Chapter 1
Equality and Equivalence, Intuitionistically

Wim Veldman

For Mohammad Ardeshir
Solem enim e mundo tollere videntur qui amicitiam e vita tollunt.
They take away the sun from the world, surely, those who take away friendship from life.

Cicero, *de Amicitia, XIII 47*

Abstract We show that the intuitionistic first-order theory of equality has continuum many complete extensions. We also study the Vitali equivalence relation and show there are many intuitionistically precise versions of it.

Keywords Brouwer's continuity principle · Apartness · Toy spread · Decidable point of a spread · Perhapsive extensions

1.1 Introduction

We want to contribute to L. E. J. Brouwer's program of doing mathematics *intuitionistically*.

We follow his advice to interpret the logical constants constructively.

A conjunction $A \wedge B$ is considered proven if and only if one has a proof of A and also a proof of B.

A disjunction $A \vee B$ is considered proven if and only if either A or B is proven.

An implication $A \to B$ is considered proven if and only if there is a proof of B using the assumption A.

A negation $\neg A$ is considered proven if and only if there is a proof of $A \to 0 = 1$.

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An existential statement $\exists x \in V[P(x)]$ is considered proven if and only an element x_0 is produced together with a proof of the associated statement $P(x_0)$.

A universal statement $\forall x \in V[P(x)]$ is considered proven if and only if a method is given that produces, given any x in V, a proof of the associated statement $P(x)$.

We also use some axioms proposed by Brouwer: his *Continuity Principle*, our Axiom 1, a slightly stronger version of it, the *First Axiom of Continuous Choice*, our Axiom 2, and his *Thesis on Bars in* $\mathcal{N}$, our Axiom 4.

In some of our proofs, we use an Axiom of Countable Choice, our Axiom 3. Intuitionistic mathematicians, who accept infinite step-by-step constructions not determined by a rule, consider this axiom a reasonable proposal.

Finally, we believe that generalized inductive definitions, like our Definition 25, fall within the compass of intuitionistic mathematics.

Our subject is the (intuitionistic) first-order theory of equality. By considering structures $(\mathcal{X}, =)$ where $\mathcal{X}$ is a subset of Baire space $\mathcal{N} = \omega^\omega$ and $=$ the usual equality relation on $\mathcal{N}$, we find that the theory has an uncountable and therefore astonishing[1] variety of elementarily different infinite models and, as a consequence, an astonishing variety of complete extensions, see Theorem 15. The key observation[2] leading to this result is the recognition that, in a *spread*,[3] an *isolated* point is the same as a *decidable* point.[4] It follows that the set of the non-isolated points of a spread is a definable subset of the spread. In spreads that are *transparent*,[5] the set of the non-isolated points of the spread coincides with the *coherence* of the spread,[6] and the coherence itself is spread. It may happen that the coherence of a transparent spread is transparent itself and then the coherence of the coherence also is a definable subset of the spread. And so on.

Any structure $(\mathcal{N}, R)$, where R is an equivalence relation on $\mathcal{N}$, is a model of the theory of equality. We study the *Vitali equivalence relation*, see Sect. 1.9, as an example. This equivalence relation, in contrast to the equality relation on $\mathcal{N}$, is not *stable*,[7] see Theorem 16.

There is a host of binary relations on $\mathcal{N}$ that, from a classical point of view, all would be the same as the Vitali equivalence relation, see Sects. 1.10 and 1.11, and especially Definition 25, Corollary 3 and Definition 28. It turned out to be difficult to find differences between them that are first-order expressible. We did find some such differences, however, by studying structures $(\mathcal{N}, =, R)$, where R is an intuitionistic version of the Vitali equivalence relation and $=$ the usual equality, see Sect. 1.12.

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[1] Classically, all infinite models of the first-order theory of equality are elementarily equivalent.

[2] This observation has been made earlier in Veldman (2001, Sect. 5). The first part of the present paper elaborates part of Veldman (2001, Sect. 5).

[3] Every *spread* is a closed subset of $\mathcal{N}$, see Sect. 1.4.

[4] See Lemma 3. $\alpha \in \mathcal{X} \subseteq \mathcal{N}$ is a *decidable* point of $\mathcal{X}$ if and only if $\forall \beta \in \mathcal{X}[\alpha = \beta \ \vee \ \neg(\alpha = \beta)]$.

[5] See Definition 8.

[6] The *coherence* of a closed set is the set of its limit points, see Definition 7.

[7] $R \subseteq \mathcal{N} \times \mathcal{N}$ is called *stable* if $\forall \alpha \forall \beta[\neg\neg \alpha R \beta \rightarrow \alpha R \beta]$, see Definition 22.

The paper is divided into 13 Sections and consists roughly of two parts. Sections 1.2, 1.3, 1.4, 1.5, 1.6, 1.7 and 1.8 lead up to the result that the theory of equality has continuum many complete extensions, see Theorem 15. Sections 1.9, 1.10, 1.11 and 1.12 treat the Vitali equivalence relations. Section 1.13 lists some notations and conventions and may be used by the reader as a reference.

1.2 Intuitionistic Model Theory

Given a relational structure $\mathfrak{A} = (A, R_0, R_1, \ldots, R_{n-1})$, we construct a first-order language $\mathcal{L}$ with basic formulas $\mathsf{R}_i(\mathsf{x}_0, \mathsf{x}_1, \ldots, \mathsf{x}_{l_i-1})$, where $i < n$ and l_i is the arity of R_i. The formulas of $\mathcal{L}$ are obtained from the basic formulas by using $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\exists$, $\forall$ in the usual way.

For every formula $\varphi = \varphi(\mathsf{x}_0, \mathsf{x}_1, \ldots, \mathsf{x}_{m-1})$ of $\mathcal{L}$, for all $a_0, a_1, \ldots, a_{m-1}$ in A, we define the statement:

$$\mathfrak{A} \models \varphi[a_0, a_1, \ldots, a_{m-1}]$$

($\mathfrak{A}$ *realizes* φ *if* $\mathsf{x}_0, \mathsf{x}_1, \ldots, \mathsf{x}_{m-1}$ *are interpreted by* $a_0, a_1, \ldots, a_{m-1}$, *respectively*), as Tarski did it, with the proviso that connectives and quantifiers are interpreted intuitionistically.

A formula φ of $\mathcal{L}$ without free variables will be called a *sentence*.

A *theory (in* $\mathcal{L}$*)* is a set of sentences of $\mathcal{L}$.

Given a theory Γ in $\mathcal{L}$ and a structure $\mathfrak{A}$, we define: $\mathfrak{A}$ *realizes* Γ if and only if, for every φ in Γ, $\mathfrak{A} \models \varphi$.

Given a structure $\mathfrak{B}$ that has the same signature as $\mathfrak{A}$, so that the formulas of $\mathcal{L}$ may be interpreted in $\mathfrak{B}$ as well as in $\mathfrak{A}$, we let $Th(\mathfrak{B})$, the *theory of* $\mathfrak{B}$, be the set of all sentences φ of $\mathcal{L}$ such that $\mathfrak{B} \models \varphi$.

A theory Γ in $\mathcal{L}$ will be called a *complete theory* if and only if there exists a structure $\mathfrak{B}$ such that $\Gamma = Th(\mathfrak{B})$.

This agrees with one of the uses of the expression '*complete theory*' in classical, that is: usual, non-intuitionistic, model theory, see Hodges (1993, p. 43). Note that one may be unable to decide, for a given sentence φ and a given structure $\mathfrak{B}$, whether or not $\mathfrak{B} \models \varphi$. Intuitionistically, it is not true that, for every complete theory Γ and every sentence φ, *either* $\varphi \in \Gamma$ *or* $\neg\varphi \in \Gamma$.

Complete theories Γ, Δ are *positively different* if one may point out a sentence ψ such that $\psi \in \Gamma$ and $\neg\psi \in \Delta$.[8]

Structures $\mathfrak{A}$, $\mathfrak{B}$ are *elementarily equivalent* if and only if $Th(\mathfrak{A}) = Th(\mathfrak{B})$ and *(positively) elementarily different* if $Th(\mathfrak{A})$ is positively different from $Th(\mathfrak{B})$.

Let Γ be a theory in $\mathcal{L}$. A good question is the following:

How many complete theories Δ *can one find extending* Γ?

[8] If $\psi \in \Gamma$ and $\neg\psi \in \Delta$, then $\neg\psi \in \Delta$ and $\neg\neg\psi \in \Gamma$: the relation *positively different* is symmetric.

We will say: Γ *admits countably many complete extensions* if and only if there exists an infinite sequence $\Delta_0, \Delta_1, \ldots$ of complete theories extending Γ such that, for all m, n, if $m \neq n$, then Δ_m, Δ_n are (positively) different, and

Γ *admits continuum many complete extensions* if and only if there exists a function $\alpha \mapsto \Delta_\alpha$ associating to every element α of $\mathcal{C} = 2^\omega$ a complete theory extending Γ such that for all α, β, if[9] $\alpha \# \beta$, then Δ_α, Δ_β are (positively) different.

A main result of this paper is that the first-order theory of equality admits continuum many complete extensions.

1.3 Equality May Be Undecidable

The first-order theory EQ of equality consists of the following three axioms:

1. $\forall \mathsf{x}[\mathsf{x} = \mathsf{x}]$,
2. $\forall \mathsf{x} \forall \mathsf{y}[\mathsf{x} = \mathsf{y} \rightarrow \mathsf{y} = \mathsf{x}]$ and
3. $\forall \mathsf{x} \forall \mathsf{y} \forall \mathsf{z}[(\mathsf{x} = \mathsf{y} \wedge \mathsf{y} = \mathsf{z}) \rightarrow \mathsf{x} = \mathsf{z}]$.

A model of EQ is a structure of the form (V, R), where V is a set and R is an equivalence relation on V, possibly, but not necessarily, the equality relation belonging to V.

Classically, every complete extension of EQ is realized in one of the structures from the list: $(\{0\}, =)$, $(\{0, 1\}, =)$, $(\{0, 1, 2\}, =)$, $\ldots$ and $(\omega, =)$. This shows that, classically, EQ admits of (no more than) countably many complete extensions.

Intuitionistically, however, we have to observe that all structures on this list satisfy the sentence

$\forall \mathsf{x} \forall \mathsf{y}[\mathsf{x} = \mathsf{y} \vee \neg(\mathsf{x} = \mathsf{y})]$,

that is: the equality relation, on each of these sets, is a *decidable* relation.

Turning to the set $\mathcal{N}$, we note that, if we define an element α of $\mathcal{N}$ by stipulating:

$$\forall n[\alpha(n) \neq 0 \leftrightarrow \forall i < 99[d(n + i) = 9]],$$

where $d : \mathbb{N} \rightarrow \{0, 1, \ldots, 9\}$ is the decimal expansion of π, then, at this moment, we have no proof of:

$$\alpha = \underline{0} \vee \neg(\alpha = \underline{0}).$$

This is because, if $\alpha = \underline{0}$, then $\neg\exists n \forall i < 99[\alpha(n + i) = 9]$, and, if $\neg(\alpha = \underline{0})$, then $\neg\neg\exists n \forall i < 99[d(n + i) = 9]$, and we have no proof of either alternative.

This example shows us that the statement $\forall \alpha[\alpha = \underline{0} \vee \neg(\alpha = \underline{0})]$, for a constructive mathematician, who interprets the disjunction strongly, is a *reckless* statement.[10]

[9] $\alpha \# \beta \leftrightarrow \alpha \perp \beta \leftrightarrow \exists n[\alpha(n) \neq \beta(n)]$, see Sect. 1.13.

[10] A statement is *reckless* if the classical mathematician holds it is true while the intuitionistic mathematician, at this point of time, has no proof for his constructive reading of it.

The following axiom, used by Brouwer,[11] implies that the statement $\forall\alpha[\alpha = \underline{0} \vee \neg(\alpha = \underline{0})]$ even leads to a contradiction.

Axiom 1 (Brouwer's Continuity Principle)
For all $R \subseteq \mathcal{N} \times \omega$, if $\forall\alpha\exists n[\alpha R n]$, then $\forall\alpha\exists m\exists n\forall\beta[\overline{\alpha}m \sqsubset \beta \rightarrow \beta R n]$.

An immediate consequence is:

Lemma 1 (Brouwer's Continuity Principle, the case of disjunction)
For all $P_0, P_1 \subseteq \mathcal{N}$, *if* $\forall\alpha[\alpha \in P_0 \vee \alpha \in P_1]$, *then* $\forall\alpha\exists m[\forall\beta[\overline{\alpha}m \sqsubset \beta \rightarrow \beta \in P_0] \vee \forall\beta[\overline{\alpha}m \sqsubset \beta \rightarrow \beta \in P_1]]$.

Proof Define $R := \{(\alpha, n) \mid n < 2 \wedge \alpha \in P_n\}$ and apply Axiom 1. □

Theorem 1 (i) $(\mathcal{N}, =) \models \forall x\neg\forall y[x = y \vee \neg(x = y)]$.
(ii) $(\mathcal{N}, =) \models \neg\forall x\forall y[x = y \vee \neg(x = y)]$.

Proof (i) Let α be given and assume: $\forall\beta[\alpha = \beta \vee \neg(\alpha = \beta)]$.
Using Lemma 1, find m such that
either $\forall\beta[\overline{\alpha}m \sqsubset \beta \rightarrow \alpha = \beta]$ *or* $\forall\beta[\overline{\alpha}m \sqsubset \beta \rightarrow \neg(\alpha = \beta)]$.
Consider $\beta := \overline{\alpha}m * \langle\alpha(m) + 1\rangle * \underline{0}$ (for the first alternative) and $\beta := \alpha$ (for the second one) and conclude that both alternatives are false.
(ii) This is an immediate consequence of (i).

□

Definition 1 For each n, we let ψ_n be the sentence
$\exists x_0\exists x_1 \ldots \exists x_n[\bigwedge_{i<j<n} \neg(x_i = x_j)]$.
$T_{inf} := EQ \cup \{\psi_n \mid n \in \omega\}$.

ψ_n expresses that a set has at least $n + 1$ elements.

Note that, in classical mathematics, T_{inf} has only one complete extension.

Intuitionistically, however, T_{inf} has (at least) two positively different complete extensions, $Th\big((\mathcal{N}, =)\big)$ and $Th\big((\omega, =)\big)$.

The next Theorem reflects the fact that, in classical model theory, all models of T_{inf} are elementarily equivalent.

Theorem 2 *The theory* $T_{inf} \cup \{\forall x\forall y[x = y \vee \neg(x = y)]\}$ *has only one complete extension.*

Proof For each n, consider the first n variables of our language: $x_0, x_1, \ldots, x_{n-1}$. A formula $\varepsilon = \varepsilon(x_0, x_1, \ldots, x_{n-1})$ is called an *equality type* if and only if it is of the form $\bigwedge_{i<j<n} \sigma_{ij}$ where each σ_{ij} either is the formula $x_i = x_j$ or the formula $\neg(x_i = x_j)$.[12] One may prove: for all structures $(V_0, R_0), (V_1, R_1)$, both realizing $T_{inf} \cup \{\forall x\forall y[x = y \vee \neg(x = y)]\}$, for each formula $\varphi = \varphi(x_0, x_1, \ldots, x_{n-1})$, for each equality type $\varepsilon = \varepsilon(x_0, x_1, \ldots, x_{n-1})$, $(V_0, R_0) \models \forall x_0\forall x_1 \ldots \forall x_{n-1}[\varepsilon \rightarrow \varphi]$ if

[11] See Veldman (2001).

[12] *Inconsistent* equality types may be annoying but do not cause difficulties.

and only if $(V_1, R_1) \models \forall \mathsf{x}_0 \forall \mathsf{x}_1 \ldots \forall \mathsf{x}_{n-1}[\varepsilon \to \varphi]$. The proof is by induction on the complexity of the formula φ.

It follows that any two models (V_0, R_0), (V_1, R_1), both realizing $T_{inf} \cup \{\forall \mathsf{x} \forall \mathsf{y}[\mathsf{x} = \mathsf{y} \vee \neg(\mathsf{x} = \mathsf{y})]\}$, are elementarily equivalent. □

From here on, we restrict attention to infinite models of EQ, that is, to models of T_{inf}. The hackneyed question to make a survey of models that are *finite*, or at least *not infinite*, and of models for which one can not decide if they are finite or infinite, is left for another occasion. That the job is not an easy one will be clear to readers of Veldman (1995).

1.4 Spreads

Definition 2 Let β be given. β is called a *spread-law*, $Spr(\beta)$, if and only if $\forall s[\beta(s) = 0 \leftrightarrow \exists n[\beta(s * \langle n \rangle) = 0]]$.

For every β, we define: $\mathcal{F}_\beta := \{\alpha \mid \forall n[\beta(\overline{\alpha}n) = 0]\}$.

$\mathcal{X} \subseteq \mathcal{N}$ is *closed* if and only if $\exists \beta[\mathcal{X} = \mathcal{F}_\beta]$.

$\mathcal{X} \subseteq \mathcal{N}$ is a *spread* if and only if $\exists \beta[Spr(\beta) \wedge \mathcal{X} = \mathcal{F}_\beta]$.

If $Spr(\beta)$ and $\beta(\langle\ \rangle) \neq 0$, then $\mathcal{F}_\beta = \emptyset$.

If $Spr(\beta)$ and $\beta(\langle\ \rangle) = 0$, then $\mathcal{F}_\beta$ is inhabited.[13] One may define α such that $\forall n[\alpha(n) = \mu p[\beta(\overline{\alpha}n * \langle p \rangle) = 0]]$ and observe: $\forall n[\beta(\overline{\alpha}n) = 0]$, that is: $\alpha \in \mathcal{F}_\beta$.

Is every closed set a spread?

Define β such that $\forall s[\beta(s) = 0 \leftrightarrow \neg \forall i < 99[d(n + i) = 9]]$, where $d : \mathbb{N} \to \{0, 1, \ldots, 9\}$ is the decimal expansion of π.

If $\mathcal{F}_\beta$ is a spread, that is $\exists \gamma[Spr(\gamma) \wedge \mathcal{F}_\gamma = \mathcal{F}_\beta]$, then *either* $\mathcal{F}_\beta$ is inhabited and $\neg \exists s \forall i < 99[d(s + i) = 9]$ *or* $\mathcal{F}_\beta = \emptyset$ and $\neg\neg \exists s \forall i < 99[d(s + i) = 9]$.

For this β, the statement '$\mathcal{F}_\beta$ is a spread' thus turns out to be reckless.

Brouwer's Continuity Principle enables one to obtain a stronger conclusion.

Theorem 3 $\neg \forall \beta \exists \gamma[Spr(\gamma) \wedge \mathcal{F}_\gamma = \mathcal{F}_\beta]$.

Proof Assume: $\forall \beta \exists \gamma[Spr(\gamma) \wedge \mathcal{F}_\gamma = \mathcal{F}_\beta]$.

Then $\forall \beta[\exists \alpha[\alpha \in \mathcal{F}_\beta] \vee \neg \exists \alpha[\alpha \in \mathcal{F}_\beta]]$. Using Lemma 1, find m such that *either* $\forall \beta[\overline{\underline{0}}m \sqsubset \beta \to \exists \alpha[\alpha \in \mathcal{F}_\beta]]$ *or* $\forall \beta[\overline{\underline{0}}m \sqsubset \beta \to \neg \exists \alpha[\alpha \in \mathcal{F}_\beta]]$.

Both alternatives are false, as we see by considering $\beta = \overline{\underline{0}}m * \underline{1}$ (for the first alternative), and $\beta = \underline{0}$ (for the second one). □

Lemma 2 (Brouwer's Continuity Principle extends to spreads)

Let β be given such that $Spr(\beta)$. Then, for all $R \subseteq \mathcal{N} \times \omega$, if $\forall \alpha \in \mathcal{F}_\beta \exists n[\alpha R n]$, then $\forall \alpha \in \mathcal{F}_\beta \exists m \exists n \forall \gamma \in \mathcal{F}_\beta[\overline{\alpha}m \sqsubset \gamma \to \gamma R n]$.

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[13] $\mathcal{X} \subseteq \mathcal{N}$ is *inhabited* if and only if $\exists \alpha[\alpha \in \mathcal{X}]$.

Proof Assume: $Spr(\beta)$. If $\beta(\langle\,\rangle) \neq 0$, then $\mathcal{F}_\beta = \emptyset$ and there is nothing to prove.
Assume $\beta(\langle\,\rangle) = 0$. Define σ such that $\sigma(\langle\,\rangle) = \langle\,\rangle$ and, for all s, for all n,

1. if $\beta(s * \langle n\rangle) = 0$, then $\sigma(s * \langle n\rangle) = s * \langle n\rangle$, and,
2. if $\beta(s * \langle n\rangle) \neq 0$, then $\sigma(s * \langle n\rangle) = \sigma(s) * \langle \mu p[\beta(\sigma(s) * \langle p\rangle) = 0]\rangle$.

Note: $\forall s[\beta(\sigma(s)) = 0]$ and $\forall s \forall t[s \sqsubset t \rightarrow \sigma(s) \sqsubset \sigma(t)]$.
Define $\rho : \mathcal{N} \rightarrow \mathcal{N}$ such that $\forall\alpha\forall n[\sigma(\overline{\alpha}n) \sqsubset \rho|\alpha]$.
Note: $\forall\alpha[\rho|\alpha \in \mathcal{F}_\beta] \;\wedge\; \forall\alpha \in \mathcal{F}_\beta[\rho|\alpha = \alpha]$.
The function ρ is called a *retraction* of $\mathcal{N}$ onto $\mathcal{F}_\beta$.

Now assume: $\forall\alpha \in \mathcal{F}_\beta \exists n[\alpha R n]$. Conclude: $\forall\alpha\exists n[(\rho|\alpha) R n]$.
Let α in $\mathcal{F}_\beta$ be given. Using Axiom 1, find m, n such that $\forall\gamma[\overline{\alpha}m \sqsubset \gamma \rightarrow (\rho|\gamma) R n]$. Conclude: $\forall\gamma \in \mathcal{F}_\beta[\overline{\alpha}m \sqsubset \gamma \rightarrow \gamma R n]$.
We thus see: $\forall\alpha \in \mathcal{F}_\beta \exists m \exists n \forall\gamma \in \mathcal{F}_\beta[\overline{\gamma}m \sqsubset \alpha \rightarrow \gamma R n]$. □

Recall that, for all α, β, $\alpha \# \beta \leftrightarrow \alpha \perp \beta \leftrightarrow \exists n[\alpha(n) \neq \beta(n)]$, and $\alpha = \beta \leftrightarrow \forall n[\alpha(n) = \beta(n)] \leftrightarrow \neg(\alpha \# \beta)$, and $\alpha \neq \beta \leftrightarrow \neg\forall n[\alpha(n) = \beta(n)]$.

The constructive *apartness relation* $\#$ is more useful than the negative *inequality relation* $\neq$.

Markov's Principle, in the form: $\forall\alpha[\neg\neg\exists n[\alpha(n) = 0] \rightarrow \exists n[\alpha(n) = 0]]$,[14] is equivalent to the statement that the two relations coincide: $\forall\alpha\forall\beta[\alpha \neq \beta \rightarrow \alpha \# \beta]$.

The intuitionistic mathematician does not accept Markov's Principle.

Definition 3 We let $AP = AP(\mathsf{x}, \mathsf{y})$ be the formula $\forall \mathsf{z}[\neg(\mathsf{z} = \mathsf{x}) \;\vee\; \neg(\mathsf{z} = \mathsf{y})]$.

The following theorem reformulates a well-known fact.

Theorem 4 (Apartness is definable) *For all β such that $Spr(\beta)$, for all α, δ in $\mathcal{F}_\beta$, $\alpha \# \delta$ if and only if $(\mathcal{F}_\beta, =) \models AP[\alpha, \delta]$.*

Proof First, assume $\alpha \# \delta$. Find n such that $\overline{\alpha}n \neq \overline{\delta}n$. Note: for every γ in $\mathcal{F}_\beta$, *either*: $\overline{\gamma}n \neq \overline{\alpha}n$ and $\gamma \# \alpha$, *or*: $\overline{\gamma}n \neq \delta n$ and $\gamma \# \delta$. Conclude: $(\mathcal{F}_\beta, =) \models AP[\alpha, \delta]$.

Next, assume $(\mathcal{F}_\beta, =) \models AP[\alpha, \delta]$, that is $\forall\gamma \in \mathcal{F}_\beta[\gamma \neq \alpha \;\vee\; \gamma \neq \delta]$.

Applying Lemma 2, find m such that *either* $\forall\gamma \in \mathcal{F}_\beta[\overline{\alpha}m \sqsubset \gamma \rightarrow \gamma \neq \alpha]$ *or* $\forall\gamma \in \mathcal{F}_\beta[\overline{\alpha}m \sqsubset \gamma \rightarrow \gamma \neq \delta]$. The first alternative is clearly wrong (take $\gamma := \alpha$). The second alternative implies: $\overline{\alpha}m \perp \delta$ (if $\overline{\alpha}m \sqsubset \delta$, one could take $\gamma := \delta$), and thus: $\alpha \# \delta$. □

Definition 4 For each n, we let ψ_n^+ be the sentence $\exists \mathsf{x}_0 \exists \mathsf{x}_1 \ldots \exists \mathsf{x}_\mathsf{n}[\bigwedge_{i<j<n} AP(\mathsf{x}_i, \mathsf{x}_j)]$.
$T_{inf}^+ := EQ \cup \{\psi_n^+ \mid n \in \omega\}$.

ψ_n^+ expresses that a set has at least $n+1$ elements that are mutually apart.
Every model of T_{inf}^+ realizes T_{inf}. In the second part of the paper we will meet a structure that realizes T_{inf} but not T_{inf}^+, see Theorem 17 in Sect. 1.9.

The theory $T_{inf}^+ \cup \{\forall \mathsf{x}\forall \mathsf{y}[\mathsf{x} = \mathsf{y} \;\vee\; \neg(\mathsf{x} = \mathsf{y})]\}$ has only one complete extension, the same as the one and only complete extension of $T_{inf} \cup \{\forall \mathsf{x}\forall \mathsf{y}[\mathsf{x} = \mathsf{y} \;\vee\; \neg(\mathsf{x} = \mathsf{y})]\}$, see Theorem 2.

[14] A. A. Markov enuntiated this principle for *primitive recursive* α only.

1.5 Spreads with a Decidable Equality

Definition 5 We let $D = D(\mathsf{x})$ be the formula: $\forall \mathsf{y}[\mathsf{x} = \mathsf{y} \ \vee \ \neg(\mathsf{x} = \mathsf{y})]$.

Definition 6 Assume $Spr(\beta)$ and $\alpha \in \mathcal{F}_\beta$.

α is an *isolated* point of $\mathcal{F}_\beta$ if and only if $\exists n \forall \gamma \in \mathcal{F}_\beta[\overline{\alpha}n \sqsubset \gamma \to \alpha = \gamma]$, or, equivalently, $\exists n \forall s[\big(\overline{\alpha}n \sqsubset s \ \wedge \ \beta(s) = 0\big) \to s \sqsubset \alpha]$.

α is a *decidable* point of $\mathcal{F}_\beta$ if and only if $\forall \gamma \in \mathcal{F}_\beta[\alpha = \gamma \ \vee \ \neg(\alpha = \gamma)]$, or, equivalently, $(\mathcal{F}_\beta, =) \models D[\alpha]$.

$\mathcal{I}(\mathcal{F}_\beta)$ is the set of the isolated points of $\mathcal{F}_\beta$.

Cantor called $\mathcal{I}(\mathcal{F}_\beta)$ the *adherence* of $\mathcal{F}_\beta$.

Lemma 3 *Assume* $Spr(\beta)$.

(i) *For each* α *in* $\mathcal{F}_\beta$, α *is an isolated point of* $\mathcal{F}_\beta$ *if and only if* α *is a decidable point of* $\mathcal{F}_\beta$.

(ii) $\mathcal{I}(\mathcal{F}_\beta)$ *is a definable subset of* $\mathcal{F}_\beta$.

Proof (i) Let α be an isolated point of $\mathcal{F}_\beta$.
Find n such that $\forall \gamma \in \mathcal{F}_\beta[\overline{\alpha}n \sqsubset \gamma \to \alpha = \gamma]$.
Note: for each γ in $\mathcal{F}_\beta$, *either* $\overline{\alpha}n \sqsubset \gamma$ and $\alpha = \gamma$, *or* $\overline{\alpha}n \perp \gamma$ and $\alpha \neq \gamma$.
Conclude: $\forall \gamma \in \mathcal{F}_\beta[\alpha = \gamma \ \vee \ \neg(\alpha = \gamma)]$, that is: α is a decidable point of $\mathcal{F}_\beta$.
Now assume: α is a decidable point of $\mathcal{F}_\beta$, that is:
$\forall \gamma \in \mathcal{F}_\beta[\alpha = \gamma \ \vee \ \neg(\alpha = \gamma)]$.
Apply Lemma 2 and find m such that *either* $\forall \gamma \in \mathcal{F}_\beta[\overline{\alpha}m \sqsubset \gamma \to \alpha = \gamma]$ *or* $\forall \gamma \in \mathcal{F}_\beta[\overline{\alpha}m \sqsubset \gamma \to \neg(\alpha = \gamma)]$. As the second alternative does not hold (take $\gamma = \alpha$), conclude: $\forall \gamma \in \mathcal{F}_\beta[\overline{\alpha}m \sqsubset \gamma \to \alpha = \gamma]$, and: α is an isolated point of $\mathcal{F}_\beta$.

(ii) Using (i), note: $\mathcal{I}(\mathcal{F}_\beta) = \{\alpha \in \mathcal{F}_\beta \mid (\mathcal{F}_\beta, =) \models D[\alpha]\}$.

□

Definition 7 Assume $Spr(\beta)$ and $\alpha \in \mathcal{F}_\beta$.

α is a *limit point* of $\mathcal{F}_\beta$ if and only if $\forall n \exists \delta \in \mathcal{F}_\beta[\overline{\alpha}n \sqsubset \delta \ \wedge \ \alpha \perp \delta]$, or, equivalently, $\forall n \exists s[\overline{\alpha}n \sqsubset s \ \wedge \ \beta(s) = 0 \ \wedge \ \alpha \perp s]$.

$\mathcal{L}(\mathcal{F}_\beta)$ is the set of the limit points of $\mathcal{F}_\beta$.

Cantor called $\mathcal{L}(\mathcal{F}_\beta)$ the *coherence* of $\mathcal{F}_\beta$.

Lemma 4 $\forall \beta[Spr(\beta) \to \mathcal{L}(\mathcal{F}_\beta) \subseteq \mathcal{F}_\beta \setminus \mathcal{I}(\mathcal{F}_\beta)]$, *that is: in all spreads, every limit point is a non-isolated point.*

Proof Obvious. □

Theorem 5 *The following are equivalent:*

(i) *Markov's Principle:* $\forall \alpha[\neg\neg\exists n[\alpha(n) = 0] \to \exists n[\alpha(n) = 0]]$.

(ii) $\forall\beta[Spr(\beta) \to \mathcal{F}_\beta \setminus \mathcal{I}(\mathcal{F}_\beta) \subseteq \mathcal{L}(\mathcal{F}_\beta)]$, *that is: in all spreads, every non-isolated point is a limit point.*

Proof (i) $\Rightarrow$ (ii). Let β be given such that $Spr(\beta)$. Assume α is not an isolated point of $\mathcal{F}_\beta$, that is: $\neg\exists n\forall s[(\overline{\alpha}n \sqsubset s \ \wedge \ \beta(s) = 0) \to s \sqsubset \alpha]$.

Let n be given.

Define δ such that $\forall s[\delta(s) = 0 \leftrightarrow (\overline{\alpha}n \sqsubset s \ \wedge \ \beta(s) = 0 \ \wedge \ s \perp \alpha)]$.

Then $\neg\forall s[\delta(s) \neq 0]$ and: $\neg\neg\exists s[\delta(s) = 0]$.

Using *Markov's Principle*, we conclude: $\exists s[\delta(s) = 0]$.

We thus see: $\forall n\exists s[\overline{\alpha}s \sqsubset s \ \wedge \ \beta(s) = 0 \ \wedge \ s \perp \alpha]$, and: α is a limit point of $\mathcal{F}_\beta$.

(ii) $\Rightarrow$ (i). Let us assume: $\forall\beta[Spr(\beta) \to \mathcal{F}_\beta \setminus \mathcal{I}(\mathcal{F}_\beta) \subseteq \mathcal{L}(\mathcal{F}_\beta)]$,

Let α be given such that $\neg\neg\exists n[\alpha(n) = 0]$.

Define β such that

$\forall s[\beta(s) = 0 \leftrightarrow \forall m < length(s)[s(m) \neq 0 \to \exists n \leq m[\alpha(n) = 0]]]$.

Note: $Spr(\beta)$ and $\underline{0} \in \mathcal{F}_\beta$, and: if $\exists n[\alpha(n) = 0]$, then $\underline{0}$ is a limit point of $\mathcal{F}_\beta$.

Conclude: if $\underline{0}$ is an isolated point of $\mathcal{F}_\beta$, then $\neg\exists n[\alpha(n) = 0]$.

As $\neg\neg\exists n[\alpha(n) = 0]$, conclude: $\underline{0}$ is not an isolated point of $\mathcal{F}_\beta$.

By our assumption, $\underline{0}$ thus is a limit point of $\mathcal{F}_\beta$.

Find s such that $\beta(s) = 0$ and $s \perp \underline{0}$. Conclude: $\exists n \leq length(s)[\alpha(n) = 0]$.

Conclude: $\forall\alpha[\neg\neg\exists n[\alpha(n) = 0] \to \exists n[\alpha(n) = 0]]$, that is: Markov's Principle.□

We thus see that the converse of Lemma 4, being equivalent to Markov's Principle, is not an intuitionistic theorem.

We could not answer the question if, in general, $\mathcal{L}(\mathcal{F}_\beta)$ is a definable subset of $(\mathcal{F}_\beta, =)$. In some special cases, however, it is, and the following definition is useful.

Definition 8 Assume $Spr(\beta)$. $\mathcal{F}_\beta$ is called *transparent* if and only if there exists γ such that $Spr(\gamma)$ and $\mathcal{F}_\gamma = \mathcal{L}(\mathcal{F}_\beta)$ and $\forall\alpha \in \mathcal{F}_\beta[\exists n[\gamma(\overline{\alpha}n) \neq 0] \to \alpha \in \mathcal{I}(\mathcal{F}_\beta)]$.

Note that, for each β such that $Spr(\beta)$, if $\mathcal{F}_\beta$ is transparent, then $\mathcal{F}_\beta \setminus \mathcal{I}(\mathcal{F}_\beta) \subseteq \mathcal{L}(\mathcal{F}_\beta)$. The statement that every spread $\mathcal{F}_\beta$ is transparent thus is seen to imply Markov's Principle.

In Sect. 1.7 we will see many examples of transparent spreads.

The fact that not every spread is a transparent spread is one of the reasons that Brouwer did not succeed in finding a nice intuitionistic version of Cantor's Main Theorem,[15] see Brouwer (1919).

Definition 9 Let β satisfy $Spr(\beta)$ and let φ be given.

We define: $\varphi : \mathcal{F}_\beta \to \omega$ if and only if $\forall\alpha \in \mathcal{F}_\beta\exists p[\varphi(\overline{\alpha}p) \neq 0]$.

If $\varphi : \mathcal{F}_\beta \to \omega$, then we define, for each α in $\mathcal{F}_\beta$, $\varphi(\alpha)$ as the number z such that $\varphi(\overline{\alpha}q) = z + 1$, where $q = \mu p[\varphi(\overline{\alpha}p) \neq 0]$.

We define: φ is an injective map from $\mathcal{F}_\beta$ into ω, notation: $\varphi : \mathcal{F}_\beta \hookrightarrow \omega$, if and only if $\varphi : \mathcal{F}_\alpha \to \omega$ and $\forall\alpha \in \mathcal{F}_\beta\forall\delta \in \mathcal{F}_\beta[\alpha \# \delta \to \varphi(\alpha) \neq \varphi(\delta)]$.

[15] Cantor's Main Theorem nowadays is called the Perfect Set Theorem: *every closed subset of $\mathcal{N}$ is the union of a perfect set and an at most countable set.*

We define: $\varphi : \mathcal{F}_\beta \to \mathcal{N}$ if and only if $\forall n[\varphi^n : \mathcal{F}_\beta \to \omega]$.

If $\varphi : \mathcal{F}_\beta \to \mathcal{N}$, then we define, for each α in $\mathcal{F}_\beta$, $\varphi|\alpha$ as the element δ of $\mathcal{N}$ such that $\forall n[\delta(n) = \varphi^n(\alpha)]$.

We define: φ is an injective map from $\mathcal{F}_\beta$ into $\mathcal{N}$, notation: $\varphi : \mathcal{F}_\beta \hookrightarrow \mathcal{N}$, if and only if $\varphi : \mathcal{F}_\alpha \to \mathcal{N}$ and $\forall \alpha \in \mathcal{F}_\beta \forall \delta \in \mathcal{F}_\beta[\alpha \# \delta \to \varphi|\alpha \# \varphi|\delta]$.

For every $\mathcal{X} \subseteq \mathcal{N}$, $\mathcal{F}_\beta$ *embeds into* $\mathcal{X}$ if and only if there exists an injective map from $\mathcal{F}_\beta$ into $\mathcal{X}$.

The following axiom is, at least at first sight, a little bit stronger than Brouwer's Continuity Principle.

Axiom 2 (First Axiom of Continuous Choice) For all $R \subseteq \mathcal{N} \times \omega$, if $\forall \alpha \exists n[\alpha R n]$, then $\exists \varphi : \mathcal{N} \to \omega \forall \alpha[\alpha R \varphi(\alpha)]$.

Lemma 5 (The First Axiom of Continuous Choice extends to spreads) *Let β be given such that $Spr(\beta)$. Then, for all $R \subseteq \mathcal{F}_\beta \times \omega$, if $\forall \alpha \in \mathcal{F}_\beta \exists n[\alpha R n]$, then $\exists \varphi : \mathcal{F}_\beta \to \omega \forall \alpha \in \mathcal{F}_\beta[\alpha R \varphi(\alpha)]$.*

Proof Assume: $Spr(\beta)$ and $\beta(\langle\,\rangle) = 0$. As in the proof of Lemma 2, define $\rho : \mathcal{N} \to \mathcal{F}_\beta$ such that $\forall \alpha[\rho|\alpha \in \mathcal{F}_\beta] \wedge \forall \alpha \in \mathcal{F}_\beta[\rho|\alpha = \alpha]$.

Now assume $\forall \alpha \in \mathcal{F}_\beta \exists n[\alpha R n]$. Conclude: $\forall \alpha \exists n[(\rho|\alpha) R n]$.
Applying Axiom 2, find $\varphi : \mathcal{N} \to \omega$ such that $\forall \gamma[(\rho|\gamma) R \varphi(\gamma)]$.
Conclude: $\varphi : \mathcal{F}_\beta \to \omega$ and $\forall \gamma \in \mathcal{F}_\beta[\gamma R \varphi(\gamma)]$. □

Theorem 6 *Assume $Spr(\beta)$. $(\mathcal{F}_\beta, =) \models \forall \mathsf{x}[D(\mathsf{x})]$ if and only if $\exists \varphi[\varphi : \mathcal{F}_\beta \hookrightarrow \omega]$.*

Proof First assume: $(\mathcal{F}_\beta, =) \models \forall \mathsf{x}[D(\mathsf{x})]$. Then, by Lemma 3, $\forall \alpha \in \mathcal{F}_\beta \exists n \forall \gamma \in \mathcal{F}_\beta[\overline{\alpha}n \sqsubset \gamma \to \alpha = \gamma]$. Using Lemma 5, find $\varphi : \mathcal{F}_\beta \to \omega$ such that $\forall \alpha \in \mathcal{F}_\beta \forall \gamma \in \mathcal{F}_\beta[\overline{\alpha}\varphi(\alpha) \sqsubset \gamma \to \alpha = \gamma]$.
Define $\psi : \mathcal{F}_\beta \to \omega$ such that $\forall \alpha \in \mathcal{F}_\beta[\psi(\alpha) = \overline{\alpha}\varphi(\alpha)]$. Clearly, $\psi : \mathcal{F}_\odot \hookrightarrow \omega$.

Now assume: $\varphi : \mathcal{F}_\beta \hookrightarrow \omega$. Note: $\forall \alpha \in \mathcal{F}_\beta \forall \delta \in \mathcal{F}_\beta[\alpha = \delta \leftrightarrow \varphi(\alpha) = \varphi(\delta)]$. Also: $\forall \alpha \in \mathcal{F}_\beta \forall \delta \in \mathcal{F}_\beta[\varphi(\alpha) = \varphi(\delta) \vee \neg\big(\varphi(\alpha) = \varphi(\delta)\big)]$.
Therefore: $\forall \alpha \in \mathcal{F}_\beta \forall \delta \in \mathcal{F}_\beta[\alpha = \delta \vee \neg(\alpha = \delta)]$. Conclude: $(\mathcal{F}_\beta, =) \models \forall \mathsf{x} [D(\mathsf{x})]$. □

Definition 10 Assume $Spr(\beta)$. $\mathcal{F}_\beta$ is *enumerable* if and only if either $\mathcal{F}_\beta = \emptyset$ or $\exists \delta[\forall n[\delta^n \in \mathcal{F}_\beta] \wedge \forall \alpha \in \mathcal{F}_\beta \exists n[\alpha = \delta^n]]$.

Lemma 6 *Assume $Spr(\beta)$. $\mathcal{F}_\beta$ is enumerable if and only if $\exists \varphi[\varphi : \mathcal{F}_\beta \hookrightarrow \omega]$.*

Proof Assume $\mathcal{F}_\beta$ is enumerable and $\beta(\langle\,\rangle) = 0$.
Find δ such that $\forall n[\delta^n \in \mathcal{F}_\beta]$ and $\forall \alpha \in \mathcal{F}_\beta \exists n[\alpha = \delta^n]$.
Using Lemma 5, find $\varphi : \mathcal{F}_\beta \to \omega$ such that $\forall \alpha \in \mathcal{F}_\beta[\alpha = \delta^{\varphi(\alpha)}]$.
Note: $\varphi : \mathcal{F}_\beta \hookrightarrow \omega$.

Now assume: $\varphi : \mathcal{F}_\beta \hookrightarrow \omega$.

We make a preliminary observation.

Let s, n be given such that $\beta(s) = 0$ and $\varphi(s) = n + 1$ and $\forall t \sqsubset s[\varphi(t) = 0]$. Note: $\forall \alpha \in \mathcal{F}_\beta[s \sqsubset \alpha \rightarrow \varphi(\alpha) = n]$ and, therefore: $\forall \alpha \in \mathcal{F}_\beta \forall \delta \in \mathcal{F}_\beta[(s \sqsubset \alpha \wedge s \sqsubset \delta) \rightarrow \alpha = \delta]$.

Now let γ be the element of $\mathcal{F}_\beta$ satisfying $\forall n[\gamma(n) := \mu p[\beta(\overline{\gamma} n * \langle p \rangle) = 0]]$. Define δ such that, for all s, *if* $\beta(s) = 0$ and $\varphi(s) \neq 0$ and $\forall t \sqsubset s[\varphi(t) = 0]$, then $s \sqsubset \delta^s$ and $\delta^s \in \mathcal{F}_\beta$, and *if not*, then $\delta^s = \gamma$.

Note: $\forall s[\delta^s \in \mathcal{F}_\beta]$ and $\forall \alpha \in \mathcal{F}_\beta \exists s[\alpha = \delta^s]$. □

Corollary 1 *Assume* $Spr(\beta)$.
$(\mathcal{F}_\beta, =) \models \forall \mathsf{x}[D(\mathsf{x})]$ *if and only if* $\mathcal{F}_\beta$ *is enumerable.*

Proof Use Theorem 6 and Lemma 6. □

1.6 Spreads with Exactly One Undecidable Point

Definition 11 We let τ_2 be the element of $\mathcal{C}$ satisfying: $\forall s[\tau_2(s) = 0 \leftrightarrow \forall i < length(s)[s(i) < 2 \wedge \big(i + 1 < length(s) \rightarrow s(i) \leq s(i+1)\big)]]$. We define: $\mathcal{T}_2 := \mathcal{F}_{\tau_2}$.

Note: τ_2 is a spread-law and $\mathcal{T}_2$ is a spread.

Let us take a closer look at $\mathcal{T}_2$.

Observe: $\forall \alpha[\alpha \in \mathcal{T}_2 \leftrightarrow \forall i[\alpha(i) \leq \alpha(i+1) < 2]]$.

For each n, we define $n^* := \overline{\underline{0}} n * \underline{1}$.

The infinite sequence $\underline{0}, 0^*, 1^*, 2^*, \ldots$ is a list of elements of $\mathcal{T}_2$ and a classical mathematician might think it is the list of all elements of $\mathcal{T}_2$. The intuitionistic mathematician knows better. He defines α in $\mathcal{T}_2$ such that

$$\forall n[\alpha(n) = 1 \leftrightarrow \exists k \leq n \forall i < 99[d(k+i) = 9]],$$

where $d : \mathbb{N} \rightarrow \{0, 1, \ldots, 9\}$ is the decimal expansion of π. As yet, one has no proof of the statement '$\alpha = \underline{0}$', as this statement implies: $\forall k \exists i < 99]d(k+i) = 9]$. As yet, one also has no proof of the statement: '$\exists n[\alpha = n^*]$' as this statement implies: $\exists n \forall i < 99[d(n+i) = 9]$. The statement that α occurs in the above list is a reckless one.

For each n, n^* is an isolated and a decidable point of $\mathcal{T}_2$, and $\underline{0}$ is a non-isolated and an undecidable point of $\mathcal{T}_2$. It follows, by Lemma 3 and Corollary 1, that $\mathcal{T}_2$ is not an enumerable spread. In particular, the statement that the list $\underline{0}, 0^*, 1^*, 2^*, \ldots$ is a complete list of the elements of $\mathcal{T}_2$, leads to a contradiction, as appears again from the following Theorem.

Theorem 7 (i) $\neg \forall \alpha \in \mathcal{T}_2[\alpha = \underline{0} \vee \exists n[\alpha = n^*]]$.
(ii) $\forall \alpha \in \mathcal{T}_2[\alpha \# \underline{0} \rightarrow \exists n[\alpha = n^*]]$.

Proof (i) Assume $\forall\alpha \in \mathcal{T}_2[\alpha = \underline{0} \vee \exists n[\alpha = n^*]]$. Using Lemma 2, find m, n such that *either* $\forall\alpha \in \mathcal{T}_2[\overline{\underline{0}}m \sqsubset \alpha \to \alpha = \underline{0}]$ *or* $\forall\alpha \in \mathcal{T}_2[\overline{\underline{0}}m \sqsubset \alpha \to \alpha = n^*]$. Note that both alternatives are false.
Conclude: $\neg\forall\alpha \in \mathcal{T}_2[\alpha = \underline{0} \vee \exists n[\alpha = n^*]]$.

(ii) Let α in $\mathcal{T}_2$ be given such that $\alpha \# \underline{0}$. Define $n := \mu m[\overline{\alpha}(m + 1) \perp \underline{0}]$. Note: $\overline{\alpha}(n + 1) = \overline{\underline{0}}n * \langle 1\rangle$ and $\alpha = n^*$.

Definition 12 Assume $Spr(\beta)$. $\mathcal{F}_\beta$ is *almost-enumerable* if and only if either $\mathcal{F}_\beta = \emptyset$ or $\exists\delta[\forall n[\delta^n \in \mathcal{F}_\beta] \wedge \forall\alpha \in \mathcal{F}_\beta\forall\varepsilon\exists n[\overline{\alpha\varepsilon(n)} = \overline{\delta^n\varepsilon(n)}]]$.

This definition deserves some explanation. If $\mathcal{F}_\beta$ is almost-enumerable and inhabited, we are able to come forward with an infinite sequence $\delta^0, \delta^1, \ldots$ of elements of $\mathcal{F}_\beta$ such that, for every α in $\mathcal{F}_\beta$, every attempt ε to prove that α is apart from all elements of the infinite sequence $\delta^0, \delta^1, \ldots$, ($\varepsilon$ expresses the guess: $\forall n[\overline{\alpha\varepsilon(n)} \perp \overline{\delta^n\varepsilon(n)}]$), will positively fail.

Almost-enumerable spreads are studied in Veldman (2018, Sect. 9), where they are called *almost-countable located and closed subsets of* $\mathcal{N}$.

Theorem 8 *$\mathcal{T}_2$ is almost-enumerable.*

Proof Define δ such that $\delta^0 = \underline{0}$ and, for each n, $\delta^{n+1} = n^* = \overline{\underline{0}}n * \underline{1}$. Note: $\forall n[\delta^n \in \mathcal{T}_2]$. Let ε be given. If $\overline{\alpha\varepsilon(0)} = \overline{\delta^0\varepsilon(0)}$, we are done. If not, then $\alpha \perp \underline{0}$ and we may determine n such that $\alpha = \delta^{n+1}$ and $\overline{\alpha\varepsilon(n + 1)} = \overline{\delta^{n+1}\varepsilon(n + 1)}$. □

Axiom 3 (Second Axiom of Countable Choice)
For every $R \subseteq \mathbb{N} \times \mathcal{N}$, if $\forall n\exists\alpha[nR\alpha]$, then $\exists\alpha\forall n[nR\alpha^n]$.

Theorem 9 (i) $(\mathcal{T}_2, =) \models \exists \mathsf{x}[\neg D(\mathsf{x}) \wedge \forall \mathsf{y}[AP(\mathsf{x}, \mathsf{y}) \to D(\mathsf{y})]]$.
(ii) *For all β such that $Spr(\beta)$,*
if $(\mathcal{F}_\beta, =) \models \exists \mathsf{x}[\neg D(\mathsf{x}) \wedge \forall \mathsf{y}[AP(\mathsf{x}, \mathsf{y}) \to D(\mathsf{y})]]$, then $\mathcal{F}_\beta$ embeds into $\mathcal{T}_2$.

Proof (i) $\underline{0}$ is not an isolated point of $\mathcal{T}_2$, and, therefore, not a decidable point of $\mathcal{T}_2$. Also, by Theorem 7 (ii), $\forall\alpha \in \mathcal{T}_2[\alpha \# \underline{0} \to \exists n[\alpha = n^*]]$, and, for each n, for each α in $\mathcal{T}_2$, $\alpha = n^* \leftrightarrow \overline{\underline{0}}n * \langle 1\rangle \sqsubset \alpha$, so one may decide: $\alpha = n^*$ or $\neg(\alpha = n^*)$, and: n^* is a decidable point of $\mathcal{T}_2$.
We thus see: $(\mathcal{T}_2, =) \models \neg D(\mathsf{x}) \wedge \forall \mathsf{y}[AP(\mathsf{x}, \mathsf{y}) \to D(\mathsf{y})][\underline{0}]$, and are done.

(ii) Assume: $Spr(\beta)$ and $(\mathcal{F}_\beta, =) \models \exists \mathsf{x}[\neg D(\mathsf{x}) \wedge \forall \mathsf{y}[AP(\mathsf{x}, \mathsf{y}) \to D(\mathsf{y})]]$.
Find α in $\mathcal{F}_\beta$ such that α is not an isolated point of $\mathcal{F}_\beta$.
Note: for each s such that $\beta(s) = 0$, the set $\mathcal{F}_\beta \cap s := \{\delta \in \mathcal{F}_\beta \mid s \sqsubset \delta\}$ is a spread, and, if $s \perp \alpha$, then $\mathcal{F}_\beta \cap s$ consists of isolated points of $\mathcal{F}_\beta \cap s$ only, and thus, by Theorem 6, embeds into ω.
Using Axiom 3, we find φ such that, for each s, if $\beta(s) = 0$ and there exist n, i such that $s = \overline{\alpha}n * \langle i\rangle$ and $i \neq \alpha(n)$, then $\varphi^s : \mathcal{F}_\beta \cap s \hookrightarrow \omega$.
We now define $\psi : \mathcal{F}_\beta \to \mathcal{T}_2$ such that $\psi|\alpha = \underline{0}$ and, for each δ in $\mathcal{F}_\beta$, if $\delta \# \alpha$, then $\psi|\delta = \overline{\underline{0}}\big(\overline{\delta}n, \varphi^{\overline{\delta}n}(\delta)\big) * \underline{1}$ where $n := \mu i[\overline{\delta}i \perp \alpha]$. □

1.7 More and More Undecidable Points: The Toy Spreads

Definition 13 For each n, we let τ_n be the element of $\mathcal{C}$ satisfying: $\forall s[\tau_n(s) = 0 \leftrightarrow \forall i < length(s)[s(i) < n \ \wedge \ \big(i + 1 < length(s) \rightarrow s(i) \leq s(i + 1)\big)]]$.
We also define: $\mathcal{T}_n := \mathcal{F}_{\tau_n}$.

For each n, τ_n is a spread-law and $\mathcal{T}_n$ and $\mathcal{T}_n = \{\alpha \mid \forall i[\alpha(i) \leq \alpha(i + 1) < n]\}$ is a spread.

In this paper, the spreads $\mathcal{T}_0, \mathcal{T}_1, \ldots$ will be called the *toy spreads*.

Note: $\mathcal{T}_0 = \emptyset$ and $\mathcal{T}_1 = \{\underline{0}\}$.

Definition 14 For each $s \neq \langle\, \rangle$, we let $s^\dagger$ be the element of $\mathcal{N}$ satisfying $s \sqsubset s^\dagger$ and $\forall i \geq length(s)[s^\dagger(i) = s^\dagger(i - 1)]$.

Note that, for each n, for each s, if $s \neq \langle\, \rangle$ and $\tau_n(s) = 0$, then $s^\dagger \in \mathcal{T}_n$.

Theorem 10 *For each $n > 0$, $\mathcal{T}_n$ is almost-enumerable.*

Proof Let $n > 0$ be given. Define δ such that, for each s, *if* $s \neq \langle\, \rangle$ and $\tau_n(s) = 0$, then $\delta^s = s^\dagger$, and *if not*, then $\delta^s = \underline{0}$.

We claim: $\forall \alpha \in \mathcal{T}_n \forall \varepsilon \exists s[\overline{\alpha}\varepsilon(s) = \overline{\delta^s}\varepsilon(s)]$.

We establish this claim by proving, for each $k < n$, $\forall \alpha \in \mathcal{T}_n[\exists i[\alpha(i) \geq k] \rightarrow \forall \varepsilon \exists s[\overline{\alpha}\varepsilon(s) = \overline{\delta^s}\varepsilon(s)]]$, and we do so by backwards induction, starting with the case $k = n - 1$.

The case $k = n - 1$ is treated as follows. If $\exists i[\alpha(i) = n - 1]$, find $i_0 := \mu i[\alpha(i) = n - 1]$ and consider $s := \overline{\alpha}(i_0 + 1)$.
Note: $\alpha = s^\dagger = \delta^s$ and, therefore, for every ε: $\overline{\alpha}\varepsilon(s) = \overline{\delta^s}\varepsilon(s)$.

Now assume $k < n - 1$ is given such that $\forall \alpha \in \mathcal{T}_n[\exists i[\alpha(i) \geq k + 1] \rightarrow \forall \varepsilon \exists s[\overline{\alpha}\varepsilon(s) = \overline{\delta^s}\varepsilon(s)]]$.

We have to prove: $\forall \alpha \in \mathcal{T}_n[\exists i[\alpha(i) = k] \rightarrow \forall \varepsilon \exists s[\overline{\alpha}\varepsilon(s) = \overline{\delta^s}\varepsilon(s)]]$.

Let α be given such that $\exists i[\alpha(i) = k]$. Let also ε be given.

Define $i_0 := \mu i[\alpha(i) = k]$ and define $s := \overline{\alpha}(i_0 + 1)$.

There are two cases to consider.

Case (i): $\overline{\alpha}\varepsilon(s) = \overline{s^\dagger}\varepsilon(s) = \overline{\delta^s}\varepsilon(s)$. We are done.

Case (ii): $\overline{\alpha}\varepsilon(s) \perp \overline{s^\dagger}\varepsilon(s)$. Then $\exists i < \varepsilon(s)[\alpha(i) \geq k + 1]$.

Using the induction hypothesis, we conclude: $\exists s[\overline{\alpha}\varepsilon(s) = \overline{\delta^s}\varepsilon(s)]$. □

Theorem 11 (i) *For each n, for all α in $\mathcal{T}_n$, $\alpha \in \mathcal{I}(\mathcal{T}_n)$ if and only if $\exists m[\alpha(m) + 1 = n]$.*
(ii) *For each n, $\mathcal{T}_{n+1} \setminus \mathcal{I}(\mathcal{T}_{n+1}) = \mathcal{T}_n = \mathcal{L}(\mathcal{T}_{n+1})$.*
(iii) *For each n, $\mathcal{T}_n = \{\alpha \in \mathcal{T}_{n+1} \mid (\mathcal{T}_{n+1}, =) \models \neg D[\alpha]\}$.*

Proof The proof uses Lemma 3 and is left to the reader. □

Definition 15 We define an infinite sequence $D_0, D_1, \ldots$ of formulas, as follows.
$D_0 := \forall \mathsf{y}[\mathsf{x} = \mathsf{y} \ \vee \ \neg(\mathsf{x} = \mathsf{y})]$,
$D_1 := \neg D_0(\mathsf{x}) \ \wedge \ \forall \mathsf{y}[\neg D_0(\mathsf{y}) \rightarrow \big(\mathsf{x} = \mathsf{y} \ \vee \ \neg(\mathsf{x} = \mathsf{y})\big)]$,

$D_2 := \neg D_0(\mathsf{x}) \wedge \neg D_1(\mathsf{x}) \wedge$
$\forall \mathsf{y}[(\neg D_0(\mathsf{y}) \wedge \neg D_1(\mathsf{y})) \to (\mathsf{x}=\mathsf{y} \vee \neg(\mathsf{x}=\mathsf{y}))]$,
and, more generally for each $m > 0$,
$D_m := \bigwedge_{i<m} \neg D_i(\mathsf{x}) \wedge \forall \mathsf{y}[(\bigwedge_{i<m} \neg D_i(\mathsf{y})) \to (\mathsf{x}=\mathsf{y} \vee \neg(\mathsf{x}=\mathsf{y}))]$.
We also define, for each $m > 0$, sentences ψ_m and ρ_m, as follows:
$\psi_m := \exists \mathsf{x}[D_m(\mathsf{x})]$ and $\rho_m := \exists \mathsf{x}[D_m(\mathsf{x}) \wedge \forall \mathsf{y}[D_m(\mathsf{y}) \to \mathsf{y}=\mathsf{x}]]$.

Definition 16 Assume $Spr(\beta)$.
α in $\mathcal{F}_\beta$ is a *limit point of order* 0 of $\mathcal{F}_\beta$ if and only if α is an isolated point of $\mathcal{F}_\beta$.
For each m, α is a *limit point of order* $m+1$ of $\mathcal{F}_\beta$ if and only if, for each p, there exists a limit point γ of order m such that $\overline{\alpha}p \sqsubset \gamma$ and $\alpha \perp \gamma$.

Assume $n > 0$ and $\alpha \in \mathcal{T}_n$. Note the following:

1. $(\mathcal{T}_n, =) \models D_0[\alpha]$ if and only if α is an isolated point of $\mathcal{T}_n$ if and only if either: $n = 1$ or: $n > 1$ and $\exists p[\alpha(p) = n-1]$.
2. $(\mathcal{T}_n, =) \models \neg D_0[\alpha]$ if and only if α is a limit point (of order 1) of $\mathcal{T}_n$ if and only if $n > 1$ and $\alpha \in \mathcal{T}_{n-1}$.
3. $(\mathcal{T}_n, =) \models D_1[\alpha]$ if and only if α is an isolated point among the limit points (of order 1) of $\mathcal{T}_n$ if and only if $n > 1$ and $\alpha \in \mathcal{T}_{n-1}$ and $\exists p[\alpha(p) = n-2]$.
4. $(\mathcal{T}_n, =) \models \neg D_0 \wedge \neg D_1[\alpha]$, if and only if α is a limit point of order 2 of $\mathcal{T}_n$ if and only if $n > 2$ and $\alpha \in \mathcal{T}_{n-2}$.
5. For each $m > 0$, $(\mathcal{T}_n, =) \models D_2[\alpha]$ if and only if α is an isolated point among the limit points of order 2 if and only if $n > 2$ and $\alpha \in \mathcal{T}_{n-2}$ and $\exists p[\alpha(p) = n-3]$.
6. For each $m > 0$, $(\mathcal{T}_n, =) \models \bigwedge_{i<m} \neg D_i[\alpha]$ if and only if α is a limit point of order m of $\mathcal{T}_n$ if and only if $n > m$ and $\alpha \in \mathcal{T}_{n-m}$.
7. For each $m > 0$, $(\mathcal{T}_n, =) \models D_m[\alpha]$ if and only if α is an isolated point among the limit points of order m if and only if $n > m$ and $\alpha \in \mathcal{T}_{n-m}$ and $\exists p[\alpha(p) = n-m-1]$.
8. For each $m > 0$, $\mathcal{T}_n \models \psi_m$ if and only if $\mathcal{T}_n$ contains an isolated point of $\mathcal{T}_{n-m}$ if and only if $n > m$.
9. For each $m > 0$, $\mathcal{T}_n \models \rho_m$ if and only if $\mathcal{T}_n$ contains exactly one isolated point of $\mathcal{T}_{n-m}$ if and only if $\mathcal{T}_{n-m} = \{\underline{0}\}$ if and only if $n = m+1$.

After these preliminary observations, the following Theorem is easy to understand:

Theorem 12 (i) *For each n, $\mathcal{T}_n$ is a transparent*[16] *spread and, if $n > 0$, then $\mathcal{I}(\mathcal{T}_n) = \{\alpha \in \mathcal{T}_n \mid \exists p[\alpha(p) + 1 = n]\}$ and $\mathcal{L}(\mathcal{T}_n) = \mathcal{T}_{n-1}$.*
(ii) *For all n, for all $m > 0$, $\mathcal{T}_n = \{\alpha \in \mathcal{T}_{n+m} \mid (\mathcal{T}_{n+m}, =) \models \bigwedge_{i<m} \neg D_i[\alpha]\}$.*
(iii) *For all m, $\{\underline{0}\} = \mathcal{T}_1 = \{\alpha \in \mathcal{T}_{m+1} \mid (\mathcal{T}_{m+1}, =) \models \bigwedge_{i<m} \neg D_i[\alpha]\}$.*
(iv) *For all $n > 0$, for all $m > 0$, $(\mathcal{T}_n, =) \models \psi_m$ if and only if $m + 1 \le n$.*
(v) *For all $n > 0$, for all $m > 0$, $(\mathcal{T}_n, =) \models \rho_m$ if and only if $m + 1 = n$.*

Proof Use the preliminary observations preceding this Theorem. □

[16]See Definition 8.

Corollary 2 *For all* n, m, *if* $n \neq m$, *then there exists a sentence* ψ *such that* $(\mathcal{T}_m, =) \models \psi$ *and* $(\mathcal{T}_n, =) \models \neg\psi$.

1.8 Finite and Infinite Sums of Toy Spreads

1.8.1 A Main Result

Definition 17 Assume $Spr(\beta)$, $Spr(\gamma)$.

We define: $\mathcal{F}_\beta \uplus \mathcal{F}_\gamma := \{\langle 0\rangle * \delta \mid \delta \in \mathcal{F}_\beta\} \cup \{\langle 1\rangle * \delta \mid \delta \in \mathcal{F}_\gamma\}$.

For each m, we define: $m \otimes \mathcal{F}_\beta := \{\langle i\rangle * \delta \mid i < m, \delta \in \mathcal{F}_\beta\}$.

We also define: $\omega \otimes \mathcal{F}_\beta := \{\langle i\rangle * \delta \mid i \in \omega, \delta \in \mathcal{F}_\beta\}$.

Note that $\mathcal{F}_\beta \uplus \mathcal{F}_\gamma$, $m \otimes \mathcal{F}_\beta$ and $\omega \otimes \mathcal{F}_\beta$ are spreads again.

We also define, for all $m, n > 0$, sentences ψ^n_m and ρ^n_m, as follows:

$\psi^n_m := \exists \mathsf{x}_0 \exists \mathsf{x}_1 \ldots \exists \mathsf{x}_{n-1}[\bigwedge_{i<j<n}[AP(\mathsf{x}_i, \mathsf{x}_j) \;\wedge\; \bigwedge_{i<n} \bigwedge_{j<m} \neg D_j(\mathsf{x}_i)]$.

and $\quad \rho^n_m := \exists \mathsf{x}_0 \exists \mathsf{x}_1 \ldots \exists \mathsf{x}_{n-1}[\bigwedge_{i<j<n}[AP(\mathsf{x}_i, \mathsf{x}_j) \;\wedge\; \bigwedge_{i<n} \bigwedge_{j<m} \neg D_j(\mathsf{x}_i) \;\wedge\; \forall \mathsf{z}[\bigwedge_{j<m} \neg D_j(\mathsf{z}) \to \bigvee_{i<n} \mathsf{z} = \mathsf{x}_i]]$.

The sentence ψ^n_m expresses: '*there exist (at least) n limit points of order m that are mutually apart*'.

The sentence ρ^n_m expresses: '*there exist exactly n limit points of order m that are mutually apart*'.

Theorem 13 (i) *For all* $m, n, p, q > 0$,

$(n \otimes \mathcal{T}_m, =) \models \psi^q_p$ *if and only if either:* $p + 1 < m$ *or:* $p + 1 = m$ *and* $q \leq n$.

(ii) *For all* $m, n, p, q > 0$, $(n \otimes \mathcal{T}_m, =) \models \rho^q_p$ *if and only if* $p + 1 = m$ *and* $n = q$.

(iii) *For all* $m, p, q > 0$, $(\omega \otimes \mathcal{T}_m, =) \models \psi^q_p$ *if and only if* $p < m$.

Proof (i) Note the following:

If $p + 1 < m$ and $n > 0$, then $\mathcal{T}_m$ and also $n \otimes \mathcal{T}_m$ contain infinitely many limit points of order p that are mutually apart.

If $p + 1 = m$ and $n > 0$, then $n \otimes \mathcal{T}_m$ contains exactly n limit points of order p that are mutually apart: the points $\langle i\rangle * \underline{0}$, where $i < n$, so $(n \otimes \mathcal{T}_m, =) \models \psi^q_p$ if and only if $q \leq n$.

If $p < m$, then $\omega \times \mathcal{T}_m$ contains infinitely many limit points of order p that are mutually apart.

The proofs of (i), (ii) and (iii) follow easily from these observations. □

Definition 18 For each k, for each s in ω^k, we define: $\mathcal{V}_s = \bigcup_{i<k}\{\langle i\rangle * \delta \mid \delta \in \mathcal{T}_{s(i)}\}$.

Definition 19 Let $\mathcal{F}_0, \mathcal{F}_1 \subseteq \mathcal{N}$ and assume $\varphi : \mathcal{F}_0 \to \mathcal{F}_1$.

φ is a *(surjective)* map from $\mathcal{F}_0$ *onto* $\mathcal{F}_1$ if and only if $\forall \beta \in \mathcal{F}_1 \exists \alpha \in \mathcal{F}_0[\varphi|\alpha = \beta]$.

$\mathcal{F}_0$ *is equivalent to* $\mathcal{F}_1$, notation: $\mathcal{F}_0 \sim \mathcal{F}_1$, if and only if there exists $\varphi : \mathcal{F}_0 \to \mathcal{F}_1$ that is both injective[17] and surjective.

[17] See Definition 9.

Theorem 14 (i) *For each* m, $\mathcal{T}_m \oplus \mathcal{T}_{m+1} \sim \mathcal{T}_{m+1}$.
(ii) *For all* m, n, *if* $m < n$, *then* $\mathcal{T}_m \oplus \mathcal{T}_n \sim \mathcal{T}_n$.
(iii) *For all* k, *for all* s *in* ω^k, *there exist* m, n *such that* $\mathcal{V}_s \sim n \otimes \mathcal{T}_m$.

Proof (i) Let m be given. Define $\varphi : \mathcal{T}_m \oplus \mathcal{T}_{m+1} \to \mathcal{T}_{m+1}$ such that, for all δ in $\mathcal{T}_m$, $\varphi|\langle 0\rangle * \delta = \langle 1\rangle * S \circ \delta$, and, for each δ in $\mathcal{T}_{m+1}$, $\varphi|\langle 1\rangle * \delta = \langle 0\rangle * \delta$. Clearly, φ is a one-to-one function mapping $\mathcal{T}_m \oplus \mathcal{T}_{m+1}$ onto $\mathcal{T}_{m+1}$.
(ii) Let m be given. We use induction on n. The case $n = m + 1$ has been treated in (i). Now let n be given such that $m < n$ and $\mathcal{T}_m \oplus \mathcal{T}_n \sim \mathcal{T}_n$.
Then $\mathcal{T}_m \oplus \mathcal{T}_{n+1} \sim \mathcal{T}_m \oplus (\mathcal{T}_n \oplus \mathcal{T}_{n+1}) \sim (\mathcal{T}_m \oplus \mathcal{T}_n) \oplus \mathcal{T}_{n+1} \sim \mathcal{T}_n \oplus \mathcal{T}_{n+1} \sim \mathcal{T}_{n+1}$.
(iii) We use induction on k. If $s \in \omega^0$, then $s = \langle\,\rangle$ and $\emptyset = \mathcal{V}_s = 0 \otimes \mathcal{T}_1$.
Now let k be given such that,
for all s in ω^k, there exist m, n such that $\mathcal{V}_s = n \otimes \mathcal{T}_m$.
Let $s = t * \langle p\rangle$ in ω^{k+1} be given. Find m, n such that $\mathcal{V}_t = n \otimes \mathcal{T}_m$.
Note: $\mathcal{V}_s \sim \mathcal{V}_t \oplus \mathcal{T}_p$ and consider several cases.
Case (1): $t = \langle\,\rangle$. Then $\mathcal{V}_s = 1 \otimes \mathcal{T}_p$.
Case (2): $t \neq \langle\,\rangle$ and $p < m$. Then, by (ii): $\mathcal{V}_s \sim \mathcal{V}_t \sim n \otimes \mathcal{T}_m$.
Case (3): $t \neq \langle\,\rangle$ and $p = m$. Then: $\mathcal{V}_s \sim \mathcal{V}_t \oplus \mathcal{T}_m \sim (n + 1) \otimes \mathcal{T}_m$.
Case (4): $t \neq \langle\,\rangle$ and $p > m$. Then, by (ii):

$$\mathcal{V}_s \sim \mathcal{V}_t \oplus \mathcal{T}_p \sim \underbrace{\mathcal{T}_m \oplus \ldots \oplus \mathcal{T}_m}_{n} \oplus \mathcal{T}_p \sim \mathcal{T}_p \sim 1 \otimes \mathcal{T}_p.$$

□

Definition 20 For each α, we define: $\mathcal{V}_\alpha := \bigcup_i \{\langle i\rangle * \delta \mid \delta \in \mathcal{T}_{\alpha(i)}\}$.

Theorem 15 (EQ has continuum many complete extensions[18])

(i) *For each* α, $\mathcal{I}(\mathcal{V}_\alpha) = \bigcup_i \{\langle i\rangle * \delta \mid \delta \in \mathcal{T}_{\alpha(i)} \wedge \exists p[\delta(p) + 1 = \alpha(i)]\}$.
(ii) *For all* α, *for all* n, $(\mathcal{V}_\alpha, =) \models \psi_n$ *if and only if* $\exists i[\alpha(i) > n]$.
(iii) *For all* α, *for all* n, $(\mathcal{V}_\alpha, =) \models \rho_n$ *if and only if* $\exists i[\alpha(i) = n + 1 \wedge \forall j[\alpha(j) = n + 1 \to i = j]]$.
(iv) *For all* ζ, η *in* $[\omega]^\omega$, *if* $\zeta \perp \eta$ *and* $\zeta(0) = \eta(0) = 2$, *then there exists a sentence* ψ *such that* $(\mathcal{V}_\zeta, =) \models \psi$ *and* $(\mathcal{V}_\eta, =) \models \neg\psi$.

Proof (i) Use Theorem 12 (i).
(ii) Note that, for each α, for each n, $(\mathcal{V}_\alpha, =) \models \psi_n$ if and only if $\mathcal{V}_\alpha$ contains a limit point of order n if and only if $\exists i[\alpha(i) > n]$.
(iii) Note that, for each α, for each n, $(\mathcal{V}_\alpha, =) \models \rho_n$ if and only if $\mathcal{V}_\alpha$ contains exactly one limit point of order n if and only if
$\exists i[\alpha(i) = n + 1 \wedge \forall j[\alpha(j) = n + 1 \to i = j]]$.
(iv) Using (iii), note that, for all ζ in $[\omega]^\omega$, if $\zeta(0) > 1$, then $\forall n[(\mathcal{V}_\zeta, =) \models \rho_n$ if and only if $\exists p[\zeta(p) = n + 1]$.

[18] Note that there exists an embedding $\rho : \mathcal{C} \hookrightarrow \{\zeta \in [\omega]^\omega \mid \zeta(0) = 2\}$.

Conclude that, for all ζ, η in $[\omega]^\omega$, for all p, if $\zeta(0) = \eta(0) = 2$ and $\zeta \perp \eta$ and $p := \mu i[\zeta(i) \neq \eta(i)]$ and $\zeta(p) < \eta(p)$, then $\neg\exists i[\eta(i) = \zeta(p)]$, and, therefore, $(\mathcal{V}_\zeta, =) \models \psi_{\zeta(p)-1}$ and $(\mathcal{V}_\eta, =) \models \neg\psi_{\zeta(p)-1}$.

□

1.8.2 Finitary Spreads Suffice

Definition 21 Assume $Spr(\beta)$. β is called a *finitary spread-law* or a *fan-law* if and only if $\exists\gamma\forall s[\beta(s) = 0 \to \forall n[\beta(s * \langle n\rangle) = 0 \to n \leq \gamma(s)]]$.

$\mathcal{X} \subseteq \mathcal{N}$ is a *fan* if and only if there exists a fan-law β such that $\mathcal{X} = \mathcal{F}_\beta$.

Note that the toy spreads $\mathcal{T}_0, \mathcal{T}_1, \ldots$ are fans.

The set $\mathcal{V}_\alpha$, however, is a spread but, in general, not a fan.

Define, for each α, $\mathcal{V}^*_\alpha := \overline{\bigcup_n \underline{\overline{0}}n * \langle 1\rangle * \mathcal{T}_{\alpha(n)}}$.[19]

Note that, for each α, $\mathcal{V}^*_\alpha$ is a fan.

One may prove a statement very similar to Theorem 15 (iv):

For all ζ, η in $[\omega]^\omega$, if $\zeta \perp \eta$ and $\zeta(0) = \eta(0) = 2$, then there exists a sentence ψ such that $(\mathcal{V}^*_\zeta, =) \models \psi$ and $(\mathcal{V}^*_\eta, =) \models \neg\psi$.

1.8.3 Comparison with an Older Theorem

The first-order theory DLO of *dense linear orderings without endpoints* is formulated in a first-order language with binary predicate symbols $=$ and $\sqsubset$ and consists of the following axioms:

1. $\forall \mathsf{x}[\mathsf{x} \sqsubset \mathsf{x}]$,
2. $\forall \mathsf{x}\forall \mathsf{y}\forall \mathsf{z}[(\mathsf{x} \sqsubset \mathsf{y} \ \wedge\ \mathsf{y} \sqsubset \mathsf{z}) \to \mathsf{x} \sqsubset \mathsf{z}]$,
3. $\forall \mathsf{x}\forall \mathsf{y}[\big(\neg(\mathsf{x} \sqsubset \mathsf{y}) \ \wedge\ \neg(\mathsf{y} \sqsubset \mathsf{x})\big) \to \mathsf{x} = \mathsf{y}]$.
4. $\forall \mathsf{x}\forall \mathsf{y}[\mathsf{x} \sqsubset \mathsf{y} \to \forall \mathsf{z}[\mathsf{x} \sqsubset \mathsf{z} \ \vee\ \mathsf{z} \sqsubset \mathsf{y}]]$,
5. $\forall \mathsf{x}\exists \mathsf{y}[\mathsf{x} \sqsubset \mathsf{y}] \ \wedge\ \forall \mathsf{x}\exists \mathsf{y}[\mathsf{y} \sqsubset \mathsf{x}]$,
6. $\forall \mathsf{x}\forall \mathsf{y}[\mathsf{x} \sqsubset \mathsf{y} \to \exists \mathsf{z}[\mathsf{x} \sqsubset \mathsf{z} \ \wedge\ \mathsf{z} \sqsubset \mathsf{y}]]$, and
7. axioms of equality.

$(\mathcal{R}, =_\mathcal{R}, <_\mathcal{R})$ realizes DLO.

Let DLO^- be the theory one obtains from DLO by leaving out axiom (4). If one defines a relation $<'_\mathcal{R}$ on $\mathcal{R}$ by: $\forall x\forall y[x <'_\mathcal{R} y \leftrightarrow \neg\neg(x <_\mathcal{R} y)]$, then $(\mathcal{R}, =_\mathcal{R}, <'_\mathcal{R})$ realizes DLO^- but not DLO.

In Veldman and Janssen (1990, Theorem 2.4) one constructs a function $\alpha \mapsto A_\alpha$ associating to each element α of $2^\omega = \mathcal{C}$ a subset A_α of the set $\mathcal{R}$ of the real numbers

[19]For each $\mathcal{X} \subseteq \mathcal{N}$, $\overline{\mathcal{X}} := \{\alpha \mid \forall n\exists\beta \in \mathcal{X}[\overline{\alpha}n \sqsubset \beta]\}$ is the *closure* of $\mathcal{X}$.
$\bigcup_n \underline{\overline{0}}n * \langle 1\rangle * \mathcal{T}_{\alpha(n)}$, in general, is not a spread, but its closure is.

such that, for each α in $\mathcal{C}$, A_α is dense in $(\mathcal{R}, <_\mathcal{R})$, and, for all α, β in $\mathcal{C}$, if $\alpha \perp \beta$, then there exists a sentence ψ such that $(A_\alpha, <_\mathcal{R}) \models \psi$ and $(A_\beta, <_\mathcal{R}) \models \neg\psi$.

Note: each structure $(A_\alpha, <_\mathcal{R})$ realizes DLO. The (intuitionistic) theory DLO thus has continuum many complete extensions.[20]

One may obtain the result of Theorem 15 (iv) for subsets of $\mathcal{R}$ as well as for subsets of $\mathcal{N}$. Define an infinite sequence $\mathcal{U}_0, \mathcal{U}_1, \dots$ of subsets of $\mathcal{R}$ by:

$\mathcal{U}_0 := \emptyset$ and $\mathcal{U}_1 := \{0_\mathcal{R}\}$, and for each $m > 0$, $\mathcal{U}_{m+1} = \overline{\bigcup_n \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} \cdot_\mathcal{R} \mathcal{U}_m}$.[21]

For each m, one may define $\varphi : \mathcal{T}_m \to \mathcal{U}_m$ such that φ is surjective and satisfies: $\forall \delta \in \mathcal{T}_m \forall \zeta \in \mathcal{T}_m[\delta \perp \zeta \leftrightarrow \varphi|\delta \#_\mathcal{R} \varphi|\zeta]$.

It follows that, for each m, the structures $(\mathcal{T}_m, =)$ and $(\mathcal{U}_m, =_\mathcal{R})$ are elementarily equivalent.

Define, for each α in $[\omega]^\omega$, $A_\alpha := \bigcup_n n +_\mathcal{R} \mathcal{U}_{\alpha(n)}$.

Note: for all α, β in $[\omega]^\omega$, if $\alpha \perp \beta$, then there exists a sentence ψ such that $(A_\alpha, =_\mathcal{R}) \models \psi$ and $(A_\beta, =_\mathcal{R}) \models \neg\psi$.

We thus obtain a result similar to Veldman and Janssen (1990, Theorem 2.4), this time using not the ordering relation $<_\mathcal{R}$ but only the equality relation $=_\mathcal{R}$.

Note that, for all α, β in $\mathcal{R}$, $\alpha =_\mathcal{R} \beta \leftrightarrow \big(\neg(\alpha <_\mathcal{R} \beta) \wedge \neg(\beta <_\mathcal{R} \alpha)\big)$.

Conclude: the relation $=_\mathcal{R}$ is definable in the structure $(\mathcal{R}, <_\mathcal{R})$.

Conclude: for all subsets $\mathcal{T}, \mathcal{U}$ of $\mathcal{R}$, if there exists a sentence ψ such that $(\mathcal{T}, =_\mathcal{R}) \models \psi$ and $(\mathcal{U}, =_\mathcal{R}) \models \neg\psi$, then there also exists a sentence ψ^* such that $(\mathcal{T}, <_\mathcal{R}) \models \psi^*$ and $(\mathcal{U}, <_\mathcal{R}) \models \neg\psi^*$.

The conclusion of Veldman and Janssen (1990, Theorem 2.4) might have been obtained as a corollary of Theorem 15 (iv).

1.9 The Vitali Equivalence Relation

For all α, β, we define

$$\alpha \sim_V \beta \leftrightarrow \exists n \forall m > n[\alpha(m) = \beta(m)].$$

The relation $\sim_V$ will be called the *Vitali equivalence relation*.

This is because the relation $\sim_V$ on $\mathcal{N}$ resembles the relation $\sim_\mathbb{Q}$ on the set $\mathcal{R}$ of the real numbers defined by:

$$x \sim_\mathbb{Q} y \leftrightarrow \exists q \in \mathbb{Q}[x -_\mathcal{R} y = q].$$

The relation $\sim_\mathbb{Q}$ has played an important rôle in classical set theory.

If one constructs, using the axiom of choice, within the interval $[0, 1]$, a *transversal* for this equivalence relation, that is: a complete set of mutually inequivalent

[20] Classically, $Th\big((\mathbb{Q}, <)\big)$ is the one and only complete extension of DLO.

[21] For each $\mathcal{X} \subseteq \mathcal{R}$, $\overline{\mathcal{X}} := \{x \in \mathcal{R} \mid \forall n \exists y \in \mathcal{X}[|x - y| < \frac{1}{2^n}]\}$ is the *closure* of $\mathcal{X}$.

representatives, one obtains a set that is not Lebesgue measurable. This discovery is due to G. Vitali.

Note: $(\mathcal{N}, \sim_V) \models EQ$.

The following theorem brings to light an important difference between $(\mathcal{N}, =)$ and $(\mathcal{N}, \sim_V)$.

Definition 22 A proposition P is *stable* if and only if $\neg\neg P \to P$.

A binary relation $\sim$ on $\mathcal{N}$ is *stable* if and and only if $\forall\alpha\forall\beta[\neg\neg(\alpha \sim \beta) \to \alpha \sim \beta]$.[22]

Theorem 16 (Equality is stable but the Vitali equivalence relation is not stable)

(i) $(\mathcal{N}, =) \models \forall \mathsf{x}\forall \mathsf{y}[\neg\neg(\mathsf{x} = \mathsf{y}) \to \mathsf{x} = \mathsf{y}]$.
(ii) $(\mathcal{N}, \sim_V) \models \forall \mathsf{x}\neg\forall \mathsf{y}[\neg\neg(\mathsf{x} = \mathsf{y}) \to \mathsf{x} = \mathsf{y}]$.

Proof (i) Note: for all α, β, $\alpha = \beta \leftrightarrow \neg(\alpha \# \beta)$, and, therefore:
$\neg\neg(\alpha = \beta) \leftrightarrow \neg\neg\neg(\alpha \# \beta) \leftrightarrow \neg(\alpha \# \beta) \leftrightarrow \alpha = \beta$.

(ii) Let γ be given.
Consider $\mathcal{F}^\gamma := \{\alpha \mid \forall m \forall n[\big(\alpha(m) \neq \gamma(m) \;\wedge\; \alpha(n) \neq \gamma(n)\big) \to m = n]$.
$\mathcal{F}^\gamma$ is the set of all α that differ at at most one place from γ.
Note that $\mathcal{F}^\gamma$ is a spread.
We have two claims.
First claim: $\forall \alpha \in \mathcal{F}^\gamma[\neg\neg(\alpha \sim_V \gamma)]$.
The proof is as follows. Let α in $\mathcal{F}^\gamma$ be given. Distinguish two cases.
Case (1). $\exists n[\alpha(n) \neq \gamma(n)]$. Find n such that $\alpha(n) \neq \gamma(n)$ and conclude: $\forall m > n[\alpha(m) = \gamma(m)]$ and $\alpha \sim_V \gamma$.
Case (2). $\neg\exists n[\alpha(n) \neq \gamma(n)]$. Conclude: $\forall n[\alpha(n) = \gamma(n)]$ and $\alpha \sim_V \gamma$.
We thus see: if $\exists n[\alpha(n) \neq \gamma(n)] \;\vee\; \neg\exists n[\alpha(n) \neq \gamma(n)]$, then $\alpha \sim_V \gamma$.
As $\neg\neg(\exists n[\alpha(n) \neq \gamma(n)] \;\vee\; \neg\exists n[\alpha(n) \neq \gamma(n)])$, also $\neg\neg(\alpha \sim_V \gamma)$.
Second claim: $\neg\forall \alpha \in \mathcal{F}^\gamma[\alpha \sim \gamma]$.
In order to see this, assume: $\forall \alpha \in \mathcal{F}^\gamma[\alpha \sim \gamma]$, that is:
$\forall \alpha \in \mathcal{F}\exists n \forall m > n[\alpha(m) = \gamma(m)]$. Using Lemma 2, find p, n such that $\forall \alpha \in \mathcal{F}^\gamma[\overline{\gamma}p \sqsubset \alpha \to \forall m > n[\alpha(m) = \gamma(m)]]$. Define $m := \max(p, n+1)$ and define α such that $\forall n[\alpha(n) \neq \gamma(n) \leftrightarrow n = m]$. Note: $\overline{\gamma}p \sqsubset \alpha$ and $m > n$ and $\alpha(m) \neq \gamma(m)$. Contradiction.
Combining our two claims, we see:
not: for all α, if $\neg\neg(\alpha \sim_V \gamma)$ then $\alpha \sim_V \gamma$.
Conclude: $(\mathcal{N}, \sim_V) \models \forall \mathsf{x}\neg\forall \mathsf{y}[\neg\neg(\mathsf{x} = \mathsf{y}) \to \mathsf{x} = \mathsf{y}]$.

□

It follows from Theorem 16 that there is no relation $\#_V$ on $\mathcal{N}$ satisfying the requirements of an apartness relation[23] with respect to $\sim_V$:

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[22]The term '*stable*' has been introduced by D. Van Dantzig, who hoped to be able to reconstruct 'classical', non-intuitionistic mathematics within the stable part of intuitionistic mathematics, see van Dantzig (1947).

[23]See Troelstra and van Dalen (1988, p. 256).

(i) $\forall\alpha\forall\beta[\neg(\alpha \#_V \beta) \leftrightarrow \alpha \sim_V \beta]$
(ii) $\forall\alpha\forall\beta[\alpha \#_V \beta \rightarrow \beta \#_V \alpha]$
(iii) $\forall\alpha\forall\beta[\alpha \#_V \beta \rightarrow \forall\gamma[\alpha \#_V \gamma \vee \gamma \#_V \beta]]$.

The existence of an apartness $\#_V$ would imply, by the first one of these requirements, that $\sim_V$ is a stable relation, as, for any proposition P, $\neg\neg\neg P \leftrightarrow \neg P$.

The next Theorem now is no surprise:

Theorem 17 $(\mathcal{N}, \sim_V) \models \forall \mathsf{x}\forall \mathsf{y}[\neg AP(\mathsf{x}, \mathsf{y})]$.

Proof Let α, β be given.

Assume $(\mathcal{N}, \sim_V) \models AP[\alpha, \beta]$, that is, $\forall\gamma[\gamma \nsim_V \alpha \vee \gamma \nsim_V \beta]$.

Applying Lemma 1, find p such that
either $\forall\gamma[\overline{\alpha}p \sqsubset \gamma \rightarrow \gamma \nsim_V \alpha]$ or $\forall\gamma[\overline{\alpha}p \sqsubset \gamma \rightarrow \gamma \nsim_V \beta]$.

The first of these two alternatives is wrong, as $\overline{\alpha}p \sqsubset \alpha \wedge \alpha \sim_V \alpha$.

Conclude: $\forall\gamma[\overline{\alpha}p \sqsubset \gamma \rightarrow \gamma \nsim_V \beta]$.

Define γ such that $\overline{\alpha}p \sqsubset \gamma$ and $\forall i > p[\gamma(i) = \beta(i)]$.

Note: $\overline{\alpha}p \sqsubset \gamma \wedge \gamma \sim_V \beta$.

Contradiction.

Conclude: $(\mathcal{N}, =_V) \models \neg AP[\alpha, \beta]$.

We thus see: $(\mathcal{N}, =_V) \models \forall \mathsf{x}\forall \mathsf{y}[\neg AP(\mathsf{x}, \mathsf{y})]$. □

Clearly, the relation defined by the formula AP in the structure $(\mathcal{N}, \sim_V)$ fails to satisfy the first requirement for an apartness relation with respect to $\sim_V$.

It follows from Theorem 17 that $(\mathcal{N}, \sim_V)$, while realizing T_{inf}, does not realize T^+_{inf}, see Definitions 1 and 4.

1.10 A First Vitali Variation

There are many intuitionistic versions of the classical Vitali equivalence relation. This is obvious to someone who knows that there are many variations upon the notion of a finite and decidable subset of $\mathbb{N}$, see Veldman (1995) and Veldman (2005, Sect. 3).

Definition 23 We define an infinite sequence $\sim^0_V, \sim^1_V, \ldots$ of relations on $\mathcal{N}$ such that $\sim^0_V = \sim_V$ and, for each i,

$$\alpha \sim^{i+1}_V \beta \leftrightarrow \exists n \forall m > n[\alpha(m) \neq \beta(m) \rightarrow \alpha \sim^i_V \beta].$$

We also define:

$$\alpha \sim^\omega_V \beta \leftrightarrow \exists i[\alpha \sim^i_V \beta].$$

Theorem 18 (i) $\forall i \forall n \forall s \in \omega^n \forall t \in \omega^n \forall\alpha\forall\beta[s * \alpha \sim^i_V t * \beta \leftrightarrow \alpha \sim^i_V \beta]$.
(ii) $\forall i\forall\alpha\forall\beta[\alpha \sim^i_V \beta \rightarrow \alpha \sim^{i+1}_V \beta]$.
(iii) $\forall i\forall\gamma\neg\forall\alpha[\alpha \sim^{i+1}_V \gamma \rightarrow \alpha \sim^i_V \gamma]$.

(iv) $\forall i \forall j \forall \alpha \forall \beta \forall \gamma[(\alpha \sim_V^i \beta \wedge \beta \sim_V^j \gamma) \rightarrow \alpha \sim_V^{i+j} \gamma]$.
(v) $\sim_V^\omega$ *is an equivalence relation on* $\mathcal{N}$.

Proof (i) One proves this easily by induction.
(ii) Obvious.
(iii) Let γ be given.
For each i, define $\mathcal{F}_i^\gamma := \{\alpha \mid \forall s \in [\omega]^{i+1} \exists j < i+1[\alpha \circ s(j) = \gamma \circ s(j)]\}$.
Note: for each i, $\mathcal{F}_i^\gamma$ is a spread, and $\mathcal{F}_i^\gamma \subsetneq \mathcal{F}_{i+1}^\gamma$.
For each i, $\mathcal{F}_i^\gamma$ consists of all α that assume at most i times a value different from the value assumed by γ. In particular, $\mathcal{F}_\gamma^0 = \{\gamma\}$.
Note: for all i, m, α, β,
if $m = \mu n[\alpha(n) \neq \gamma(n)]$ and $\alpha = \overline{\alpha}(m+1) * \beta$, then $\alpha \in \mathcal{F}_{i+1}^\gamma \leftrightarrow \beta \in \mathcal{F}_i^\gamma$.
We have two claims.
First claim: $\forall i \forall \alpha \in \mathcal{F}_i^\gamma[\alpha \sim_V^i \gamma]$.
We prove this claim by induction.
The starting point of the induction is the observation:
$\forall \alpha \in \mathcal{F}_0^\gamma[\alpha = \gamma]$, so $\forall \alpha \in \mathcal{F}_0^\gamma[\alpha \sim_V^0 \gamma]$.
Now assume i is given such that $\forall \alpha \in \mathcal{F}_i^\gamma[\alpha \sim_V^i \gamma]$.
Assume $\alpha \in \mathcal{F}_{i+1}^\gamma$ and $\exists n[\alpha(n) \neq \gamma(n)]$. Find n such that $\alpha(n) \neq \gamma(n)$. Find β such that $\alpha = \overline{\alpha}(n+1) * \beta$, and note: $\beta \in \mathcal{F}_i^\gamma$ and thus, by the induction hypothesis, $\beta \sim_V^i \gamma$. Conclude, using (i): $\alpha \sim_V^i \gamma$.
We thus see:
$\forall \alpha \in \mathcal{F}_{i+1}^\gamma[\exists n[\alpha(n) \neq \gamma(n)] \rightarrow \alpha \sim_V^i \gamma]$, that is: $\forall \alpha \in \mathcal{F}_{i+1}^\gamma[\alpha \sim_V^{i+1} \gamma]$.
This completes the proof of the induction step.
Second claim: $\forall i \neg \forall \alpha \in \mathcal{F}_{i+1}^\gamma[\alpha \sim_V^i \gamma]$.
We again use induction.
We first prove: $\neg \forall \alpha \in \mathcal{F}_1^\gamma[\alpha \sim_V \gamma]$. Assume $\forall \alpha \in \mathcal{F}_1^\gamma[\alpha \sim_V \gamma]$, that is: $\forall \alpha \in \mathcal{F}_1 \exists n \forall m > n[\alpha(m) = \gamma(m)]$.
Note: $\gamma \in \mathcal{F}_1^\gamma$ and $\mathcal{F}_\gamma^1$ is a spread.
Using Lemma 2, find p, n such that
$\forall \alpha \in \mathcal{F}_1[\overline{\gamma}p \sqsubset \alpha \rightarrow \forall m > n[\alpha(m) = \gamma(m)]$.
Define $m := \max(n+1, p)$ and define α such that $\forall n[\alpha(n) = \gamma(n) \leftrightarrow n \neq m]$.
Note: $\alpha \in \mathcal{F}_1$ and $\overline{\gamma}p \sqsubset \alpha$ and $\alpha(m) \neq \gamma(m)$ and $m > n$. Contradiction.
Conclude: $\neg \forall \alpha \in \mathcal{F}_1^\gamma[\alpha \sim_V \gamma]$.
Now let i be given such that $\neg \forall \alpha \in \mathcal{F}_{i+1}^\gamma[\alpha \sim_V^i \gamma]$.
We want to prove: $\neg \forall \alpha \in \mathcal{F}_{i+2}^\gamma[\alpha \sim_V^{i+1} \gamma]$.
Assume: $\forall \alpha \in \mathcal{F}_{i+2}^\gamma[\alpha \sim_V^{i+1} \gamma]$, that is:
$\forall \alpha \in \mathcal{F}_{i+2}^\gamma \exists n \forall m > n[\alpha(m) \neq \gamma(m) \rightarrow \alpha \sim_V^{i+1} \gamma]$. Using Lemma 2, find p, n such that $\forall \alpha \in \mathcal{F}_{i+2}^\gamma[(\overline{\gamma}p \sqsubset \alpha \wedge m > n \wedge \alpha(m) \neq \gamma(m)) \rightarrow \alpha \sim_V^i \gamma]$. Define $m := \max(n+1, p)$. Let β in $\mathcal{F}_{i+1}^\gamma$ be given. Define α such that $m = \mu n[\alpha(n) \neq \gamma(n)]$ and $\forall n > m[\alpha(n) = \beta(n)]$. Note: $\alpha \in \mathcal{F}_{i+2}^\gamma$ and $\alpha(m) \neq \gamma(m)$ and $m > n$, so $\alpha \sim_V^i \gamma$, and, therefore, by (i), $\beta \sim_V^i \gamma$. We thus see: $\forall \beta \in \mathcal{F}_{i+1}^\gamma[\beta \sim_V^i \gamma]$ and, by the induction hypothesis, obtain a contradiction.
This completes the proof of the induction step.

Taking our first and second claim together, we obtain the conclusion: $\forall\gamma\forall i\neg\forall\alpha[\alpha \sim_V^{i+1} \gamma \to \alpha \sim_V^i \gamma]$.

(iv) We have to prove:
for all i, for all j, $\forall\alpha\forall\beta\forall\gamma[(\alpha \sim_V^i \beta \wedge \beta \sim_V^j \gamma) \to \alpha \sim_V^{i+j} \gamma]$.
We use induction on $i + j$ and distinguish four cases.
Case (1): $i = j = 0$. Assume $\alpha \sim_V^0 \beta$ and $\beta \sim_V^0 \gamma$. Find n, p such that $\forall m > n[\alpha(m) = \beta(m)]$ and $\forall m > p[\beta(m) = \gamma(m)]$. Define $q := \max(n, p)$ and note: $\forall m > q[\alpha(m) = \gamma(m)]$. Conclude: $\alpha \sim_V^0 \gamma$.
Case (2): $i = 0$ and $j > 0$. Assume $\alpha \sim_V^0 \beta$ and $\beta \sim_V^j \gamma$. Find n, p such that $\forall m > n[\alpha(m) = \beta(m)]$ and $\forall m > p[\beta(m) \neq \gamma(m) \to \beta \sim_V^{j-1} \gamma]$.
Define $q := \max(n, p)$.
Assume $m > q$ and note: if $\alpha(m) \neq \gamma(m)$, then $\beta(m) \neq \gamma(m)$ and $\beta \sim_V^{j-1} \gamma$.
Using the induction hypothesis, conclude: $\alpha \sim_V^{j-1} \gamma$.
We thus see: $\forall m > q[\alpha(m) \neq \gamma(m) \to \alpha \sim_V^{j-1} \gamma]$, that is: $\alpha \sim_V^j \gamma$.
Case (3): $i > 0$ and $j = 0$. Assume $\alpha \sim_V^i \beta$ and $\beta \sim_V^0 \gamma$. Find n, p such that $\forall m > n[\alpha(m) \neq \beta(m) \to \alpha \sim_V^{i-1} \beta]$ and $\forall m > p[\beta(m) = \gamma(m)]$.
Define $q := \max(n, p)$.
Assume $m > q$ and note: if $\alpha(m) \neq \gamma(m)$, then $\alpha(m) \neq \beta(m)$ and $\alpha \sim_V^{i-1} \beta$.
Using the induction hypothesis, conclude: $\alpha \sim_V^{i-1} \gamma$.
We thus see: $\forall m > q[\alpha(m) \neq \gamma(m) \to \alpha \sim_V^{i-1} \gamma]$, that is: $\alpha \sim_V^i \gamma$.
Case (4): $i > 0$ and $j > 0$. Assume $\alpha \sim_V^i \beta$ and $\beta \sim_V^j \gamma$. Find n, p such that $\forall m > n[\alpha(m) \neq \beta(m) \to \alpha \sim_V^{i-1} \beta]$ and
$\forall m > p[\beta(m) \neq \gamma(m) \to \beta \sim_V^{j-1} \gamma]$. Define $q := \max(n, p)$.
Assume $m > q$ and $\alpha(m) \neq \gamma(m)$. Then *either:* $\alpha(m) \neq \beta(m)$ and $\alpha \sim^{i-1} \beta$, and, by the induction hypothesis, $\alpha \sim_V^{i+j-1} \gamma$, *or:* $\beta(m) \neq \gamma(m)$ and $\beta \sim^{j-1} \gamma$ and, by the induction hypothesis, $\alpha \sim^{i+j-1} \gamma$.
We thus see: $\forall m > q[\alpha(m) \neq \gamma(m) \to \alpha \sim^{i+j-1} \gamma]$. Conclude: $\alpha \sim^{i+j} \gamma$.

(v) is an easy consequence of (iv).

The next Theorem shows that the structures $(\mathcal{N}, \sim_V)$ and $(\mathcal{N}, \sim_V^\omega)$ have a property in common.

Theorem 19 ($\sim_V^\omega$ is not stable)
$(\mathcal{N}, \sim_V^\omega) \models \forall \mathsf{x}\neg\forall \mathsf{y}[\neg\neg(\mathsf{x} = \mathsf{y}) \to \mathsf{x} = \mathsf{y}]$.

Proof Let γ be given.
We repeat a definition we gave in the proof of Theorem 18 (iii).
For each i, $\mathcal{F}_i^\gamma := \{\alpha \mid \forall s \in [\omega]^{i+1}\exists j < i + 1[\alpha \circ s(j) = \gamma \circ s(j)]\}$.
In the proof of Theorem 18 (iii), we saw: $\forall i\forall\alpha \in \mathcal{F}_i^\gamma[\alpha \sim_V^i \gamma]$.
Conclude: $\forall i\forall\alpha \in \mathcal{F}_i^\gamma[\alpha \sim_V^\omega \gamma]$.
We now define: $\mathcal{F}_\omega^\gamma := \{\alpha \mid \forall i[i = \mu n[\alpha(n) \neq \gamma(n)] \to \alpha \in \mathcal{F}_{i+1}]\}$.
Like each $\mathcal{F}_i^\gamma$, $\mathcal{F}_\omega^\gamma$ is a spread, and $\gamma \in \mathcal{F}_\omega^\gamma$.
We have two claims.
First claim: $\forall\alpha \in \mathcal{F}_\omega^\gamma[\neg\neg(\alpha \sim_V^\omega \gamma)]$.
The argument is as follows. Let α in $\mathcal{F}_\omega^\gamma$ be given and distinguish two cases.

Case (1): $\neg\exists n[\alpha(n) \neq \gamma(n)]$. Then $\alpha = \gamma$ and $\alpha \sim_V^\omega \gamma$.
Case (2): $\exists n[\alpha(n) \neq \gamma(n)]$. Find $i := \mu n[\alpha(n) \neq \gamma(n)]$.
Note: $\alpha \in \mathcal{F}_{i+1}^\gamma$ and $\alpha \sim_V^\omega \gamma$.
As $\neg\neg(\exists n[\alpha(n) \neq \gamma(n)] \vee \neg\exists n[\alpha(n) \neq \gamma(n)])$, also $\neg\neg(\alpha \sim_V^\omega \gamma)$.
Second claim: $\neg\forall\alpha \in \mathcal{F}_\omega^\gamma[\alpha \sim_V^\omega \gamma]$.
In order to see this, assume: $\forall\alpha \in \mathcal{F}_\omega^\gamma[\alpha \sim_V^\omega \gamma]$, that is: $\forall\alpha \in \mathcal{F}_\omega\exists i[\alpha \sim_V^i \gamma]$.
Using Lemma 2, find p, i such that $\forall\alpha \in \mathcal{F}_\omega^\gamma[\overline{\gamma}p \sqsubset \alpha \to \alpha \sim_V^i \gamma]$.
Define $q := \max(p, i+1)$. Let α in $\mathcal{F}_q^\gamma$ be given. Define β such that $\forall n < q[\beta(n) = \gamma(n)]$ and $\beta(q) \neq \gamma(q)$ and $\forall n > q[\beta(n) = \alpha(n)]$.
Note: $\beta \in \mathcal{F}_{q+1}$ and $q = \mu n[\beta(n) \neq \gamma(n)]$, and, therefore, $\beta \in \mathcal{F}_\omega^\gamma$.
As $\overline{\gamma}q \sqsubset \beta$, we conclude: $\beta \sim_V^i \gamma$.
As $\beta \sim_V^0 \alpha$, also $\alpha \sim_V^i \gamma$.
We thus see: $\forall\alpha \in \mathcal{F}_q[\alpha \sim_V^i \gamma]$.
As $q > i$, this contradicts the Second claim in the proof of Theorem 18 (iii).
Taking our two claims together, we conclude:
$\forall\gamma\neg\forall\alpha \in \mathcal{F}_\omega^\gamma[\neg\neg(\alpha \sim_V^\omega \gamma) \to \alpha \sim_V^\omega \gamma]$.
Conclude: $(\mathcal{N}, \sim_V^\omega) \models \forall \mathsf{x}\neg\forall \mathsf{y}[\neg\neg(\mathsf{x} = \mathsf{y}) \to \mathsf{x} = \mathsf{y}]$. □

We did not succeed in finding a sentence ψ such that $(\mathcal{N}, \sim_V) \models \psi$ and $(\mathcal{N}, \sim_V^\omega) \models \neg\psi$.

1.11 More and More Vitali Relations

In Veldman (1995), Veldman (1999) and Veldman (2005, Sect. 3), one studies the set

$$\mathbf{Fin} := \{\alpha \mid \alpha \sim_V \underline{0}\} = \{\alpha \mid \exists n\forall m > n[\alpha(m) = 0]\}.$$

For each α, $\alpha \in \mathbf{Fin}$ if and only if $D_\alpha := \{m \mid \alpha(m) \neq 0\}$ is a *finite* subset of $\mathbb{N}$.

For each i, the set $\{\alpha \mid \alpha \sim_V^i \underline{0}\}$ is called, in Veldman (1999) and Veldman (2005), the i-th *perhapsive extension* of the set **Fin**. It is shown, in Veldman (1995), Veldman (1999) and Veldman (2005), that the process of building perhapsive extensions of **Fin** can be carried on into the transfinite.

In a similar way, the Vitali equivalence relation $\sim_V$ admits of transfinitely many extensions.

The relation $\sim_V^\omega$ is only a *first* extension of $\sim_V$. Let us consider a second one.
Recall: $\forall\alpha\forall\beta[\alpha \sim_V^\omega \leftrightarrow i[\alpha \sim_V^i \beta]]$.

Definition 24 We define an infinite sequence $\sim_V^{\omega+0} = \sim_V^\omega, \sim_V^{\omega+1}, \sim_V^{\omega+2}, \ldots$ of relations on $\mathcal{N}$, such that, for each $i > 0$,

$$\alpha \sim_V^{\omega+i+1} \beta \leftrightarrow \exists n\forall m > n[\alpha(m) \neq \beta(m) \to \alpha \sim_V^{\omega+i} \beta].$$

We also define:

$$\alpha \sim_V^{\omega+\omega} \beta \leftrightarrow \exists i[\alpha \sim_V^{\omega+i} \beta].$$

One may prove analogues of Theorems 18 and 19 and conclude:

$\sim_V^{\omega+\omega}$ is an equivalence relation on $\mathcal{N}$, properly extending $\sim_V^{\omega}$, that, like $\sim_V$ and $\sim_V^{\omega}$, is not stable in the sense of Theorem 19.

One may continue and define $\sim_V^{\omega+\omega+\omega}$, and $\sim_V^{\omega+\omega+\omega+\omega}$ and so on.

The process of building such extensions leads further into the transfinite, as follows.

Definition 25 Let R be a binary relation on $\mathcal{N}$.

We define a binary relation R^+ on $\mathcal{N}$ by:

$$\alpha R^+ \beta \leftrightarrow \exists n \forall m > n[\alpha(m) \neq \beta(m) \to \alpha R \beta].$$

We let $\mathcal{E}$ be the least class of binary relations on $\mathcal{N}$ such that

(i) the Vitali equivalence relation $\sim_V$ belongs to $\mathcal{E}$, and,
(ii) for every R in $\mathcal{E}$, also $R^+ \in \mathcal{E}$, and,
(iii) for every infinite sequence $R_0, R_1, \ldots$ of elements of $\mathcal{E}$, also $\bigcup_i R_i \in \mathcal{E}$.

The elements of $\mathcal{E}$ are the *extensions of the Vitali equivalence relation*.

Note that $<_V^{\omega}$ and $<_V^{\omega+\omega}$ are in $\mathcal{E}$.

In general, a relation R in $\mathcal{E}$ is not transitive. One may prove, for instance, that the relation $<_V^1$, while belonging to $\mathcal{E}$, is not transitive.

The next Theorem shows that $\mathcal{E}$ contains many transitive relations.

Theorem 20 ($\mathcal{E}$ contains many transitive relations)

(i) $\sim_V$ *is transitive.*
(ii) *Given any transitive R in $\mathcal{E}$, there exists a transitive T in $\mathcal{E}$ such that $R^+ \subseteq T$.*
(iii) *Given any infinite and increasing sequence $R_0 \subseteq R_1 \subseteq \ldots$ of transitive relations in $\mathcal{E}$, also $\bigcup_i R_i$ is a transitive relation in $\mathcal{E}$.*

Proof (i) Obvious.

(ii) We take our inspiration from Theorem 18 (iv) and (v).
Let a transitive R in $\mathcal{E}$ be given.
Define an infinite sequence $R^0, R^1, \ldots$ of elements of $\mathcal{E}$ such that $R^0 = R$ and, for each i, $R^{i+1} = (R^i)^+$.
One may prove: for all i, for all j, $\forall\alpha\forall\beta\forall\gamma[(\alpha R^i \beta \wedge \beta R^i \gamma) \to \alpha R^{i+j} \gamma]$, as it is done for the special case $R = \sim_V$ in the proof of Theorem 18 (iv).
Define $T := \bigcup_i R^i$ and note: $T \in \mathcal{E}$, $R^+ \subseteq T$ and T is transitive.

(iii) Note: for every increasing sequence $R_0 \subseteq R_1 \subseteq \ldots$ of transitive relations on $\mathcal{N}$, also $\bigcup_i R_i$ is transitive.

Theorem 20 will gain significance after Corollary 3, which shows that, for every R in $\mathcal{E}$, $R \subseteq R^+$ and $\neg(R^+ \subseteq R)$.

We did not succeed in proving that every R in $\mathcal{E}$ extends to a transitive T in $\mathcal{E}$.

Definition 26 A binary relation R on $\mathcal{N}$ is *shift-invariant* if and only if $\forall\alpha\forall\beta[\alpha R\beta \leftrightarrow (\alpha \circ S)R(\beta \circ S)]$.

Lemma 7 *Every R in $\mathcal{E}$ is shift-invariant.*

Proof The proof is a straightforward exercise in induction on $\mathcal{E}$. Note:

(I) $\sim_V$ is shift-invariant.
(II) For every binary relation R on $\mathcal{N}$, if R is shift-invariant, then R^+ is shift-invariant.
(III) For every infinite sequence $R_0, R_1, \ldots$ of binary relations on $\mathcal{N}$, if each R_n is shift-invariant, then $\bigcup_i R_i$ is shift-invariant.
Conclude: every R in $\mathcal{E}$ is shift-invariant. □

Definition 27 We let $\mathcal{E}^*$ be the least class of binary relations on $\mathcal{N}$ such that

(i) the Vitali equivalence relation $\sim_V$ belongs to $\mathcal{E}^*$, and
(ii) for every infinite sequence $R_0, R_1, \ldots$ of elements of $\mathcal{E}^*$, also $(\bigcup_i R_i)^+ \in \mathcal{E}^*$.

Lemma 8 *$\mathcal{E}^* \subseteq \mathcal{E}$ and, for all R in $\mathcal{E}$, there exists T in $\mathcal{E}^*$ such that $R \subseteq T$.*

Proof The proofs of the two statements are straightforward, by induction on $\mathcal{E}^*$ and $\mathcal{E}$, respectively. □

Theorem 21 *For each R in $\mathcal{E}^*$, $R \subseteq R^+$ and $\neg(R^+ \subseteq R)$.*

Proof For each R in $\mathcal{E}$, we define $Fin_R := \{\alpha \mid \alpha R\underline{0}\}$.[24]

We prove for each R in $\mathcal{E}^*$ there exists a fan $\mathcal{F}$ such that $\mathcal{F} \subseteq Fin_{R^+}$ and $\neg(\mathcal{F} \subseteq Fin_R)$.

We do so by induction on $\mathcal{E}^*$.

(I) Define $\mathcal{F} := \{\alpha \mid \forall m\forall n[(\alpha(m) \neq 0 \ \wedge\ \alpha(n) \neq 0) \rightarrow m = n]\}$.

Note that $\mathcal{F}$ is a fan.

For each α in $\mathcal{F}$, for each n, if $\alpha(n) \neq 0$ then: $\forall m > n[\alpha(m) = 0]$ and $\alpha \in Fin_{\sim_V}$. Conclude: for each $\alpha \in \mathcal{F}$, if $\exists n[\alpha(n) \neq 0]$, then $\alpha \in Fin_{\sim_V}$, that is: $\alpha \in Fin_{(\sim_V)^+}$. Conclude: $\mathcal{F} \subseteq Fin_{(\sim_V)^+}$.

Now assume $\mathcal{F} \subseteq Fin_{\sim_V}$, that is: $\forall\alpha \in \mathcal{F}\exists n\forall m > n[\alpha(m) = 0]$. Using Lemma 2, find p, n such that $\forall\alpha \in \mathcal{F}[\overline{\underline{0}}p \sqsubset \alpha \rightarrow \forall m > n[\alpha(m) = 0]]$. Define $q := \max(p, n+1)$ and consider $\alpha := \overline{\underline{0}}q * \langle 1\rangle * \underline{0}$. Contradiction.

Conclude: $\neg(\mathcal{F} \subseteq Fin_{\sim_V})$.

(II) Let $R_0, R_1, \ldots$ be an infinite sequence of elements of $\mathcal{E}$.

Let $\mathcal{F}_0, \mathcal{F}_1, \ldots$ be an infinite sequence of fans such that, for each n, $\mathcal{F}_n \subseteq Fin_{(R_n)^+}$ and $\neg(\mathcal{F}_n \subseteq Fin_{R_n})$.

Consider $R := (\bigcup_i R_i)^+$.

Define $\mathcal{F} := \{\alpha \mid \forall n[n = \mu i[\alpha(i) \neq 0] \rightarrow \exists\beta \in \mathcal{F}_{n'}[\alpha = \overline{\alpha}(n+1) * \beta]]\}$.[25]

Note that $\mathcal{F}$ is a fan.

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[24] In Veldman (1995), $\mathcal{X} \subseteq \mathcal{N}$ is called a *notion of finiteness* if $\mathbf{Fin} \subseteq \mathcal{X} \subseteq \mathbf{Fin}^{\neg\neg}$. For every R in $\mathcal{E}$, Fin_R is a notion a finiteness.

[25] For each n, $n = (n', n'')$, see Sect. 1.13.

We now prove: $\mathcal{F} \subseteq Fin_{R^+}$ and $\neg(\mathcal{F} \subseteq Fin_R)$.

Note that, for each $\alpha \in \mathcal{F}$, for each n, if $n = \mu i[\alpha(i) \neq 0]$, then there exists β in $\mathcal{F}_{n'}$ such that $\alpha = \overline{\alpha}(n+1) * \beta$.

As, for each n, $\mathcal{F}_n \subseteq Fin_{(R_n)^+} \subseteq Fin_{\bigcup_i (R_i)^+}$, and $\bigcup_i (R_i)^+ \subseteq \left(\bigcup_i R_i\right)^+ = R$ and R is shift-invariant, conclude: $\forall \alpha \in \mathcal{F}[\exists n[\alpha(n) \neq 0] \rightarrow \alpha \in Fin_R]$, that is: $\mathcal{F} \subseteq Fin_{R^+}$.

Now assume $\mathcal{F} \subseteq Fin_R$, that is:
$\forall \alpha \in \mathcal{F} \exists n \forall m > n[\alpha(m) \neq 0] \rightarrow \exists i[\alpha \in Fin_{R_i}]]$. Using Lemma 2, find p, n such that
$\forall \alpha \in \mathcal{F}[\underline{\overline{0}}p \sqsubset \alpha \rightarrow \forall m > n[\alpha(m) \neq 0 \rightarrow \exists i[\alpha \in Fin_{R_i}]]$.

Define $q := \max(p, n+1)$ and note: $\forall \alpha \in \mathcal{F}[\underline{\overline{0}}q * \langle 1 \rangle \sqsubset \alpha \rightarrow \exists i[\alpha \in \mathcal{F}_i]]$.

Using Lemma 2 again, find r, i such that $\forall \alpha \in \mathcal{F}[\underline{\overline{0}}q * \langle 1 \rangle * \underline{\overline{0}}r \sqsubset \alpha \rightarrow \alpha \in \mathcal{F}_i]$.

Find $n \geq q + r + 1$ such that $n' = i$ and define $t := n - (q+1)$.

Note: $t \geq r$ and conclude: $\forall \beta \in \mathcal{F}_i[\underline{\overline{0}}q * \langle 1 \rangle * \underline{\overline{0}}t * \langle 1 \rangle * \beta \in Fin_{R_i}]$.

As R_i is shift-invariant, conclude: $\mathcal{F}_i \subseteq Fin_{R_i}$.

Contradiction, as $\neg(\mathcal{F}_i \subseteq Fin_{R_i})$.

Conclude: $\neg(\mathcal{F} \subseteq Fin_R)$. □

Corollary 3 *For each R in $\mathcal{E}$, $R \subseteq R^+$ and $\neg(R^+ \subseteq R)$.*

Proof Assume we find R in $\mathcal{E}$ such that $R = R^+$.

Conclude, by induction on $\mathcal{E}$: for all U in $\mathcal{E}$, $U \subseteq R$.

Using Lemma 8, find T in $\mathcal{E}^*$ such that $R \subseteq T$.

By Theorem 21, $T \subseteq T^+$ and $\neg(T^+ \subseteq T)$.

On the other hand, $T^+ \subseteq R \subseteq T$.

Contradiction. □

Definition 28 We define binary relations $\sim_V^{\neg\neg}$ and $\sim_V^{almost}$ on $\mathcal{N}$, as follows.

For all α, β, $\alpha \sim_V^{\neg\neg} \beta \leftrightarrow \neg\neg\exists n \forall m > n[\alpha(n) = \beta(n)] \leftrightarrow \neg\neg(\alpha \sim_V \beta)$, and $\alpha \sim_V^{almost} \beta \leftrightarrow \forall \zeta \in [\omega]^\omega \exists n[\alpha \circ \zeta(n) = \beta \circ \zeta(n)]$.

$\alpha \sim_V^{almost} \beta$ if and only if the set $\{n \mid \alpha(n) \neq \beta(n)\}$ is *almost***-finite* in the sense used in Veldman (2005, Section 0.8.2).

The following axiom is a form of Brouwer's famous *Thesis on bars in* $\mathcal{N}$, see Veldman (2006).

Axiom 4 (The Principle of Bar Induction)

For all $B, C \subseteq \mathbb{N}$, if $\forall \alpha \exists n[\overline{\alpha}n \in B]$ and $B \subseteq C$ and $\forall s[s \in C \leftrightarrow \forall n[s * \langle n \rangle \in C]]$, then $\langle\ \rangle \in C$,

or, equivalently,

for all $B, C \subseteq [\omega]^{<\omega}$, if $\forall \zeta \in [\omega]^\omega \exists n[\overline{\zeta}n \in B]$ and $B \subseteq C$ and $\forall s \in [\omega]^{<\omega}[s \in C \leftrightarrow \forall n[s * \langle n \rangle \in [\omega]^{<\omega} \rightarrow s * \langle n \rangle \in C]]$, then $\langle\ \rangle \in C$.

Theorem 22 (i) *$\sim_V^{\neg\neg}$ and $\sim_V^{almost}$ are equivalence relations on $\mathcal{N}$.*

(ii) *For all R in $\mathcal{E}$, $\sim_V \subseteq R \subseteq \sim_V^{\neg\neg}$.*

(iii) *For all R in $\mathcal{E}$, $R \subseteq \sim_V^{almost}$.*

(iv) $\forall\alpha\forall\beta[\alpha \sim_V^{almost} \beta \to \exists R \in \mathcal{E}[\alpha R\, \beta]$.
(v) $\forall\alpha\forall\beta[\alpha \sim_V^{almost} \beta \to \alpha \sim_V^{\neg\neg} \beta]$.

Proof (i) One easily proves that $\sim_V^{\neg\neg}$ is an equivalence relation. One needs the fact that, for all propositions P, Q, $(\neg\neg P \wedge \neg\neg Q) \to \neg\neg(P \wedge Q)$.

We prove that $\sim_V^{almost}$ is a transitive relation.

Let α, β, γ be given such that $\alpha \sim_V^{almost} \beta$ and $\beta \sim_V^{almost} \gamma$.

Let ζ in $[\omega]^\omega$ be given. Find η in $[\omega]^\omega$ such that $\forall n[\alpha \circ \zeta \circ \eta(n) = \beta \circ \zeta \circ \eta(n)]$. Find p such that $\beta \circ \zeta \circ \eta(p) = \gamma \circ \zeta \circ \eta(p)$. Define $n := \eta(p)$ and note: $\alpha \circ \zeta(n) = \gamma \circ \zeta(n)$.

We thus see: $\forall\zeta \in [\omega]^\omega \exists n[\alpha \circ \zeta(n) = \gamma \circ \zeta(n)]$, that is: $\alpha \sim_V^{almost} \gamma$.

(ii) The proof is by (transfinite) induction on $\mathcal{E}$. We only prove: for all R in $\mathcal{E}$, $R \subseteq \sim_V^{\neg\neg}$ as the statement: for all R in $\mathcal{E}$, $\sim_V \subseteq R$ is very easy to prove.

(I) Our starting point is the trivial observation: $\forall\alpha\forall\beta[\alpha \sim_V \beta \to \neg\neg(\alpha \sim_V \beta)]$.

(II) Now let R in $\mathcal{E}$ be given such that $\forall\alpha\forall\beta[\alpha R\beta \to \neg\neg(\alpha \sim_V \beta)]$.

We have to prove: $\forall\alpha\forall\beta[\alpha R^+\beta \to \neg\neg(\alpha \sim_V \beta)]$.

We do so as follows.

Let α, β be given such that $\alpha R^+\beta$.

Find n such that $\forall m > n[\alpha(m) \neq \beta(m) \to \alpha R\beta]$ and consider two special cases.

Case (1): $\exists m > n[\alpha(m) \neq \beta(m)$. Then $\alpha R\, \beta$, and, therefore: $\neg\neg(\alpha \sim_V \beta)$.

Case (2): $\neg\exists m > n[\alpha(m) \neq \beta(m)$. Then $\forall m > n[\alpha(m) = \beta(m)]$ and $\alpha \sim_V \beta$.

In both cases, we find: $\neg\neg(\alpha \sim_V \beta)$.

Conclude[26]: $\neg\neg(\alpha \sim_V \beta)$.

(III) Now let $R_0, R_1, \ldots$ be an infinite sequence of elements of $\mathcal{E}$ such that, for all n, $\forall\alpha\forall\beta[\alpha R_n\beta \to \neg\neg(\alpha \sim_V \beta)]$.

Define $R := \bigcup_n R_n$ and note: $\forall\alpha\forall\beta[\alpha R\beta \to \neg\neg(\alpha \sim_V \beta)]$.

(iii) The proof is by (transfinite) induction on $\mathcal{E}$.

(I) Our starting point is the observation: $\forall\alpha\forall\beta[a \sim_V^0 \beta \to \alpha \sim_V^{almost} \beta]$.

We prove this as follows:

Let α, β be given such that $\alpha \sim_V^0 \beta$. Find n such that $\forall m > n[\alpha(m) = \beta(m)]$. Note: $\forall\zeta \in [\omega]^\omega][\zeta(n+1) > n \wedge \alpha \circ \zeta(n+1) = \beta \circ \zeta(n+1)]$. Conclude: $\alpha \sim_V^{almost} \beta$.

(II) Now let R in $\mathcal{E}$ be given such that $\forall\alpha\forall\beta[\alpha R\beta \to \alpha \sim_V^{almost} \beta]$.

We have to prove: $\forall\alpha\forall\beta[aR^+\beta \to \alpha \sim_V^{almost} \beta]$.

We do so as follows.

Let α, β be given such that $\alpha R^+\beta$.

Find n such that $\forall m > n[\alpha(m) \neq \beta(m) \to \alpha R\beta]$. Let ζ in $[\omega]^\omega$ be given. Consider $\zeta(n+1)$ and note $\zeta(n+1) > n$. There now are two cases.

Either $\alpha \circ \zeta(n+1) = \beta \circ \zeta(n+1)$ *or* $\alpha \circ \zeta(n+1) \neq \beta \circ \zeta(n+1)$.

In the first case we are done, and in the second case we conclude $\alpha R\beta$, and, using the induction hypothesis, find p such that $\alpha \circ \zeta(p) = \beta \circ \zeta(p)$.

In both cases we conclude: $\exists q[\alpha \circ \zeta(q) = \beta \circ \zeta(q)]$.

We thus see: $\forall\zeta \in [\omega]^\omega \exists q[\alpha \circ \zeta(q) = \beta \circ \zeta(q)]$, that is $\alpha \sim_V^{almost} \beta$.

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[26] using the scheme: if $P \to Q$ and $\neg P \to Q$, then $\neg\neg Q$.

Clearly then: $\forall\alpha\forall\beta[[\alpha R^+\beta \to \alpha \sim_V^{almost} \beta]$.

(III) Now let $R_0, R_1, \ldots$ be an infinite sequence of elements of $\mathcal{E}$ such that, for all n, $\forall\alpha\forall\beta[\alpha R_n\beta \to \alpha \sim_V^{almost} \beta]$.

Define $R := \bigcup_n R_n$ and note: $\forall\alpha\forall\beta[\alpha R\beta \to \alpha \sim_V^{almost} \beta]$.

(iv) Let α, β be given such that $\alpha \sim^{almost} \beta$, that is:
$\forall\zeta \in [\omega]^\omega \exists n[\alpha \circ \zeta(n) = \beta \circ \zeta(n)]$.
Using Axiom 4, we shall prove: there exists R in $\mathcal{E}$ such that $\alpha R\beta$.

Define $B := \bigcup_k \{s \in [\omega]^{k+1} \mid \alpha \circ s(k) = \beta \circ s(k)\}$ and note: B is a bar in $[\omega]^\omega$, that is: $\forall\zeta \in [\omega]^\omega \exists n[\overline{\zeta}n \in B]$.

Define $C := \bigcup_k \{s \in [\omega]^k \mid \exists n < k[\alpha \circ s(n) = \beta \circ s(n)] \ \vee\ \exists R \in \mathcal{E}[\alpha R\beta]\}$.

Note: $C = \bigcup_k \{s \in [\omega]^k \mid \forall n < k[\alpha \circ s(n) \neq \beta \circ s(n)] \to \exists R \in \mathcal{E}[\alpha R\beta]\}$.

Note: $B \subseteq C$ and: C is *monotone*, that is:
$\forall s \in [\omega]^{<\omega}[s \in C \to \forall n[s * \langle n\rangle \in [\omega]^{<\omega} \to s * \langle n\rangle \in C]]$.

We still have to prove that C is what one calls *inductive* or *hereditary*.

Let s in $[\omega]^{<\omega}$ be given such that $\forall n[s * \langle n\rangle \in [\omega]^{<\omega} \to s * \langle n\rangle \in C]$.
We want to prove: $s \in C$.

Find k such that $s \in [\omega]^k$. In case $\exists n < k[\alpha \circ s(n) = \beta \circ s(n)]$, $s \in C$ and we are done, so we assume: $\forall n < k[\alpha \circ s(n) \neq \beta \circ s(n)]$.

Find a sequence[27] $R_0, R_1, \ldots$ of elements of $\mathcal{E}$ such that, for each n, if $s * \langle n\rangle \in [\omega]^\omega$ and $\alpha(n) \neq \beta(n)$, then $\alpha R_n\beta$.

Define $R := (\bigcup_i R_i)^+$ and note: $R \in \mathcal{E}$.

We claim: $\alpha R\beta$.

We establish this claim as follows.

Define p such that, if $k = 0$, then $p := 0$ and, if $k > 0$, then $p := s(k-1) + 1$.

Assume we find $n \geq p$ such that $\alpha(n) \neq \beta(n)$.

Note: $s * \langle n\rangle \in [\omega]^{k+1}$ and $\forall i < k+1[\alpha \circ (s * \langle n\rangle)(i) \neq \beta \circ (s * \langle n\rangle)(i)]$ and $s * \langle n\rangle \in C$. Conclude: $\alpha R_n\beta$ and $\alpha(\bigcup_i R_i)\beta$.

We thus see: $\forall n \geq p[\alpha(n) \neq \beta(n) \to \alpha(\bigcup_i R_i)\beta]$.

Conclude: $\alpha(\bigcup_i R_i)^+\beta$, that is: $\alpha R\beta$, and, therefore: $s \in C$.

We thus see that C is inductive.

Using Axiom 4, we conclude: $\langle\,\rangle \in C$, that is: $\exists R \in \mathcal{E}[\alpha R\beta]$.

(v) Let α, β be given such that $\alpha \sim^{almost} \beta$, that is:
$\forall\zeta \in [\omega]^\omega \exists n[\alpha \circ \zeta(n) = \beta \circ \zeta(n)]$.
Using Axiom 4, we prove: $\neg\neg\exists p\forall n > p[\alpha(n) = \beta(n)]$.

Define $B := \bigcup_k \{s \in [\omega]^{k+1} \mid \alpha \circ s(k) = \beta \circ s(k)\}$ and note: B is a bar in $[\omega]^\omega$, that is: $\forall\zeta \in [\omega]^\omega \exists n[\overline{\zeta}n \in B]$. Define
$C := \bigcup_k \{s \in [\omega]^k \mid \exists n < k[\alpha \circ s(n) = \beta \circ s(n)] \ \vee\ \neg\neg\exists p\forall n > p[\alpha(n) = \beta(n)]\}$.
Note: $C = \bigcup_k \{s \in [\omega]^k \mid \forall n < k[\alpha \circ s(n) \neq \beta \circ s(n)] \to \neg\neg\exists p\forall n > p[\alpha(n) = \beta(n)]\}$.

Note: $B \subseteq C$ and C is monotone, that is:
$\forall s \in [\omega]^{<\omega}[s \in C \to \forall n[s * \langle n\rangle \in [\omega]^{<\omega} \to s * \langle n\rangle \in C]]$.

We still have to prove that C is inductive.

[27] This application of countable choice may be reduced to Axiom 3. One may define $\mathcal{B} \subseteq \mathcal{N}$ and a *coding* mapping $\alpha \mapsto R_\alpha$ such that $\mathcal{E} = \{R_\alpha \mid \alpha \in \mathcal{B}\}$.

Let s in $[\omega]^{<\omega}$ be given such that $\forall n[s * \langle n \rangle \in [\omega]^{<\omega} \rightarrow s * \langle n \rangle \in C]]$.
We want to prove: $s \in C$.

Find k such that $s \in [\omega]^k$. In case $\exists n < k[\alpha \circ s(n) = \beta \circ s(n)]$, $s \in C$, and we are done, so we assume $\forall n < k[\alpha \circ s(n) \neq \beta \circ s(n)]$.

Define q such that $q := 0$ if $k = 0$ and $q := s(k-1)$ if $k > 0$.

Consider two special cases:

Case (1): $\exists n > q[\alpha(n) \neq \beta(n)]$.
Find such n, note: $s * \langle n \rangle \in [\omega]^\omega$ and $\forall i < k+1[\alpha \circ (s * \langle n \rangle)(i) \neq \beta \circ (s * \langle n \rangle)(i)]$ and $s * \langle n \rangle \in C$, and conclude: $\neg\neg\exists p \forall n > p[\alpha(n) = \beta(n)]$.

Case (2): $\neg\exists n > q[\alpha(n) \neq \beta(n)]$, and, therefore, $\forall n > q[\alpha(n) = \beta(n)]$.

In both cases, we find: $\neg\neg\exists p \forall n > p[\alpha(n) = \beta(n)]$.

Conclude[28]: $\neg\neg\exists p \forall n > p[\alpha(n) = \beta(n)]$, and: $s \in C$.

We thus see that C is inductive.

Using Axiom 4, we conclude: $\langle\ \rangle \in C$, and, therefore,
$\neg\neg\exists p \forall n > p[\alpha(n) = \beta(n)]$, that is: $\neg\neg(\alpha \sim_V \beta)$. □

Corollary 4 (i) $(\mathcal{N}, \sim_V^{\neg\neg}) \models \forall x \forall y[\neg\neg(x = y) \rightarrow x = y]$.
(ii) *For each R in $\mathcal{E}$,* $(\mathcal{N}, R) \models \forall x \neg\forall y[\neg\neg(x = y) \rightarrow x = y]$.

Proof (i) Obvious, as, for any proposition P, $\neg\neg\neg\neg P \leftrightarrow \neg\neg P$.
(ii) Assume $R \in \mathcal{E}$.
We first prove: $(\mathcal{N}, R) \models \neg\forall x \forall y[\neg\neg(x = y) \rightarrow x = y]$.
Assume $\forall\alpha\forall\beta[\neg\neg(\alpha R \beta) \rightarrow \alpha R \beta]$.
Note: $\forall\alpha\forall\beta[\alpha \sim_V \beta \rightarrow \alpha R \beta]$ and, therefore: $\forall\alpha\forall\beta[\neg\neg(\alpha \sim_V \beta) \rightarrow \neg\neg(\alpha R \beta)]$.
Conclude: $\sim_V^{\neg\neg} \subseteq R$.
By Theorem 22 (ii), $R^+ \subseteq \sim_V^{\neg\neg}$, so $R^+ \subseteq R$. This contradicts Corollary 3.
The stronger statement announced in the Theorem may be proven in a similar way. Inspection of he proof of Theorem 22 enables one to conclude:
$(\mathcal{N}, R) \models \neg\forall y[\neg\neg(x = y) \rightarrow x = y][\underline{0}]$.
One easily generalizes this conclusion to:
for each α, $(\mathcal{N}, R) \models \neg\forall y[\neg\neg(x = y) \rightarrow x = y][\alpha]$.
Conclude: $(\mathcal{N}, R) \models \forall x \neg\forall y[\neg\neg(x = y) \rightarrow x = y]$. □

Markov's Principle has been mentioned in Sect. 1.4. Markov's Principle is not accepted in intuitionistic mathematics, but the following observation still is of interest.

Corollary 5 *The following are equivalent.*

(i) *Markov's Principle:* $\forall\alpha[\neg\neg\exists n[\alpha(n) = 0] \rightarrow \exists n[\alpha(n) = 0]]$.
(ii) $\sim_V^{\neg\neg} \subseteq \sim_V^{almost}$.
(iii) $\sim_V^{almost}$ *is stable.*

[28]Using the scheme: If $P \rightarrow Q$ and $\neg P \rightarrow Q$, then $\neg\neg Q$.

Proof (i) $\Rightarrow$ (ii). Assume $\neg\neg(\alpha \sim_V \beta)$, that is $\neg\neg\exists n\forall m > n[\alpha(m) = \beta(m)]$.
Let $\zeta \in [\omega]^\omega$ be given.
Assume: $\neg\exists n[\alpha \circ \zeta(n) = \beta \circ \zeta(n)]$.
Then $\forall n[\zeta(n+1) > n \ \wedge \ \alpha \circ \zeta(n) \neq \beta \circ \zeta(n)]$, so $\forall n\exists m > n[\alpha(m) \neq \beta(m)]$. Contradiction.
Conclude: $\neg\neg\exists n[\alpha \circ \zeta(n) = \beta \circ \zeta(n)]$ and, by Markov's Principle, $\exists n[\alpha \circ \zeta(n) = \beta \circ \zeta(n)]$.
We thus see $\forall \zeta \in [\omega]^\omega \exists n[\alpha \circ \zeta(n) = \beta \circ \zeta(n)]$, that is: $\alpha \sim_V^{almost} \beta$.
(ii) $\Rightarrow$ (iii). By Theorem 22 (v), $\sim_V^{almost} \subseteq \sim_V^{\neg\neg}$. Therefore: $(\sim_V^{almost})^{\neg\neg} \subseteq \sim_V^{\neg\neg}$. Using (ii), we conclude: $(\sim_V^{almost})^{\neg\neg} \subseteq \sim_V^{almost}$, that is: $\sim_V^{almost}$ is stable.
(iii) $\Rightarrow$ (i). Let α be given such that $\neg\neg\exists n[\alpha(n) \neq 0]$.
Define β such that $\forall m[\beta(m) = 0 \leftrightarrow \exists n \leq m[\alpha(n) = 0]]$.
Note: $\neg\neg(\beta \sim_V \underline{0})$ and, therefore: $\neg\neg(\beta \sim_V^{almost} \underline{0})$.
Conclude, using (iii), $\beta \sim_V^{almost} \underline{0}$.
Define ζ such that $\forall n[\zeta(n) = n]$.
Find m such that $\beta \circ \zeta(m) = \beta(m) = 0$ and, therefore, $\exists n \leq m[\alpha(n) = 0]$.
We thus see: $\forall \alpha[\neg\neg\exists n[\alpha(n) = 0] \rightarrow \exists n[\alpha(n) = 0]]$, that is: Markov's Principle. □

1.12 Equality and Equivalence

We did not succeed in finding a sentence ψ such that $(\mathcal{N}, \sim_V) \models \psi$ and $(\mathcal{N}, \sim_V^\omega) \models \neg\psi$. We now want to compare the structures $(\mathcal{N}, =, \sim_V)$ and $(\mathcal{N}, =, \sim_V^\omega)$. We need a first order language with two binary relation symbols: $=$ and $\sim$. The symbol $=$ will denote the equality relation and the symbol $\sim$ will denote, in the first structure, the relation $\sim_V$ and, in the second structure, the relation $\sim_V^\omega$. The reader hopefully will not be confused by the fact that, in the earlier sections, where we used the first order language with a single binary relation symbol, $=$, the symbol $=$ denoted the relations $\sim_V$ and $\sim_V^\omega$.

The next Theorem makes us see that equality is decidable on each equivalence class of $\sim_V$ whereas, on each equivalence class of $\sim_V^\omega$, it is not decidable.

Theorem 23 (i) $(\mathcal{N}, =, \sim_V) \models \forall \mathsf{x}\forall \mathsf{y}[\mathsf{x} \sim \mathsf{y} \rightarrow (\mathsf{x} = \mathsf{y} \ \vee \ \neg(\mathsf{x} = \mathsf{y}))]$.
(ii) $(\mathcal{N}, =, \sim_V^\omega) \models \forall \mathsf{x}\neg\forall \mathsf{y}[\mathsf{x} \sim \mathsf{y} \rightarrow (\mathsf{x} = \mathsf{y} \ \vee \ \neg(\mathsf{x} = \mathsf{y}))]$.

Proof (i) Let γ, α be given such that $\gamma \sim_V \alpha$.
Find n such that $\forall m > n[\gamma(m) = \alpha(m)]$ and distinguish two cases.
Either $\overline{\gamma}(m+1) = \overline{\alpha}(m+1)$ and $\gamma = \alpha$, *or* $\overline{\gamma}(m+1) \neq \overline{\alpha}(m+1)$ and $\neg(\gamma = \alpha)$.
Conclude: $\forall \gamma \forall \alpha[\gamma \sim_V \alpha \rightarrow (\gamma = \alpha \ \vee \ \neg(\gamma = \alpha))]$.
(ii) Let γ be given.
Consider $\mathcal{F}_1^\gamma := \{\alpha \mid \forall m \forall n[(\alpha(m) \neq \gamma(m) \ \wedge \ \alpha(n) \neq \gamma(n)) \rightarrow m = n]\}$.

Note: $\mathcal{F}_1^\gamma$ is a spread. Also: $\forall \alpha \in \mathcal{F}_1^\gamma[\gamma \sim_V^1 \alpha]$ [29] and, therefore, $\forall \alpha \in \mathcal{F}_1^\gamma[\gamma \sim_V^\omega \alpha]$. Assume $\forall \alpha \in \mathcal{F}_1^\gamma[\gamma = \alpha \ \vee\ \neg(\gamma = \alpha)]$. Applying Lemma 1, find p such that *either* $\forall \alpha \in \mathcal{F}_1^\gamma[\overline{\gamma} p \sqsubset \alpha \to \gamma = \alpha]$ *or* $\forall \alpha[\overline{\gamma} p \sqsubset \alpha \to \neg(\gamma = \alpha)]$, and note that both alternatives are false.
Conclude: $\forall \gamma \neg \forall \alpha[\gamma \sim_V^\omega \alpha \ \vee \neg(\gamma = \alpha)]$.

□

Lemma 9 $(\sim_V^{\neg\neg})^+ \ \subseteq\ \sim_V^{\neg\neg}$ *and* $(\sim_V^{almost})^+ \ \subseteq\ \sim_V^{almost}$.[30]

Proof Assume $\alpha(\sim_V^{\neg\neg})^+\beta$.
Find n such that $\forall m > n[\alpha(m) \neq \beta(m) \to \alpha \sim_V^{\neg\neg} \beta]$.
Note: if $\exists m > n[\alpha(m) \neq \beta(m)]$, then $\alpha \sim_V^{\neg\neg} \beta$, and if $\neg\exists m > n[\alpha(m) \neq \beta(m)]$, then $\forall m > n[\alpha(m) = \beta(m)]$ and $\alpha \sim_V \beta$ and also $\alpha \sim_V^{\neg\neg} \beta$.
Conclude: $\neg\neg(\alpha \sim_V^{\neg\neg} \beta)$, and, therefore, $\alpha \sim_V^{\neg\neg} \beta$.
Assume $\alpha(\sim_V^{almost})^+\beta$.
Find n such that $\forall m > n[\alpha(m) \neq \beta(m) \to \alpha \sim_V^{almost} \beta]$.
Let ζ in $[\omega]^\omega$ be given. Note: $\zeta(n+1) > n$.
Either: $\alpha \circ \zeta(n+1) = \beta \circ \zeta(n+1)$
or: $\alpha \sim_V^{almost} \beta$ and $\exists p[\alpha \circ \zeta(p) = \beta \circ \zeta(p)]$.
We thus see: $\forall \zeta \in [\omega]^\omega \exists n[\alpha \circ \zeta(n) = \beta \circ \zeta(n)]$, that is: $\alpha \sim_V^{almost} \beta$. □

Lemma 10 *For every shift-invariant binary relation R on $\mathcal{N}$,*
$R^+ \subseteq R$ *if and only if* $(\mathcal{N}, =, R) \models \forall \mathsf{x} \forall \mathsf{y}[\big(AP(\mathsf{x},\mathsf{y}) \to \mathsf{x} \sim \mathsf{y}\big) \to \mathsf{x} \sim \mathsf{y}]$.

Proof First assume $R^+ \subseteq R$.
Assume $\alpha \# \beta \to \alpha R \beta$.
Then: $\forall m > 0[\alpha(m) \neq \beta(m) \to \alpha R \beta]$, so: $\alpha R^+ \beta$, and, therefore: $\alpha R \beta$.
We thus see: $(\mathcal{N}, =, R) \models \forall \mathsf{x} \forall \mathsf{y}[\big(AP(\mathsf{x},\mathsf{y}) \to \mathsf{x} \sim \mathsf{y}\big) \to \mathsf{x} \sim \mathsf{y}]$.
Now assume $(\mathcal{N}, =, R) \models \forall \mathsf{x} \forall \mathsf{y}[\big(AP(\mathsf{x},\mathsf{y}) \to \mathsf{x} \sim \mathsf{y}\big) \to \mathsf{x} \sim \mathsf{y}]$.
Assume $\alpha R^+ \beta$. Find n such that $\forall m > n[\alpha(m) \neq \beta(m) \to \alpha R \beta]$.
Define γ, δ such that $\forall m[\gamma(m) = \alpha(n+1+m) \ \wedge\ \delta(m) = \beta(n+1+m)]$.
Note: $\gamma \# \delta \to \alpha R \beta$, and, as R is shift-invariant, also: $\gamma \# \delta \to \gamma R \delta$, and, therefore: $\gamma R \delta$, and also: $\alpha R \beta$.
We thus see: $R^+ \subseteq R$. □

Corollary 6 (i) $(\mathcal{N}, =, \sim_V^{\neg\neg}) \models \forall \mathsf{x} \forall \mathsf{y}[\big(AP(\mathsf{x},\mathsf{y}) \to \mathsf{x} \sim \mathsf{y}\big) \to \mathsf{x} \sim \mathsf{y}]$.
(ii) $(\mathcal{N}, =, \sim_V^{almost}) \models \forall \mathsf{x} \forall \mathsf{y}[\big(AP(\mathsf{x},\mathsf{y}) \to \mathsf{x} \sim \mathsf{y}\big) \to \mathsf{x} \sim \mathsf{y}]$.
(iii) *For each R in $\mathcal{E}$,* $(\mathcal{N}, =, R) \models \neg \forall \mathsf{x} \forall \mathsf{y}[\big(AP(\mathsf{x},\mathsf{y}) \to \mathsf{x} \sim \mathsf{y}\big) \to \mathsf{x} \sim \mathsf{y}]$.

Proof Use Lemmas 9 and 10 and Corollary 3. □

[29]See the proof of Theorem 18 (iii).

[30]Following the terminology in Veldman (1995), a binary relation R on $\mathcal{N}$ should be called *perhapsive* if $R^+ \subseteq R$.

1.13 Notations and Conventions

We use $m, n, \ldots$ as variables over the set $\omega = \mathbb{N}$ of the natural numbers.

For every $P \subseteq \mathbb{N}$ such that $\forall n[P(n) \vee \neg P(n)]$, for all m,
$m = \mu n[P(n)] \leftrightarrow \big(P(m) \wedge \forall n < m[\neg P(n)]\big)$.

$(m, n) \mapsto J(m, n)$ is a one-to-one surjective mapping from $\omega \times \omega$ onto ω.

$K, L : \omega \to \omega$ are its inverse functions, so $\forall n[J\big(K(n), L(n)\big) = n]$.

For each n, $n' := K(n)$ and $n'' := L(n)$.

$(n_0, n_1, \ldots, n_{k-1}) \mapsto \langle n_0, n_1, \ldots, n_{k-1}\rangle$ is a one-to-one surjective mapping from the set of finite sequences of natural numbers to the set of the natural numbers.

$\langle n_0, n_1, \ldots, n_{k-1}\rangle$ is the *code* of the finite sequence $(n_0, n_1, \ldots, n_{k-1})$.

$s \mapsto length(s)$ is is the function that, for each s, gives the length of the finite sequence coded by s.

$s, n \mapsto s(n)$ is the function that, for all s, n, gives the value of the finite sequence coded by s at n. If $n \geq length(s)$, then $s(n) = 0$.

For all s, k, if $length(s) = k$, then $s = \langle s(0), s(1), \ldots s(k-1)\rangle$.

$0 = \langle\ \rangle$ codes the empty sequence of natural numbers,
the unique finite sequence s such that $length(s) = 0$.

$\omega^k := \{s \mid length(s) = k\}$.

$[\omega]^k := \{s \in \omega^k \mid \forall i[i + 1 < k \to s(i) < s(i+1)]\}$.

$[\omega]^{<\omega} := \bigcup_k [\omega]^k$.

For all s, k, t, l, if $s \in \omega^k$ and $t \in \omega^l$, then $s * t$ is the element u of ω^{k+l} such that $\forall i < k[u(i) = s(i)]$ and $\forall j < l[u(k + j) = t(j)]$.

$s \sqsubseteq t \leftrightarrow \exists u[s * u = t]$.

$s \sqsubset t \leftrightarrow (s \sqsubseteq t \wedge s \neq t)$.

We use $\alpha, \beta, \ldots$ as variables over *Baire space*, the set $\omega^\omega := \mathcal{N}$ of functions from $\mathbb{N}$ to $\mathbb{N}$.

$(\alpha, n) \mapsto \alpha(n)$ is the function that associates to all α, n, the value of α at n.

For all α, β, $\alpha \circ \beta$ is the element γ of $\mathcal{N}$ such that $\forall n[\gamma(n) = \alpha\big(\beta(n)\big)]$.

$2^\omega := \mathcal{C} := \{\alpha \mid \forall n[\alpha(n) < 2]\}$ is *Cantor space*.

For all α, for all k, for all s in ω^k, $\alpha \circ s$ is the element t of ω^k satisfying $\forall n < k[t(k) = \alpha\big(s(k)\big)]$.

For each s, k, if $s \in \omega^k$, then, for each α, $s * \alpha$ is the element β of $\mathcal{N}$ such that $\forall i < k[\beta(i) = s(i)]$ and $\forall i[\beta(k + i) = \alpha(i)]$.

For each s, for each $\mathcal{X} \subseteq \mathcal{N}$, $s * \mathcal{X} := \{s * \alpha \mid \alpha \in \mathcal{X}\}$.

For each α, for each n, α^n is the element of $\mathcal{N}$ satisfying $\forall m[\alpha^n(m) = \alpha\big(J(n, m)\big)]$.

For each m, $\underline{m} \in \mathcal{N}$ is the element of $\mathcal{N}$ satisfying $\forall n[\underline{m}(n) = m]$.

S is the element of $\mathcal{N}$ satisfying $\forall n[S(n) = n + 1]$.

$\forall n[\alpha'(n) = \big(\alpha(n)\big)' \wedge \alpha''(n) = \big(\alpha(n)\big)'']$.

$\overline{\alpha}n := \langle \alpha(0), \alpha(1), \ldots \alpha(n-1)\rangle$.

$s \sqsubset \alpha \leftrightarrow \exists n[\overline{\alpha}n = s]$.

$\alpha \perp \beta \leftrightarrow \alpha \# \beta \leftrightarrow \exists n[\alpha(n) \neq \beta(n)]$.

$[\omega]^\omega := \{\zeta \in \mathcal{N} \mid \forall i[\zeta(i) < \zeta(i+1)]\}$.

$\mathbb{Q}$, the set of the rationals, may be defined as a subset of ω, with accompanying relations $=_{\mathbb{Q}}$, $<_{\mathbb{Q}}$, $\leq_{\mathbb{Q}}$ and operations $+_{\mathbb{Q}}$, $-_{\mathbb{Q}}$, $\cdot_{\mathbb{Q}}$.

$\mathcal{R} := \{\alpha \mid \forall n[\alpha'(n) \in \mathbb{Q} \ \wedge \ \alpha''(n) \in \mathbb{Q}] \ \wedge \forall n[\alpha'(n) \leq_{\mathbb{Q}} \alpha'(n+1) \leq_{\mathbb{Q}} \alpha''(n+1) \leq_{\mathbb{Q}} \alpha''(n)] \ \wedge \ \forall m \exists n[\alpha''(n) -_{\mathbb{Q}} \alpha'(n) <_{\mathbb{Q}} \frac{1}{2^m}]\}$.

For all α, β in $\mathcal{R}$,

$\alpha <_{\mathcal{R}} \beta \leftrightarrow \exists n[\alpha''(n) <_{\mathbb{Q}} \beta'(n)]$ and $\alpha =_{\mathcal{R}} \beta \leftrightarrow \big(\neg(\alpha <_{\mathcal{R}} \beta) \ \wedge \ \neg(\beta <_{\mathcal{R}} \alpha)\big)$.

Operations $+_{\mathcal{R}}$, $-_{\mathcal{R}}$ are defined straightforwardly.

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