Two Positive Solutions for a Nonlinear Neumann Problem Involving the Discrete p-Laplacian

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Abstract This paper is devoted to study of existence of at least two positive solutions for a nonlinear Neumann boundary value problem involving the discrete p-Laplacian.

Keywords Multiple solutions · Difference equations · Neumann problem

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1 Introduction

In this paper, we investigate the existence of two positive solutions for the following nonlinear discrete Neumann boundary value problem

$$\begin{cases} -\Delta(\phi_p(\Delta u(k-1))) + q(k)\phi_p(u(k)) = \lambda f(k, u(k)), & k \in [1, N], \\ \Delta u(0) = \Delta u(N) = 0, \end{cases} \qquad N_{\lambda,\underline{f}}$$

where λ is a positive parameter, N is a fixed positive integer, $[0, N+1]$ is the discrete interval $\{0, ..., N+1\}$, $\phi_p(s) := |s|^{p-2}s$, $1 < p < +\infty$ and for all $k \in [0, N+1]$, $q(k) > 0$, $\Delta u(k) := u(k+1) - u(k)$ denotes the forward difference operator and $f : [0, N+1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

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The theory of difference equations employs numerical analysis, fixed point methods, upper an lower solutions methods (see, for instance, [3, 5, 7, 23]). The variational approach represents an important advance as it allows to prove multiplicity results, usually, under a suitable condition on the nonlinearities, see [1, 2, 7–11, 14–22, 24, 25].

In the present paper, we study the problem $(N_{\lambda,\underline{f}})$ following a variational approach, based on a recent result of Bonanno and D'Aguì (see [6]), that assures the existence of at least two non trivial critical points for a certain class of functionals defined on infinite-dimensional Banach space. This theorem is obtained by combining a local minimum result given in [13], together with the Ambrosetti-Rabinowitz theorem (see [4]). In the application of the mountain pass theorem, to prove the Palais-Smale condition of the energy functional associated to the nonlinear differential problems, the Ambrosetti-Rabinowitz condition is requested on the nonlinear term, in particular this means that the nonlinear term has to be more than p-superlinear at infinity.

In this paper, exploiting that the variational framework of the problem $(N_{\lambda,\underline{f}})$ is defined in a finite-dimensional space, we prove that the p-superlinearity at infinity of the primitive on the nonlinearity is enough to prove the Palais-Smale condition. For a complete overview on variational methods on finite Banach spaces and discrete problems, see [12]. We obtain, here, Theorem 2, which gived the existence of two positive solutions, by requiring an algebraic condition on the nonlinearity (we mean (6) in 2).

The paper is so organized: Sect. 2, contains basic definitions and main results on difference equations and some critical point tools, in addition, Lemma 2 is given in order to prove the Palais-Smale condition of the functional associated to problem $(N_{\lambda,\underline{f}})$. Section 3 is devoted to our main result. In particular, our main theorem allows us to obtain two positive solutions with only one hypothesis on the primitive of the nonlinear term f without any asymptotic behaviour at zero. Moreover, a consequence (Corollary 1) (requiring the p-superlinearity at infinity and the p-sublinearity at zero on the primitive of f) of our main result is presented in order to show the applicability of our results.

2 Mathematical Background

In the $N+2$-dimensional Banach space

$$X = \{u : [0, N+1] \to \mathbb{R} : \Delta u(0) = \Delta u(N) = 0\},$$

we consider the norm

$$\|u\| := \left(\sum_{k=1}^{N+1} |\Delta u(k-1)|^p + \sum_{k=1}^{N} q(k)|u(k)|^p \right)^{1/p} \quad \forall u \in X.$$

Moreover, we will use also the equivalent norm

$$\|u\|_\infty := \max_{k\in[0,N+1]} |u(k)|, \quad \forall u \in X.$$

For our purpose, it will be useful the following inequality

$$\|u\|_\infty \leq \|u\| q^{-1/p}, \quad \forall u \in X, \quad \text{where} \quad q := \min_{k\in[1,N]} q_k. \tag{1}$$

Moreover, we mention the classical Hölder norm on X.

$$\|u\|_p = \left(\sum_{k=0}^{N+1} |u(k)|^p \right)^{\frac{1}{p}}.$$

We observe that being X a finite dimensional Banach space, all norms defined on it are equivalent and in particular, there exist two positive constants L_1 and L_2 such that

$$L_1 \|u\|_p \leq \|u\| \leq L_2 \|u\|_p. \tag{2}$$

To describe the variational framework of problem $(N_{\lambda,\underline{f}})$, we introduce the following two functions

$$\Phi(u) := \frac{\|u\|^p}{p} \quad \text{and} \quad \Psi(u) := \sum_{k=1}^{N} F(k, u(k)), \quad \forall u \in X, \tag{3}$$

where $F(k,t) := \int_0^t f(k,\xi)d\xi$ for every $(k,t) \in [1,N] \times \mathbb{R}$. Clearly, Φ and Ψ are two functionals of class $C^1(X,\mathbb{R})$ whose Gâteaux derivatives at the point $u \in X$ are given by

$$\Phi'(u)(v) = \sum_{k=1}^{N+1} \phi_p\left(\Delta u\left(k-1\right)\right) \Delta v\left(k-1\right) + q(k) \left|u\left(k\right)\right|^{p-2} u\left(k\right) v\left(k\right),$$

and

$$\Psi'(u)(v) = \sum_{k=1}^{N} f\left(k, u\left(k\right)\right) v(k),$$

for all $u, v \in X$. Taking into account that

$$-\sum_{k=1}^{N}\Delta(\phi_p(\Delta u(k-1)))v(k)=\sum_{k=1}^{N+1}\phi_p(\Delta u(k-1))\Delta v(k-1),\quad \forall\, u\ \ v, \in X,$$

it is easy to verify, see also [25], that

Lemma 1. *A vector $u \in X$ is a solution of problem $(N_{\lambda,\underline{f}})$ if and only if u is a critical point of the function $I_\lambda = \Phi - \lambda\Psi$.*

Let $(X, \|\cdot\|)$ be a Banach space and let $I \in C^1(X,\mathbb{R})$. We say that I satisfies the Palais-Smale condition (in short (PS)-condition), if any sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that

1. $\{I(u_n)\}_{n\in\mathbb{N}}$ is bounded,
2. $\{I'(u_n)\}_{n\in\mathbb{N}}$ converges to 0 in X^*,

admits a subsequence which is convergent in X.

Here, we recall the abstract result established in [6], on the existence of two non-zero critical points.

Theorem 1. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two functionals of class C^1 such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that*

$$\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}, \tag{4}$$

and, for each

$$\lambda \in \Lambda = \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}\right[,$$

the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the (PS)-condition and it is unbounded from below.

Then, for each $\lambda \in \Lambda$, the functional I_λ admits at least two non-zero critical points $u_{\lambda,1}$, $u_{\lambda,2}$ such that $I(u_{\lambda,1}) < 0 < I(u_{\lambda,2})$.

Here and in the sequel we suppose $f(k,0) \geq 0$ for all $k \in [1,N]$. We assume that $f(k,x) = f(k,0)$ for all $x < 0$ and for all $k \in [1,N]$. Put

$$L_\infty(k) := \liminf_{s\to+\infty}\frac{F(k,s)}{s^p},\quad L_\infty := \min_{k\in[1,N]} L_\infty(k).$$

We give the following lemma.

Lemma 2. *If* $L_\infty > 0$ *then* I_λ *satisfies* (PS)*-condition and it is unbounded from below for all* $\lambda \in \left]\frac{L_2^p}{pL_\infty}, +\infty\right[$, *where* L_2 *is given in* (2).

Proof. Since $L_\infty > 0$ we put $\lambda > \frac{L_2^p}{pL_\infty}$ and l such that $L_\infty > l > \frac{L_2^p}{p\lambda}$. Let $\{u_n\}$ be a sequence such that $\lim_{n\to+\infty} I_\lambda(u_n) = c$ and $\lim_{n\to+\infty} I'_\lambda(u_n) = 0$. Put $u_n^+ = \max\{u_n, 0\}$ and $u_n^- = \max\{-u_n, 0\}$ for all $n \in \mathbb{N}$. We have that $\{u_n^-\}$ is bounded. In fact, one has

$$\left|\Delta u_n^-(k-1)\right|^p \le -\phi_p\left(\Delta u_n(k-1)\right)\Delta u_n^-(k-1),$$

for all $k \in [1, N+1]$, and

$$q(k)\left|u_n^-(k)\right|^p = -q(k)\left|u_n(k)\right|^{p-2}u_n(k)u_n^-(k),$$

for all $k \in [1, N+1]$.
So we have,

$$\begin{aligned}&\sum_{k=1}^{N+1}\left(\left|\Delta u_n^-(k-1)\right|^p + q(k)\left|u_n^-(k)\right|^p\right)\\ &\le -\sum_{k=1}^{N+1}\left(\phi_p\left(\Delta u_n(k-1)\right)\Delta u_n^-(k-1) + q(k)\left|u_n(k)\right|^{p-2}u_n(k)u_n^-(k)\right).\end{aligned}$$

So,

$$\begin{aligned}\|u_n^-\|^p &= \sum_{k=1}^{N+1}\left(\left|\Delta u_n^-(k-1)\right|^p + q(k)\left|u_n^-(k)\right|^p\right)\\ &\le -\sum_{k=1}^{N+1}\left(\phi_p\left(\Delta u_n(k-1)\right)\Delta u_n^-(k-1) + q(k)\left|u_n(k)\right|^{p-2}u_n(k)u_n^-(k)\right)\\ &= -\Phi'(u_n)(u_n^-).\end{aligned}$$

By definition of u_n^- and taking into account that $f(k, x) = f(k, 0)$ for all $x < 0$ and for all $k \in [1, N]$, we have

$$\Psi'(u_n)(u_n^-) = \sum_{k=1}^{N} f\left(k, u_n(k)\right)u_n^-(k) \ge 0.$$

So, we get

$$\|u_n^-\|^p \le -\Phi'(u_n)(u_n^-) \le -\Phi'(u_n)(u_n^-) + \lambda\Psi'(u_n)(u_n^-),$$

that is

$$\|u_n^-\|^p \le -I'_\lambda(u_n)(u_n^-), \tag{5}$$

for all $n \in \mathbb{N}$. Now, from $\lim_{n\to+\infty} I'_\lambda(u_n) = 0$, one has $\lim_{n\to+\infty} \frac{I'_\lambda(u_n)(u_n^-)}{\|u_n^-\|} = 0$, for which, taking (5) into account, gives $\lim_{n\to+\infty} \|u_n^-\| = 0$. So, we obtain the claim. And, there is $M > 0$ such that $\|u_n^-\| \le M$, $\|u_n^-\|_p \le \frac{M}{L_1} = L$, $0 \le u_n^-(k) \le L$ for all $k \in [1, N]$ for all $n \in \mathbb{N}$.

At this point, by contradiction argument, assume that $\{u_n\}$ is unbounded (that is, $\{u_n^+\}$ is unbounded).

From $\liminf_{s\to+\infty} \frac{F(k,s)}{s^p} = L_\infty(k) \ge L_\infty > l$ there is $\delta_k > 0$ such that $F(k,s) > ls^p$ for all $s > \delta_k$. Moreover,

$$\begin{aligned} F(k,s) &\ge \min_{s\in[-L,\delta_k]} F(k,s) \ge ls^p - l\,(\max\{\delta_k, L\})^p + \min_{s\in[-L,\delta_k]} F(k,s) \\ &\ge ls^p - \max\{l\,(\max \delta_k, L)^p - \min_{s\in[-L,\delta_k]} F(k,s), 0\} = ls^p - Q(k) \end{aligned}$$

for all $s \in [-L, \delta_k]$. Hence, $F(k,s) \ge ls^p - Q(k)$ for all $s \ge -L$. It follows that $F\,(k, u_n(k)) \ge l\,(u_n(k))^p - Q(k)$ for all $n \in \mathbb{N}$ and for all $k \in [1, N]$, $\sum_{k=1}^N F(k, u_n(k)) \ge \sum_{k=1}^N \left[l\,(u_n(k))^p - Q(k)\right] = l\|u_n\|_p^p - \sum_{k=1}^N Q(k) = l\|u_n\|_p^p - \overline{Q}$, that is,

$$\Psi(u_n) \ge l\|u_n\|_p^p - \overline{Q},$$

for all $n \in \mathbb{N}$. Therefore, one has

$$I_\lambda(u_n) = \Phi(u_n) - \lambda\Psi(u_n) = \frac{1}{p}\|u_n\|^p - \lambda\Psi(u_n) \le \frac{L_2^p}{p}\|u_n\|_p^p - \lambda l\|u_n\|_p^p + \lambda\overline{Q},$$

that is

$$I_\lambda(u_n) \le \left(\frac{L_2^p}{p} - \lambda l\right)\|u_n\|_p^p + \lambda\overline{Q},$$

for all $n \in \mathbb{N}$. Since $\|u_n\|_p \to +\infty$ and $\frac{L_2^p}{p} - \lambda l < 0$, one has $\lim_{n\to+\infty} I_\lambda(u_n) = -\infty$ and this is absurd. Hence, I_λ satisfies (PS)-condition.

Finally, we get that I_λ is unbounded from below. Let $\{u_n\}$ be such that $\{u_n^-\}$ is bounded and $\{u_n^+\}$ is unbounded. As before, we obtain $\Psi(u_n) \ge l\|u_n\|_p^p - \overline{Q}$, for all

$n \in \mathbb{N}$ and, consequently, $I_\lambda(u_n) \leq \left(\frac{L_2^p}{p} - \lambda l\right) \|u_n\|_p^p + \lambda \overline{Q}$, for all $n \in \mathbb{N}$. Hence, $\lim_{n\to+\infty} I_\lambda(u_n) = -\infty$ and the proof is complete.

3 Main Results

In this section, we present the main existence result of our paper. We start putting

$$Q = \sum_{k=1}^{N} q(k).$$

Theorem 2. *Let $f : [1, N] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(k, 0) \geq 0$ for all $k \in [1, N]$, and $f(k, 0) \neq 0$ for some $k \in [1, N]$. Assume also that there exist two positive constants c and d with $d < c$ such that*

$$\frac{\sum_{k=1}^{N} \max_{|\xi|\leq c} F(k, \xi)}{c^p} < q \min\left\{\frac{1}{Q}\frac{\sum_{k=1}^{N} F(k, d)}{d^p}, \frac{L_\infty}{L_2^p}\right\}. \tag{6}$$

Then, for each $\lambda \in \bar{\Lambda}$ with

$$\bar{\Lambda} = \left]\max\left\{\frac{Q}{p}\frac{d^p}{\sum_{k=1}^{N} F(k, d)}, \frac{L_2^p}{pL_\infty}\right\}, \frac{q}{p}\frac{c^p}{\sum_{k=1}^{N} \max_{|\xi|\leq c} F(k, \xi)}\right[,$$

the problem $(N_{\lambda,\underline{f}})$ admits at least two positive solutions.

Proof. We consider the functionals Φ and Ψ given in (3). Φ and Ψ satisfy all regularity assumptions requested in Theorem 1, moreover we have that any critical point in X of the functional I_λ is exactly a solution of problem $(N_{\lambda,\underline{f}})$. Furthermore, $\inf_S \Phi = \Phi(0) = \Psi(0) = 0$. In order to prove our result, we need to verify condition (4) of Theorem 1. Fix $\lambda \in \bar{\Lambda}$, from (6) one has that $L_\infty > 0$ and $\bar{\Lambda}$ is non-degenerate. From Lemma 2, the functional I_λ satisfies the (PS)-condition for each $\lambda > \frac{L_2^p}{pL_\infty}$, and it is unbounded from below. Now, put $r = \frac{qc^p}{p}$, an condier $u \in \Phi^{-1}(]-\infty, r])$; so such a u satisfies

$$\frac{1}{p}\|u\|^p \leq r,$$

so

$$\|u\| \leq (pr)^{\frac{1}{p}}.$$

One has

$$|u| \leq \frac{1}{q^{\frac{1}{p}}}\|u\| \leq \left(\frac{pr}{q}\right)^{\frac{1}{p}} = c.$$

So,

$$\Psi(u) = \sum_{k=1}^{N} F(k, u(k)) \leq \sum_{k=1}^{N} \max_{|\xi|\leq c} F(k, \xi),$$

for all $u \in X$ such that $u \in \Phi^{-1}(]-\infty, r])$.
Hence,

$$\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}{r} \leq \frac{p}{q} \frac{\sum\limits_{k=1}^{N} \max\limits_{|\xi|\leq c} F(k, \xi)}{c^p}. \tag{7}$$

Now, let be $\tilde{u} \in \mathbb{R}^{N+2}$ be such that $\tilde{u}(k) = d$ for all $k \in [0, N+1]$. Clearly, $\tilde{u} \in X$ and it holds

$$\Phi(\tilde{u}) = \frac{Qd^p}{p}, \tag{8}$$

and so, we have

$$\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} = \frac{p}{Q} \frac{\sum\limits_{k=1}^{N} F(k, d)}{d^p}. \tag{9}$$

Therefore, from (7), (9) and assumption (6) one has

$$\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}.$$

Moreover, taking into account that $0 < d < c$ and again by (6), we have that

$$0 < d < \left(\frac{q}{Q}\right)^{\frac{1}{p}} c. \tag{10}$$

Indeed, by contradiction, if we suppose that $d \geq \left(\frac{q}{Q}\right)^{\frac{1}{p}} c$, we have

$$\frac{\sum_{k=1}^{N} \max_{|\xi|\leq c} F(k,\xi)}{c^p} \geq \frac{\sum_{k=1}^{N} F(k,d)}{c^p} \geq \frac{q}{Q} \frac{\sum_{k=1}^{N} F(k,d)}{d^p},$$

which contradicts (6). Hence by (8) and (10) we get $0 < \Phi(\tilde{u}) < r$.

So, finally we obtain that I_λ admits at least two non-zero critical points and then, for all $\lambda \in \bar{\Lambda} \subset \Lambda$, these are non zero solutions of $(N_{\lambda,\underline{f}})$.

Since we are interested to obtain a positive solution for problem $(N_{\lambda,\underline{f}})$, we adopt the following truncation on the functions $f(k,s)$,

$$f^+(k,s) = \begin{cases} f(k,s), & \text{if } s \geq 0; \\ f(k,0), & \text{if } s < 0. \end{cases}$$

Fixed $\lambda \in \Lambda_c^+$. Working with the truncations $f^+(k,s)$, since we have that $f(k(0,s) \neq 0$ for some $k \in [1,N]$, let u a non trivial solution guaranteed in the first part of the proof, now, to prove the u is nonnegative, we exploit the u is a critical point of the energy functional $I_\lambda = \Phi - \lambda\Psi$ associated to problem (N_{λ,f^+}). In other words, we have that $u \in X$ satisfies the following condition

$$\sum_{k=1}^{N+1} \phi_p(\Delta u(k-1))\Delta v(k-1) + \sum_{k=1}^{N} q(k)\phi_p(u(k))v(k) = \sum_{k=1}^{N} f^+(k,u(k))v(k), \ \forall u,v \in X. \tag{11}$$

From this, taking as test function $v = -u^-$, it is a simple computation to prove that $\|u^-\| = 0$, that is u is nonnegative. Moreover, arguing by contradiction, we show that u is also a positive solution of problem $(N_{\lambda,\underline{f}})$. Suppose that $u(k) = 0$ for some $k \in [1,N]$. Being u a solution of problem $(N_{\lambda,\underline{f}})$ we have

$$\phi_p(\Delta u(k-1)) - \phi_p(\Delta u(k)) = f(k,0) \geq 0,$$

which implies that

$$0 \geq -|u(k-1)|^{p-2}u(k-1) - |u(k+1)|^{p-2}u(k+1) \geq 0.$$

So, we have that $u(k-1) = u(k+1) = 0$. Hence, iterating this process, we get that $u(k) = 0$ for every $k \in [1, N]$, which contradicts that u is nontrivial and this completes the proof.

Now, we present a particular case of Theorem 2.

Corollary 1. *Assume that f is a continuous function such that $f(k, 0) > 0$ for all $k \in [0, N]$ and*

$$\limsup_{t \to 0^+} \frac{F(k,t)}{t^p} = +\infty, \tag{12}$$

and

$$\lim_{t \to +\infty} \frac{F(k,t)}{t^p} = +\infty,$$

for all $k \in [0, N]$, and put $\lambda^* = \dfrac{q}{p} \sup_{c>0} \dfrac{c^p}{\sum_{k=1}^{N} \max_{|\xi| \le c} F(k, \xi)}$.

Then, for each $\lambda \in]0, \lambda^*[$, the problem $(N_{\lambda, \underline{f}})$ admits at least two positive solutions.

Proof. First, note that $L_\infty = +\infty$. Then, fix $\lambda \in]0, \lambda^*[$ and $c > 0$ such that

$$\lambda < \frac{q}{p} \frac{c^p}{\sum_{k=1}^{N} \max_{|\xi| \le c} F(k, \xi)}.$$

From (12) we have

$$\limsup_{t \to 0^+} \frac{\sum_{k=1}^{N} F(k,t)}{t^p} = +\infty,$$

then there is $d > 0$ with $d < c$ such that $\dfrac{p}{Q} \dfrac{\sum_{k=1}^{N} F(k,d)}{d^p} > \dfrac{1}{\lambda}$. Hence, Theorem 2 ensures the conclusion.

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