Multiple Periodic Solutions for a Duffing Type Equation with One-Sided Sublinear Nonlinearity: Beyond the Poincaré-Birkhoff Twist Theorem

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Abstract We prove the existence of multiple periodic solutions for a planar Hamiltonian system generated from the second order scalar ODE of Duffing type $x'' + q(t)g(x) = 0$ with g satisfying a one-sided condition of sublinear type. We consider the classical approach based on the Poincaré-Birkhoff fixed point theorem as well as some refinements on the side of the theory of bend-twist maps and topological horseshoes. We focus our analysis to the case of a stepwise weight function, in order to highlight the underlying geometrical structure.

1 Introduction

The Poincaré-Birkhoff fixed point theorem deals with a planar homeomorphism Ψ defined on an annular region A, such that Ψ is area-preserving, leaves the boundary of A invariant and rotates the two components of ∂A in opposite directions (twist condition). Under these assumptions, in 1912 Poincaré conjectured (and proved in some particular cases) the existence of at least two fixed points for Ψ, a result known as "the Poincaré last geometric theorem". A proof for the existence of at least one fixed point (and actually two in a non-degenerate situation) was obtained by Birkhoff in 1913 [3]. In the subsequent years Birkhoff reconsidered the theorem as well as its possible extensions to a more general setting, for instance, removing the assumption of boundary invariance, or proposing some hypotheses of topological nature instead of the area-preserving condition, thus opening a line of research that is still active today (see for example [4, 9], and the references therein). The skepticism of some mathematicians about the correctness of the proof of the second fixed point motivated Brown and Neumann to present in [7] a full detailed proof, adapted from Birkhoff's 1913 paper, in order to eliminate previous possible controversial aspects. Another approach for the proof of the second fixed point has been proposed in [47], coupling [3] with a result for removing fixed points of zero index.

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In order to express the twist condition in a more precise manner, the statement of the Poincaré-Birkhoff theorem is usually presented in terms of the lifted map $\tilde{\Psi}$. Let us first introduce some notation. Let $D(R)$ and $D[R]$ be, respectively, the open and the closed disc of center the origin and radius $R > 0$ in $\mathbb{R}^2$ endowed with the Euclidean norm $||\cdot||$. Let also $C_R := \partial D(R)$. Given $0 < r < R$, we denote by A or $A[r, R]$ the closed annulus $A[r, R] := D[R] \setminus D(r)$. Hence the area-preserving (and orientation-preserving) homeomorphism $\Psi : A \to \Psi(A) = A$ is lifted to a map $\tilde{\Psi} : \tilde{A} \to \tilde{A}$, where $\tilde{A} := \mathbb{R} \times [r, R]$ is the covering space of A via the covering projection $\Pi : (\theta, \rho) \mapsto (\rho\cos\theta, \rho\sin\theta)$ and

$$\tilde{\Psi} : (\theta, \rho) \mapsto (\theta + 2\pi\mathscr{J}(\theta, \rho), \mathscr{R}(\theta, \rho)), \tag{1}$$

with the functions $\mathscr{J}$ and $\mathscr{R}$ being 2π-periodic in the θ-variable. Then, the classical (1912–1913) Poincaré-Birkhoff fixed point theorem can be stated as follows (see [7]).

Theorem 1. *Let $\Psi : A \to \Psi(A) = A$ be an area preserving homeomorphism such that the following two conditions are satisfied:*

$$\mathscr{R}(\theta, r) = r, \quad \mathscr{R}(\theta, R) = R, \quad \forall\,\theta \in \mathbb{R}, \tag{$PB1$}$$

$$\exists\, j \in \mathbb{Z} : \ (\mathscr{J}(\theta, r) - j)(\mathscr{J}(\theta, R) - j) < 0, \quad \forall\,\theta \in \mathbb{R}. \tag{$PB2$}$$

Then Ψ has at least two fixed points z_1, z_2 in the interior of A and $\mathscr{J}(\theta, \rho) = j$ for $\Pi(\theta, \rho) = z_i$.

We refer to condition $(PB1)$ as to the "boundary invariance" and we call $(PB2)$ the "twist condition". The function $\mathscr{J}$ can be regarded as a rotation number associated with the points. In the original formulation of the theorem it is $j = 0$, however any integer j can be considered.

The Poincaré-Birkhoff theorem is a fundamental result in the areas of fixed point theory and dynamical systems, as well as in their applications to differential equations. General presentations can be found in [28, 35, 37]. There is a large literature on the subject and certain subtle and delicate points related to some controversial extensions of the theorem have been settled only in recent years (see [29, 33, 46]). In the applications to the study of periodic non-autonomous planar Hamiltonian systems, the map Ψ is often the Poincaré map (or one of its iterates). In this situation the condition of boundary invariance is usually not satisfied, or very difficult to prove: as a consequence, variants of the Poincaré-Birkhoff theorem in which the hypothesis $(PB1)$ is not required turn out to be quite useful for the applications (see [14] for a general discussion on this topic). As a step in this direction we present the next result, following from Ding in [18].

Theorem 2. *Let $\Psi : D[R] \to \Psi(D[R]) \subseteq \mathbb{R}^2$ be an area preserving homeomorphism with $\Psi(0) = 0$ and such that the twist condition $(PB2)$ holds. Then Ψ has at least two fixed points z_1, z_2 in the interior of A and $\mathscr{J}(\theta, \rho) = j$ for $\Pi(\theta, \rho) = z_i$.*

The proof in [18] (see also [17, Appendix]) relies on the Jacobowitz version of the Poincaré-Birkhoff theorem for a pointed topological disk [25, 26] which was corrected in [29], since the result is true for strictly star-shaped pointed disks and not valid in general, as shown by a counterexample in the same article. Another (independent) proof of Theorem 2 was obtained by Rebelo in [46], who brought the proof back to that of Theorem 1 and thus to the "safe" version of Brown and Neumann [7]. Other versions of the Poincaré-Birkhoff theorem giving Theorem 2 as a corollary can be found in [23, 24, 32, 45] (see also [20, Introduction] for a general discussion about these delicate aspects). For Poincaré maps associated with Hamiltonian systems there is a much more general version of the theorem due to Fonda and Ureña in [21, 22], which holds in higher dimension, too.

In [15, 16], Ding proposed a variant of the Poincaré-Birkhoff theorem, by introducing the concept of "bend-twist map". Given a continuous map $\Psi : A \to \Psi(A) \subseteq \mathbb{R}^2 \setminus \{0\}$, which admits a lifting $\tilde{\Psi}$ as in (1), we define

$$\Upsilon(\theta, \rho) := \mathscr{R}(\theta, \rho) - \rho.$$

We call Ψ a *bend-twist map* if it Ψ satisfies the twist condition and Υ changes its sign on a non-contractible Jordan closed curve Γ contained in the set of points in the interior of A where $\mathscr{J} = j$. The original treatment was given in [15] for analytic maps. There are extensions to continuous maps as well [43, 44]. The bend-twist map condition is difficult to check in practice, due to the lack of information about the curve Γ (which, in the non-analytic case, may not even be a curve). For this reason, one can rely on the following corollary [15, Corollary 7.3] which also follows from the Poincaré-Miranda theorem (as observed in [43]).

Theorem 3. *Let $\Psi : A = A[r, R] \to \Psi(A) \subseteq \mathbb{R}^2 \setminus \{0\}$ be a continuous map such that the twist condition $(PB2)$ holds. Suppose that there are two disjoint arcs α, β contained in A, connecting the inner with the outer boundary of the annulus and such that*

$$\Upsilon > 0 \text{ on } \alpha \text{ and } \Upsilon < 0 \text{ on } \beta. \tag{BT1}$$

Then Ψ has at least two fixed points z_1, z_2 in the interior of A and $\mathscr{J}(\theta, \rho) = j$ for $\Pi(\theta, \rho) = z_i$.

A simple variant of the above theorem considers $2n$ pairwise disjoint simple arcs α_i and β_i (for $i = 1, \dots, n$) contained in A and connecting the inner with the outer boundary. We label these arcs in cyclic order so that each β_i is between α_i and α_{i+1} and each α_i is between β_{i-1} and β_i (with $\alpha_{n+1} = \alpha_1$ and $\beta_0 = \beta_n$) and suppose that

$$\Upsilon > 0 \text{ on } \alpha_i \text{ and } \Upsilon < 0 \text{ on } \beta_i\,, \quad \forall\, i = 1, \dots, n. \tag{BTn}$$

Then Ψ has at least $2n$ fixed points z_i in the interior of A and $\mathscr{J}(\theta, \rho) = j$ for $\Pi(\theta, \rho) = z_i$. These results also apply in the case of a topological annulus (namely, a compact planar set homeomorphic to A) and do not require that Ψ is area-preserving and also the assumption of Ψ being a homeomorphism is not required, as continuity is enough. Moreover, since the fixed points are obtained in regions with index ± 1, the results are robust with respect to small (continuous) perturbations of the map Ψ.

A special case in which condition $(BT1)$ holds is when $\Psi(\alpha) \in D(r)$ and $\Psi(\beta) \in \mathbb{R}^2 \setminus D[R]$, namely, the annulus A, under the action of the map Ψ, is not only twisted, but also strongly stretched, in the sense that there is a portion of the annulus around the curve α which is pulled inward near the origin inside the disc $D(r)$, while there is a portion of the annulus around the curve β which is pushed outside the disc $D[R]$. This special situation where a strong bend and twist occur is reminiscent of the geometry of the *Smale horseshoe maps* [36, 48] and, indeed, we will show how to enter in a variant of the theory of *topological horseshoes* in the sense of Kennedy and Yorke [27]. To this aim, we recall a few definitions which are useful for the present setting. By a *topological rectangle* we mean a subset $\mathscr{R}$ of the plane which is homeomorphic to the unit square. Given an arbitrary topological rectangle $\mathscr{R}$ we can define an orientation, by selecting two disjoint compact arcs on its boundary. The union of these arcs is denoted by $\mathscr{R}^-$ and the pair $\widehat{\mathscr{R}} := (\mathscr{R}, \mathscr{R}^-)$ is called an *oriented rectangle*. Usually the two components of $\mathscr{R}^-$ are labelled as the left and the right sides of $\widehat{\mathscr{R}}$. Given two oriented rectangles $\widehat{\mathscr{A}}$, $\widehat{\mathscr{B}}$, a continuous map Ψ and a compact set $H \subseteq \mathrm{dom}(\Psi) \cap \mathscr{A}$, the notation $(H, \Psi) : \widehat{\mathscr{A}} \Leftrightarrow\!\!\!\!\rightarrow \widehat{\mathscr{B}}$ means that the following "stretching along the paths" (SAP) property is satisfied: *any path γ, contained in $\mathscr{A}$ and joining the opposite sides of $\mathscr{A}^-$, contains a sub-path σ in H such that the image of σ through Ψ is a path contained in $\mathscr{B}$ which connects the opposite sides of $\mathscr{B}^-$*. We also write $\Psi : \widehat{\mathscr{A}} \Leftrightarrow\!\!\!\!\rightarrow \widehat{\mathscr{B}}$ when $H = \mathscr{A}$. By a path γ we mean a continuous map defined on a compact interval. When, loosely speaking, we say that a path is contained in a given set we actually refer to its image $\bar{\gamma}$. Sometimes it will be useful to consider a relation of the form $\Psi : \widehat{\mathscr{A}} \Leftrightarrow\!\!\!\!\rightarrow^k \widehat{\mathscr{B}}$, for $k \geq 2$ a positive integer, which means that there are at least k compact subsets $H_1, \ldots, H_k$ of $\mathscr{A}$ such that $(H_i, \Psi) : \widehat{\mathscr{A}} \Leftrightarrow\!\!\!\!\rightarrow \widehat{\mathscr{B}}$ for all $i = 1, \ldots, k$. From the results in [40, 41] we have that Ψ has a fixed point in H whenever $(H, \Psi) : \widehat{\mathscr{R}} \Leftrightarrow\!\!\!\!\rightarrow \widehat{\mathscr{R}}$. If for a rectangle $\mathscr{R}$ we have that $\Psi : \widehat{\mathscr{R}} \Leftrightarrow\!\!\!\!\rightarrow^k \widehat{\mathscr{R}}$, for $k \geq 2$, then Ψ has at least k fixed points in $\mathscr{R}$. In this latter situation, one can also prove the presence of chaotic-like dynamics of coin-tossing type (this will be briefly discussed later).

The aim of this paper is to analyze, under these premises, the second order scalar equation of Duffing type

$$x'' + q(t)g(x) = 0 \qquad (DE)$$

with $q(t)$ being a periodic sign-changing weight. The prototypical nonlinearity we consider is a function which changes sign at zero and is bounded only on one-side, such as $g(x) = -1 + \exp(x)$. We prove the presence of periodic solutions coming in pairs (Theorem 4 in Sect. 2, following the Poincaré-Birkhoff theorem) or coming in quadruplets (Theorem 5 in Sect. 2, following bend-twist maps and SAP techniques), the latter depending on the intensity of the negative part of $q(\cdot)$.

2 Statement of the Main Results

We express (DE) as a sign-indefinite nonlinear first order planar systems of the form

$$x' = y, \qquad y' = -a_{\lambda,\mu}(t)g(x). \tag{2}$$

Throughout the article, we suppose that $g : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function satisfying the following assumptions:

$$g(0) = 0, \quad g(x)x > 0 \ \text{ for all } x \neq 0, \quad g_0 := \liminf_{|x|\to 0} \frac{g(x)}{x} > 0. \tag{C_0}$$

We also suppose that *at least one* of the two following conditions holds:

$$(g_-) \quad g \text{ is bounded on } \ \mathbb{R}^-, \qquad (g_+) \quad g \text{ is bounded on } \ \mathbb{R}^+.$$

The weight function $q(t) := a_{\lambda,\mu}(t)$ is defined starting from a T-periodic sign-changing map $a : \mathbb{R} \to \mathbb{R}$ by setting

$$a_{\lambda,\mu}(t) = \lambda a^+(t) - \mu a^-(t), \qquad \lambda, \mu > 0,$$

where $a^+ := (a + |a|)/2$ is the positive part of $a(\cdot)$ and $a^- := a^+ - a$ is the negative one. Given an interval I, we denote by $a \succ 0$ on I the condition $a(t) \geq 0$ for almost every $t \in I$ with $a > 0$ on a subset of I of positive measure. Similarly, $a \prec 0$ on I means that $-a \succ 0$ on I. We suppose that, in a period, the weight function $a(t)$ displays one positive hump followed by one negative hump, i.e. there are t_0 and $T_1 \in]0, T[$ such that

$$a \succ 0 \quad \text{on } \ [t_0, t_0 + T_1] \quad \text{and} \quad a \prec 0 \quad \text{on } \ [t_0 + T_1, t_0 + T].$$

Due to the T-periodicity of the weight function, it is not restrictive to take $t_0 = 0$ and we shall assume it for the rest of the paper. As for the regularity of the weight function, we suppose that $a(\cdot)$ is continuous or piecewise-continuous (more general Carathéodory assumptions could be considered, too).

We consider the Poincaré map associated with system (2), namely

$$\Phi_{t_0}^{t}(z) := (x(t; t_0, z), y(t; t_0, z))$$

where $(x(\cdot\,; t_0, z), y(\cdot\,; t_0, z))$ is the solution of (2) satisfying the initial condition $z = (x(t_0), y(t_0))$ and set $\Phi(z) := \Phi_0^T(z)$. Since (2) has a Hamiltonian structure, the associated Poincaré map is an area-preserving homeomorphism, defined on a open set $\Omega := \text{dom}\Phi \subseteq \mathbb{R}^2$, with $(0, 0) \in \Omega$. In view of the Introduction, a possible method to prove the existence (and multiplicity) of T-periodic solutions makes use of the Poincaré-Birkhoff theorem. Accordingly, we look for a suitable annulus around the origin with radii $0 < r_0 < R_0$ such that for some $\mathfrak{a} < \mathfrak{b}$ the twist condition

$$\text{rot}_z(T) > \mathfrak{b} \;\; \forall z : ||z|| = r_0, \quad \text{rot}_z(T) < \mathfrak{a} \;\; \forall z : ||z|| = R_0 \qquad (TC)$$

holds, where $\text{rot}_z(T)$ is the rotation number on the interval $[0, T]$ associated with $z \in \mathbb{R}^2 \setminus \{(0, 0)\}$. In this setting, a standard definition of the rotation number is given by $\text{rot}_z(T) := \text{rot}_z(0, T)$, where

$$\text{rot}_z(t_1, t_2) := \frac{1}{2\pi} \int_{t_1}^{t_2} \frac{y(t)^2 + a_{\lambda,\mu}(t)x(t)g(x(t))}{x^2(t) + y^2(t)} dt, \tag{3}$$

being $(x(t), y(t))$ the solution of (2) with $(x(t_1), y(t_1)) = z \neq (0, 0)$. Notice that in (3) the angular displacement is positive when the rotations around the origin are performed in the clockwise sense.

Under these assumptions, the Poincaré-Birkhoff theorem, in the version of [46, Corollary 2], guarantees that for each integer $j \in [\mathfrak{a}, \mathfrak{b}]$, there exist *at least two* T-periodic solutions of system (2), having j as associated rotation number. It turns out that these solutions have precisely $2j$ simple transversal crossings with the y-axis in the interval $[0, T[$ (see, for instance, [30, Theorem A]). Equivalently, for any periodic solution $(x(t), y(t))$, we have that x has precisely $2j$ simple zeros in the interval $[0, T[$.

We stress that, to apply this approach, the Poincaré map must be well defined on the annulus—actually, on the whole closed disc $D[R_0]$, that is $D[R_0] \subseteq \Omega$. As shown in [13], for the superlinear equation $x'' + q(t)x^{2n+1} = 0$ (with $n \geq 1$), even for a positive weight $q(t)$ the global existence of the trajectories is not guaranteed, due to the presence of solutions which blow-up in finite time with infinitely many winds around the origin. In our case, the boundedness assumption at infinity, given by one among (g_-) or (g_+), prevents such highly oscillatory phenomenon and guarantees the continuability on $[0, T_1]$. In the time intervals where the weight function is negative, we cannot prevent blow-up phenomena (see [8]) unless we impose some growth restrictions on the vector field (for instance, assuming both (g_-) and (g_+)).

At this point, if we are willing to assume the global continuability for the solutions of (2), the following result can be stated.

Theorem 4. *Assume* (C_0) *and* (g_-) *or* (g_+)*. Then, for each positive integer* k*, there exists* $\Lambda_k > 0$ *such that for each* $\lambda > \Lambda_k$ *and* $j = 1, \ldots, k$*, the equation* (DE) *has at least* two T*-periodic solutions having exactly* $2j$*-zeros in the interval* $[0, T[$.

Notice that no condition on the parameter $\mu > 0$ is required. On the other hand, we are forced to suppose the global continuability of the solutions. The next result overcomes the difficulties related to the Poincaré-Birkhoff approach, by using a different fixed point theorem which requires μ to be sufficiently large.

Theorem 5. *Assume* (C_0) *and* (g_-) *or* (g_+)*. Then, for each positive integer* k*, there exists* $\Lambda_k > 0$ *such that for each* $\lambda > \Lambda_k$ *there exists* $\mu^* = \mu^*(\lambda)$ *such that for each* $\mu > \mu^*$ *and* $j = 1, \ldots, k$*, the equation* (DE) *has at least* four T*-periodic solutions having exactly* $2j$*-zeros in the interval* $[0, T[$.

In [19] the general proofs of Theorem 4 and Theorem 5 are given directly for a class of planar systems including (2). Theorem 4 is related to a previous work by Boscaggin [5], dealing with subharmonic solutions. Concerning Theorem 5, we propose in the next section a different proof in the special case of a stepwise weight function. The simplified form of the weight allows us to display the geometric features of the problem and to provide more detailed information on the distribution of the zeros.

3 Proofs. A Simplified Geometric Framework

We focus on the particular case in which $g : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function satisfying (C_0) along with (g_-). A possible choice could be $g(x) = e^x - 1$, but we stress that we do not ask for g to be unbounded on $\mathbb{R}^+$. Recalling the choice of $q(t)$, we rewrite (DE) as

$$x'' + a_{\lambda,\mu}(t)g(x) = 0. \tag{4}$$

In order to illustrate quantitatively the main ideas of the proof we choose a stepwise T-periodic function $a(\cdot)$ which takes value $a(t) = 1$ on an interval of length T_1 and value $a(t) = -1$ on a subsequent interval of length $T_2 = T - T_1$, so that $a_{\lambda,\mu}$ is defined as

$$a_{\lambda,\mu}(t) = \begin{cases} \lambda & \text{for } t \in [0, T_1[\\ -\mu & \text{for } t \in [T_1, T_1 + T_2[\end{cases} \qquad T_1 + T_2 = T. \tag{5}$$

With this particular choice of $a(t)$, the planar system associated with (4) turns out to be a periodic switched system [2]. Such kind of systems are widely studied in control theory.

For our analysis we first take into account the interval of positivity for the weight, where (2) becomes

$$x' = y, \qquad y' = -\lambda g(x). \tag{6}$$

For this system the origin is a local center, which is global if $\mathscr{G}(x) \to +\infty$ as $x \to \pm\infty$, where $\mathscr{G}(x)$ is the primitive of $g(x)$ such that $\mathscr{G}(0) = 0$. The associated energy function is given by

$$E_1(x, y) := \frac{1}{2}y^2 + \lambda\mathscr{G}(x).$$

For any constant c with $0 < c < \min\{\mathscr{G}(-\infty), \mathscr{G}(+\infty)\}$, the level line of (6) of positive energy λc is a closed orbit Γ which intersects the x-axis in the phase-plane at two points $(x_-, 0)$ and $(x_+, 0)$ such that $x_- < 0 < x_+$, and $c := \mathscr{G}(x_-) = \mathscr{G}(x_+) > 0$. We call $\tau(c)$ the period of Γ, which is given by

$$\tau(c) = \tau^+(c) + \tau^-(c),$$

where

$$\tau^+(c) := \sqrt{\frac{2}{\lambda}} \int_0^{x_+} \frac{d\xi}{\sqrt{(c - \mathscr{G}(\xi))}}, \quad \tau^-(c) := \sqrt{\frac{2}{\lambda}} \int_{x_-}^{0} \frac{d\xi}{\sqrt{(c - \mathscr{G}(\xi))}}$$

The maps $c \mapsto \tau^{\pm}(c)$ are continuous. To proceed with our discussion, we suppose that $\mathscr{G}(-\infty) \leq \mathscr{G}(+\infty)$ (the other situation can be treated symmetrically). Then $\tau^-(c) \to +\infty$ as $c \to \mathscr{G}(-\infty)$ (this follows from the fact that $g(x)/x$ goes to zero as $x \to -\infty$, see [38]). We can couple this result with an estimate near the origin

$$\limsup_{c \to 0^+} \tau(c) \leq 2\pi / \sqrt{\lambda g_0}$$

which follows from classical and elementary arguments.

Proposition 1. *For each* $\lambda > 0$*, the time-mapping* τ *associated with system* (6) *is continuous and its range includes the interval* $]2\pi/\sqrt{\lambda g_0}, +\infty[$.

Showing the monotonicity of the whole time-map $\tau(c)$ is, in general, a difficult task. However, for the exponential case $g(x) = e^x - 1$ this has been proved in [11] (see also [10]).

On the interval of negativity of $a_{\lambda,\mu}(t)$, system (2) becomes

$$x' = y, \qquad y' = \mu g(x), \tag{7}$$

with $g(x)$ as above. For this system the origin is a global saddle with unbounded stable and unstable manifolds contained in the zero level set of the energy

$$E_2(x, y) := \frac{1}{2} y^2 - \mu \mathscr{G}(x).$$

If we start from a point $(0, y_0)$ with $y_0 > 0$ we can explicitly evaluate the blow-up time as follows. First of all we compute the time needed to reach the level $x = \kappa > 0$ along the trajectory of (7), which is the curve of fixed energy $E_2(x, y) = E_2(0, y_0)$ with $y > 0$. Equivalently, we have

$$y = x' = \sqrt{y_0^2 + 2\mu \mathscr{G}(x)}$$

from which

$$t = \int_0^{\kappa} \frac{dx}{\sqrt{y_0^2 + 2\mu \mathscr{G}(x)}}$$

follows. Therefore, the blow-up time is given by

$$T(y_0) = \int_0^{+\infty} \frac{dx}{\sqrt{y_0^2 + 2\mu \mathscr{G}(x)}}.$$

Standard theory guarantees that if the Keller-Osserman condition

$$\int^{+\infty} \frac{dx}{\sqrt{\mathscr{G}(x)}} < +\infty \tag{8}$$

holds, then the blow-up time is always finite and $T(y_0) \searrow 0$ for $y_0 \nearrow +\infty$. On the other hand, $T(y_0) \nearrow +\infty$ for $y_0 \searrow 0^+$. Hence there exists $\bar{y} > 0$ such that $T(y_0) > T_2$ for $y_0 \in]0, \bar{y}[$ and hence there is no blow-up in $[T_1, T]$.

If we start with null derivative, i.e. from a point $(x_0, 0)$, then similar calculations return

$$t = \int_{x_0}^{\kappa} \frac{dx}{\sqrt{2\mu(\mathscr{G}(x) - \mathscr{G}(x_0))}}$$

and, since $\mathscr{G}(x) - \mathscr{G}(x_0) \sim g(x_0)(x - x_0)$ for $|x - x_0| \ll 1$, the improper integral at x_0 is finite. Therefore, the blow-up time is given by

$$T(x_0) = \int_{x_0}^{+\infty} \frac{dx}{\sqrt{2\mu(\mathscr{G}(x) - \mathscr{G}(x_0))}}.$$

If (8) is satisfied, then the blow-up time is always finite. Moreover, $T(x_0) \to +\infty$ as $x_0 \to 0^+$. A similar but more refined result can be found in [39, Lemma 3].

Now we describe how to obtain Theorem 4 and Theorem 5 for system (2) in the special case of a T-periodic stepwise function as in (5). As we already observed, due to the special form of the weight function, equation (2) is a periodic switched system and therefore its associated Poincaré map Φ on the interval $[0, T]$ splits as $\Phi = \Phi_2 \circ \Phi_1$ where Φ_1 is the Poincaré map on the interval $[0, T_1]$ associated with system (6) and Φ_2 is the Poincaré map on the interval $[0, T_2]$ associated with system (7).

(I). Proof of Theorem 4 for the Stepwise Weight

Proof. We start by selecting a closed orbit Γ^0 near the origin of (6) at a level energy λc_0 and fix λ sufficiently large, say $\lambda > \Lambda_k$, so that in view of Proposition 1

$$\tau(c_0) < \frac{T_1}{k+1}. \tag{9}$$

Next, for the given (fixed) λ, we consider a second energy level λc_1 with $c_1 > c_0$ such that

$$\tau^-(c_1) > 2T_2 \tag{10}$$

and denote by Γ^1 the corresponding closed orbit. Let also

$$\mathscr{A} := \{(x, y) : 2\lambda c_0 \le y^2 + 2\lambda\mathscr{G}(x) \le 2\lambda c_1\}$$

be the planar annular region enclosed between Γ^0 and Γ^1. If we assume that the Poincaré map Φ_2 is defined on $\mathscr{A}$, then the complete Poincaré map Φ associated with system (2) is a well defined area-preserving homeomorphism of the annulus $\mathscr{A}$ onto its image $\Phi(\mathscr{A}) = \Phi_2(\mathscr{A})$. In fact the annulus is invariant under the action of Φ_1.

During the time interval $[0, T_1]$, each point $z \in \Gamma^0$ performs $\lfloor T_1/\tau(c_0) \rfloor$ complete turns around the origin in the clockwise sense. This implies that

$$\mathrm{rot}_z(0, T_1) \geq \left\lfloor \frac{T_1}{\tau(c_0)} \right\rfloor, \quad \forall z \in \Gamma^0.$$

On the other hand, from [6, Lemma 3.1] we know that

$$\mathrm{rot}_z(T_1, T) = \mathrm{rot}_z(0, T_2) > -\frac{1}{2}, \quad \forall z \neq (0, 0).$$

We conclude that $\mathrm{rot}_z(T) > k$, for all $z \in \Gamma^0$.

During the time interval $[0, T_1]$, each point $z \in \Gamma^1$ is unable to complete a full revolution around the origin, because the time needed to cross either the second or the third quadrant is larger than T_1. Using this information in connection to the fact that the first and the third quadrants are positively invariant for the flow associated with (7), we find that $\mathrm{rot}_z(T) < 1$, for all $z \in \Gamma^1$.

Thus we have condition (TC) matched with $\mathfrak{b} = k$ and $\mathfrak{a} = 1$. An application of the Poincaré-Birkhoff fixed point theorem [46] (this time for a topological annulus with strictly star-shaped boundaries) guarantees for each $j = 1, \ldots, k$ the existence of at least two fixed points $u_j = (u_x^j, u_y^j)$, $v_j = (v_x^j, v_y^j)$ of the Poincaré map, with u_j, v_j in the interior of $\mathscr{A}$ and such that $\mathrm{rot}_{u_j}(T) = \mathrm{rot}_{v_j}(T) = j$. This in turns implies the existence of at least two T-periodic solutions of Eq. (4) with $x(\cdot)$ having exactly $2j$-zeros in the interval $[0, T]$. □

In this manner, we have proved Theorem 4 for system (2) in the special case of a stepwise weight function $a_{\lambda,\mu}$ as in (5). Notice that no assumption on $\mu > 0$ is required. On the other hand, we have to suppose that Φ_2 is globally defined on $\mathscr{A}$.

Remark 1. From (9) and the formulas for the period τ it is clear that assuming T_1 fixed and λ large is equivalent to suppose λ fixed and T_1 large. This also follows from general considerations concerning the fact that equation $x'' + \lambda g(x) = 0$ is equivalent to $u'' + \varepsilon^2 \lambda g(u) = 0$ for $u(\xi) := x(\varepsilon \xi)$. ◁

(II). An Intermediate Step

Now we show how to improve the previous result if we add the condition that μ is sufficiently large. First of all, we take Γ^0 and Γ^1 as before and $\lambda > \Lambda_k$ in order to produce the desired twist for Φ at the boundary of $\mathscr{A}$. Then we observe that the derivative of the energy E_1 along the trajectories of system (7) is given by $(\lambda + \mu) y g(x)$, so it increases on the first and the third quadrant and decreases on the second and the fourth. Hence, if μ is sufficiently large, we can find four arcs $\varphi_i \subseteq \mathscr{A}$, each one in the open i-th quadrant, with φ_i joining Γ^0 and Γ^1 such that $\Phi_2(\varphi_i)$ is outside the region bounded by Γ^1 for $i = 1, 3$ and $\Phi_2(\varphi_i)$ is inside the region bounded by Γ^0 for $i = 2, 4$. The corresponding position of $\mathscr{A}$ and $\Phi_2(\mathscr{A})$ is illustrated in Fig. 1.

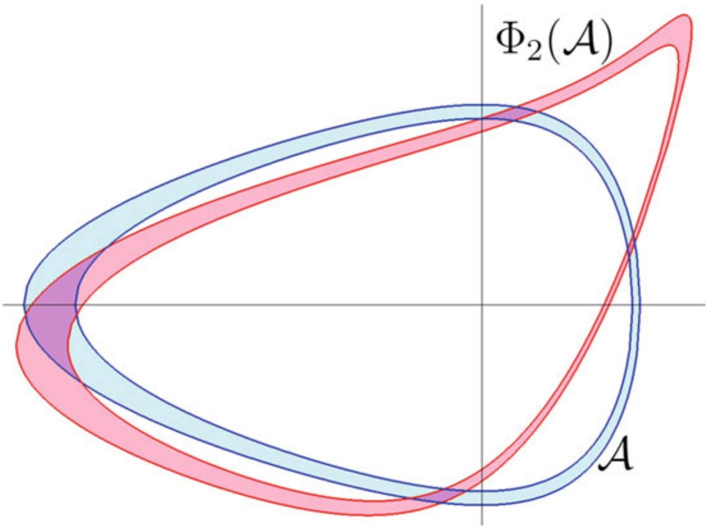

Fig. 1 A possible configuration of $\mathscr{A}$ and $\Phi_2(\mathscr{A})$. The example is obtained for $g(x) = -1 + \exp x$, $\lambda = \mu = 0.1$ and $T_2 = 1$. The inner and outer boundary Γ^0 and Γ^1 of the annulus $\mathscr{A}$ are the energy level lines $E_1(x, y) = E_1(2, 0)$ and $E_1(x, y) = E_1(2.1, 0)$. To produce this geometry, the value of T_1 is not relevant because the annulus is invariant for system (6). Since $\tau(c_0) < \tau(c_1)$, to have a desired twist condition, we need to assume T_1 large enough

At this point, we enter in the setting of bend-twist maps. The arcs $\Phi_1^{-1}(\varphi_i)$ divide $\mathscr{A}$ into four regions, homeomorphic to rectangles. The boundary of each of these regions can be split into two opposite sides contained in Γ^0 and Γ^1 and two other opposite sides made by $\Phi_1^{-1}(\varphi_i)$ and $\Phi_1^{-1}(\varphi_{i+1})$ (in cyclic order). On Γ^0 and Γ^1 we have the previously proved twist condition on the rotation numbers, while on the other two sides we have $E_1(\Phi(P)) > E_1(P)$ for $P \in \Phi_1^{-1}(\varphi_i)$ with $i = 1, 3$ and $E_1(\Phi(P)) < E_1(P)$ for $P \in \Phi_1^{-1}(\varphi_i)$ with $i = 2, 4$. Thus, using the Poincaré-Miranda theorem, we obtain the existence of at least one fixed point of the Poincaré map Φ in the interior of each of these regions. In this manner, under an additional hypothesis of the form $\mu > \mu^*(\lambda)$, we improve Theorem 4 (for system (2) and again in the special case of a stepwise weight), finding at least four solutions with a given rotation number j for $j = 1, \dots, k$. On the other hand, we still suppose that Φ_2 is globally defined on $\mathscr{A}$. The version of the bend-twist map theorem that we apply here is robust for small perturbations of the Poincaré map, therefore the result holds also for some non-Hamiltonian systems whose vector field is close to that of (4). □

(III). Proof of Theorem 5 for the Stepwise Weight

Proof. First of all, we start with the same construction as in **(I)** and choose Γ^0, $\lambda > \Lambda_k$ according to (9) and Γ^1 so that (10) is satisfied. Consistently with the previously introduced notation, we take

$$x_-^1 < x_-^0 < 0 < x_+^0 < x_+^1, \quad \text{with } \mathscr{G}(x_-^i) = c_i = \mathscr{G}(x_+^i),\ i = 0, 1.$$

Notice that the closed curves Γ^i intersect the coordinate axes at the points $(x^i_{\pm}, 0)$ and $(0, \pm\sqrt{2\lambda c_i})$. Next we choose $x^{\mu}_{\pm}$ and y_0 with

$$x^0_- < x^{\mu}_- < 0 < x^{\mu}_+ < x^0_+ , \quad \text{and } 0 < y_0 < \sqrt{2\lambda c_0}$$

and define the orbits

$$\mathscr{X}_{\pm} := \gamma(x^{\mu}_{\pm}, 0), \quad \mathscr{Y}_{\pm} := \gamma(0, \pm y_0),$$

where we denote by $\gamma(P)$ the complete orbit of the system passing through the point $P \in \mathbb{R}^2$.

Setting

$$\mathscr{T}(\mathscr{X}_{\pm}) := \pm 2 \int_{x^{\mu}_{\pm}}^{x^1_{\pm}} \frac{dx}{\sqrt{2\mu(\mathscr{G}(x) - \mathscr{G}(x^{\mu}_{\pm}))}} , \quad \mathscr{T}(\mathscr{Y}) := \int_{x^1_-}^{x^1_+} \frac{dx}{\sqrt{y_0^2 + 2\mu\mathscr{G}(x)}}$$

we tune the values $x^{\mu}_{\pm}$, y_0 and μ so that

$$\max\{\mathscr{T}(\mathscr{X}_{\pm}), \mathscr{T}(\mathscr{Y})\} < T_2.$$

Clearly, given the other parameters, we can always choose μ sufficiently large, say $\mu > \mu^*$, so that the above condition is satisfied.

Finally, we introduce the stable and unstable manifolds, W^s and W^u, for the origin as saddle point of system (7). More precisely, we define the sets

$$W^s_+ := \{(x, y) : E_2(x, y) = 0, x > 0, y < 0\}, \; W^s_- := \{(x, y) : E_2(x, y) = 0, x < 0, y > 0\},$$

$$W^u_+ := \{(x, y) : E_2(x, y) = 0, x > 0, y > 0\}, \; W^u_- := \{(x, y) : E_2(x, y) = 0, x < 0, y < 0\},$$

so that $W^s = W^s_- \cup W^s_+$ and $W^u = W^u_- \cup W^u_+$. The resulting configuration is illustrated in Fig. 2.

The closed trajectories Γ^0, Γ^1 together with $\mathscr{X}_{\pm}$, $\mathscr{Y}_{\pm}$, $W^s_{\pm}$ and $W^u_{\pm}$ determine eight regions that we denote by $\mathscr{A}_i$ and $\mathscr{B}_i$ for $i = 1, \dots, 4$, as in Fig. 3.

Each of the regions $\mathscr{A}_i$ and $\mathscr{B}_i$ is homeomorphic to the unit square and thus is a topological rectangle. In this setting, we give an orientation to $\mathscr{A}_i$ by choosing $\mathscr{A}_i^- := \mathscr{A}_i \cap (\Gamma^0 \cup \Gamma^1)$. We take as $\mathscr{B}_i^-$ the closure of $\partial\mathscr{B}_i \setminus (\Gamma^0 \cup \Gamma^1)$.

We can now apply a result in the framework of the theory of topological horseshoes as presented in [42] and [31]. Indeed, by the previous choice of $\lambda > \Lambda_k$ we obtain that

$$\Phi_1 : \widehat{\mathscr{A}_i} \Longleftrightarrow\!\!\!\!\!\!\!\longrightarrow^k \widehat{\mathscr{B}_i} , \quad \forall i = 1, \dots, 4,$$

On the other hand, from $\mu > \mu^*$ it follows that

$$\Phi_2 : \widehat{\mathscr{B}_i} \Longleftrightarrow\!\!\!\!\!\!\!\longrightarrow \widehat{\mathscr{A}_i} , \quad \forall i = 1, \dots, 4.$$

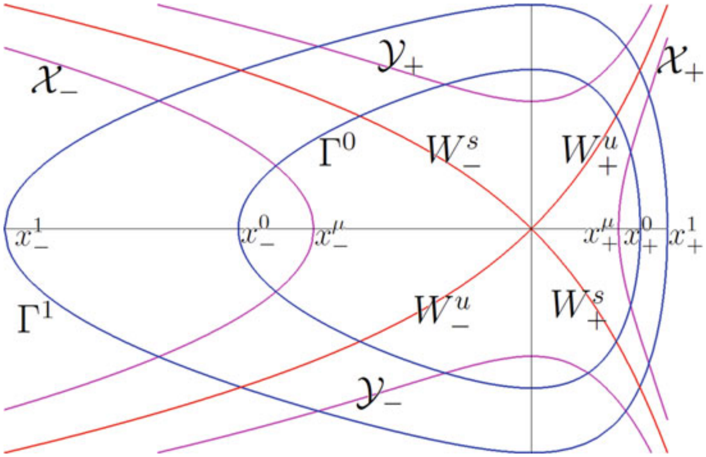

Fig. 2 The present figure shows the appropriate overlapping of the phase-portraits of systems (6) and (7)

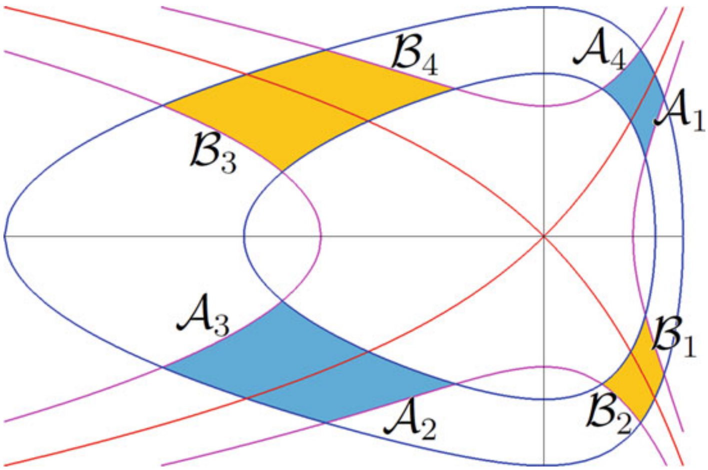

Fig. 3 The present figure shows the regions $\mathscr{A}_i$ and $\mathscr{B}_i$. We have labelled the regions following a clockwise order, which is useful from the point of view of the dynamics

Then [42, Theorem 3.1] (see also [31, Theorem 2.1]) ensures the existence of at least k fixed points for $\Phi = \Phi_2 \circ \Phi_1$ in each of the regions $\mathscr{A}_i$. This, in turns, implies the existence of $4k$ T-periodic solutions for system (2).

Such solutions are topologically different and can be classified, as follows: for each $j = 1, \dots, k$ there is a solution (x, y) with

- $(x(0), y(0)) \in \mathscr{A}_1$ with $x(t)$ having $2j$ zeros in $]0, T_1[$ and strictly positive in $[T_1, T]$;
- $(x(0), y(0)) \in \mathscr{A}_2$ with $x(t)$ having $2j - 1$ zeros in $]0, T_1[$ and one zero in $]T_1, T[$;

- $(x(0), y(0)) \in \mathscr{A}_3$ with $x(t)$ having $2j$ zeros in $]0, T_1[$ and strictly negative in $[T_1, T]$;
- $(x(0), y(0)) \in \mathscr{A}_4$ with $x(t)$ having $2j - 1$ zeros in $]0, T_1[$ and one zero in $]T_1, T[$.

In conclusion, for each $j = 1, \dots, k$ we find at least four T-periodic solutions having precisely $2j$-zeros in $[0, T[$. □

Remark 2. Having assumed that g is bounded on $\mathbb{R}^-$, we can also prove the existence of a T-periodic solution with $(x(0), y(0)) \in \mathscr{A}_3$ and such that $x(t) < 0$ for all $t \in [0, T]$ while $y(t) = x'(t)$ has two zeros in $[0, T[$. Moreover, the results from [31, 42] guarantee also that each of the regions $\mathscr{A}_i$ contains a compact invariant set where Φ is chaotic in the sense of Block and Coppel (see [1, 34]). At last, we also mention that the result (from Theorem 5) is robust with respect to small perturbations. In particular, it applies to a perturbed Hamiltonian system of the form

$$x' = y + F_1(t, x, y, \varepsilon), \quad y' = -a_{\lambda,\mu} g(x) + F_2(t, x, y, \varepsilon) \tag{11}$$

with $F_1, F_2 \to 0$ as $\varepsilon \to 0$, uniformly in t, and for (x, y) on compact sets. Observe that system (11) has not necessarily a Hamiltonian structure and therefore it is no more guaranteed that the associated Poincaré map is area-preserving. ◁

Remark 3. We further observe that, for Eq. (4) the same results hold if condition (g_-) is relaxed to

$$\lim_{x \to -\infty} \frac{g(x)}{x} = 0. \tag{12}$$

Under the same condition at infinity, four T-periodic solutions are obtained also in [6]. However, we stress that, the assumptions at the origin are completely different. Indeed, in [6] a one-sided superlinear condition in zero, of the form $g'(0^+) = 0$ or $g'(0^-) = 0$ was required. As a consequence, for λ large, one could prove the existence of four (or $4k$) T-periodic solutions with prescribed nodal properties which come in pair, namely two "small" and two "large". In our case, if in place of $g_0 > 0$ we assume $g'(0^+) = 0$ or $g'(0^-) = 0$, with the same approach we could prove the existence of eight (or $8k$) T-periodic solutions, four "small" and four "large". ◁

Remark 4. We conclude this note by observing that if we want to produce the same results for non-autonomous perturbations of the more general system

$$x' = h(y), \qquad y' = -g(x), \tag{13}$$

then we cannot replace (g_-) (or (g_+) with a weaker condition of the form of (12). In fact, a crucial step in our proof is to have a twist condition, that is a gap in the period between a fast orbit (like Γ^0) and slow one (like Γ^1). This is no more guaranteed for an autonomous system of the form (13) if $g(x)$ satisfies a sublinear condition at infinity as (12). Indeed, the slow decay of g at infinity could be compensated by a fast growth of h at infinity. In [12] the Authors provide examples of isochronous centers for planar Hamiltonian systems even in the case when one of the two components is sublinear at infinity. See [19] for perturbations of system (13) with a periodic sign-changing weight on the second equation. ◁

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