

Monochromatic Random Waves for General Riemannian Manifolds



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1 Introduction

This is a survey article on the recent developments on monochromatic random waves for general Riemannian manifolds obtained in [7–10, 30]. Let (M, g) be a compact, smooth, Riemannian manifold without boundary of dimension $n \geq 2$, and write Δ_g for the corresponding positive definite Laplace–Beltrami operator. Consider an orthonormal basis $\{\varphi_{\lambda_j}\}_{j=1}^{\infty}$ of $L^2(M, g)$ consisting of real-valued eigenfunctions

$$\Delta_g \varphi_{\lambda_j} = \lambda_j^2 \varphi_{\lambda_j},$$

with eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \nearrow \infty$, normalized so that $\|\varphi_{\lambda_j}\|_{L^2} = 1$.

Laplace eigenfunctions have been a common object of study for the mathematical physics community since they encode how heat and waves propagate across M . From a quantum mechanics point of view, $|\varphi_{\lambda_j}(x)|^2$ is the probability density for finding a quantum particle of energy λ_j^2 at the point x . It is therefore a natural problem to try to understand how φ_{λ_j} behaves. For example, one would like to understand how many minimums and maximums φ_{λ_j} has, as they are the most likely places for the quantum particles to be found at. See Figure 1. Similarly, understanding the geometry of the zero set of φ_{λ_j} would yield information on the structure of the least likely places for the quantum particles. See Figure 2.

However, it is often the case that studying such questions for φ_{λ_j} defined on a general manifold is quite hard, as eigenfunctions cannot be computed explicitly.

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N. Anantharaman et al. (eds.), *Frontiers in Analysis and Probability*,
https://doi.org/10.1007/978-3-030-56409-4_1



Fig. 1 Zeros and critical points for an eigenfunction on a torus. The function takes positive values on the shaded black areas, and negative values on the white areas. The maximums for the function are attained at the red points, while the minimums occur at the blue points. This picture was created by E. Vouga

Fig. 2 Nodal domains of a monochromatic random wave on the round sphere. Picture created by D. Beliaev



Even more so, for high energies, numerical methods cannot approximate the eigenfunctions in an efficient way that would keep track of features such as the structure of their zero set. It is then natural to randomize the problem and to study how the eigenfunctions behave on average.

It is then natural to work with *monochromatic random waves* of frequency λ . These are random fields on M defined by

$$\phi_\lambda := \frac{1}{\sqrt{\dim H_{\eta,\lambda}}} \sum_{\lambda_j \in [\lambda, \lambda + \eta(\lambda)]} a_j \varphi_{\lambda_j}, \quad (1)$$

where the coefficients $a_j \sim N(0, 1)$ are real valued, i.i.d, standard Gaussian random variables, $\eta = \eta_\lambda = \eta(\lambda)$ is a non-negative function satisfying $\eta(\lambda) = o(\lambda)$ as $\lambda \rightarrow \infty$, and

$$H_{\eta,\lambda} := \bigoplus_{\lambda_j \in [\lambda, \lambda + \eta_\lambda]} \ker(\Delta_g - \lambda_j^2 \text{Id}).$$

We write

$$\phi_\lambda \in \text{RW}_\lambda(M, g, \eta)$$

for short. The ensembles ϕ_λ are Gaussian models for eigenfunctions of the Laplacian with eigenvalue approximately equal to λ^2 on a compact Riemannian manifold (M, g) . In the setting of a general smooth manifold, the ensembles RW_λ were first defined by Zelditch in [36]. Zelditch was inspired in large part by the influential work of Berry [3], which proposes that random planar waves on Euclidean space and flat tori are good semiclassical models for high frequency eigenfunctions in quantum systems whose classical dynamics are chaotic. As we will see in Section 2, when properly scaled, the waves $\phi_\lambda \in \text{RW}_\lambda(M, g, \eta)$ behave like random planar waves. Random planar waves are Laplace eigenfunctions with eigenvalue 1, and since their frequency is fixed to be 1 they are said to be monochromatic. The fact that the scaled ϕ_λ behave like random planar waves as $\lambda \rightarrow \infty$ is the reason why waves in $\text{RW}_\lambda(M, g, \eta)$ are said to be monochromatic.

On round spheres and flat tori the Laplace eigenvalues occur with large multiplicity. Indeed, $\dim H_{0,\lambda}$ grows like λ^{n-1} when λ is an eigenvalue. Therefore, in these cases, one typically takes $\eta \equiv 0$ so that $\phi_\lambda \in \text{RW}_\lambda(M, g, 0)$ is an exact eigenfunction and $\lambda \in \{\lambda_j\}$. However, for a generic metric on any smooth compact manifold M , the eigenvalues λ_j^2 are simple. It is then natural to take η so that $\dim H_{\eta,\lambda}$ has the same rate of growth in powers of λ as the dimension of the eigenspaces for a round sphere. In particular, it is known [7] that if (M, g) has at least one non self-focal point (that is, there exists $x \in M$ so that $|\mathcal{L}_{x,x}| = 0$, see (11)), then for every $c > 0$ there exists $C > 0$ such that $\dim H_{c,\lambda}$ grows like $C\lambda^{n-1}$ as $\lambda \rightarrow \infty$. Since the existence of a non self-focal point is a very weak condition, it is customary to work with random waves in $\text{RW}_\lambda(M, g, c)$ for some $c > 0$.

This survey article focuses on the results of [7–10, 30]. The results in [7, 9] were the first ones to allow for the treatment of monochromatic random waves to take place on general manifolds by establishing that, when properly rescaled, the waves have a universal behavior. Prior to these results, monochromatic random waves had only been studied for the torus or the sphere. The article [8] is the first one in the literature pertaining statistics of the size of zero set and of the numbers of critical points for monochromatic random waves on general Riemannian manifolds. The results in [10, 30] deal with the study of the diffeomorphism types of the components of the zero sets of the monochromatic random waves, and of the nesting configurations of the components. These results build on the ground breaking work of Nazarov–Sodin [31].

In this article we discuss the following aspects of $\phi_\lambda \in \text{RW}_\lambda(M, g, \eta)$.

- **Section 2:** Universal behavior of ϕ_λ .
- **Section 3:** Number of critical points and size of the zero set of ϕ_λ .
- **Section 4:** Structure of the zero set of ϕ_λ .

The literature about random waves is extensive and rapidly evolving. This survey by no means attempts to give an overall account of every known result. There are numerous works directly related to the topics of this survey, including [4–6, 11, 14–16, 19–22, 25–29, 31, 33, 34].

2 Universal Behavior of ϕ_λ

By the Kolmogorov Consistency Theorem, the law of $\phi_\lambda \in \text{RW}_\lambda(M, g, \eta)$, which is a centered smooth Gaussian field, is completely characterized by its covariance kernel

$$\Pi_{\eta, \lambda}(x, y) := \text{Cov}(\phi_\lambda(x), \phi_\lambda(y)) = \frac{1}{\dim H_{\eta, \lambda}} \sum_{\lambda_j \in [\lambda, \lambda + \eta\lambda]} \varphi_{\lambda_j}(x) \varphi_{\lambda_j}(y),$$

where $x, y \in M$. The function $\Pi_{\eta, \lambda}(x, y)$ is the Schwartz kernel for the orthogonal projection operator $\Pi_{\eta, \lambda} : L^2(M, g) \rightarrow H_{\eta, \lambda}$, normalized to have unit trace. The study of local quantities, such as the size of the zero set of ϕ_λ , or the number of critical points of ϕ_λ , hinges on understanding the statistics of ϕ_λ , as $\lambda \rightarrow \infty$, restricted to “wavelength balls” of radius $\approx \lambda^{-1}$ around a fixed point $x \in M$. After rescaling by $1/\lambda$, the function ϕ_λ has frequency approximately equal to 1 on such balls in the sense that it solves the approximate local eigenvalue equation

$$\Delta_{T_x M} \phi_\lambda(x + \frac{u}{\lambda}) \approx \phi_\lambda(x + \frac{u}{\lambda}), \quad (2)$$

where $\Delta_{T_x M}$ denotes the *flat* Laplacian on the tangent space at x , $T_x M$. One could therefore expect, after the scaling, for the Gaussian random wave $u \mapsto \phi_\lambda(x + \frac{u}{\lambda})$ to behave like a Gaussian random wave ϕ_∞ on $\mathbb{R}^n \cong T_x M$ satisfying

$$\Delta_{\mathbb{R}^n} \phi_\infty = \phi_\infty.$$

The latter is called a random planar wave, and we discuss them in Section 2.1. Moreover, we shall see in Section 2.2, that for a generic Riemannian metric on M , the rescaled covariance kernel $\Pi_{\eta, \lambda}$ of $\phi_\lambda \in \text{RW}_\lambda(M, g, \eta)$ converges in the C^∞ topology to that of a random planar wave ϕ_∞ on $\mathbb{R}^n \cong T_x M$.

2.1 Random Planar Waves

Let $\sigma_{S^{n-1}}$ be the Haar measure on the round sphere S^{n-1} , normalized so that $\sigma_{S^{n-1}}(S^{n-1}) = 1$. Using that the transformation $\xi \mapsto -\xi$ preserves S^{n-1} , choose a real-valued orthonormal basis $\{\psi_j\}_{j=1}^\infty$ of $L^2(S^{n-1}, \sigma_{S^{n-1}})$ satisfying

$$\psi_j(-\xi) = (-1)^{\epsilon_j} \psi_j(\xi), \quad \epsilon_j \in \{0, 1\}. \quad (3)$$

A *random planar wave* is defined to be the random real-valued function ϕ_∞ on \mathbb{R}^n given by

$$\phi_\infty(u) = \sum_{j=1}^{\infty} b_j i^{\eta_j} \widehat{\psi}_j(u), \quad (4)$$

where

$$\widehat{\psi}_j(u) = \int_{\mathbb{R}^n} \psi_j(\xi) e^{-i\langle u, \xi \rangle} d\sigma_{S^{n-1}}(\xi), \quad (5)$$

and the b_j 's are i.i.d, real valued, standard Gaussian random variables. We write

$$\phi_\infty \in \text{RW}_1(\mathbb{R}^n, g_{\mathbb{R}^n}),$$

for short, where $g_{\mathbb{R}^n}$ is the Euclidean metric. We note that the fields in $\text{RW}_1(\mathbb{R}^n, g_{\mathbb{R}^n})$ do not depend on the choice of the orthonormal basis $\{\psi_j\}$. In addition, since the Euclidean Laplacian is $\Delta_{g_{\mathbb{R}^n}} = -\sum_{k=1}^n \partial_{u_k}^2$, and $\Delta_{g_{\mathbb{R}^n}} e^{-i\langle u, \xi \rangle} = e^{-i\langle u, \xi \rangle} \sum_{k=1}^n \xi_k^2$, it is immediate that

$$\Delta_{g_{\mathbb{R}^n}} \phi_\infty = \phi_\infty.$$

As explained in the introduction, random planar waves are often called monochromatic random waves because their frequency (the square root of their eigenvalue) is *equal to 1*.

Next, note that the distributional identity $\sum_{j=1}^{\infty} \phi_j(\xi) \phi_j(\eta) = \delta(\xi - \eta)$ on S^{n-1} together with (3) lead to the explicit expression for the covariance function:

$$\Pi_\infty(u, v) := \text{Cov}(\phi_\infty(u), \phi_\infty(v)) = \int_{\mathbb{R}^n} e^{i\langle u-v, \xi \rangle} d\sigma_{S^{n-1}}(\xi), \quad (6)$$

where $u, v \in \mathbb{R}^n$. From (4) it follows that almost all ϕ_∞ 's are analytic in u [1]. It is also known that

$$\Pi_\infty(u, v) = \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{J_\nu(|u-v|)}{|u-v|^\nu}, \quad (7)$$

where J_ν is the Bessel function of index $\nu := \frac{n-2}{2}$.

There is a natural choice of a basis for $L^2(S^{n-1}, d\sigma_{S^{n-1}})$ given by spherical harmonics. Let $\{Y_m^\ell\}_{m=1}^{d_{\ell,n}}$ be a real-valued basis for the space of spherical harmonics $\mathcal{E}_\ell(S^{n-1})$ of eigenvalue $\ell(\ell + n - 2)$, where $d_{\ell,n} = \dim \mathcal{E}_\ell(S^{n-1})$. In [10, Corollary 2.2] we prove that the monochromatic Gaussian ensembles ϕ_∞ 's take the form

$$\phi_\infty(u) = (2\pi)^{\frac{n}{2}} \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_{\ell,n}} b_{\ell,m} Y_m^\ell \left(\frac{u}{|u|} \right) \frac{J_{\ell+v}(|u|)}{|u|^\nu},$$

where the $b_{\ell,m}$'s are i.i.d standard Gaussian variables.

2.2 Points of Isotropic Scaling

The discussion around (2) shows that it is natural to study ϕ_λ by fixing $x \in M$ and considering the rescaled pullback of ϕ_λ to the tangent space $T_x M$. We denote this pullback by

$$\phi_\lambda^x(u) := \phi_\lambda \left(\exp_x \left(\frac{u}{\lambda} \right) \right), \quad (8)$$

where $\exp_x : T_x M \rightarrow M$ is the exponential map. The dilated functions ϕ_λ^x are centered Gaussian fields on $T_x M$, and we denote their scaled covariance kernel by

$$\Pi_{\eta,\lambda}^x(u, v) := \text{Cov}(\phi_\lambda^x(u), \phi_\lambda^x(v)) = \Pi_{\eta,\lambda} \left(\exp_x \left(\frac{u}{\lambda} \right), \exp_x \left(\frac{v}{\lambda} \right) \right).$$

When x is a point of isotropic scaling (see Definition 1 below), we shall see that the kernels $\Pi_{\eta,\lambda}^x$ converge to the covariance kernel of a random planar wave

$$\phi_\infty^x \in \text{RW}_1(T_x M, g_x).$$

Here, g_x denotes the constant coefficient metric obtained by freezing g at x . By the Kolmogorov Extension Theorem, together with (6), the random wave ϕ_∞^x is completely characterized by its two point correlation function kernel

$$\Pi_\infty^x(u, v) = (2\pi)^{\frac{n}{2}} \frac{J_{\frac{n-2}{2}}(\|u-v\|_{g_x})}{\|u-v\|_{g_x}^{\frac{n-2}{2}}} = \int_{S_x M} e^{i\langle u-v, \xi \rangle_{g_x}} d\sigma_{S_x M}(\xi). \quad (9)$$

Here J_ν denotes a Bessel function of the first kind with index ν , $S_x M$ is the unit sphere in $T_x M$ with respect to g_x , and $d\sigma_{S_x M}$ is the hypersurface measure on $S_x M$.

Definition 1 A point $x \in M$ is a *point of isotropic scaling*, denoted $x \in \mathcal{IS}(M, g, \eta)$, if for every non-negative function r_λ satisfying $r_\lambda = o(\lambda)$ as $\lambda \rightarrow \infty$, and all $\alpha, \beta \in \mathbb{N}^n$, we have

$$\sup_{u, v \in B_{r_\lambda}} \left| \partial_u^\alpha \partial_v^\beta \left[\Pi_{\eta,\lambda}^x(u, v) - \Pi_\infty^x(u, v) \right] \right| = o_{\alpha,\beta}(1) \quad (10)$$

as $\lambda \rightarrow \infty$, where the rate of convergence depends on α, β and B_r denotes a ball of radius r centered at $0 \in T_x M$. We also say that M is a *manifold of isotropic scaling* if

$$M = \mathcal{IS}(M, g, \eta)$$

and if the convergence in (10) is uniform over $x \in M$ for each $\alpha, \beta \in \mathbb{N}^n$.

Verifying that $x \in M$ belongs to $\mathcal{IS}(M, g, \eta)$ is difficult to do directly, except on simple examples such as the flat torus. We briefly recall several settings in which $\mathcal{IS}(M, g, \eta)$ is known to be large.

- Let S^n be the n -sphere equipped with the round metric g_{S^n} . The Mehler–Heine asymptotics [24] imply that

$$\mathcal{IS}(S^n, g_{S^n}, 0) = S^n,$$

when the limit in (10) is taken along the sequence of eigenvalues $\lambda_j \rightarrow \infty$ for the sphere. In this case, the ϕ_λ 's are known as random spherical harmonics.

- Let \mathbb{T}^n be the n -dimensional torus equipped with the flat metric $g_{\mathbb{T}^n}$. When $n \geq 5$ we have that $\mathcal{IS}(\mathbb{T}^n, g_{\mathbb{T}^n}, 0) = \mathbb{T}^n$. For $2 \leq n \leq 4$, the asymptotics (10) hold at every $x \in \mathbb{T}^n$ but only for a density one subsequence of eigenvalues [13]. In this case, the ϕ_λ 's are known as random trigonometric polynomials.
- The pointwise Weyl law [17] implies that if $\lim_{\lambda \rightarrow \infty} \eta_\lambda = \infty$, then $\mathcal{IS}(M, g, \eta) = M$.

In addition, it is very likely that if (M, g) has no conjugate points, then the condition

$$\lim_{\lambda \rightarrow \infty} \log(\lambda) \cdot \eta_\lambda = \infty$$

implies $\mathcal{IS}(M, g, \eta) = M$. This was proved by B. Keeler in [18], but with the convergence in (10) only holding for $\alpha = \beta = 0$. Note that if (M, g) has negative sectional curvature everywhere, then it has no conjugate points and all points are non self-focal. In contrast, there exist smooth perturbations of the round metric on S^2 for which $\mathcal{IS}(S^2, g, 1) \subsetneq S^2$ (see [23, 35]).

For $x, y \in M$ let

$$\mathcal{L}_{x,y} = \{\xi \in S_x M : \exists t > 0 \text{ s.t. } \exp_x(t\xi) = y\} \tag{11}$$

be the set of directions that generate geodesic arcs from x to y . The set $\mathcal{L}_{x,y}$ is contained in $S_x M$ and $S_x M$ is endowed with the Liouville measure. The corresponding volume of $\mathcal{L}_{x,y}$ is denoted by $|\mathcal{L}_{x,y}|$.

The main result of this section is the following, and it was proved in [7, 9].

Theorem 1 *Let (M, g) be a compact, smooth, Riemannian manifold, with no boundary. Let η be a non-negative function with $\liminf_{\lambda \rightarrow \infty} \eta_\lambda > 0$. Let $x \in M$ be so that $|\mathcal{L}_{x,x}| = 0$. Then,*

$$x \in \mathcal{IS}(M, g, \eta). \quad (12)$$

By [32, Lem 6.1], the condition that $|\mathcal{L}_{x,x}| = 0$ for all $x \in M$ is generic in the space of Riemannian metrics on a fixed compact smooth manifold M .

Definition 1 gives that if $x \in \mathcal{IS}(M, g, \eta)$, then the scaling limit of waves in $\text{RW}_\lambda(M, g, \eta)$ around x is universal in the sense that it depends only on the dimension of M . In the language of Nazarov–Sodin [31] the asymptotics (10) imply that if $M = \mathcal{IS}(M, g, \eta)$, then the ensembles $\text{RW}_\lambda(M, g, \eta)$ have translation invariant local limits.

3 Number of Critical Points and Size of the Zero Set

Define the measures of integration over the zero set $\{\phi_\lambda = 0\}$ and the set of critical points $\{d\phi_\lambda = 0\}$ by

$$Z_\lambda(\psi) := \int_{\phi_\lambda^{-1}(0)} \psi(x) d\sigma_{Z_\lambda}(x) \quad \text{and} \quad \text{Crit}_\lambda(\psi) := \sum_{d\phi_\lambda(x)=0} \psi(x),$$

where $\psi : M \rightarrow \mathbb{R}$ and σ_{Z_λ} is the $(n-1)$ -dimensional Hausdorff measure over $\{\phi_\lambda = 0\}$. This section is divided into two parts. In Section 3.1 we give asymptotics for $\mathbb{E}[Z_\lambda]$ and $\mathbb{E}[\text{Crit}_\lambda]$, and bounds for their variances. The results in Section 3.1 rely heavily on a careful analysis of what happens for the scaled random waves ϕ_λ^x . The results for the localized waves are discussed in Section 3.2.

Previous results on the Hausdorff measure of the zero sets focus primarily on exactly solvable examples. On round spheres, for instance, Bérard [2] proved (14) (example (1) on p.3). Later, in the same setting, Neuheisel [25] and Wigman [33] obtained upper bounds for the variance that are of polynomial order in λ . Further, on S^2 , Wigman [34] found that the variance actually grows like $\lambda^{-2} \log \lambda$ as $\lambda \rightarrow \infty$. On flat tori \mathbb{T}^n (for exact eigenfunctions) Rudnick and Wigman [29] computed the expected value of the total Hausdorff measure of the zero set and gave an upper bound of the form $\lambda^2 (\dim(H_{0,\lambda}))^{-1/2}$ on its variance. Subsequently, on \mathbb{T}^2 , Krishnapur, Kurlberg, and Wigman [19] found that the variance is asymptotic to a constant, while Marinucci, Peccati, Rossi, and Wigman proved that the size of the zero set converges to a limiting distribution that is not Gaussian and depends on the angular distribution of lattice points on circles [22].

The behavior of the number of critical points has been studied in detail on S^2 . Nicolaescu [26] studied the expected value of the number of critical points, obtaining (15). The variance was studied by Cammarota, Marinucci, and Wigman [6].

They obtain a polynomial upper bound. This upper bound was later improved by Cammarota and Wigman [5] who proved that the variance grows like $\lambda^2 \log \lambda$ (as opposed to our $\lambda^{7/2}$ estimate) as $\lambda \rightarrow \infty$. Finally, for smooth domain in \mathbb{R}^2 , Nourdin–Peccati–Rossi [27] prove that both for real and complex random waves, the Hausdorff measure of the nodal set is asymptotically normal in the high frequency limit.

3.1 Global Statistics

The main result in this section gives asymptotics for the expected value, and estimates for the variance, of the linear statistics of $Z_\lambda, \text{Crit}_\lambda$ that are valid for generic Riemannian metrics on M . For the estimates about the means of $Z_\lambda(\psi), \text{Crit}_\lambda(\psi)$ one needs to ask that (M, g) be a manifold of isotropic scaling (see Definition 1). This is true for any manifold with negative curvature, or with no conjugate points. The variance estimates are more delicate, so one needs to ask in addition that the restrictions of ϕ_λ to small balls centered at different points become asymptotically uncorrelated. This is the following definition.

Definition 2 The random waves $\phi_\lambda \in \text{RW}_\lambda(M, g, \eta)$ are said to have *short-range correlations* if for each $\varepsilon > 0$ and every $\alpha, \beta \in \mathbb{N}$

$$\sup_{\{x, y: d_g(x, y) \geq \lambda^{-1+\varepsilon}\}} \left| \nabla_x^\alpha \nabla_y^\beta \Pi_{\eta, \lambda}(x, y) \right| = o_\varepsilon(\lambda^{\alpha+\beta}), \quad (13)$$

as $\lambda \rightarrow \infty$, where ∇_x, ∇_y are covariant derivatives.

This condition is again generic in the space of Riemannian metrics on (M, g) and is satisfied for example if for any pair of points $x, y \in M$ the measure of geodesic arcs joining them is zero. That is, if $|\mathcal{L}_{x, y}| = 0$ for all $x, y \in M$, then the random waves in $\text{RW}_\lambda(M, g, \eta)$ have short-range correlations.

The condition that $|\mathcal{L}_{x, y}| = 0$ for all $x, y \in M$ is known to happen on manifolds of negative curvature, or more generally, with no conjugate points (see [8, Section 1.5]). It is likely that a similar argument would show that $|\mathcal{L}_{x, y}| = 0$ for all $x, y \in M$ is also generic but have not checked the details. It is known, however, that $|\mathcal{L}_{x, y}| = 0$ holds for all $x, y \in M$ if (M, g) is negatively curved or, more generally, has no conjugate points.

We are ready to state the main theorem of this section. This result was proved in [8].

Theorem 2 *Let (M, g) be a smooth, compact, Riemannian manifold of dimension $n \geq 2$ with no boundary. Let $\eta = \eta(\lambda)$ be a non-negative function satisfying $\eta(\lambda) = o(\lambda)$ as $\lambda \rightarrow \infty$. Let $\phi_\lambda \in \text{RW}_\lambda(M, g, \eta)$ and suppose that M is a manifold of isotropic scaling (Definition 1). Then, for any bounded measurable function $\psi : M \rightarrow \mathbb{R}$,*

$$\lim_{\lambda \rightarrow \infty} \mathbb{E} \left[\lambda^{-1} Z_\lambda(\psi) \right] = \frac{1}{\sqrt{\pi n}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \int_M \psi(x) dv_g(x) \quad (14)$$

and

$$\lim_{\lambda \rightarrow \infty} \mathbb{E} \left[\lambda^{-n} \text{Crit}_\lambda(\psi) \right] = C_n \int_M \psi(x) dv_g(x), \quad (15)$$

where C_n is a positive constant that depends only on n . Suppose further that ϕ_λ has short-range correlations in the sense of (13). Then,

$$\text{Var} \left[\lambda^{-1} Z_\lambda(\psi) \right] = O\left(\lambda^{-\frac{n-1}{2}}\right) \quad (16)$$

and

$$\text{Var} \left[\lambda^{-n} \text{Crit}_\lambda(\psi) \right] = O\left(\lambda^{-\frac{n-1}{2}}\right), \quad (17)$$

as $\lambda \rightarrow \infty$.

Theorem 2 is the first result with a non-trivial variance estimate for the Hausdorff measure of the nodal set of random waves for a generic smooth Riemannian manifold (for real analytic (M, g) a weaker estimate was given in [36, Cor. 2]). A version of (14) was also stated, with a heuristic proof, in [36, Prop. 2.3] for both Zoll and aperiodic manifolds.

We also note that the test function ψ in Theorem 2 can be replaced by a function $\psi(x) = \psi(x, \phi_\lambda(x), D^2\phi_\lambda(x), \dots)$ depending on the jets of ϕ_λ provided $\psi : \mathbb{R}^n \times C^0(\mathbb{R}^n, \mathbb{R}^k) \rightarrow \mathbb{R}$ is bounded and continuous when $C^0(\mathbb{R}^n, \mathbb{R}^k)$ is equipped with the topology of uniform convergence on compact sets. Hence, for example, we could study the distribution of critical values by taking $\psi(u, \phi_\lambda) = \mathbf{1}_{\{\phi_\lambda^x \geq \alpha\}}(u)$, for $\alpha \in \mathbb{R}$.

In addition, the proof of Theorem 2 actually shows that (14) holds as soon as almost every point is a point of isotropic scaling. That is, it holds provided

$$\text{vol}_g(M \setminus \mathcal{IS}(M, g, \eta)) = 0,$$

(see Definition 1).

Furthermore, by the Borel–Cantelli Lemma, if $n \geq 4$ and ϕ_j are independent frequency $j \in \mathbb{N}$ random waves on (M, g) , then (16) shows that the total nodal set measure $j^{-1} Z_j(\psi) - \mathbb{E} [j^{-1} Z_j(\psi)]$ converges almost surely to 0.

Finally, when $n = 2$ we have $C_2 = \mathbb{E} [\text{Crit}_{\infty,1}] = \frac{1}{4\pi\sqrt{6}}$ where C_2 is the dimensional constant in (15), see (25).

Theorem 2 hinges on a careful study of the statistics of ϕ_λ when restricted to “wavelength balls” of radius $\approx \lambda^{-1}$ around a fixed point $x \in M$ of isotropic scaling. The results that describe the behavior of Z_λ or Crit_λ restricted to these

shrinking balls are described in Sections 3.2.1 and 3.2.2, respectively. The results are “glued” to obtain Theorem 2. Glueing variance estimates is a delicate matter. It is instrumental to the proof that the waves have short-range correlations.

3.2 Local Statistics

In this section we discuss the behavior of the zero sets and of the critical points for the scaled waves ϕ_λ^x . When x is a point of isotropic scaling the behavior of the scaled random wave ϕ_λ^x converges to that of the random planar wave $\phi_\infty^x \in \text{RW}_1(T_x M, g_x)$. One can therefore prove much stronger results on statistics for ϕ_λ^x than ϕ_λ .

3.2.1 Local Universality of Zeros

Consider the rescaled random wave ϕ_λ^x for $x \in \mathcal{IS}(M, g, \eta)$ and denote by Z_λ^x its Riemannian hypersurface (i.e. Hausdorff) measure:

$$Z_\lambda^x(A) := \sigma_{Z_\lambda^x}((\phi_\lambda^x)^{-1}(0) \cap A), \quad \forall A \subseteq T_x M \text{ measurable.}$$

The main result concerns the restriction of Z_λ^x to various balls B_r of radius r centered at $0 \in T_x M$. We set

$$Z_{\lambda,r}^x := \frac{\mathbf{1}_{B_r} \cdot Z_\lambda^x}{\text{vol}(B_r)} \quad \text{and} \quad Z_{\infty,r}^x := \frac{\mathbf{1}_{B_r} \cdot Z_\infty^x}{\text{vol}(B_r)}. \quad (18)$$

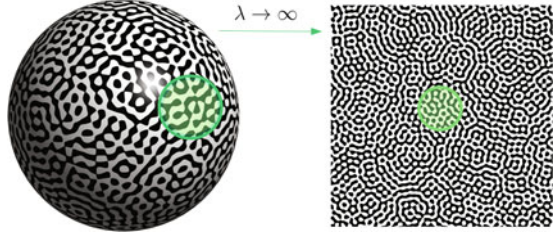
We have denoted by $\mathbf{1}_{B_r}$ the characteristic function of the ball B_r and by Z_∞^x the hypersurface measure on $(\phi_\infty^x)^{-1}(0)$ for $\phi_\infty^x \in \text{RW}_1(T_x M, g_x)$. Again, for various measures μ , we write $\mu(\psi)$ for integration of a measurable function ψ against μ . In particular,

$$Z_{\lambda,r}^x(1) = \frac{\mathcal{H}^{n-1}((\phi_\lambda^x)^{-1}(0) \cap B_r)}{\text{vol}(B_r)}.$$

The following result is proved in [8]. See Figure 3 for a depiction of the statement.

Theorem 3 (Weak Convergence of Zero Set Measures) *Let (M, g) be a smooth, compact, Riemannian manifold of dimension $n \geq 2$ with no boundary. Let $\eta = \eta(\lambda)$ be a non-negative function satisfying $\eta(\lambda) = o(\lambda)$ as $\lambda \rightarrow \infty$. Fix a non-negative function r_λ that satisfies $r_\lambda = o(\lambda)$ as $\lambda \rightarrow \infty$. Let $\phi_\lambda \in \text{RW}_\lambda(M, g, \eta)$ and $x \in \mathcal{IS}(M, g, \eta)$. Suppose $\lim_{\lambda \rightarrow \infty} r_\lambda$ exists and equals $r_\infty \in (0, \infty]$.*

Fig. 3 Depiction of the universal behavior displayed by monochromatic random waves. The zero set measure for the monochromatic random wave on the sphere (left) converges to the zero set measure for the random planar wave (right)



Case 1 ($r_\infty < \infty$): *The measures Z_{λ, r_λ}^x converge to Z_{∞, r_∞}^x weakly in distribution. That is, for any bounded, measurable function $\psi : T_x M \rightarrow \mathbb{R}$*

$$Z_{\lambda, r_\lambda}^x(\psi) \xrightarrow{d} Z_{\infty, r_\infty}^x(\psi) \quad (19)$$

as $\lambda \rightarrow \infty$, where \xrightarrow{d} denotes convergence in distribution.

Case 2 ($r_\infty = \infty$): *We have the following convergence in probability to a constant:*

$$Z_{\lambda, r_\lambda}^x(1) \xrightarrow{p} \frac{1}{\sqrt{\pi n}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}, \quad (20)$$

as $\lambda \rightarrow \infty$. In particular,

$$\lim_{\lambda \rightarrow \infty} \text{Var} [Z_{\lambda, r_\lambda}^x(1)] = 0. \quad (21)$$

The function ψ in (19) can be allowed to depend on the jets $D^j \phi_\lambda$, $j \geq 1$. More precisely, $\psi(u)$ can be replaced by $\psi(u, W(u))$, where W is a random field so that $u \mapsto (\phi_\lambda^x(u), W(u))$ is a continuous Gaussian field with values in \mathbb{R}^{1+k} and $\psi : \mathbb{R}^n \times C^0(\mathbb{R}^n, \mathbb{R}^k) \rightarrow \mathbb{R}$ is bounded and continuous when $C^0(\mathbb{R}^n, \mathbb{R}^k)$ is equipped with the topology of uniform convergence on compact sets. Since $(\phi_\lambda^x(u), D\phi_\lambda^x(u), D^2\phi_\lambda^x(u), \dots)$ is a smooth Gaussian field, we may take $W(u) = (D^j \phi_\lambda(u), j \geq 1)$. Similarly, in (20) and (21), the function $1 = 1(u)$ can be replaced by $\psi(W(u))$ where again $\psi : C^0(\mathbb{R}^n, \mathbb{R}^k) \rightarrow \mathbb{R}$ is bounded and continuous in the topology of uniform convergence on compact sets. The only difference is that (20) then reads

$$Z_{\lambda, r_\lambda}^x(\psi) - \mathbb{E} [Z_{\infty, r_\lambda}^x(\psi)] \xrightarrow{p} 0.$$

The relations (20) and (21) hold even if the balls B_{r_λ} in the definition of Z_{λ, r_λ}^x are replaced by any λ -dependent sets A_{λ, r_λ} for which the diameter is bounded above and below by constant times r_λ , and whose volume tends to infinity when $r_\lambda \rightarrow \infty$.

The rates of convergence in (19)–(21) - even after the generalizations indicated above- are uniform as x varies over a compact set $S \subset \mathcal{IS}(M, g, \eta)$ as long as the convergence in (10) is uniform over S .

3.2.2 Local Universality of Critical Points

Let $x \in M$ and for each $r > 0$ define the normalized counting measure

$$\text{Crit}_{\lambda,r}^x := \frac{1}{\text{vol}(B_r)} \sum_{\substack{d\phi_\lambda^x(u)=0 \\ u \in B_r}} \delta_u \quad (22)$$

of critical points in a ball of radius r . We define $\text{Crit}_{\infty,r}^x$ in the same way as $\text{Crit}_{\lambda,r}^x$ but with ϕ_λ^x replaced by $\phi_\infty^x \in \text{RW}_1(T_x M, g_x)$, and continue to write $\mu(\psi)$ for the pairing of a measure μ with a function ψ . For example,

$$\text{Crit}_{\lambda,r}^x(1) = \frac{\#\{u \in B_r : d\phi_\lambda^x(u) = 0\}}{\text{vol}(B_r)}.$$

Theorem 4 *Let (M, g) be a smooth, compact, Riemannian manifold of dimension $n \geq 2$ with no boundary. Let $\eta = \eta(\lambda)$ be a non-negative function satisfying $\eta(\lambda) = o(\lambda)$ as $\lambda \rightarrow \infty$. Fix a non-negative function r_λ that satisfies $r_\lambda = o(\lambda)$ as $\lambda \rightarrow \infty$. Let $\phi_\lambda \in \text{RW}_\lambda(M, g, \eta)$ and $x \in \mathcal{IS}(M, g, \eta)$. Suppose that $\lim_{\lambda \rightarrow \infty} r_\lambda$ exists and equals $r_\infty \in (0, \infty]$.*

Case 1. ($r_\infty < \infty$): *For $k = 1, 2$ and each bounded measurable function $\psi : T_x M \rightarrow \mathbb{R}$*

$$\lim_{\lambda \rightarrow \infty} \mathbb{E} \left[\text{Crit}_{\lambda,r_\lambda}^x(\psi)^k \right] = \mathbb{E} \left[\text{Crit}_{\infty,r_\infty}^x(\psi)^k \right]. \quad (23)$$

Case 2. ($r_\infty = \infty$): *We have*

$$\lim_{\lambda \rightarrow \infty} \text{Var}[\text{Crit}_{\lambda,r_\lambda}^x(1)] = \mathbb{E}[\text{Crit}_{\infty,1}^x(1)]. \quad (24)$$

This limit is the expected number of critical points in a ball of radius 1 for frequency 1 random waves on \mathbb{R}^n , which is independent of x .

The moments $\mathbb{E}[(\text{Crit}_{\infty,r_\infty}^x(\psi))^k]$ are finite for $k = 1, 2$. In particular, if $\dim(M) = 2$, then for every $x \in M$

$$\mathbb{E}[\text{Crit}_{\infty,1}^x(1)] = \frac{1}{4\pi\sqrt{6}}. \quad (25)$$

The balls B_{r_λ} in (24) can be replaced by any λ -dependent sets A_{λ, r_λ} for which the diameter is bounded above and below by a constant times r_λ and whose volume tends to infinity with r_λ .

Both ψ in (23) and the function 1 being integrated against $\text{Crit}_{\lambda, r_\lambda}^x$ in (24) can be replaced by a bounded continuous function of the jets of ϕ_λ , giving information for instance about critical points filtered by critical value.

Also, the rates of convergence in (23) and (24) are uniform over $x \in S \subset \mathcal{IS}(M, g, \eta)$ if (10) is uniform over S .

On the n -dimensional flat torus, Nicolaescu [26] obtained several results related to Theorem 4 in the $r_\infty < \infty$ case.

4 Structure of the Zero Set

Let (M, g) be a Riemannian manifold, and let ϕ be an eigenfunction for the Laplace operator. The zero set $\phi^{-1}(0) = \{x \in M : \phi(x) = 0\}$ decomposes into a collection of connected components which we denote by $\mathcal{C}(\phi)$. See Figure 4. Our interest is in the diffeomorphism types of the components in $\mathcal{C}(\phi)$. For generic ϕ the components of $\mathcal{C}(\phi)$ are smooth $(n - 1)$ -dimensional manifolds. The connected components of $M \setminus \phi^{-1}(0)$ are the nodal domains of ϕ and our interest is in their nesting properties, again for generic ϕ .

The results presented in this section build on the ground breaking work of Nazarov–Sodin [31]. They studied the number of nodal domains for monochromatic random waves on manifolds with isotropic scaling. They proved that there exists a positive constant C so that the mean number of nodal domains for ϕ_λ grows like $C\lambda^n$. The approach of [10, 30] to study the diffeomorphism types of the zero set components is very similar in spirit to the work [31] as the rationale is that one is counting components of the zero set with a given diffeomorphism type. A similar argument is carried to deal with the nesting configurations.

The argument developed by Nazarov–Sodin hinges on the fact that most zero set components lie within a ball of radius R/λ for $R > 0$ large enough. One can therefore count the number of components of ϕ_λ^x within the ball $B(0, R) \subset T_x M$. The latter is done using the universal behavior of ϕ_λ^x guaranteed by the fact that M is a manifold of isotropic scaling.

The works of Gayet–Welshinger [14–16] are also very related to the results described in this section, only that they are not applicable to monochromatic random waves.

4.1 Diffeomorphism Types

Let \mathcal{D}_{n-1} denote the (countable and discrete) set of diffeomorphism classes of compact connected smooth $(n - 1)$ -dimensional manifolds that can be embedded

in \mathbb{R}^n . The compact components c in $\mathcal{C}(\phi)$ give rise to elements $D(c)$ in \mathcal{D}_{n-1} (here we are assuming that ϕ is generic with respect to a Gaussian measure so that $\phi^{-1}(0)$ is smooth).

Let $\phi_\lambda \in \text{RW}(M, g, \eta)$. The diffeomorphism types exhibited by the components of $\phi_\lambda^{-1}(0)$ are described by the probability measure $\mu_{\mathcal{D}(\phi_\lambda)}$ on \mathcal{D}_{n-1} given by

$$\mu_{\mathcal{D}(\phi_\lambda)} := \frac{1}{|\mathcal{C}(\phi_\lambda)|} \sum_{c \in \mathcal{C}(\phi_\lambda)} \delta_{D(c)},$$

where δ_D is a point mass at $D \in \mathcal{D}_{n-1}$. The following is part of the main theorem in [30, Theorem 1.1].

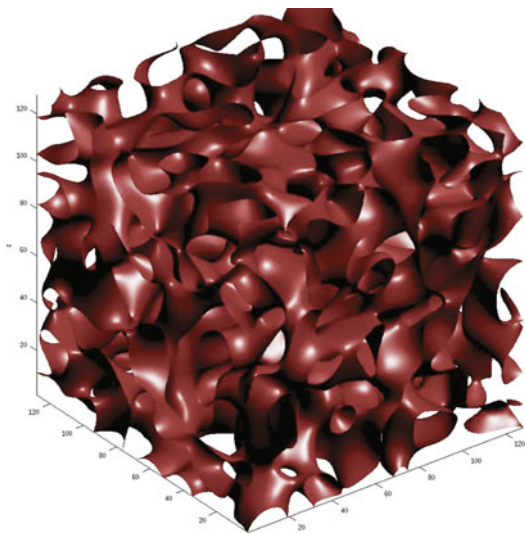
Theorem 5 *There exists a probability measure $\mu_{\mathcal{D}}$ supported on \mathcal{D}_{n-1} such that the following holds. Let (M, g) be a smooth, compact, Riemannian manifold of dimension $n \geq 2$ with no boundary. Let $\eta = \eta(\lambda)$ be a non-negative function satisfying $\eta(\lambda) = o(\lambda)$ as $\lambda \rightarrow \infty$. Suppose that M is a manifold of isotropic scaling. Then, for any given $D \in \mathcal{D}_{n-1}$ and $\varepsilon > 0$,*

$$\lim_{\lambda \rightarrow \infty} \mathbb{P}\left(\phi_\lambda \in \text{RW}_\lambda(M, g, \eta) : |\mu_{\mathcal{D}(\phi_\lambda)}(D) - \mu_{\mathcal{D}}(D)| > \varepsilon\right) = 0.$$

The theorem asserts that there exists a probability measure $\mu_{\mathcal{D}}$ on \mathcal{D}_{n-1} to which $\mu_{\mathcal{D}(\phi)}$ approaches as $\lambda \rightarrow \infty$, for almost all ϕ . The probability measure $\mu_{\mathcal{D}}$ is universal in that it only depends on the dimension n of M .

For $n \geq 4$, little is known about the space \mathcal{D}_{n-1} . In particular, there is no classification for the diffeomorphism types of $(n - 1)$ -dimensional smooth

Fig. 4 Zero set of a random planar wave in \mathbb{R}^3 . Picture created by A. Barnett



manifolds. This makes it difficult to study the support of $\mu_{\mathcal{D}}$. Remarkably, it is possible to prove that the support of $\mu_{\mathcal{D}}$ is all of \mathcal{D}_{n-1} . This result is proved in [10].

Theorem 6 *Every atom of \mathcal{D}_{n-1} is positively charged by $\mu_{\mathcal{D}}$. That is,*

$$\text{supp}(\mu_{\mathcal{D}}) = \mathcal{D}_{n-1}.$$

Theorem 6 asserts that every diffeomorphism type that can occur will do so with a positive probability for the universal distribution of topological types of random monochromatic waves in [30].

The proof of Theorem 6 relies on the fact that for a manifold of isotropic scaling the statistics of ϕ_{λ}^x converge to those of ϕ_{∞}^x for every $x \in M$. Indeed, the proof reduces to establishing the following result.

Theorem 7 *Given $D \in \mathcal{D}_{n-1}$ there exists $\phi \in \ker(\Delta_{\mathbb{R}^n} - Id)$ and $c \in \mathcal{C}(\phi)$ for which $D(c) = D$.*

Theorem 7 is of basic interest in the understanding of the possible shapes of nodal sets and domains of eigenfunctions in \mathbb{R}^n (it applies equally well to any eigenfunction with eigenvalue $\lambda^2 > 0$ instead of 1). To prove Theorem 7 one applies Whitney’s approximation Theorem to realize c as an embedded real analytic submanifold of \mathbb{R}^n . Then, following some techniques in [12] one can find suitable approximations of $\phi \in \ker(\Delta_{g_{\mathbb{R}^n}} - 1)$ and whose zero set contains a diffeomorphic copy of c . The construction of ϕ hinges on the Lax–Malgrange Theorem and Thom’s Isotopy Theorem.

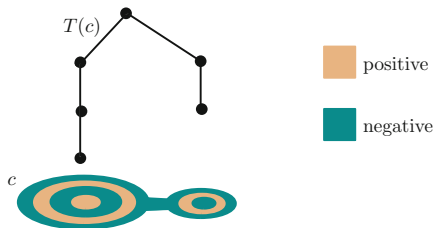
The reduction from Theorem 7 to Theorem 6 is abstract and is based on the “soft” techniques in [30, 31]. In particular, it offers us no lower bounds for these probabilities. Developing such lower bounds is an interesting problem.

4.2 Nesting Configurations

Let ϕ be a Laplace eigenfunction for a Riemannian manifold (M, g) . The connected components of $M \setminus \phi^{-1}(0)$ are the nodal domains of ϕ and our interest is in their nesting properties, again for generic ϕ . Let \mathcal{U} be a coordinate patch for M . The components of $\mathcal{C}(\phi)$ that are contained in \mathcal{U} are denoted by $\mathcal{C}_{\mathcal{U}}(\phi)$. To each compact $c \in \mathcal{C}_{\mathcal{U}}(\phi)$ we associate a finite connected rooted tree as follows. By the Jordan–Brouwer separation Theorem each component $c \in \mathcal{C}(\phi)$ has an exterior and interior. We choose the interior to be the end that is contained within \mathcal{U} . The nodal domains of ϕ , which are in the interior of c , are taken to be the vertices of a graph. Two vertices share an edge if the respective nodal domains have a common boundary component (unique if there is one). This gives a finite connected rooted tree denoted $T(c)$; the root being the domain adjacent to c (see Figure 5).

The reason for working in a coordinate patch \mathcal{U} for M is that for general (M, g) there is no global way to define a tree that describes the nesting configuration of the

Fig. 5 This picture shows a nodal domain configuration, where positive nodal domains are depicted in orange and negative nodal domains are green. The corresponding rooted tree is shown



zero set in all of M , for all $c \in C(\phi)$. The reason is that a zero set component may not divide M into two different regions. It is important to note that in a coordinate patch this is always the case. However, according to [31] almost all c 's localize to small coordinate patches. This inconvenience is the reason why [10] is written for $M = S^n$ the round sphere. By the Jordan–Brouwer separation Theorem, on S^n every component of the zero set separates S^n into two distinct components. This gives that the nesting graph for the zero sets is a rooted tree well defined without the need for a coordinate patch.

Let \mathcal{T} be the collection (countable and discrete) of finite connected rooted trees. The distribution of nested ends of nodal domains of ϕ that lie within \mathcal{U} is described by the measure $\mu_{\mathcal{T}(\phi), \mathcal{U}}$ on \mathcal{T} given by

$$\mu_{\mathcal{T}(\phi), \mathcal{U}} := \frac{1}{|\mathcal{C}_{\mathcal{U}}(\phi)|} \sum_{c \in \mathcal{C}_{\mathcal{U}}(\phi)} \delta_{T(\phi)},$$

where δ_T is the point mass at $T \in \mathcal{T}$.

The following is part of the main theorem in [30, Theorem 1.1]. Also, see [10, Remark 2].

Theorem 8 *There exists a probability measure $\mu_{\mathcal{T}}$ supported on \mathcal{T} such that the following holds. Let (M, g) be a smooth, compact, Riemannian manifold of dimension $n \geq 2$ with no boundary. Let $\eta = \eta(\lambda)$ be a non-negative function satisfying $\eta(\lambda) = o(\lambda)$ as $\lambda \rightarrow \infty$. Suppose that M is a manifold of isotropic scaling and let \mathcal{U} be a coordinate patch for M . Then, for any given $T \in \mathcal{T}$ and $\varepsilon > 0$,*

$$\lim_{\lambda \rightarrow \infty} \mathbb{P}(\phi_{\lambda} \in \text{RW}_{\lambda}(M, g, \eta) : |\mu_{\mathcal{T}(\phi), \mathcal{U}}(T) - \mu_{\mathcal{T}}(T)| > \varepsilon) = 0.$$

Theorem in [30] asserts that there exists a probability measure $\mu_{\mathcal{T}}$ on \mathcal{T} to which $\mu_{\mathcal{T}(\phi), \mathcal{U}}$ approaches as $\lambda \rightarrow \infty$, for almost all ϕ provided M is a manifold of isotropic scaling.

The probability measure $\mu_{\mathcal{T}}$ is universal in that it only depends on the dimension n of M . The following result is part of theorem in [10] and deals with the support of $\mu_{\mathcal{T}}$.

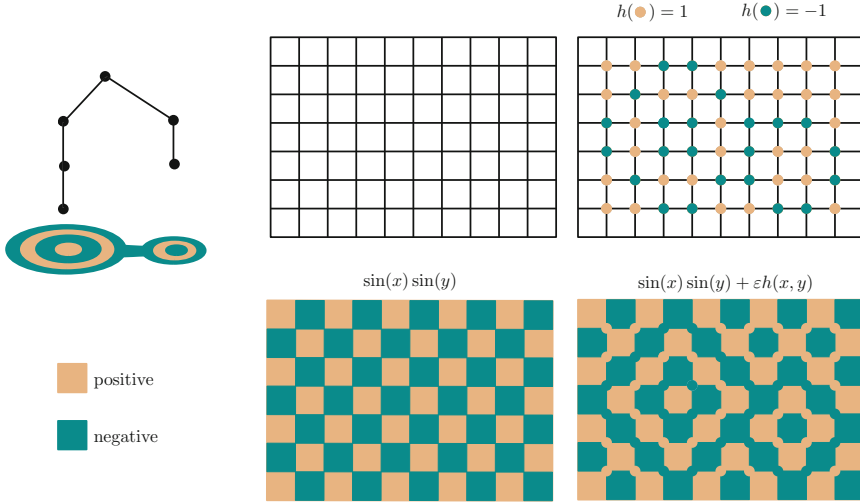


Fig. 6 This picture shows how to perturb the zero set of $\psi(x, y) = \sin(x) \sin(y)$ by adding $h \in \ker(\Delta_{\mathbb{R}^n} - I)$ that we prescribe on the singularities of ψ that lie in its zero set so that the zero set of $\phi = \psi + \varepsilon h$, for $\varepsilon > 0$ small, has the correct nesting configuration

Theorem 9 *Every atom of \mathcal{T} is positively charged by $\mu_{\mathcal{T}}$. That is,*

$$\text{supp}(\mu_{\mathcal{T}}) = \mathcal{T}.$$

The proof of Theorem 9 hinges on the fact that any rooted tree can be realized by elements of $\ker(\Delta_{\mathbb{R}^n} - I)$ as described by the following result.

Theorem 10 *Given $T \in \mathcal{T}$ there exists $\phi \in \ker(\Delta_{\mathbb{R}^n} - Id)$ and $c \in \mathcal{C}(\phi)$ for which $T(c) = T$.*

As far as Theorem 10, the case $n = 2$ is resolved in [30] using a deformation of $\sin(\pi x) \sin(\pi y)$ and a combinatorial chess board type argument. This is described in Figure 6. In higher dimensions, for example $n = 3$, one proceeds by deforming

$$\psi(x, y, z) = \sin(\pi x) \sin(\pi y) \sin(\pi z).$$

This ψ has enough complexity to produce all elements in \mathcal{T} after deformation. However, it is much more difficult to study than the 2-dimensional case. Unlike $\sin(\pi x) \sin(\pi y)$, the zero set $\psi^{-1}(0)$ has point and 1-dimensional edge singularities. The analysis of its resolution under deformation requires a lot of care, especially as far as engineering elements of \mathcal{T} . The pay off as we noted is that it is rich enough to prove Theorem 10.

Acknowledgments The author is very grateful to her collaborators B. Hanin and P. Sarnak. The author would also like to thank the Alfred P. Sloan Foundation for their support.

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