Notes on Sub-Gaussian Random Elements

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Abstract We give a short survey concerning sub-Gaussian random elements in a Banach space and prove a statement about the induced operator of a bounded random element in a Hilbert space.

1 Sub-Gaussian and Related Random Variables

The sub-Gaussian random variables were explicitly defined by Kahane in [1] (see also [2]). They were further studied by Buldygin and Kozachenko in [3, 4] (see also [5, Chap. 3] and [6]).

A real valued random variable ξ given on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is called *sub-Gaussian* if there exists $a \geq 0$ such that

$$\mathbb{E}\, e^{t\xi} \leq e^{\frac{1}{2}t^2a^2}, \quad \text{for every} \quad t \in \mathbb{R}\,.$$

To a random variable ξ let us associate a quantity $\tau(\xi) \in [0, +\infty]$ defined by the equality:

$$\tau(\xi) = \inf\{a \geq 0 : \mathbb{E}\, e^{t\xi} \leq e^{\frac{1}{2}t^2a^2} \quad \textit{for every} \quad t \in \mathbb{R}\},$$

and call it the Gaussian standard of ξ [3] (it is called *the Gaussian deviation* ("écart de Gauss") of ξ in [1]).

Dedicated to 90th birthday anniversary of Nicholas Vakhania (1930–2014).

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Lemma 1 ([1, 4]; see also, [6, Proposition 2.1 and Corollary 2.1]) *For a real valued random variable ξ the following statements are equivalent:*
(i) ξ is sub-Gaussian.
(ii) $\tau(\xi) < +\infty$ and $\mathbb{E}\,\xi = 0$.
(iii) There is $\lambda > 0$ such that $\mathbb{E}\exp(\lambda\xi^2) < +\infty$ and $\mathbb{E}\,\xi = 0$.
Moreover, if (i) holds, then

$$\mathbb{E}\,e^{\lambda\xi^2} \le \frac{1}{\sqrt{1-2\lambda\tau^2(\xi)}} < \infty \quad for every \quad \lambda \in \left[0, \frac{1}{2\tau^2(\xi)}\right[,$$

and

$$(\mathbb{E}\,|\xi|^p)^{\frac{1}{p}} \le \beta_p \tau(\xi) \quad for every \quad p \in]0, \infty[\,,$$

where $\beta_p = 1$ if $p \in]0, 2]$ and $\beta_p = 2^{\frac{1}{p}}(\frac{p}{e})^{\frac{1}{2}}$ if $p \in]2, \infty[$.
In particular we have

$$\mathbb{E}\,\xi = 0 \quad and \quad \mathbb{E}\,\xi^2 \le \tau^2(\xi)\,.$$

Remark 1 An interesting application of implication $(i) \Longrightarrow (iii)$ of Lemma 1 is the following observation: *if ξ is sub-Gaussian random variable with infinitely divisible distribution, then ξ is (possibly degenerate) Gaussian.* This can be derived e.g. from [7, Theorem 2], or from [8, Theorem 1(a)] or (more directly) from [9, Theorem 2] which asserts in particular that if for a random variable ξ *with infinitely divisible distribution* we have

$$\mathbb{E}\,\exp(\alpha|\xi|\ln(|\xi|+1)) < \infty \quad \text{for every} \quad \alpha > 0\,,$$

then it is Gaussian.

A sub-Gaussian random variable ξ with $\tau(\xi) \le 1$ is called in [2, p. 67] *subnormal*.
For a centered Gaussian random variable ξ clearly $\tau^2(\xi) = \mathbb{E}\,\xi^2$.

A random variable ξ is called *strictly sub-Gaussian* if it is sub-Gaussian and $\tau^2(\xi) = \mathbb{E}\,\xi^2$.

Let $SG(\Omega)$ be the set of all sub-Gaussian random variables $\xi : \Omega \to \mathbb{R}$. It is known that $SG(\Omega)$ is a vector space with respect to the natural point-wise operations, the functional $\tau(\cdot)$ is a norm on $SG(\Omega)$ (provided the random variables which coincide a.s. are identified) and, moreover, $(SG(\Omega), \tau(\cdot))$ is a Banach space [3, 4]. It follows, that if ξ_1 and ξ_2 *are centered Gaussian random variables* (not necessarily jointly Gaussian) then the random variable $\xi_1 + \xi_2$ is sub-Gaussian, but in general $\xi_1 + \xi_2$ *may not be strictly sub-Gaussian* (even if $\mathbb{E}\,\xi_1\xi_2 = 0$) [6, Example 3.7 (d)].

From Lemma 1 we can conclude that for every $p \in]0, +\infty[$ we have

$$SG(\Omega) \subset L_p(\Omega)$$

and the norm of the inclusion mapping $\le \beta_p$.

2 Sub-Gaussian Random Elements

Below X will be a real normed space with the dual space X^*.

We recall that a mapping $\eta : \Omega \to X$ is a *random element* (in X) if

$$\langle x^*, \eta \rangle := x^* \circ \eta$$

is a random variable for every $x^* \in X^*$.

A random element $\eta : \Omega \to X$ is called *Gaussian* if for every $x^* \in X^*$ the random variable $\langle x^*, \eta \rangle$ is Gaussian.

Such a definition of a Gaussian random element goes back to Kolmogorov [10] and Fréchet [11]. For a Gaussian random element we have the following important integrability result (Vakhania [12] for $X = lp$, $1 \le p < +\infty$; Fernique [13], Landau-Shepp [14], Skorokhod [15] in general; see [16, Corollary 2 of Proposition V.5.5, p. 329–330] for a proof):

Theorem 1 *Let η be a separably valued Gaussian random element in a normed space X. Then there is $\lambda > 0$ such that $\mathbb{E}\exp(\lambda\|\eta\|^2) < +\infty$.*

A random element $\eta : \Omega \to X$ is called *weakly sub-Gaussian* if for every $x^* \in X^*$ the random variable $\langle x^*, \eta \rangle$ is sub-Gaussian (cf. [6, 17]).

In [17] it was shown that an analogue of Theorem 1 may fail for weakly sub-Gaussian random elements (see also [6, Theorem 4.2 and Remark 4.1]).

Let us call a random element $\eta : \Omega \to X$ *strictly sub-Gaussian* if for every $x^* \in X^*$ the random variable $\langle x^*, \eta \rangle$ is strictly sub-Gaussian.

Definition 1 ([18]) A random element $\eta : \Omega \to X$ is called *sub-Gaussian*, if there is a finite constant $C_\eta \ge 0$ such that

$$\tau(\langle x^*, \eta \rangle) \le C_\eta \left(\mathbb{E}\,|\langle x^*, \eta \rangle|^2\right)^{\frac{1}{2}} < +\infty \quad \text{for every} \quad x^* \in X^* .$$

We call a random element $\eta : \Omega \to X$ satisfying conditions of Definition 1 *sub-Gaussian in Fukuda's sense*, or *F-sub-Gaussian*.

An analogue of Theorem 1 remains true for F-sub-Gaussian random elements with values in $X = L_p$ with $1 \le p < +\infty$ [18, Theorem 4.3]; however, it may fail for $X = c_0$ (S. Kwapien, personal communication).

In [18] (motivating by [19, Theorem 15 (p. 120)], where a similar concept is implicitly used) a random element $\eta : \Omega \to X$ is called *γ-sub-Gaussian* if there exists a centered Gaussian random element ζ in X such that

$$\mathbb{E}\,e^{\langle x^*, \eta \rangle} \le \mathbb{E}\,e^{\langle x^*, \zeta \rangle} \quad \text{for every} \quad x^* \in X^* .$$

We call a γ-sub-Gaussian random element *sub-Gaussian in Talagrand's sense* or *T-sub-Gaussian*. In [20, Remark 4] the definition of a γ-sub-Gaussian random element in a Hilbert space is attributed to [19].

An analogue of Theorem 1 remains true for γ-sub-Gaussian random elements in a Banach space [18, Theorem 3.4].

If $X = \mathbb{R}$ then the notion of a T-sub-Gaussian, as well as the notion of a F-sub-Gaussian random element coincides with the notion of a sub-Gaussian random variable and the notion of a F-sub-Gaussian random variable ξ with the constant $C_\xi = 1$ coincides with the notion of a strictly sub-Gaussian random variable.

If X is a finite-dimensional Banach space then weakly sub-Gaussian random elements are γ-sub-Gaussian (see [6, Proposition 4.4]). In every infinite-dimensional Banach space there exists a weakly sub-Gaussian random element, which is not γ-sub-Gaussian (see [6, Theorem 4.4]).

In what follows H will denote an infinite-dimensional separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle$.

Definition 2 ([20, Definition 2.1]) Let $\mathbf{e} := \{e_n, n \in \mathbb{N}\}$ be an orthonormal basis of H. A random element η with values in H is *subgaussian with respect to* $\mathbf{e}$ if the following conditions are satisfied:

(1) For every $x \in H$ the real valued random variable $\langle x, \eta \rangle$ is sub-Gaussian (i.e. η is weakly sub-Gaussian),

(2) $\sum_{n=1}^{\infty} \tau^2(\langle e_n, \eta \rangle) < \infty$.

Using the terminology of the definition we have obtained (see [21, Theorem 1.6]) the following characterization of weakly sub-Gaussian random elements in a separable Hilbert space which are γ-sub-Gaussian.

Theorem 2 *For a random element η with values in H the following statements are equivalent:*

(i) η is γ-sub-Gaussian.

(ii) For every orthonormal basis $\mathbf{e} := \{e_n, n \in \mathbb{N}\}$ *of H the random element η is subgaussian with respect to* $\mathbf{e}$.

For a weakly sub-Gaussian random element η in a Banach space X let

$$T_\eta : X^* \to SG(\Omega)$$

be *the induced operator*, which sends each $x^* \in X^*$ to the element $\langle x^*, \ \eta \rangle \in SG(\Omega)$ (the continuity and other related properties of induced operators can be seen in [6, Proposition 4.2]).

Theorem 2 in [21] is derived from the following general result (the definitions of a 2-summing operator and a type 2 space can be seen e.g.. in [16]):

Theorem 3 *For a weakly sub-Gaussian random element η with values in a Banach space X consider the assertions:*

(i) η is γ-sub-Gaussian;

(ii) $T_\eta : X^* \to SG(\Omega)$ is a 2-summing operator.

Then $(i) \Rightarrow (ii)$. The implication $(ii) \Rightarrow (i)$ is true when X is a reflexive type 2 space.

The following statement, which is a refinement of a similar assertion contained in [5, Chap. 3], shows in particular that the implication $(i) \Longrightarrow (ii)$ of Theorem 2 may fail for a bounded symmetrically distributed elementary random element η.

Proposition 1 *Let* $\mathbf{e} := \{e_n, n \in \mathbb{N}\}$ *be an orthonormal basis of* H*. Then there exists a symmetric bounded random element* $\eta : \Omega \to H$ *with a countable range, such that*

(a) $\sum_{i=1}^{\infty} \|\langle \eta, e_i \rangle\|_{L_p}^2 < \infty$ *for every* $p \in]0, \infty[$*;*

(b) $\sum_{i=1}^{\infty} (\tau(\langle \eta, e_i \rangle))^2 = \infty$ *and hence* η *is not subgaussian with respect to* $\mathbf{e}$.

Proof (a). Denote

$$I_n = \{2^n - 1, \dots, 2^{n+1} - 2\}, \ n = 1, 2, \dots$$

and

$$b_n = 2^{-n} \sum_{k \in I_n} e_k, \ n = 1, 2, \dots .$$

Observe that

$$\sum_{k=1}^{\infty} \|b_k\|^2 = \sum_{n=1}^{\infty} \sum_{k \in I_n} \|b_k\|^2 = \sum_{n=1}^{\infty} 2^{-2n} \cdot 2^n = 1 .$$

Thus we can define a probability measure $\mathbb{P}$ on $\Omega := \mathbb{N}$ and a random element $\eta : \Omega \to H$ by setting:

$$\mathbb{P}(\{2n-1\}) = \mathbb{P}(\{2n\}) = \frac{1}{2}\|b_n\|^2, \ n = 1, 2, \dots$$

and

$$\eta(2n-1) = -\frac{b_n}{\|b_n\|}, \ \eta(2n) = \frac{b_n}{\|b_n\|}, \ n = 1, 2, \dots .$$

Fix now $p \in]0, \infty[$ and $i \in \mathbb{N}$. Clearly,

$$\mathbb{E}|\langle \eta, e_i \rangle|^p = \sum_{n=1}^{\infty} \left(\sum_{k \in I_n} \langle e_k, e_i \rangle \right) \frac{1}{2^{n(1+p/2)}} .$$

Hence

$$\mathbb{E}|\langle \eta, e_i \rangle|^p = \frac{1}{2^{n(1+p/2)}} \quad \text{for every} \quad i \in I_n, \ n = 1, 2, \dots$$

and so

$$\sum_{i=1}^{\infty} \|T_\eta e_i\|_{L_p}^2 = \left(\mathbb{E}|\langle \eta, e_i \rangle|^p\right)^{2/p} = \sum_{n=1}^{\infty} \sum_{k \in I_n} \frac{1}{2^{n(1+2/p)}} =$$

$$\sum_{n=1}^{\infty} \frac{2^n}{2^{n(1+2/p)}} = \sum_{n=1}^{\infty} \frac{1}{2^{2n/p}} < \infty .$$

(b). To a (real-valued) random variable ξ let us associate a quantity $\vartheta_2(\xi) \in [0, +\infty]$ defined by the equality:

$$\vartheta_2(\xi) = \sup_{m\in\mathbb{N}} \frac{\left(\mathbb{E}\,|\xi|^{2m}\right)^{1/2m}}{\sqrt{m}} .$$

According to [6, Proposition 2.9(b)] we have:

$$\vartheta_2(\xi) \le \frac{2}{\sqrt{e}}\tau(\xi) \qquad \text{for every} \qquad \xi \in SG(\Omega).$$

So, it is sufficient to show that

$$\sum_{i=1}^{\infty} \left(\vartheta_2(T_\eta e_i)\right)^2 = \infty . \tag{2.1}$$

We have for every $n \in \mathbb{N}$ and $i \in I_n$:

$$\vartheta_2(\langle \eta, e_i\rangle) = \sup_m \frac{\left(\mathbb{E}\,|\langle \eta, e_i\rangle|^{2m}\right)^{1/2m}}{\sqrt{m}} = \sup_m \frac{1}{2^{n(1/2+1/2m)}\sqrt{m}} \ge$$

$$\frac{1}{2^{n(1/2+1/2n)}\sqrt{n}} .$$

Hence

$$\sum_{i=1}^{\infty} \vartheta_2^2(\langle \eta, e_i\rangle) = \sum_{n=1}^{\infty}\sum_{i\in I_n} \vartheta_2^2(\langle \eta, e_i\rangle) \ge \sum_{n=1}^{\infty} 2^n \left(\frac{1}{2^{n(1/2+1/2n)}\sqrt{n}}\right)^2 =$$

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n} = \infty$$

and (2.1) is proved.

The authors do not know whether the following conjecture related with Proposition 1 is true.

Conjecture 1 *There exists a symmetric bounded random element* $\eta : \Omega \to H$ *such that*

(a) $\sum_{i=1}^{\infty} \|\langle \eta, e_i\rangle\|_{L_p}^2 < \infty$ for every $p \in]0, \infty[$ *and for every orthonormal basis* $\mathbf{e} := \{e_n, n \in \mathbb{N}\}$ of H;

(b) $\sum_{i=1}^{\infty} (\tau(\langle \eta, e_i \rangle))^2 = \infty$ *for some orthonormal basis* $\mathbf{e} := \{e_n, n \in \mathbb{N}\}$ of H.

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References

1. Kahane, J.P.: Propriétés locales des fonctions à séries de Fourier aleatoires. Stud. Math. **19**, 1–25 (1960)
2. Kahane, J.P.: Some Random Series of Functions, 2nd edn. Cambridge University Press, Cambridge (1985)
3. Buldygin, V.V., Kozachenko, YuV.: Subgaussian random variables. Ukrainian Math. J. **32**, 723–730 (1980)
4. Buldygin, V.V., Kozachenko, Yu.V.: Metric Characterization of Random Variables and Random Processes. Translations of Mathematical Monographs, vol. 188. AMS, Providence R.I. (2000)
5. Kvaratskhelia, V.V.: Unconditional Convergence of Functional Series in Problems of Infinite-Dimensional Probability Theory. Doctoral Dissertation, Tbilisi (2002)
6. Vakhania, N.N., Kvaratskhelia, V.V., Tarieladze, V.I.: Weakly Sub-Gaussian random elements in Banach spaces. Ukrainian Math. J. **57**, 1187–1208 (2005)
7. Ruegg, A.: A characterization of certain infinitely divisible laws. Ann. Math. Stat. **41**, 1354–1356 (1974)
8. Horn, R.A.: On necessary and sufficient conditions for an infinitely divisible distribution to be normal or degenerate. Z. Wahrscheinlichkeitstheorie verw. Geb. **21**, 179–187 (1972)
9. Kruglov, V.M.: Characterization of a class of infinitely divisible distributions in a Hilbert space. Mat. Zametki **16**, 777–782 (1974)
10. Kolmogorov, A.N.: La transformation de Laplace dans les espaces linéaires. C.R. Acad. Sc. Paris **200**, 1717–1718 (1935)
11. Fréchet, M.: Généralisation de la loi de probabilité de Laplace. Ann. Inst. H. Poincare **12**(1), 1–29 (1951)
12. Vakhania, N.N.: Sur les répartitions de probabilités dans les espaces de suites numériques. C.R. Acad. Sc. Paris **260**, 1560–1562 (1965)
13. Fernique, X.: Intégrabilité des vecteurs gaussiens. C.R. Acad. Sc. Paris, Série A **270**, 1698–1699 (1970)
14. Landau, H.J., Shepp, L.A.: On the supremum of Gaussian process. Sankhya, Série A **32**(4), 369–378 (1970)
15. Skorokhod, A.V.: A note on Gaussian measures in a Banach space. Theory Probab. Appl. **15**(3), 508–508 (1970)
16. Vakhania, N.N., Tarieladze, V.I., Chobanyan, S.A.: Probability distributions on Banach spaces. D. Reidel Publishing Company, Dordrecht (1987)
17. Vakhania, N.: Subgaussian random vectors in normed spaces. Bull. Georg. Acad. Sci. **163**, 8–11 (2001)
18. Fukuda, R.: Exponential integrability of sub-Gaussian vectors. Probab. Theory Relat. Fields **85**, 505–521 (1990)
19. Talagrand, M.: Regularity of gaussian processes. Acta Math. **159**, 99–149 (1987)
20. Antonini, R.G.: Subgaussian random variables in Hilbert spaces. Rend. Sem. Mat. Univ. Padova **98**, 89–99 (1997)
21. Kvaratskhelia, V., Tarieladze, V., Vakhania, N.: Characterization of γ-subgaussian random elements in a Banach space. J. Math. Sci. **216**, 564–568 (2016)