Chapter 8
Bounded Distributive Lattices with Two Subordinations

Sergio Celani and Ramon Jansana

Abstract In this paper we consider the notion of subordination on distributive lattices, equivalent to that of quasi-modal operator for distributive lattices introduced by Castro and Celani in 2004. We provide topological dualities for categories of distributive lattices with a subordination and then for some categories of distributive lattices with two subordinations, structures that we name bi-subordination lattices. We investigate three classes of bi-subordination lattices. In particular that of positive bi-subordination lattices.

Keywords Subordination relations on distributive lattices · Contact relations · Distributive lattices · Distributive lattices with operators · Quasi-modal operators

8.1 Introduction

Subordination algebras and contact algebras originate in the duality for compact Hausdorff spaces developed by de Vries (1962) where the algebraic duals of the spaces are complete Boolean algebras with a proximity relation. The relations on arbitrary Boolean algebras that satisfy the conditions in the definition of de Vries proximity relation are known as compingent relations. Deleting some of the conditions we have the subordination relations of Bezhanishvili et al. (2016). These relations also originate in the Region-based theory of space, where precontact rela-

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tions on a Boolean algebra, as are called in Dimov and Vakarelov (2006c), were introduced by Düntsch and Vakarelov (2003, 2007) under the name of proximity relations as a weakening of the relations of the contact algebras studied in Dimov and Vakarelov (2006a, b). The precontact relations and the subordination relations of Bezhanishvili et al. (2016) are dual notions (a is related to b in one relation if and only if a is not related to the complement of b in the other). An equivalent concept to those of subordination relation and precontact relation is that of quasi-modal operator introduced by Celani in Celani (2001), where a topological duality for Boolean algebras with a quasi-modal operator is given.

The definition of subordination relation on a Boolean algebra does not mention the complement operation and therefore it can be considered for bounded distributive lattices as well as its equivalent concept of quasi-modal operator. This is done in Castro and Celani (2004) where the concept of quasi-modal operator for bounded distributive lattices is studied and a topological Priestley duality is given for bounded distributive lattices with two quasi-modal operators. From the results proved in Castro and Celani (2004) one easily obtains a duality for bounded distributive lattices with a subordination.

In this paper we study three kinds of distributive lattices with two subordination relations that we call bi-subordination lattices: the bi-subordination lattices where one subordination is included in the other, the bi-subordination lattices where one subordination is the converse of the other, and the positive bi-subordination lattices where the relation between one subordination and the other is similar to that between the box operation and the diamond operator in positive modal algebras. We present topological dualities for these classes of bi-subordination lattices. In order to be able to do it we introduce in detail topological dualities for several categories of bounded distributive lattices with a subordination given by different choices of morphisms between them. The dual objects are Priestley spaces endowed with a binary relation. Some of the results we report can be found in Castro and Celani (2004) but for completeness we decided to present them with full proofs, besides phrasing them in terms of subordination relations instead of quasi-modal operators.

After the preliminaries section we present in Sect. 8.3 the concepts of subordination, Δ-quasi-modal operator and ∇-quasi-modal operator for bounded distributive lattices as well as the concept of bi-subordination lattice. We also introduce some tools necessary for the dualities we present in Sect. 8.5. In Sect. 8.4 we discuss some examples of bi-subordination lattices and define the concept of positive bi-subordination lattice. In Sect. 8.5 we present the dualities for different categories of subordination lattices. The objects of the dual categories are Priestley spaces with two binary relations, one for each subordination. We extend the dualities to bi-subordination lattices in the natural way. Finally, in Sect. 8.6 we first discus the dualities for positive bi-subordination lattices that naturally result when we dualize each subordination by a relation. Then we present a different duality where the objects are Priestley spaces with a single binary relation in a similar way as one can obtain a Priestley duality for positive modal algebras by considering only one relation on the Priestley space dual to the distributive lattice reduct instead of considering one relation for the box operation and another one for the diamond operator.

8.2 Preliminaries

In this preliminaries section we introduce the most basic concepts and notation we need related to posets, lattices, and binary relations. The other concepts assumed to be known in the paper, like Priestley space, will be introduced when needed.

Let $\langle X, \leq \rangle$ be a partially ordered set (or poset). A set $U \subseteq X$ is an *upset* of X if for every $x, y \in X$, if $x \in U$ and $x \leq y$, then $y \in U$. The dual notion is that of downset, that is, a set $V \subseteq X$ is a *downset* of X if for every $x, y \in X$ such that $x \in V$ and $y \leq x$, we have $y \in V$.

We assume knowledge of bounded distributive lattices (Balbes and Dwinger 1974; Davey and Priestley 2002; Grätzer 2009). Let L be a bounded distributive lattice. Recall that a *filter* of L is a nonempty subset of L that is an upset w.r.t. the order of the lattice and is closed under the operation of meet. Dually, an *ideal* of L is a nonempty subset of L that is a downset w.r.t. the order of the lattice and is closed under the operation of join. A filter F of L is said to be *prime* if for every $a, b \in L$ such that $a \vee b \in F$ it holds that $a \in F$ or $b \in F$. If L is a Booelan lattice (i.e. a lattice where every element has a complement) the prime filters are known as *ultrafilters*. The filter generated by a set $H \subseteq L$ will be denoted by $[H)$ or by $\mathrm{Fg}(H)$ and the ideal generated by H by $(H]$ or $\mathrm{Ig}(H)$. Given $a \in L$, we write $[a)$ or $\mathrm{Fg}(a)$ for the filter generated by $\{a\}$ and $(a]$ or $\mathrm{Ig}(a)$ for the ideal generated by $\{a\}$. The set, and the lattice, of ideals of L will be denoted by $\mathrm{Id}(L)$ and that of its filters by $\mathrm{Fi}(L)$.

For every set X, we use $\mathcal{P}(X)$ to denote the powerset of X as well as the powerset lattice and the powerset Booelan algebra of (the subsets of) X.

If X is an arbitrary set and R a binary relation on X, then for every $x \in X$ we let

$$R(x) := \{y \in X : \langle x, y\rangle \in R\} \qquad \text{and} \qquad R^{-1}(x) := \{y \in X : \langle y, x\rangle \in R\}$$

and for every set $Y \subseteq X$ we let

$$\begin{aligned} R[Y] &:= \{y \in X : (\exists x \in Y)\, xRy\}, \\ R^{-1}[Y] &:= \{x \in X : (\exists y \in Y)\, xRy\}, \\ \Box_R(Y) &:= \{x \in X : R(x) \subseteq Y\}. \end{aligned}$$

Note that

$$R[Y] = \bigcup \{R(y) : y \in Y\} \quad \text{and} \quad R^{-1}[Y] = \{x \in X : R(x) \cap Y \neq \emptyset\}.$$

We also refer to $R^{-1}[Y]$ by $\Diamond_R(Y)$. Note that then $\Box_R(Y) = [\Diamond_R(Y^c)]^c$ and $\Diamond_R(Y) = [\Box_R(Y^c)]^c$. Moreover, we denote by R^{-1} the converse of the relation R, i.e., $R^{-1} = \{\langle x, y\rangle : yRx\}$.

8.3 Subordination Relations and Quasi-modal Operators on Distributive Lattices

The notion of subordination on a Boolean algebra defined in Bezhanishvili et al. (2016) is equivalent to the notion of precontact or proximity relation on a Boolean algebra given in Dimov and Vakarelov (2006c) and Düntsch and Vakarelov (2007). It can be exported to bounded distributive lattices since it does not involve the operation of complement.

Definition 1 A *subordination* on a bounded distributive lattice L is a binary relation $\prec$ on L satisfying the following conditions for every $a, b, c, d \in L$:

(S1) $0 \prec 0$ and $1 \prec 1$;
(S2) $a \prec b, c$ implies $a \prec b \wedge c$;
(S3) $a, b \prec c$ implies $a \vee b \prec c$;
(S4) $a \leq b \prec c \leq d$ implies $a \prec d$.

A *subordination lattice* is a pair $\langle L, \prec \rangle$ where L is a bounded distributive lattice and $\prec$ a subordination on L. A *bi-subordination lattice* is a triple $\langle L, \prec, \lhd \rangle$ where L is a bounded distributive lattice and $\prec, \lhd$ are subordinations on L.

We will denote by SLat the class of subordination lattices and by BSLat the class of bi-subordination lattices.

In the case of Boolean algebras, the subordination relations are equivalent to the quasi-modal operators of Celani (2001). Similarly, on bounded distributive lattices they are equivalent to the quasi-modal operators on bounded distributive lattices introduced in Castro and Celani (2004).

Definition 2 (Castro and Celani 2004) A Δ-*quasi-modal operator* on a bounded distributive lattice L is a map $\Delta : L \to \mathrm{Id}(L)$ satisfying the conditions:

(QM1) $\Delta(a \wedge b) = \Delta(a) \cap \Delta(b)$, for every $a, b \in L$,
(QM2) $\Delta(1) = L$,

that is, it is a meet-homomorphism (preserving also the top element) from the lattice L to the lattice of its ideals.

Dually, a ∇-*quasi-modal operator* on a bounded distributive lattice L is a map $\nabla : L \to \mathrm{Fi}(L)$ satisfying the conditions:

(QM3) $\nabla(a \vee b) = \nabla(a) \cap \nabla(b)$, for every $a, b \in L$,
(QM4) $\nabla(0) = L$,

that is, it is a join-homomorphism (preserving also the bottom element) from L to the dual of the lattice of the filters of L.

Remark 1 A dual modal operator $\Box$ on a bounded distributive lattice L is a unary operation on L that is a meet-homomorphism from L to L preserving the top element. The map that sends every element of L to the principal ideal it generates is an embedding from L to the lattice of the ideals of L. Thus we can look at a dual modal operator $\Box$ on a bounded distributive lattice L as a meet-homomorphism from L to the lattice $\mathrm{Id}(L)$ of the ideals of L that preserves also the top element and has the property that the elements of its range are principal ideals. In this way, the concept of Δ-quasi-modal operator on a bounded distributive lattice is a natural generalization of the notion of dual modal operator. Dually, an operator $\Diamond$ on a bounded distributive lattice L is a unary operation on L that is a join-homomorphism from L to L that preserves the bottom element and since L is dually embeddable into the lattice of the filters of L by the map that sends every element of L to the principal filter it generates, an operator $\Diamond$ on a bounded distributive lattice L can be seen as a join-homomorphism from L to the dual lattice of the lattice $\mathrm{Fi}(L)$ of the filters of L that in addition preserves the bottom element. Therefore, the concept of ∇-quasi-modal operator on a bounded distributive lattice is a natural generalization of the notion of modal operator.

Quasi-modal operators and subordination relations are strictly connected in the way we proceed to describe. Recall the well-known fact that any map $f : L \to \mathcal{P}(L)$ determines two relations $R_f, R_f^+ \subseteq L \times L$, one the converse of the other, defined by the conditions

$$aR_f b \quad \text{iff} \quad a \in f(b) \qquad \text{and} \qquad aR_f^+ b \quad \text{iff} \quad b \in f(a).$$

Conversely, every relation $R \subseteq L \times L$ determines two maps $f_R, f_R^+ : L \to \mathcal{P}(L)$ defined by the conditions

$$f_R(a) := R^{-1}(a) = \{b \in L : bRa\} \qquad \text{and} \qquad f_R^+(a) := R(a) = \{b \in L : aRb\}.$$

It is immediate to see that if $f : L \to \mathcal{P}(L)$, then $f_{R_f} = f$ and $f^+_{R_f^+} = f$ and that if $R \subseteq L \times L$, then $R_{f_R} = R$ and $R^+_{f_R^+} = R$.

We apply these facts to Δ-quasi-modal operators, ∇-quasi-modal operators and subordinations on L.

Let $f : L \to \mathcal{P}(L)$ be a map. It is easy to see that f is a Δ-quasi-modal operator if and only if its associated relation R_f is a subordination on L, and that f is a ∇-quasi-modal operator if and only if R_f^+ is a subordination on L.

If $\Delta : L \to \mathcal{P}(L)$ is a Δ-quasi-modal operator, then we denote the relation R_Δ by $\prec_\Delta$. Thus for every $a, b \in L$

$$a \prec_\Delta b \quad \text{iff} \quad a \in \Delta(b).$$

Analogously, if $\nabla : L \to \mathcal{P}(L)$ is a ∇-quasi-modal operator, then we denote the relation R_∇^+ by $\prec_\nabla$ and we have for every $a, b \in L$

$$a \prec_\nabla b \quad \text{iff} \quad b \in \nabla(a).$$

Consider now a binary relation R on L. It is easy to see that the function $f_R : L \to \mathcal{P}(L)$ is a Δ-quasi-modal operator on L if and only if R is a subordination and that this holds if and only if $f_R^+ : L \to \mathcal{P}(L)$ is a ∇-quasi-modal operator on L.

If $\prec$ is a subordination on L, then we denote the map $f_\prec$ by $\Delta_\prec$ and the map $f_\prec^+$ by $\nabla_\prec$. Hence, for every $a \in L$

$$\Delta_\prec(a) := \{b \in B : b \prec a\} \qquad \text{and} \qquad \nabla_\prec(a) := \{b \in B : a \prec b\}\,.$$

Since Δ-quasi-modal operators correspond to subordinations and these to ∇-quasi-modal operators, the procedures just described above allow us to associate with every Δ-quasi-modal operator a ∇-quasi-modal operator and conversely, in the following way.

Let L be a bounded distributive lattice and Δ a Δ-quasi-modal operator on L. The ∇-quasi-modal operator $\nabla_{\prec_\Delta}$ of the subordination $\prec_\Delta$ is then given for each $a \in L$ by

$$\nabla_{\prec_\Delta}(a) := \{b \in L : a \in \Delta(b)\}\,.$$

In a similar way, given a ∇-quasi-modal operator ∇, the Δ-quasi-modal operator of the subordination $\prec_\nabla$ is given for each $a \in L$ by

$$\Delta_{\prec_\nabla}(a) := \{b \in L : a \in \nabla(b)\}\,.$$

It immediately follows that $\Delta_{\nabla_{\prec_\Delta}} = \Delta$ and $\nabla_{\Delta_{\prec_\nabla}} = \nabla$.

Note that due to the equivalence between subordinations and Δ-(∇-)modal operators, Remark 1 shows that subordinations can be taken as generalizations of modal operators.

Remark 2 If L is a bounded distributive lattice, $\Box$ a dual modal operator on L and $\Diamond$ a modal operator on L, then it is easy to see that the binary relations $\prec_\Box$ and $\prec_\Diamond$ defined on L by setting for every $a, b \in L$

$$a \prec_\Box b \iff a \leq \Box b$$

and

$$a \prec_\Diamond b \iff \Diamond a \leq b$$

are subordinations on L.

The Δ-quasi-modal operator $\Delta_{\prec_\Box}$ associated with $\prec_\Box$ satisfies that $\Delta_{\prec_\Box}(a) = (\Box a]$ for all $a \in L$. The ∇-quasi-modal operator $\nabla_{\prec_\Box}$ of $\prec_\Box$ is then given by the condition $b \in \nabla_{\prec_\Box}(a)$ if and only if $a \leq \Box b$. Therefore, $\nabla_{\prec_\Box}(a) = \Box^{-1}[[a)]$ for every $a \in B$.

Similarly, the ∇-quasi-modal operator associated with $\prec_\Diamond$ satisfies for every $a \in L$ that $\nabla_{\prec_\Diamond}(a) = [\Diamond a)$. The Δ-quasi-modal operator of $\prec_\Diamond$ is then given for every $a, b \in L$ by the condition $b \in \Delta_{\prec_\Diamond}(a)$ if and only if iff $\Diamond b \leq a$. Thus, for every $a \in L$ we have $\Delta_{\prec_\Diamond}(a) = \Diamond^{-1}[(a]]$.

Being the notions of subordination relation and Δ-quasi-modal operator equivalent, as well as equivalent to that of ∇-quasi-modal operator, we can take any of them as a primitive notion. We decided to take the notion of subordination as primitive in this paper; nevertheless we will make use of the associated quasi-modal operators on some proofs and statements.

In Castro and Celani (2004) the authors introduce and study quasi-modal lattices which consist of a bounded distributive lattice together with both a Δ-quasi-modal operator and a ∇-quasi-modal operator. Thus they consider in disguise bounded distributive lattices with two subordinations, i.e., bi-subordination lattices.

We proceed to introduce in the remaining part of this section some tools that are essential to the presentation of the results in the paper.

8.3.1 Two Maps on the Power Set of a Subordination Lattice Determined by the Subordination Relation

Given a bounded distributive lattice with a subordination we define two maps on the poset of all subsets of the lattice determined by the subordination and present the properties we need. One is a modal operator and the other its dual. Using them we will define two relations on the set of prime filters of a bounded distributive lattice with a subordination.

Let L be a bounded distributive lattice and $\prec$ a subordination on L. The maps $\Delta_{\prec}^{-1} : \mathcal{P}(L) \to \mathcal{P}(L)$ and $\nabla_{\prec}^{-1} : \mathcal{P}(L) \to \mathcal{P}(L)$ are defined by setting for every $C \subseteq L$:

1. $\Delta_{\prec}^{-1}(C) := \{a \in L : \Delta_{\prec}(a) \cap C \neq \emptyset\}$,
2. $\nabla_{\prec}^{-1}(C) := \{a \in L : \nabla_{\prec}(a) \subseteq C\}$.

These two maps are obviously monotone (w.r.t. inclusion), $\nabla_{\prec}^{-1}$ distributes over intersections, $\Delta_{\prec}^{-1}$ over unions, $\Delta_{\prec}^{-1}(\emptyset) = \emptyset$, and $\nabla_{\prec}^{-1}(L) = L$. It is easy to see that for every $C \subseteq L$,

$$\Delta_{\prec}^{-1}(C) = (\nabla_{\prec}^{-1}(C^c))^c \quad \text{and} \quad \nabla_{\prec}^{-1}(C) = (\Delta_{\prec}^{-1}\left(C^c\right))^c.$$

Hence, $\Delta_{\prec}^{-1}$ is a modal operator on the powerset lattice $\mathcal{P}(L)$ and $\nabla_{\prec}^{-1}$ is its dual.

The first two items of the next lemma are proved in Castro and Celani (2004).

Lemma 1 *For every filter F, every ideal I, and every prime filter P of L:*

1. *$\Delta_{\prec}^{-1}(F)$ is a filter,*
2. *$\nabla_{\prec}^{-1}(I)$ is an ideal,*
3. *$\left(\nabla_{\prec}^{-1}(P)\right)^c$ is an ideal.*

Proof We abbreviate all along the proof $\Delta_{\prec}$ by Δ and $\nabla_{\prec}$ by ∇.

(1) Suppose that $a, b \in \Delta^{-1}(F)$. Then $\Delta(a) \cap F \neq \emptyset$ and $\Delta(b) \cap F \neq \emptyset$. Let $c \in \Delta(a) \cap F$ and $d \in \Delta(b) \cap F$. Then $c \wedge d \in F$ and $c \wedge d \in \Delta(a) \cap \Delta(b)$, because

these sets are ideals. Hence $c \wedge d \in \Delta(a \wedge b)$. Therefore, $\Delta(a \wedge b) \cap F \neq \emptyset$ and so $a \wedge b \in \Delta^{-1}(F)$. Suppose now that $a \in \Delta^{-1}(F)$ and $a \leq b$. Then $\Delta(a) \cap F \neq \emptyset$. Since $\Delta(a) \subseteq \Delta(b)$, $\Delta(b) \cap F \neq \emptyset$. Hence $b \in \Delta^{-1}(F)$.

(2) Suppose that $a, b \in \nabla^{-1}(I)$. Then $\nabla(a) \subseteq I$ and $\nabla(b) \subseteq I$. Therefore $\nabla(a \vee b) = \nabla(a) \cap \nabla(b) \subseteq I$. Hence $a \vee b \in \nabla^{-1}(I)$. Suppose that $a \in \nabla^{-1}(I)$ and $b \leq a$. Then $\nabla(a) \subseteq I$ and $\nabla(a) \subseteq \nabla(b)$. Therefore, $\nabla(b) \subseteq I$. Thus, $b \in \nabla^{-1}(I)$.

(3) Let P be a prime filter of L. First note that since $\nabla(0) = L$ and $P \neq L$, we have $\nabla(0) \nsubseteq P$. Therefore $0 \notin \nabla^{-1}(P)$. Suppose now that $a, b \notin \nabla^{-1}(P)$. Hence there are $c \in \nabla(a)$ and $d \in \nabla(b)$ such that $c, d \notin P$. Since P is a prime filter it follows that $c \vee d \notin P$. But since $\nabla(a), \nabla(b)$ are filters $c \vee d \in \nabla(a) \cap \nabla(b) = \nabla(a \vee b)$. Hence $\nabla(a \vee b) \nsubseteq P$ and therefore $a \vee b \notin \nabla^{-1}(P)$. Finally, suppose that $a \notin \nabla^{-1}(P)$ and $b \leq a$. Then $\nabla(a) \nsubseteq P$. But since $\nabla(a) \subseteq \nabla(b)$, $\nabla(b) \nsubseteq P$ which implies that $b \notin \nabla^{-1}(P)$. □

8.3.2 The Two Relations on the Set of Prime Filters of a Lattice Determined by a Subordination

Let L be a bounded distributive lattice and $\prec$ a subordination on L. We define the binary relation $R^{\Delta}_{\prec}$ on the set $X(L)$ of the prime filters of L by setting for every $P, Q \in X(L)$

$$(P, Q) \in R^{\Delta}_{\prec} \iff \Delta^{-1}_{\prec}(P) \subseteq Q.$$

In a similar way, we define the binary relation $R^{\nabla}_{\prec}$ on $X(L)$ by setting for every $P, Q \in X(L)$

$$(P, Q) \in R^{\nabla}_{\prec} \iff Q \subseteq \nabla^{-1}_{\prec}(P).$$

Proposition 1 *Let $\prec$ be a subordination on a bounded distributive lattice L. Then $R^{\Delta}_{\prec}$ is the converse of the relation $R^{\nabla}_{\prec}$.*

Proof Suppose that $P R^{\Delta}_{\prec} Q$, i.e., that $\{a \in L : \Delta_{\prec}(a) \cap P \neq \emptyset\} \subseteq Q$. To prove that $Q R^{\nabla}_{\prec} P$ we have to show that $P \subseteq \{a \in L : \nabla_{\prec}(a) \subseteq Q\}$. Suppose that $a \in P$ and $\nabla_{\prec}(a) \nsubseteq Q$. Let $b \in \nabla_{\prec}(a)$ be such that $b \notin Q$. Thus $b \notin \{a \in L : \Delta_{\prec}(a) \cap P \neq \emptyset\}$, that is, $\Delta_{\prec}(b) \cap P = \emptyset$. Note that since $b \in \nabla_{\prec}(a)$, $a \in \Delta_{\prec}(b)$. Therefore $\Delta_{\prec}(b) \cap P \neq \emptyset$. Hence $b \in Q$, a contradiction.

Conversely, suppose that $Q R_{\nabla_{\prec}} P$, so that $P \subseteq \{a \in L : \nabla_{\prec}(a) \subseteq Q\}$. To prove that $P R^{\Delta}_{\prec} Q$, suppose that $\Delta_{\prec}(a) \cap P \neq \emptyset$ and let $b \in \Delta_{\prec}(a) \cap P$. Then $a \in \nabla_{\prec}(b)$ and $\nabla_{\prec}(b) \subseteq Q$. Therefore, $a \in Q$. □

Lemma 2 *Let L be a bounded distributive lattice and $\prec$ a subordination on L. The relations $R^{\Delta}_{\prec}$ and $R^{\nabla}_{\prec}$ satisfy the following conditions:*

1. $R^{\Delta}_{\prec} = (\subseteq \circ R^{\Delta}_{\prec})$,
2. $R^{\Delta}_{\prec} = (R^{\Delta}_{\prec} \circ \subseteq)$,
3. $R^{\nabla}_{\prec} = (\subseteq^{-1} \circ R^{\nabla}_{\prec})$,
4. $R^{\nabla}_{\prec} = (R^{\nabla}_{\prec} \circ \subseteq^{-1})$.

Proof (1) The inclusion $R^{\Delta}_{\prec} \subseteq (\subseteq \circ R^{\Delta}_{\prec})$ is obvious. To prove the other inclusion assume that $P \subseteq Q'$ and $Q' R^{\Delta}_{\prec} Q$. Then $\{a \in L : \Delta_{\prec}(a) \cap Q' \neq \emptyset\} \subseteq Q$. Since $P \subseteq Q'$, we have $\{a \in L : \Delta_{\prec}(a) \cap P \neq \emptyset\} \subseteq Q$, and we are done. (2) follows easily form the definitions involved. (3) and (4) follow from (2) and (1) respectively using that $R^{\Delta}_{\prec}$ is the converse of $R^{\nabla}_{\prec}$. $\square$

Lemma 2 is basically Lemma 5 in Castro and Celani (2004) and the next lemma is Lemma 6 in Castro and Celani (2004).

Lemma 3 *Let L be a bounded distributive lattice and $\prec$ a subordination on L. Let $a \in L$ and $P \in X(L)$. Then*

1. $a \in \Delta^{-1}_{\prec}(P)$ *iff* $(\forall Q \in X(L))($*if* $\Delta^{-1}_{\prec}(P) \subseteq Q$, *then* $a \in Q)$,
2. $a \in \nabla^{-1}_{\prec}(P)$ *iff* $(\exists Q \in X(L))(Q \subseteq \nabla^{-1}_{\prec}(P)$ *and* $a \in Q)$.

Proof (1) follows from the fact that $\Delta^{-1}_{\prec}(P)$ is a filter. (2) follows from the fact that $\nabla^{-1}_{\prec}(P)^c$ is an ideal. The direction from right to left is obvious. Assume that $a \in \nabla^{-1}_{\prec}(P)$. Hence $a \notin \nabla^{-1}_{\prec}(P)^c$. Thus since this last set is an ideal, there exist $Q \in X(L)$ such that $a \in Q$ and $\nabla^{-1}_{\prec}(P)^c \cap Q = \emptyset$. Hence $Q \subseteq \nabla^{-1}_{\prec}(P)$. $\square$

8.4 Some Kinds of Bi-Subordination Lattices

We are interested in some kinds of bi-subordination lattices $L = \langle L, \prec, \lhd \rangle$. In one kind $\prec \subseteq \lhd$, in another $\lhd = \prec^{-1}$. Finally, we are interested in positive bi-subordination lattices where the link between the subordinations $\prec$ and $\lhd$ is similar to the link between the $\Box$ and $\Diamond$ in positive modal algebras.

Definition 3 A bi-subordination lattice $L = \langle L, \prec, \lhd \rangle$ is a *positive bi-subordination lattice* if the following conditions hold for all $a, b, c \in L$:

(P1) $c \prec a \vee b \implies (\forall d \in L)(a \lhd d \Rightarrow (\exists e \in L)(e \prec b \;\&\; c \leq e \vee d))$
(P2) $a \wedge b \lhd c \implies (\forall d \in L)(d \prec a \Rightarrow (\exists e \in L)(b \lhd e \;\&\; e \wedge d \leq c))$.

The conditions (P1) and (P2) can be stated in an equivalent form using the operators $\Delta_{\prec}$ and $\nabla_{\lhd}$. To do it we need to introduce the following operations between filters and ideals and between ideals and filters of a bounded distributive lattice.

Let L be a bounded distributive lattice, $F \in \mathrm{Fi}(L)$ and $I \in \mathrm{Id}(L)$. We define the following ideal and filter, respectively

$$F \odot I := \bigcap \{(I \cup \{f\}] : f \in F\}$$

and

$$I \oplus F := \bigcap \{[F \cup \{i\}) : i \in I\}.$$

In terms of the operators $\Delta_{\prec}$ and $\nabla_{\lhd}$ the conditions (P1) and (P2) respectively say that for all $a, b \in L$,

1. $\Delta_{\prec}(a \vee b) \subseteq \nabla_{\lhd}(a) \odot \Delta_{\prec}(b)$,
2. $\nabla_{\lhd}(a \wedge b) \subseteq \Delta_{\prec}(a) \oplus \nabla_{\lhd}(b)$.

We proceed to provide examples of the three kinds of bi-subordination lattices we are interested in.

Example 1 Let $\langle X, \tau\rangle$ be a topological space. The relations $\prec$ and $\lhd$ defined on $\mathcal{P}(X)$ by

$$U \prec V \;\Leftrightarrow\; U \subseteq \mathrm{int}(V)$$

and

$$U \lhd V \;\Leftrightarrow\; \mathrm{cl}(U) \subseteq V$$

are easily seen to be subordinations. Thus $\langle \mathcal{P}(X), \prec, \lhd\rangle$ is bi-subordination lattice. We note that the quasi-modal operators $\Delta_{\prec}$ and $\nabla_{\lhd}$ satisfy that

$$\Delta_{\prec}(U) = (\mathrm{int}(U)]$$

and

$$\nabla_{\lhd}(U) = [\mathrm{cl}(U))$$

for each $U \in \mathcal{P}(X)$.

If we restrict $\prec$ and $\lhd$ respectively to the distributive lattices of the open sets of X and of the closed sets of X we obtain bounded distributive lattices with two subordinations, which are one included in the other. Indeed, if U, V are closed then

$$U \prec V \;\Leftrightarrow\; U \subseteq \mathrm{int}(V) \;\Leftrightarrow\; \mathrm{cl}(U) \subseteq \mathrm{int}(V) \;\Rightarrow\; U \lhd V.$$

Also, if U, V are open, then

$$U \lhd V \;\Leftrightarrow\; \mathrm{cl}(U) \subseteq V \;\Leftrightarrow\; \mathrm{cl}(U) \subseteq \mathrm{int}(V) \;\Rightarrow\; U \subseteq \mathrm{int}(V) \;\Leftrightarrow\; U \prec V.$$

Example 2 Recall that a distributive double p-algebra $\langle L, \vee, \wedge, ^{*}, ^{+}, 0, 1\rangle$, see Katriňák (1973), is a double *Stone algebra* if $a^{*} \vee a^{**} = 1$ and $a^{+} \wedge a^{++} = 0$. In a double Stone algebra L the following properties are valid:

1. $a^{*} \leq a^{+}$.
2. $a^{+*} = a^{++} \leq a \leq a^{**} = a^{*+}$.
3. $(a \wedge b)^{*} = a^{*} \vee b^{*}$ and $(a \vee b)^{+} = a^{+} \wedge b^{+}$.

Double Stone algebras are considered by Katriňák in Katriňák (1974) and in several papers by the same author. For information on Stone algebras see Grätzer (2009) and for double Stone algebras see also Balbes and Dwinger (1974).

If $\langle L, \vee, \wedge, ^*, ^+, 0, 1\rangle$ is distributive double p-algebra it is easily seen that the relation $\prec$ on L defined by

$$a \prec b \;\Leftrightarrow\; a^* \vee b = 1,$$

is a subordination and that the relation $\lhd$ defined by

$$a \lhd b \;\Leftrightarrow\; b^+ \wedge a = 0$$

is also a subordination.

On a double Stone algebra both subordination relations are equal. In fact, a distributive double p-algebra L is a double Stone algebra if and only if $\prec = \lhd$.

Proposition 2 *Let $\langle L, \vee, \wedge, ^*, ^+, 0, 1\rangle$ be a distributive double p-algebra. Then L is a double Stone algebra if and only if for every $a, b \in L$,*

$$a^* \vee b = 1 \;\Leftrightarrow\; b^+ \wedge a = 0$$

Proof Suppose that L is a double Stone algebra. Then for every $a, b \in L$ we have:

$$\begin{aligned} a^* \vee b = 1 \;&\Rightarrow\; (a^* \vee b)^+ = 1^+ \\ &\Leftrightarrow\; a^{*+} \wedge b^+ = 0 \\ &\Rightarrow\; a \wedge b^+ = 0 \end{aligned}$$

and

$$\begin{aligned} a \wedge b^+ = 0 \;&\Rightarrow\; (a \wedge b^+)^* = 0^* \\ &\Leftrightarrow\; (a^* \vee b^{+*}) = 1 \\ &\Rightarrow\; a^* \vee b = 1. \end{aligned}$$

Now assume that for every $a, b \in L$, $a^* \vee b = 1$ if and only if $b^+ \wedge a = 0$. Let $a \in L$. Since $a^{*+} \wedge a^* = 0$, we obtain that $a^{**} \vee a^* = 1$. And since $a^{+*} \vee a^+ = 1$ we obtain $a^{++} \wedge a^+ = 0$. Hence, L is a double Stone algebra. □

The quasi-modal operators $\Delta_{\prec}$ and $\nabla_{\prec}$ associated with $\prec$ have the following description:

$$\Delta_{\prec}(a) = \left\{x \in L : x^* \vee a = 1\right\}$$

$$\nabla_{\prec}(a) = \left\{x \in L : x^+ \wedge a = 0\right\}.$$

Proposition 3 *Let L be a double Stone algebra. Then the bi-subordination lattice $\langle L, \prec, \prec\rangle$ is a positive bi-subordination lattice.*

Proof We proceed to prove that it satisfies the conditions (P1) and (P2) in Definition 3. We will work with the equivalent conditions stated in terms of the delta and nabla operators.

To prove that the condition (P1) holds, suppose that $a, b, c \in L$ are such that $c \in \Delta_{\prec}(a \vee b)$ but $c \notin \nabla_{\prec}(a) \odot \Delta_{\prec}(b)$. Then there exists $d \in \nabla_{\prec}(a)$ such that $c \notin (\Delta_{\prec}(b) \cup \{d\}]$. So, there exists $P \in X(L)$ such that $c \in P$, $\Delta_{\prec}(b) \cap P = \emptyset$ and $d \notin P$. Since $d \vee d^{+} = 1 \in P$, we get $d^{+} \in P$, and as $d \in \nabla_{\prec}(a)$, we have $d^{+} \wedge a = 0$. So

$$0 = 0^{**} = (d^{+} \wedge a)^{**} = d^{+**} \wedge a^{**} = d^{+++} \wedge a^{**} = d^{+++} \wedge a^{**} = d^{+} \wedge a^{**}.$$

Then, since $d^{+} \in P$, $a^{**} \notin P$. So $a^{*} \in P$, because L is a Stone algebra, and since $c \in P$, we get $c^{**} \in P$. Thus, $a^{*} \wedge c^{**} \in P$. Moreover,

$$1 = c^{*} \vee a \vee b \leq b \vee a^{**} \vee c^{*} = b \vee (a^{*} \wedge c^{**})^{*},$$

so that $b \vee (a^{*} \wedge c^{**})^{*} = 1$ and therefore $a^{*} \wedge c^{**} \in \Delta_{\prec}(b)$. Now since $\Delta_{\prec}(b) \cap P = \emptyset$, it follows that $a^{*} \wedge c^{**} \notin P$, which is a contradiction.

Now, to prove that the condition (P2) holds, suppose that there are elements $a, b, c \in L$ such that $c \in \nabla_{\prec}(a \wedge b)$, but $c \notin \Delta_{\prec}(a) \oplus \nabla_{\prec}(b)$. Therefore $c^{+} \wedge (a \wedge b) = 0$ and there exists $P \in X(L)$ and $d \in \Delta_{\prec}(a)$ such that $c \notin P$, $\nabla(b) \subseteq P$, and $d \in P$. So, $1 = d^{*} \vee a$, and therefore $1 = 1^{++} = (d^{*} \vee a)^{++} = d^{*++} \vee a^{++} = d^{*} \vee a^{++} \in P$. Since $d \in P$, $d^{+} \notin P$. Therefore, $a^{++} \in P$. We note that $c^{+} \in P$, because $c \notin P$. So, $a \vee c^{+} \in P$ and therefore, $a^{+} \vee c^{++} \notin P$. As $c \in \nabla_{\prec}(a \wedge b)$,

$$\begin{aligned} 0 &= a \wedge b \wedge c^{+} = a^{++} \wedge c^{+} \wedge b \\ &= (a^{+} \vee c^{++})^{+} \wedge b. \end{aligned}$$

Then $a^{+} \vee c^{++} \in \nabla_{\prec}(b) \subseteq P$, which is a contradiction.

Thus we have that for every double Stone algebra L the bi-subordination lattice $\langle L, \prec, \prec \rangle$ is a positive bi-subordination lattice. □

Example 3 This example is given in Bezhanishvili (2013) for bounded sublattices of Boolean algebras. It can be extended to bounded sublattices of bounded distributive lattices. Let L be a bounded distributive lattice and let S be a bounded sublattice of L. We consider the relations $\prec_S$ and $\lhd_S$ defined by

$$a \prec_S b \iff (\exists c \in S)\; a \leq c \leq b$$

and

$$a \lhd_S b \iff (\exists c \in S)\; b \leq c \leq a.$$

These two relations are easily seen to be subordination relations and each one is the converse relation of the other.

The operators associated with the relations $\prec_S$ and $\lhd_S$ are given by

$$\Delta_{\prec_S}(a) = \{b \in L : S \cap [b) \cap (a] \neq \emptyset\}$$

and

$$\nabla_{\lhd_S}(a) = \{b \in L : S \cap [a) \cap (b] \neq \emptyset\}.$$

An element a of a bounded lattice L is said to be *complemented* if there is $b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$. The set of all complemented elements of L is called the *center* of L. The center of L contains 0 and 1. Moreover, if L is distributive, the complements when they exist are unique. This implies that the center of a bounded distributive lattice L is a bounded sublattice of L and a Boolean lattice.

Proposition 4 *Let L be a bounded distributive lattice. If S is a bounded sublattice of the center of L, then $\langle L, \prec_S, \lhd_S\rangle$ is a positive bi-subordination lattice.*

Proof We note that S is a bounded sublattice of the center of L if and only if it is a Boolean lattice. Thus, $P \cap S$ is an ultrafilter of S for each prime filter P of L.

We note that $\Delta^{-1}_{\prec_S}(P) \subseteq Q$ if and only if $P \cap S \subseteq Q$, for all $P, Q \in X(L)$. Indeed: If $a \in P \cap S$, then $a \in \Delta_{\prec_S}(a)$, and so $\Delta_{\prec_S}(a) \cap P \neq \emptyset$ having then that $a \in Q$. Conversely, if $\Delta_{\prec_S}(a) \cap P \neq \emptyset$, there exists $b \in \Delta_{\prec_S}(a) \cap P$ and there exits $s \in S$ such that $b \leq s \leq a$. Then $s \in P \cap S \subseteq Q$, and thus $a \in Q$.

Suppose that there are elements $a, b, c \in L$ such that $c \in \Delta_{\prec_S}(a \vee b)$, but $c \notin \nabla_{\lhd_S}(a) \odot \Delta_{\prec_S}(b)$. So, there exists $d \in \nabla_{\lhd_S}(a)$ such that $c \notin \left(\Delta_{\prec_S} b \cup \{d\}\right]$. Then there exists $P \in X(L)$ such that $c \in P$, $\Delta_{\prec_S} b \cap P = \emptyset$ and $d \notin P$. Therefore, there exists $Q \in X(L)$ such that $\Delta^{-1}_{\prec_S}(P) \subseteq Q$ and $b \notin Q$., i.e., $P \cap S \subseteq Q$. As S is a Boolean lattice, $P \cap S = Q \cap S$. Since $c \in \Delta_{\prec_S}(a \vee b) \cap P$ and $b \notin Q$, we have $a \in Q$. And since $d \in \nabla_{\lhd_S}(a)$, there exists $e \in S$ such that $a \leq e \leq y$. So, $s' \in Q \cap S = P \cap S$, and thus $d \in P$, which is impossible. Therefore $\Delta_{\prec_S}(a \vee b) \subseteq \nabla_{\lhd_S}(a) \odot \Delta_{\prec_S} b$, for all $a, b \in L$. The proof of the inclusion $\nabla_{\lhd_S}(a \wedge b) \subseteq \Delta_{\prec_S} a \oplus \nabla(b)$ is similar. $\square$

8.5 Duality for Subordination Lattices and Bi-Subordination Lattices

We recall first the Priestley topological duality between bounded distributive lattices and Priestley spaces (see for example Davey and Priestley 2002) and then we expand it to subordination lattices and Priestley subordination spaces. The duality for subordiantion lattices we present can be extracted from that in Castro and Celani (2004) which is for distributive lattices with a Δ and a ∇ quasi-modal operator and Priestley spaces with two binary relations. A duality for bi-subordination lattices, which is basically the duality obtained in Castro and Celani (2004), easily follows from the duality we describe for subordination lattices. For completeness we opted to give the details.

A *totally order-disconnected* topological space is a triple $X = \langle X, \leq, \tau_X\rangle$ where $\langle X, \leq\rangle$ is a poset, $\langle X, \tau_X\rangle$ is a topological space, and given $x, y \in X$ such that $x \nleq y$ there exists a clopen upset U of X such that $x \in U$ and $y \notin U$. A *Priestley space* is a compact and totally order-disconnected topological space.

If X is a Priestley space, the set of all clopen upsets of X is denoted by $D(X)$. It is well-known that $D(X) = \langle D(X), \cup, \cap, \emptyset, X \rangle$ is a bounded distributive lattice, which is a sublattice of the complete lattice $\mathcal{P}_u(X)$ of all the upsets of X. The lattice $D(X)$ is the dual of the Priestley space X.

If $L = \langle L, \vee, \wedge, 0, 1 \rangle$ is a bounded distributive lattice, we denote the set of all prime filters of L by $X(L)$ and recall that we denote the families of all ideals and all filters of L by $\mathrm{Id}(L)$ and $\mathrm{Fi}(L)$, respectively. Given a bounded distributive lattice L, the representation map is the function $\sigma_L : L \to \mathcal{P}_u(X(L))$ given for every $a \in L$ by

$$\sigma_L(a) := \{P \in X(L) : a \in L\}.$$

It is a one-to-one lattice homomorphism, i.e. $L \cong \sigma_L[L]$. Moreover, the topological space $\left\langle X(L), \subseteq, \tau_{X(L)} \right\rangle$ where the topology $\tau_{X(L)}$ has the set

$$\sigma[L] \cup \{X(L) \smallsetminus \sigma(a) : \sigma(a) \in \sigma[L]\}$$

as a subbase, is a Priestley space such that the domain of $D(X(L))$ is $\sigma_L[L]$ and therefore the map σ_L establishes an isomorphism between L and the lattice $D(X(L))$. The Priestley space $X(L) := \left\langle X(L), \subseteq, \tau_{X(L)} \right\rangle$ is the dual of L.

Let X be a Priestley space. The map $\varepsilon_X : X \to X(D(X))$ defined for every $x \in X$ by

$$\varepsilon_X(x) := \{U \in D(X) : x \in U\}$$

is a homeomorphism between the Priestley space X and the Priestley space $X(D(X))$ of the bounded distributive lattice of the clopen upsets of X and it is also an isomorphism between the posets $\langle X, \leq \rangle$ and $\langle X(D(X)), \subseteq \rangle$.

A *homomorphism* from a bounded distributive lattice L_1 to a bounded distributive lattice L_2 is a map that preserves the infimums of finite sets and the supremums of finite sets; thus it preserves the bounds. A *Priestley morphism* from a Priestley space $\langle X_1, \leq_1, \tau_1 \rangle$ to a Priestley space $\langle X_2, \leq_2, \tau_2 \rangle$ is a continuous map from $\langle X_1, \tau_1 \rangle$ to $\langle X_2, \tau_2 \rangle$ that is order preserving w.r.t. the orders $\leq_1$ and $\leq_2$.

Let L_1, L_2 be bounded distributive lattices and $h : L_1 \to L_2$ a homomorphism. The map $X(h) : X(L_2) \to X(L_1)$ defined for every $P \in X(L_2)$ by

$$X(h)(P) := h^{-1}[P]$$

is a continuos and order preserving function, thus a morphism from $X(L_2)$ to $X(L_1)$.

If X_1 and X_2 are Priestley spaces and $f : X_1 \to X_2$ is a Priestley morphism, then the map $D(f) : D(X_2) \to D(X_1)$ defined for every $U \in D(X_2)$ by

$$D(f)(U) := f^{-1}[U]$$

is a homomorphism from $D(X_2)$ to $D(X_1)$.

Let PriSp be the category with objects the Priestley spaces and arrows the Priestley morphisms and let DLat be the cateogry of the bounded distributive lattices with the homomorphisms as its arrowss. The Priestley duality says that the maps D from PriSp to DLat and X from DLat to PriSp given by the definitions above are functors that establish a dual equivalence between the two categories whit natural transformations given by the maps σ_L and ε_X.

We proceed to expand the duality between DLat and PriSp to a duality between subordination lattices and Priestley spaces augmented with a binary relation.

We recall some facts we need on Priestley duality. Let L be a bounded distributive lattice. We denote by $\mathcal{O}_u(X(L))$ the set of all open upsets of $X(L)$, which is a lattice when ordered by inclusion, and by $\mathcal{C}_d(X(L))$ the set of all closed downsets sets of $X(L)$, which is also a lattice when ordered by inclusion. The map $\varphi : \mathrm{Id}(L) \to \mathcal{O}_u(X(L))$ given by

$$\varphi(I) := \{P \in X(L) : P \cap I \neq \emptyset\},$$

for every $I \in \mathrm{Id}(L)$ is a lattice isomorphism. Similarly, the function $\psi : \mathrm{Fi}(L) \to \mathcal{C}_d(X(L))$ given by

$$\psi(F) := \{P \in X(L) : F \subseteq P\},$$

for every $F \in \mathrm{Fi}(L)$ is a dual lattice isomorphism. These functions can be expressed in terms of σ_L as follows:

$$\varphi(I) = \bigcup \{\sigma_L(a) : a \in I\}$$

for each $I \in \mathrm{Id}(L)$ and

$$\psi(F) = \bigcap \{\sigma_L(a) : a \in F\}$$

for each $F \in \mathrm{Fi}(L)$.

If $L = \langle L, \prec \rangle$ is a subordination lattice, the relations $R_{\prec}^{\Delta}$ and $R_{\prec}^{\nabla}$ on $X(L)$ will be used to obtain the dual structures of L. We proceed to see the relevant topological properties that they have on the Priestley space $X(L)$. To this end we first note that Lemma 3 can be stated using $R_{\prec}^{\Delta}$ and $R_{\prec}^{\nabla}$ as follows:

Lemma 4 *Let L be a bounded distributive lattice and $\prec$ a subordination on L. Let $a \in L$ and $P \in X(L)$. For every $P \in X(L)$ and $a \in L$*

1. $R_{\prec}^{\Delta}(P) \subseteq \sigma_L(a)$ *iff* $a \in \Delta_{\prec}^{-1}(P)$,
2. $P \in (R_{\prec}^{\nabla})^{-1}[\sigma_L(a)]$ *iff* $a \in \nabla_{\prec}^{-1}(P)$.

The lemma implies the next corollary, which is Lemma 7 in Castro and Celani (2004).

Corollary 1 *Let L be a bounded distributive lattice and $\prec$ a subordination on L. Then*

1. $\Box_{R_{\prec}^{\Delta}}(\sigma_L(a)) = \varphi(\Delta_{\prec}(a))$,
2. $\Diamond_{R_{\prec}^{\nabla}}(\sigma_L(a)) = \psi(\nabla_{\prec}(a))$.

Proof (1) Let $P \in X(L)$. Then $P \in \Box_{R^{\Delta}_{\prec}}(\sigma_L(a))$ if and only if $R^{\Delta}_{\prec}(P) \subseteq \sigma_L(a)$. By Lemma 4 the last condition holds if and only if $a \in \Delta^{-1}_{\prec}(P)$, which means that $\Delta_{\prec}(a) \cap P \neq \emptyset$. This can be restated saying that $P \in \varphi(\Delta_{\prec}(a))$, because $\Delta_{\prec}(a)$ is an ideal.

(2) Let $P \in X(L)$. Then $P \in \Diamond_{R^{\nabla}_{\prec}}(\sigma_L(a))$ if and only if $a \in \nabla^{-1}_{\prec}(P)$. This holds if and only if $\nabla_{\prec}(a) \subseteq P$, which is equivalent to say that $P \in \psi(\nabla_{\prec}(a))$, because $\nabla_{\prec}(a)$ is a filter. □

Let X be a Priestley space. A binary relation R on X is *point-closed* if $R(x)$ is a closed set for every $x \in X$. We say that R is *up point-closed* if it is point-closed and for every $x \in X$ the set $R(x)$ is an upset of X. Similarly, we say that R is *down point-closed* if it is point-closed and $R(x)$ is a downset of X, for every $x \in X$.

Proposition 5 *If L is a bounded distributive lattice and $\prec$ a subordination on L, then*

1. *$R^{\Delta}_{\prec}$ is an up point-closed relation on $X(L)$,*
2. *$\Box_{R^{\Delta}_{\prec}}(U)$ is an open upset for each $U \in D(X(L))$,*
3. *$R^{\nabla}_{\prec}$ is a down point-closed relation on $X(L)$,*
4. *$\Diamond_{R^{\nabla}_{\prec}}(U)$ is a closed upset for each $U \in D(X(L))$.*

Proof (1) Let $P \in X(L)$. Then $R^{\Delta}_{\prec}(P) = \{Q \in X(L) : \Delta^{-1}_{\prec}(P) \subseteq Q\}$. Since $\Delta^{-1}_{\prec}(P)$ is a filter, $R^{\Delta}_{\prec}(P) = \psi(\Delta^{-1}_{\prec}(P))$. Hence, $R^{\Delta}_{\prec}(P)$ is a closed upset of $X(L)$.

(2) If $U \in D(X(L))$, then $U = \sigma_L(a)$ for some $a \in L$. Since $\Delta_{\prec}(a)$ is an ideal and $\Box_{R^{\Delta}_{\prec}}(U) = \varphi(\Delta_{\prec}(a))$ we obtain that $\Box_{R^{\Delta}_{\prec}}(U)$ is an open upset.

(3) Let $P \in X(L)$. Then $R^{\nabla}_{\prec}(P) = \{Q \in X(L) : Q \subseteq \nabla^{-1}_{\prec}(P)\} = \{Q \in X(L) : Q \cap \nabla^{-1}(P)^c = \emptyset\}$. Therefore $R_{\nabla}(P)^c = \{Q \in X(L) : Q \cap \nabla^{-1}(P)^c \neq \emptyset\}$. Since $\nabla^{-1}(P)^c$ is an ideal, $R_{\nabla}(P)^c = \varphi(\nabla^{-1}(P)^c)$ and hence it is an open upset. Therefore, $R_{\nabla}(P)^c$ is a closed downset.

(4) If $U \in D(X(L))$, then $U = \sigma_L(a)$ for some $a \in L$. Hence, since then $\Diamond_{R^{\nabla}_{\prec}}(U) = \psi(\nabla_{\prec}(a))$ and moreover $\nabla_{\prec}(a)$ is an ideal, $\Diamond_{R^{\nabla}_{\prec}}(U)$ is a closed upset. □

Definition 4 Let X be a Priestley space. We say that a binary relation R on X is the Δ*-dual of a subordination* if R is up point-closed and $\Box_R(U)$ is an open upset for every $U \in D(X)$. Similarly, we say that a binary relation R on X is the ∇*-dual of a subordination* if R is down point-closed and $\Diamond_R(U)$ is a closed upset for every $U \in D(X)$.

We have two choices to obtain the dual objects of subordination lattices. One is to consider Priestley spaces X endowed with a binary relation R which is the Δ-dual of a subordination and the other is to take Priestley spaces X endowed with a binary relation which is the ∇-dual of a subordination. In this way we will end up with two equivalent categories for every choice of morphisms between subordination lattices we take. We will see that the functor that transforms an object of one category into an object of the other simply changes the relation to its converse.

For every Priestley space X and binary relation R on X note that $R(x)$ is an upset for every $x \in X$ if and only if $(R \circ \leq) = R$, and that $R(x)$ is an downset for every

$x \in X$ if and only if $(R \circ \leq^{-1}) = R$. In general, $R \subseteq (R \circ \leq)$ and $R \subseteq (R \circ \leq^{-1})$ because $\leq$ is reflexive.

Lemma 5 *Let X be a Priestley space and R a binary relation on X.*

1. *If R is the Δ-dual of a subordination, then*
   a. $(\leq \circ R) = (R \circ \leq) = R$,
   b. *if $x \leq y$, then $R(y) \subseteq R(x)$, for all $x, y \in X$.*
2. *If R is the ∇-dual of a subordination, then*
   a. $(\leq^{-1} \circ R) = (R \circ \leq^{-1}) = R$,
   b. *if $x \leq y$, then $R(x) \subseteq R(y)$, for all $x, y \in X$.*

Proof (1). We first prove (a). Assume that $x, y, z \in X$ are such that $x \leq y$ and $(y, z) \in R$. This implies that $R(x) \neq \emptyset$. Otherwise, since $R(x) \subseteq \emptyset$ and $\emptyset \in D(X)$ we have $x \in \Box_R(\emptyset)$. Therefore, $y \in \Box_R(\emptyset)$ so that $R(y) \subseteq \emptyset$ and this is not possible since $z \in R(y)$. Suppose in search of a contradiction that $w \not\leq z$ for all $w \in R(x)$. Then for each $w \in R(x)$ there exists $U_w \in D(X)$ such that $w \in U_w$ and $z \notin U_w$. So, $R(x) \subseteq \bigcup\{U_w : w \in R(x)\}$. As $R(x)$ is closed, and hence compact, there exists a finite family $\{U_1, \ldots, U_n\}$ such that $R(x) \subseteq U_1 \cup \cdots \cup U_n = U$. Thus $x \in \Box_R(U)$ and $U \in D(X)$. As $\Box_R(U)$ is an upset, $y \in \Box_R(U)$. This yields $R(y) \subseteq U$, and since $(y, z) \in R$, $z \in U$, which is impossible. Thus there exists $w \in X$ such that $(x, w) \in R$ and $w \leq z$. We conclude that $(\leq \circ R) \subseteq (R \circ \leq)$. The inclusion $(R \circ \leq) \subseteq R$ follows from the assumption that $R(x)$ is an upset for every $x \in X$. And $R \subseteq (\leq \circ R)$ follows from the fact that $\leq$ is reflexive.

(b) follows from (a). Let $x \leq y$ and $z \in R(y)$, so that $(x, z) \in \leq \circ R$. Hence, by (a), $(x, z) \in R$, i.e., $z \in R(x)$.

(2). To prove (a) let $x, y \in X$ be such that $(x, y) \in \leq^{-1} \circ R$. Then there exists $z \in X$ such that $z \leq x$ and $(z, y) \in R$. It follows that $R(x) \neq \emptyset$. Otherwise, $R(x) \cap X = \emptyset$ and therefore $x \notin \Diamond_R(X)$. Thus, since this set is an upset, $z \notin \Diamond_R(X)$ so that $R(z) \cap X = \emptyset$ which is not possible because $y \in R(z)$. Suppose now that $w \not\leq y$ for all $w \in R(x)$. Then for each $w \in R(x)$, there exists $U_w \in D(X)$ such that $w \notin U_i$ and $y \in U_w$. So, $R(x) \subseteq \bigcup \left\{U_w^c : w \in R(x)\right\}$. As $R(x)$ is closed, and hence compact, there exists a finite family $\{U_1, \ldots, U_n\}$ such that $R(x) \subseteq U_1^c \cup \cdots \cup U_n^c = U^c$. Hence, $x \notin \Diamond_R(U)$. Since $U = U_1 \cap \cdots \cap U_n \in D(X)$, $\Diamond_R(U)$ is an upset by assumption; thus $z \notin \Diamond_R(U)$. i.e., $R(z) \cap U = \emptyset$. But $y \in R(z) \cap U$, which is a contradiction. Thus there exists $w \in X$ such that $(x, w) \in R$ and $y \leq w$. We conclude that $(\leq^{-1} \circ R) \subseteq (R \circ \leq^{-1})$. The inclusion $R \circ \leq^{-1} \subseteq R$ follows from the fact that $R(x)$ is a downset for every $x \in X$. Item (b) follows from (a). $\Box$

In the next proof we use Esakia's lemma (2019) that says that if X is a Priestley space and R is a point-closed relation on X, then for every down-directed family C of closed sets of X,

$$R^{-1}[\bigcap C] = \bigcap\{R^{-1}[U] : U \in C\}.$$

Lemma 6 *Let X be a Priestley space and R a binary relation on X.*

1. *The following two conditions are equivalent:*

   a. *R is the Δ-dual of a subordination.*
   b. *For every closed set Y in X, $R[Y]$ is a closed upset, and for every closed downset Z of X, $R^{-1}[Z]$ is a closed downset of X.*

2. *The following two conditions are equivalent:*

   a. *R is the ∇-dual of a subordination,*
   b. *For every closed subset Y of X, $R[Y]$ is a closed downset, and for every closed upset Z of X, $R^{-1}[Z]$ is a closed upset of X.*

Proof (1). Assume (a). Let Y be a closed subset of X. If $Y = \emptyset$, then since $R[Y] = \emptyset$ we are done. If $Y \neq \emptyset$, suppose that $x \notin R[Y]$. Then $x \notin R(y)$ for all $y \in Y$. Therefore, as R is up point-closed, for each $y \in Y$ there exists $U_y \in D(X)$ such that $R(y) \subseteq U_y$ and $x \notin U_y$. We fix such an U_y for each $y \in Y$. So, $y \in \Box_R(U_y)$, for every $y \in Y$. Therefore, $Y \subseteq \bigcup \{\Box_R(U_y) : y \in Y\}$, and as Y is closed and X is compact, Y is compact. Using that from the assumption, for every $y \in Y$ the set $\Box_R(U_y)$ is open and the fact that Y is compat, there exist $y_1, \dots, y_n \in Y$ such that

$$Y \subseteq \Box_R(U_{y_1}) \cup \cdots \cup \Box_R(U_{y_n}) \subseteq \Box_R(U_{y_1} \cup \cdots \cup U_{y_n}).$$

We choose $y_1, \dots, y_n \in Y$ with that property and we let $U_x = U_{y_1} \cup \cdots \cup U_{y_n}$. Then $Y \subseteq \Box_R(U_x)$ and therefore $R[Y] \subseteq U_x$. Moreover, $x \notin U_x$. It easily follows that $R[Y] = \bigcap \{U_x : x \notin R[Y]\}$. Thus, $R[Y]$ is a closed upset of X.

Let now Z be a closed downset of X. Then

$$Z = \bigcap \{U^c : Z \subseteq U^c \text{ and } U \in D(X)\}.$$

Note that the family $\{U^c : Z \subseteq U^c \text{ and } U \in D(X)\}$ is a downdirected family of closed sets. Thus by Esakia's lemma we have

$$R^{-1}[Z] = \bigcap \{R^{-1}[U^c] : Z \subseteq U^c \text{ and } U \in D(X)\}.$$

But by assumption $\Box_R(U) = (R^{-1}[U^c])^c$ is an open upset for every $U \in D(X)$. Thus $R^{-1}[U^c]$ is a closed downset for every $U \in D(X)$. This implies that $R^{-1}[Z]$ is a closed downset, as desired.

Assume now (b). As X is Hausdorff, for every $x \in X$, $\{x\}$ is closed. Thus, by (b), $R(x) = R[\{x\}]$ is a closed upset. For each $U \in D(X)$ we have $\Box_R(U) = R^{-1}[U^c]^c$. Thus, $\Box_R(U)$ is an open upset, because U^c is a closed downset.

(2). Assume (a) and let Y be a closed subset of X. If $Y = \emptyset$, then since $R[Y] = \emptyset$ we are done. Assume that $Y \neq \emptyset$ and that $x \notin R[Y] = \bigcup \{R(y) : y \in Y\}$. Then $x \notin R(y)$ for all $y \in Y$. As R is point-closed, we have that for every $y \in Y$ there exists $U_y \in D(X)$ such that $R(y) \cap U_y = \emptyset$ and $x \in U_y$. We fix one such U_y for every $y \in Y$. So, $y \in \Diamond_R(U_y)^c$, for every $y \in Y$, i.e., $Y \subseteq \bigcup \{\Diamond_R(U_y)^c : y \in Y\}$.

As Y is closed and X is compact, Y is compact. Thus there exists $y_1, \ldots, y_n \in Y$ such that

$$Y \subseteq \Diamond_R(U_{y_1})^c \cup \cdots \cup \Diamond_R(U_{y_n})^c \subseteq \Diamond_R(U_{y_1} \cap \cdots \cap U_{y_n})^c.$$

We choose $y_1, \ldots, y_n \in Y$ with that property and we let $U_x = U_{y_1} \cap \cdots \cap U_{y_n}$. Then, $Y \subseteq \Diamond_R(U_x)^c$ and so $Y \cap \Diamond_R(U_x) = \emptyset$. Therefore, $R[Y] \cap U_x = \emptyset$, i.e., $R[Y] \subseteq U_x^c$. Moreover, $x \in U_x$. It easily follows that $R[Y] = \bigcap \left\{U_x^c : x \notin R[Y]\right\}$. Thus, $R[Y]$ is a closed downset.

Let now Z be a closed upset of X. Then $Z = \bigcap \{U : Z \subseteq U \in D(X)\}$. Therefore,

$$R^{-1}[Z] = R^{-1}\left[\bigcap \{U \in D(X) : Z \subseteq U\}\right].$$

Note that the set $\{U : Z \subseteq U \in D(X)\}$ is a filter of $D(X)$, thus a downdirected family of closed sets. By Esakia's lemma we have

$$R^{-1}\left[\bigcap \{U \in D(X) : Z \subseteq U\}\right] = \bigcap \left\{R^{-1}[U] : Z \subseteq U \in D(X)\right\}.$$

As $R^{-1}[U] = \Diamond_R(U)$ is a closed upset for every closed upset U, we obtain that $R^{-1}[Z]$ is a closed upset.

Now we assume (b). As X is Hausdorff, $\{x\}$ is closed. Thus $R(x) = R[\{x\}]$ is a closed downset. For each $U \in D(X)$ we get that $R^{-1}[U]$ is a closed upset, because U is a closed upset. □

Lemma 7 *Let X be a Priestley space and R a binary relation on X. The following statements are equivalent:*

1. *R^{-1} is the Δ-dual of a subordination,*
2. *R is the ∇-dual of a subordination.*

Proof Assume (2). Note that for every $x \in X$, the set $(x] = \{y \in X : y \leq x\}$ is a closed downset. Using (1) in Lemma 5, we have $R^{-1}(x) = R^{-1}[\{x\}] = R^{-1}[(x]]$. Then using (1) in Lemma 6 we obtain that $R^{-1}(x)$ is closed and a downset. Now, given $U \in D(X)$, note that $\Diamond_{R^{-1}}(U) = R[U]$. Using again (1) in Lemma 5 we obtain that $\Diamond_{R^{-1}}(U)$ is a closed upset.

The proof of the implication from (1) to (2) is similar, using now (2) in Lemmas 5 and 6. □

Proposition 6 *Let X be a Priestley space and R a binary relation on X. If R is the Δ-dual of a subordination or the ∇-dual of a subordination, then R is a closed relation (i.e., a closed set of the product space).*

Proof Suppose that R is up point-closed and $\Box_R(U)$ is an open upset for each $U \in D(X)$. Suppose that $\langle x, y\rangle \notin R$. Using Lemma 5 it is easy to see that $R(x) = R[[x)]$. Moreover, the set $[x)$ is closed, so applying Lemma 6 it follows that $R(x)$

is a closed upset. Hence there is $U \in D(X)$ such that $R(x) \subseteq U$ and $y \notin U$. It follows that $x \in \Box_R(U)$. Consider now the complement U^c of U, which is a clopen downset and $y \in U^c$. By Lemma 6, $R^{-1}[U^c]$ is a closed downset and therefore $(R^{-1}[U^c])^c$ an open upset. Note that $z \in (R^{-1}[U^c])^c$ if and only if $R(z) \subseteq U$. Let $O := (R^{-1}[U^c])^c$. Then $\langle x, y\rangle \in O \times U^c$ and $O \times U^c$ is an open set in the product topology. We show that $R \cap (O \times U^c) = \emptyset$. If $\langle u, v\rangle \in R \cap (O \times U^c)$, then since $u \in O$ we have $v \in R(u) \subseteq U$ and $v \notin U$, a contradiction.

Now suppose that R is down point-closed and $\Diamond_R(U)$ is a closed downset for each $U \in D(X)$. Then by Lemma 7, R^{-1} is up point-closed and $\Box_{R^{-1}}(U)$ is an open upset for each $U \in D(X)$. Therefore by the first part of the proof, R^{-1} is a closed relation. It is easy to see that the converse of a closed relation is a closed relation. Thus R is a closed relation. □

Remark 3 A closed relation on a Priestley space need not be up point-closed nor down point-closed.

We proceed to see how a binary relation on a set determines two subordinations on its powerset lattice. Thus, a binary relation R on a Priestley space X determines two subordinations on the lattice of the clopen upsets of X by restricting to this lattice the subordinations determined by R on the powerset of X.

Let X be a set and R a binary relation on X. The map $\Box_R : \mathcal{P}(X) \to \mathcal{P}(X)$ is a dual modal operator and the map $\Diamond_R : \mathcal{P}(X) \to \mathcal{P}(X)$ a modal operator. That is, we have for all $U, V \subseteq X$ that $\Box_R(U \cap V) = \Box_R(U) \cap \Box_R(V)$, $\Box_R(X) = X$, $\Diamond_R(U \cup V) = \Diamond_R(U) \cap \Diamond_R(V)$, and $\Diamond_R(\emptyset) = \emptyset$. Considering Remark 2, the relations $\prec_R$ and $\prec_R^*$ defined on $\mathcal{P}(X)$ by

$$U \prec_R V \iff U \subseteq \Box_R(V)$$

and

$$U \prec_R^* V \iff \Diamond_R(U) \subseteq V.$$

are subordination relations. Note that since $\Diamond_R(U) = \Box_R(U^c)^c$ for every $U \subseteq X$, we have

$$U \prec_R V \iff V^c \prec_R^* U^c.$$

Remark 4 It is well known, and easy to check, that the maps $\Diamond_{R^{-1}}$ and $\Box_R$ form an adjoint pair, that is, for every $U, V \subseteq X$, $U \subseteq \Box_R(V)$ if and only if $\Diamond_{R^{-1}}(U) \subseteq V$. Therefore, $U \prec_R V$ if and only if $U \prec_{R^{-1}}^* V$. Hence, $\prec_R = \prec_{R^{-1}}^*$ and, similarly, $\prec_R^* = \prec_{R^{-1}}$.

If X is a Priestley space and R a binary relation on X, the subordinations $\prec_R$ and $\prec_R^*$ on $\mathcal{P}(X)$ restrict to subordinations on the sublattice $D(X)$ of $\mathcal{P}(X)$. Therefore, give a Priestley space X and a binary relation R on X we have two natural ways to obtain a subordination on $D(X)$.

In the next two propositions we proceed to find a necessary and sufficient condition that R has to satisfy in order that the homeomorphism $\varepsilon : X \to X(D(X))$ is an isomorphism between $\langle X, R\rangle$ and $\langle X(D(X)), R^{\Delta}_{\prec_R}\rangle$ and a necessary and sufficient condition to be an isomorphism between $\langle X, R\rangle$ and $\langle X(D(X)), R^{\nabla}_{\prec^*_R}\rangle$.

Proposition 7 *Let X be a Priestley space and R a binary relation on X. The following statements are equivalent:*

1. *R is the Δ-dual of a subordination.*
2. *for every $x, y \in X$, xRy if and only if $\varepsilon_X(x)R^{\Delta}_{\prec_R}\varepsilon_X(y)$.*

Proof We omit the subscript X in ε_X all along the proof.

(1) $\Rightarrow$ (2). Assume that xRy. We have to prove that $\Delta^{-1}_{\prec_R}(\varepsilon(x)) \subseteq \varepsilon(y)$. Suppose that $U \in \Delta^{-1}_{\prec_R}(\varepsilon(x))$. Hence, $\Delta_{\prec_R}(U) \cap \varepsilon(x) \neq \emptyset$. Thus, there is $V \in D(X)$ such that $V \subseteq \Box_R(U)$ and $x \in V$. Therefore, $R(x) \subseteq U$. This implies that $y \in U$ and so $U \in \varepsilon(y)$. Conversely, suppose that $\varepsilon(x)R^{\Delta}_{\prec_R}\varepsilon(y)$ and $x \not R y$. Then, since $R(x)$ is a closed upset, there is $V \in D(X)$ such that $y \notin V$ and $R(x) \subseteq V$. Hence, $V \notin \varepsilon(y)$ and $x \in \Box_R(V)$. Thus, $V \notin \Delta^{-1}_{\prec_R}(\varepsilon(x))$, which means that $\Delta_{\prec_R}(V) \cap \varepsilon(x) = \emptyset$. But $\Box_R(V)$ is an open upset. So $\Box_R(V) = \bigcup\{U \in D(X) : U \subseteq \Box_R(V)\}$. Hence, there is $U \in D(X)$ such that $x \in U$ and $U \subseteq \Box_R(V)$. Therefore, $V \in \Delta^{-1}_{\prec_R}(\varepsilon(x))$. Since $V \notin \varepsilon(y)$, it follows that $\varepsilon(x) \not R^{\Delta}_{\prec_R}\varepsilon(y)$. (2) $\Rightarrow$ (1). Let $x, y \in X$ be such that $y \in \mathrm{Cl}(R(x))$ and $y \notin R(x)$. Then, by (2), $\varepsilon(y) \notin R^{\Delta}_{\prec_R}(\varepsilon(x))$. Therefore, there exist $U \in D(X)$ such that $\Delta_{\prec_R}(U) \cap \varepsilon(x) \neq \emptyset$ and $y \notin U$. Let $V \in \Delta_{\prec_R}(U) \cap \varepsilon(x)$. Then $V \in D(X)$, $V \subseteq \Box_R(U)$, and $x \in V$. Therefore $R(x) \subseteq U$. Hence, $y \notin \mathrm{Cl}(R(x))$, a contradiction. Thus R is point-closed. Now to prove that it is an upset, suppose that $y \in R(x)$ and $y \leq z$. Since ε is an order isomorphism, $\varepsilon(y) \subseteq \varepsilon(z)$. Moreover, since xRy, by the assumption (2) we have $\Delta^{-1}_{\prec}(\varepsilon(x)) \subseteq \varepsilon(y)$. Thus, $\Delta^{-1}_{\prec}(\varepsilon(x)) \subseteq \varepsilon(z)$. This, again by the assumption (2), implies that xRz.

Now let $U \in D(X)$. Using (2), the definitions involved, and the fact that ε is a bijection, it is easy to see that $\Box_{R^{\Delta}_{\prec}}(\varepsilon[U]) = \varepsilon[\Box_R(U)]$. Hence, considering that $\Box_{R^{\Delta}_{\prec_R}}(\varepsilon[U])$ is an open upset of $X(D(X))$, because $\prec_R$ is a subordination on $D(X)$, and the fact that ε is an order isomorphism and a homeomorphism we obtain that $\Box_R(U)$ is an open upset. $\Box$

The next proposition is proved in Castro and Celani (2004).

Proposition 8 *Let X be a Priestley space and R a binary relation on X. The following are equivalent:*

1. *R is the ∇-dual of a subordination,*
2. *for every $x, y \in X$, xRy if and only if $\varepsilon_X(x)R^{\nabla}_{\prec^*_R}\varepsilon_X(y)$.*

Proof As in the last proof, we omit the subscript X in ε_X.

(1) $\Rightarrow$ (2). Suppose that xRy. We have to prove that $\varepsilon(y) \subseteq \nabla^{-1}_{\prec^*_R}(\varepsilon(x))$. Suppose that $U \in \varepsilon(y)$. To prove that $U \in \nabla^{-1}_{\prec^*_R}(\varepsilon(x))$ we have to show that $\nabla_{\prec_R}(U) \subseteq \varepsilon(x)$. To this end suppose that $U \prec^*_R V$ which means that $\Diamond_R(U) \subseteq V$. Since $y \in U$ and xRy, $x \in \Diamond_R(U)$. Therefore, $V \in \varepsilon(x)$ and we are done. Conversely, suppose that $x \not R y$. Then since $R(x)$ is a closed downset there is $U \in D(X)$ such that $y \in U$ and $R(x) \cap U = \emptyset$. Then $x \notin \Diamond_R(U)$ and $U \in \varepsilon(y)$. Since $\Diamond_R(U)$ is a closed upset,

there exists $V \in D(X)$ such that $\Diamond_R(U) \subseteq V$ and $x \notin V$. Then $V \in \nabla_{\prec_R^*}(U)$ and therefore $\nabla_{\prec_R^*}(U) \nsubseteq \varepsilon(x)$, which implies that $U \notin \nabla_{\prec_R^*}^{-1}(\varepsilon(x))$. Hence we obtain that $\varepsilon(x) \not R^{\nabla}_{\prec_R^*} \varepsilon(y)$.

(2) $\Rightarrow$ (1). Let $x, y \in X$ be such that $y \in \mathrm{Cl}(R(x))$ and $y \notin R(x)$. Then, by (2), $\varepsilon(y) \notin R^{\nabla}_{\prec_R^*}(\varepsilon(x))$. Therefore, there exist $U \in D(X)$ such that $y \in U$ and $\nabla_{\prec_R^*}(U) \nsubseteq \varepsilon(x)$. Let then $V \in \nabla_{\prec_R^*}(U)$, namely that $\Diamond_R(U) \subseteq V$, and such that $x \notin V$. Hence, $x \notin \Diamond_R(U)$ and so $R(x) \cap U = \emptyset$. Since $R(x)$ is a closed downset and $y \in U$, $y \notin \mathrm{Cl}(R(x))$, a contradiction. Thus $R(x)$ is closed. Now to prove that $R(x)$ is a downset, suppose that $y \in R(x)$ and $z \leq y$. Then $\varepsilon(z) \subseteq \varepsilon(y)$. By the assumption (2), since xRy we have $\varepsilon(x) R^{\nabla}_{\prec_R^*} \varepsilon(y)$, namely $\varepsilon(y) \subseteq \nabla_{\prec_R^*}^{-1}(\varepsilon(x))$. It follows that $\varepsilon(z) R^{\nabla}_{\prec_R^*} \varepsilon(y)$ and therefore that $z \in R(x)$.

Now let $U \in D(X)$. Using (2), the definitions involved, and the fact that ε is a bijection, it is easy to see that $\Diamond_{R^{\nabla}_{\prec_R^*}}(\varepsilon[U]) = \varepsilon[\Diamond_R(U)]$. Hence, considering that $\Diamond_{R^{\nabla}_{\prec_R^*}}(\varepsilon[U])$ is a closed upset of $X(D(X))$, because $\prec_R^*$ is a subordination on $D(X)$, and the fact that ε is an order isomorphism and a homeomorphism we obtain that $\Diamond_R(U)$ is a closed upset. $\square$

Proposition 9 *Let L be a bounded distributive lattice and $\prec$ a subordination on L. The representation isomorphism $\sigma_L : L \to D(X(L))$ satisfies for every $a, b \in L$ the following two conditions:*

$$b \prec a \iff \sigma_L(b) \prec_{R^{\Delta}_{\prec}} \sigma_L(a) \quad \textit{and} \quad b \prec a \iff \sigma_L(b) \prec^*_{R^{\nabla}_{\prec}} \sigma_L(a).$$

In terms of the associated Δ-quasi-modal operators the conditions say that for every $a \in L$, $\Delta_{\prec_{R^{\Delta}_{\prec}}}(\sigma_L(a)) = \sigma_L[\Delta_{\prec}(a)]$ and $\nabla_{\prec^*_{R^{\nabla}_{\prec}}}(\sigma_L(b)) = \sigma_L[\nabla_{\prec}(b)]$.

Proof First note that by Corollary 1, $\Box_{R^{\Delta}_{\prec}}(\sigma_L(a)) = \varphi(\Delta_{\prec}(a))$ and $\Diamond_{R^{\nabla}_{\prec}}(\sigma_L(a)) = \psi(\nabla_{\prec}(a))$. Now for every $b \in L$,

$$\begin{aligned} \sigma_L(b) \prec_{R^{\Delta}_{\prec}} \sigma_L(a) &\Leftrightarrow \sigma_L(b) \subseteq \Box_{R^{\Delta}_{\prec}}(\sigma_L(a)) \\ &\Leftrightarrow \sigma_L(b) \subseteq \varphi(\Delta_{\prec}(a)) \\ &\Leftrightarrow b \in \Delta_{\prec}(a) \\ &\Leftrightarrow b \prec a. \end{aligned}$$

The equivalence before the last one holds because if $b \in \Delta_{\prec}(a)$, then by the definition of φ, $\sigma_L(b) \subseteq \varphi(\Delta_{\prec}(a))$, and if $b \notin \Delta_{\prec}(a)$, then there exists $P \in X(L)$ such that $b \in P$ and $\Delta_{\prec}(a) \cap P = \emptyset$, which implies that $P \notin \varphi(\Delta_{\prec}(a))$ and hence we have that $\sigma_L(b) \nsubseteq \varphi(\Delta_{\prec}(a))$. This proves the first condition.

To prove the second condition we have for every $b \in L$,

$$\begin{aligned}\sigma_L(b) \prec^*_{R^\nabla_\prec} \sigma_L(a) &\Leftrightarrow \Diamond_{R^\nabla_\prec}(\sigma_L(b)) \subseteq \sigma_L(a) \\ &\Leftrightarrow \psi(\nabla_\prec(b)) \subseteq \sigma_L(a) \\ &\Leftrightarrow a \in \nabla_\prec(b) \\ &\Leftrightarrow b \prec a.\end{aligned}$$

The equivalence before the last one holds because if $a \in \nabla_\prec(b)$, then by the definition of ψ, $\psi(\nabla_\prec(b)) \subseteq \sigma_L(a)$, and if $a \notin \nabla_\prec(b)$, then, since $\nabla_\prec(b)$ is a filter, there exists $P \in X(L)$ such that $a \notin P$ and $\nabla_\prec(b) \subseteq P$, which implies that $P \in \psi(\nabla_\prec(b))$ and hence we have $\psi(\nabla_\prec(b)) \nsubseteq \sigma_L(a)$. □

Definition 5 We say that a pair $\langle X, R\rangle$ is a *Priestley* Δ-*subordination space* (a Priestley Δ-space, for short) if X is a Priestley space and R is the Δ-dual of a subordination. Similarly, we say a pair $\langle X, R\rangle$ is a *Priestley* ∇-*subordination space* (a Priestley ∇-space, for short) if X is a Priestley space and R is the ∇-dual of a subordination.

Proposition 5 establishes that if $L = \langle L, \prec\rangle$ is a subordination lattice, then $\langle X(L), R^\Delta_\prec\rangle$ is a Priestley Δ-subordination space and $\langle X(L), R^\nabla_\prec\rangle$ is a Priestley ∇-subordination space. Moreover, Proposition 9 shows that the map σ_L is an isomorphism between the subordination lattices $\langle L, \prec\rangle$ and $\langle D(X(L)), \prec_{R^\Delta_\prec}\rangle$ and between $\langle L, \prec\rangle$ and $\langle D(X(L)), \prec^*_{R^\nabla_\prec}\rangle$. Conversely, Proposition 7 shows that if $\langle X, R\rangle$ is a Priestley Δ-subordination space, then $\langle D(X), \prec_R\rangle$ is a subordination lattice such that the map ε_X an isomorphism between $\langle X, R\rangle$ and $\langle X(D(X)), R^\Delta_{\prec_R}\rangle$ and Proposition 8 shows that if $\langle X, R\rangle$ is a Priestley ∇-subordination space, then $\langle D(X), \prec^*_R\rangle$ is a subordination lattice such that the map ε_X an isomorphism between $\langle X, R\rangle$ and $\langle X(D(X)), R^\nabla_{\prec^*_R}\rangle$.

To complete the duality we have to introduce the morphisms. We will consider three kinds of morphisms on subordination lattices and four kinds of morphisms on Priestley spaces with a binary relation.

Definition 6 Let L_1 and L_2 be subordination lattices. A *subordination homomorphism* from L_1 to L_2 is a homomorphism $h : L_1 \to L_2$ such that for every $a, b \in L_1$, if $a \prec_1 b$, then $h(a) \prec_2 h(b)$. A subordination homomorphism h from L_1 to L_2 is *strong* if for every $a \in L_1$ and $c \in L_2$, if $c \prec_2 h(a)$, then there exists $b \in L_1$ such that $b \prec_1 a$ and $c \leq h(b)$. We say that it is *dually strong* if for every $a \in L_1$ and $c \in L_2$, if $h(a) \prec_2 c$, then there exists $b \in L_1$ such that $a \prec_1 b$ and $h(b) \leq c$.

Remark 5 It is easy to see that if $\prec$ is a subordination on a lattice L, then $\prec^{-1}$ is a subordination on the dual lattice L^∂ of L. The condition that defines dually strong subordination homomorphism in the definition above is then the same as that for strong subordination homomorphism but between L_1 and $\langle L_2^\partial, \prec_2^{-1}\rangle$.

Definition 7 Let X_1 and X_2 be Priestley spaces with binary relations R_1 and R_2 respectively. A Priestley morphism $f : X_1 \to X_2$ is *stable* if for every $x, y \in X_1$ such that xR_1y we have $f(x)R_2 f(y)$, and it is *strongly stable* if in addition for every $x \in X_1$ and $y \in X_2$, if $f(x)R_2y$, then there exists $z \in X_1$ such that xR_1z and $f(z) \leq_2 y$. Moreover, we say that f is *reversely strongly stable* if it is stable and for every $x \in X_1$ and $y \in X_2$, if $yR_2f(x)$, then there exists $z \in X_1$ such that zR_1x and $f(z) \leq_2 y$. Also we say that f is *dually strongly stable* if it is stable and for every $x \in X_1$ and $y \in X_2$, if $yR_2f(x)$, then there exists $z \in X_1$ such that zR_1x and $y \leq_2 f(z)$.

The strongly stable Priestley morphisms are the morphisms considered in the duality for quasi-modal distributive lattices given in Castro and Celani (2004).

Proposition 10 *Let L_1, L_2 be subordination lattices. For every map $h : L_1 \to L_2$ the following conditions are equivalent:*

1. *h is a (strong) subordination homomorphism from L_1 to L_2,*
2. *the dual map $X(h)$ is a (strongly) stable Priestley morphism from the Priestley Δ-space $\langle X(L_2), R^{\Delta}_{\prec_2}\rangle$ to the Priestley Δ-space $\langle X(L_1), R^{\Delta}_{\prec_1}\rangle$,*
3. *the dual map $X(h)$ is a (reversely strongly) stable Priestley morphism from the Priestley ∇-space $\langle X(L_2), R^{\nabla}_{\prec_2}\rangle$ to the Priestley ∇-space $\langle X(L_1), R^{\nabla}_{\prec_1}\rangle$.*

Proof (1) $\Rightarrow$ (2). Assume that h is a subordination homomorphism from L_1 to L_2. Suppose that $\langle P, Q\rangle \in R^{\Delta}_{\prec_2}$. Then $\Delta^{-1}_{\prec_2}(P) \subseteq Q$. We prove that $\Delta^{-1}_{\prec_1}(h^{-1}[P]) \subseteq h^{-1}[Q]$. Suppose that $a \in \Delta^{-1}_{\prec_1}(h^{-1}[P])$. Then $\Delta_{\prec_1}(a) \cap h^{-1}[P] \neq \emptyset$. Let then $b \in \Delta_{\prec_1}(a) \cap h^{-1}[P]$. Thus $b \prec_1 a$ and $h(b) \in P$. Hence $h(b) \prec_2 h(a)$ and we obtain that $h(b) \in \Delta_{\prec_1}(h(a)) \cap P$. Thus, $h(a) \in \Delta^{-1}_{\prec_2}(P)$. It follows that $b \in h^{-1}[Q]$. If h is in addition strong, then suppose that $P \in X(L_2)$ and $Q \in X(L_1)$ are such that $h^{-1}[P]R^{\Delta}_{\prec_1}Q$, i.e., $\Delta^{-1}_{\prec_1}(h^{-1}[P]) \subseteq Q$. We prove that $\Delta^{-1}_{\prec_2}(P) \cap (h[L_1 \setminus Q]] = \emptyset$. On the contrary, let $a \in L_2$ and $b \in L_1 \setminus Q$ be such that $a \in \Delta^{-1}_{\prec_2}(P)$ and $a \leq_2 h(b)$. Then $\Delta_{\prec_2}(a) \cap P \neq \emptyset$. So, let $c \prec_2 a$ be such that $c \in P$. It follows that $c \prec_2 h(b)$. Since h is strong, there exists $d \in L_1$ such that $d \prec_1 b$ and $c \leq_2 h(d)$. Thus $h(d) \in P$ and $d \in h^{-1}[P]$, therefore $\Delta_{\prec_1}(b) \cap h^{-1}[P] \neq \emptyset$. Hence $b \in \Delta^{-1}_{\prec_1}(h^{-1}[P])$ and so $b \in Q$, a contradiction. By the Prime filter theorem there is $P' \in X(L_2)$ such that $\Delta^{-1}_{\prec_2}(P) \subseteq P'$ and $h^{-1}[P'] \subseteq Q$. Hence, there is $P' \in X(L_2)$ such that $PR^{\Delta}_{\prec_2}P'$ and $h^{-1}[P'] \subseteq Q$. This shows that $X(h)$ is strong.

(2) $\Rightarrow$ (1). Suppose now that $X(h)$ is a stable Priestley morphism from the space $\langle X(L_2), R^{\Delta}_{\prec_2}\rangle$ to $\langle X(L_1), R^{\Delta}_{\prec_1}\rangle$ and suppose that $a, b \in L_1$ are such that $a \prec_1 b$. If $h(a) \not\prec_2 h(b)$, then, since $h(a) \notin \Delta_{\prec_2}(h(b))$ and $\Delta_{\prec_2}(h(b))$ is an ideal, there is $P \in X(L_2)$ such that $h(a) \in P$ and $P \cap \Delta_{\prec_2}(h(b)) = \emptyset$. Hence, $h(b) \notin \Delta^{-1}_{\prec_2}(P)$. Since $\Delta^{-1}_{\prec_2}(P)$ is a filter, there is $Q \in X(L_2)$ such that $\Delta^{-1}_{\prec_2}(P) \subseteq Q$ and $h(b) \notin Q$. Thus, $PR^{\Delta}_{\prec_2}Q$. The stability of $X(h)$ implies that $h^{-1}[P]R^{\Delta}_{\prec_1}h^{-1}[Q]$. Since $b \notin h^{-1}[Q]$, we have $b \notin \Delta^{-1}_{\prec_1}(h^{-1}[P])$. And since $a \in h^{-1}[P]$ it follows that $a \notin \Delta_{\prec_1}(b)$, which is not possible because by assumption $a \prec_1 b$. If $X(h)$ is in addition strongly stable, to prove that h is strong assume that $a \in L_1$ and $c \in L_2$ are such that $c \prec_2 h(a)$. Consider the ideal I of L_2 generated by $h[\Delta_{\prec_1}(a)]$. Assume that $c \notin I$. Then let $P \in X(L_2)$ be such that $c \in P$ and $I \cap P = \emptyset$. Thus $\Delta_{\prec_1}(a) \cap h^{-1}[P] = \emptyset$. It follows that $a \notin \Delta^{-1}_{\prec_1}(h^{-1}[P])$. Thus there exists $Q \in X(L_1)$ such that $\Delta^{-1}_{\prec_1}(h^{-1}[P]) \subseteq Q$, so that $h^{-1}[P]R^{\Delta}_{\prec_1}Q$ and $a \notin Q$. Since $X(h)$ is strongly stable, there exists $P' \in X(L_2)$

such that $P R^{\Delta}_{\prec_2} P'$ and $h^{-1}[P'] \subseteq Q$. Since $c \in P$ and $c \prec_2 h(a)$, $h(a) \in \Delta^{-1}_{\prec_2}(P)$; therefore, $h(a) \in P'$ and $a \in h^{-1}[P']$. Hence $a \in Q$, a contradiction. We conclude that $c \in I$. Therefore there exists $b \in \Delta_{\prec_1}(a)$ such that $c \leq_2 h(b)$ and we are done.

The equivalence between (3) and (2) can be proved using Proposition 1. □

Proposition 11 *Let $\langle X_1, R_1\rangle$ and $\langle X_2, R_2\rangle$ be two Priestley Δ-subordination spaces and $f : X_1 \to X_2$ a map. Then f is a (strongly) stable Priestley morphism if and only if the map $D(f) : D(X_2) \to D(X_1)$ is a (strong) subordination homomorphism from $\langle D(X_2), \prec_{R_2}\rangle$ to $\langle D(X_1), \prec_{R_1}\rangle$.*

Proof From Priestley duality we have that f is a Priestley morphism if and only if $D(f) : D(X_2) \to D(X_1)$ is a homomorphism. Moreover, for every $x \in X_1$ it holds that $X(D(f))(\varepsilon_1(x)) = \varepsilon_2(f(x))$. Thus, using Propositions 7 and 10 we have that f is (strongly) stable if and only is $D(f)$ a (strong) subordination homomorphism. □

In a similar way we have:

Proposition 12 *Let $\langle X_1, R_1\rangle$ and $\langle X_2, R_2\rangle$ be two Priestley ∇-subordination spaces and $f : X_1 \to X_2$ a map. Then f is a (reversely strongly) stable Priestley morphism if and only if the map $D(f) : D(X_2) \to D(X_1)$ is a (strong) subordination homomorphism from $\langle D(X_2), \prec^*_{R_2}\rangle$ to $\langle D(X_1), \prec^*_{R_1}\rangle$.*

Proof From Priestley duality we have that f is a Priestley morphism if and only if $D(f) : D(X_2) \to D(X_1)$ is a homomorphism. And moreover for every $x \in X_1$ it holds that $X(D(f))(\varepsilon_1(x)) = \varepsilon_2(f(x))$. Thus, using Propositions 8 and 10 we have that f is (reversely strongly) stable if and only if $D(f)$ is a (strong) subordination homomorphism. □

Lemma 8 *Let $\langle X_1, R_1\rangle$ and $\langle X_2, R_2\rangle$ be Priestley Δ-subordination spaces. A Priestley morphism $f : X_1 \to X_2$ is dually strongly stable if and only if for every $U \in D(X_2)$,*

$$\Diamond_{R_1^{-1}}(f^{-1}[U]) = f^{-1}[\Diamond_{R_2^{-1}}(U)].$$

Proof Assume that f is a Priestley morphism that is dually strongly stable from $\langle X_1, R_1\rangle$ to $\langle X_2, R_2\rangle$. Let $U \in D(X_2)$. If $x \in \Diamond_{R_1^{-1}}(f^{-1}[U])$, then there exists $y \in f^{-1}[U]$ such that $x R_1^{-1} y$. Therefore $y R_1 x$ and, since f is stable, $f(y) R_2 f(x)$. Since $f(y) \in U$ it follows that $f(x) \in \Diamond_{R_2^{-1}}(U)$ and hence $x \in f^{-1}[\Diamond_{R_2^{-1}}(U)]$. Conversely, if $x \in f^{-1}[\Diamond_{R_2^{-1}}(U)]$, we have $f(x) R_2^{-1} y$ for some $y \in U$, so that $y R_2 f(x)$. Applying that f is dually strongly stable, there is $z \in X_1$ such that $z R_1 x$ and $y \leq_2 f(z)$. Since U is an upset, $z \in f^{-1}[U]$. Hence, since $x R_1^{-1} z$ we obtain that $x \in \Diamond_{R_1^{-1}}(f^{-1}[U])$.

Suppose now that for every $U \in D(X_2)$, $\Diamond_{R_1^{-1}}(f^{-1}[U]) = f^{-1}[\Diamond_{R_2^{-1}}(U)]$. Assume that $x, y \in X_1$ are such that $x R_1 y$ and $f(x) \not R_2 f(y)$, so that $f(x) \notin R_2^{-1}(f(y))$. Since $R_2^{-1}(f(y))$ is a closed set and a downset, there exists $U \in D(X_2)$ such that $R_2^{-1}(f(y)) \subseteq U^c$ and $f(x) \in U$, and therefore $x \in f^{-1}[U]$. Hence, $y \in$

$\Diamond_{R_1^{-1}}(f^{-1}[U])$. The assumption implies that $f(y) \in \Diamond_{R_2^{-1}}(U)$, a contradiction with $R_2^{-1}(f(y)) \subseteq U^c$. This shows that f is stable. To prove that f is dually strong assume that $x \in X_1$ and $y \in X_2$ are such that $yR_2 f(x)$. Let $U \in D(X_2)$ be such that $y \in U$. Then $x \in f^{-1}[\Diamond_{R_2^{-1}}(U)]$ and therefore $x \in \Diamond_{R_1^{-1}}(f^{-1}[U])$. This implies that $R_1^{-1}(x)$ is nonempty. Suppose that $R_1^{-1}(x) \cap f^{-1}[\uparrow y] = \emptyset$. For every $z \in R_1^{-1}(x)$, since $y \nleq_2 f(z)$, let $U_z \in D(X_2)$ be such that $y \in U_z$ and $f(z) \notin U_z$ so that $z \in f^{-1}[U_z^c]$. Then $R_1^{-1}(x) \subseteq \bigcup\{f^{-1}[U_z^c] : z \in R_1^{-1}(x)\}$. Since $R_1^{-1}(x) = R_1^{-1}[(x]]$, because $R_1 \circ \leq_1 = R_1$ and $(x]$ is a closed downset, we have that $R_1^{-1}(x)$ is a closed downset. Therefore $R_1^{-1}(x)$ is compact. Thus there are $z_1, \ldots, z_n \in R_1^{-1}(x)$ with $R_1^{-1}(x) \subseteq f^{-1}[U_{z_1}^c] \cup \cdots \cup f^{-1}[U_{z_n}^c]$. Let $U = U_{z_1} \cap \cdots \cap U_{z_n}$. Then $U \in D(X_2)$, $f^{-1}[U_{z_1}^c] \cup \cdots \cup f^{-1}[U_{z_n}^c] = f^{-1}[U^c]$, and $y \in U$. Therefore, $R_1^{-1}(x) \cap f^{-1}[U] = \emptyset$. If follows that $x \notin \Diamond_{R_1^{-1}}(f^{-1}[U])$ but since $yR_2 f(x)$ and $y \in U$, $x \in f^{-1}[\Diamond_{R_2^{-1}}(U)]$, a contradiction. □

Proposition 13 *Let L_1, L_2 be subordination lattices and $h : L_1 \to L_2$ a map. If h is a dually strong subordination homomorphism from L_1 to L_2, then the dual map $X(h)$ is a dually strongly stable Priestley morphism from the Priestley Δ-space $\langle X(L_2), R^{\Delta}_{\prec_2}\rangle$ to the Priestley Δ-space $\langle X(L_1), R^{\Delta}_{\prec_1}\rangle$.*

Proof Suppose that $h : L_1 \to L_2$ is a dually strong subordination homomorphism. Since it is a subordination homomorphism we know that $X(h)$ is a stable Priestley morphism from $\langle X(L_2), R^{\Delta}_{\prec_2}\rangle$ to $\langle X(L_1), R^{\Delta}_{\prec_1}\rangle$. To prove that $X(h)$ is dually strongly stable assume that $QR^{\Delta}_{\prec_1} h^{-1}[P]$. Then $\Delta^{-1}_{\prec_1}(Q) \subseteq h^{-1}[P]$. Let I be the ideal generated by $\bigcup_{a \in L_2 \setminus P} \Delta_{\prec_2}(a)$ and let F be the filter generated by $h[Q]$. We claim that $F \cap I = \emptyset$. Assume the contrary and let $d \in F \cap I$. Then let $b \in Q$ such that $h(b) \leq_2 d$ and let $a \in L_2 \setminus P$ and $c \in \Delta_{\prec_2}(a)$ such that $d \leq_2 c$. It follows that $h(b) \prec_2 a$. Then by (1) there exists $e \in L_1$ such that $b \prec_1 e$ and $h(e) \leq_2 a$. Since $b \in Q$ we have $e \in \Delta^{-1}_{\prec_1}(Q)$ and therefore $e \in h^{-1}[P]$. Thus, $h(e) \in P$ and hence $a \in P$, a contradiction. We conclude that $F \cap I = \emptyset$. Let then $P' \in X(L_2)$ be such that $F \subseteq P'$ and $I \cap P' = \emptyset$. It follows that $\Delta^{-1}_{\prec_2}(P') \subseteq P$ and $Q \subseteq h^{-1}[P']$. Thus $P' R^{\Delta}_{\prec_2} P$ and $Q \subseteq h^{-1}[P']$. □

Proposition 14 *Let $\langle X_1, R_1\rangle, \langle X_2, R_2\rangle$ be Priestley Δ-subordination spaces and $f : X_1 \to X_2$ a Priestley morphism that is dually strongly stable from $\langle X_1, R_1\rangle$ to $\langle X_2, R_2\rangle$ if and only if $D(f) : D(X_2) \to D(X_1)$ is a dually strong subordination homomorphism from $\langle D(X_2), \prec_{R_2}\rangle$ to $\langle D(X_1), \prec_{R_1}\rangle$.*

Proof Assume that f is a Priestley morphism that is dually strongly stable from $\langle X_1, R_1\rangle$ to $\langle X_2, R_2\rangle$. We know from Lemma 8 that for every $U \in D(X_2)$,

$$\Diamond_{R_1^{-1}}(f^{-1}[U]) = f^{-1}[\Diamond_{R_2^{-1}}(U)].$$

We show that $D(f)$ is a subordination homomorphism from the subordination lattice $\langle D(X_2), \prec_{R_2}\rangle$ to $\langle D(X_1), \prec_{R_1}\rangle$. Suppose that $U, V \in D(X_2)$ are such that $U \prec_{R_2} V$. By Remark 4 we have $\Diamond_{R_2^{-1}}(U) \subseteq V$. Then $f^{-1}[\Diamond_{R_2^{-1}}(U)] \subseteq f^{-1}[V]$. Therefore,

$\Diamond_{R_1^{-1}}(f^{-1}[U]) \subseteq f^{-1}[V]$ and using Remark 4 again we have $f^{-1}[U] \prec_{R_1} f^{-1}[V]$. Now we prove that $D(f)$ is dually strong. Suppose that $U \in D(X_2)$ and $V \in D(X_1)$ are such that $f^{-1}[U] \prec_{R_1} V$. Thus, using Remark 4, $\Diamond_{R_1^{-1}}(f^{-1}[U]) \subseteq V$ and therefore $f^{-1}[\Diamond_{R_2^{-1}}(U)] \subseteq V$. The set $\Diamond_{R_2^{-1}}(U)$ is a closed upset. Therefore there is a family $\{U_j : j \in J\} \subseteq D(X_2)$ such that $\Diamond_{R_2^{-1}}(U) = \bigcap_{j\in J} U_j$ and hence $f^{-1}[\bigcap_{j\in J} U_j] \subseteq V$. Thus $\bigcap_{j\in J} f^{-1}[U_j] \subseteq V$. Since the sets $f^{-1}[U_j]$ are closed and V is open, by compactness of the space follows that there exists a finite $J' \subseteq J$ such that $\bigcap_{j\in J'} f^{-1}[U_j] \subseteq V$. Let $U' = \bigcap_{j\in J'} U_j$. Then $f^{-1}[U'] \subseteq V$ and since $\Diamond_{R_2^{-1}}(U) \subseteq U'$ we obtain, using Remark 4, that $U \prec_{R_2} U'$. Hence, we conclude that $D(f) : D(X_2) \to D(X_1)$ is a dually strong subordination homomorphism from $\langle D(X_2), \prec_{R_2}\rangle$ to $\langle D(X_1), \prec_{R_1}\rangle$.

Conversely, assume that $D(f) : D(X_2) \to D(X_1)$ is a dually strong subordination homomorphism from $\langle D(X_2), \prec_{R_2}\rangle$ to $\langle D(X_1), \prec_{R_1}\rangle$. Then Proposition 13 implies that $X(D(f))$ is a dually strongly stable Priestley morphism from the Priestley Δ-s-space $\langle X(D(L_1)), R^{\Delta}_{\prec_{R_1}}\rangle$ to the Priestley Δ-s-space $\langle X(D(L_2)), R^{\Delta}_{\prec_{R_2}}\rangle$. Using Proposition 7 and Priestley duality it easily follows that f is a dually strongly stable Priestley morphism from $\langle X_1, R_1\rangle$ to $\langle X_2, R_2\rangle$. □

Proposition 15 *Let L_1, L_2 be subordination lattices and $h : L_1 \to L_2$ a map. Then h is a dually strong subordination homomorphism from L_1 to L_2 if and only if the dual map $X(h)$ is a dually strongly stable Priestley morphism from the Priestley Δ-space $\langle X(L_2), R^{\Delta}_{\prec_2}\rangle$ to the Priestley Δ-space $\langle X(L_1), R^{\Delta}_{\prec_1}\rangle$.*

Proof The implication from left to right is Proposition 13. To prove the other implication, if $X(h)$ is a dually strongly stable Priestley morphism from the Priestley Δ-space $\langle X(L_2), R^{\Delta}_{\prec_2}\rangle$ to the Priestley Δ-space $\langle X(L_1), R^{\Delta}_{\prec_1}\rangle$, then by Proposition 14, $D(X(h))$ is a dually strong subordination homomorphism from the subordination lattice $\langle D(X(L_2)), \prec_{R^{\Delta}_{\prec_2}}\rangle$ to $\langle D(X(L_1)), \prec_{R^{\Delta}_{\prec_1}}\rangle\rangle$. By Proposition 9 the map σ_{L_i} is an isomorphism between $\langle L_i, \prec_i\rangle$ and $\langle D(X(L_i)), \prec_{R^{\Delta}_{\prec_i}}\rangle$ for $i = 1, 2$. Using Priestley duality, it follows that h is a dually strong subordination homomorphism from L_1 to L_2. □

We consider the following categories:

- ΔPriSp: the category of Priestley Δ-subordination spaces with the stable Priestley morphisms as its arrows.
- $\Delta\mathsf{PriSp}^s$: the category of Priestley Δ-subordination spaces with the strongly stable Priestley morphisms as its arrows.
- $\Delta\mathsf{PriSp}^{ds}$: the category of Priestley Δ-subordination spaces with the dually strongly stable Priestley morphisms as its arrows.
- ∇PriSp: the category of Priestley ∇-subordination spaces with the stable Priestley morphisms as its arrows.
- $\nabla\mathsf{PriSp}^s$: the category of Priestley ∇-subordination spaces with the reversely strongly stable Priestley morphisms as its arrows.
- SLat: the category of the subordination lattices with the subordination homomorphisms as its arrows.

- SLat^s: the category of the subordination lattices with the strong subordination homomorphisms as its arrows.
- SLat^{ds}: the category of the subordination lattices with the dually strong subordination homomorphisms as its arrows.

Remark 6 The categories $\Delta\mathsf{PriSp}$ and $\nabla\mathsf{PriSp}$ are equivalent as well as the categories $\Delta\mathsf{PriSp}^s$ and $\nabla\mathsf{PriSp}^s$. The functors that witness the equivalence are defined as follows. The functor from $\Delta\mathsf{PriSp}$ to $\nabla\mathsf{PriSp}$ maps a Priestley Δ-subordination space $\langle X, R\rangle$ to the Priestley ∇-subordination space $\langle X, R^{-1}\rangle$, and the functor from $\nabla\mathsf{PriSp}$ to $\Delta\mathsf{PriSp}$ does the same. For morphisms the functors leave the functions as they are. The same happens with $\Delta\mathsf{PriSp}^s$ and $\nabla\mathsf{PriSp}^s$.

The results above show that the functor D from the category PriSp to the category DLat can be expanded to a functor from $\Delta\mathsf{PriSp}$ to SLat, to a functor from $\Delta\mathsf{PriSp}^s$ to SLat^s and to a functor from $\Delta\mathsf{PriSp}^{ds}$ to SLat^{ds} by mapping any Δ-Priestley space $\langle X, R\rangle$ to its subordination lattice $\langle D(X), \prec_R\rangle$ and every morphism in the corresponding category of spaces to its Priestley dual. Also the results above show that the functor X from DLat to PriSp can be expanded to a functor from SLat to $\Delta\mathsf{PriSp}$, to a functor from SLat^s to $\Delta\mathsf{PriSp}^s$, and to a functor from SLat^{ds} to $\Delta\mathsf{PriSp}^{ds}$ by sending a subordination lattice $\langle L, \prec\rangle$ to the Priestley Δ-subordination space $\langle X(L), R^{\Delta}_{\prec}\rangle$ and every morphism in the corresponding category of subordination lattices to its Priestley dual. Doing it, we have that the two categories in the pairs $(\Delta\mathsf{PriSp}, \mathsf{SLat})$, $(\Delta\mathsf{PriSp}^s, \mathsf{SLat}^s)$, and $(\Delta\mathsf{PriSp}^{ds}, \mathsf{SLat}^{ds})$ are dually equivalent.

In a similar way, the functor D from PriSp to DLat can be expanded to a functor from $\nabla\mathsf{PriSp}$ to SLat and to a functor from $\Delta\mathsf{PriSp}^s$ to SLat^s by mapping any ∇-Priestley space $\langle X, R\rangle$ to its subordination lattice $\langle D(X), \prec^*_R\rangle$. The functor X from DLat to PriSp can also be expanded to a functor from SLat to $\nabla\mathsf{PriSp}$ and to a functor from SLat^s to $\nabla\mathsf{PriSp}^s$ that sends a subordination lattice $\langle L, \prec\rangle$ to the Priestley ∇-subordination space $\langle X(L), R^{\nabla}_{\prec}\rangle$. The morphisms are mapped in each case to their Priestley duals.

From the dualities discussed above for categories of subordination lattices we can obtain dualities for categories of bi-subordination lattices in the natural way. Let us introduce the dual objects of bi-subordination lattices.

Definition 8 We say that a triple $\langle X, R, S\rangle$ is a *Priestley bi-subordination space* if X is a Priestley space and R and S are binary relations on X each one of which is the Δ-dual of a subordination.

Note that $\langle X, R, S\rangle$ is a Priestley bi-subordination space if and only if $\langle X, R, S^{-1}\rangle$ is a quasi-modal space in the terminology of Castro and Celani (2004).

By combining the properties of subordination homomorphism, strong subordination homomorphism and dually strong subordination homomorphism we can consider several categories of bi-subordination lattices by taking as morphisms maps that are of one of these kinds for the first subordination and of another for the second. Similarly, we can consider several categories of Priestley bi-subordination spaces. Once we fix a choice of morphisms for a category of bi-subordination lattices we can consider the category of Priestley bi-subordination spaces with the corresponding choice of morphisms and in this way we obtain two categories that are dually equivalent.

For example, the category of bi-subordination lattices with morphisms the lattice homomorphisms that are a subordination homomorphism w.r.t. the first subordination and a strong subordination homomorphism w.r.t. the second subordination is dually equivalent to the category of Priestley bi-subordination spaces with the Priestley morphisms that are a stable morphism w.r.t. the first relation and a strongly stable morphism w.r.t. the second relation.

Now we turn to discuss the duals of the bi-subordination lattices with the properties we considered in the examples given in Sect. 8.3.

First we consider the bi-subordination lattices where the first subordination is included in the second. They include the bi-subordinations lattices in Example 1.

Proposition 16 *Let* $\langle L, \prec, \lhd \rangle$ *be a bi-subordination lattice. Then*

$$\prec \subseteq \lhd \iff R_{\lhd} \subseteq R_{\prec},$$

(where $R_{\lhd} = R_{\Delta_{\lhd}}$ *and* $R_{\prec} = R_{\Delta_{\prec}}$*).*

Proof Assume that $\leq \subseteq \lhd$. Suppose that $P, Q \in X(L)$ are such that $P R_{\lhd} Q$. To prove that $P R_{\prec} Q$, suppose that $a \in \Delta_{\prec}^{-1}(P)$. Then $\Delta_{\prec}(a) \cap P \neq \emptyset$. So, let $b \in \Delta_{\prec}(a) \cap P$. Then $b \prec a$ and therefore $b \lhd a$. Therefore, $b \in \Delta_{\lhd}(a) \cap P$ and so $a \in \Delta_{\prec}^{-1}(P)$. Since $P R_{\lhd} Q$ it follows that $a \in Q$. We conclude that $\Delta_{\prec}^{-1}(P) \subseteq Q$, which by definition implies that $P R_{\lhd} Q$.

Assume now that $R_{\lhd} \subseteq R_{\prec}$, that $a \prec b$, and that it is not the case that $a \lhd b$. Then $a \notin \Delta_{\lhd}(b)$. Therefore there exists $P \in X(L)$ such that $a \in P$ and $P \cap \Delta_{\lhd}(b) = \emptyset$. Then $b \notin \Delta_{\lhd}^{-1}(P)$. Let then $Q \in X(L)$ such that $\Delta_{\lhd}^{-1}(P) \subseteq Q$ and $b \notin Q$. So we have $P R_{\lhd} Q$ and hence, by the assumption, $P R_{\prec} Q$, namely $\Delta_{\prec}^{-1}(P) \subseteq Q$. Since $a \in \Delta_{\prec}(b) \cap P$, $b \in \Delta_{\prec}^{-1}(P)$. Therefore $b \in Q$, a contradiction. We conclude that $a \lhd b$. Hence, $\prec \subseteq \lhd$. □

The proposition allows us to consider categories of Priestley bi-subordination spaces where the first relation is included in the second and obtain dualities for the categories of bi-subordination lattices with the first subordination included in the second.

Now we can consider the bi-subordination lattices where the second subordination is the converse of the first. They include the bi-subordination lattices in Example 3.

Let $L = \langle L, \prec, \lhd \rangle$ be a bi-subordination lattice such that $\lhd$ is the converse relation of $\prec$. Then, since for every $a, b \in L$

$$\sigma(a) \prec_{R_{\prec}} \sigma(b) \Leftrightarrow \sigma(a) \subseteq \Box_{R_{\prec}}(\sigma(b))$$

and

$$\sigma(a) \prec_{R_{\lhd}} \sigma(b) \Leftrightarrow \sigma(a) \subseteq \Box_{R_{\lhd}}(\sigma(b)),$$

we have

$$\sigma(a) \subseteq \Box_{R_{\prec}}(\sigma(b)) \Leftrightarrow \sigma(b) \subseteq \Box_{R_{\lhd}}(\sigma(a)).$$

This suggest considering the Priestley bi-subordination spaces $\langle X, R_1, R_2\rangle$ where R_1 and R_2 satisfy for all clopen upsets U, V the following condition:

$$U \subseteq \Box_{R_1}(V) \Leftrightarrow V \subseteq \Box_{R_2}(U).$$

Using this observation we can obtain dualities for the categories of bi-subordination lattices with each subordination being the converse of the other by takin as duals of these bi-subordination lattices the Priestley bi-subordination spaces that satisfy the above condition.

In the next section we discuss the dualities for positive bi-subordination lattices.

8.6 Positive Bi-Subordination Lattices

In this section we present first the dualities for positive subordination lattices that follow from the general facts described in the previous section. Then we present a different duality where positive subordination lattices are represented by a Priestley space endowed with a single binary relation.

The conditions that define positive bi-subordination lattices in Definition 3 can be characterized by properties of the relations associated with the subordinations as shown in the next two propositions.

Proposition 17 *Let $\langle L, \prec, \lhd\rangle$ be a bi-subordination lattice. The following conditions are equivalent:*

1. $\Delta_{\prec}(a \vee b) \subseteq \nabla_{\lhd}(a) \odot \Delta_{\prec}(b)$, *for all* $a, b \in L$,
2. $R^{\Delta}_{\prec} = (R_L \circ \subseteq)$,

where $R_L = R^{\Delta}_{\prec} \cap R^{\nabla}_{\lhd}$.

Proof (1) $\Rightarrow$ (2). To prove the inclusion $(R_L \circ \subseteq) \subseteq R^{\Delta}_{\prec}$, note that $(R_L \circ \subseteq) \subseteq R^{\Delta}_{\prec} \circ \subseteq$ and that by Lemma 2, $R^{\Delta}_{\prec} \circ \subseteq = R^{\Delta}_{\prec}$. To prove the other inclusion, suppose that $P, Q \in X(L)$ are such that $\Delta^{-1}_{\prec}(P) \subseteq Q$. We recall, by Lemma 1, that the set $\Delta^{-1}_{\prec}(P)$ is a filter of L. Consider the ideal $\left(Q^c \cup \nabla^{-1}_{\lhd}(P)^c\right]$. We prove that

$$\Delta^{-1}_{\prec}(P) \cap \left(Q^c \cup \nabla^{-1}_{\lhd}(P)^c\right] = \emptyset. \tag{8.1}$$

We assume the contrary. Then let $c \in \Delta^{-1}_{\prec}(P)$, $b \notin Q$ and $a \notin \nabla^{-1}_{\lhd}(P)$ such that $c \leq a \vee b$. We note that as $b \notin Q$, we have $b \notin \Delta^{-1}_{\prec}(P)$ and hence

$$\Delta_{\prec}(b) \cap P = \emptyset. \tag{8.2}$$

Also, as $a \notin \nabla^{-1}_{\lhd}(P)$, there exists $d \in L$ such that

$$d \in \nabla_{\lhd}(a) \text{ and } d \notin P. \tag{8.3}$$

Since $c \leq a \vee b$, and $\Delta_{\prec}$ is monotonic, $\Delta_{\prec}(c) \subseteq \Delta_{\prec}(a \vee b)$, and thus $\Delta_{\prec}(a \vee b) \cap P \neq \emptyset$, i.e., there exists $e \in \Delta_{\prec}(a \vee b)$ such that $e \in P$. Since by the hypothesis, $\Delta_{\prec}(a \vee b) \subseteq \nabla_{\lhd}(a) \odot \Delta_{\prec}(b)$ we have $e \in \nabla_{\lhd}(a) \odot \Delta_{\prec}(b)$ and as $d \in \nabla_{\lhd}(a)$, there exists $w \in \Delta_{\prec}(b)$ such that $e \leq d \vee w$. Since $e \in P$ and $d \notin P$, it follows that $w \in P$; hence $\Delta_{\prec}(b) \cap P \neq \emptyset$, in contradiction with (8.2). Thus, we obtain (8.1). Then there exists $D \in X(L)$ such that

$$\Delta_{\prec}^{-1}(P) \subseteq D \subseteq \nabla_{\lhd}^{-1}(P) \text{ and } D \subseteq Q.$$

This implies that $(P, Q) \in (R_L \circ \subseteq)$.

(2) $\Rightarrow$ (1) Assume that there exists $c \in \Delta_{\prec}(a \vee b)$ such that $c \notin \nabla_{\lhd}(a) \odot \Delta_{\prec}(b)$. Then there exists $d \in \nabla_{\lhd}(a)$ such that $c \notin (\Delta_{\prec}(b) \cup \{d\}]$. Therefore there exists $P \in X(L)$ satisfying that $c \in P$, $\Delta_{\prec}(b) \cap P = \emptyset$, and $d \notin P$. So, there exists $Q \in X(L)$ such that $\Delta_{\prec}^{-1}(P) \subseteq Q$ and $b \notin Q$. By hypothesis, there exists $D \in X(L)$ such that $\Delta_{\prec}^{-1}(P) \subseteq D \subseteq \nabla_{\lhd}^{-1}(P)$ and $D \subseteq Q$. As $c \in \Delta_{\prec}(a \vee b) \cap P$ and $\Delta_{\prec}^{-1}(P) \subseteq D$, we get that $a \vee b \in D \subseteq Q$, but since $b \notin Q$, it follows that $b \notin D$. Therefore, $a \in D$ and hence $a \in \nabla_{\lhd}^{-1}(P)$, which means that $\nabla_{\lhd}(a) \subseteq P$. Since $d \in \nabla_{\lhd}(a)$, it follows that $d \in P$, which is a contradiction. $\square$

Proposition 18 *Let $\langle L, \prec, \lhd \rangle$ be a bi-subordination lattice. The following conditions are equivalent:*

1. $\nabla_{\lhd}(a \wedge b) \subseteq \Delta_{\prec}(a) \oplus \nabla_{\lhd}(b)$, *for all* $a, b \in L$,
2. $R_{\lhd}^{\nabla} = (R_L \circ \subseteq^{-1})$,

where $R_L = R_{\prec}^{\Delta} \cap R_{\lhd}^{\nabla}$.

Proof (1) $\Rightarrow$ (2) The inclusion $(R_L \circ \subseteq^{-1}) \subseteq R_{\lhd}^{\nabla}$ follows from Lemma 2. To prove the other inclusion, assume that $P, Q \in X(L)$ are such that $Q \subseteq \nabla_{\lhd}^{-1}(P)$. By Lemma 1, the set $\nabla_{\lhd}^{-1}(P)^c$ is an ideal of L. Consider the filter $\left[\Delta^{-1}(P) \cup Q\right)$. We prove that.

$$\left[\Delta_{\prec}^{-1}(P) \cup Q\right) \cap \nabla_{\lhd}^{-1}(P)^c = \emptyset. \tag{8.4}$$

Suppose the contrary. Then let $a \in \Delta_{\prec}^{-1}(P)$, $b \in Q$ and $c \notin \nabla_{\lhd}^{-1}(P)$ such that $a \wedge b \leq c$. Since $\nabla_{\lhd}$ is antimonotonic, $\nabla_{\lhd}(c) \subseteq \nabla_{\lhd}(a \wedge b)$, and since $c \notin \nabla_{\lhd}^{-1}(P)$, $\nabla_{\lhd}(c) \nsubseteq P$; hence $\nabla_{\lhd}(a \wedge b) \nsubseteq P$. Thus, let $d \in \nabla_{\lhd}(a \wedge b)$ be such that $d \notin P$. By hypothesis, $\nabla_{\lhd}(a \wedge b) \subseteq \Delta_{\prec}(a) \oplus \nabla_{\lhd}(b)$, so $d \in \Delta_{\prec}(a) \oplus \nabla_{\lhd}(b)$. Since $a \in \Delta_{\prec}^{-1}(P)$, there exists $e \in \Delta_{\prec}(a) \cap P$. Then there exists $w \in \nabla_{\lhd}(b)$ such that $e \wedge w \leq d$. Since $b \in Q \subseteq \nabla_{\lhd}^{-1}(P)$, we have $\nabla_{\lhd}(b) \subseteq P$. Thus $w \in P$, and hence, since then $e \wedge w \in P$, we have $d \in P$, which is a contradiction. Therefore, (8.4) is valid. Then there exists $D \in X(L)$ such that

$$\Delta_{\prec}^{-1}(P) \subseteq D \subseteq \nabla_{\lhd}^{-1}(P) \text{ and } Q \subseteq D,$$

i.e., $(P, Q) \in (R_L \circ \subseteq^{-1})$.

(2) ⇒ (1) Assume (2) and suppose that $a, b, c \in L$ are such that $c \in \nabla_{\lhd}(a \wedge b)$. Suppose that $c \notin \Delta_{\prec}(a) \oplus \nabla_{\lhd}(b)$. Then there exists $d \in \Delta_{\prec}(a)$ such that $c \notin [\nabla_{\lhd}(b) \cup \{d\})$. Then there exists $P \in X(L)$ such that $\nabla_{\lhd}(b) \subseteq P$, $d \in P$ and $c \notin P$. By Lemma 3 there exists $Q \in X(L)$ such that $Q \subseteq \nabla_{\lhd}^{-1}(P)$ and $b \in Q$. By hypothesis, there exists $D \in X(L)$ such that $\Delta_{\prec}^{-1}(P) \subseteq D \subseteq \nabla_{\lhd}^{-1}(P)$ and $Q \subseteq D$. As $d \in \Delta_{\prec}(a) \cap P$, we have $a \in D$, and as $b \in Q$, we get that $a \wedge b \in D$. So, $\nabla_{\lhd}(a \wedge b) \subseteq P$, but this implies that $c \in P$, which is impossible. Therefore, $\nabla_{\lhd}(a \wedge b) \subseteq \Delta_{\prec}(a) \oplus \nabla_{\lhd}(b)$. □

Corollary 2 *Let $\langle L, \prec, \lhd \rangle$ be a bi-subordination lattice. Then L is a positive bi-subordination lattice if and only if the following two conditions hold*

1. $R_{\prec}^{\Delta} = (R_L \circ \subseteq)$,
2. $R_{\lhd}^{\nabla} = (R_L \circ \subseteq^{-1})$,

where $R_L = R_{\prec}^{\Delta} \cap R_{\lhd}^{\nabla}$. Equivalently, if and only if

1. $R_{\prec}^{\Delta} = (R_L \circ \subseteq)$,
2. $(R_{\lhd}^{\Delta})^{-1} = (R_L \circ \subseteq^{-1})$,

where $R_L = R_{\prec}^{\Delta} \cap (R_{\lhd}^{\Delta})^{-1}$.

The corollary motivates the next definition.

Definition 9 A Priestley bi-subordination space $\langle X, R_1, R_2 \rangle$ is *positive* if R_1 and R_2 satisfy the following conditions:

1. $R_1 = (R_1 \cap R_2^{-1}) \circ \leq$,
2. $R_2^{-1} = (R_1 \cap R_2^{-1}) \circ \leq^{-1}$.

From the results obtained up to now we easily can prove the next two propositions.

Proposition 19 *Let L be a bi-subordination lattice. Then L is a positive subordination lattice if and only if the Priestley bi-subordination space $\langle X(L), R_{\prec}^{\Delta}, R_{\lhd}^{\Delta} \rangle$ is positive.*

Proof It follows from Corollary 2. □

Proposition 20 *A Priestley bi-subordination space $\langle X, R_1, R_2 \rangle$ is positive if and only if $\langle D(X), \prec_{R_1}, \prec_{R_2} \rangle$ is a positive subordination lattice.*

Proof It follows from Proposition 7, Corollary 2, and Proposition 9. □

As for other classes of bi-subordination lattices, once we fix as objects the positive subordination lattices we obtain different categories by taking as arrows maps that are of one kind of morphism (subordination homomorphism, strong subordination homomorphism and dually strong subordination homomorphism) for the first subordination and of another for the second. Then, moving to the corresponding categories of positive Priestley bi-subordination spaces, the results in Sect. 8.4 provide us with the corresponding duality results.

Now we turn to find a category of Priestley spaces with a single binary relation dually equivalent to the category of the positive subordination lattices with arrows the maps that are a subordination homomorphism for both subordinations and also a category of Priestley spaces with a single binary relation dually equivalent to the category of the positive subordination lattices with arrows the maps that are a strong subordination homomorphism for the first subordination and a dually strong subordination homomorphism for the second subordination.

Proposition 21 *If $L = \langle L, \prec, \lhd\rangle$ is a positive bi-subordination lattice, $a \in L$, and $P \in X(L)$, then using the relation $R_L = R^{\Delta}_{\prec} \cap R^{\nabla}_{\lhd} = R^{\Delta}_{\prec} \cap (R^{\Delta}_{\prec})^{-1}$ we have:*

1. *$\Delta_{\prec}(a) \cap P = \emptyset$ iff there exists $Q \in X(L)$ such that $(P, Q) \in R_L$ and $a \notin Q$.*
2. *$\nabla_{\lhd}(a) \subseteq P$ iff there exists $Q \in X(L)$ such that $(P, Q) \in R_L$ and $a \in Q$.*

Proof (1) Assume that $\Delta_{\prec}(a) \cap P = \emptyset$. By Lemma 3 there exists $Q \in X(L)$ such that $P R^{\Delta}_{\prec} Q$ and $a \notin Q$. By Corollary 2, there exists $Q' \in X(L)$ such that $P R_L Q'$ and $Q' \subseteq Q$. The converse follows from the fact that $R_L \subseteq R^{\Delta}_{\prec}$.

The proof of (2) is similar. □

Lemma 9 *If $\langle X, R_1, R_2\rangle$ is a positive Priestley bi-subordination space and $R = R_1 \cap R_2^{-1}$, then*

1. *$R(x)$ is a closed subset of X, for each $x \in X$,*
2. *$R = (R \circ \leq) \cap (R \circ \leq^{-1})$,*
3. *$\Box_R(U) = \Box_{R_1}(U)$, for each $U \in D(X)$,*
4. *$\Diamond_R(U) = \Diamond_{R_2^{-1}}(U)$, for each $U \in D(X)$.*

Proof The proof of (1) is immediate because both $R_1(x)$ and $R_2^{-1}(x)$ are closed sets. For (2) we note that as $R_1 = (R_1 \cap R_2^{-1}) \circ \leq = R \circ \leq$ and $R_2^{-1} = (R_1 \cap R_2^{-1}) \circ \leq^{-1} = R \circ \leq^{-1}$, then $R = R_1 \cap R_2^{-1} = (R \circ \leq) \cap (R \circ \leq^{-1})$.

For the proof of (3) assume that $x \in \Box_R(U)$. If $x \notin \Box_{R_1}(U)$, then there exists $y \in X$ such that $(x, y) \in R_1$ but $y \notin U$. As $R_1 = R \circ \leq$, there exist $z \in X$ such that $(x, z) \in R$ and $z \leq y$. So, since $x \in \Box_R(U)$, we have $z \in U$, and since U is an upset, $y \in U$, which is a contradiction. Suppose now that $x \in \Box_{R_1}(U)$. Note that $R \subseteq R \circ \leq = R_1$. Therefore $R(x) \subseteq R_1(x)$. It follows that $x \in \Box_R(U)$.

To prove (4) assume that $x \in \Diamond_R(U)$. Then $R(x) \cap U \neq \emptyset$. Therefore, $R_2^{-1}(x) \cap U \neq \emptyset$ and so $x \in \Diamond_{R_2^{-1}}(U)$. Conversely, if $x \in \Diamond_{R_2^{-1}}(U)$, let $y \in R_2^{-1}(x) \cap U$. Then, since $R_2^{-1} = (R_1 \cap R_2^{-1}) \circ \leq^{-1}$, there is $u \in (R_1 \cap R_2^{-1})(x)$ such that $y \leq u$. Since U is an upset, it follows that $u \in R(x) \cap U$; therefore $x \in \Diamond_R(U)$. □

Proposition 22 *Let $\langle X, R\rangle$ be a relational structure such that X is a Priestley space and R is a binary relation on X satisfying the following conditions:*

1. *For every $x \in X$, $R(x)$ is a closed set,*
2. *$R = (R \circ \leq) \cap (R \circ \leq^{-1})$,*
3. *$\Box_R(U)$ is an open upset for every $U \in D(X)$, and $\Diamond_R(U)$ is closed upset for every $U \in D(X)$.*

Then the structure $\langle X, R_1, (R_2)^{-1}\rangle$ *is a positive bi-subordination space, where* R_1 *and* R_2 *are defined as* $R_1 = R \circ \leq$ *and* $R_2 = R \circ \leq^{-1}$*, respectively.*

Proof We prove that $R_1(x)$ is a closed set and an upset. If it is empty it is clear. If it is nonempty, suppose that $y \notin R_1(x)$. Then, since $R_1 = R \circ \leq$, for each $z \in R(x)$ we have $z \not\leq y$. Also $R(x)$ is nonempty. So, for each $z \in R(x)$ there exists $U_z \in D(X)$ such that $z \in U_z$, and $y \notin U_z$. Then, $R(x) \subseteq \bigcup\{U_z : z \in R(x)\}$ and since $R(x)$ is closed, there exists a finite subfamily $\{U_{z_1}, \ldots, U_{z_n}\}$ of $\{U_z : z \in R(x)\}$ such that $R(x) \subseteq U_{z_1} \cup \cdots \cup U_{z_n}$. Let $U = U_{z_1} \cup \cdots \cup U_{z_n}$. It is clear that $y \notin U$ and that U is an upset. Therefore, $R_1(x) \subseteq U$. Thus we have proved that for each $y \notin R_1(x)$ there exists $U \in D(X)$ such that $R_1(x) \subseteq U$ and $y \notin U$. It follwos that $R_1(x)$ is closed and the definition of R_1 implies that it is an upset. Similarly, we can prove that $R_2(x)$ is a closed downset.

Now we prove that for every $U \in D(X)$, $\Box_R(U) = \Box_{R_1}(U)$ and $\Diamond_R(U) = \Diamond_{R_2}(U)$. Since every $U \in D(X)$ is an upset, it easily follows that for every $U \in D(X)$ and $x \in X$, $R(x) \subseteq U$ if and only if $(R \circ \leq)(x) \subseteq U$. Hence, $\Box_R(U) = \Box_{R_1}(U)$ for every $U \in D(X)$. Moreover, for every $U \in D(X)$ it also holds that $\Diamond_R(U) = \Diamond_{R_2}(U)$. Indeed, since $R_2 = R \circ \leq^{-1}$, for every $x \in X$ and $U \in D(X)$, since U is an upset it follows that $R(x) \cap U \neq \emptyset$ if and only if $R_2(x) \cap U \neq \emptyset$. Therefore $\Diamond_R(U) = \Diamond_{R_2}(U)$.

Using the fact we have just proved, (3) of the assumption implies that R_1 is the Δ-dual of a subordination and R_2 is the ∇-dual of a subordination. Therefore, R_2^{-1} is the Δ-dual of a subordination. Then (2) of the assumption implies that $R = R_1 \cap R_2$. Therefore $R_1 = R \circ \leq = (R_1 \cap R_2) \circ \leq$ and $R_2 = R \circ \leq^{-1} = (R_1 \cap R_2) \circ \leq^{-1}$. It follows form the definition of positive bi-subordination space that $\langle X, R_1, (R_2)^{-1}\rangle$ is a positive bi-subordination space. □

Definition 10 A *positive Priestley space* is a pair $\langle X, R\rangle$ where X is a Priestley space and R is a relation on X that satisfies the conditions of Proposition 22.

By the above results, if $\langle X, R\rangle$ is a positive Priestley space, then the structure $\langle X, R_1, (R_2)^{-1}\rangle$ defined as in Proposition 22 is a positive Priestley bi-subordination space such that the pair $\langle X, R_1 \cap R_2\rangle$ satisfies the conditions in Proposition 22. Conversely, if $\langle X, R_1, R_2\rangle$ is a positive Priestley bi-subordination space, then the structure $\langle X, R_1 \cap R_2^{-1}\rangle$ satisfies the conditions in Proposition 22, and therefore it is a positive Priestley space such that the triple $\langle X, (R_1 \cap R_2^{-1}) \circ \leq, (R_1 \cap R_2^{-1}) \circ \leq^{-1}\rangle$ is a positive Priestley bi-subordination space where $R_1 = (R_1 \cap R_2^{-1}) \circ \leq$ and $R_2 = (R_1 \cap R_2^{-1}) \circ \leq^{-1}$. Thus, we have that there exists a bijective correspondence between positive Priestley bi-subordination spaces and positive Priestley spaces.

Definition 11 Let $\langle X, R\rangle$ and $\langle Y, S\rangle$ be positive Priestley spaces. A Priestley morphism $f : X \to Y$ from $\langle X, R\rangle$ to $\langle Y, S\rangle$ is *stable* if for every $x, y \in X$ such that xRy it holds that $f(x)Sf(y)$ and it is *doubly strongly stable* if it is stable and for all $x \in X$ and all $y \in Y$, if $f(x)Sy$ then there exist $z_1, z_2 \in X$ such that xRz_1, xRz_2 and $f(z_1) \leq y \leq f(z_2)$.

Proposition 23 *Let $\langle X, R\rangle$ and $\langle Y, S\rangle$ be positive Priestley spaces and consider the relations $R_1 = R \circ \leq$, $R_2 = R \circ \leq^{-1}$, $S_1 = S \circ \leq$, and $S_2 = S \circ \leq^{-1}$. Then $f : X \to Y$ is a (doubly strongly) stable Priestley morphism from $\langle X, R\rangle$ to $\langle Y, S\rangle$ if and only if f is a (strongly) stable morphism from $\langle X, R_1\rangle$ to $\langle Y, S_1\rangle$ and a (dually strongly) stable morphism from $\langle X, R_2^{-1}\rangle$ to $\langle Y, S_2^{-1}\rangle$.*

Proof Let $f : X \to Y$ be a Priestley morphism. Suppose that f is stable from $\langle X, R\rangle$ to $\langle Y, S\rangle$. To prove that f is stable from $\langle X, R_1\rangle$ to $\langle Y, S_1\rangle$ suppose that $x, y \in X$ are such that xR_1y. Then let $z \in X$ be such that xRz and $z \leq y$. Hence, by the assumption of stability, $f(x)Sf(z)$ and since f is a Priestley morphism $f(z) \leq f(y)$. Therefore $f(x)S_1f(y)$. A similar proof shows that f is stable from $\langle X, R_2^{-1}\rangle$ to $\langle Y, S_2^{-1}\rangle$. Assume now that f is doubly strongly stable. We proceed to prove that f is strongly stable from $\langle X, R_1\rangle$ to $\langle Y, S_1\rangle$. Assume that $x \in X$ and $y \in Y$ are such that $f(x)S_1y$. Then there is $u \in Y$ such that $f(x)Su$ and $u \leq y$. Thus, since f is double strongly stable there are $z_1, z_2 \in X$ such that xRz_1, xRz_2 and $f(z_1) \leq u \leq f(z_2)$. Then, as $u \leq y$, xRz_1 and $f(z_1) \leq y$. This shows that f is strongly stable from $\langle X, R_1\rangle$ to $\langle Y, S_1\rangle$. We now show that f is dually strongly stable from $\langle X, R_2^{-1}\rangle$ to $\langle Y, S_2^{-1}\rangle$. Suppose now that $x \in X$ and $y \in Y$ are such that $f(x)S_2y$. Then there is $u \in Y$ such that $f(x)Su$ and $y \leq u$. Since f is double strongly stable there are $z_1, z_2 \in X$ such that xRz_1, xRz_2 and $f(z_1) \leq u \leq f(z_2)$. Then, as $y \leq u$, xRz_2 and $y \leq f(z_2)$. Thus we obtain that f is dually strongly stable from $\langle X, R_2^{-1}\rangle$ to $\langle Y, S_2^{-1}\rangle$.

Conversely, assume that f is a stable morphism from $\langle X, R_1\rangle$ to $\langle Y, S_1\rangle$ and a stable morphism from $\langle X, R_2\rangle$ to $\langle Y, S_2\rangle$. To prove that f is stable from $\langle X, R\rangle$ to $\langle Y, S\rangle$, suppose that $x, y \in X$ are such that xRy. Note that since $\langle X, R\rangle$ and $\langle Y, S\rangle$ are positive Priestley spaces $R = R_1 \cap R_2$ and $S = S_1 \cap S_2$. Hence, xR_1y and xR_2y. Therefore the assumption implies that $f(x)S_1f(y)$ and $f(x)S_2f(y)$. Thus we have $f(x)Sf(y)$. Suppose now that f is a strongly stable morphism from $\langle X, R_1\rangle$ to $\langle Y, S_1\rangle$ and a dually strongly stable morphism from $\langle X, R_2^{-1}\rangle$ to $\langle Y, S_2^{-1}\rangle$. Assume that $x \in X$ and $y \in Y$ are such that $f(x)Sy$. Then $f(x)S_1y$ and $f(x)S_2y$. Thus, let $z_1, z_2 \in X$ such that xR_1z_1 and $f(z_1) \leq y$ and xR_2z_2 and $y \leq f(z_2)$. Let then $u_1, u_2 \in X$ such that xRu_1, $u_1 \leq z_1$, xRu_2, and $z_2 \leq u_2$. Then xRu_1, xRu_2 $f(u_1) \leq f(z_1) \leq y$, and $y \leq f(z_2) \leq f(u_2)$. We obtain the desired conclusion. □

Let PBiSLat be the category with objects the positive subordination lattices and arrows the maps between them that are a subordination homomorphism w.r.t. the two subordinations. Let PBiSLat$^{\mathsf{s}}$ be the category with objects the positive subordination lattices and arrows the maps between them that are a strong subordination homomorphism w.r.t. the first subordination and a dual strong subordination homomorphism w.r.t. the second. Similarly, let PPriSp be the category with objects the positive Priestley spaces and arrows the stable Priestley morphisms and let PPriSp$^{\mathsf{s}}$ be the category with objects the positive Priestley spaces and arrows the doubly strongly stable Priestley morphisms. From the results above the next theorem follows.

Proposition 24 *The categories* PBiSLat *and* PPriSp *are dually equivalent as well as the categories* PBiSLat$^{\mathsf{s}}$ *and* PPriSp$^{\mathsf{s}}$.

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