Chapter 3
Regularity and Inviscid Limits in Hydrodynamic Models

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Abstract We discuss the vanishing viscosity limit and low regularity bounds, uniform in viscosity, for vorticity in Yudovich class in 2D. We also show that multiscale steady solutions of Navier–Stokes equations with power law energy spectrum, including K41, can be constructed in any domain in 3D

3.1 Introduction

The three-dimensional incompressible Navier–Stokes equations are the basic equations of mathematical fluid mechanics. The equations

$$\partial_t u + u \cdot \nabla u + \nabla p - \nu \Delta u = f, \tag{3.1.1}$$

with the incompressibility constraint

$$\nabla \cdot u = 0, \tag{3.1.2}$$

describe the motion of a fluid of uniform density (taken above to be identically 1), with velocity $u = u(x, t) \in \mathbb{R}^d$ with $x \in \mathbb{R}^d$, $t \geq 0$, in $d = 2$ or $d = 3$ dimensions. The scalar unknown $p = p(x, t)$ represents the hydrodynamic pressure, arising in response to the constraint of incompressibility (3.1.2). The positive number ν represents the kinematic viscosity, and f are body forces.

The Euler equations,

$$\partial_t u + u \cdot \nabla u + \nabla p = f, \tag{3.1.3}$$

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together with the incompressibility condition (3.1.2) are obtained by formally setting $\nu = 0$ in the Navier–Stokes equations. The pressure enforces the incompressibility condition, and if the forces are divergence-free, the pressure must satisfy

$$-\Delta p = \nabla \cdot (u \cdot \nabla u)\,. \tag{3.1.4}$$

The subject of these lectures is motivated by questions arising in turbulence, one of the greatest challenges in physics. A law of turbulence states that the average rate of dissipation of kinetic energy per unit mass does not vanish in the limit of infinite Reynolds numbers.

$$-\lim_{Re\to\infty} \langle \frac{dE}{dt} \rangle = \epsilon > 0$$

This law is experimentally well verified. Another important law of turbulence theory is the K41 spectrum, or Kolmogorov–Obukhov spectrum,

$$E(k) = C\epsilon^{\frac{2}{3}} k^{-\frac{5}{3}},$$

which states that the energy per wave number k has a universal power law behavior for a range of scales, called the inertial range. This range extends from low wave numbers, where the energy injection typically occurs, to a viscosity dependent cutoff wave number, which converges to infinity in the limit of zero viscosity. This again is very well verified experimentally. The physical literature on the subject is vast. A lucid presentation is given in [1].

The mathematical description of these two laws requires a more precise formulation. The laws are not in any way mathematical statements, and formulations can be given so that they invalid. The more challenging task is to understand why they are observed in nature, and how are they related to the fundamental underlying equations. In these lectures we present negative results, results in which the vanishing viscosity limit is conservative, and results in which non-turbulent Navier–Stokes stationary solutions exhibit power law scaling behavior.

3.2 Inviscid Limit

If we consider the issue of the limit of energy dissipation, we certainly can find cases in which the limit vanishes. These are cases in which the solutions of the Navier–Stokes equations converge to solutions of Euler equations, and the latter are smooth enough to conserve energy. This situation occurs, as it is very well known, if we are considering spatially periodic solutions and solutions of the Euler equations which belong to $H^s(\mathbb{T}^d)$, $s > \frac{d}{2} + 1$ [2, 3].

The difference between solutions vanishes in the inviscid limit, in strong norms, at a rate proportional to the difference between coefficients, that is, linearly with viscosity. This rate changes if we consider less smooth solutions of Euler equations, even in 2D. This was first investigated in [4] and [5] for vortex patches, a class of weak solutions of Euler equations in 2D. We describe below recent results [6] extending the earlier work.

3.2.1 Yudovich Class

We discuss here the connection between Yudovich solutions of the Euler equations [27]

$$\partial_t \omega + u \cdot \nabla \omega = g, \tag{3.2.1}$$

with bounded forcing $g \in L^\infty(0,T; L^\infty(\mathbb{T}^2))$, and initial data

$$\omega(0) = \omega_0 \in L^\infty(\mathbb{T}^2), \tag{3.2.2}$$

and the vanishing viscosity limit ($\lim_{\nu\to 0}$) of solutions of the Navier–Stokes equations,

$$\partial_t \omega^\nu + u^\nu \cdot \nabla \omega^\nu = \nu \Delta \omega^\nu + g, \tag{3.2.3}$$

with initial data

$$\omega^\nu(0) = \omega_0^\nu \in L^\infty(\mathbb{T}^2), \tag{3.2.4}$$

and the same forcing g. We consider uniformly bounded initial data

$$\sup_{\nu>0} \|\omega_0^\nu\|_{L^\infty(\mathbb{T}^2)} \le \Omega_{0,\infty} < \infty. \tag{3.2.5}$$

The solutions of (3.2.1), (3.2.2), (3.2.3), and (3.2.4) are uniformly bounded in $L^\infty(\mathbb{T}^2)$:

$$\sup_{\nu\ge 0} \sup_{0\le t\le T} \|\omega^\nu(t)\|_{L^\infty(\mathbb{T}^2)} \le \Omega_\infty = \Omega_{0,\infty} + \int_0^T \|g(t)\|_{L^\infty(\mathbb{T}^2)} dt. \tag{3.2.6}$$

This bound is valid in $\mathbb{T}^2$ or $\mathbb{R}^2$ but is not available if boundaries are present or in 3D. The bound will be used repeatedly below.

The vorticity distribution function $\pi_{\omega^\nu(t)}(\mathrm{d}y)$ is defined by

$$\int f(y)\pi_{\omega^\nu(t)}(\mathrm{d}y) = \int f(\omega^\nu(t,x))\mathrm{d}x, \tag{3.2.7}$$

for all continuous functions (observables) f. If $\omega_0^\nu \to \omega_0$ we the distributions convergence

$$\pi_{\omega^\nu(t)}(\mathrm{d}y) \xrightarrow{\nu\to 0} \pi_{\omega(t)}(\mathrm{d}y) = \pi_{\omega_0}(\mathrm{d}y), \tag{3.2.8}$$

where the time invariance of the vorticity distribution function for the Euler equations follows from Lagrangian transport $\omega(t) = \omega_0 \circ X_t^{-1}$ and volume preservation of the homeomorphism $A_t = X_t^{-1}$. The statement (3.2.8) is a consequence of the strong convergence of the vorticity in $L^\infty(0, T; L^p(\mathbb{T}^2))$ for all $p \in [1, \infty)$ and for any $T > 0$. This fact was proved in [6], extending previous work for vortex patch solutions with smooth boundary [4], and removing additional assumptions on the Euler path [5]. This result has implications for equilibrium theories [28] of decaying two-dimensional turbulence [7, 8, 29] The result of [6] is:

Theorem 3.1 *Let ω be the unique Yudovich weak solution of the Euler equations with initial data $\omega_0 \in L^\infty(\mathbb{T}^2)$ and forcing $g \in L^\infty(0, T; L^\infty(\mathbb{T}^2))$. Let ω^ν be the solution of the Navier–Stokes equation with the same forcing and initial data $\omega_0^\nu \to \omega_0$ strongly in $L^2(\mathbb{T}^2)$. Then, for any $T > 0$ and $p \in [1, \infty)$, the inviscid limit $\omega^\nu \to \omega$ holds strongly in $L^\infty(0, T; L^p(\mathbb{T}^2))$:*

$$\lim_{\nu\to 0} \sup_{0\le t\le T} \|\omega^\nu(t) - \omega(t)\|_{L^p(\mathbb{T}^2)} = 0. \tag{3.2.9}$$

Consequently, the distributions converge,

$$\lim_{\nu\to 0} \pi_{\omega^\nu(t)}(\mathrm{d}y) = \pi_{\omega_0}(\mathrm{d}y), \tag{3.2.10}$$

for all $t \in [0, T]$.

Remark 3.1 The result is sharp, in several ways. First, there can be no infinite time result as the Euler solution is conservative and the Navier–Stokes solution is dissipative. Secondly, there can be no rate without additional regularity assumptions on ω_0, as is the case for the heat equation. Thirdly, there can be no strong convergence in L^∞ because ω_0 may not be continuous while ω^ν is smooth for any $t > 0$. And, finally there can be no strong convergence for $p > 1$ in domains with boundaries, if the boundary condition of the Navier–Stokes solutions is no slip, and the Euler solution has non-vanishing tangential velocity at the boundary, in other words, if there are boundary layers [9].

The method of proof of Theorem 3.1 yielded also the continuity of the Yudovich solution map $\omega(t) = S(t)(\omega_0)$ in the L^p topology when restricted to fixed balls in L^∞.

Theorem 3.2 *For any $\omega_0, \omega_0^n \in L^\infty(\mathbb{T}^2)$ such that ω_0^n is uniformly bounded in $L^\infty(\mathbb{T}^2)$ and $\omega_0^n \to \omega_0$ as $n \to \infty$ strongly in $L^2(\mathbb{T}^2)$ we have*

$$\lim_{n\to\infty} \|S(t)(\omega_0^n) - S(t)(\omega_0)\|_{L^p(\mathbb{T}^2)} = 0 \tag{3.2.11}$$

for each time $t > 0$.

If additional smoothness is assumed on the data then some degree of fractional smoothness in L^p can be propagated uniformly in viscosity [6]:

Theorem 3.3 *Suppose $\omega_0 \in (L^\infty \cap B^s_{p,\infty})(\mathbb{T}^2)$ for some $s > 0$ and some $p \geq 1$. Then the solutions of the Navier–Stokes equations satisfy $\omega^\nu(t) \in (L^\infty \cap B^{s(t)}_{p,\infty})(\mathbb{T}^2)$ uniformly in ν, where*

$$s(t) = s\exp(-Ct\|\omega_0\|_{L^\infty(\mathbb{T}^2)})$$

for some universal constant $C > 0$.

The proof of Theorem 3.3 relied on the fact that the velocity is log-Lipschitz uniformly in ν and showed that the exponential estimate with loss of [10] holds uniformly in viscosity. The proof given in [6] used the stochastic Lagrangian representation formula of [11]

$$\mathrm{d}X_t(x) = u^\nu(X_t(x), t)\mathrm{d}t + \sqrt{2\nu}\,\mathrm{d}W_t, \qquad X_0(x) = x, \tag{3.2.12}$$

yielding the representation formula

$$\omega^\nu(t) = \mathbb{E}\left[\omega_0 \circ A_t\right] \tag{3.2.13}$$

where back-to-labels map is defined as $A_t = X_t^{-1}$. The noisy Lagrangian picture allowed for adaptation of ideas of [10, 12] to the viscous case. Uniform Sobolev regularity could be established by similar arguments; if $\omega_0 \in (L^\infty \cap W^{s,p})(\mathbb{T}^2)$ then $\omega^\nu(t) \in (L^\infty \cap W^{s(t),p})(\mathbb{T}^2)$ with uniformly bounded norms.

The uniform regularity of Theorem 3.3was used to deduce

Corollary 3.1 *Let $\omega_0 \in (L^\infty \cap B^s_{2,\infty})(\mathbb{T}^2)$ with $s > 0$ and let ω and ω^ν solve respectively (3.2.1) and (3.2.3), with the same initial data $\omega_0^\nu = \omega_0$. Then the L^p convergence of vorticity, for any $p \in [1, \infty)$ and any finite time $T > 0$, occurs at the rate*

$$\sup_{t\in[0,T]} \|\omega^\nu(t) - \omega(t)\|_{L^p(\mathbb{T}^2)} \lesssim (\nu T)^{\frac{s\exp(-2CT\|\omega_0\|_\infty)}{p(1+s\exp(-CT\|\omega_0\|_\infty))}-}, \tag{3.2.14}$$

with the universal constant $C > 0$ in Proposition 3.3.

Corollary 3.1 applies in particular to the to inviscid limits of vortex patches with non-smooth boundary. Indeed, Lemma 3.2 of [5] shows that if $\omega_0 = \chi_\Omega$ is the characteristic function of a bounded domain whose boundary has box-counting (fractal) dimension D not larger than the dimension of space $d = 2$, i.e. $d_F(\partial\Omega) := D < 2$, then $\omega_0 \in B^{(2-D)/p}_{p,\infty}(\mathbb{T}^2)$. Proposition 3.3 then shows that some degree of fractional Besov regularity of the solution $\omega^\nu(t)$ is retained uniformly in viscosity for any finite time $T < \infty$ and Corollary 3.1 provides a rate depending only D, T and p at which the vanishing viscosity limit holds, removing therefore the need for the additional assumptions on the solution imposed in [5].

The proof of Theorem 3.1, adapted from [6], is given below. It is based on a number of properties of Yudovich class solutions, in particular the exponential integrability of gradients and the fact that linear transport by Yudovich solutions has a short time uniformly controlled loss of regularity: it maps bounded sets in $W^{1,p},\ p > 2$ to bounded sets in H^1, uniformly in viscosity.

We give further a proof of a uniform propagation of regularity result, Theorem 3.4, a version of Theorem 3.3 which does not use the stochastic representation.

We start the proof of Theorem 3.1 with the exponential integrability of gradients of velocities obtained via the Biot–Savart law in dimension two.

Lemma 3.1 *Let $\omega \in L^\infty(\mathbb{T}^2)$ and let u be obtained from ω by the Biot–Savart law*

$$u = K[\omega] = \nabla^\perp(\Delta)^{-1}\omega. \tag{3.2.15}$$

There exists a non-dimensional constant $\gamma > 0$ and a constant C_K with units of area such that

$$\int_{\mathbb{T}^2} \exp\{\beta|\nabla u(x)|\}\,\mathrm{d}x \le C_K \tag{3.2.16}$$

holds for any $\beta > 0$ such that

$$\beta\|\omega\|_{L^\infty(\mathbb{T}^2)} \le \gamma. \tag{3.2.17}$$

Proof The bound (3.2.16) holds due to the fact that Calderon–Zygmund operators map L^∞ to BMO [13], $\omega \in L^\infty \mapsto \nabla u = \nabla K[u] \in BMO$, and from the John–Nirenberg inequality [14] for BMO functions. We provide below a direct and elementary argument (modulo a fact about norms of singular integral operators), for the sake of completeness.

We recall that there exists a constant C_* so that for all $p \ge 2$,

$$\|\nabla K[v]\|_{L^p(\mathbb{T}^2)} = \|\nabla \otimes \nabla(-\Delta)^{-1}v\|_{L^p(\mathbb{T}^2)} \le C_* p\|v\|_{L^p(\mathbb{T}^2)}. \tag{3.2.18}$$

(See [13]). The dependence of (3.2.18) on p is the important point. Thus,

$$\begin{aligned}&\int_{\mathbb{T}^2} e^{\beta|\nabla u|}\mathrm{d}x = \sum_{p=0}^{\infty} \beta^p \frac{\|\nabla u\|_{L^p(\mathbb{T}^2)}^p}{p!}\\ &\leq \sum_{p=0}^{\infty} \frac{\left(C_*\beta\|\omega\|_{L^p(\mathbb{T}^2)}\right)^p p^p}{p!} \leq |\mathbb{T}^2| \sum_{p=0}^{\infty} \frac{\left(C_*\beta\|\omega\|_{L^\infty(\mathbb{T}^2)}\right)^p p^p}{p!}.\end{aligned}$$

This is a convergent series provided $C_*\beta\|\omega\|_{L^\infty(\mathbb{T}^2)} < 1/e$. Indeed, this can be seen using Stirling's bound $n! \geq \sqrt{2\pi}n^{n+1/2}e^{-n}$ which yields

$$\sum_{p=0}^{\infty} \frac{c^p p^p}{p!} \leq 1 + \sum_{p=1}^{\infty} \frac{p^{-1/2}}{\sqrt{2\pi}}(ce)^p \leq \frac{1}{1-ce}, \quad \text{provided} \quad c \in [0, 1/e)$$

where $c := C_*\beta\|\omega\|_{L^\infty(\mathbb{T}^2)}$. In (3.2.16) we may take thus

$$\gamma = (2C_*e)^{-1}, \quad C_K = 2\left|\mathbb{T}^2\right|. \tag{3.2.19}$$

The constant γ depends on the Biot–Savart kernel and is non-dimensional, the constant C_K then is proportional to the area of the domain.

The next result establishes strong convergence of the velocity in $L^2(0, T; L^2(\mathbb{T}^2))$. If $g = 0$ and $u_0^\nu = u_0$, this is a consequence of Theorem 1.4 of [15]. Below is a generalization of [15] which applies in our setting and is proved by a different argument.

Lemma 3.2 *Let $\omega_0 \in L^\infty(\mathbb{T}^2)$. There exist constants U, Ω_2 and K (see below (3.2.23), (3.2.24), and (3.2.39)) depending on norms of the initial data and of the forcing such that the difference $v = u^\nu - u$ of velocities of solutions (3.2.1) and (3.2.3) obeys*

$$\|v(t)\|_{L^2}^2 \leq 3U^2 K^{\frac{5(t-t_0)\Omega_\infty}{\gamma}} \left(\frac{\|v(t_0)\|_{L^2(\mathbb{T}^2)}^2}{U^2} + \gamma\frac{\Omega_2^2}{U^2\Omega_\infty}\nu\right)^{1-\frac{5(t-t_0)\Omega_\infty}{\gamma}} \tag{3.2.20}$$

for all $0 \leq t_0 \leq t$. By iterating the above, we obtain

$$\|v(t)\|_{L^2}^2 \leq 20U^2 K^{1-e^{-10t\Omega_\infty/\gamma}} \left(\frac{\|v(0)\|_{L^2(\mathbb{T}^2)}^2}{U^2} + \gamma\frac{\Omega_2^2}{U^2\Omega_\infty}\nu\right)^{e^{-\frac{10t\Omega_\infty}{\gamma}}} \tag{3.2.21}$$

provided that $\|v(0)\|_{L^2(\mathbb{T}^2)}^2 + \gamma\nu\Omega_2^2/\Omega_\infty \leq 9KU^2$.

Remark 3.2 (Continuity of Solution Map) At zero viscosity, Lemma 3.2 establishes Hölder continuity of the Yudovich (velocity) solution map. Specifically, denoting $S(t)(u_0)$ the velocity with initial data u_0 and $\nu = 0$, a consequence of Lemma 3.2

is that $\|S(t)v(u_0) - S(t)(u_0')\|_{L^2(\mathbb{T}^2)} \leq C\|u_0 - u_0'\|_{L^2(\mathbb{T}^2)}^{\alpha(t)}$ where $\alpha(t) := e^{-ct}$ and $c, C > 0$ are appropriate constants. This fact is used to prove Theorem 3.2. It is worth further remarking that the condition on the data $\|v(0)\|_{L^2(\mathbb{T}^2)}^2 \leq 9KU^2$ required for the above estimate to hold is $O(1)$ (data need not be taken very close).

Proof The proof of Lemma 3.2 proceeds in two steps.

Step 1: Short Time Bound The proof of the lemma starts from the equation obeyed by the difference v,

$$\partial_t v + u^\nu \cdot \nabla v + v \cdot \nabla u + \nabla p = \nu\Delta v + \nu\Delta u$$

leading to the inequality

$$\frac{d}{dt}\|v\|_{L^2}^2 + \nu\|\nabla v\|_{L^2}^2 \leq \nu\|\nabla u\|_{L^2}^2 + 2\int |\nabla u||v|^2 \mathrm{d}x \tag{3.2.22}$$

which is a straightforward consequence of the equation, using just integration by parts. We use the bound Ω_∞ (3.2.6) for the vorticity of the Euler solution. We also use a bound for the L^2 norms

$$\sup_{0\leq t\leq T} \left(\|u^\nu(t)\|_{L^2(\mathbb{T}^2)} + \|u(t)\|_{L^2(\mathbb{T}^2)}\right) \leq U, \tag{3.2.23}$$

which is easily obtained from energy balance. We use also bounds for L^p norms of vorticity,

$$\Omega_p = \sup_{0\leq t\leq T} \|\omega(t)\|_{L^p(\mathbb{T}^2)}. \tag{3.2.24}$$

We split the integral

$$\int |\nabla u||v|^2 \mathrm{d}x = \int_B |\nabla u||v|^2 \mathrm{d}x + \int_{\mathbb{T}^2\setminus B} |\nabla u||v|^2 \mathrm{d}x$$

where

$$B = \{x \mid \ |v(x,t)| \geq MU\}$$

with M to be determined below. Although B depends in general on time, it has small measure if M is large,

$$|B| \leq M^{-2}.$$

The constant M has dimensions of inverse length. We bound

$$2\int_B |\nabla u||v|^2 \mathrm{d}x \le 2\|\nabla u\|_{L^2}\|v\|_{L^4}^2 \le 2|B|^{\frac{1}{4}}\|\nabla u\|_{L^4}\|v(t)\|_{L^4}^2 \tag{3.2.25}$$

where we used $\int_B |\nabla u|^2 dx \le |B|^{\frac{1}{2}}\|\nabla u\|_{L^4}^2$. We now use the fact that we are in Yudovich class and Ladyzhenskaya inequality to deduce

$$\|v(t)\|_{L^4}^2 \le C\|v(t)\|_{L^2}[\|\omega_0\|_{L^2} + \|g\|_{L^1(0,T;L^2)}] \le CU\Omega_2$$

and we use also

$$\|\nabla u\|_{L^4} \le [C\|\omega_0\|_{L^4} + \|g\|_{L^1(0,T;L^4)}] = \Omega_4$$

to bound (3.2.25) by

$$2\int_B |\nabla u||v|^2 \mathrm{d}x \le CU\Omega_2\Omega_4 M^{-\frac{1}{2}}, \tag{3.2.26}$$

We non-dimensionalize by dividing by U^2 and we multiply by $\beta = \gamma/\Omega_\infty$. The quantity

$$y(t) = \frac{\|v(t)\|_{L^2(\mathbb{T}^2)}^2}{U^2} \tag{3.2.27}$$

obeys the inequality

$$\beta\frac{\mathrm{d}y}{\mathrm{d}t} \le \beta\nu\frac{\Omega_2^2}{U^2} + C\beta\Omega_4\frac{\Omega_2}{U}M^{-\frac{1}{2}} + 2\int_{\mathbb{T}^2\setminus B} \beta|\nabla u|\frac{|v|^2}{U^2}\mathrm{d}x. \tag{3.2.28}$$

We write the term

$$2\int_{\mathbb{T}^2\setminus B} \beta|\nabla u||v|^2 U^{-2}\mathrm{d}x = 2\int_{\mathbb{T}^2\setminus B} (\beta|\nabla u| + \log\epsilon + \log\frac{1}{\epsilon})|v|^2 U^{-2}\mathrm{d}x \tag{3.2.29}$$

with ϵ (with units of inverse area) to be determined below. We use the inequality (3.2.56) and Lemma 3.1 with

$$a = \beta|\nabla u| + \log\epsilon, \qquad b = \frac{|v|^2}{U^2}$$

to deduce

$$2\int_{\mathbb{T}^2\setminus B} \beta|\nabla u||v|^2 U^{-2}\mathrm{d}x \le 2\epsilon C_K + 2\log\frac{M^2}{\epsilon}y(t). \tag{3.2.30}$$

Inserting (3.2.30) in (3.2.28) we obtain

$$\beta \frac{\mathrm{d}y}{\mathrm{d}t} \le F + \log\left(\frac{M^2}{\epsilon}\right) y(t) \tag{3.2.31}$$

with

$$F = \beta\nu \frac{\Omega_2^2}{U^2} + C\beta\Omega_4 \frac{\Omega_2}{U} M^{-\frac{1}{2}} + 2\epsilon C_K. \tag{3.2.32}$$

Note that F and $\frac{M^2}{\epsilon}$ are non-dimensional. From (3.2.31) we obtain immediately

$$y(t) \le \left(\frac{M^2}{\epsilon}\right)^{\frac{t-t_0}{\beta}} y(t_0) + \frac{F}{\log\left(\frac{M^2}{\epsilon}\right)} \left(\left(\frac{M^2}{\epsilon}\right)^{\frac{t-t_0}{\beta}} - 1\right). \tag{3.2.33}$$

We choose M such that

$$C\beta\Omega_4 \frac{\Omega_2}{U} M^{-\frac{1}{2}} = \beta\nu \frac{\Omega_2^2}{U^2} + y(t_0) \tag{3.2.34}$$

and we choose ϵ such that

$$2\epsilon C_K = \beta\nu \frac{\Omega_2^2}{U^2} + y(t_0). \tag{3.2.35}$$

These choices imply

$$F = 3\beta\nu \frac{\Omega_2^2}{U^2} + 2y(t_0). \tag{3.2.36}$$

Then we see that

$$\Gamma = \frac{M^2}{\epsilon} = 2C_K \left(C\beta\Omega_4 \frac{\Omega_2}{U}\right)^4 \times \left(\beta\nu \frac{\Omega_2^2}{U^2} + y(t_0)\right)^{-5}. \tag{3.2.37}$$

Taking without loss of generality $\log\Gamma \ge 1$, we have from (3.2.33)

$$\begin{aligned} y(t) &\le 3\left(y(t_0) + \beta\nu \frac{\Omega_2^2}{U^2}\right) \Gamma^{\frac{t-t_0}{\beta}} \\ &\le 3\left(y(t_0) + \beta\nu \frac{\Omega_2^2}{U^2}\right)^{1-\frac{5(t-t_0)}{\beta}} \times \left(2C_K \left(C\beta\Omega_4 \frac{\Omega_2}{U}\right)^4\right)^{\frac{5(t-t_0)}{\beta}}. \end{aligned} \tag{3.2.38}$$

Recalling that $\beta = \gamma/\Omega_\infty$ and denoting the non-dimensional constant

$$K = 2C_K \left(C\beta\Omega_4 \frac{\Omega_2}{U} \right)^4 \tag{3.2.39}$$

we established

$$\frac{\|v(t)\|^2}{U^2} \le 3K^{\frac{5(t-t_0)\Omega_\infty}{\gamma}} \left(\frac{\|v(t_0)\|^2_{L^2(\mathbb{T}^2)}}{U^2} + \beta\nu \frac{\Omega_2^2}{U^2} \right)^{1-\frac{5(t-t_0)\Omega_\infty}{\gamma}} . \tag{3.2.40}$$

Thus, we established (3.2.20).

Step 2: Long Time Bound With (3.2.20) established, we now prove (3.2.21). Let $c = 5\Omega_\infty/\gamma$, $\Delta t = 1/2c$ and $t_i = t_{i-1} + \Delta t$ and $a_i = \|v(t_i)\|^2_{L^2}/U^2$ for $i \in \mathbb{N}$. Then (3.2.20) states

$$a_i \le C_1 \left(a_{i-1} + C_2\nu\right)^{1/2}, \qquad i = 1, 2, \ldots \tag{3.2.41}$$

with $C_1 = 3K^{\frac{5\Omega_\infty}{2c\gamma}} = 3K^{\frac{1}{2}}$ and $C_2 = \beta\frac{\Omega_2^2}{U^2}$. We set

$$\delta_n = \frac{a_i + C_2\nu}{C_1^2} \tag{3.2.42}$$

and observe that (3.2.41) is

$$\delta_n \le \sqrt{\delta_{n-1}} + \widetilde{\nu} \tag{3.2.43}$$

where

$$\widetilde{\nu} = \frac{C_2\nu}{C_1^2} \tag{3.2.44}$$

is a non-dimensional inverse Reynolds number. It follows then by induction that

$$\delta_n \le (\delta_0)^{2^{-n}} + \sum_{i=0}^{n-1} (\widetilde{\nu})^{2^{-i}} . \tag{3.2.45}$$

Indeed, the induction step follows from

$$\delta_{n+1} \le \sqrt{\delta_n} + \widetilde{\nu} \tag{3.2.46}$$

and the subadditivity of $\lambda \mapsto \sqrt{\lambda}$. If

$$\widetilde{\nu} \le \frac{1}{\sqrt{5} - 1} \tag{3.2.47}$$

then the iteration (3.2.43) starting from $0 < \delta_0 < r$ where r is the positive root of the equation $x^2 - x - \widetilde{\nu} = 0$, remains in the interval $(0, r)$, and for any n, δ_n obeys (3.2.45). We observe that

$$\sum_{i=0}^{n-1} (\widetilde{\nu})^{2^{-i}} = (\widetilde{\nu})^{2^{-n+1}} \left(1 + \cdots + (\widetilde{\nu})^{2^{n-1}}\right) \leq \frac{1}{1-\widetilde{\nu}} (\widetilde{\nu})^{2^{-n+1}} \tag{3.2.48}$$

and therefore (3.2.21) follows from (3.2.45). We note that the iteration defined with equality in (3.2.43) converges as $n \to \infty$ to r. Fixing any $t > 0$ and letting $n = \lceil t/\Delta t \rceil = \lceil 2ct \rceil = \lceil 10t\Omega_\infty/\gamma \rceil$ establishes the bound.

The next useful result concerns scalars transported and amplified by a velocity with bounded curl in two dimensions.

Lemma 3.3 *Let $u := u(x,t)$ be divergence free and $\omega := \nabla^\perp \cdot u \in L^\infty(0,T; L^\infty(\mathbb{T}^2))$ with*

$$\sup_{0 \leq t \leq T} \|\omega(t)\|_{L^\infty(\mathbb{T}^2)} \leq \Omega_\infty. \tag{3.2.49}$$

Consider a nonnegative scalar field $\theta := \theta(x,t)$ satisfying the differential inequality

$$\partial_t \theta + u \cdot \nabla\theta - \nu\Delta\theta \leq |\nabla u|\theta + f, \tag{3.2.50}$$

with initial data $\theta|_{t=0} = \theta_0 \in L^\infty(\mathbb{T}^2)$, and forcing $f \in L^\infty(0,T; L^\infty(\mathbb{T}^2))$. Let $\gamma > 0$ be the constant from Lemma 3.1. Then, for any $p > 1$ and the time $T(p) = \frac{\gamma(p-1)}{2p\Omega_\infty}$ it holds that

$$\sup_{t \in [0,T(p)]} \|\theta(t)\|_{L^2(\mathbb{T}^2)} \leq C_1 \|\theta_0\|^p_{L^{2p}(\mathbb{T}^2)} + C_2 \tag{3.2.51}$$

for some constants C_1, C_2 depending only on p, Ω_∞ and $\|f\|_{L^\infty(0,T;L^\infty(\mathbb{T}^2))}$.

Proof Let $p := p(t)$ with $p(0) = p_0$ and time dependence of $p(t)$ to be specified below. Consider

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2} |\theta|^{2p(t)}\mathrm{d}x &= p'(t)\int_{\mathbb{T}^2} \ln|\theta||\theta|^{2p(t)}\mathrm{d}x + p(t)\int_{\mathbb{T}^2} |\theta|^{2p(t)-2}\theta\partial_t\theta\mathrm{d}x \\
&\leq p'(t)\int_{\mathbb{T}^2} \ln|\theta||\theta|^{2p(t)}\mathrm{d}x - p(t)\int_{\mathbb{T}^2} |\theta|^{2p(t)-2}\theta u\cdot\nabla\theta\mathrm{d}x \\
&\quad + \nu p(t)\int_{\mathbb{T}^2} |\theta|^{2p(t)-2}\theta\Delta\theta\mathrm{d}x + p(t)\int_{\mathbb{T}^2} |\theta|^{2p(t)-2}|\nabla u|\theta^2\mathrm{d}x \\
&\quad + p(t)\int_{\mathbb{T}^2} |\theta|^{2p(t)-2}\theta f\mathrm{d}x.
\end{aligned} \tag{3.2.52}$$

We now use the following facts

$$\int_{\mathbb{T}^2} |\theta|^{2p-2}\theta f \mathrm{d}x \le C\|f\|_{L^\infty(0,T;L^\infty(\mathbb{T}^2))}\|\theta\|_{2p}^{2p-1}, \tag{3.2.53}$$

$$p\int_{\mathbb{T}^2} |\theta|^{2p-2}\theta u\cdot\nabla\theta \mathrm{d}x = \frac{1}{2}\int_{\mathbb{T}^2} u\cdot\nabla(|\theta|^{2p})\mathrm{d}x = 0, \tag{3.2.54}$$

$$\nu\int_{\mathbb{T}^2} |\theta|^{2p-2}\theta\Delta\theta \mathrm{d}x = -\nu(2p-1)\int_{\mathbb{T}^2} |\theta|^{2p-2}|\nabla\theta|^2 \mathrm{d}x \le 0. \tag{3.2.55}$$

In the second equality we used the fact that the velocity is divergence free. Altogether we find thus

$$\begin{aligned}&\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|\theta(t)\|_{2p(t)}^{2p(t)}\mathrm{d}x\\ &\le p'(t)\textstyle\int_{\mathbb{T}^2}\ln|\theta||\theta|^{2p(t)}\mathrm{d}x + p(t)\int_{\mathbb{T}^2}|\theta|^{2p(t)}|\nabla u|\mathrm{d}x + p(t)\|f\|_{L^\infty}\|\theta\|_{2p}^{2p-1}.\end{aligned}$$

We now use the following elementary inequality: for $a, b > 0$,

$$ab \le e^a + b\ln b - b. \tag{3.2.56}$$

In fact, we use only that $ab \le e^a + b\ln b$. The inequality (3.2.56) is proved via calculus and follows because the Legendre transform of the convex function $b\ln b - b + 1$ is $e^a - 1$. Setting $a = \beta|\nabla u|$ and $b = \frac{1}{\beta}|\theta|^{2p}$, applying (3.2.56) and Lemma 3.1 we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta(t)\|_{2p(t)}^{2p(t)} &\le p'(t)\int_{\mathbb{T}^2}\ln|\theta||\theta|^{2p}\mathrm{d}x + \frac{p(t)}{\beta}\int_{\mathbb{T}^2}\ln(\beta^{-1}|\theta|^{2p})|\theta|^{2p}\mathrm{d}x\\ &\quad + p(t)\int_{\mathbb{T}^2} e^{\beta|\nabla u|}\mathrm{d}x + Cp(t)\|f\|_{L^\infty}\|\theta\|_{2p}^{2p-1}\\ &\le \left(p'(t) + \frac{2p(t)^2}{\beta}\right)\int_{\mathbb{T}^2}\ln|\theta||\theta|^{2p}\mathrm{d}x + \frac{p(t)}{\beta}\ln(\beta^{-1})\|\theta(t)\|_{2p}^{2p}\\ &\quad + p(t)C_K + Cp(t)\|f\|_{L^\infty}\|\theta\|_{2p}^{2p-1},\end{aligned} \tag{3.2.57}$$

where C_K is the constant from Lemma 3.1 and $\beta = \frac{\gamma}{\Omega_\infty}$ depends on the bound for $\|\omega(t)\|_{L^\infty}$. We now choose p to evolve according to

$$p'(t) = -2\beta^{-1}p(t)^2, \quad p(0) = p_0 \quad \Longrightarrow \quad p(t) = \frac{\beta p_0}{\beta + 2p_0 t}. \tag{3.2.58}$$

Note that $p(t)$ is a positive monotonically decreasing function of t. Let the time t_* defined by $t_* = T(p_0) := \beta(p_0 - 1)/2p_0$ be such that $p(t_*) = 1$. Then $p(t) \in [1, p_0]$ for all $t \in [0, t_*]$. Note also from (3.2.58) that

$$\int_0^t p(s)ds = \log\left(\frac{p_0}{p(t)}\right)^{2\beta} = \log\left(1 + \frac{2p_0 t}{\beta}\right)^{\frac{2}{\beta}}.$$

Defining $m(t) = \frac{1}{2}\|\theta(t)\|_{2p(t)}^{2p(t)}$ and using (3.2.58) we have the differential inequality

$$m'(t) \le p(t)(C_1 m(t) + C_2) \implies C_1 m(t) + C_2 = (C_1 m_0 + C_2)\left(1 + \frac{2p_0 t}{\beta}\right)^{\frac{2C_1}{\beta}} \tag{3.2.59}$$

with C_1 and C_2 depending on $\|f\|_{L^\infty(0,T;L^\infty(\mathbb{T}^2))}$, p_0, C_K and β. Thus

$$m(t) \le m_0\left(1 + \frac{2p_0 t}{\beta}\right)^{\frac{2C_1}{\beta}} + \frac{C_2}{C_1}\left[\left(1 + \frac{2p_0 t}{\beta}\right)^{\frac{2C_1}{\beta}} - 1\right]$$

Note that $p_0/p(t) = 1 + 2p_0\beta^{-1}t$ is increasing on $[0, t_*]$ from 1 to $p_0/p(t_*) = p_0$. Consequently

$$\|\theta(t)\|_{2p(t)} \le C_1\|\theta_0\|_{2p_0}^{p_0} + C_2 \tag{3.2.60}$$

where the constants C_1 and C_2 have been redefined but the dependence on parameters is the same. As $p(t) \in [1, p_0]$ for all $t \in [0, t_*]$ we have that $\|\theta(t)\|_2 \le \|\theta(t)\|_{2p(t)}$ and we obtain

$$\sup_{t\in[0,t_*]} \|\theta(t)\|_2 \le C_1\|\theta_0\|_{2p_0}^{p_0} + C_2, \tag{3.2.61}$$

which completes the proof.

We prove now a short time inviscid limit result, in which the time of convergence importantly depends only on L^∞ initial vorticity bounds.

Proposition 3.1 *Let ω and ω^ν solve (3.2.1) and (3.2.3) respectively, with initial data (3.2.2) and (3.2.4). Assume that the Navier–Stokes initial data converge uniformly in $L^2(\mathbb{T}^2)$*

$$\lim_{\nu\to 0}\|\omega_0^\nu - \omega_0\|_{L^2(\mathbb{T}^2)} = 0. \tag{3.2.62}$$

Assume also that there exists a constant Ω_∞ such that the initial data are uniformly bounded in $L^\infty(\mathbb{T}^2)$:

$$\sup_{\nu>0} \|\omega_0^\nu\|_{L^\infty(\mathbb{T}^2)} \leq \Omega_\infty. \tag{3.2.63}$$

Then there exists a constant C_* such that the vanishing viscosity limit holds

$$\lim_{\nu\to 0} \sup_{t\in[0,T_*]} \|\omega^\nu(t) - \omega(t)\|_{L^2(\mathbb{T}^2)} = 0 \tag{3.2.64}$$

on the time interval $[0, T_*]$ *where*

$$T_* = (C_*\Omega_\infty)^{-1}. \tag{3.2.65}$$

Proof For the proof we introduce functions ω_ℓ and ω_ℓ^ν which are the unique solutions of the following *linear* problems. We fix $\ell > 0$ and let

$$\partial_t\omega_\ell + u\cdot\nabla\omega_\ell = \varphi_\ell * g, \qquad \omega_\ell(0) = \varphi_\ell * \omega_0, \tag{3.2.66}$$

$$\partial_t\omega_\ell^\nu + u^\nu\cdot\nabla\omega_\ell^\nu = \nu\Delta\omega_\ell^\nu + \varphi_\ell * g, \qquad \omega_\ell^\nu(0) = \varphi_\ell * \omega_0^\nu, \tag{3.2.67}$$

where φ_ℓ is a standard mollifier at scale ℓ and where u and u^ν are respectively the unique solutions of Euler and Navier–Stokes equations. Note that the solutions to the linear problems (3.2.66) and (3.2.67) exist globally and are unique because the Yudovich velocity field u is log-Lipshitz. We observe that we have

$$\begin{aligned}&\|\omega^\nu(t) - \omega(t)\|_{L^2(\mathbb{T}^2)}\\ &\leq \|\omega(t) - \omega_\ell(t)\|_{L^2(\mathbb{T}^2)} + \|\omega^\nu(t) - \omega_\ell^\nu(t)\|_{L^2(\mathbb{T}^2)} + \|\omega_\ell^\nu(t) - \omega_\ell(t)\|_{L^2(\mathbb{T}^2)}.\end{aligned} \tag{3.2.68}$$

Because the equations for $\omega_\ell, \omega_\ell^\nu$ and, respectively ω, ω^ν share the same incompressible velocities, we find

$$\|\omega(t) - \omega_\ell(t)\|_{L^2(\mathbb{T}^2)} \leq \|\omega_0 - \varphi_\ell * \omega_0\|_{L^2(\mathbb{T}^2)} + \int_0^t \|g(s) - \varphi_\ell * g(s)\|_{L^2(\mathbb{T}^2)}\mathrm{d}s, \tag{3.2.69}$$

$$\|\omega^\nu(t) - \omega_\ell^\nu(t)\|_{L^2(\mathbb{T}^2)} \leq \|\omega_0^\nu - \varphi_\ell * \omega_0^\nu\|_{L^2(\mathbb{T}^2)} + \int_0^t \|g(s) - \varphi_\ell * g(s)\|_{L^2(\mathbb{T}^2)}\mathrm{d}s. \tag{3.2.70}$$

As mollification can be removed strongly in L^p, the two terms in the right hand sides converge to zero as $\ell, \nu \to 0$, in any order.

It remains to show that

$$\lim_{\nu\to 0}\sup_{t\in[0,T_*]}\|\omega_\ell^\nu(t)-\omega_\ell(t)\|_{L^2(\mathbb{T}^2)}\to 0 \tag{3.2.71}$$

for fixed ℓ. We show now that the two-dimensional linearized Euler and Navier–Stokes equations have uniformly bounded vorticity gradients for short time. This is done in the following Lemma.

Lemma 3.4 *Fix $\ell > 0$ and let ω_ℓ and ω_ℓ^ν solve (3.2.66) and (3.2.67) respectively. Then there exists a constant C_* and a constant $C_\ell < \infty$ depending only on ℓ, the forcing norm $\|g\|_{L^\infty(0,T;L^\infty(\mathbb{T}^2))}$, and the uniform bound on solutions given in (3.2.6) such that for $T_* \le (C_*\Omega_\infty)^{-1}$, we have that*

$$\sup_{t\in[0,T_*]}\left(\|\omega_\ell(t)\|_{H^1}+\|\omega_\ell^\nu(t)\|_{H^1}\right)\le C_\ell. \tag{3.2.72}$$

For the proof of this lemma we provide a viscosity independent bound for $\|\omega_\ell^\nu(t)\|_{H^1}$. The proof for $\|\omega_\ell(t)\|_{H^1}$ is the same, setting $\nu = 0$. We show that $|\nabla\omega_\ell^\nu|$ obeys (3.2.50). Differentiating (3.2.67), we find

$$(\partial_t + u^\nu\cdot\nabla)\nabla\omega_\ell^\nu + \nabla u^\nu\cdot\nabla\omega_\ell^\nu = \nu\Delta(\nabla\omega_\ell^\nu)+\nabla(\varphi_\ell * g). \tag{3.2.73}$$

A standard computation shows that $|\nabla\omega_\ell^\nu|$ satisfies

$$(\partial_t + u^\nu\cdot\nabla-\nu\Delta)|\nabla\omega_\ell^\nu| \le |\nabla u||\nabla\omega_\ell^\nu| + |\nabla(\varphi_\ell * g)| \tag{3.2.74}$$

which is a particular case of the scalar inequality (3.2.50) with $\theta = |\nabla\omega_\ell^\nu|$, initial data $\theta_0 = |\nabla(\varphi_\ell * \omega_0^\nu)| \in L^\infty(\mathbb{T}^2)$ and forcing $f = |\nabla(\varphi_\ell * g)| \in L^\infty(0,T;L^\infty(\mathbb{T}^2))$, as claimed. Applying Lemma 3.3, we find that for any $p > 1$ (e.g. $p = 2$) we have

$$\begin{aligned}&\sup\nolimits_{t\in[0,T_*]}\|\omega_\ell^\nu(t)\|_{H^1} = C_1\tfrac{1}{\ell^p}\left(\textstyle\int_{\mathbb{T}^2}|\omega_0^\nu *(\nabla\varphi)_\ell|^{2p}\mathrm{d}x\right)^{1/2}+C_2\\ &\le C_\ell\|\omega_0^\nu\|^p_{L^\infty(\mathbb{T}^2)}\le C_\ell\Omega_\infty^p.\end{aligned} \tag{3.2.75}$$

The constant C_ℓ diverges with the mollification scale ℓ, through the prefactor ℓ^{-p} and through the dependence on $\|\nabla(\varphi_\ell * g)\|_{L^\infty} \lesssim \ell^{-1}\|g\|_{L^\infty}$. The important point however is that (3.2.75) holds uniformly in viscosity, completing the proof of Lemma 3.4. Using it, the difference enstrophy obeys

$$\begin{aligned}&\tfrac{\mathrm{d}}{\mathrm{d}t}\|\omega_\ell^\nu-\omega_\ell\|^2_{L^2(\mathbb{T}^2)} = -\textstyle\int_{\mathbb{T}^2}(u^\nu-u)\cdot\nabla\omega_\ell^\nu(\omega_\ell^\nu-\omega_\ell)\mathrm{d}x - \nu\int_{\mathbb{T}^2}|\nabla\omega_\ell^\nu|^2\mathrm{d}x\\ &+\nu\textstyle\int_{\mathbb{T}^2}\nabla\omega_\ell^\nu\cdot\nabla\omega_\ell\mathrm{d}x \le 4\Omega\|u^\nu-u\|_{L^2(\mathbb{T}^2)}\|\nabla\omega_\ell^\nu\|_{L^2(\mathbb{T}^2)}+\nu\|\nabla\omega_\ell^\nu\|_{L^2(\mathbb{T}^2)}\|\nabla\omega_\ell\|_{L^2(\mathbb{T}^2)}\\ &\lesssim C_\ell\|u^\nu-u\|_{L^\infty(0,T;L^2(\mathbb{T}^2))}+\nu C_\ell^2.\end{aligned} \tag{3.2.76}$$

Integrating we find

$$\|\omega_\ell^\nu - \omega_\ell\|_{L^2(\mathbb{T}^2)}^2 \lesssim \|\varphi_\ell * (\omega_0^\nu - \omega_0)\|_{L^2(\mathbb{T}^2)}^2 + C_\ell T \|u^\nu - u\|_{L^\infty(0,T;L^2(\mathbb{T}^2))} + \nu C_\ell^2 T. \tag{3.2.77}$$

To conclude the proof we must show that, at fixed $\ell > 0$, we have $\lim_{\nu\to 0} \|\omega_\ell^\nu - \omega_\ell\|_{L^2(\mathbb{T}^2)} = 0$. Recall that by our assumption (3.2.62) we have that $\lim_{\nu\to 0} \|\omega_0^\nu - \omega_0\|_{L^2(\mathbb{T}^2)} \to 0$. Due to assumption (3.2.62) we have that $\lim_{\nu\to 0} \|u_0^\nu - u_0\|_{L^2(\mathbb{T}^2)} \to 0$. Lemma 3.2 then allows us to conclude from (3.2.77) that $\lim_{\nu\to 0} \sup_{t\in[0,T_*]} \|\omega_\ell^\nu - \omega_\ell\|_{L^2(\mathbb{T}^2)} \to 0$ at fixed $\ell > 0$ and the proof of Proposition 3.1 is complete.

Proof of Theorem 3.1 It suffices to prove that $\lim_{\nu\to 0} \sup_{t\in[0,T]} \|\omega^\nu(t) - \omega(t)\|_{L^2(\mathbb{T}^2)} = 0$. Indeed, convergence in L^p for any $p \in [2, \infty)$ then follows from interpolation and boundedness in L^∞:

$$\|\omega^\nu(t) - \omega(t)\|_{L^p(\mathbb{T}^2)} \leq 2\Omega_\infty^{\frac{p-2}{p}} \|\omega^\nu(t) - \omega(t)\|_{L^2(\mathbb{T}^2)}^{\frac{2}{p}}. \tag{3.2.78}$$

In order to establish strong $L_t^\infty L_x^2$ convergence for arbitrary finite times T, it is enough to the convergence for a short time which depends only on a uniform L^∞ bound on the initial vorticity. The proof of Theorem 3.1 follows by dividing the time interval $[0, T]$ in subintervals

$$[0, T] = [0, T_*] \cup [T_*, 2T_*] \cup \cdots$$

where T_* is determined from the uniform bound (3.2.6), and applying Proposition 3.1 to each interval, with initial data $\omega(nT_*)$, and respectively $\omega^\nu(nT_*)$. As there is no required rate of convergence for the initial data in Proposition 3.1, Theorem 3.1 follows.

3.2.2 Uniform Regularity

In this section we consider for simplicity the unforced case in $\mathbb{R}^2$. We study propagation of low regularity, uniform in viscosity. Let us consider the Navier–Stokes equation in $\mathbb{R}^2$

$$\partial_t \omega + u \cdot \nabla\omega - \nu\Delta\omega = 0, \tag{3.2.79}$$

with initial vorticity $\omega_0 \in \mathbb{Y}$ where

$$\mathbb{Y} = L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2). \tag{3.2.80}$$

The velocity u is given by the Biot–Savart law, (3.2.15). The main result of this section is the following.

Theorem 3.4 *Let* $1 < p < \infty$. *Let* $\omega_0 \in \mathbb{Y} \cap B^s_{p,1}(\mathbb{R}^2)$. *There exist constants* C_Ω *and* Ω_∞ *depending only on the norm of the initial data in* $\mathbb{Y}$ *such that the solution of the Navier–Stokes equations (3.2.79) with initial data* ω_0 *(3.2.92) satisfies, uniformly in* ν,

$$\|\omega(t)\|_{B^{s(t)}_{p,1}(\mathbb{R}^2)} \le e^{Ct\Omega_\infty}\|\omega_0\|_{B^s_{p,1}(\mathbb{R}^2)} \tag{3.2.81}$$

with

$$s(t) = s - (5\log 2C_\Omega)t \tag{3.2.82}$$

for $0 \le t \le (5\log 2C_\Omega)^{-1}s$.

Remark 3.3 Note that in view of the embeddings

$$B^{s'}_{p,\infty}(\mathbb{R}^n) \subset B^s_{p,1}(\mathbb{R}^n) \subset B^s_{p,\infty}(\mathbb{R}^n)$$

for $0 \le s < s'$ we can track the regularity of solutions with initial data in $B^{s'}_{p,\infty}(\mathbb{R}^2)$, and hence that of vortex patches with rough boundaries, of positive codimension.

We recall the fact that Biot–Savart velocities of Yudovich class vorticities are log-Lipschitz:

Proposition 3.2 *Let* $u = K[\omega]$ *be given by the Biot–Savart law (3.2.15) and let* $\omega \in \mathbb{Y}$. *There exists a constant* C *such that*

$$|u(x+h) - u(x)| \le C\Omega_\infty |h|\left[1 + \log\left(1 + \frac{L}{|h|}\right)\right] \tag{3.2.83}$$

holds for $x, h \in \mathbb{R}^2$, *where* $L = \sqrt{\frac{\Omega_1}{\Omega_\infty}}$ *and* Ω_p *are the* $L^p(\mathbb{R}^2)$ *norms of* ω.

Proof We write

$$u(x+h)-u(x) = \int_{\mathbb{R}^2}(k(x-y+h)-k(x-y))\omega(y)dy = \int_{\mathbb{R}^2}(k(z+h)-k(z))\omega(x-z)dz,$$

where

$$k(z) = \frac{1}{2\pi}\frac{z^\perp}{|z|^2}. \tag{3.2.84}$$

We split the integral in two, corresponding to $|z| \le 2|h|$ and $|z| \ge 2|h|$. We have

$$\left|\int_{|z|\le 2|h|} |k(z+h)|\omega(x-z)|dz\right| + \left|\int_{|z|\le 2|h|} |k(z)|\omega(x-z)|dz\right| \le C|h|\|\omega\|_{L^\infty(\mathbb{R}^2)}$$

by passing to polar coordinates centered at $-h$ and respectively at 0, and using $|k(x)| \le \frac{1}{2\pi|x|}$. The second integral we bound by

$$\begin{aligned} &\int_{|z|\ge 2|h|} |k(z+h)-k(z)|\,|\omega(x-z)|dz \\ &\le C|h|\int_0^1 d\lambda \int_{|z+\lambda h|\ge |h|} |z+\lambda h|^{-2}|\omega(x-z)|dz \end{aligned}$$

here we used $|\nabla k(x)| \le C|x|^{-2}$. Now we split the integral again, for $|z+\lambda h| \le L$ and $|z+\lambda h| \ge L$. In the first integral we use L^∞ bounds on ω and obtain a logarithmic dependence, $\|\omega\|_{L^\infty} \log\frac{L}{|h|}$ and in the second integral we use L^1 bounds on ω and we obtain $L^{-2}\|\omega\|_{L^1}$.

We recall some facts about the Littlewood–Paley decomposition. We start with a smooth, nonincreasing, radial nonnegative function $\phi(r)$ satisfying

$$\begin{cases} \phi(r) = 1, & \text{for } 0 \le r \le a, \\ \phi(r) = 0, & \text{for } b \le r, \\ 0 < a < b. \end{cases}$$

We define

$$\psi_0(r) = \phi\left(\frac{r}{2}\right) - \phi(r),$$

$$(\Delta_{-1}u)(x) = (\phi(D)u)(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\phi(|\xi|)\widehat{u}(\xi)d\xi, \tag{3.2.85}$$

$$(\Delta_0 u)(x) = (\psi_0(D)u)(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\psi_0(|\xi|)\widehat{u}(\xi)d\xi, \tag{3.2.86}$$

$$\psi_j(r) = \psi_0(2^{-j}r)$$

and

$$(\Delta_j u)(x) = (\psi_j(D)u)(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\psi_j(|\xi|)\widehat{u}(\xi)d\xi, \tag{3.2.87}$$

where

$$\widehat{u}(\xi) = \mathcal{F}u(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(x)dx.$$

We choose $a = \frac{1}{2}$, $b = \frac{5}{8}$. We set also

$$S_k(u) = \sum_{j=-1}^{k} \Delta_j(u) \tag{3.2.88}$$

Proposition 3.3 *If $u \in \mathcal{S}'(\mathbb{R}^n)$, then*

$$u = \sum_{j=-1}^{\infty} \Delta_j u,$$

$$\mathrm{supp}\mathcal{F}(\Delta_j u) \subset 2^j \left[\frac{1}{2}, \frac{5}{4}\right],$$

for $j \geq 0$, and in particular

$$\Delta_j \Delta_k \neq 0 \Rightarrow |j-k| \leq 1, \quad \text{for } j, k \geq 0.$$

Moreover,

$$(\Delta_j + \Delta_{j+1} + \Delta_{j+2})\Delta_{j+1} = \Delta_{j+1},$$

for $j \geq 0$,

$$\Delta_j \left(S_{k-2}(u)\Delta_k(v)\right) \neq 0 \Rightarrow k \in [j-2, j+2]$$

for $j \geq 2$, $k \geq 2$.

Proposition 3.4 (Bernstein Inequalities)

$$\|\Delta_j u\|_{L^q(\mathbb{R}^n)} \leq C 2^{j(\frac{n}{p}-\frac{n}{q})} \|\Delta_j u\|_{L^p(\mathbb{R}^n)}, \quad q \geq p \geq 1,$$

$$\|S_j u\|_{L^q(\mathbb{R}^n)} \leq C 2^{j(\frac{n}{p}-\frac{n}{q})} \|S_j u\|_{L^p(\mathbb{R}^n)}, \quad q \geq p \geq 1,$$

and

$$2^{jm}\|\Delta_j u\|_{L^p(\mathbb{R}^n)} \leq C \sum_{|\alpha|=m} \|\partial^\alpha \Delta_j u\|_{L^p(\mathbb{R}^n)} \leq C 2^{jm}\|\Delta_j u\|_{L^p(\mathbb{R}^n)}$$

We introduce the inhomogeneous Besov space with norm

$$\|u\|_{B^s_{p,q}(\mathbb{R}^n)} = \left\| \left\{ 2^{sj} \|\Delta_j u\|_{L^p(\mathbb{R}^n)} \right\}_j \right\|_{\ell^q(\mathbb{N})}$$

Proposition 3.5 (Littlewood–Paley) *Let* $1 < p < \infty$. *Then* $(\mathbb{I} - \Delta)^{\frac{s}{2}} u \in L^p(\mathbb{R}^n)$ *if and only if* $\Delta_j u \in L^p(\mathbb{R}^n)$ *for all* $j \geq -1$ *and*

$$\|u\|_{W^{s,p}(\mathbb{R}^n)} \sim \left\| \sqrt{\sum_{j\geq -1} 2^{2js} |\Delta_j(u)|^2} \right\|_{L^p(\mathbb{R}^n)}$$

Proposition 3.6 *Embeddings:*

$$B^s_{p,r}(\mathbb{R}^n) \subset B^{s-\left(\frac{n}{p}-\frac{n}{q}\right)}_{q,r}(\mathbb{R}^n), \qquad q \geq p \geq 1,$$

$$B^0_{p,2}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \subset B^0_{p,p}(\mathbb{R}^n) \qquad p \geq 2,$$

$$B^0_{p,p}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \subset B^0_{p,2}(\mathbb{R}^n) \qquad p \leq 2.$$

Products Consider two functions, $u = \sum_{k\geq -1} \Delta_k u$ and $v = \sum_{l\geq -1} \Delta_l(v)$. Then we have the Bony decomposition

$$\Delta_j(uv) = I_j(u, v) + I_j(v, u) + R_j(u, v) \tag{3.2.89}$$

with

$$I_j(u, v) = \sum_{k\in[j-2,j+2]} \Delta_j(S_{k-2}(u)\Delta_k(v)) \tag{3.2.90}$$

and

$$R_j(u, v) = \sum_{|k-l|\leq 1} \Delta_j(\Delta_k u \Delta_l v). \tag{3.2.91}$$

Proof of Theorem 3.4 We consider the Navier–Stokes vorticity evolution is the $B^s_{p,1}$ space, with $s > 0$ and $1 < p < \infty$. We take initial vorticity

$$\omega_0 \in \mathbb{Y} \cap B^s_{p,1}(\mathbb{R}^2) \tag{3.2.92}$$

and look first at the evolution of $\Delta_j\omega$ in L^p, using the Bony decomposition.

$$\frac{1}{p}\frac{d}{dt}\|\Delta_j\omega\|_{L^p} \leq A_j + B_j + C_j \tag{3.2.93}$$

for $j \geq 5$ where

$$A_j = \left\| \sum_{k\in[j-2,j+2]} \left[S_{k-2}(u), \Delta_j\right] \cdot \nabla \Delta_k \omega \right\|_{L^p(\mathbb{R}^2)}, \tag{3.2.94}$$

$$B_j = \left\| \sum_{k\in[j-2,j+2]} \Delta_j \left(\Delta_k(u), \cdot \nabla S_{k-2}\omega\right) \right\|_{L^p(\mathbb{R}^2)}, \tag{3.2.95}$$

and

$$C_j = \left\| \sum_{|k-l|\leq 1, k\geq j-2} \Delta_j \left(\Delta_l u \cdot \nabla \Delta_k \omega\right) \right\|_{L^p(\mathbb{R}^2)}, \tag{3.2.96}$$

The commutator appears in A_j because of the property $\Delta_j \sum_{k\in[j-2,j+2]} \Delta_k = \Delta_j$ and the fact that $S_{j-2}u$ is divergence-free. We discarded the nonnegative term due to the viscosity. We use the fact that $S_{k-2}(u)$ are uniformly log-Lipschitz:

$$\begin{aligned}
&\left|\left[S_{k-2}(u), \Delta_j\right] f(x)\right| \\
&\leq C\Omega_\infty 2^{2j} \int_{\mathbb{R}^2} |\Psi_0(2^j(x-y))||x-y| \log\left(1 + \tfrac{L}{|x-y|}\right) |f(y)| dy \\
&\leq j 2^{-j} \int_{\mathbb{R}^2} \widetilde{\Psi}(z)|f|(x - 2^{-j}z) dz
\end{aligned}$$

where Ψ_0 is a Schwartz function, Fourier inverse of ψ_0, $\mathcal{F}\Psi_0 = \psi_0$, and

$$\widetilde{\Psi} = C\Omega_\infty |x| \left(\log\left(1 + \frac{L}{|x|}\right) + \log 2\right) |\Psi_0(x)| \tag{3.2.97}$$

is rapidly decaying, and hence belongs in $L^1(\mathbb{R}^2)$. Here we used the fact that S_{k-2} commute with translation and are uniformly bounded in all L^p, and hence Ω_∞ and L are bounded independently of k and t.

$$\| \left[S_{k-2}(u), \Delta_j\right] f \|_{L^p(\mathbb{R}^2)} \leq C_\Omega j 2^{-j} \|f\|_{L^p(\mathbb{R}^2)}$$

and where C_Ω is the L^1 norm of $\widetilde{\Psi}$. It follows that

$$A_j \leq j C_\Omega \sum_{k\in[j-2,j+2]} \|\Delta_k \omega\|_{L^p(\mathbb{R}^2)} \tag{3.2.98}$$

The bound of B_j is more straightforward,

$$B_j \leq C\Omega_\infty \sum_{k\in[j-2,j+2]} \|\Delta_k \omega\|_{L^p(\mathbb{R}^2)} \tag{3.2.99}$$

and uses Bernstein inequalities and the boundedness of ∇K in L^p spaces, where K is the Biot–Savart operator. The remaining term is bounded also using Bernstein inequalities

$$C_j \leq C\Omega_\infty \sum_{l \geq j-3} \|\omega_l\|_{L^p(\mathbb{R}^2)} \tag{3.2.100}$$

We consider now the norm

$$\|\omega(t)\|_{B^{s(t)}_{p,1}} \tag{3.2.101}$$

and arrange the decay of the exponent so that it counter balances the logarithmic growth of the term A_j. In order to do so, we observe that (3.2.98) implies the bound

$$2^{sj} A_j \leq C_\Omega \sum_{k \in [j-2, j+2]} k 2^{sk} \|\Delta_k \omega\|_{L^p(\mathbb{R}^2)} \tag{3.2.102}$$

as long as $s \leq 1$, with a slightly larger C_Ω. Similarly, from (3.2.99) and from (3.2.100) we obtain

$$2^{sj} B_j \leq C\Omega_\infty \sum_{k \in [j-2, j+2]} 2^{sk} \|\Delta_k \omega\|_{L^p(\mathbb{R}^2)} \tag{3.2.103}$$

and

$$2^{sj} C_j \leq C\Omega_\infty \sum_{l \geq j-3} 2^{s(j-l)} 2^{sl} \|\omega_l\|_{L^p(\mathbb{R}^2)}. \tag{3.2.104}$$

Imposing

$$\frac{ds}{dt} = -5 \log 2 C_\Omega \tag{3.2.105}$$

where C_Ω is the constant in (3.2.102), we deduce

$$\frac{d}{dt} \|\omega(t)\|_{B^{s(t)}_{p,1}} \leq C\Omega_\infty \|\omega(t)\|_{B^{s(t)}_{p,1}} \tag{3.2.106}$$

This concludes the proof of Theorem 3.4.

3.3 Multiscale Solutions

We describe constructions of solutions of inviscid equations given in [16] which were inspired by work of [17].

For the incompressible 3D Euler equations, if u is Beltrami, i.e. if the curl of the velocity

$$\omega = \nabla \times u \tag{3.3.1}$$

is parallel to the velocity, and if $u \in L^2(\mathbb{R}^3)$, then u must be identically zero [18, 19]. In fact, Liouville theorems which assert the vanishing of solutions which have constant behavior at infinity are often true for systems of the sort we are discussing. In contrast, vortex rings are examples of solutions of the 3D Euler equations with compactly supported vorticity [20]. However, they have nonzero constant velocities at infinity. Because of the Biot–Savart law

$$u(x,t) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \omega(y,t)dy, \tag{3.3.2}$$

if ω is compactly supported, it is hard to imagine that u can also be compactly supported. In view of these considerations, the following result of Gavrilov [17] was surprising.

Theorem 3.5 *(Gavrilov [17]) There exist nontrivial time independent solutions* $u \in \left(C_0^\infty(\mathbb{R}^3)\right)^3$ *of the three-dimensional incompressible Euler equations.*

The paper [16] described a construction inspired by the result of Gavrilov but based on Grad–Shafranov equations, classical equations arising in the study of plasmas [21, 22] augmented by a *localizability* condition (see (3.3.17)). This point of view yielded a general method which was applied to several other hydrodynamic equations, revealing a number of universal features. The 3D incompressible Euler equations result which extends Theorem 3.5 is stated in Theorem 3.6. An application providing multiscale steady solutions which are locally smooth, vanish at $\partial\Omega$, but globally belong only to Hölder classes $C^\alpha(\Omega)$ is given in Theorem 3.7. Such solutions can be constructed so that they belong to $L^2(\Omega) \cap C^{\frac{1}{3}}(\Omega)$ but not to any $C^\alpha(\Omega)$ with $\alpha > \frac{1}{3}$, they have vanishing local dissipation $u \cdot \nabla(\frac{|u|^2}{2} + p) = 0$, but have arbitrary large $\||\nabla u||u|^2\|_{L^\infty(\Omega)}$. These solutions conserve energy, as they are stationary in time, and they have the regularity of the dissipative solutions recently constructed in connection with the Onsager conjecture (see review papers [23, 24]). Compactly supported weak solutions which belong to $C^\alpha(\Omega)$ but not to $C^\beta(\Omega)$, $0 < \alpha < \beta \le 1$ can also be constructed.

3.3.1 Steady Axisymmetric Euler Equations

The stationary 3D axisymmetric Euler equations are solved via the Grad–Shafranov ansatz

$$u = \frac{1}{r}(\partial_z \psi)e_r - \frac{1}{r}(\partial_r \psi)e_z + \frac{1}{r}F(\psi)e_\phi \tag{3.3.3}$$

where $\psi = \psi(r, z)$ is a smooth function of $r > 0, z \in \mathbb{R}$, and the swirl F is a smooth function of ψ alone. It is known that smooth compactly supported velocities solving stationary axisymmetric 3D Euler equations must vanish identically if the swirl F vanishes [25]. Above e_r, e_z, e_ϕ are the orthonormal basis of cylindrical coordinates r, z, ϕ with the orientation convention $e_r \times e_\phi = e_z$, $e_r \times e_z = -e_\phi$, $e_\phi \times e_z = e_r$. Note that u is automatically divergence-free,

$$\nabla \cdot u = 0, \tag{3.3.4}$$

and also that, by construction,

$$u \cdot \nabla \psi = 0. \tag{3.3.5}$$

The vorticity $\omega = \nabla \times u$ is given by

$$\omega = -\frac{1}{r}(\partial_z \psi)F'(\psi)e_r + \frac{1}{r}(\partial_r \psi)F'(\psi)e_z + \frac{\Delta^* \psi}{r}e_\phi \tag{3.3.6}$$

where $F' = \frac{dF}{d\psi}$ and the Grad–Shafranov operator Δ^* is

$$\Delta^* \psi = \partial_r^2 \psi - \frac{1}{r}\partial_r \psi + \partial_z^2 \psi. \tag{3.3.7}$$

In view of (3.3.3) and (3.3.6), the vorticity can be written as

$$\omega = -F'(\psi)u + \frac{1}{r}\left(\Delta^* \psi + \frac{1}{2}(F^2)'\right)e_\phi. \tag{3.3.8}$$

As it is very well known, the steady Euler equations

$$u \cdot \nabla u + \nabla p = 0 \tag{3.3.9}$$

can be written as

$$\omega \times u + \nabla\left(\frac{|u|^2}{2} + p\right) = 0, \tag{3.3.10}$$

and therefore the axisymmetric Euler equations are solved if ψ solves the Grad–Shafranov equation [21, 22]

$$-\Delta^*\psi = \partial_\psi\left(\frac{F^2}{2} + r^2 P\right) \tag{3.3.11}$$

where the function $P = P(\psi)$ represents the plasma pressure:

$$\omega \times u = \nabla P. \tag{3.3.12}$$

The analogy with the steady MHD equations $u \leftrightarrow B$, $\omega \leftrightarrow J$ motivates the name. Both the swirl F and the plasma pressure P are arbitrary functions of ψ. The plasma pressure and the hydrodynamic pressure are related via

$$p + \frac{|u|^2}{2} + P = \text{constant}. \tag{3.3.13}$$

The constant should be time independent if we are studying time independent solutions, and it may be taken without loss of generality to be zero.

If

$$u \cdot \nabla p = 0, \tag{3.3.14}$$

then, together with a solution u, p of (3.3.9, 3.3.4), any function

$$\widetilde{u} = \phi(p)u \tag{3.3.15}$$

with ϕ smooth is again a solution of (3.3.9, 3.3.4) with pressure given by

$$\nabla\widetilde{p} = \phi^2(p)\nabla p. \tag{3.3.16}$$

This can be used to localize solutions. In his construction Gavrilov obtained solutions u defined in the neighborhood of a circle, obeying the Euler equations near the circle, and having a relationship $|u|^2 = 3p$ between the velocity magnitude and the hydrodynamic pressure.

This motivates us to consider the overdetermined system formed by the Grad–Shafranov equation for ψ (3.3.11) coupled with the requirement

$$\frac{|u|^2}{2} = A(\psi). \tag{3.3.17}$$

This requirement is the constraint of *localizability* of the Grad–Shafranov equation, and it severely curtails the freedom of choice of functions F and P. This localizability is in fact the essence and the novelty of the method. Without this constraint many solutions (3.3.3) with ψ solving the Grad–Shafranov equation (3.3.11) exist, including explicit ones [26]), but they cannot be localized in space.

The method we are describing consists thus in seeking functions F, P, A of ψ such that the system

$$\begin{cases} -\Delta^*\psi = \partial_\psi \left(\frac{1}{2}F^2(\psi) + r^2 P(\psi)\right), \\ |\nabla\psi|^2 + F^2(\psi) = 2r^2 A(\psi), \end{cases} \tag{3.3.18}$$

is solved. Then the function u given in the ansatz (3.3.3), and the pressure

$$p = -P(\psi) - A(\psi) \tag{3.3.19}$$

together satisfy the steady 3D Euler equations (3.3.9, 3.3.4), and are *localizable*, meaning that (3.3.17) is satisfied. It is important to observe that it is enough to find smooth functions F, P, A of ψ and a smooth function ψ in an open set. This open set need not be simply connected, but once u and p are found using this construction, any function $\phi(p)u$ is again a solution of steady Euler equations, and it is sometimes possible to extend this solution to the whole space.

3.3.2 Construction

The construction of solutions of (3.3.18) starts with a hodograph transformation. We seek functions $U(r, \psi)$ and $V(r, \psi)$ defined in an open set in the (r, ψ) plane and a smooth function $\psi(r, z)$ defined in an open set of the (r, z) plane such that the equations

$$\partial_r \psi(r, z) = U(r, \psi(r, z)), \tag{3.3.20}$$

$$\partial_z \psi(r, z) = V(r, \psi(r, z)) \tag{3.3.21}$$

are satisfied. This clearly requires the compatibility

$$V\partial_\psi U = U\partial_\psi V + \partial_r V. \tag{3.3.22}$$

Once the compatibility is satisfied then the solution ψ exists locally (in simply connected components). The system (3.3.18) becomes

$$\begin{cases} \partial_r U + U\partial_\psi U + V\partial_\psi V - \frac{1}{r}U = -F\partial_\psi F - r^2\partial_\psi P \\ U^2 + V^2 + F^2 = 2r^2 A. \end{cases} \tag{3.3.23}$$

We traded a system of two equations in two independent variables (r, z) of total degree three, (3.3.18), for a system of three first order equations (3.3.22, 3.3.23) in two independent variables (r, ψ). We integrate this locally. We start by noticing that the first equation of (3.3.23) is

$$\partial_r U - \frac{1}{r}U + \frac{1}{2}\partial_\psi \left(U^2 + V^2 + F^2\right) = -r^2 \partial_\psi P, \tag{3.3.24}$$

which, in view of the second equation in (3.3.23), becomes

$$\partial_r U - \frac{U}{r} = -r^2 \partial_\psi (A + P), \tag{3.3.25}$$

and, using (3.3.19) we see that

$$\partial_\psi p = \frac{1}{r}\partial_r \left(\frac{U}{r}\right), \tag{3.3.26}$$

which then can be used to determine p from knowledge of U. We observe that in order to have $p = p(\psi)$ a function of ψ alone, from (3.3.26) we have to have

$$U = r^3 M(\psi) + r N(\psi). \tag{3.3.27}$$

for some functions M, N of ψ. Let us denote

$$Q_2(r, \psi) = 2r^2 A(\psi) - F^2(\psi), \tag{3.3.28}$$

$$Q_3(r, \psi) = r^3 M(\psi) + r N(\psi), \tag{3.3.29}$$

and

$$Q_6(r, \psi) = Q_2(r, \psi) - (Q_3(r, \psi))^2 \tag{3.3.30}$$

polynomials of degree 2, 3 and 6 in r with smooth and yet unknown coefficients depending only on ψ. We note that, in view of (3.3.27),

$$U = Q_3, \tag{3.3.31}$$

and that the second equation in (3.3.23) yields

$$V^2 = Q_6. \tag{3.3.32}$$

Multiplying (3.3.22) by V results in

$$\partial_r Q_6 + Q_3 \partial_\psi Q_6 - 2(\partial_\psi Q_3) Q_6 = 0. \tag{3.3.33}$$

Identifying coefficients in the 9th order polynomial equation (3.3.33) we observe that only odd powers appear, the equations for powers 9 and 7 are tautological, and the remaining three equations become a system of 3 first order ODEs with four unknown functions which is equivalent to the compatibility relation (3.3.22). In order to localize the sought solution u in (r, z) space we need the pressure p to take a value at a point (r_0, z_0) which is strictly separated from all the values it takes on a circle in (r, z) around that point. We seek then conditions which result in a strict local minimum for the function ψ at the chosen point (r_0, z_0), and then a similar behavior for the resulting p. Without loss of generality we may take this local minimum value of ψ to be zero. Because U and V represent derivatives of ψ we are lead to the requirement that the polynomials Q_3 and Q_6 both vanish at the point $(r_0, 0)$ in the (r, ψ) plane, $Q_3(r_0, 0) = 0$ and $Q_2(r_0, 0) = 0$. This results in singular, non-Lipschitz ODEs. They do have nontrivial solutions though, and the consequence given in [16] is

Theorem 3.6 *Let $\ell > 0$, $\tau > 0$ be given. There exists $\epsilon > 0$ and a function $\psi \in C^\infty(B)$, where $B = \{(r, z) \mid \ |r - \ell|^2 + |z|^2 < \epsilon^2\ell^2\}$ satisfying $\psi(\ell, 0) = 0$, $\psi > 0$ in B and such that (3.3.18) holds with A, P and F^2 real analytic functions of ψ. The Grad–Shafranov equation (3.3.11) is solved pointwise and has classical solutions in B. The associated velocity u given by the Grad–Shafranov ansatz (3.3.3) is Hölder continuous in B. The Euler equation (3.3.9, 3.3.4) holds weakly in B. The pressure is given by $p = \frac{1}{\ell\tau}\psi$. The vorticity is bounded, $\omega \in L^\infty(B)$ and (3.3.10) holds a.e. in B.*

We note that $F(\psi)$ vanishes like $\sqrt{\psi}$. Therefore, while the ansatz (3.3.3) gives a bounded swirl and a Hölder continuous velocity, the vorticity is not smooth. In fact, in view of (3.3.8) the vorticity equals

$$\omega(r, z) = -F'(\psi)u(r, z) + \text{smooth}. \tag{3.3.34}$$

Thus, $\omega \in L^\infty(B)$, because u vanishes to first order at $(\ell, 0)$, but the r derivative of the z component of vorticity is infinite there.

Once ψ has been constructed so that it has a local minimum at $(\ell, 0)$, then p has also a local minimum there, because, by (3.3.26),

$$p = \frac{\psi}{\ell\tau} \tag{3.3.35}$$

is monotonic in ψ.

Theorem 3.5 holds because the cutoff can be chosen so that the point $(\ell, 0)$ is omitted. By choosing a suitable cutoff function $\phi_\epsilon(p)$, the solution $\widetilde{u} = \phi_\epsilon(p)u$ is supported in the region $A = \{(r, z) \mid \ \frac{1}{2}\ell^2\epsilon^2 < |r - \ell|^2 + |z|^2 < \epsilon^2\ell^2\}$. A consequence of Theorem 3.6 is the following.

Theorem 3.7 *Let $0 < \alpha < 1$. In any domain $\Omega \subset \mathbb{R}^3$ there exist steady solutions of Euler equations belonging to $C^\alpha(\Omega)$ and vanishing at $\partial\Omega$, but such that they do not* *belong to $C^{\beta}(\Omega)$ for $\beta > \alpha$. There exist such solutions which are locally smooth, meaning that for every $x \in \Omega$ there exists a neighborhood of x where the solution is C^{∞}. For any $\Gamma > 0$, there exist steady solutions u which belong to $L^2(\Omega) \cap C^{\frac{1}{3}}(\Omega)$, vanish at $\partial\Omega$, are locally smooth and have*

$$\sup_{x\in\Omega} |\nabla u(x)||u(x)|^2 \geq \Gamma,$$

while the local dissipation vanishes, i.e. $u \cdot \nabla(\frac{|u|^2}{2} + p) = 0$ in the sense of distributions. There exist steady solutions which are locally smooth and whose Lagrangian trajectories have arbitrary linking numbers. For any $0 < \alpha < \beta \leq 1$ there exist weak solutions which are compactly supported in Ω, belong to $C^{\alpha}(\Omega)$ but not to $C^{\beta}(\Omega)$.

3.3.3 Steady Multiscale Navier–Stokes Solutions

Proof of Theorem 3.7 is based on a construction which has consequences for the Navier–Stokes equations as well. We describe them here. We take a basic solution of the Euler equations u_B, p_B solving

$$u_B(x) \cdot \nabla u_B(x) + \nabla p_B(x) = 0, \quad \nabla \cdot u_B = 0 \tag{3.3.36}$$

in the unit annulus $A = \{x = (r, z) \mid \frac{1}{2} < |r-1|^2 + |z|^2 < 1\}$ with

$$u_B \in (C_0^{\infty}(A))^3, \quad \nabla p_B \in C_0^{\infty}(A) \tag{3.3.37}$$

constructed by the method of Theorem 3.6. We take an open domain $\Omega \subset \mathbb{R}^3$ and take a sequence of points $x_j \in \Omega$, rotations $R_j \in O(3)$, and numbers $L > 0$, $T > 0$, $\ell > 0$ $\tau > 0$, with associated length scales

$$\ell_j = L2^{-\ell j} \tag{3.3.38}$$

and time scales

$$\tau_j = T2^{-\tau j}, \tag{3.3.39}$$

for $j = 1, 2, \ldots,$ such that functions

$$u_j(x) = \frac{L}{T} 2^{(\tau-\ell)j} R_j u_B \left(2^{\ell j} \frac{R_j^*(x - x_j)}{L} \right) \tag{3.3.40}$$

have disjoint supports

$$A_j = x_j + \ell_j R_j(A) \subset \Omega \tag{3.3.41}$$

in Ω. These are annuli which are rotated, dilated and translated versions of A. Note that the supports of the corresponding pressure gradients

$$\nabla p_j(x) = \frac{L}{T^2} R_j(\nabla p_B)\left(2^{\ell j}\frac{R_j^*(x-x_j)}{L}\right) \tag{3.3.42}$$

are also A_j, and thus disjoint as well, and because of the well known invariance with respect of rotations of the Euler equations we have that

$$u_j \cdot \nabla u_j + \nabla p_j = 0, \qquad \nabla \cdot u_j = 0 \tag{3.3.43}$$

holds in A_j. Let us consider now

$$u(x) = \sum_{j=1}^{N} u_j(x). \tag{3.3.44}$$

Note that $u \in C_0^\infty(\Omega)$, and because the supports of u_j are disjoint, we have

$$\|\partial^\alpha u\|_{L^p(\Omega)} = \left(\sum_{j=1}^{N} 2^{pj[(m-1-\frac{3}{p})\ell+\tau]}\right)^{\frac{1}{p}} L^{1+\frac{3}{p}-m} T^{-1} \|\partial^\alpha u_B\|_{L^p(\Omega)} \tag{3.3.45}$$

for any multiindex α of length $|\alpha| = m \geq 0$. In particular, if we demand that

$$a = \frac{\tau}{\ell} \tag{3.3.46}$$

obeys

$$\frac{3}{2} < a < \frac{5}{2}, \tag{3.3.47}$$

then we have that

$$L^{-3}\|\nabla u\|^2_{L^2(\Omega)} = \frac{1}{T^2} 2^{N(2\tau-3\ell)} C_1 \tag{3.3.48}$$

and

$$L^{-3}\|u\|^2_{L^2(\Omega)} = \frac{L^2}{T^2} C_0. \tag{3.3.49}$$

It is natural to consider the wave number scales

$$k = L^{-1}2^{\ell j}. \tag{3.3.50}$$

The energy spectrum $E(k)$ is by definition the contribution of the kinetic energy at scale k, per unit mass and per scale:

$$E(k) = L^{-3}k^{-1}\|u_j\|^2_{L^2(\Omega)} \tag{3.3.51}$$

so, it follows from our construction of u_j that

$$E(k) = \frac{L^3}{T^2}(kL)^{2a-6}. \tag{3.3.52}$$

The range of scales is limited, the smallest length scale is $L2^{-N\ell}$. If we define a viscosity by

$$\nu = \frac{L^2}{T}2^{-N(2\tau-3\ell)} \tag{3.3.53}$$

then from (3.3.48) we have that

$$\epsilon = \nu L^{-3}\|\nabla u\|^2_{L^2(\Omega)} = \frac{L^2}{T^3}C_1. \tag{3.3.54}$$

Inserting in (3.3.52) we have thus

$$E(k) = C_1^{-\frac{2}{3}}\epsilon^{\frac{2}{3}}L^{\frac{5}{3}}(kL)^{2a-6}. \tag{3.3.55}$$

The Kolmogorov–Obukhov spectrum

$$E(k) = C_K\epsilon^{\frac{2}{3}}k^{-\frac{5}{3}} \tag{3.3.56}$$

is the only spectrum in this family of spectra that does not depend on L. It is obtained at the value

$$a = \frac{13}{6} \tag{3.3.57}$$

which is admissible in view of (3.3.47). If we express the viscosity ν of (3.3.53) in terms of the smallest length scale, (the "dissipation scale") $\ell_d = L2^{-N\ell}$ and in terms of the quantity ϵ of (3.3.54) we obtain

$$\nu = C_1^{-\frac{1}{3}}\epsilon^{\frac{1}{3}}L^{\frac{4}{3}-(2a-3)}(\ell_d)^{2a-3}. \tag{3.3.58}$$

Denoting by $k_d = (\ell_d)^{-1}$ the dissipation wave number scale, (largest wave number scale before exponential decay), we have

$$k_d = C_1^{-\frac{1}{3(2a-3)}} \epsilon^{\frac{1}{3(2a-3)}} L^{\frac{4}{3(2a-3)}-1} \nu^{-\frac{1}{2a-3}}. \tag{3.3.59}$$

Again, the only case which does not depend on L is the Kolmogorov–Obukhov spectrum case $a = \frac{13}{6}$ and in that case we obtain the familiar expression

$$k_d^{-1} = \ell_d = c\nu^{\frac{3}{4}} \epsilon^{-\frac{1}{4}}. \tag{3.3.60}$$

We have proved thus, in particular

Theorem 3.8 *Let Ω be an open set in $\mathbb{R}^3$. There exist smooth stationary solutions of the forced Navier–Stokes equations*

$$u \cdot \nabla u + \nabla p = \nu \Delta u + F, \qquad \nabla \cdot u = 0 \tag{3.3.61}$$

with $u \in C_0^\infty(\Omega)$, $\nabla p \in C_0^\infty(\Omega)$, and such that $\nu \|\nabla u\|^2_{L^2(\Omega)}$ is bounded below uniformly as in (3.3.54) as $\nu \to 0$. There is an inertial range of wave number scales $k \in [k_0, k_d]$ and an exponent $x \in (-3, -1)$, $x = 2a - 6$ with a of (3.3.46), such that the dissipation wave number scale $k_d \sim \nu^{-\frac{1}{x+3}}$ (see (3.3.59)) diverges with $\nu \to 0$ and the energy spectrum $E(k)$ obeys

$$E(k) \sim k^x \tag{3.3.62}$$

(see (3.3.52)) in the inertial range. The force F is smooth, compactly supported, and converges to zero as $\nu \to 0$ in $L^p(\Omega)$ for some p (depending on choice of parameter x) with $1 \le p < 2$.

The proof was given in the computation above, because of the tautology

$$u \cdot \nabla u + \nabla p - \nu \Delta u = F \tag{3.3.63}$$

with

$$F = -\nu \Delta u. \tag{3.3.64}$$

We have

$$x = 2a - 6 \tag{3.3.65}$$

with a given in (3.3.46). The only computation that remains to be shown uses (3.3.45) and (3.3.53), and yields

$$\|F\|_{L^p(\Omega)} \leq C_p \frac{L^{1+\frac{3}{p}}}{T^2} 2^{-N\ell(2a-3)} \tag{3.3.66}$$

which follows if $\frac{3}{p} > 1 + a$, or

$$\|F\|_{L^p(\Omega)} \leq C_p \frac{L^{1+\frac{3}{p}}}{T^2} 2^{N\ell(4-a-\frac{3}{p})} \tag{3.3.67}$$

if $\frac{3}{p} \in [4-a, 1+a]$. For each fixed a, we have $p \in [1, \frac{3}{1+a}]$ when $a \in (\frac{3}{2}, 2]$ and $p \in [1, \frac{3}{4-a}]$ when $a \in [2, \frac{5}{2})$.

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