harmonic boundaries are equal. See [24, Corollary 5.10]. However, for our current purpose, only the corollary will be required.

Corollary 4.9 *Assume Γ has IS_d and that $d > p$. Then $f \in \mathsf{D}^p(\Gamma)$ represents a trivial class in $\underline{\ell^p H}^1(\Gamma)$ if and only if f is constant at infinity.*

The proof is essentially the same as Proposition 4.5 and Corollary 4.6 above.

There are two very useful consequences of this result.

Note that for $q < p$, $\mathsf{D}^q(\Gamma) \subset \mathsf{D}^p(\Gamma)$. This means that the identity map $\underline{\ell^q H}^1(\Gamma) \to \underline{\ell^p H}^1(\Gamma)$ is a quotient map (since one quotients out by a larger subspace in $\underline{\ell^p H}^1(\Gamma)$).

Theorem 4.10 *Assume Γ has IS_d and that $d > p$. Then, for $1 \leq q < p$, the natural quotient map $\underline{\ell^q H}^1(\Gamma) \to \underline{\ell^p H}^1(\Gamma)$ is injective.*

Proof According to Corollary 4.9 (or Proposition 3.1 if $q = 1$), if there is a function $f \in \mathsf{D}^q(\Gamma)$ such that $[f] \neq 0 \in \underline{\ell^q H}^1(\Gamma)$, then f is not constant at ∞. But since f is not constant at infinity, Corollary 4.9 implies that f is not in the trivial class in $\underline{\ell^p H}^1(\Gamma)$ too. Consequently, the map is injective. □

This is very effective in the realm of groups since:

- either the group is nilpotent, and in that case Proposition 1.5 shows that $\underline{\ell^p H}^1(\Gamma)$ is trivial for any $p \in]1, \infty[$.
- or the group is not nilpotent, and in that case it has IS_d for any $d \geq 1$. Hence $\underline{\ell^q H}^1(\Gamma) \to \underline{\ell^p H}^1(\Gamma)$ is injective for any $1 \leq q < p$.

This also shows that for any amenable group and for all $p \in]1, 2]$, $\underline{\ell^p H}^1(\Gamma)$ is trivial. Indeed, Cheeger & Gromov [12] showed that $\underline{\ell^2 H}^1(\Gamma)$ is trivial for any amenable group.

Theorem 4.10 is also counter intuitive if one thinks in terms of p-harmonic functions. Indeed, there is *a priori* no reason to believe that the absence of non-constant p-harmonic function implies the absence of non-constant q-harmonic function. These are different nonlinear equations.

Another powerful consequence of this result is

Theorem 4.11 *Assume Γ has a spanning connected subgraph Γ' such that: Γ' has* IS_d *and $\underline{\ell^p H}^1(\Gamma')$ is trivial for some $p < d$. Then $\underline{\ell^p H}^1(\Gamma)$ is trivial for any $q \le p$.*

Proof It follows from the definition of IS_d that Γ has IS_d too. Take any $f \in \mathsf{D}^q(\Gamma)$. Then $f \in \mathsf{D}^p(\Gamma')$ too (since $q \le p$ and there are less edges in Γ' so the norm of the gradient can only be smaller). By Corollary 4.9 (applied to Γ'), f is constant at infinity. But then Corollary 4.9 (applied to Γ) tells us that f must have a trivial class in $\underline{\ell^q H}^1(\Gamma)$. □

There are many applications of this simple fact. It can be used to show that many wreath products and Cartesian products of graph have trivial $\underline{\ell^p H}^1$ for all $p \in [1, \infty[$. In fact, the Cartesian product of any two groups $G = G_1 \times G_2$ has trivial $\underline{\ell^p H}^1$ for all $p \in [1, \infty[$. See [22] for details.

4.3 Harmonic Functions

Harmonic functions come naturally into play not only because of the result of Lohoué [37]. Because of the maximum principle, a harmonic function which is constant at infinity is constant. Hence

Lemma 4.12 *Assume Γ has* IS_d. *Assume either that*

- *Γ is vertex-transitive and $d > p$.*
- *or $d > 2p$.*

If $\mathscr{H}\mathsf{D}^p(\Gamma)$ contains a non-constant harmonic function then $\underline{\ell^p H}^1(\Gamma)$ is not trivial.

The reverse implication is essentially a question of Pansu [46, Question 6 in §1.9] (Pansu restricts the question to groups which are not nilpotent). A short answer (and the best to date) is "almost yes, because we lose a bit of regularity":

Lemma 4.13 *Let Γ be a graph with* IS_d *and $1 \le p < d/2$. For any $g \in \mathsf{D}^p(\Gamma)$, there is a function $\tilde{g}$ such that:*

- *$\tilde{g}$ is harmonic.*
- *$\tilde{g}$ is bounded if g is.*
- *$\tilde{g} - g \in \ell^r(X)$ for any $r > \frac{dp}{d-2p}$.*

Proof It turns out $\tilde{g}$ is in the most obvious function which could fit the bill. Indeed, given g one can make it "more harmonic" by replacing the value at a vertex by the average if its values at neighbouring vertices. Since harmonic functions are exactly those which have the mean-value property, repeating this process infinitely many times, one finds the desired function $\tilde{g}$.

So, let R be the random walk operator, *i.e.* given a function $g : X \to \mathbb{R}$, $Rg(x) = \sum_{y \in N(x)} g(y)$ (where $N(x)$ are the neighbours of x). We want to show that $\tilde{g} = \lim_{n\to\infty} R^n g$ is a well-defined function with all the above properties. Actually the two first properties are essentially automatic (if the limit converges even just in the point-wise sense).

The operator R and its iterations R^n are given by very simple kernels. Recall that $P_x^n(y)$ is the probability that a simple random walk from x lands at y after n steps. Then $R^n g(x) = \sum_{y \in X} P_x^n(y) g(y)$.

Write:

$$\tilde{g} - g = \lim_{n\to\infty} R^n g - g = \sum_{i\geq 0} (R^{i+1} g - R^i g) = \sum_{i \geq 0} R^i (R - \mathrm{Id}) g,$$

where Id is the identity operator. Let $h = (R - \mathrm{Id})g$, then $h(x)$ is a finite average of values of the gradient of g. Since $g \in \mathsf{D}^p(\Gamma)$ then $h \in \ell^p(X)$. In fact $\|h\|_{\ell^p(X)} \leq 2\|\nabla g\|_{\ell^p(E)}$.

Since R^i are operators defined by a kernel one may use Young's inequality (see *e.g.* Sogge's book [57, Theorem 0.3.1]): for $r > p$ and $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$,

$$\begin{aligned}\|\tilde{g} - g\|_{\ell^r(X)} = \Big\| \big(\sum_{i\geq 0} R^i\big) h \Big\|_{\ell^r(X)} &\leq \sup_{x\in X} \Big\| \sum_{i \geq 0} P_x^i \Big\|_{\ell^q(X)} \|h\|_{\ell^p(X)} \\ &\leq 2 \sup_{x \in X} \Big\| \sum_{i\geq 0} P_x^i \Big\|_{\ell^q(X)} \|\nabla g\|_{\ell^p(E)}.\end{aligned}$$

We are done if one can show that $\sup_{x\in X} \| \sum_{i \geq 0} P_x^i \|_{\ell^q(X)} < +\infty$ for all $q' < d/2$. Indeed this would mean that $\tilde{g} - g \in \ell^r(X)$ (for all $r > \frac{dp}{d-2p}$). This shows the convergence (and existence of $\tilde{g}$) and concludes the proof.

Fortunately, there are very good estimates at hand for $\|P_x^i\|_{\ell^q(X)}$, which rely only on isoperimetric profiles (see proof of Lemma 4.7). Indeed, if Γ has IS_d, then

$$\exists K > 0, \quad \forall x, y \in X, \quad P_x^n(y) \leq K n^{-d/2}.$$

Obviously $\|P_x^n\|_{\ell^1(X)} = 1$ (because it is a probability distribution). By Hölder's inequality,

$$\|P_x^n\|_{\ell^q(X)} \leq \|P_x^n\|_{\ell^1(X)}^{1/q} \|P_x^n\|_{\ell^\infty(X)}^{1/q'}.$$

Hence $\|P_x^{(n)}\|_{\ell^q(X)} \leq K' n^{-d/2q'}$ uniformly in x, for some $K' > 0$. The condition $\frac{d}{2q'} > 1$ translates as $q > \frac{d}{d-2}$. Plunging this in $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ yields $r > \frac{pd}{d-2p}$. □

Theorem 4.14 *If Γ has* IS_d*, $p < \frac{d}{2}$ and $q > \frac{dp}{d-2p}$ then $(4_q) \implies (1_p)$.*

Proof We will show the contrapositive. So assume $\neg(1_p)$, *i.e.* there is $f \in \mathsf{D}^p(\Gamma)$ which is not trivial in $\underline{\ell^p H}^1(\Gamma)$. Then, by Lemma 4.2 one may assume f is actually bounded (otherwise, consider some truncation of f). By Corollary 4.9, f is not constant at infinity. By Lemma 4.13, there is a harmonic bounded function $\tilde{f}$ which differs from f by an element of $\ell^q(X)$.

Since $\ell^q(X) \subset c_0(X)$, $\tilde{f}$ is not constant at infinity either and hence not constant. Lastly, since $\nabla : \ell^q(X) \to \ell^q(E)$ is bounded, $\tilde{f} \in \mathsf{D}^q(\Gamma)$.

To sum up $\tilde{f}$ is not constant, bounded, harmonic and its gradient is in ℓ^q. So $f \in \mathsf{B}\mathscr{H}\mathsf{D}^q(\Gamma)$. This shows $\neg(4_q)$ as claimed. □

The most effective application of Theorem 4.14 are the two following corollaries:

Corollary 4.15 *Assume a graph has the Liouville property (*i.e. *there are no non-constant bounded harmonic functions) and satisfies* IS_d *for some $d > 2$. Then $\underline{\ell^p H}^1(\Gamma)$ is trivial for any $p < \frac{d}{2}$.*

This is again very effective in the realm of groups since one may assume IS_d for any d. Also, there are many amenable groups which are known to have the Liouville property (in some Cayley graph). Hence the previous corollary covers a lot of amenable groups (for all p).

Corollary 4.16 *Assume Γ has* IS_d*. If $\underline{\ell^p H}^1(\Gamma)$ is not trivial for some $p < \frac{d}{2}$, then any graph quasi-isometric to Γ has a non-trivial Poisson boundary.*

As mentioned before, this contrasts with the fact that the triviality of the Poisson boundary is not invariant under quasi-isometries.

5 Epilogue

5.1 Further Results

Let us summarise some of the results in the realm of groups. It is known that the reduced ℓ^p-cohomology in degree one is trivial in degree 1 for the following groups ($1 < p < \infty$):

1. G has an infinite FC-centre (see Kappos [32, Theorem 6.4], Martin & Valette [40, Theorem 4.3], Puls [52, Theorem 5.3], Tessera [59, Proposition 3] or [20, Theorem 3.2])
2. G has a finitely supported measure with the Liouville property, *i.e.* no bounded μ-harmonic functions (see [21, Theorem 1.2 or Corollary 3.14]). This includes all polycyclic groups (for such groups, see also Tessera [59])
3. G is a direct product of two infinite finitely generated groups (see [22, Corollary 3]).

4. G is a wreath product with infinite base group (see [22, Proposition 1] and Martin & Valette [40, Theorem.(iv)]) unless the base group has infinitely many ends and the lamp group is amenable. Arguments from Georgakopoulos [17] show that this also holds for finite lamp groups (even if the base group has infinitely many ends).
5. G is some specific type of semi-direct product $N \rtimes H$ with N not finitely generated (see [23] for the full hypothesis).
6. L^p-cohomology can be defined for groups which are not endowed with the discrete topology. Amenable groups can then be non-unimodular. For such groups results of Tessera show the L^p-cohomology in degree one is trivial, see [59].

It is also trivial in any amenable group for any $1 < p \leq 2$ (see [21] or Theorem 4.10 above).

Lastly:

- (see [24, Corollary 1.3]) if G is finitely generated and there is a finitely generated subgroup K so that (a) either $\underline{\ell^p H}^1(K)$ is trivial or K has an infinite FC-centraliser, (b) K has growth at least polynomial of degree $d > p$, and (c) K is not contained in an almost-malnormal strict subgroup of G, then $\underline{\ell^p H}^1(G)$ is trivial.
- (see [24, Corollary 5.11] or Bourdon, Martin & Valette [6, Theorem 1.1)] for a weaker version) if $K < G$ is an infinite subgroup and $\ell^p H^1(K) = \{0\}$, then either $\ell^p H^1(G) = 0$ or there is an almost-malnormal subgroup $H \lneq G$ so that $K < H$.

In particular, Baumslag–Solitar groups also have trivial reduced ℓ^p-cohomology for all $p \in [1, \infty[$.

These last two can actually be interpreted as a trichotomy (resp. a dichotomy) which resembles a result of Gaboriau [16, Théorème 6.8] (in the case $p = 2$). Gaboriau presents [16, Théorème 6.8] as a generalisation of a result of Schreier [16, ¶ after Théorème 6.8 in §0]. Gaboriau's result cannot be generalised to $p > 2$: Bourdon in [3, paragraph **4)** in §1.6] gives an example to this effect.

As for groups where the ℓ^p-cohomology is not trivial:

1. any hyperbolic group or relatively hyperbolic group has a p_0 so that $\ell^p H^1(G)$ is not trivial for any $p > p_0$ (see §2.2.2 for details).
2. there are torsion groups (of infinite exponent) for which $\ell^p H^1(G)$ is not trivial for all $p > 2$. These groups have no free subgroups, yet are not amenable. They do not have a finite presentation. See Osin [42].

However, there are acylindrically hyperbolic groups for which $\ell^p H^1$ is trivial for all $p \in [1, \infty[$ (see §2.2.2 for details).

As for graphs it is easy to construct graphs which are amenable and have non-trivial $\underline{\ell^p H}^1$. Indeed, take any graph Γ which has IS_d for some $d > 2$ and more than two ends. By Proposition 3.1, $\underline{\ell^1 H}^1(\Gamma)$ is not trivial. By Theorem 4.10, for any $p \in [1, \frac{d}{2}[$, $\underline{\ell^p H}^1(\Gamma)$ is also non-trivial.

To make the example slightly more specific, take two copies of a Cayley graph of some group which is amenable but not nilpotent. Join these two copies by an edge. Then it fits the description of the previous paragraph and has IS_d for any d.

5.2 Questions

It was shown in §3 that the reduced ℓ^1-cohomology in degree one identifies to the space of functions on the ends modulo constant functions. This is an isomorphism of vector space, but the norm on the space of functions is probably related to how "large" the ends are and how they are connected.

Question 5.1 *Describe the norm on the vector space* $\underline{\ell^1 H}^1(\Gamma) \simeq \mathcal{N} = \mathbb{R}^{\mathrm{ends}(\Gamma)}/$*constants.*

A question dating back at least to Gromov [28, §8.A_1.(A_2), p.226]:

Question 5.2 *Let G be an amenable group, is it true that for one (and hence all) Cayley graph Γ and all $1 < p < \infty$, $\underline{\ell^p H}^1(\Gamma) = 0$?*

The original question concerns cohomology in all degrees.

Of course, this brings up the question what should be the cohomology of a graph in higher degree. The only results (beyond Cheeger & Gromov [12]) are those of Kappos [32] (in the discrete case) and those of Bourdon & Rémy [8], Pansu & Rumin [50], and Pansu & Tripaldi [51] (in the continuous case).

One of the problems is that there are many possibilities (and that unlike in degree one, they do not coincide). The simplest possibility pops up in the case of groups. One considers the left-regular representation on $\ell^p(G)$. There are standard definitions to speak of the cohomology of this representation in higher degree.

Another simple definition is in the continuous set-up (*i.e.* the cohomology of manifolds). We dealt almost exclusively with the case of graphs, but for manifolds ℓ^p-cohomology in degree k can be defined as "$(k-1)$-forms ω so that $\mathrm{d}\omega \in \mathsf{L}^p$"/"$(k-1)$-forms ω in L^p".

In the case of graphs (which are not necessarily Cayley graphs), one possibility (for degree two) is to look at the space of cycles C. There are some technicalities in finding the reasonable (*e.g.* countable) basis of this space so as to make it tractable. For example, in the Cayley graph of a group with a finite presentation, the presentation gives a good basis for the space of cycles. One can then define the "rotational" as follows: if c is a[n oriented] cycle given by following the oriented edges $e_1, \ldots, e_n$ and g is a function on the edges, then $\mathrm{rot} g(c) = \sum_i g(e_i)$. Assuming the rotational gives a bounded operator (in the case groups, this amounts to the fact that the presentation is finite), a possibility for the cohomology in degree two is then given by taking the quotient "functions on the edges with rotational in $\ell^p(C)$"/"functions which are in $\ell^p(E)$".

Question 5.3 *Given a Cayley graph of a finitely presented amenable group, are the triviality of both definitions above equivalent? invariant under quasi-isometry? for which class of groups are they trivial?*

Élek [15] showed that the following three definition of ℓ^p-cohomology coincide for groups which possess a finite $K(\pi, 1)$:

- the coarse ℓ^p-cohomology of finitely generated groups defined by Élek himself in [15];
- the singular ℓ^p-cohomology for any countable group defined as the ℓ^2-cohomology from Cheeger & Gromov [12];
- Pansu's asymptotic L^p-cohomology defined for any measured metric space (see [45]).

Note that Pansu's definition can also be used for graphs (in degree two, it should coincide with the definition given above using the rotational).

Here is a conjecture motivated by Osin [43, Problem 3.3] (do $\underline{\ell^2 H}^1(\Gamma) \neq 0$ and finite presentation imply acylindrically hyperbolic)

Conjecture 5.4 *Assume Γ is a torsion-free finitely presented group. If, for some $p \in]1, \infty[$, $\underline{\ell^p H}^1(\Gamma) \neq \{0\}$ then Γ contains a free subgroup (of rank 2).*

One could also strengthen the hypothesis to "finite $K(\Gamma, 1)$". Osin [42] showed that there are (non-amenable) groups without free subgroups (in fact, infinite torsion groups), whose reduced ℓ^2-cohomology in degree one is not trivial.

Note that these groups also show that groups whose ℓ^p-cohomology is not trivial can have a trivial Floyd boundary (a natural question coming from Puls [54]). Indeed, Karlsson [33] showed that groups with a non-trivial Floyd boundary contain free subgroups.

The next step for a positive answer to question 5.2 would be:

Question 5.5 *If G is a finitely generated solvable group, does $\underline{\ell^p H}^1(G) = \{0\}$ for any $1 < p < \infty$?*

Already the metabelian (derived length 2) case is not clear. In fact the special case "locally nilpotent not finitely generated"-by-Abelian would probably suffice to answer the question.

An interesting strengthening of Question 5.2 is

Question 5.6 *Can an amenable group have a Cayley graph with a non-constant harmonic function with gradient in c_0?*

The case of nilpotent group (more generally, groups with an infinite centre) is already treated in [25, Proposition 1.5 and Lemma 2.7].

Note that this is not the same thing as reduced c_0-cohomology in degree one. In fact, it is not too hard to see that reduced c_0-cohomology in degree one is always trivial, while reduced ℓ^∞-cohomology in degree one is never trivial.

As mentioned in Corollary 4.16, $\underline{\ell^p H}^1(\Gamma)$ can be a great way to see that some harmonic functions may not disappear after a quasi-isometry. However, because there is a small loss in the exponent in Theorem 4.1, the following remains open:

Question 5.7 *Are there two graphs Γ and Γ' which are quasi-isometric but so that $\mathscr{H}\mathsf{D}^p(\Gamma)$ contains only the constant function, while $\mathscr{H}\mathsf{D}^p(\Gamma')$ contains more than just these functions? In other words, is the triviality of $\mathscr{H}\mathsf{D}^p$ invariant under quasi-isometries?*

The same question could be asked with $\mathsf{B}\mathscr{H}\mathsf{D}^p$ (with the chance of a negative answer being higher).

In a similar vein, one could ask, in the spirit of a question of Pansu [46, Question 6 in §1.9], whether there is a harmonic function in each equivalence class of $\underline{\ell^p H}^1(G)$. The uniqueness up to a constant can be easily obtained: if h_1 and h_2 are two such functions, then $h_1 - h_2$ is harmonic and belongs to the trivial class; by Corollary 4.9, $h_1 - h_2$ is harmonic and constant at infinity, hence constant everywhere.

The referee pointed out the following interesting question, related to Corollary 4.16:

Question 5.8 *When (*i.e. *for which groups and which p) can the ℓ^p-cohomology be used to define a boundary of the random walk (*i.e. *a quotient of the Poisson boundary)?*

In the hyperbolic set-up, this question should admit a positive answer. Indeed, Bourdon & Pajot [7] showed that when p is larger than the conformal dimension of the boundary, functions in different cohomology classes can separate points on the boundary. It is to be expected that the Gromov boundary is a quotient of the p-harmonic boundary (see Puls [55])

Simple cases of groups which are not hyperbolic but have non-trivial [reduced] ℓ^p-cohomology are groups of the form $\mathbb{Z}^n * \mathbb{Z}^m$ with $n + m \geq 3$ (and $m, n > 0$). To see that the ℓ^p-cohomology is non-trivial for any $p \in [1, \infty[$, note that there are infinitely many ends and use the embedding of $\underline{\ell^p H}^1$ in $\underline{\ell^q H}^1$ for $p < q$; to see that it is not hyperbolic, use the fact it contains $\mathbb{Z}^2$ as a subgroup.

Let me conclude with a technical question:

Question 5.9 *If Γ is the Cayley graph of a group, can one relax the condition $p < \frac{d}{2}$ in Lemma 4.13 to $p < d$?*

Indeed, Proposition 4.5 shows that the second condition is sufficient. Note that the condition $p < \frac{d}{2}$ of Lemma 4.13 comes from the same estimates as those of Lemma 4.7 (which themselves do not really require $p < \frac{d}{2}$, see Remark 4.8).

On the one hand, the proof of Lemma 4.13 uses very crude estimates, so an improvement seems likely. On the other hand, one really looks for an estimate on the functions $\tau_n \in \mathbb{R}^E$ on the edges so that $\nabla^* \tau_n = \delta_x - P_n^x$. For $p = 2$, the only point where an improvement might occur is to avoid the triangle inequality (as

$\tau_n = \nabla(\sum_{i=0}^{n-1} P_x^i)$ is the function minimising the $\ell^2 E$-norm). For other p, there might be more room for improvement.

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