



Study on Elastic Rod Oscillations Considering Material Relaxation Properties

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Abstract. A mathematical model for elastic oscillations of a longitudinal rod has been developed on the basis of relaxation terms in the Newton's second law. An exact analytical solution of the corresponding boundary value problem has been obtained using the method of separation of variables. The analysis of the obtained solution showed that taking into account the medium relaxation properties has a significant effect on the oscillatory process: the amplitude of the oscillations and the shape of the wave profile. Taking into account relaxation coefficients leads to the smoothing of the wave, eliminating jumps in the unknown displacement function. The author estimated for the first time, the influence of high-order derivatives in a modified equation of motion on an oscillatory process. It is shown that high-order derivatives, at a sufficiently large value of the relaxation coefficients, reduce the intensity of the oscillatory process. In this case, the delay of the displacement function in time occurs (compared to the case when the relaxation properties are not taken into account). The theoretical and experimental studies performed made it possible to determine the values of the relaxation and resistance coefficients.

Keywords: Material relaxation properties · Relaxation coefficients · Resistance coefficient · Longitudinal oscillations · An elastic rod · Analytical solution

1 Introduction

Most technical processes are characterized by oscillatory motions (oscillations) of some elements of equipment, mechanisms, and structures. Vibrations can occur due to the reciprocating movement of machine parts and assemblies; during rotation of turbine rotors, compressors; due to their imbalance, etc. Resonance phenomena are of particular interest [1–10]. The accuracy of the mathematical description of the above listed processes depends on the coefficients used in the calculations that characterize the material mechanical properties. For example, the ability of a material to resist stretching, compression during elastic deformation is characterized by the Young's modulus; the intensity of oscillation damping is determined by the coefficient of medium resistance, etc. In this paper, it is shown that a more accurate description of oscillatory processes needs taking into account the material relaxation properties. The effect of taking into account relaxation coefficients on oscillatory processes is considered using the example of damped longitudinal oscillations of an elastic rod.

The elastic deformation of a solid caused by some disturbance propagates with a speed depending on the medium properties. In this case, the wave process of medium oscillations is not characterized by the substance movement. The equations (of the hyperbolic type) describing these processes are derived using Hooke's law.

$$\sigma = E \frac{\partial U}{\partial x}, \quad (1)$$

and Newton's second law in the form of the motion equation [1–4, 11]

$$\rho \frac{\partial^2 U}{\partial t^2} = \frac{\partial \sigma}{\partial x}. \quad (2)$$

where σ —normal stress, N/m^2 ; U —displacement, m ; x —coordinate, m ; t —time, s ; ρ —density, kg/m^3 ; E —modulus of normal elasticity (Young's modulus), Pa ; $\varepsilon = \partial U / \partial x$ —deformation.

In order to take into account the resistance, assume that the resistance force is proportional to the displacement change in time

$$F_r = k \frac{\partial U}{\partial t},$$

where k —coefficient of proportionality.

Given that the resistance force F_r has a direction opposite to the displacement, and refers to volume forces, Eq. (2) can be written

$$\rho \frac{\partial^2 U}{\partial t^2} = \frac{\partial \sigma}{\partial x} - \frac{k}{V} \frac{\partial U}{\partial t}. \quad (3)$$

Substituting (1) into (3), we obtain the classical wave equation describing undamped (at $F_r = 0$) or damped (at $F_r \neq 0$) longitudinal oscillations of an elastic rod

$$\rho \frac{\partial^2 U}{\partial t^2} = E \frac{\partial^2 U}{\partial x^2} - \frac{k}{V} \frac{\partial U}{\partial t}.$$

In the formula (1), there is no cause–effect relationship of the phenomena [12]. The cause (effective force) here is deformation $\varepsilon = \partial U / \partial x$, and the effect is stress σ . Cause and effect in this case are not separated in time. Therefore, the effect with a change in the cause occurs instantaneously (step change). However, the propagation velocity of the potentials of any physical fields cannot take infinite values. In a real body, the process of their change occurs with a certain delay in time due to the material relaxation properties, which are taken into account by relaxation coefficients. Therefore, it should be noted that a jump-like change in stress over time occurs in classical models.

2 Mathematical Statement of the Problem

This paper presents the development results of a mathematical model for longitudinal oscillations of an elastic rod considering the relaxation properties and material resistance.

When deriving a differential equation describing this process, the author proposes to use a modified Newton’s law

$$\rho \frac{\partial^2}{\partial t^2} \left(U + \sum_{k=1}^N \tau_k^k \frac{\partial^k U}{\partial t^k} \right) = \frac{\partial}{\partial x} \left(\sigma + \sum_{k=1}^N r_k^k \frac{\partial^k \sigma}{\partial t^k} \right) - \frac{k}{V} \frac{\partial}{\partial t} \left(U + \sum_{k=1}^N \tau_k^k \frac{\partial^k U}{\partial t^k} \right). \quad (4)$$

Dividing the left- and right-hand sides of Eq. (4) by ρ and substituting $\sigma = E \partial U / \partial x$ (Hooke’s law formula) into (4), we obtain the equation of rod oscillations, taking into account the delay in stress and displacement functions.

$$\frac{\partial^2}{\partial t^2} \left(U + \sum_{k=1}^N \tau_k^k \frac{\partial^k U}{\partial t^k} \right) + \eta \frac{\partial}{\partial t} \left(U + \sum_{k=1}^N \tau_k^k \frac{\partial^k U}{\partial t^k} \right) = e^2 \frac{\partial}{\partial x} \left(\frac{\partial U}{\partial x} + \sum_{k=1}^N r_k^k \frac{\partial^{k+1} U}{\partial x \partial t^k} \right). \quad (5)$$

where $\eta = k / (\rho V)$ —resistance coefficient having dimension, l/s ; $e = \sqrt{E/\rho}$ —propagation velocity of the longitudinal disturbance.

It should be noted that, depending on the accepted values of the relaxation coefficients, the differential Eq. (5) reduces to the well-known ones: hyperbolic equation of undamped oscillations (at $\tau_k^k = r_k^k = \eta = 0$); classical hyperbolic equation with damping (at $\tau_k^k = r_k^k = 0$; $\eta \neq 0$); equation of two-phase delay—Maxwell and Kelvin–Voigt model of the viscoelastic body (at $N = 1$) [9, 10].

The paper deals with the effect of high-order derivatives on the oscillatory process $N = 2$. So, an analytical solution of the problem for longitudinal oscillations of a rod, one end of which is fixed, and the second one is free to move, is presented. The mathematical statement of the problem in this case is as follows

$$\begin{aligned} \frac{\partial^2 U}{\partial t^2} + \tau_1 \frac{\partial^3 U}{\partial t^3} + \tau_2^2 \frac{\partial^4 U}{\partial t^4} + \eta \frac{\partial U}{\partial t} + \eta \tau_1 \frac{\partial^2 U}{\partial t^2} \\ + \eta \tau_2^2 \frac{\partial^3 U}{\partial t^3} = e^2 \left[\frac{\partial^2 U}{\partial x^2} + r_1 \frac{\partial^3 U}{\partial x^2 \partial t} + r_2^2 \frac{\partial^4 U}{\partial x^2 \partial t^2} \right], \end{aligned}$$

or, after some transformations:

$$\begin{aligned} \tau_2^2 \frac{\partial^4 U}{\partial t^4} + (\tau_1 + \eta \tau_2^2) \frac{\partial^3 U}{\partial t^3} + (1 + \eta \tau_1) \frac{\partial^2 U}{\partial t^2} \\ + \eta \frac{\partial U}{\partial t} = e^2 \left[\frac{\partial^2 U}{\partial x^2} + r_1 \frac{\partial^3 U}{\partial x^2 \partial t} + r_2^2 \frac{\partial^4 U}{\partial x^2 \partial t^2} \right] \quad (6) \end{aligned}$$

Let us find a solution to the boundary value problem of rod oscillations, one end of which is rigidly fixed (at $x = \delta$), and the second one is free to move. At the initial moment of time, the rod is deformed according to a linear law. The boundary conditions to Eq. (6) in this case are as follows

$$\begin{aligned} U(x, 0) = b(\delta - x); \quad \frac{\partial U(x, 0)}{\partial t} = \frac{\partial^2 U(x, 0)}{\partial t^2} = \frac{\partial^3 U(x, 0)}{\partial t^3} = 0; \\ U(\delta, t) = 0; \quad \frac{\partial U(0, t)}{\partial x} = 0. \quad (7) \end{aligned}$$

where δ —length of the rod, m ; b —coefficient considering the initial displacement of the rod.

Problem (6) and (7) can be reduced to a dimensionless form. To do this, we introduce dimensionless variables and parameters:

$$\Theta = \frac{U}{U_0}; \quad \xi = \frac{x}{\delta}; \quad \text{Fo} = \frac{et}{\delta}; \quad \text{Fo}_k = \frac{e\tau_k}{\delta}; \quad R_k = \frac{er_k}{\delta}; \quad \gamma = \frac{\delta\eta}{e},$$

where Θ —dimensionless displacement; ξ —dimensionless coordinate; Fo —Fourier number (dimensionless time); $U_0 = b\delta$; Fo_k , R_k —dimensionless relaxation coefficients; γ —dimensionless coefficient of medium resistance.

In view of the introduced notation, problem (6) and (7) can be written

$$\begin{aligned} & \text{Fo}_2^2 \frac{\partial^4 \Theta(\xi, \text{Fo})}{\partial \text{Fo}^4} + (\text{Fo}_1 + \gamma \text{Fo}_2^2) \frac{\partial^3 \Theta(\xi, \text{Fo})}{\partial \text{Fo}^3} + (1 + \gamma \text{Fo}_1) \frac{\partial^2 \Theta(\xi, \text{Fo})}{\partial \text{Fo}^2} \\ & + \gamma \frac{\partial \Theta(\xi, \text{Fo})}{\partial \text{Fo}} = \frac{\partial^2 \Theta(\xi, \text{Fo})}{\partial \xi^2} + R_1 \frac{\partial^3 \Theta(\xi, \text{Fo})}{\partial \xi^2 \partial \text{Fo}} \\ & + R_2^2 \frac{\partial^4 \Theta(\xi, \text{Fo})}{\partial \xi^2 \partial \text{Fo}^2} \quad (\text{Fo} > 0; \quad 0 < \xi < 1) \end{aligned} \tag{8}$$

$$\Theta(\xi, 0) = 1 - \xi; \quad \frac{\partial \Theta(\xi, 0)}{\partial \text{Fo}} = \frac{\partial^2 \Theta(\xi, 0)}{\partial \text{Fo}^2} = \frac{\partial^3 \Theta(\xi, 0)}{\partial \text{Fo}^3} = 0; \tag{9}$$

$$\Theta(1, \text{Fo}) = 0; \quad \frac{\partial \Theta(0, \text{Fo})}{\partial \xi} = 0. \tag{10}$$

3 Problem Solving Method

According to the Fourier method, a solution to problem (8)–(10) can be found as a product of two functions

$$\Theta(\xi, \text{Fo}) = \phi(\text{Fo}) \psi(\xi). \tag{11}$$

where $\phi(\text{Fo})$ —unknown time function; $\psi(\xi)$ —unknown coordinate function.

Substituting (11) into (8), we find

$$\begin{aligned} & \text{Fo}_1^2 \frac{d^4 \varphi}{d \text{Fo}^4} + \text{Fo}_1 \text{Fo}_4 \frac{d^3 \varphi}{d \text{Fo}^3} + (\text{Fo}_2^2 \nu_k + \text{Fo}_4) \frac{d^2 \varphi}{d \text{Fo}^2} \\ & + (\nu_k \text{Fo}_2 + \text{Fo}_3) \frac{d \varphi}{d \text{Fo}} + \nu_k \varphi = 0; \end{aligned} \tag{12}$$

$$\frac{d^2 \psi}{d \xi^2} + \nu_k \psi = 0, \tag{13}$$

where $\nu_k = \text{const}$.

The boundary conditions to Eq. (13), according to (10), are as follows

$$\psi(1) = 0; \frac{d\psi(0)}{d\xi} = 0. \tag{14}$$

The solution to the Sturm–Liouville problem (13)–(14) is presented as follows

$$\psi(\xi) = \cos\left((2k - 1)\frac{\pi}{2}\xi\right) (k = \overline{1, \infty}). \tag{15}$$

Conditions (14) and (15) are satisfied by relation (15). Substituting (15) into (13) we find the formula for calculating the eigenvalues

$$\nu_k = \frac{\pi^2(2k - 1)^2}{4} (k = \overline{1, \infty}). \tag{16}$$

The characteristic equation to differential Eq. (12) is as follows

$$\begin{aligned} & \text{Fo}_1^2 z^4 + \text{Fo}_1 \text{Fo}_4 z^3 + (\text{Fo}_2^2 \nu_k + \text{Fo}_4) z^2 \\ & + (\nu_k \text{Fo}_2 + \text{Fo}_3) z + \nu_k \phi = 0. \end{aligned} \tag{17}$$

Particular solutions of Eq. (57) are written as

$$\phi_k(\text{Fo}) = \sum_{j=1}^4 C_{jk} \exp(z_{jk} \text{Fo}) (k = \overline{1, \infty}), \tag{18}$$

where C_{jk} —unknown coefficients; z_{jk} —roots of the characteristic Eq. (17), determined numerically.

Substituting (15) and (18) into (11), we obtain

$$\Theta_k(\xi, \text{Fo}) = \sum_{j=1}^4 C_{jk} \exp(z_{jk} \text{Fo}) \cos\left((2k - 1)\frac{\pi}{2}\xi\right) (k = \overline{1, \infty}). \tag{19}$$

All particular solutions (19) satisfy Eq. (8) and conditions (10). However, none of them satisfy the initial conditions (9). The sum of particular solutions is presented as follows

$$\Theta(\xi, \text{Fo}) = \sum_{k=1}^{\infty} \left\{ \sum_{j=1}^4 C_{jk} \exp(z_{jk} \text{Fo}) \cos\left[(2k - 1)\frac{\pi}{2}\xi\right] \right\}. \tag{20}$$

To find the unknown coefficients C_{jk} , we find the residual of the initial conditions (9) and require the residual orthogonality to all eigenfunctions. Due to the orthogonality of cosines, the solving for the resulting system of $4k$ equations for C_{jk} , reduces to solving four algebraic equations for each value k

$$\begin{cases} \int_0^1 (C_{1k} + C_{2k} + C_{3k} + C_{4k}) \cos\left[(2k - 1)\frac{\pi}{2}\xi\right]^2 d\xi - \int_0^1 (1 - \xi) \cos\left[(2k - 1)\frac{\pi}{2}\xi\right] d\xi = 0; \\ \int_0^1 (C_{1k} z_{1k}^m + C_{2k} z_{2k}^m + C_{3k} z_{3k}^m + C_{4k} z_{4k}^m) \cos\left[(2k - 1)\frac{\pi}{2}\xi\right]^2 d\xi = 0 \quad (m = 1, 2, 3). \end{cases}$$

After calculating the unknown coefficients C_{jk} , the solution to problem (8)–(10) is found from (20).

4 Results and Discussions

Calculation results by formula (20) are presented in Figs. 1 and 2. It follows from the analysis of the results that taking into account the medium relaxation properties leads to a significant change in the wave profile. Figure 1a shows the calculation results of displacements using formula (20) at a point $\xi = 0, 4$. The analysis shows that taking into account relaxation terms in the equation of motion leads to the wave profile smoothing, eliminating jumps in the unknown function. Moreover, high-order derivatives (at sufficiently large values of relaxation coefficients) also effect the oscillatory process.

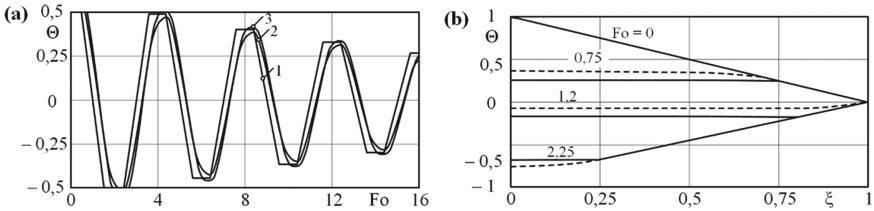


Fig. 1 a Rod oscillations ($\gamma = 0, 1$): 1— $Fo_k = R_k = 0$ ($k = 1, 2$); 2— $Fo_1 = R_1 = 0, 3, Fo_2 = R_2 = 0$; 3— $Fo_1 = R_1 = 0, 3, Fo_2 = R_2 = 0, 21$; (b) — — without relaxation; - - - with relaxation ($Fo_1 = R_1 = 0, 3, Fo_2 = R_2 = 0, 21$)

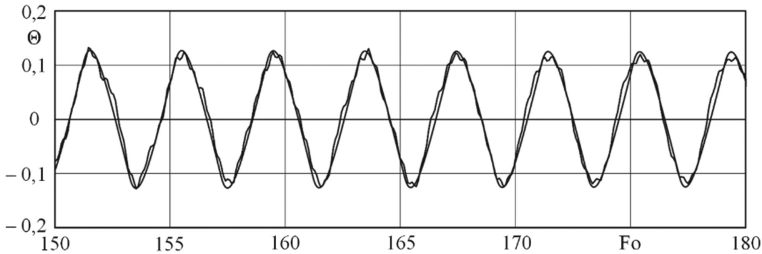


Fig. 2 Rod oscillations in the area of linear amplitude variation (on a larger scale): 1—calculation at $Fo_1 = R_1 = 0, 5, \gamma = 0, 003$; 2—experimental data

Figure 1b shows the displacement distribution along the coordinate at various points in time. It should be noted that when the material relaxation properties are taken into account, a delay in the displacement function at each moment of time occurs (compared to the case when relaxation properties are not taken into account).

In order to verify the developed models, a series of experiments was also performed. The studies were carried out on specialized equipment. The results obtained practically coincide with the experimental data [1] (Fig. 2).

5 Conclusion

1. An exact analytical solution to the boundary value problem on the elastic rod oscillations (one end of which is rigidly fixed, and the second one is free to move) is

- obtained. The values of relaxation and resistance coefficients are determined on the basis of a comparative analysis of the results of theoretical and experimental studies.
2. A mathematical model of damped oscillations of elastic bodies taking into account the medium resistance and the material relaxation properties was developed. Analysis of the results makes it possible to conclude that taking into account the relaxation terms in the Hooke's law enables approaching the description of real processes that do not allow infinite stresses at the initial time under boundary conditions of the first kind, as well as jumps in the unknown displacement function.

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