Logic, Epistemology, and the Unity of Science 49

Mojtaba Mojtahedi
Shahid Rahman
Mohammad Saleh Zarepour *Editors*

Mathematics, Logic, and their Philosophies

Essays in Honour of Mohammad Ardeshir

Springer

Logic, Epistemology, and the Unity of Science

Volume 49

Logic, Epistemology, and the Unity of Science aims to reconsider the question of the unity of science in light of recent developments in logic. At present, no single logical, semantical or methodological framework dominates the philosophy of science. However, the editors of this series believe that formal frameworks, for example, constructive type theory, deontic logics, dialogical logics, epistemic logics, modal logics, and proof-theoretical semantics, have the potential to cast new light on basic issues in the discussion of the unity of science.

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ISSN 2214-9775 ISSN 2214-9783 (electronic)
Logic, Epistemology, and the Unity of Science
ISBN 978-3-030-53653-4 ISBN 978-3-030-53654-1 (eBook)
https://doi.org/10.1007/978-3-030-53654-1

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

The present volume celebrates the outstanding intellectual heritage of Prof. Mohammad Ardeshir by collecting papers related to the different aspects of his research interests. Mohammad Ardeshir is a Full Professor of mathematical logic at the Department of Mathematical Sciences, Sharif University of Technology (SUT), Tehran, Iran, where he also did parts of his university studies. He completed B.Sc. in Electrical Engineering (1980), followed by M.Sc. in Mathematics (1990) both at SUT. He obtained his Ph.D. from Marquette University, Milwaukee, USA (May 1995). His dissertation, entitled *Aspects of Basic Logic*, was supervised by Wim Ruitenburg. Ardeshir is internationally known in the first place for his prominent works in Basic Logic, Algebraic Logic, Constructive Arithmetic, and Constructive Analysis. His areas of interest are, however, much broader and include topics in the Intuitionistic Philosophy of Mathematics and History of Philosophy of Logic and Mathematics in the Medieval Islamic World. Putting his different interests together, we can see that all the research projects with which Ardeshir has been engaged in his career lie in the areas where mathematics meets logic and/or philosophy. Hence, the title of this volume: *Mathematics, Logic, and Their Philosophies*.

Some of Ardeshir's most important works are in Basic Logic, which is Intuitionistic Logic without the *modus ponens* rule. The technical aspect of this logic was first introduced by Albert Visser (Visser, 1981). Wim Ruitenburg (Ruitenburg, 1991) later defended Basic Logic from a philosophical point of view and provided a sequent calculus for it. Ardeshir, in his Ph.D. thesis (Ardeshir, 1995) and its subsequent papers (Ardeshir & Ruitenburg, 1998, 2001), explored various aspects of the proof theory and Kripke semantics of Basic Logic and introduced several sequent calculi and, most notably, a cut-free axiomatization for it. Another sequent calculus of Basic Logic with the subformula property was later presented in (Aghaei & Ardeshir, 2001). One of Adeshir's other innovations in this area is to introduce a translation, which embeds Intuitionistic Logic in Basic Logic (Ardeshir, 1999). Ardeshir's translation is comparable to Gödel's negative translation, which embeds Classical Logic in the Intuitionistic Logic. A strict upper bound of the translation from Intuitionistic Logic into Basic Logic, when it is restricted to the propositional case, was introduced in (Aghaei & Ardeshir, 2000). Ardeshir's latest contribution to this area, (Ardeshir & Vaezian, 2012), was to introduce a logic, called U, which is weaker than both Visser's Basic Logic (Visser, 1981) and Sambin's Basic Logic (Battilotti & Sambin, 1999).

Algebraic Logic is another area with which Ardeshir has been engaged since the time of his Ph.D. Algebraic Logic studies possible relations between Algebra and Logic. It enables logicians to employ advanced algebraic tools and results to study Logic. In his Ph.D. dissertation, Ardeshir introduced algebraic models—currently known as Visser Algebras—for Basic Propositional Logic. Ardeshir also proved Stone's Representation Theorem for those algebras. Another achievement of Ardeshir's thesis was to put forward a model for Basic Logic based on Heyting Algebras which are equipped with an additional conjunction-preserving operator. This part of Ardeshir's thesis was a source of inspiration for later works in this field. See, for example, (Alizadeh, 2004; Celani & Jansana, 2005; Suzuki, Wolter, & Zakharyaschev, 1998). Mainly in collaboration with Wim Ruitenburg and Majid Alizadeh, Ardeshir's fruitful works in Algebraic Logic has continued to this day.

The third significant aspect of Ardeshir's works pertains to his research in Constructive Arithmetic. Consider a Kripke model for Heyting Arithmetic (HA), which is Peano Arithmetic (PA) with Intuitionistic Logic as its base logic. Corresponding to each node of the Kripke model, one may easily define a classical structure. An immediate question that could be raised is: Is every Kripke model of HA, locally PA? Equivalently, is this the case that the classical structures assigned to each node of a Kripke model of HA are models of PA? (van Dalen, Mulder, Krabbe, & Visser, 1986) gives a positive answer to this question, albeit for a restricted class of Kripke models called finite-depth models. This positive answer was later extended, by (Wehmeier, 1996), to the class of ω-frame Kripke models. The positive answer is generalized to the class of the rooted narrow Kripke models —i.e., finite models with some ω-tails—by (Ardeshir & Hesaam, 2002) and to the class of the semi-narrow Kripke models by (Mojtahedi, 2019). An interesting relevant result, proved by (Ardeshir, Ruitenburg, & Salehi, 2003), is that HA is strongly complete for the class of end-extension Kripke models. A Kripke model is end-extension, if for every two nodes $u \preccurlyeq v$ in the model, the classical structure assigned to v is an end-extension of the model assigned to u.

Basic Arithmetic (BA), which is the Basic Logic variant of HA was introduced by (Ruitenburg, 1998). That BA is closed under a restricted form of Markov rule—according to which BA $\vdash \neg\neg\exists xA$ implies BA $\vdash \exists xA$, for all A without $\rightarrow$ and $\forall$—was shown by (Ardeshir & Hesaam, 2008). They also introduce a Basic Logic variant for the seriality axiom schema in modal logic: $\top \rightarrow \bot \Rightarrow \bot$. The *propositional logic* of a first-order theory T is defined to be the set of all propositions A such that for every substitution α of atomic variables with sentences in the language of T, we have $T \vdash \alpha(A)$. It can easily be shown that for a theory T over Classical Logic, the propositional logic of T is Classical Logic. However, things get complicated when it comes to *non-classical* first-order theories like HA. It is proved by (de Jongh, 1970) that the propositional logic of HA is Intuitionistic Logic. In the same vein, it is shown by (Ardeshir & Mojtahedi, 2014) that the propositional logic of BA is Basic Logic.

The *provability logic* of a first-order theory *T* is defined as the set of all *modal* propositions *A* such that for every substitution α of propositional variables with sentences in the language of *T*, we have $T \vdash \alpha(A)$, in which $\Box$ is *interpreted* as provability in *T*. One of the early remarkable results in this area, proved by (Solovay, 1976), states that GL := K4+$\Box(\Box A \rightarrow A) \rightarrow \Box A$ is the provability logic of PA. A few years later it was shown by (Visser, 1982) that Σ_1-provability logic of PA is GL plus the completeness principle for atomic variables: $p \rightarrow \Box p$. Σ_1-provability logic of PA is a special instance of the provability logic in which we restrict the substitutions to Σ_1-substitutions. That the arithmetical completeness of the provability logic of PA is propositionally reducible to that of its Σ_1-provability logic is shown by (Ardeshir & Mojtahedi, 2015). Another important result of (Visser, 1982) is that although HA is a sub-theory of PA, the provability logic of HA is not a sub-theory of the provability logic of PA. Indeed, it was later established by (Artemov & Beklemishev, 2004) that the characterization of the provability logic of HA is remarkably difficult. An outstanding result in this area was obtained by (Ardeshir & Mojtahedi, 2018). They characterize and prove the decidability of the Σ_1-provability logic of HA. The Σ_1-provability logic of HA*—which is the self-completion of the HA introduced by (Visser, 1982)—is characterized in (Ardeshir & Mojtahedi, 2019). Conjoining what we described so far with Ardeshir's works in Constructive Analysis, we will have an overall picture of his contributions to mathematics and logic. Ardeshir has conducted several impressive research projects in Constructive Analysis, mainly in collaboration with two of his former Ph.D. students, Zahra Ghafouri and Rasoul Ramezanian.

Now it seems to be time to say a few words about Ardeshir's engagement with the philosophy of logic and mathematics. It was thanks to Ardeshir's writings and translations that the Intuitionistic Philosophy of Mathematics was introduced to the Iranian academic community in the last years of the previous century. Ardeshir has authored, in Persian, several articles about Intuitionism and its founder, L. E. J. Brouwer (whose big framed photo, hanged on the wall, is the first thing which catches your eye when you enter Ardeshir's office at SUT). The faithful soldier of Intuitionism—as Ardeshir once described himself in a talk at the Iranian Institute of Philosophy, February 2006—has also translated, from English into Persian, one book and several articles on Intuitionism. These publications together with his *Mathematical Logic*—a Persian textbook which, up until now, has been printed five times and is the standard textbook of the courses on mathematical logic in various universities around Iran—have been highly praised by Iranian scholars. A sign of the national appreciation of Ardeshir's Persian works is that his *Mathematical Logic* was the winner of the 23rd Iran's Book of the Year Award (2005).

Ardeshir has always had deep interests in Philosophy of Logic and Mathematics in the Medieval Islamic World. His brilliant paper, "Ibn Sīnā's Philosophy of Mathematics" (Ardeshir, 2008), was the first study on this subject and the main trigger for Mohammad Saleh Zarepour's Cambridge Ph.D. dissertation (Zarepour, 2019). Ardeshir's "Brouwer's Notion of Intuition and Theory of Knowledge by Presence" is an important—tough unfortunately neglected—work which bridges his interests in Intuitionism, on the one hand, and Islamic philosophy, on the other.

The significance of Ardeshir's career is not fairly recognized unless we consider his phenomenal role in educating Iranian students. He has taught generations of students for more than a quarter century and supervised numerous M.Sc. and Ph.D. students most of whom have become successful scholars. His students will never forget either his strict work disciplines or his kind, modest, gentle, and wise personality. It is by no means an exaggeration to say that many Iranian logicians have been, either directly or indirectly, under the influence of Ardeshir's works. Putting all these achievements together leaves no doubt that dedicating a collection of papers to him is one of the least things we can do to appreciate Ardeshir's career.

Tehran, Iran — Mojtaba Mojtahedi
Villeneuve d'Ascq, France — Shahid Rahman
München, Germany — Mohammad Saleh Zarepour

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Acknowledgements

We would like to express our gratitude to the reviewers of the papers contributed to this volume: Majid Alizadeh, Bahram Assadian, Michel Crubellier, Danko Ilik, Chajda Ivan, Ramon Jansana, Mousa Mohammadian, Stephen Ogden, Hiroakira Ono, Graham Priest, Mehrnoosh Sadrzadeh, Thomas Schindler, Luca Spada, Wim Veldman, Albert Visser, and Mostafa Zare. We are also thankful to Majid Alizadeh and Zahra Ghafouri from whose reports on Ardeshir's works we benefited in writing the preface to this volume. While coediting the present volume, Mohammad Saleh Zarepour benefited from a Humboldt Research Fellowship at LMU Munich. We are grateful to Alexander von Humboldt Foundation for their support. Last but not least, we owe special thanks to Leone Gazziero (STL), Laurent Cesalli (Genève), and Tony Street (Cambridge), leaders of the ERC-Generator project "Logic in Reverse: Fallacies in the Latin, the Islamic and the Hebrew traditions," and to Claudio Majolino (STL), associated researcher to that project, for fostering the edition of the present volume.

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May, 1995
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Contents

1 **Equality and Equivalence, Intuitionistically** 1
Wim Veldman

2 **Binary Modal Companions for Subintuitionistic Logics** 35
Dick de Jongh and Fatemeh Shirmohammadzadeh Maleki

3 **Extension and Interpretability** 53
Albert Visser

4 **Residuated Expansions of Lattice-Ordered Structures** 93
Majid Alizadeh and Hiroakira Ono

5 **Everyone Knows that Everyone Knows** 117
Rahim Ramezanian, Rasoul Ramezanian, Hans van Ditmarsch, and Malvin Gattinger

6 **Fuzzy Generalised Quantifiers for Natural Language in Categorical Compositional Distributional Semantics** 135
Mătej Dostál, Mehrnoosh Sadrzadeh, and Gijs Wijnholds

7 **Implication via Spacetime** 161
Amirhossein Akbar Tabatabai

8 **Bounded Distributive Lattices with Two Subordinations** 217
Sergio Celani and Ramon Jansana

9 **Hard Provability Logics** 253
Mojtaba Mojtahedi

10 **On PBZ*–Lattices** 313
Roberto Giuntini, Claudia Mureşan, and Francesco Paoli

11 **From Intuitionism to Many-Valued Logics Through Kripke Models** 339
Saeed Salehi

12 Non-conditional Contracting Connectives 349
Luis Estrada-González and Elisángela Ramírez-Cámara

13 Deflationary Reference and Referential Indeterminacy 365
Bahram Assadian

14 The Curious Neglect of Geometry in Modern Philosophies of Mathematics 379
Siavash Shahshahani

15 De-Modalizing the Language 391
Kaave Lajevardi

16 On Descriptional Propositions in Ibn Sīnā: Elements for a Logical Analysis 411
Shahid Rahman and Mohammad Saleh Zarepour

17 Avicenna on Syllogisms Composed of Opposite Premises 433
Behnam Zolghadr

18 Is Avicenna an Empiricist? 443
Seyed N. Mousavian

Correction to: Binary Modal Companions for Subintuitionistic Logics C1
Dick de Jongh and Fatemeh Shirmohammadzadeh Maleki

Author Index 475

Subject Index 481

Chapter 1
Equality and Equivalence, Intuitionistically

Wim Veldman

For Mohammad Ardeshir
Solem enim e mundo tollere videntur qui amicitiam e vita tollunt.
They take away the sun from the world, surely, those who take away friendship from life.

Cicero, *de Amicitia, XIII 47*

Abstract We show that the intuitionistic first-order theory of equality has continuum many complete extensions. We also study the Vitali equivalence relation and show there are many intuitionistically precise versions of it.

Keywords Brouwer's continuity principle · Apartness · Toy spread · Decidable point of a spread · Perhapsive extensions

1.1 Introduction

We want to contribute to L. E. J. Brouwer's program of doing mathematics *intuitionistically*.

We follow his advice to interpret the logical constants constructively.

A conjunction $A \wedge B$ is considered proven if and only if one has a proof of A and also a proof of B.

A disjunction $A \vee B$ is considered proven if and only if either A or B is proven.

An implication $A \to B$ is considered proven if and only if there is a proof of B using the assumption A.

A negation $\neg A$ is considered proven if and only if there is a proof of $A \to 0 = 1$.

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M. Mojtahedi et al. (eds.), *Mathematics, Logic, and their Philosophies*,
Logic, Epistemology, and the Unity of Science 49,
https://doi.org/10.1007/978-3-030-53654-1_1

An existential statement $\exists x \in V[P(x)]$ is considered proven if and only an element x_0 is produced together with a proof of the associated statement $P(x_0)$.

A universal statement $\forall x \in V[P(x)]$ is considered proven if and only if a method is given that produces, given any x in V, a proof of the associated statement $P(x)$.

We also use some axioms proposed by Brouwer: his *Continuity Principle*, our Axiom 1, a slightly stronger version of it, the *First Axiom of Continuous Choice*, our Axiom 2, and his *Thesis on Bars in* $\mathcal{N}$, our Axiom 4.

In some of our proofs, we use an Axiom of Countable Choice, our Axiom 3. Intuitionistic mathematicians, who accept infinite step-by-step constructions not determined by a rule, consider this axiom a reasonable proposal.

Finally, we believe that generalized inductive definitions, like our Definition 25, fall within the compass of intuitionistic mathematics.

Our subject is the (intuitionistic) first-order theory of equality. By considering structures $(\mathcal{X}, =)$ where $\mathcal{X}$ is a subset of Baire space $\mathcal{N} = \omega^\omega$ and $=$ the usual equality relation on $\mathcal{N}$, we find that the theory has an uncountable and therefore astonishing[1] variety of elementarily different infinite models and, as a consequence, an astonishing variety of complete extensions, see Theorem 15. The key observation[2] leading to this result is the recognition that, in a *spread*,[3] an *isolated* point is the same as a *decidable* point.[4] It follows that the set of the non-isolated points of a spread is a definable subset of the spread. In spreads that are *transparent*,[5] the set of the non-isolated points of the spread coincides with the *coherence* of the spread,[6] and the coherence itself is spread. It may happen that the coherence of a transparent spread is transparent itself and then the coherence of the coherence also is a definable subset of the spread. And so on.

Any structure $(\mathcal{N}, R)$, where R is an equivalence relation on $\mathcal{N}$, is a model of the theory of equality. We study the *Vitali equivalence relation*, see Sect. 1.9, as an example. This equivalence relation, in contrast to the equality relation on $\mathcal{N}$, is not *stable*,[7] see Theorem 16.

There is a host of binary relations on $\mathcal{N}$ that, from a classical point of view, all would be the same as the Vitali equivalence relation, see Sects. 1.10 and 1.11, and especially Definition 25, Corollary 3 and Definition 28. It turned out to be difficult to find differences between them that are first-order expressible. We did find some such differences, however, by studying structures $(\mathcal{N}, =, R)$, where R is an intuitionistic version of the Vitali equivalence relation and $=$ the usual equality, see Sect. 1.12.

[1] Classically, all infinite models of the first-order theory of equality are elementarily equivalent.

[2] This observation has been made earlier in Veldman (2001, Sect. 5). The first part of the present paper elaborates part of Veldman (2001, Sect. 5).

[3] Every *spread* is a closed subset of $\mathcal{N}$, see Sect. 1.4.

[4] See Lemma 3. $\alpha \in \mathcal{X} \subseteq \mathcal{N}$ is a *decidable* point of $\mathcal{X}$ if and only if $\forall \beta \in \mathcal{X}[\alpha = \beta \vee \neg(\alpha = \beta)]$.

[5] See Definition 8.

[6] The *coherence* of a closed set is the set of its limit points, see Definition 7.

[7] $R \subseteq \mathcal{N} \times \mathcal{N}$ is called *stable* if $\forall \alpha \forall \beta[\neg\neg \alpha R \beta \to \alpha R \beta]$, see Definition 22.

The paper is divided into 13 Sections and consists roughly of two parts. Sections 1.2, 1.3, 1.4, 1.5, 1.6, 1.7 and 1.8 lead up to the result that the theory of equality has continuum many complete extensions, see Theorem 15. Sections 1.9, 1.10, 1.11 and 1.12 treat the Vitali equivalence relations. Section 1.13 lists some notations and conventions and may be used by the reader as a reference.

1.2 Intuitionistic Model Theory

Given a relational structure $\mathfrak{A} = (A, R_0, R_1, \ldots, R_{n-1})$, we construct a first-order language $\mathcal{L}$ with basic formulas $\mathsf{R}_i(\mathsf{x}_0, \mathsf{x}_1, \ldots, \mathsf{x}_{l_i-1})$, where $i < n$ and l_i is the arity of R_i. The formulas of $\mathcal{L}$ are obtained from the basic formulas by using $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\exists$, $\forall$ in the usual way.

For every formula $\varphi = \varphi(\mathsf{x}_0, \mathsf{x}_1, \ldots, \mathsf{x}_{m-1})$ of $\mathcal{L}$, for all $a_0, a_1, \ldots, a_{m-1}$ in A, we define the statement:

$$\mathfrak{A} \models \varphi[a_0, a_1, \ldots, a_{m-1}]$$

($\mathfrak{A}$ *realizes* φ *if* $\mathsf{x}_0, \mathsf{x}_1, \ldots, \mathsf{x}_{m-1}$ *are interpreted by* $a_0, a_1, \ldots, a_{m-1}$, *respectively*), as Tarski did it, with the proviso that connectives and quantifiers are interpreted intuitionistically.

A formula φ of $\mathcal{L}$ without free variables will be called a *sentence*.

A *theory (in $\mathcal{L}$)* is a set of sentences of $\mathcal{L}$.

Given a theory Γ in $\mathcal{L}$ and a structure $\mathfrak{A}$, we define: $\mathfrak{A}$ *realizes* Γ if and only if, for every φ in Γ, $\mathfrak{A} \models \varphi$.

Given a structure $\mathfrak{B}$ that has the same signature as $\mathfrak{A}$, so that the formulas of $\mathcal{L}$ may be interpreted in $\mathfrak{B}$ as well as in $\mathfrak{A}$, we let $Th(\mathfrak{B})$, the *theory of* $\mathfrak{B}$, be the set of all sentences φ of $\mathcal{L}$ such that $\mathfrak{B} \models \varphi$.

A theory Γ in $\mathcal{L}$ will be called a *complete theory* if and only if there exists a structure $\mathfrak{B}$ such that $\Gamma = Th(\mathfrak{B})$.

This agrees with one of the uses of the expression '*complete theory*' in classical, that is: usual, non-intuitionistic, model theory, see Hodges (1993, p. 43). Note that one may be unable to decide, for a given sentence φ and a given structure $\mathfrak{B}$, whether or not $\mathfrak{B} \models \varphi$. Intuitionistically, it is not true that, for every complete theory Γ and every sentence φ, *either* $\varphi \in \Gamma$ *or* $\neg\varphi \in \Gamma$.

Complete theories Γ, Δ are *positively different* if one may point out a sentence ψ such that $\psi \in \Gamma$ and $\neg\psi \in \Delta$.[8]

Structures $\mathfrak{A}$, $\mathfrak{B}$ are *elementarily equivalent* if and only if $Th(\mathfrak{A}) = Th(\mathfrak{B})$ and *(positively) elementarily different* if $Th(\mathfrak{A})$ is positively different from $Th(\mathfrak{B})$.

Let Γ be a theory in $\mathcal{L}$. A good question is the following:

How many complete theories Δ can one find extending Γ?

---

[8] If $\psi \in \Gamma$ and $\neg\psi \in \Delta$, then $\neg\psi \in \Delta$ and $\neg\neg\psi \in \Gamma$: the relation *positively different* is symmetric.

We will say: Γ *admits countably many complete extensions* if and only if there exists an infinite sequence $\Delta_0, \Delta_1, \dots$ of complete theories extending Γ such that, for all m, n, if $m \neq n$, then Δ_m, Δ_n are (positively) different, and

Γ *admits continuum many complete extensions* if and only if there exists a function $\alpha \mapsto \Delta_\alpha$ associating to every element α of $\mathcal{C} = 2^\omega$ a complete theory extending Γ such that for all α, β, if[9] $\alpha \# \beta$, then Δ_α, Δ_β are (positively) different.

A main result of this paper is that the first-order theory of equality admits continuum many complete extensions.

1.3 Equality May Be Undecidable

The first-order theory EQ of equality consists of the following three axioms:

1. $\forall \mathsf{x}[\mathsf{x} = \mathsf{x}]$,
2. $\forall \mathsf{x} \forall \mathsf{y}[\mathsf{x} = \mathsf{y} \rightarrow \mathsf{y} = \mathsf{x}]$ and
3. $\forall \mathsf{x} \forall \mathsf{y} \forall \mathsf{z}[(\mathsf{x} = \mathsf{y} \ \wedge \ \mathsf{y} = \mathsf{z}) \rightarrow \mathsf{x} = \mathsf{z}]$.

A model of EQ is a structure of the form (V, R), where V is a set and R is an equivalence relation on V, possibly, but not necessarily, the equality relation belonging to V.

Classically, every complete extension of EQ is realized in one of the structures from the list: $(\{0\}, =)$, $(\{0, 1\}, =)$, $(\{0, 1, 2\}, =)$, $\dots$ and $(\omega, =)$. This shows that, classically, EQ admits of (no more than) countably many complete extensions.

Intuitionistically, however, we have to observe that all structures on this list satisfy the sentence

$\forall \mathsf{x} \forall \mathsf{y}[\mathsf{x} = \mathsf{y} \ \vee \ \neg(\mathsf{x} = \mathsf{y})]$,

that is: the equality relation, on each of these sets, is a *decidable* relation.

Turning to the set $\mathcal{N}$, we note that, if we define an element α of $\mathcal{N}$ by stipulating:

$$\forall n[\alpha(n) \neq 0 \leftrightarrow \forall i < 99[d(n+i) = 9]],$$

where $d : \mathbb{N} \to \{0, 1, \dots, 9\}$ is the decimal expansion of π, then, at this moment, we have no proof of:

$$\alpha = \underline{0} \ \vee \neg(\alpha = \underline{0}).$$

This is because, if $\alpha = \underline{0}$, then $\neg \exists n \forall i < 99[\alpha(n+i) = 9]$, and, if $\neg(\alpha = \underline{0})$, then $\neg\neg\exists n \forall i < 99[d(n+i) = 9]$, and we have no proof of either alternative.

This example shows us that the statement $\forall \alpha[\alpha = \underline{0} \ \vee \ \neg(\alpha = \underline{0})]$, for a constructive mathematician, who interprets the disjunction strongly, is a *reckless* statement.[10]

[9] $\alpha \# \beta \leftrightarrow \alpha \perp \beta \leftrightarrow \exists n[\alpha(n) \neq \beta(n)]$, see Sect. 1.13.

[10] A statement is *reckless* if the classical mathematician holds it is true while the intuitionistic mathematician, at this point of time, has no proof for his constructive reading of it.

The following axiom, used by Brouwer,[11] implies that the statement $\forall\alpha[\alpha = \underline{0} \vee \neg(\alpha = \underline{0})]$ even leads to a contradiction.

Axiom 1 (Brouwer's Continuity Principle)
For all $R \subseteq \mathcal{N} \times \omega$, if $\forall\alpha\exists n[\alpha R n]$, then $\forall\alpha\exists m\exists n\forall\beta[\overline{\alpha}m \sqsubset \beta \to \beta R n]$.

An immediate consequence is:

Lemma 1 (Brouwer's Continuity Principle, the case of disjunction)
For all $P_0, P_1 \subseteq \mathcal{N}$, *if* $\forall\alpha[\alpha \in P_0 \vee \alpha \in P_1]$, *then* $\forall\alpha\exists m[\forall\beta[\overline{\alpha}m \sqsubset \beta \to \beta \in P_0] \vee \forall\beta[\overline{\alpha}m \sqsubset \beta \to \beta \in P_1]]$.

Proof Define $R := \{(\alpha, n) \mid n < 2 \wedge \alpha \in P_n\}$ and apply Axiom 1. □

Theorem 1 (i) $(\mathcal{N}, =) \models \forall \mathsf{x}\neg\forall \mathsf{y}[\mathsf{x} = \mathsf{y} \vee \neg(\mathsf{x} = \mathsf{y})]$.
(ii) $(\mathcal{N}, =) \models \neg\forall \mathsf{x}\forall \mathsf{y}[\mathsf{x} = \mathsf{y} \vee \neg(\mathsf{x} = \mathsf{y})]$.

Proof (i) Let α be given and assume: $\forall\beta[\alpha = \beta \vee \neg(\alpha = \beta)]$.
Using Lemma 1, find m such that
either $\forall\beta[\overline{\alpha}m \sqsubset \beta \to \alpha = \beta]$ *or* $\forall\beta[\overline{\alpha}m \sqsubset \beta \to \neg(\alpha = \beta)]$.
Consider $\beta := \overline{\alpha}m * \langle\alpha(m) + 1\rangle * \underline{0}$ (for the first alternative) and $\beta := \alpha$ (for the second one) and conclude that both alternatives are false.
(ii) This is an immediate consequence of (i).

□

Definition 1 For each n, we let ψ_n be the sentence
$\exists \mathsf{x}_0\exists \mathsf{x}_1 \ldots \exists \mathsf{x}_n[\bigwedge_{i<j<n} \neg(\mathsf{x}_i = \mathsf{x}_j)]$.
$T_{inf} := EQ \cup \{\psi_n \mid n \in \omega\}$.

ψ_n expresses that a set has at least $n + 1$ elements.

Note that, in classical mathematics, T_{inf} has only one complete extension.

Intuitionistically, however, T_{inf} has (at least) two positively different complete extensions, $Th\big((\mathcal{N}, =)\big)$ and $Th\big((\omega, =)\big)$.

The next Theorem reflects the fact that, in classical model theory, all models of T_{inf} are elementarily equivalent.

Theorem 2 *The theory* $T_{inf} \cup \{\forall \mathsf{x}\forall \mathsf{y}[\mathsf{x} = \mathsf{y} \vee \neg(\mathsf{x} = \mathsf{y})]\}$ *has only one complete extension.*

Proof For each n, consider the first n variables of our language: $\mathsf{x}_0, \mathsf{x}_1, \ldots, \mathsf{x}_{n-1}$. A formula $\varepsilon = \varepsilon(\mathsf{x}_0, \mathsf{x}_1, \ldots, \mathsf{x}_{n-1})$ is called an *equality type* if and only if it is of the form $\bigwedge_{i<j<n} \sigma_{ij}$ where each σ_{ij} either is the formula $\mathsf{x}_i = \mathsf{x}_j$ or the formula $\neg(\mathsf{x}_i = \mathsf{x}_j)$.[12] One may prove: for all structures (V_0, R_0), (V_1, R_1), both realizing $T_{inf} \cup \{\forall \mathsf{x}\forall \mathsf{y}[\mathsf{x} = \mathsf{y} \vee \neg(\mathsf{x} = \mathsf{y})]\}$, for each formula $\varphi = \varphi(\mathsf{x}_0, \mathsf{x}_1, \ldots, \mathsf{x}_{n-1})$, for each equality type $\varepsilon = \varepsilon(\mathsf{x}_0, \mathsf{x}_1, \ldots, \mathsf{x}_{n-1})$, $(V_0, R_0) \models \forall \mathsf{x}_0\forall \mathsf{x}_1 \ldots \forall \mathsf{x}_{n-1}[\varepsilon \to \varphi]$ if

[11] See Veldman (2001).

[12] *Inconsistent* equality types may be annoying but do not cause difficulties.

and only if $(V_1, R_1) \models \forall \mathsf{x}_0 \forall \mathsf{x}_1 \dots \forall \mathsf{x}_{n-1}[\varepsilon \to \varphi]$. The proof is by induction on the complexity of the formula φ.

It follows that any two models (V_0, R_0), (V_1, R_1), both realizing $T_{inf} \cup \{\forall \mathsf{x} \forall \mathsf{y}[\mathsf{x} = \mathsf{y} \ \vee \ \neg(\mathsf{x} = \mathsf{y})]\}$, are elementarily equivalent. □

From here on, we restrict attention to infinite models of EQ, that is, to models of T_{inf}. The hackneyed question to make a survey of models that are *finite*, or at least *not infinite*, and of models for which one can not decide if they are finite or infinite, is left for another occasion. That the job is not an easy one will be clear to readers of Veldman (1995).

1.4 Spreads

Definition 2 Let β be given. β is called a *spread-law*, $Spr(\beta)$, if and only if $\forall s[\beta(s) = 0 \leftrightarrow \exists n[\beta(s * \langle n \rangle) = 0]]$.

For every β, we define: $\mathcal{F}_\beta := \{\alpha \mid \forall n[\beta(\overline{\alpha}n) = 0]\}$.

$\mathcal{X} \subseteq \mathcal{N}$ is *closed* if and only if $\exists \beta[\mathcal{X} = \mathcal{F}_\beta]$.

$\mathcal{X} \subseteq \mathcal{N}$ is a *spread* if and only if $\exists \beta[Spr(\beta) \ \wedge \ \mathcal{X} = \mathcal{F}_\beta]$.

If $Spr(\beta)$ and $\beta(\langle \ \rangle) \neq 0$, then $\mathcal{F}_\beta = \emptyset$.

If $Spr(\beta)$ and $\beta(\langle \ \rangle) = 0$, then $\mathcal{F}_\beta$ is inhabited.[13] One may define α such that $\forall n[\alpha(n) = \mu p[\beta(\overline{\alpha}n * \langle p \rangle) = 0]]$ and observe: $\forall n[\beta(\overline{\alpha}n) = 0]$, that is: $\alpha \in \mathcal{F}_\beta$.

Is every closed set a spread?

Define β such that $\forall s[\beta(s) = 0 \leftrightarrow \neg \forall i < 99[d(n + i) = 9]]$, where $d : \mathbb{N} \to \{0, 1, \dots, 9\}$ is the decimal expansion of π.

If $\mathcal{F}_\beta$ is a spread, that is $\exists \gamma[Spr(\gamma) \ \wedge \ \mathcal{F}_\gamma = \mathcal{F}_\beta]$, then *either* $\mathcal{F}_\beta$ is inhabited and $\neg \exists s \forall i < 99[d(s + i) = 9]$ *or* $\mathcal{F}_\beta = \emptyset$ and $\neg\neg \exists s \forall i < 99[d(s + i) = 9]$.

For this β, the statement '$\mathcal{F}_\beta$ is a spread' thus turns out to be reckless.

Brouwer's Continuity Principle enables one to obtain a stronger conclusion.

Theorem 3 $\neg \forall \beta \exists \gamma[Spr(\gamma) \ \wedge \ \mathcal{F}_\gamma = \mathcal{F}_\beta]$.

Proof Assume: $\forall \beta \exists \gamma[Spr(\gamma) \ \wedge \ \mathcal{F}_\gamma = \mathcal{F}_\beta]$.

Then $\forall \beta[\exists \alpha[\alpha \in \mathcal{F}_\beta] \ \vee \ \neg \exists \alpha[\alpha \in \mathcal{F}_\beta]]$. Using Lemma 1, find m such that *either* $\forall \beta[\overline{\underline{0}}m \sqsubset \beta \to \exists \alpha[\alpha \in \mathcal{F}_\beta]]$ *or* $\forall \beta[\overline{\underline{0}}m \sqsubset \beta \to \neg \exists \alpha[\alpha \in \mathcal{F}_\beta]]$.

Both alternatives are false, as we see by considering $\beta = \overline{\underline{0}}m * \underline{1}$ (for the first alternative), and $\beta = \underline{0}$ (for the second one). □

Lemma 2 (Brouwer's Continuity Principle extends to spreads)

Let β be given such that $Spr(\beta)$. Then, for all $R \subseteq \mathcal{N} \times \omega$, if $\forall \alpha \in \mathcal{F}_\beta \exists n[\alpha R n]$, then $\forall \alpha \in \mathcal{F}_\beta \exists m \exists n \forall \gamma \in \mathcal{F}_\beta[\overline{\alpha}m \sqsubset \gamma \to \gamma R n]$.

---

[13] $\mathcal{X} \subseteq \mathcal{N}$ is *inhabited* if and only if $\exists \alpha[\alpha \in \mathcal{X}]$.

Proof Assume: $Spr(\beta)$. If $\beta(\langle\ \rangle) \neq 0$, then $\mathcal{F}_\beta = \emptyset$ and there is nothing to prove.

Assume $\beta(\langle\ \rangle) = 0$. Define σ such that $\sigma(\langle\ \rangle) = \langle\ \rangle$ and, for all s, for all n,

1. if $\beta(s * \langle n\rangle) = 0$, then $\sigma(s * \langle n\rangle) = s * \langle n\rangle$, and,
2. if $\beta(s * \langle n\rangle) \neq 0$, then $\sigma(s * \langle n\rangle) = \sigma(s) * \langle \mu p[\beta(\sigma(s) * \langle p\rangle) = 0]\rangle$.

Note: $\forall s[\beta(\sigma(s)) = 0]$ and $\forall s \forall t[s \sqsubset t \rightarrow \sigma(s) \sqsubset \sigma(t)]$.

Define $\rho : \mathcal{N} \rightarrow \mathcal{N}$ such that $\forall \alpha \forall n[\sigma(\overline{\alpha}n) \sqsubset \rho|\alpha]$.

Note: $\forall \alpha[\rho|\alpha \in \mathcal{F}_\beta] \ \wedge\ \forall \alpha \in \mathcal{F}_\beta[\rho|\alpha = \alpha]$.

The function ρ is called a *retraction* of $\mathcal{N}$ onto $\mathcal{F}_\beta$.

Now assume: $\forall \alpha \in \mathcal{F}_\beta \exists n[\alpha R n]$. Conclude: $\forall \alpha \exists n[(\rho|\alpha) R n]$.

Let α in $\mathcal{F}_\beta$ be given. Using Axiom 1, find m, n such that $\forall \gamma[\overline{\alpha}m \sqsubset \gamma \rightarrow (\rho|\gamma) R n]$. Conclude: $\forall \gamma \in \mathcal{F}_\beta[\overline{\alpha}m \sqsubset \gamma \rightarrow \gamma R n]$.

We thus see: $\forall \alpha \in \mathcal{F}_\beta \exists m \exists n \forall \gamma \in \mathcal{F}_\beta[\overline{\gamma}m \sqsubset \alpha \rightarrow \gamma R n]$. □

Recall that, for all α, β, $\alpha \# \beta \leftrightarrow \alpha \perp \beta \leftrightarrow \exists n[\alpha(n) \neq \beta(n)]$, and $\alpha = \beta \leftrightarrow \forall n[\alpha(n) = \beta(n)] \leftrightarrow \neg(\alpha \# \beta)$, and $\alpha \neq \beta \leftrightarrow \neg \forall n[\alpha(n) = \beta(n)]$.

The constructive *apartness relation* # is more useful than the negative *inequality relation* $\neq$.

Markov's Principle, in the form: $\forall \alpha[\neg\neg\exists n[\alpha(n) = 0] \rightarrow \exists n[\alpha(n) = 0]]$,[14] is equivalent to the statement that the two relations coincide: $\forall \alpha \forall \beta[\alpha \neq \beta \rightarrow \alpha \# \beta]$.

The intuitionistic mathematician does not accept Markov's Principle.

Definition 3 We let $AP = AP(\mathsf{x}, \mathsf{y})$ be the formula $\forall \mathsf{z}[\neg(\mathsf{z} = \mathsf{x}) \ \vee\ \neg(\mathsf{z} = \mathsf{y})]$.

The following theorem reformulates a well-known fact.

Theorem 4 (Apartness is definable) *For all β such that $Spr(\beta)$, for all α, δ in $\mathcal{F}_\beta$, $\alpha \# \delta$ if and only if $(\mathcal{F}_\beta, =) \models AP[\alpha, \delta]$.*

Proof First, assume $\alpha \# \delta$. Find n such that $\overline{\alpha}n \neq \overline{\delta}n$. Note: for every γ in $\mathcal{F}_\beta$, *either*: $\overline{\gamma}n \neq \overline{\alpha}n$ and $\gamma \# \alpha$, *or*: $\overline{\gamma}n \neq \overline{\delta}n$ and $\gamma \# \delta$. Conclude: $(\mathcal{F}_\beta, =) \models AP[\alpha, \delta]$.

Next, assume $(\mathcal{F}_\beta, =) \models AP[\alpha, \delta]$, that is $\forall \gamma \in \mathcal{F}_\beta[\gamma \neq \alpha \ \vee\ \gamma \neq \delta]$.

Applying Lemma 2, find m such that *either* $\forall \gamma \in \mathcal{F}_\beta[\overline{\alpha}m \sqsubset \gamma \rightarrow \gamma \neq \alpha]$ *or* $\forall \gamma \in \mathcal{F}_\beta[\overline{\alpha}m \sqsubset \gamma \rightarrow \gamma \neq \delta]$. The first alternative is clearly wrong (take $\gamma := \alpha$). The second alternative implies: $\overline{\alpha}m \perp \delta$ (if $\overline{\alpha}m \sqsubset \delta$, one could take $\gamma := \delta$), and thus: $\alpha \# \delta$. □

Definition 4 For each n, we let ψ_n^+ be the sentence $\exists \mathsf{x}_0 \exists \mathsf{x}_1 \ldots \exists \mathsf{x}_\mathsf{n}[\bigwedge_{i<j<n} AP(\mathsf{x}_i, \mathsf{x}_j)]$.

$T_{inf}^+ := EQ \cup \{\psi_n^+ \mid n \in \omega\}$.

ψ_n^+ expresses that a set has at least $n + 1$ elements that are mutually apart.

Every model of T_{inf}^+ realizes T_{inf}. In the second part of the paper we will meet a structure that realizes T_{inf} but not T_{inf}^+, see Theorem 17 in Sect. 1.9.

The theory $T_{inf}^+ \cup \{\forall \mathsf{x} \forall \mathsf{y}[\mathsf{x} = \mathsf{y} \ \vee\ \neg(\mathsf{x} = \mathsf{y})]\}$ has only one complete extension, the same as the one and only complete extension of $T_{inf} \cup \{\forall \mathsf{x} \forall \mathsf{y}[\mathsf{x} = \mathsf{y} \ \vee\ \neg(\mathsf{x} = \mathsf{y})]\}$, see Theorem 2.

[14] A. A. Markov enuntiated this principle for *primitive recursive* α only.

1.5 Spreads with a Decidable Equality

Definition 5 We let $D = D(\mathsf{x})$ be the formula: $\forall \mathsf{y}[\mathsf{x} = \mathsf{y} \ \vee \ \neg(\mathsf{x} = \mathsf{y})]$.

Definition 6 Assume $Spr(\beta)$ and $\alpha \in \mathcal{F}_\beta$.

α is an *isolated* point of $\mathcal{F}_\beta$ if and only if $\exists n \forall \gamma \in \mathcal{F}_\beta[\overline{\alpha}n \sqsubset \gamma \rightarrow \alpha = \gamma]$, or, equivalently, $\exists n \forall s[\big(\overline{\alpha}n \sqsubset s \ \wedge \ \beta(s) = 0\big) \rightarrow s \sqsubset \alpha]$.

α is a *decidable* point of $\mathcal{F}_\beta$ if and only if $\forall \gamma \in \mathcal{F}_\beta[\alpha = \gamma \ \vee \ \neg(\alpha = \gamma)]$, or, equivalently, $(\mathcal{F}_\beta, =) \models D[\alpha]$.

$\mathcal{I}(\mathcal{F}_\beta)$ is the set of the isolated points of $\mathcal{F}_\beta$.

Cantor called $\mathcal{I}(\mathcal{F}_\beta)$ the *adherence* of $\mathcal{F}_\beta$.

Lemma 3 *Assume* $Spr(\beta)$.

(i) *For each* α *in* $\mathcal{F}_\beta$*,* α *is an isolated point of* $\mathcal{F}_\beta$ *if and only if* α *is a decidable point of* $\mathcal{F}_\beta$.

(ii) $\mathcal{I}(\mathcal{F}_\beta)$ *is a definable subset of* $\mathcal{F}_\beta$.

Proof (i) Let α be an isolated point of $\mathcal{F}_\beta$.

Find n such that $\forall \gamma \in \mathcal{F}_\beta[\overline{\alpha}n \sqsubset \gamma \rightarrow \alpha = \gamma]$.

Note: for each γ in $\mathcal{F}_\beta$, *either* $\overline{\alpha}n \sqsubset \gamma$ and $\alpha = \gamma$, *or* $\overline{\alpha}n \perp \gamma$ and $\alpha \neq \gamma$.

Conclude: $\forall \gamma \in \mathcal{F}_\beta[\alpha = \gamma \ \vee \ \neg(\alpha = \gamma)]$, that is: α is a decidable point of $\mathcal{F}_\beta$.

Now assume: α is a decidable point of $\mathcal{F}_\beta$, that is:

$\forall \gamma \in \mathcal{F}_\beta[\alpha = \gamma \ \vee \ \neg(\alpha = \gamma)]$.

Apply Lemma 2 and find m such that *either* $\forall \gamma \in \mathcal{F}_\beta[\overline{\alpha}m \sqsubset \gamma \rightarrow \alpha = \gamma]$ *or* $\forall \gamma \in \mathcal{F}_\beta[\overline{\alpha}m \sqsubset \gamma \rightarrow \neg(\alpha = \gamma)]$. As the second alternative does not hold (take $\gamma = \alpha$), conclude: $\forall \gamma \in \mathcal{F}_\beta[\overline{\alpha}m \sqsubset \gamma \rightarrow \alpha = \gamma]$, and: α is an isolated point of $\mathcal{F}_\beta$.

(ii) Using (i), note: $\mathcal{I}(\mathcal{F}_\beta) = \{\alpha \in \mathcal{F}_\beta \mid (\mathcal{F}_\beta, =) \models D[\alpha]\}$.

□

Definition 7 Assume $Spr(\beta)$ and $\alpha \in \mathcal{F}_\beta$.

α is a *limit point* of $\mathcal{F}_\beta$ if and only if $\forall n \exists \delta \in \mathcal{F}_\beta[\overline{\alpha}n \sqsubset \delta \ \wedge \ \alpha \perp \delta]$, or, equivalently, $\forall n \exists s[\overline{\alpha}n \sqsubset s \ \wedge \ \beta(s) = 0 \ \wedge \ \alpha \perp s]$.

$\mathcal{L}(\mathcal{F}_\beta)$ is the set of the limit points of $\mathcal{F}_\beta$.

Cantor called $\mathcal{L}(\mathcal{F}_\beta)$ the *coherence* of $\mathcal{F}_\beta$.

Lemma 4 $\forall \beta[Spr(\beta) \rightarrow \mathcal{L}(\mathcal{F}_\beta) \subseteq \mathcal{F}_\beta \setminus \mathcal{I}(\mathcal{F}_\beta)]$, *that is: in all spreads, every limit point is a non-isolated point.*

Proof Obvious. □

Theorem 5 *The following are equivalent:*

(i) *Markov's Principle:* $\forall \alpha[\neg\neg\exists n[\alpha(n) = 0] \rightarrow \exists n[\alpha(n) = 0]]$.

(ii) $\forall\beta[Spr(\beta) \rightarrow \mathcal{F}_\beta \setminus \mathcal{I}(\mathcal{F}_\beta) \subseteq \mathcal{L}(\mathcal{F}_\beta)]$, *that is:*
in all spreads, every non-isolated point is a limit point.

Proof (i) $\Rightarrow$ (ii). Let β be given such that $Spr(\beta)$. Assume α is not an isolated point of $\mathcal{F}_\beta$, that is: $\neg\exists n\forall s[(\overline{\alpha}n \sqsubset s \ \wedge\ \beta(s) = 0) \rightarrow s \sqsubset \alpha]$.

Let n be given.

Define δ such that $\forall s[\delta(s) = 0 \leftrightarrow (\overline{\alpha}n \sqsubset s \ \wedge\ \beta(s) = 0 \ \wedge\ s \perp \alpha)]$.

Then $\neg\forall s[\delta(s) \neq 0]$ and: $\neg\neg\exists s[\delta(s) = 0]$.

Using *Markov's Principle*, we conclude: $\exists s[\delta(s) = 0]$.

We thus see: $\forall n\exists s[\overline{\alpha}s \sqsubset s \ \wedge\ \beta(s) = 0 \ \wedge\ s \perp \alpha]$, and: α is a limit point of $\mathcal{F}_\beta$.

(ii) $\Rightarrow$ (i). Let us assume: $\forall\beta[Spr(\beta) \rightarrow \mathcal{F}_\beta \setminus \mathcal{I}(\mathcal{F}_\beta) \subseteq \mathcal{L}(\mathcal{F}_\beta)]$,

Let α be given such that $\neg\neg\exists n[\alpha(n) = 0]$.

Define β such that
$\forall s[\beta(s) = 0 \leftrightarrow \forall m < length(s)[s(m) \neq 0 \rightarrow \exists n \leq m[\alpha(n) = 0]]]$.

Note: $Spr(\beta)$ and $\underline{0} \in \mathcal{F}_\beta$, and: if $\exists n[\alpha(n) = 0]$, then $\underline{0}$ is a limit point of $\mathcal{F}_\beta$.

Conclude: if $\underline{0}$ is an isolated point of $\mathcal{F}_\beta$, then $\neg\exists n[\alpha(n) = 0]$.

As $\neg\neg\exists n[\alpha(n) = 0]$, conclude: $\underline{0}$ is not an isolated point of $\mathcal{F}_\beta$.

By our assumption, $\underline{0}$ thus is a limit point of $\mathcal{F}_\beta$.

Find s such that $\beta(s) = 0$ and $s \perp \underline{0}$. Conclude: $\exists n \leq length(s)[\alpha(n) = 0]$.

Conclude: $\forall\alpha[\neg\neg\exists n[\alpha(n) = 0] \rightarrow \exists n[\alpha(n) = 0]]$, that is: Markov's Principle.□

We thus see that the converse of Lemma 4, being equivalent to Markov's Principle, is not an intuitionistic theorem.

We could not answer the question if, in general, $\mathcal{L}(\mathcal{F}_\beta)$ is a definable subset of $(\mathcal{F}_\beta, =)$. In some special cases, however, it is, and the following definition is useful.

Definition 8 Assume $Spr(\beta)$. $\mathcal{F}_\beta$ is called *transparent* if and only if there exists γ such that $Spr(\gamma)$ and $\mathcal{F}_\gamma = \mathcal{L}(\mathcal{F}_\beta)$ and $\forall\alpha \in \mathcal{F}_\beta[\exists n[\gamma(\overline{\alpha}n) \neq 0] \rightarrow \alpha \in \mathcal{I}(\mathcal{F}_\beta)]$.

Note that, for each β such that $Spr(\beta)$, if $\mathcal{F}_\beta$ is transparent, then $\mathcal{F}_\beta \setminus \mathcal{I}(\mathcal{F}_\beta) \subseteq \mathcal{L}(\mathcal{F}_\beta)$. The statement that every spread $\mathcal{F}_\beta$ is transparent thus is seen to imply Markov's Principle.

In Sect. 1.7 we will see many examples of transparent spreads.

The fact that not every spread is a transparent spread is one of the reasons that Brouwer did not succeed in finding a nice intuitionistic version of Cantor's Main Theorem,[15] see Brouwer (1919).

Definition 9 Let β satisfy $Spr(\beta)$ and let φ be given.

We define: $\varphi : \mathcal{F}_\beta \rightarrow \omega$ if and only if $\forall\alpha \in \mathcal{F}_\beta\exists p[\varphi(\overline{\alpha}p) \neq 0]$.

If $\varphi : \mathcal{F}_\beta \rightarrow \omega$, then we define, for each α in $\mathcal{F}_\beta$, $\varphi(\alpha)$ as the number z such that $\varphi(\overline{\alpha}q) = z + 1$, where $q = \mu p[\varphi(\overline{\alpha}p) \neq 0]$.

We define: φ is an injective map from $\mathcal{F}_\beta$ into ω, notation: $\varphi : \mathcal{F}_\beta \hookrightarrow \omega$, if and only if $\varphi : \mathcal{F}_\alpha \rightarrow \omega$ and $\forall\alpha \in \mathcal{F}_\beta\forall\delta \in \mathcal{F}_\beta[\alpha \# \delta \rightarrow \varphi(\alpha) \neq \varphi(\delta)]$.

[15] Cantor's Main Theorem nowadays is called the Perfect Set Theorem: *every closed subset of $\mathcal{N}$ is the union of a perfect set and an at most countable set.*

We define: $\varphi : \mathcal{F}_\beta \to \mathcal{N}$ if and only if $\forall n[\varphi^n : \mathcal{F}_\beta \to \omega]$.

If $\varphi : \mathcal{F}_\beta \to \mathcal{N}$, then we define, for each α in $\mathcal{F}_\beta$, $\varphi|\alpha$ as the element δ of $\mathcal{N}$ such that $\forall n[\delta(n) = \varphi^n(\alpha)]$.

We define: φ is an injective map from $\mathcal{F}_\beta$ into $\mathcal{N}$, notation: $\varphi : \mathcal{F}_\beta \hookrightarrow \mathcal{N}$, if and only if $\varphi : \mathcal{F}_\alpha \to \mathcal{N}$ and $\forall \alpha \in \mathcal{F}_\beta \forall \delta \in \mathcal{F}_\beta[\alpha \# \delta \to \varphi|\alpha \# \varphi|\delta]$.

For every $\mathcal{X} \subseteq \mathcal{N}$, $\mathcal{F}_\beta$ *embeds into* $\mathcal{X}$ if and only if there exists an injective map from $\mathcal{F}_\beta$ into $\mathcal{X}$.

The following axiom is, at least at first sight, a little bit stronger than Brouwer's Continuity Principle.

Axiom 2 (First Axiom of Continuous Choice) For all $R \subseteq \mathcal{N} \times \omega$, if $\forall \alpha \exists n[\alpha R n]$, then $\exists \varphi : \mathcal{N} \to \omega \forall \alpha[\alpha R \varphi(\alpha)]$.

Lemma 5 (The First Axiom of Continuous Choice extends to spreads) *Let β be given such that $Spr(\beta)$. Then, for all $R \subseteq \mathcal{F}_\beta \times \omega$, if $\forall \alpha \in \mathcal{F}_\beta \exists n[\alpha R n]$, then $\exists \varphi : \mathcal{F}_\beta \to \omega \forall \alpha \in \mathcal{F}_\beta[\alpha R \varphi(\alpha)]$.*

Proof Assume: $Spr(\beta)$ and $\beta(\langle\,\rangle) = 0$. As in the proof of Lemma 2, define $\rho : \mathcal{N} \to \mathcal{F}_\beta$ such that $\forall \alpha[\rho|\alpha \in \mathcal{F}_\beta] \wedge \forall \alpha \in \mathcal{F}_\beta[\rho|\alpha = \alpha]$.

Now assume $\forall \alpha \in \mathcal{F}_\beta \exists n[\alpha R n]$. Conclude: $\forall \alpha \exists n[(\rho|\alpha) R n]$.
Applying Axiom 2, find $\varphi : \mathcal{N} \to \omega$ such that $\forall \gamma[(\rho|\gamma) R \varphi(\gamma)]$.
Conclude: $\varphi : \mathcal{F}_\beta \to \omega$ and $\forall \gamma \in \mathcal{F}_\beta[\gamma R \varphi(\gamma)]$. □

Theorem 6 *Assume $Spr(\beta)$. $(\mathcal{F}_\beta, =) \models \forall \mathsf{x}[D(\mathsf{x})]$ if and only if $\exists \varphi[\varphi : \mathcal{F}_\beta \hookrightarrow \omega]$.*

Proof First assume: $(\mathcal{F}_\beta, =) \models \forall \mathsf{x}[D(\mathsf{x})]$. Then, by Lemma 3, $\forall \alpha \in \mathcal{F}_\beta \exists n \forall \gamma \in \mathcal{F}_\beta[\overline{\alpha} n \sqsubset \gamma \to \alpha = \gamma]$. Using Lemma 5, find $\varphi : \mathcal{F}_\beta \to \omega$ such that $\forall \alpha \in \mathcal{F}_\beta \forall \gamma \in \mathcal{F}_\beta[\overline{\alpha}\varphi(\alpha) \sqsubset \gamma \to \alpha = \gamma]$.
Define $\psi : \mathcal{F}_\beta \to \omega$ such that $\forall \alpha \in \mathcal{F}_\beta[\psi(\alpha) = \overline{\alpha}\varphi(\alpha)]$. Clearly, $\psi : \mathcal{F}_\odot \hookrightarrow \omega$.

Now assume: $\varphi : \mathcal{F}_\beta \hookrightarrow \omega$. Note: $\forall \alpha \in \mathcal{F}_\beta \forall \delta \in \mathcal{F}_\beta[\alpha = \delta \leftrightarrow \varphi(\alpha) = \varphi(\delta)]$.
Also: $\forall \alpha \in \mathcal{F}_\beta \forall \delta \in \mathcal{F}_\beta[\varphi(\alpha) = \varphi(\delta) \vee \neg\big(\varphi(\alpha) = \varphi(\delta)\big)]$.
Therefore: $\forall \alpha \in \mathcal{F}_\beta \forall \delta \in \mathcal{F}_\beta[\alpha = \delta \vee \neg(\alpha = \delta)]$. Conclude: $(\mathcal{F}_\beta, =) \models \forall \mathsf{x} [D(\mathsf{x})]$. □

Definition 10 Assume $Spr(\beta)$. $\mathcal{F}_\beta$ is *enumerable* if and only if either $\mathcal{F}_\beta = \emptyset$ or $\exists \delta[\forall n[\delta^n \in \mathcal{F}_\beta] \wedge \forall \alpha \in \mathcal{F}_\beta \exists n[\alpha = \delta^n]]$.

Lemma 6 *Assume $Spr(\beta)$. $\mathcal{F}_\beta$ is enumerable if and only if $\exists \varphi[\varphi : \mathcal{F}_\beta \hookrightarrow \omega]$.*

Proof Assume $\mathcal{F}_\beta$ is enumerable and $\beta(\langle\,\rangle) = 0$.
Find δ such that $\forall n[\delta^n \in \mathcal{F}_\beta]$ and $\forall \alpha \in \mathcal{F}_\beta \exists n[\alpha = \delta^n]$.
Using Lemma 5, find $\varphi : \mathcal{F}_\beta \to \omega$ such that $\forall \alpha \in \mathcal{F}_\beta[\alpha = \delta^{\varphi(\alpha)}]$.
Note: $\varphi : \mathcal{F}_\beta \hookrightarrow \omega$.

Now assume: $\varphi : \mathcal{F}_\beta \hookrightarrow \omega$.
We make a preliminary observation.

Let s, n be given such that $\beta(s) = 0$ and $\varphi(s) = n + 1$ and $\forall t \sqsubset s[\varphi(t) = 0]$. Note: $\forall \alpha \in \mathcal{F}_\beta[s \sqsubset \alpha \to \varphi(\alpha) = n]$ and, therefore: $\forall \alpha \in \mathcal{F}_\beta \forall \delta \in \mathcal{F}_\beta[(s \sqsubset \alpha \wedge s \sqsubset \delta) \to \alpha = \delta]$.

Now let γ be the element of $\mathcal{F}_\beta$ satisfying $\forall n[\gamma(n) := \mu p[\beta(\overline{\gamma}n * \langle p \rangle) = 0]]$. Define δ such that, for all s, *if* $\beta(s) = 0$ and $\varphi(s) \neq 0$ and $\forall t \sqsubset s[\varphi(t) = 0]$, then $s \sqsubset \delta^s$ and $\delta^s \in \mathcal{F}_\beta$, and *if not*, then $\delta^s = \gamma$.

Note: $\forall s[\delta^s \in \mathcal{F}_\beta]$ and $\forall \alpha \in \mathcal{F}_\beta \exists s[\alpha = \delta^s]$. □

Corollary 1 *Assume* $Spr(\beta)$.
$(\mathcal{F}_\beta, =) \models \forall \mathsf{x}[D(\mathsf{x})]$ *if and only if* $\mathcal{F}_\beta$ *is enumerable.*

Proof Use Theorem 6 and Lemma 6. □

1.6 Spreads with Exactly One Undecidable Point

Definition 11 We let τ_2 be the element of $\mathcal{C}$ satisfying: $\forall s[\tau_2(s) = 0 \leftrightarrow \forall i < length(s)[s(i) < 2 \wedge \big(i + 1 < length(s) \to s(i) \leq s(i+1)\big)]]$. We define: $\mathcal{T}_2 := \mathcal{F}_{\tau_2}$.

Note: τ_2 is a spread-law and $\mathcal{T}_2$ is a spread.

Let us take a closer look at $\mathcal{T}_2$.

Observe: $\forall \alpha[\alpha \in \mathcal{T}_2 \leftrightarrow \forall i[\alpha(i) \leq \alpha(i+1) < 2]]$.

For each n, we define $n^* := \overline{\underline{0}}n * \underline{1}$.

The infinite sequence $\underline{0}, 0^*, 1^*, 2^*, \ldots$ is a list of elements of $\mathcal{T}_2$ and a classical mathematician might think it is the list of all elements of $\mathcal{T}_2$. The intuitionistic mathematician knows better. He defines α in $\mathcal{T}_2$ such that

$$\forall n[\alpha(n) = 1 \leftrightarrow \exists k \leq n \forall i < 99[d(k+i) = 9]],$$

where $d : \mathbb{N} \to \{0, 1, \ldots, 9\}$ is the decimal expansion of π. As yet, one has no proof of the statement '$\alpha = \underline{0}$', as this statement implies: $\forall k \exists i < 99]d(k+i) = 9]$. As yet, one also has no proof of the statement: '$\exists n[\alpha = n^*]$' as this statement implies: $\exists n \forall i < 99[d(n+i) = 9]$. The statement that α occurs in the above list is a reckless one.

For each n, n^* is an isolated and a decidable point of $\mathcal{T}_2$, and $\underline{0}$ is a non-isolated and an undecidable point of $\mathcal{T}_2$. It follows, by Lemma 3 and Corollary 1, that $\mathcal{T}_2$ is not an enumerable spread. In particular, the statement that the list $\underline{0}, 0^*, 1^*, 2^*, \ldots$ is a complete list of the elements of $\mathcal{T}_2$, leads to a contradiction, as appears again from the following Theorem.

Theorem 7 (i) $\neg \forall \alpha \in \mathcal{T}_2[\alpha = \underline{0} \vee \exists n[\alpha = n^*]]$.
(ii) $\forall \alpha \in \mathcal{T}_2[\alpha \mathrel{\#} \underline{0} \to \exists n[\alpha = n^*]]$.

Proof (i) Assume $\forall\alpha \in \mathcal{T}_2[\alpha = \underline{0} \vee \exists n[\alpha = n^*]]$. Using Lemma 2, find m, n such that *either* $\forall\alpha \in \mathcal{T}_2[\overline{\underline{0}}m \sqsubset \alpha \rightarrow \alpha = \underline{0}]$ *or* $\forall\alpha \in \mathcal{T}_2[\overline{\underline{0}}m \sqsubset \alpha \rightarrow \alpha = n^*]$. Note that both alternatives are false.
Conclude: $\neg\forall\alpha \in \mathcal{T}_2[\alpha = \underline{0} \vee \exists n[\alpha = n^*]]$.

(ii) Let α in $\mathcal{T}_2$ be given such that $\alpha \# \underline{0}$. Define $n := \mu m[\overline{\alpha}(m+1) \perp \underline{0}]$. Note: $\overline{\alpha}(n+1) = \overline{\underline{0}}n * \langle 1\rangle$ and $\alpha = n^*$.

Definition 12 Assume $Spr(\beta)$. $\mathcal{F}_\beta$ is *almost-enumerable* if and only if either $\mathcal{F}_\beta = \emptyset$ or $\exists\delta[\forall n[\delta^n \in \mathcal{F}_\beta] \wedge \forall\alpha \in \mathcal{F}_\beta\forall\varepsilon\exists n[\overline{\alpha}\varepsilon(n) = \overline{\delta^n}\varepsilon(n)]]$.

This definition deserves some explanation. If $\mathcal{F}_\beta$ is almost-enumerable and inhabited, we are able to come forward with an infinite sequence $\delta^0, \delta^1, \ldots$ of elements of $\mathcal{F}_\beta$ such that, for every α in $\mathcal{F}_\beta$, every attempt ε to prove that α is apart from all elements of the infinite sequence $\delta^0, \delta^1, \ldots$, ($\varepsilon$ expresses the guess: $\forall n[\overline{\alpha}\varepsilon(n) \perp \overline{\delta^n}\varepsilon(n)]$), will positively fail.

Almost-enumerable spreads are studied in Veldman (2018, Sect. 9), where they are called *almost-countable located and closed subsets of $\mathcal{N}$*.

Theorem 8 *$\mathcal{T}_2$ is almost-enumerable.*

Proof Define δ such that $\delta^0 = \underline{0}$ and, for each n, $\delta^{n+1} = n^* = \overline{\underline{0}}n * \underline{1}$. Note: $\forall n[\delta^n \in \mathcal{T}_2]$. Let ε be given. If $\overline{\alpha}\varepsilon(0) = \overline{\delta^0}\varepsilon(0)$, we are done. If not, then $\alpha \perp \underline{0}$ and we may determine n such that $\alpha = \delta^{n+1}$ and $\overline{\alpha}\varepsilon(n+1) = \overline{\delta^{n+1}}\varepsilon(n+1)$. □

Axiom 3 (Second Axiom of Countable Choice)
For every $R \subseteq \mathbb{N} \times \mathcal{N}$, if $\forall n\exists\alpha[nR\alpha]$, then $\exists\alpha\forall n[nR\alpha^n]$.

Theorem 9 (i) $(\mathcal{T}_2, =) \models \exists \mathsf{x}[\neg D(\mathsf{x}) \wedge \forall \mathsf{y}[AP(\mathsf{x}, \mathsf{y}) \rightarrow D(\mathsf{y})]]$.

(ii) *For all β such that $Spr(\beta)$,*
if $(\mathcal{F}_\beta, =) \models \exists \mathsf{x}[\neg D(\mathsf{x}) \wedge \forall \mathsf{y}[AP(\mathsf{x}, \mathsf{y}) \rightarrow D(\mathsf{y})]]$, then $\mathcal{F}_\beta$ embeds into $\mathcal{T}_2$.

Proof (i) $\underline{0}$ is not an isolated point of $\mathcal{T}_2$, and, therefore, not a decidable point of $\mathcal{T}_2$. Also, by Theorem 7 (ii), $\forall\alpha \in \mathcal{T}_2[\alpha \# \underline{0} \rightarrow \exists n[\alpha = n^*]]$, and, for each n, for each α in $\mathcal{T}_2$, $\alpha = n^* \leftrightarrow \overline{\underline{0}}n * \langle 1\rangle \sqsubset \alpha$, so one may decide: $\alpha = n^*$ or $\neg(\alpha = n^*)$, and: n^* is a decidable point of $\mathcal{T}_2$.
We thus see: $(\mathcal{T}_2, =) \models \neg D(\mathsf{x}) \wedge \forall \mathsf{y}[AP(\mathsf{x}, \mathsf{y}) \rightarrow D(\mathsf{y})][\underline{0}]$, and are done.

(ii) Assume: $Spr(\beta)$ and $(\mathcal{F}_\beta, =) \models \exists \mathsf{x}[\neg D(\mathsf{x}) \wedge \forall \mathsf{y}[AP(\mathsf{x}, \mathsf{y}) \rightarrow D(\mathsf{y})]]$.
Find α in $\mathcal{F}_\beta$ such that α is not an isolated point of $\mathcal{F}_\beta$.
Note: for each s such that $\beta(s) = 0$, the set $\mathcal{F}_\beta \cap s := \{\delta \in \mathcal{F}_\beta \mid s \sqsubset \delta\}$ is a spread, and, if $s \perp \alpha$, then $\mathcal{F}_\beta \cap s$ consists of isolated points of $\mathcal{F}_\beta \cap s$ only, and thus, by Theorem 6, embeds into ω.
Using Axiom 3, we find φ such that, for each s, if $\beta(s) = 0$ and there exist n, i such that $s = \overline{\alpha}n * \langle i\rangle$ and $i \neq \alpha(n)$, then $\varphi^s : \mathcal{F}_\beta \cap s \hookrightarrow \omega$.
We now define $\psi : \mathcal{F}_\beta \rightarrow \mathcal{T}_2$ such that $\psi|\alpha = \underline{0}$ and, for each δ in $\mathcal{F}_\beta$, if $\delta \# \alpha$, then $\psi|\delta = \overline{\underline{0}}\big(\overline{\delta}n, \varphi^{\overline{\delta}n}(\delta)\big) * \underline{1}$ where $n := \mu i[\overline{\delta}i \perp \alpha]$. □

1.7 More and More Undecidable Points: The Toy Spreads

Definition 13 For each n, we let τ_n be the element of $\mathcal{C}$ satisfying: $\forall s[\tau_n(s) = 0 \leftrightarrow \forall i < length(s)[s(i) < n \wedge \big(i + 1 < length(s) \rightarrow s(i) \leq s(i + 1)\big)]]$.
We also define: $\mathcal{T}_n := \mathcal{F}_{\tau_n}$.

For each n, τ_n is a spread-law and $\mathcal{T}_n$ and $\mathcal{T}_n = \{\alpha \mid \forall i[\alpha(i) \leq \alpha(i + 1) < n]\}$ is a spread.

In this paper, the spreads $\mathcal{T}_0, \mathcal{T}_1, \ldots$ will be called the *toy spreads*.

Note: $\mathcal{T}_0 = \emptyset$ and $\mathcal{T}_1 = \{\underline{0}\}$.

Definition 14 For each $s \neq \langle\,\rangle$, we let $s^\dagger$ be the element of $\mathcal{N}$ satisfying $s \sqsubset s^\dagger$ and $\forall i \geq length(s)[s^\dagger(i) = s^\dagger(i - 1)]$.

Note that, for each n, for each s, if $s \neq \langle\,\rangle$ and $\tau_n(s) = 0$, then $s^\dagger \in \mathcal{T}_n$.

Theorem 10 *For each $n > 0$, $\mathcal{T}_n$ is almost-enumerable.*

Proof Let $n > 0$ be given. Define δ such that, for each s, *if* $s \neq \langle\,\rangle$ and $\tau_n(s) = 0$, then $\delta^s = s^\dagger$, and *if not*, then $\delta^s = \underline{0}$.

We claim: $\forall \alpha \in \mathcal{T}_n \forall \varepsilon \exists s[\overline{\alpha}\varepsilon(s) = \overline{\delta^s}\varepsilon(s)]$.

We establish this claim by proving, for each $k < n$,
$\forall \alpha \in \mathcal{T}_n[\exists i[\alpha(i) \geq k] \rightarrow \forall \varepsilon \exists s[\overline{\alpha}\varepsilon(s) = \overline{\delta^s}\varepsilon(s)]]$, and we do so by backwards induction, starting with the case $k = n - 1$.

The case $k = n - 1$ is treated as follows. If $\exists i[\alpha(i) = n - 1]$, find $i_0 := \mu i[\alpha(i) = n - 1]$ and consider $s := \overline{\alpha}(i_0 + 1)$.
Note: $\alpha = s^\dagger = \delta^s$ and, therefore, for every ε: $\overline{\alpha}\varepsilon(s) = \overline{\delta^s}\varepsilon(s)$.

Now assume $k < n - 1$ is given such that
$\forall \alpha \in \mathcal{T}_n[\exists i[\alpha(i) \geq k + 1] \rightarrow \forall \varepsilon \exists s[\overline{\alpha}\varepsilon(s) = \overline{\delta^s}\varepsilon(s)]]$.

We have to prove: $\forall \alpha \in \mathcal{T}_n[\exists i[\alpha(i) = k] \rightarrow \forall \varepsilon \exists s[\overline{\alpha}\varepsilon(s) = \overline{\delta^s}\varepsilon(s)]]$.

Let α be given such that $\exists i[\alpha(i) = k]$. Let also ε be given.

Define $i_0 := \mu i[\alpha(i) = k]$ and define $s := \overline{\alpha}(i_0 + 1)$.

There are two cases to consider.

Case (i): $\overline{\alpha}\varepsilon(s) = \overline{s^\dagger}\varepsilon(s) = \overline{\delta^s}\varepsilon(s)$. We are done.

Case (ii): $\overline{\alpha}\varepsilon(s) \perp \overline{s^\dagger}\varepsilon(s)$. Then $\exists i < \varepsilon(s)[\alpha(i) \geq k + 1]$.

Using the induction hypothesis, we conclude: $\exists s[\overline{\alpha}\varepsilon(s) = \overline{\delta^s}\varepsilon(s)]$. □

Theorem 11 (i) *For each n, for all α in $\mathcal{T}_n$, $\alpha \in \mathcal{I}(\mathcal{T}_n)$ if and only if $\exists m[\alpha(m) + 1 = n]$.*

(ii) *For each n, $\mathcal{T}_{n+1} \setminus \mathcal{I}(\mathcal{T}_{n+1}) = \mathcal{T}_n = \mathcal{L}(\mathcal{T}_{n+1})$.*

(iii) *For each n, $\mathcal{T}_n = \{\alpha \in \mathcal{T}_{n+1} \mid (\mathcal{T}_{n+1}, =) \models \neg D[\alpha]\}$.*

Proof The proof uses Lemma 3 and is left to the reader. □

Definition 15 We define an infinite sequence $D_0, D_1, \ldots$ of formulas, as follows.

$D_0 := \forall \mathsf{y}[\mathsf{x} = \mathsf{y} \vee \neg(\mathsf{x} = \mathsf{y})]$,

$D_1 := \neg D_0(\mathsf{x}) \wedge \forall \mathsf{y}[\neg D_0(\mathsf{y}) \rightarrow \big(\mathsf{x} = \mathsf{y} \vee \neg(\mathsf{x} = \mathsf{y})\big)]$,

$D_2 := \neg D_0(\mathsf{x}) \wedge \neg D_1(\mathsf{x}) \wedge$
$\forall \mathsf{y}[(\neg D_0(\mathsf{y}) \wedge \neg D_1(\mathsf{y})) \rightarrow (\mathsf{x}=\mathsf{y} \vee \neg(\mathsf{x}=\mathsf{y}))]$,
and, more generally for each $m > 0$,
$D_m := \bigwedge_{i<m} \neg D_i(\mathsf{x}) \wedge \forall \mathsf{y}[(\bigwedge_{i<m} \neg D_i(\mathsf{y})) \rightarrow (\mathsf{x}=\mathsf{y} \vee \neg(\mathsf{x}=\mathsf{y}))]$.
We also define, for each $m > 0$, sentences ψ_m and ρ_m, as follows:
$\psi_m := \exists \mathsf{x}[D_m(\mathsf{x})]$ and $\rho_m := \exists \mathsf{x}[D_m(\mathsf{x}) \wedge \forall \mathsf{y}[D_m(\mathsf{y}) \rightarrow \mathsf{y}=\mathsf{x}]]$.

Definition 16 Assume $Spr(\beta)$.
α in $\mathcal{F}_\beta$ is a *limit point of order* 0 of $\mathcal{F}_\beta$ if and only if α is an isolated point of $\mathcal{F}_\beta$.
For each m, α is a *limit point of order* $m+1$ of $\mathcal{F}_\beta$ if and only if, for each p, there exists a limit point γ of order m such that $\overline{\alpha}p \sqsubset \gamma$ and $\alpha \perp \gamma$.

Assume $n > 0$ and $\alpha \in \mathcal{T}_n$. Note the following:

1. $(\mathcal{T}_n, =) \models D_0[\alpha]$ if and only if α is an isolated point of $\mathcal{T}_n$ if and only if either: $n = 1$ or: $n > 1$ and $\exists p[\alpha(p) = n-1]$.
2. $(\mathcal{T}_n, =) \models \neg D_0[\alpha]$ if and only if α is a limit point (of order 1) of $\mathcal{T}_n$ if and only if $n > 1$ and $\alpha \in \mathcal{T}_{n-1}$.
3. $(\mathcal{T}_n, =) \models D_1[\alpha]$ if and only if α is an isolated point among the limit points (of order 1) of $\mathcal{T}_n$ if and only if $n > 1$ and $\alpha \in \mathcal{T}_{n-1}$ and $\exists p[\alpha(p) = n-2]$.
4. $(\mathcal{T}_n, =) \models \neg D_0 \wedge \neg D_1[\alpha]$, if and only if α is a limit point of order 2 of $\mathcal{T}_n$ if and only if $n > 2$ and $\alpha \in \mathcal{T}_{n-2}$.
5. For each $m > 0$, $(\mathcal{T}_n, =) \models D_2[\alpha]$ if and only if α is an isolated point among the limit points of order 2 if and only if $n > 2$ and $\alpha \in \mathcal{T}_{n-2}$ and $\exists p[\alpha(p) = n-3]$.
6. For each $m > 0$, $(\mathcal{T}_n, =) \models \bigwedge_{i<m} \neg D_i[\alpha]$ if and only if α is a limit point of order m of $\mathcal{T}_n$ if and only if $n > m$ and $\alpha \in \mathcal{T}_{n-m}$.
7. For each $m > 0$, $(\mathcal{T}_n, =) \models D_m[\alpha]$ if and only if α is an isolated point among the limit points of order m if and only if $n > m$ and $\alpha \in \mathcal{T}_{n-m}$ and $\exists p[\alpha(p) = n-m-1]$.
8. For each $m > 0$, $\mathcal{T}_n \models \psi_m$ if and only if $\mathcal{T}_n$ contains an isolated point of $\mathcal{T}_{n-m}$ if and only if $n > m$.
9. For each $m > 0$, $\mathcal{T}_n \models \rho_m$ if and only if $\mathcal{T}_n$ contains exactly one isolated point of $\mathcal{T}_{n-m}$ if and only if $\mathcal{T}_{n-m} = \{\underline{0}\}$ if and only if $n = m+1$.

After these preliminary observations, the following Theorem is easy to understand:

Theorem 12 (i) *For each* n, $\mathcal{T}_n$ *is a transparent*[16] *spread and, if* $n > 0$, *then* $\mathcal{I}(\mathcal{T}_n) = \{\alpha \in \mathcal{T}_n \mid \exists p[\alpha(p) + 1 = n]\}$ *and* $\mathcal{L}(\mathcal{T}_n) = \mathcal{T}_{n-1}$.
(ii) *For all* n, *for all* $m > 0$, $\mathcal{T}_n = \{\alpha \in \mathcal{T}_{n+m} \mid (\mathcal{T}_{n+m}, =) \models \bigwedge_{i<m} \neg D_i[\alpha]\}$.
(iii) *For all* m, $\{\underline{0}\} = \mathcal{T}_1 = \{\alpha \in \mathcal{T}_{m+1} \mid (\mathcal{T}_{m+1}, =) \models \bigwedge_{i<m} \neg D_i[\alpha]\}$.
(iv) *For all* $n > 0$, *for all* $m > 0$, $(\mathcal{T}_n, =) \models \psi_m$ *if and only if* $m+1 \leq n$.
(v) *For all* $n > 0$, *for all* $m > 0$, $(\mathcal{T}_n, =) \models \rho_m$ *if and only if* $m+1 = n$.

Proof Use the preliminary observations preceding this Theorem. □

[16] See Definition 8.

Corollary 2 *For all n, m, if $n \neq m$, then there exists a sentence ψ such that $(\mathcal{T}_m, =) \models \psi$ and $(\mathcal{T}_n, =) \models \neg\psi$.*

1.8 Finite and Infinite Sums of Toy Spreads

1.8.1 A Main Result

Definition 17 Assume $Spr(\beta)$, $Spr(\gamma)$.

We define: $\mathcal{F}_\beta \uplus \mathcal{F}_\gamma := \{\langle 0\rangle * \delta \mid \delta \in \mathcal{F}_\beta\} \cup \{\langle 1\rangle * \delta \mid \delta \in \mathcal{F}_\gamma\}$.

For each m, we define: $m \otimes \mathcal{F}_\beta := \{\langle i\rangle * \delta \mid i < m, \delta \in \mathcal{F}_\beta\}$.

We also define: $\omega \otimes \mathcal{F}_\beta := \{\langle i\rangle * \delta \mid i \in \omega, \delta \in \mathcal{F}_\beta\}$.

Note that $\mathcal{F}_\beta \uplus \mathcal{F}_\gamma$, $m \otimes \mathcal{F}_\beta$ and $\omega \otimes \mathcal{F}_\beta$ are spreads again.

We also define, for all $m, n > 0$, sentences ψ^n_m and ρ^n_m, as follows:

$\psi^n_m := \exists \mathsf{x}_0 \exists \mathsf{x}_1 \ldots \exists \mathsf{x}_{n-1}[\bigwedge_{i<j<n}[AP(\mathsf{x}_i, \mathsf{x}_j) \ \wedge \ \bigwedge_{i<n}\bigwedge_{j<m} \neg D_j(\mathsf{x}_i)]$.

and $\quad \rho^n_m := \exists \mathsf{x}_0 \exists \mathsf{x}_1 \ldots \exists \mathsf{x}_{n-1}[\bigwedge_{i<j<n}[AP(\mathsf{x}_i, \mathsf{x}_j) \ \wedge \ \bigwedge_{i<n}\bigwedge_{j<m} \neg D_j(\mathsf{x}_i) \ \wedge \ \forall \mathsf{z}[\bigwedge_{j<m} \neg D_j(\mathsf{z}) \to \bigvee_{i<n} \mathsf{z} = \mathsf{x}_i]]$.

The sentence ψ^n_m expresses: '*there exist (at least) n limit points of order m that are mutually apart*'.

The sentence ρ^n_m expresses: '*there exist exactly n limit points of order m that are mutually apart*'.

Theorem 13 (i) *For all $m, n, p, q > 0$,*

$(n \otimes \mathcal{T}_m, =) \models \psi^q_p$ if and only if either: $p + 1 < m$ or: $p + 1 = m$ and $q \leq n$.

(ii) *For all $m, n, p, q > 0$, $(n \otimes \mathcal{T}_m, =) \models \rho^q_p$ if and only if $p + 1 = m$ and $n = q$.*

(iii) *For all $m, p, q > 0$, $(\omega \otimes \mathcal{T}_m, =) \models \psi^q_p$ if and only if $p < m$.*

Proof (i) Note the following:

If $p + 1 < m$ and $n > 0$, then $\mathcal{T}_m$ and also $n \otimes \mathcal{T}_m$ contain infinitely many limit points of order p that are mutually apart.

If $p + 1 = m$ and $n > 0$, then $n \otimes \mathcal{T}_m$ contains exactly n limit points of order p that are mutually apart: the points $\langle i\rangle * \underline{0}$, where $i < n$, so $(n \otimes \mathcal{T}_m, =) \models \psi^q_p$ if and only if $q \leq n$.

If $p < m$, then $\omega \times \mathcal{T}_m$ contains infinitely many limit points of order p that are mutually apart.

The proofs of (i), (ii) and (iii) follow easily from these observations. □

Definition 18 For each k, for each s in ω^k, we define: $\mathcal{V}_s = \bigcup_{i<k}\{\langle i\rangle * \delta \mid \delta \in \mathcal{T}_{s(i)}\}$.

Definition 19 Let $\mathcal{F}_0, \mathcal{F}_1 \subseteq \mathcal{N}$ and assume $\varphi : \mathcal{F}_0 \to \mathcal{F}_1$.

φ is a *(surjective)* map from $\mathcal{F}_0$ *onto* $\mathcal{F}_1$ if and only if $\forall \beta \in \mathcal{F}_1 \exists \alpha \in \mathcal{F}_0[\varphi|\alpha = \beta]$.

$\mathcal{F}_0$ *is equivalent to* $\mathcal{F}_1$, notation: $\mathcal{F}_0 \sim \mathcal{F}_1$, if and only if there exists $\varphi : \mathcal{F}_0 \to \mathcal{F}_1$ that is both injective[17] and surjective.

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[17] See Definition 9.

Theorem 14 (i) *For each m,* $\mathcal{T}_m \oplus \mathcal{T}_{m+1} \sim \mathcal{T}_{m+1}$.
(ii) *For all m, n, if* $m < n$*, then* $\mathcal{T}_m \oplus \mathcal{T}_n \sim \mathcal{T}_n$.
(iii) *For all k, for all s in* ω^k*, there exist* m, n *such that* $\mathcal{V}_s \sim n \otimes \mathcal{T}_m$.

Proof (i) Let m be given. Define $\varphi : \mathcal{T}_m \oplus \mathcal{T}_{m+1} \to \mathcal{T}_{m+1}$ such that, for all δ in $\mathcal{T}_m$, $\varphi|\langle 0\rangle * \delta = \langle 1\rangle * S \circ \delta$, and, for each δ in $\mathcal{T}_{m+1}$, $\varphi|\langle 1\rangle * \delta = \langle 0\rangle * \delta$. Clearly, φ is a one-to-one function mapping $\mathcal{T}_m \oplus \mathcal{T}_{m+1}$ onto $\mathcal{T}_{m+1}$.
(ii) Let m be given. We use induction on n. The case $n = m + 1$ has been treated in (i). Now let n be given such that $m < n$ and $\mathcal{T}_m \oplus \mathcal{T}_n \sim \mathcal{T}_n$.
Then $\mathcal{T}_m \oplus \mathcal{T}_{n+1} \sim \mathcal{T}_m \oplus (\mathcal{T}_n \oplus \mathcal{T}_{n+1}) \sim (\mathcal{T}_m \oplus \mathcal{T}_n) \oplus \mathcal{T}_{n+1} \sim \mathcal{T}_n \oplus \mathcal{T}_{n+1} \sim \mathcal{T}_{n+1}$.
(iii) We use induction on k. If $s \in \omega^0$, then $s = \langle\,\rangle$ and $\emptyset = \mathcal{V}_s = 0 \otimes \mathcal{T}_1$.
Now let k be given such that,
for all s in ω^k, there exist m, n such that $\mathcal{V}_s = n \otimes \mathcal{T}_m$.
Let $s = t * \langle p\rangle$ in ω^{k+1} be given. Find m, n such that $\mathcal{V}_t = n \otimes \mathcal{T}_m$.
Note: $\mathcal{V}_s \sim \mathcal{V}_t \oplus \mathcal{T}_p$ and consider several cases.
Case (1): $t = \langle\,\rangle$. Then $\mathcal{V}_s = 1 \otimes \mathcal{T}_p$.
Case (2): $t \neq \langle\,\rangle$ and $p < m$. Then, by (ii): $\mathcal{V}_s \sim \mathcal{V}_t \sim n \otimes \mathcal{T}_m$.
Case (3): $t \neq \langle\,\rangle$ and $p = m$. Then: $\mathcal{V}_s \sim \mathcal{V}_t \oplus \mathcal{T}_m \sim (n + 1) \otimes \mathcal{T}_m$.
Case (4): $t \neq \langle\,\rangle$ and $p > m$. Then, by (ii):

$$\mathcal{V}_s \sim \mathcal{V}_t \oplus \mathcal{T}_p \sim \underbrace{\mathcal{T}_m \oplus \ldots \oplus \mathcal{T}_m}_{n} \oplus \mathcal{T}_p \sim \mathcal{T}_p \sim 1 \otimes \mathcal{T}_p.$$

□

Definition 20 For each α, we define: $\mathcal{V}_\alpha := \bigcup_i \{\langle i\rangle * \delta \mid \delta \in \mathcal{T}_{\alpha(i)}\}$.

Theorem 15 (EQ has continuum many complete extensions[18])

(i) *For each* α, $\mathcal{I}(\mathcal{V}_\alpha) = \bigcup_i \{\langle i\rangle * \delta \mid \delta \in \mathcal{T}_{\alpha(i)} \ \wedge\ \exists p[\delta(p) + 1 = \alpha(i)]\}$.
(ii) *For all* α*, for all n,* $(\mathcal{V}_\alpha, =) \models \psi_n$ *if and only if* $\exists i[\alpha(i) > n]$.
(iii) *For all* α*, for all n,* $(\mathcal{V}_\alpha, =) \models \rho_n$ *if and only if*
$\exists i[\alpha(i) = n + 1 \ \wedge\ \forall j[\alpha(j) = n + 1 \to i = j]]$.
(iv) *For all* ζ, η *in* $[\omega]^\omega$*, if* $\zeta \perp \eta$ *and* $\zeta(0) = \eta(0) = 2$*,*
then there exists a sentence ψ *such that* $(\mathcal{V}_\zeta, =) \models \psi$ *and* $(\mathcal{V}_\eta, =) \models \neg\psi$.

Proof (i) Use Theorem 12 (i).
(ii) Note that, for each α, for each n, $(\mathcal{V}_\alpha, =) \models \psi_n$ if and only if $\mathcal{V}_\alpha$ contains a limit point of order n if and only if $\exists i[\alpha(i) > n]$.
(iii) Note that, for each α, for each n, $(\mathcal{V}_\alpha, =) \models \rho_n$ if and only if $\mathcal{V}_\alpha$ contains exactly one limit point of order n if and only if
$\exists i[\alpha(i) = n + 1 \ \wedge\ \forall j[\alpha(j) = n + 1 \to i = j]]$.
(iv) Using (iii), note that, for all ζ in $[\omega]^\omega$, if $\zeta(0) > 1$, then $\forall n[(\mathcal{V}_\zeta, =) \models \rho_n$ if and only if $\exists p[\zeta(p) = n + 1]$.

[18]Note that there exists an embedding $\rho : \mathcal{C} \hookrightarrow \{\zeta \in [\omega]^\omega \mid \zeta(0) = 2\}$.

Conclude that, for all ζ, η in $[\omega]^\omega$, for all p, if $\zeta(0) = \eta(0) = 2$ and $\zeta \perp \eta$ and $p := \mu i[\zeta(i) \neq \eta(i)]$ and $\zeta(p) < \eta(p)$, then $\neg\exists i[\eta(i) = \zeta(p)]$, and, therefore, $(\mathcal{V}_\zeta, =) \models \psi_{\zeta(p)-1}$ and $(\mathcal{V}_\eta, =) \models \neg\psi_{\zeta(p)-1}$.

□

1.8.2 Finitary Spreads Suffice

Definition 21 Assume $Spr(\beta)$. β is called a *finitary spread-law* or a *fan-law* if and only if $\exists\gamma\forall s[\beta(s) = 0 \to \forall n[\beta(s * \langle n\rangle) = 0 \to n \leq \gamma(s)]]$.

$\mathcal{X} \subseteq \mathcal{N}$ is a *fan* if and only if there exists a fan-law β such that $\mathcal{X} = \mathcal{F}_\beta$.

Note that the toy spreads $\mathcal{T}_0, \mathcal{T}_1, \ldots$ are fans.

The set $\mathcal{V}_\alpha$, however, is a spread but, in general, not a fan.

Define, for each α, $\mathcal{V}^*_\alpha := \overline{\bigcup_n \underline{\overline{0}}n * \langle 1\rangle * \mathcal{T}_{\alpha(n)}}$.[19]

Note that, for each α, $\mathcal{V}^*_\alpha$ is a fan.

One may prove a statement very similar to Theorem 15 (iv):

For all ζ, η *in* $[\omega]^\omega$, *if* $\zeta \perp \eta$ *and* $\zeta(0) = \eta(0) = 2$, *then there exists a sentence* ψ *such that* $(\mathcal{V}^*_\zeta, =) \models \psi$ *and* $(\mathcal{V}^*_\eta, =) \models \neg\psi$.

1.8.3 Comparison with an Older Theorem

The first-order theory DLO of *dense linear orderings without endpoints* is formulated in a first-order language with binary predicate symbols $=$ and $\sqsubset$ and consists of the following axioms:

1. $\forall x[x \sqsubset x]$,
2. $\forall x\forall y\forall z[(x \sqsubset y \wedge y \sqsubset z) \to x \sqsubset z]$,
3. $\forall x\forall y[(\neg(x \sqsubset y) \wedge \neg(y \sqsubset x)) \to x = y]$.
4. $\forall x\forall y[x \sqsubset y \to \forall z[x \sqsubset z \vee z \sqsubset y]]$,
5. $\forall x\exists y[x \sqsubset y] \wedge \forall x\exists y[y \sqsubset x]$,
6. $\forall x\forall y[x \sqsubset y \to \exists z[x \sqsubset z \wedge z \sqsubset y]]$, and
7. axioms of equality.

$(\mathcal{R}, =_\mathcal{R}, <_\mathcal{R})$ realizes DLO.

Let DLO^- be the theory one obtains from DLO by leaving out axiom (4). If one defines a relation $<'_\mathcal{R}$ on $\mathcal{R}$ by: $\forall x\forall y[x <'_\mathcal{R} y \leftrightarrow \neg\neg(x <_\mathcal{R} y)]$, then $(\mathcal{R}, =_\mathcal{R}, <'_\mathcal{R})$ realizes DLO^- but not DLO.

In Veldman and Janssen (1990, Theorem 2.4) one constructs a function $\alpha \mapsto A_\alpha$ associating to each element α of $2^\omega = \mathcal{C}$ a subset A_α of the set $\mathcal{R}$ of the real numbers

[19] For each $\mathcal{X} \subseteq \mathcal{N}$, $\overline{\mathcal{X}} := \{\alpha \mid \forall n\exists\beta \in \mathcal{X}[\overline{\alpha}n \sqsubset \beta]\}$ is the *closure* of $\mathcal{X}$.
$\bigcup_n \underline{\overline{0}}n * \langle 1\rangle * \mathcal{T}_{\alpha(n)}$, in general, is not a spread, but its closure is.

such that, for each α in $\mathcal{C}$, A_α is dense in $(\mathcal{R}, <_\mathcal{R})$, and, for all α, β in $\mathcal{C}$, if $\alpha \perp \beta$, then there exists a sentence ψ such that $(A_\alpha, <_\mathcal{R}) \models \psi$ and $(A_\beta, <_\mathcal{R}) \models \neg\psi$.

Note: each structure $(A_\alpha, <_\mathcal{R})$ realizes DLO. The (intuitionistic) theory DLO thus has continuum many complete extensions.[20]

One may obtain the result of Theorem 15 (iv) for subsets of $\mathcal{R}$ as well as for subsets of $\mathcal{N}$. Define an infinite sequence $\mathcal{U}_0, \mathcal{U}_1, \ldots$ of subsets of $\mathcal{R}$ by:

$\mathcal{U}_0 := \emptyset$ and $\mathcal{U}_1 := \{0_\mathcal{R}\}$, and for each $m > 0$, $\mathcal{U}_{m+1} = \overline{\bigcup_n \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} \cdot_\mathcal{R} \mathcal{U}_m}$.[21]

For each m, one may define $\varphi : \mathcal{T}_m \to \mathcal{U}_m$ such that φ is surjective and satisfies: $\forall \delta \in \mathcal{T}_m \forall \zeta \in \mathcal{T}_m [\delta \perp \zeta \leftrightarrow \varphi|\delta \#_\mathcal{R} \varphi|\zeta]$.

It follows that, for each m, the structures $(\mathcal{T}_m, =)$ and $(\mathcal{U}_m, =_\mathcal{R})$ are elementarily equivalent.

Define, for each α in $[\omega]^\omega$, $A_\alpha := \bigcup_n n +_\mathcal{R} \mathcal{U}_{\alpha(n)}$.

Note: for all α, β in $[\omega]^\omega$, if $\alpha \perp \beta$, then there exists a sentence ψ such that $(A_\alpha, =_\mathcal{R}) \models \psi$ and $(A_\beta, =_\mathcal{R}) \models \neg\psi$.

We thus obtain a result similar to Veldman and Janssen (1990, Theorem 2.4), this time using not the ordering relation $<_\mathcal{R}$ but only the equality relation $=_\mathcal{R}$.

Note that, for all α, β in $\mathcal{R}$, $\alpha =_\mathcal{R} \beta \leftrightarrow \big(\neg(\alpha <_\mathcal{R} \beta) \wedge \neg(\beta <_\mathcal{R} \alpha)\big)$.

Conclude: the relation $=_\mathcal{R}$ is definable in the structure $(\mathcal{R}, <_\mathcal{R})$.

Conclude: for all subsets $\mathcal{T}, \mathcal{U}$ of $\mathcal{R}$, if there exists a sentence ψ such that $(\mathcal{T}, =_\mathcal{R}) \models \psi$ and $(\mathcal{U}, =_\mathcal{R}) \models \neg\psi$, then there also exists a sentence ψ^* such that $(\mathcal{T}, <_\mathcal{R}) \models \psi^*$ and $(\mathcal{U}, <_\mathcal{R}) \models \neg\psi^*$.

The conclusion of Veldman and Janssen (1990, Theorem 2.4) might have been obtained as a corollary of Theorem 15 (iv).

1.9 The Vitali Equivalence Relation

For all α, β, we define

$$\alpha \sim_V \beta \leftrightarrow \exists n \forall m > n[\alpha(m) = \beta(m)].$$

The relation $\sim_V$ will be called the *Vitali equivalence relation*.

This is because the relation $\sim_V$ on $\mathcal{N}$ resembles the relation $\sim_\mathbb{Q}$ on the set $\mathcal{R}$ of the real numbers defined by:

$$x \sim_\mathbb{Q} y \leftrightarrow \exists q \in \mathbb{Q}[x -_\mathcal{R} y = q].$$

The relation $\sim_\mathbb{Q}$ has played an important rôle in classical set theory.

If one constructs, using the axiom of choice, within the interval $[0, 1]$, a *transversal* for this equivalence relation, that is: a complete set of mutually inequivalent

[20] Classically, $Th\big((\mathbb{Q}, <)\big)$ is the one and only complete extension of DLO.

[21] For each $\mathcal{X} \subseteq \mathcal{R}$, $\overline{\mathcal{X}} := \{x \in \mathcal{R} \mid \forall n \exists y \in \mathcal{X}[|x - y| < \frac{1}{2^n}]\}$ is the *closure* of $\mathcal{X}$.

representatives, one obtains a set that is not Lebesgue measurable. This discovery is due to G. Vitali.

Note: $(\mathcal{N}, \sim_V) \models EQ$.

The following theorem brings to light an important difference between $(\mathcal{N}, =)$ and $(\mathcal{N}, \sim_V)$.

Definition 22 A proposition P is *stable* if and only if $\neg\neg P \to P$.

A binary relation $\sim$ on $\mathcal{N}$ is *stable* if and and only if $\forall\alpha\forall\beta[\neg\neg(\alpha \sim \beta) \to \alpha \sim \beta]$.[22]

Theorem 16 (Equality is stable but the Vitali equivalence relation is not stable)

(i) $(\mathcal{N}, =) \models \forall \mathsf{x} \forall \mathsf{y}[\neg\neg(\mathsf{x} = \mathsf{y}) \to \mathsf{x} = \mathsf{y}]$.
(ii) $(\mathcal{N}, \sim_V) \models \forall \mathsf{x} \neg \forall \mathsf{y}[\neg\neg(\mathsf{x} = \mathsf{y}) \to \mathsf{x} = \mathsf{y}]$.

Proof (i) Note: for all α, β, $\alpha = \beta \leftrightarrow \neg(\alpha \# \beta)$, and, therefore:
$\neg\neg(\alpha = \beta) \leftrightarrow \neg\neg\neg(\alpha \# \beta) \leftrightarrow \neg(\alpha \# \beta) \leftrightarrow \alpha = \beta$.

(ii) Let γ be given.
Consider $\mathcal{F}^\gamma := \{\alpha \mid \forall m \forall n[\big(\alpha(m) \neq \gamma(m) \ \wedge \ \alpha(n) \neq \gamma(n)\big) \to m = n]$.
$\mathcal{F}^\gamma$ is the set of all α that differ at at most one place from γ.
Note that $\mathcal{F}^\gamma$ is a spread.
We have two claims.
First claim: $\forall \alpha \in \mathcal{F}^\gamma[\neg\neg(\alpha \sim_V \gamma)]$.
The proof is as follows. Let α in $\mathcal{F}^\gamma$ be given. Distinguish two cases.
Case (1). $\exists n[\alpha(n) \neq \gamma(n)]$. Find n such that $\alpha(n) \neq \gamma(n)$ and conclude: $\forall m > n[\alpha(m) = \gamma(m)]$ and $\alpha \sim_V \gamma$.
Case (2). $\neg\exists n[\alpha(n) \neq \gamma(n)]$. Conclude: $\forall n[\alpha(n) = \gamma(n)]$ and $\alpha \sim_V \gamma$.
We thus see: if $\exists n[\alpha(n) \neq \gamma(n)] \ \vee \ \neg\exists n[\alpha(n) \neq \gamma(n)]$, then $\alpha \sim_V \gamma$.
As $\neg\neg(\exists n[\alpha(n) \neq \gamma(n)] \ \vee \ \neg\exists n[\alpha(n) \neq \gamma(n)])$, also $\neg\neg(\alpha \sim_V \gamma)$.
Second claim: $\neg\forall \alpha \in \mathcal{F}^\gamma[\alpha \sim \gamma]$.
In order to see this, assume: $\forall \alpha \in \mathcal{F}^\gamma[\alpha \sim \gamma]$, that is:
$\forall \alpha \in \mathcal{F} \exists n \forall m > n[\alpha(m) = \gamma(m)]$. Using Lemma 2, find p, n such that $\forall \alpha \in \mathcal{F}^\gamma[\overline{\gamma}p \sqsubset \alpha \to \forall m > n[\alpha(m) = \gamma(m)]]$. Define $m := \max(p, n+1)$ and define α such that $\forall n[\alpha(n) \neq \gamma(n) \leftrightarrow n = m]$. Note: $\overline{\gamma}p \sqsubset \alpha$ and $m > n$ and $\alpha(m) \neq \gamma(m)$. Contradiction.
Combining our two claims, we see:
not: for all α, if $\neg\neg(\alpha \sim_V \gamma)$ then $\alpha \sim_V \gamma$.
Conclude: $(\mathcal{N}, \sim_V) \models \forall \mathsf{x} \neg \forall \mathsf{y}[\neg\neg(\mathsf{x} = \mathsf{y}) \to \mathsf{x} = \mathsf{y}]$.

□

It follows from Theorem 16 that there is no relation $\#_V$ on $\mathcal{N}$ satisfying the requirements of an apartness relation[23] with respect to $\sim_V$:

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[22] The term '*stable*' has been introduced by D. Van Dantzig, who hoped to be able to reconstruct 'classical', non-intuitionistic mathematics within the stable part of intuitionistic mathematics, see van Dantzig (1947).

[23] See Troelstra and van Dalen (1988, p. 256).

(i) $\forall\alpha\forall\beta[\neg(\alpha \#_V \beta) \leftrightarrow \alpha \sim_V \beta]$
(ii) $\forall\alpha\forall\beta[\alpha \#_V \beta \rightarrow \beta \#_V \alpha]$
(iii) $\forall\alpha\forall\beta[\alpha \#_V \beta \rightarrow \forall\gamma[\alpha \#_V \gamma \ \vee \ \gamma \#_V \beta]]$.

The existence of an apartness $\#_V$ would imply, by the first one of these requirements, that $\sim_V$ is a stable relation, as, for any proposition P, $\neg\neg\neg P \leftrightarrow \neg P$.

The next Theorem now is no surprise:

Theorem 17 $(\mathcal{N}, \sim_V) \models \forall \mathsf{x} \forall \mathsf{y}[\neg AP(\mathsf{x}, \mathsf{y})]$.

Proof Let α, β be given.

Assume $(\mathcal{N}, \sim_V) \models AP[\alpha, \beta]$, that is, $\forall\gamma[\gamma \nsim_V \alpha \ \vee \ \gamma \nsim_V \beta]$.

Applying Lemma 1, find p such that
either $\forall\gamma[\overline{\alpha}p \sqsubset \gamma \rightarrow \gamma \nsim_V \alpha]$ or $\forall\gamma[\overline{\alpha}p \sqsubset \gamma \rightarrow \gamma \nsim_V \beta]$.

The first of these two alternatives is wrong, as $\overline{\alpha}p \sqsubset \alpha \ \wedge \ \alpha \sim_V \alpha$.

Conclude: $\forall\gamma[\overline{\alpha}p \sqsubset \gamma \rightarrow \gamma \nsim_V \beta]$.

Define γ such that $\overline{\alpha}p \sqsubset \gamma$ and $\forall i > p[\gamma(i) = \beta(i)]$.

Note: $\overline{\alpha}p \sqsubset \gamma \ \wedge \ \gamma \sim_V \beta$.

Contradiction.

Conclude: $(\mathcal{N}, =_V) \models \neg AP[\alpha, \beta]$.

We thus see: $(\mathcal{N}, =_V) \models \forall \mathsf{x} \forall \mathsf{y}[\neg AP(\mathsf{x}, \mathsf{y})]$. □

Clearly, the relation defined by the formula AP in the structure $(\mathcal{N}, \sim_V)$ fails to satisfy the first requirement for an apartness relation with respect to $\sim_V$.

It follows from Theorem 17 that $(\mathcal{N}, \sim_V)$, while realizing T_{inf}, does not realize T^+_{inf}, see Definitions 1 and 4.

1.10 A First Vitali Variation

There are many intuitionistic versions of the classical Vitali equivalence relation. This is obvious to someone who knows that there are many variations upon the notion of a finite and decidable subset of $\mathbb{N}$, see Veldman (1995) and Veldman (2005, Sect. 3).

Definition 23 We define an infinite sequence $\sim^0_V, \sim^1_V, \dots$ of relations on $\mathcal{N}$ such that $\sim^0_V = \sim_V$ and, for each i,

$$\alpha \sim^{i+1}_V \beta \leftrightarrow \exists n \forall m > n[\alpha(m) \neq \beta(m) \rightarrow \alpha \sim^i_V \beta].$$

We also define:

$$\alpha \sim^{\omega}_V \beta \leftrightarrow \exists i[\alpha \sim^i_V \beta].$$

Theorem 18 (i) $\forall i \forall n \forall s \in \omega^n \forall t \in \omega^n \forall\alpha\forall\beta[s * \alpha \sim^i_V t * \beta \leftrightarrow \alpha \sim^i_V \beta]$.
(ii) $\forall i \forall\alpha\forall\beta[\alpha \sim^i_V \beta \rightarrow \alpha \sim^{i+1}_V \beta]$.
(iii) $\forall i \forall\gamma\neg\forall\alpha[\alpha \sim^{i+1}_V \gamma \rightarrow \alpha \sim^i_V \gamma]$.

(iv) $\forall i \forall j \forall \alpha \forall \beta \forall \gamma[(\alpha \sim_V^i \beta \wedge \beta \sim_V^j \gamma) \to \alpha \sim_V^{i+j} \gamma]$.
(v) $\sim_V^\omega$ *is an equivalence relation on* $\mathcal{N}$.

Proof (i) One proves this easily by induction.
(ii) Obvious.
(iii) Let γ be given.
For each i, define $\mathcal{F}_i^\gamma := \{\alpha \mid \forall s \in [\omega]^{i+1} \exists j < i+1[\alpha \circ s(j) = \gamma \circ s(j)]\}$.
Note: for each i, $\mathcal{F}_i^\gamma$ is a spread, and $\mathcal{F}_i^\gamma \subsetneq \mathcal{F}_{i+1}^\gamma$.
For each i, $\mathcal{F}_i^\gamma$ consists of all α that assume at most i times a value different from the value assumed by γ. In particular, $\mathcal{F}_\gamma^0 = \{\gamma\}$.
Note: for all i, m, α, β,
if $m = \mu n[\alpha(n) \neq \gamma(n)]$ and $\alpha = \overline{\alpha}(m+1) * \beta$, then $\alpha \in \mathcal{F}_{i+1}^\gamma \leftrightarrow \beta \in \mathcal{F}_i^\gamma$.
We have two claims.
First claim: $\forall i \forall \alpha \in \mathcal{F}_i^\gamma[\alpha \sim_V^i \gamma]$.
We prove this claim by induction.
The starting point of the induction is the observation:
$\forall \alpha \in \mathcal{F}_0^\gamma[\alpha = \gamma]$, so $\forall \alpha \in \mathcal{F}_0^\gamma[\alpha \sim_V^0 \gamma]$.
Now assume i is given such that $\forall \alpha \in \mathcal{F}_i^\gamma[\alpha \sim_V^i \gamma]$.
Assume $\alpha \in \mathcal{F}_{i+1}^\gamma$ and $\exists n[\alpha(n) \neq \gamma(n)]$. Find n such that $\alpha(n) \neq \gamma(n)$. Find β such that $\alpha = \overline{\alpha}(n+1) * \beta$, and note: $\beta \in \mathcal{F}_i^\gamma$ and thus, by the induction hypothesis, $\beta \sim_V^i \gamma$. Conclude, using (i): $\alpha \sim_V^i \gamma$.
We thus see:
$\forall \alpha \in \mathcal{F}_{i+1}^\gamma[\exists n[\alpha(n) \neq \gamma(n)] \to \alpha \sim_V^i \gamma]$, that is: $\forall \alpha \in \mathcal{F}_{i+1}^\gamma[\alpha \sim_V^{i+1} \gamma]$.
This completes the proof of the induction step.
Second claim: $\forall i \neg \forall \alpha \in \mathcal{F}_{i+1}^\gamma[\alpha \sim_V^i \gamma]$.
We again use induction.
We first prove: $\neg \forall \alpha \in \mathcal{F}_1^\gamma[\alpha \sim_V \gamma]$. Assume $\forall \alpha \in \mathcal{F}_1^\gamma[\alpha \sim_V \gamma]$, that is: $\forall \alpha \in \mathcal{F}_1 \exists n \forall m > n[\alpha(m) = \gamma(m)]$.
Note: $\gamma \in \mathcal{F}_1^\gamma$ and $\mathcal{F}_\gamma^1$ is a spread.
Using Lemma 2, find p, n such that
$\forall \alpha \in \mathcal{F}_1[\overline{\gamma}p \sqsubset \alpha \to \forall m > n[\alpha(m) = \gamma(m)]$.
Define $m := \max(n+1, p)$ and define α such that $\forall n[\alpha(n) = \gamma(n) \leftrightarrow n \neq m]$.
Note: $\alpha \in \mathcal{F}_1$ and $\overline{\gamma}p \sqsubset \alpha$ and $\alpha(m) \neq \gamma(m)$ and $m > n$. Contradiction.
Conclude: $\neg \forall \alpha \in \mathcal{F}_1^\gamma[\alpha \sim_V \gamma]$.
Now let i be given such that $\neg \forall \alpha \in \mathcal{F}_{i+1}^\gamma[\alpha \sim_V^i \gamma]$.
We want to prove: $\neg \forall \alpha \in \mathcal{F}_{i+2}^\gamma[\alpha \sim_V^{i+1} \gamma]$.
Assume: $\forall \alpha \in \mathcal{F}_{i+2}^\gamma[\alpha \sim_V^{i+1} \gamma]$, that is:
$\forall \alpha \in \mathcal{F}_{i+2}^\gamma \exists n \forall m > n[\alpha(m) \neq \gamma(m) \to \alpha \sim_V^{i+1} \gamma]$. Using Lemma 2, find p, n such that $\forall \alpha \in \mathcal{F}_{i+2}^\gamma[(\overline{\gamma}p \sqsubset \alpha \wedge m > n \wedge \alpha(m) \neq \gamma(m)) \to \alpha \sim_V^i \gamma]$. Define $m := \max(n+1, p)$. Let β in $\mathcal{F}_{i+1}^\gamma$ be given. Define α such that $m = \mu n[\alpha(n) \neq \gamma(n)]$ and $\forall n > m[\alpha(n) = \beta(n)]$. Note: $\alpha \in \mathcal{F}_{i+2}^\gamma$ and $\alpha(m) \neq \gamma(m)$ and $m > n$, so $\alpha \sim_V^i \gamma$, and, therefore, by (i), $\beta \sim_V^i \gamma$. We thus see: $\forall \beta \in \mathcal{F}_{i+1}^\gamma[\beta \sim_V^i \gamma]$ and, by the induction hypothesis, obtain a contradiction.
This completes the proof of the induction step.

Taking our first and second claim together, we obtain the conclusion:
$\forall\gamma\forall i\neg\forall\alpha[\alpha \sim_V^{i+1} \gamma \to \alpha \sim_V^i \gamma]$.

(iv) We have to prove:
for all i, for all j, $\forall\alpha\forall\beta\forall\gamma[(\alpha \sim_V^i \beta \wedge \beta \sim_V^j \gamma) \to \alpha \sim_V^{i+j} \gamma]$.
We use induction on $i + j$ and distinguish four cases.
Case (1): $i = j = 0$. Assume $\alpha \sim_V^0 \beta$ and $\beta \sim_V^0 \gamma$. Find n, p such that $\forall m > n[\alpha(m) = \beta(m)]$ and $\forall m > p[\beta(m) = \gamma(m)]$. Define $q := \max(n, p)$ and note: $\forall m > q[\alpha(m) = \gamma(m)]$. Conclude: $\alpha \sim_V^0 \gamma$.
Case (2): $i = 0$ and $j > 0$. Assume $\alpha \sim_V^0 \beta$ and $\beta \sim_V^j \gamma$. Find n, p such that $\forall m > n[\alpha(m) = \beta(m)]$ and $\forall m > p[\beta(m) \neq \gamma(m) \to \beta \sim_V^{j-1} \gamma]$. Define $q := \max(n, p)$.
Assume $m > q$ and note: if $\alpha(m) \neq \gamma(m)$, then $\beta(m) \neq \gamma(m)$ and $\beta \sim_V^{j-1} \gamma$.
Using the induction hypothesis, conclude: $\alpha \sim_V^{j-1} \gamma$.
We thus see: $\forall m > q[\alpha(m) \neq \gamma(m) \to \alpha \sim_V^{j-1} \gamma]$, that is: $\alpha \sim_V^j \gamma$.
Case (3): $i > 0$ and $j = 0$. Assume $\alpha \sim_V^i \beta$ and $\beta \sim_V^0 \gamma$. Find n, p such that $\forall m > n[\alpha(m) \neq \beta(m) \to \alpha \sim_V^{i-1} \beta]$ and $\forall m > p[\beta(m) = \gamma(m)]$. Define $q := \max(n, p)$.
Assume $m > q$ and note: if $\alpha(m) \neq \gamma(m)$, then $\alpha(m) \neq \beta(m)$ and $\alpha \sim_V^{i-1} \beta$.
Using the induction hypothesis, conclude: $\alpha \sim_V^{i-1} \gamma$.
We thus see: $\forall m > q[\alpha(m) \neq \gamma(m) \to \alpha \sim_V^{i-1} \gamma]$, that is: $\alpha \sim_V^i \gamma$.
Case (4): $i > 0$ and $j > 0$. Assume $\alpha \sim_V^i \beta$ and $\beta \sim_V^j \gamma$. Find n, p such that $\forall m > n[\alpha(m) \neq \beta(m) \to \alpha \sim_V^{i-1} \beta]$ and
$\forall m > p[\beta(m) \neq \gamma(m) \to \beta \sim_V^{j-1} \gamma]$. Define $q := \max(n, p)$.
Assume $m > q$ and $\alpha(m) \neq \gamma(m)$. Then *either:* $\alpha(m) \neq \beta(m)$ and $\alpha \sim^{i-1} \beta$, and, by the induction hypothesis, $\alpha \sim_V^{i+j-1} \gamma$, *or:* $\beta(m) \neq \gamma(m)$ and $\beta \sim^{j-1} \gamma$ and, by the induction hypothesis, $\alpha \sim^{i+j-1} \gamma$.
We thus see: $\forall m > q[\alpha(m) \neq \gamma(m) \to \alpha \sim^{i+j-1} \gamma]$. Conclude: $\alpha \sim^{i+j} \gamma$.

(v) is an easy consequence of (iv).

The next Theorem shows that the structures $(\mathcal{N}, \sim_V)$ and $(\mathcal{N}, \sim_V^\omega)$ have a property in common.

Theorem 19 ($\sim_V^\omega$ is not stable)
$(\mathcal{N}, \sim_V^\omega) \models \forall \mathsf{x}\neg\forall \mathsf{y}[\neg\neg(\mathsf{x} = \mathsf{y}) \to \mathsf{x} = \mathsf{y}]$.

Proof Let γ be given.
We repeat a definition we gave in the proof of Theorem 18 (iii).
For each i, $\mathcal{F}_i^\gamma := \{\alpha \mid \forall s \in [\omega]^{i+1}\exists j < i + 1[\alpha \circ s(j) = \gamma \circ s(j)]\}$.
In the proof of Theorem 18 (iii), we saw: $\forall i\forall\alpha \in \mathcal{F}_i^\gamma[\alpha \sim_V^i \gamma]$.
Conclude: $\forall i\forall\alpha \in \mathcal{F}_i^\gamma[\alpha \sim_V^\omega \gamma]$.
We now define: $\mathcal{F}_\omega^\gamma := \{\alpha \mid \forall i[i = \mu n[\alpha(n) \neq \gamma(n)] \to \alpha \in \mathcal{F}_{i+1}]\}$.
Like each $\mathcal{F}_i^\gamma$, $\mathcal{F}_\omega^\gamma$ is a spread, and $\gamma \in \mathcal{F}_\omega^\gamma$.
We have two claims.
First claim: $\forall\alpha \in \mathcal{F}_\omega^\gamma[\neg\neg(\alpha \sim_V^\omega \gamma)]$.
The argument is as follows. Let α in $\mathcal{F}_\omega^\gamma$ be given and distinguish two cases.

Case (1): $\neg\exists n[\alpha(n) \neq \gamma(n)]$. Then $\alpha = \gamma$ and $\alpha \sim_V^\omega \gamma$.
Case (2): $\exists n[\alpha(n) \neq \gamma(n)]$. Find $i := \mu n[\alpha(n) \neq \gamma(n)]$.
Note: $\alpha \in \mathcal{F}_{i+1}^\gamma$ and $\alpha \sim_V^\omega \gamma$.
As $\neg\neg(\exists n[\alpha(n) \neq \gamma(n)] \vee \neg\exists n[\alpha(n) \neq \gamma(n)])$, also $\neg\neg(\alpha \sim_V^\omega \gamma)$.
Second claim: $\neg\forall\alpha \in \mathcal{F}_\omega^\gamma[\alpha \sim_V^\omega \gamma]$.
In order to see this, assume: $\forall\alpha \in \mathcal{F}_\omega^\gamma[\alpha \sim_V^\omega \gamma]$, that is: $\forall\alpha \in \mathcal{F}_\omega\exists i[\alpha \sim_V^i \gamma]$.
Using Lemma 2, find p, i such that $\forall\alpha \in \mathcal{F}_\omega^\gamma[\overline{\gamma}p \sqsubset \alpha \to \alpha \sim_V^i \gamma]$.
Define $q := \max(p, i+1)$. Let α in $\mathcal{F}_q^\gamma$ be given. Define β such that $\forall n < q[\beta(n) = \gamma(n)]$ and $\beta(q) \neq \gamma(q)$ and $\forall n > q[\beta(n) = \alpha(n)]$.
Note: $\beta \in \mathcal{F}_{q+1}$ and $q = \mu n[\beta(n) \neq \gamma(n)]$, and, therefore, $\beta \in \mathcal{F}_\omega^\gamma$.
As $\overline{\gamma}q \sqsubset \beta$, we conclude: $\beta \sim_V^i \gamma$.
As $\beta \sim_V^0 \alpha$, also $\alpha \sim_V^i \gamma$.
We thus see: $\forall\alpha \in \mathcal{F}_q[\alpha \sim_V^i \gamma]$.
As $q > i$, this contradicts the Second claim in the proof of Theorem 18 (iii).
Taking our two claims together, we conclude:
$\forall\gamma\neg\forall\alpha \in \mathcal{F}_\omega^\gamma[\neg\neg(\alpha \sim_V^\omega \gamma) \to \alpha \sim_V^\omega \gamma]$.
Conclude: $(\mathcal{N}, \sim_V^\omega) \models \forall \mathsf{x}\neg\forall \mathsf{y}[\neg\neg(\mathsf{x} = \mathsf{y}) \to \mathsf{x} = \mathsf{y}]$. □

We did not succeed in finding a sentence ψ such that $(\mathcal{N}, \sim_V) \models \psi$ and $(\mathcal{N}, \sim_V^\omega) \models \neg\psi$.

1.11 More and More Vitali Relations

In Veldman (1995), Veldman (1999) and Veldman (2005, Sect. 3), one studies the set

$$\mathbf{Fin} := \{\alpha \mid \alpha \sim_V \underline{0}\} = \{\alpha \mid \exists n\forall m > n[\alpha(m) = 0]\}.$$

For each α, $\alpha \in \mathbf{Fin}$ if and only if $D_\alpha := \{m \mid \alpha(m) \neq 0\}$ is a *finite* subset of $\mathbb{N}$.

For each i, the set $\{\alpha \mid \alpha \sim_V^i \underline{0}\}$ is called, in Veldman (1999) and Veldman (2005), the i-th *perhapsive extension* of the set **Fin**. It is shown, in Veldman (1995), Veldman (1999) and Veldman (2005), that the process of building perhapsive extensions of **Fin** can be carried on into the transfinite.

In a similar way, the Vitali equivalence relation $\sim_V$ admits of transfinitely many extensions.

The relation $\sim_V^\omega$ is only a *first* extension of $\sim_V$. Let us consider a second one.
Recall: $\forall\alpha\forall\beta[\alpha \sim_V^\omega \leftrightarrow i[\alpha \sim_V^i \beta]]$.

Definition 24 We define an infinite sequence $\sim_V^{\omega+0} = \sim_V^\omega, \sim_V^{\omega+1}, \sim_V^{\omega+2}, \ldots$ of relations on $\mathcal{N}$, such that, for each $i > 0$,

$$\alpha \sim_V^{\omega+i+1} \beta \leftrightarrow \exists n\forall m > n[\alpha(m) \neq \beta(m) \to \alpha \sim_V^{\omega+i} \beta].$$

We also define:

$$\alpha \sim_V^{\omega+\omega} \beta \leftrightarrow \exists i[\alpha \sim_V^{\omega+i} \beta].$$

One may prove analogues of Theorems 18 and 19 and conclude:

$\sim_V^{\omega+\omega}$ is an equivalence relation on $\mathcal{N}$, properly extending $\sim_V^{\omega}$, that, like $\sim_V$ and $\sim_V^{\omega}$, is not stable in the sense of Theorem 19.

One may continue and define $\sim_V^{\omega+\omega+\omega}$, and $\sim_V^{\omega+\omega+\omega+\omega}$ and so on.

The process of building such extensions leads further into the transfinite, as follows.

Definition 25 Let R be a binary relation on $\mathcal{N}$.

We define a binary relation R^+ on $\mathcal{N}$ by:

$$\alpha R^+ \beta \leftrightarrow \exists n \forall m > n[\alpha(m) \neq \beta(m) \rightarrow \alpha R \beta].$$

We let $\mathcal{E}$ be the least class of binary relations on $\mathcal{N}$ such that

(i) the Vitali equivalence relation $\sim_V$ belongs to $\mathcal{E}$, and,
(ii) for every R in $\mathcal{E}$, also $R^+ \in \mathcal{E}$, and,
(iii) for every infinite sequence $R_0, R_1, \ldots$ of elements of $\mathcal{E}$, also $\bigcup_i R_i \in \mathcal{E}$.

The elements of $\mathcal{E}$ are the *extensions of the Vitali equivalence relation*.

Note that $<_V^{\omega}$ and $<_V^{\omega+\omega}$ are in $\mathcal{E}$.

In general, a relation R in $\mathcal{E}$ is not transitive. One may prove, for instance, that the relation $<_V^1$, while belonging to $\mathcal{E}$, is not transitive.

The next Theorem shows that $\mathcal{E}$ contains many transitive relations.

Theorem 20 ($\mathcal{E}$ contains many transitive relations)

(i) $\sim_V$ *is transitive.*
(ii) *Given any transitive R in $\mathcal{E}$, there exists a transitive T in $\mathcal{E}$ such that $R^+ \subseteq T$.*
(iii) *Given any infinite and increasing sequence $R_0 \subseteq R_1 \subseteq \ldots$ of transitive relations in $\mathcal{E}$, also $\bigcup_i R_i$ is a transitive relation in $\mathcal{E}$.*

Proof (i) Obvious.

(ii) We take our inspiration from Theorem 18 (iv) and (v).
Let a transitive R in $\mathcal{E}$ be given.
Define an infinite sequence $R^0, R^1, \ldots$ of elements of $\mathcal{E}$ such that $R^0 = R$ and, for each i, $R^{i+1} = (R^i)^+$.
One may prove: for all i, for all j, $\forall\alpha\forall\beta\forall\gamma[(\alpha R^i \beta \wedge \beta R^i \gamma) \rightarrow \alpha R^{i+j} \gamma]$, as it is done for the special case $R = \sim_V$ in the proof of Theorem 18 (iv).
Define $T := \bigcup_i R^i$ and note: $T \in \mathcal{E}$, $R^+ \subseteq T$ and T is transitive.

(iii) Note: for every increasing sequence $R_0 \subseteq R_1 \subseteq \ldots$ of transitive relations on $\mathcal{N}$, also $\bigcup_i R_i$ is transitive.

Theorem 20 will gain significance after Corollary 3, which shows that, for every R in $\mathcal{E}$, $R \subseteq R^+$ and $\neg(R^+ \subseteq R)$.

We did not succeed in proving that every R in $\mathcal{E}$ extends to a transitive T in $\mathcal{E}$.

Definition 26 A binary relation R on $\mathcal{N}$ is *shift-invariant* if and only if $\forall\alpha\forall\beta[\alpha R\beta \leftrightarrow (\alpha \circ S)R(\beta \circ S)]$.

Lemma 7 *Every R in $\mathcal{E}$ is shift-invariant.*

Proof The proof is a straightforward exercise in induction on $\mathcal{E}$. Note:

(I) $\sim_V$ is shift-invariant.
(II) For every binary relation R on $\mathcal{N}$, if R is shift-invariant, then R^+ is shift-invariant.
(III) For every infinite sequence $R_0, R_1, \ldots$ of binary relations on $\mathcal{N}$, if each R_n is shift-invariant, then $\bigcup_i R_i$ is shift-invariant.
Conclude: every R in $\mathcal{E}$ is shift-invariant. □

Definition 27 We let $\mathcal{E}^*$ be the least class of binary relations on $\mathcal{N}$ such that

(i) the Vitali equivalence relation $\sim_V$ belongs to $\mathcal{E}^*$, and
(ii) for every infinite sequence $R_0, R_1, \ldots$ of elements of $\mathcal{E}^*$, also $(\bigcup_i R_i)^+ \in \mathcal{E}^*$.

Lemma 8 *$\mathcal{E}^* \subseteq \mathcal{E}$ and, for all R in $\mathcal{E}$, there exists T in $\mathcal{E}^*$ such that $R \subseteq T$.*

Proof The proofs of the two statements are straightforward, by induction on $\mathcal{E}^*$ and $\mathcal{E}$, respectively. □

Theorem 21 *For each R in $\mathcal{E}^*$, $R \subseteq R^+$ and $\neg(R^+ \subseteq R)$.*

Proof For each R in $\mathcal{E}$, we define $Fin_R := \{\alpha \mid \alpha R\underline{0}\}$.[24]

We prove for each R in $\mathcal{E}^*$ there exists a fan $\mathcal{F}$ such that $\mathcal{F} \subseteq Fin_{R^+}$ and $\neg(\mathcal{F} \subseteq Fin_R)$.

We do so by induction on $\mathcal{E}^*$.

(I) Define $\mathcal{F} := \{\alpha \mid \forall m\forall n[(\alpha(m) \neq 0 \ \wedge \ \alpha(n) \neq 0) \rightarrow m = n]\}$.

Note that $\mathcal{F}$ is a fan.

For each α in $\mathcal{F}$, for each n, if $\alpha(n) \neq 0$ then: $\forall m > n[\alpha(m) = 0]$ and $\alpha \in Fin_{\sim_V}$. Conclude: for each $\alpha \in \mathcal{F}$, if $\exists n[\alpha(n) \neq 0]$, then $\alpha \in Fin_{\sim_V}$, that is: $\alpha \in Fin_{(\sim_V)^+}$. Conclude: $\mathcal{F} \subseteq Fin_{(\sim_V)^+}$.

Now assume $\mathcal{F} \subseteq Fin_{\sim_V}$, that is: $\forall\alpha \in \mathcal{F}\exists n\forall m > n[\alpha(m) = 0]$. Using Lemma 2, find p, n such that $\forall\alpha \in \mathcal{F}[\overline{\underline{0}}p \sqsubset \alpha \rightarrow \forall m > n[\alpha(m) = 0]]$. Define $q := \max(p, n + 1)$ and consider $\alpha := \overline{\underline{0}}q * \langle 1\rangle * \underline{0}$. Contradiction.

Conclude: $\neg(\mathcal{F} \subseteq Fin_{\sim_V})$.

(II) Let $R_0, R_1, \ldots$ be an infinite sequence of elements of $\mathcal{E}$.

Let $\mathcal{F}_0, \mathcal{F}_1, \ldots$ be an infinite sequence of fans such that, for each n, $\mathcal{F}_n \subseteq Fin_{(R_n)^+}$ and $\neg(\mathcal{F}_n \subseteq Fin_{R_n})$.

Consider $R := (\bigcup_i R_i)^+$.

Define $\mathcal{F} := \{\alpha \mid \forall n[n = \mu i[\alpha(i) \neq 0] \rightarrow \exists\beta \in \mathcal{F}_{n'}[\alpha = \overline{\alpha}(n + 1) * \beta]]\}$.[25]

Note that $\mathcal{F}$ is a fan.

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[24] In Veldman (1995), $\mathcal{X} \subseteq \mathcal{N}$ is called a *notion of finiteness* if $\mathbf{Fin} \subseteq \mathcal{X} \subseteq \mathbf{Fin}^{\neg\neg}$. For every R in $\mathcal{E}$, Fin_R is a notion a finiteness.

[25] For each n, $n = (n', n'')$, see Sect. 1.13.

We now prove: $\mathcal{F} \subseteq Fin_{R^+}$ and $\neg(\mathcal{F} \subseteq Fin_R)$.

Note that, for each $\alpha \in \mathcal{F}$, for each n, if $n = \mu i[\alpha(i) \neq 0]$, then there exists β in $\mathcal{F}_{n'}$ such that $\alpha = \overline{\alpha}(n+1) * \beta$.

As, for each n, $\mathcal{F}_n \subseteq Fin_{(R_n)^+} \subseteq Fin_{\bigcup_i (R_i)^+}$, and $\bigcup_i (R_i)^+ \subseteq \left(\bigcup_i R_i\right)^+ = R$ and R is shift-invariant, conclude: $\forall \alpha \in \mathcal{F}[\exists n[\alpha(n) \neq 0] \to \alpha \in Fin_R]$, that is: $\mathcal{F} \subseteq Fin_{R^+}$.

Now assume $\mathcal{F} \subseteq Fin_R$, that is:
$\forall \alpha \in \mathcal{F} \exists n \forall m > n[\alpha(m) \neq 0] \to \exists i[\alpha \in Fin_{R_i}]]$. Using Lemma 2, find p, n such that
$\forall \alpha \in \mathcal{F}[\underline{\overline{0}}p \sqsubset \alpha \to \forall m > n[\alpha(m) \neq 0 \to \exists i[\alpha \in Fin_{R_i}]]$.

Define $q := \max(p, n+1)$ and note: $\forall \alpha \in \mathcal{F}[\underline{\overline{0}}q * \langle 1 \rangle \sqsubset \alpha \to \exists i[\alpha \in \mathcal{F}_i]]$.

Using Lemma 2 again, find r, i such that $\forall \alpha \in \mathcal{F}[\underline{\overline{0}}q * \langle 1 \rangle * \underline{\overline{0}}r \sqsubset \alpha \to \alpha \in \mathcal{F}_i]$.

Find $n \geq q + r + 1$ such that $n' = i$ and define $t := n - (q+1)$.

Note: $t \geq r$ and conclude: $\forall \beta \in \mathcal{F}_i[\underline{\overline{0}}q * \langle 1 \rangle * \underline{\overline{0}}t * \langle 1 \rangle * \beta \in Fin_{R_i}]$.

As R_i is shift-invariant, conclude: $\mathcal{F}_i \subseteq Fin_{R_i}$.

Contradiction, as $\neg(\mathcal{F}_i \subseteq Fin_{R_i})$.

Conclude: $\neg(\mathcal{F} \subseteq Fin_R)$. □

Corollary 3 *For each R in $\mathcal{E}$, $R \subseteq R^+$ and $\neg(R^+ \subseteq R)$.*

Proof Assume we find R in $\mathcal{E}$ such that $R = R^+$.

Conclude, by induction on $\mathcal{E}$: for all U in $\mathcal{E}$, $U \subseteq R$.

Using Lemma 8, find T in $\mathcal{E}^*$ such that $R \subseteq T$.

By Theorem 21, $T \subseteq T^+$ and $\neg(T^+ \subseteq T)$.

On the other hand, $T^+ \subseteq R \subseteq T$.

Contradiction. □

Definition 28 We define binary relations $\sim_V^{\neg\neg}$ and $\sim_V^{almost}$ on $\mathcal{N}$, as follows.

For all α, β, $\alpha \sim_V^{\neg\neg} \beta \leftrightarrow \neg\neg\exists n \forall m > n[\alpha(n) = \beta(n)] \leftrightarrow \neg\neg(\alpha \sim_V \beta)$, and
$\alpha \sim_V^{almost} \beta \leftrightarrow \forall \zeta \in [\omega]^\omega \exists n[\alpha \circ \zeta(n) = \beta \circ \zeta(n)]$.

$\alpha \sim_V^{almost} \beta$ if and only if the set $\{n \mid \alpha(n) \neq \beta(n)\}$ is *almost***-finite* in the sense used in Veldman (2005, Section 0.8.2).

The following axiom is a form of Brouwer's famous *Thesis on bars in $\mathcal{N}$*, see Veldman (2006).

Axiom 4 (The Principle of Bar Induction)

For all $B, C \subseteq \mathbb{N}$, if $\forall \alpha \exists n[\overline{\alpha}n \in B]$ and $B \subseteq C$ and $\forall s[s \in C \leftrightarrow \forall n[s * \langle n \rangle \in C]]$, then $\langle\ \rangle \in C$,

or, equivalently,

for all $B, C \subseteq [\omega]^{<\omega}$, if $\forall \zeta \in [\omega]^\omega \exists n[\overline{\zeta}n \in B]$ and $B \subseteq C$ and $\forall s \in [\omega]^{<\omega}[s \in C \leftrightarrow \forall n[s * \langle n \rangle \in [\omega]^{<\omega} \to s * \langle n \rangle \in C]]$, then $\langle\ \rangle \in C$.

Theorem 22 (i) *$\sim_V^{\neg\neg}$ and $\sim_V^{almost}$ are equivalence relations on $\mathcal{N}$.*

(ii) *For all R in $\mathcal{E}$, $\sim_V \subseteq R \subseteq \sim_V^{\neg\neg}$.*

(iii) *For all R in $\mathcal{E}$, $R \subseteq \sim_V^{almost}$.*

(iv) $\forall\alpha\forall\beta[\alpha \sim_V^{almost} \beta \rightarrow \exists R \in \mathcal{E}[\alpha R\, \beta]$.
(v) $\forall\alpha\forall\beta[\alpha \sim_V^{almost} \beta \rightarrow \alpha \sim_V^{\neg\neg} \beta]$.

Proof (i) One easily proves that $\sim_V^{\neg\neg}$ is an equivalence relation. One needs the fact that, for all propositions P, Q, $(\neg\neg P \wedge \neg\neg Q) \rightarrow \neg\neg(P \wedge Q)$.

We prove that $\sim_V^{almost}$ is a transitive relation.

Let α, β, γ be given such that $\alpha \sim_V^{almost} \beta$ and $\beta \sim_V^{almost} \gamma$.

Let ζ in $[\omega]^\omega$ be given. Find η in $[\omega]^\omega$ such that $\forall n[\alpha \circ \zeta \circ \eta(n) = \beta \circ \zeta \circ \eta(n)]$. Find p such that $\beta \circ \zeta \circ \eta(p) = \gamma \circ \zeta \circ \eta(p)$. Define $n := \eta(p)$ and note: $\alpha \circ \zeta(n) = \gamma \circ \zeta(n)$.

We thus see: $\forall\zeta \in [\omega]^\omega \exists n[\alpha \circ \zeta(n) = \gamma \circ \zeta(n)]$, that is: $\alpha \sim_V^{almost} \gamma$.

(ii) The proof is by (transfinite) induction on $\mathcal{E}$. We only prove: for all R in $\mathcal{E}$, $R \subseteq \sim_V^{\neg\neg}$ as the statement: for all R in $\mathcal{E}$, $\sim_V \subseteq R$ is very easy to prove.

(I) Our starting point is the trivial observation: $\forall\alpha\forall\beta[\alpha \sim_V \beta \rightarrow \neg\neg(\alpha \sim_V \beta)]$.

(II) Now let R in $\mathcal{E}$ be given such that $\forall\alpha\forall\beta[\alpha R \beta \rightarrow \neg\neg(\alpha \sim_V \beta)]$.

We have to prove: $\forall\alpha\forall\beta[\alpha R^+ \beta \rightarrow \neg\neg(\alpha \sim_V \beta)]$.

We do so as follows.

Let α, β be given such that $\alpha R^+ \beta$.

Find n such that $\forall m > n[\alpha(m) \neq \beta(m) \rightarrow \alpha R \beta]$ and consider two special cases.

Case (1): $\exists m > n[\alpha(m) \neq \beta(m)$. Then $\alpha R\, \beta$, and, therefore: $\neg\neg(\alpha \sim_V \beta)$.

Case (2): $\neg\exists m > n[\alpha(m) \neq \beta(m)$. Then $\forall m > n[\alpha(m) = \beta(m)]$ and $\alpha \sim_V \beta$.

In both cases, we find: $\neg\neg(\alpha \sim_V \beta)$.

Conclude[26]: $\neg\neg(\alpha \sim_V \beta)$.

(III) Now let $R_0, R_1, \ldots$ be an infinite sequence of elements of $\mathcal{E}$ such that, for all n, $\forall\alpha\forall\beta[\alpha R_n \beta \rightarrow \neg\neg(\alpha \sim_V \beta)]$.

Define $R := \bigcup_n R_n$ and note: $\forall\alpha\forall\beta[\alpha R \beta \rightarrow \neg\neg(\alpha \sim_V \beta)]$.

(iii) The proof is by (transfinite) induction on $\mathcal{E}$.

(I) Our starting point is the observation: $\forall\alpha\forall\beta[a \sim_V^0 \beta \rightarrow \alpha \sim_V^{almost} \beta]$.

We prove this as follows:

Let α, β be given such that $\alpha \sim_V^0 \beta$. Find n such that $\forall m > n[\alpha(m) = \beta(m)]$. Note: $\forall\zeta \in [\omega]^\omega][\zeta(n+1) > n \wedge \alpha \circ \zeta(n+1) = \beta \circ \zeta(n+1)]$. Conclude: $\alpha \sim_V^{almost} \beta$.

(II) Now let R in $\mathcal{E}$ be given such that $\forall\alpha\forall\beta[\alpha R \beta \rightarrow \alpha \sim_V^{almost} \beta]$.

We have to prove: $\forall\alpha\forall\beta[a R^+ \beta \rightarrow \alpha \sim_V^{almost} \beta]$.

We do so as follows.

Let α, β be given such that $\alpha R^+ \beta$.

Find n such that $\forall m > n[\alpha(m) \neq \beta(m) \rightarrow \alpha R \beta]$. Let ζ in $[\omega]^\omega$ be given. Consider $\zeta(n+1)$ and note $\zeta(n+1) > n$. There now are two cases.

Either $\alpha \circ \zeta(n+1) = \beta \circ \zeta(n+1)$ *or* $\alpha \circ \zeta(n+1) \neq \beta \circ \zeta(n+1)$.

In the first case we are done, and in the second case we conclude $\alpha R \beta$, and, using the induction hypothesis, find p such that $\alpha \circ \zeta(p) = \beta \circ \zeta(p)$.

In both cases we conclude: $\exists q[\alpha \circ \zeta(q) = \beta \circ \zeta(q)]$.

We thus see: $\forall\zeta \in [\omega]^\omega \exists q[\alpha \circ \zeta(q) = \beta \circ \zeta(q)]$, that is $\alpha \sim_V^{almost} \beta$.

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[26]using the scheme: if $P \rightarrow Q$ and $\neg P \rightarrow Q$, then $\neg\neg Q$.

Clearly then: $\forall\alpha\forall\beta[[\alpha R^+\beta \to \alpha \sim_V^{almost} \beta]$.

(III) Now let $R_0, R_1, \ldots$ be an infinite sequence of elements of $\mathcal{E}$ such that, for all n, $\forall\alpha\forall\beta[\alpha R_n\beta \to \alpha \sim_V^{almost} \beta]$.

Define $R := \bigcup_n R_n$ and note: $\forall\alpha\forall\beta[\alpha R\beta \to \alpha \sim_V^{almost} \beta]$.

(iv) Let α, β be given such that $\alpha \sim^{almost} \beta$, that is:
$\forall\zeta \in [\omega]^\omega \exists n[\alpha \circ \zeta(n) = \beta \circ \zeta(n)]$.
Using Axiom 4, we shall prove: there exists R in $\mathcal{E}$ such that $\alpha R\beta$.

Define $B := \bigcup_k \{s \in [\omega]^{k+1} \mid \alpha \circ s(k) = \beta \circ s(k)\}$ and note: B is a bar in $[\omega]^\omega$, that is: $\forall\zeta \in [\omega]^\omega \exists n[\overline{\zeta}n \in B]$.

Define $C := \bigcup_k \{s \in [\omega]^k \mid \exists n < k[\alpha \circ s(n) = \beta \circ s(n)] \ \vee \ \exists R \in \mathcal{E}[\alpha R\beta]\}$.

Note: $C = \bigcup_k \{s \in [\omega]^k \mid \forall n < k[\alpha \circ s(n) \neq \beta \circ s(n)] \to \exists R \in \mathcal{E}[\alpha R\beta]\}$.

Note: $B \subseteq C$ and: C is *monotone*, that is:
$\forall s \in [\omega]^{<\omega}[s \in C \to \forall n[s * \langle n\rangle \in [\omega]^{<\omega} \to s * \langle n\rangle \in C]]$.

We still have to prove that C is what one calls *inductive* or *hereditary*.

Let s in $[\omega]^{<\omega}$ be given such that $\forall n[s * \langle n\rangle \in [\omega]^{<\omega} \to s * \langle n\rangle \in C]$.
We want to prove: $s \in C$.

Find k such that $s \in [\omega]^k$. In case $\exists n < k[\alpha \circ s(n) = \beta \circ s(n)]$, $s \in C$ and we are done, so we assume: $\forall n < k[\alpha \circ s(n) \neq \beta \circ s(n)]$.

Find a sequence[27] $R_0, R_1, \ldots$ of elements of $\mathcal{E}$ such that, for each n, if $s * \langle n\rangle \in [\omega]^\omega$ and $\alpha(n) \neq \beta(n)$, then $\alpha R_n\beta$.

Define $R := (\bigcup_i R_i)^+$ and note: $R \in \mathcal{E}$.

We claim: $\alpha R\beta$.

We establish this claim as follows.

Define p such that, if $k = 0$, then $p := 0$ and, if $k > 0$, then $p := s(k-1) + 1$.

Assume we find $n \geq p$ such that $\alpha(n) \neq \beta(n)$.

Note: $s * \langle n\rangle \in [\omega]^{k+1}$ and $\forall i < k + 1[\alpha \circ (s * \langle n\rangle)(i) \neq \beta \circ (s * \langle n\rangle)(i)]$ and $s * \langle n\rangle \in C$. Conclude: $\alpha R_n\beta$ and $\alpha(\bigcup_i R_i)\beta$.

We thus see: $\forall n \geq p[\alpha(n) \neq \beta(n) \to \alpha(\bigcup_i R_i)\beta]$.

Conclude: $\alpha(\bigcup_i R_i)^+\beta$, that is: $\alpha R\beta$, and, therefore: $s \in C$.

We thus see that C is inductive.

Using Axiom 4, we conclude: $\langle\ \rangle \in C$, that is: $\exists R \in \mathcal{E}[\alpha R\beta]$.

(v) Let α, β be given such that $\alpha \sim^{almost} \beta$, that is:
$\forall\zeta \in [\omega]^\omega \exists n[\alpha \circ \zeta(n) = \beta \circ \zeta(n)]$.
Using Axiom 4, we prove: $\neg\neg\exists p\forall n > p[\alpha(n) = \beta(n)]$.

Define $B := \bigcup_k \{s \in [\omega]^{k+1} \mid \alpha \circ s(k) = \beta \circ s(k)\}$ and note: B is a bar in $[\omega]^\omega$, that is: $\forall\zeta \in [\omega]^\omega \exists n[\overline{\zeta}n \in B]$. Define
$C := \bigcup_k \{s \in [\omega]^k \mid \exists n < k[\alpha \circ s(n) = \beta \circ s(n)] \ \vee \ \neg\neg\exists p\forall n > p[\alpha(n) = \beta(n)]\}$.
Note: $C = \bigcup_k \{s \in [\omega]^k \mid \forall n < k[\alpha \circ s(n) \neq \beta \circ s(n)] \to \neg\neg\exists p\forall n > p[\alpha(n) = \beta(n)]\}$.

Note: $B \subseteq C$ and C is monotone, that is:
$\forall s \in [\omega]^{<\omega}[s \in C \to \forall n[s * \langle n\rangle \in [\omega]^{<\omega} \to s * \langle n\rangle \in C]]$.

We still have to prove that C is inductive.

[27] This application of countable choice may be reduced to Axiom 3. One may define $\mathcal{B} \subseteq \mathcal{N}$ and a *coding* mapping $\alpha \mapsto R_\alpha$ such that $\mathcal{E} = \{R_\alpha \mid \alpha \in \mathcal{B}\}$.

Let s in $[\omega]^{<\omega}$ be given such that $\forall n[s * \langle n\rangle \in [\omega]^{<\omega} \rightarrow s * \langle n\rangle \in C]]$. We want to prove: $s \in C$.

Find k such that $s \in [\omega]^k$. In case $\exists n < k[\alpha \circ s(n) = \beta \circ s(n)]$, $s \in C$, and we are done, so we assume $\forall n < k[\alpha \circ s(n) \neq \beta \circ s(n)]$.

Define q such that $q := 0$ if $k = 0$ and $q := s(k-1)$ if $k > 0$.

Consider two special cases:

Case (1): $\exists n > q[\alpha(n) \neq \beta(n)]$.

Find such n, note: $s * \langle n\rangle \in [\omega]^\omega$ and $\forall i < k+1[\alpha \circ (s * \langle n\rangle)(i) \neq \beta \circ (s * \langle n\rangle)(i)]$ and $s * \langle n\rangle \in C$, and conclude: $\neg\neg\exists p\forall n > p[\alpha(n) = \beta(n)]$.

Case (2): $\neg\exists n > q[\alpha(n) \neq \beta(n)]$, and, therefore, $\forall n > q[\alpha(n) = \beta(n)]$.

In both cases, we find: $\neg\neg\exists p\forall n > p[\alpha(n) = \beta(n)]$.

Conclude[28]: $\neg\neg\exists p\forall n > p[\alpha(n) = \beta(n)]$, and: $s \in C$.

We thus see that C is inductive.

Using Axiom 4, we conclude: $\langle\ \rangle \in C$, and, therefore,
$\neg\neg\exists p\forall n > p[\alpha(n) = \beta(n)]$, that is: $\neg\neg(\alpha \sim_V \beta)$. □

Corollary 4 (i) $(\mathcal{N}, \sim_V^{\neg\neg}) \models \forall \mathsf{x}\forall \mathsf{y}[\neg\neg(\mathsf{x} = \mathsf{y}) \rightarrow \mathsf{x} = \mathsf{y}]$.
(ii) *For each R in $\mathcal{E}$,* $(\mathcal{N}, R) \models \forall \mathsf{x}\neg\forall \mathsf{y}[\neg\neg(\mathsf{x} = \mathsf{y}) \rightarrow \mathsf{x} = \mathsf{y}]$.

Proof (i) Obvious, as, for any proposition P, $\neg\neg\neg\neg P \leftrightarrow \neg\neg P$.
(ii) Assume $R \in \mathcal{E}$.
We first prove: $(\mathcal{N}, R) \models \neg\forall \mathsf{x}\forall \mathsf{y}[\neg\neg(\mathsf{x} = \mathsf{y}) \rightarrow \mathsf{x} = \mathsf{y}]$.
Assume $\forall\alpha\forall\beta[\neg\neg(\alpha R\beta) \rightarrow \alpha R\beta]$.
Note: $\forall\alpha\forall\beta[\alpha \sim_V \beta \rightarrow \alpha R\beta]$ and, therefore: $\forall\alpha\forall\beta[\neg\neg(\alpha \sim_V \beta) \rightarrow \neg\neg(\alpha R\beta)]$.
Conclude: $\sim_V^{\neg\neg} \subseteq R$.
By Theorem 22 (ii), $R^+ \subseteq \sim_V^{\neg\neg}$, so $R^+ \subseteq R$. This contradicts Corollary 3.
The stronger statement announced in the Theorem may be proven in a similar way. Inspection of he proof of Theorem 22 enables one to conclude:
$(\mathcal{N}, R) \models \neg\forall \mathsf{y}[\neg\neg(\mathsf{x} = \mathsf{y}) \rightarrow \mathsf{x} = \mathsf{y}][\underline{0}]$.
One easily generalizes this conclusion to:
for each α, $(\mathcal{N}, R) \models \neg\forall \mathsf{y}[\neg\neg(\mathsf{x} = \mathsf{y}) \rightarrow \mathsf{x} = \mathsf{y}][\alpha]$.
Conclude: $(\mathcal{N}, R) \models \forall \mathsf{x}\neg\forall \mathsf{y}[\neg\neg(\mathsf{x} = \mathsf{y}) \rightarrow \mathsf{x} = \mathsf{y}]$. □

Markov's Principle has been mentioned in Sect. 1.4. Markov's Principle is not accepted in intuitionistic mathematics, but the following observation still is of interest.

Corollary 5 *The following are equivalent.*

(i) *Markov's Principle:* $\forall\alpha[\neg\neg\exists n[\alpha(n) = 0] \rightarrow \exists n[\alpha(n) = 0]]$.
(ii) $\sim_V^{\neg\neg} \subseteq \sim_V^{almost}$.
(iii) $\sim_V^{almost}$ *is stable.*

[28]Using the scheme: If $P \rightarrow Q$ and $\neg P \rightarrow Q$, then $\neg\neg Q$.

Proof (i) $\Rightarrow$ (ii). Assume $\neg\neg(\alpha \sim_V \beta)$, that is $\neg\neg\exists n \forall m > n[\alpha(m) = \beta(m)]$.

Let $\zeta \in [\omega]^\omega$ be given.

Assume: $\neg\exists n[\alpha \circ \zeta(n) = \beta \circ \zeta(n)]$.

Then $\forall n[\zeta(n+1) > n \wedge \alpha \circ \zeta(n) \neq \beta \circ \zeta(n)]$, so $\forall n \exists m > n[\alpha(m) \neq \beta(m)]$. Contradiction.

Conclude: $\neg\neg\exists n[\alpha \circ \zeta(n) = \beta \circ \zeta(n)]$ and, by Markov's Principle, $\exists n[\alpha \circ \zeta(n) = \beta \circ \zeta(n)]$.

We thus see $\forall \zeta \in [\omega]^\omega \exists n[\alpha \circ \zeta(n) = \beta \circ \zeta(n)]$, that is: $\alpha \sim_V^{almost} \beta$.

(ii) $\Rightarrow$ (iii). By Theorem 22 (v), $\sim_V^{almost} \subseteq \sim_V^{\neg\neg}$. Therefore: $(\sim_V^{almost})^{\neg\neg} \subseteq \sim_V^{\neg\neg}$. Using (ii), we conclude: $(\sim_V^{almost})^{\neg\neg} \subseteq \sim_V^{almost}$, that is: $\sim_V^{almost}$ is stable.

(iii) $\Rightarrow$ (i). Let α be given such that $\neg\neg\exists n[\alpha(n) \neq 0]$.

Define β such that $\forall m[\beta(m) = 0 \leftrightarrow \exists n \leq m[\alpha(n) = 0]]$.

Note: $\neg\neg(\beta \sim_V \underline{0})$ and, therefore: $\neg\neg(\beta \sim_V^{almost} \underline{0})$.

Conclude, using (iii), $\beta \sim_V^{almost} \underline{0}$.

Define ζ such that $\forall n[\zeta(n) = n]$.

Find m such that $\beta \circ \zeta(m) = \beta(m) = 0$ and, therefore, $\exists n \leq m[\alpha(n) = 0]$.

We thus see: $\forall \alpha[\neg\neg\exists n[\alpha(n) = 0] \rightarrow \exists n[\alpha(n) = 0]]$, that is: Markov's Principle. □

1.12 Equality and Equivalence

We did not succeed in finding a sentence ψ such that $(\mathcal{N}, \sim_V) \models \psi$ and $(\mathcal{N}, \sim_V^\omega) \models \neg\psi$. We now want to compare the structures $(\mathcal{N}, =, \sim_V)$ and $(\mathcal{N}, =, \sim_V^\omega)$. We need a first order language with two binary relation symbols: $=$ and $\sim$. The symbol $=$ will denote the equality relation and the symbol $\sim$ will denote, in the first structure, the relation $\sim_V$ and, in the second structure, the relation $\sim_V^\omega$. The reader hopefully will not be confused by the fact that, in the earlier sections, where we used the first order language with a single binary relation symbol, $=$, the symbol $=$ denoted the relations $\sim_V$ and $\sim_V^\omega$.

The next Theorem makes us see that equality is decidable on each equivalence class of $\sim_V$ whereas, on each equivalence class of $\sim_V^\omega$, it is not decidable.

Theorem 23 (i) $(\mathcal{N}, =, \sim_V) \models \forall \mathsf{x} \forall \mathsf{y}[\mathsf{x} \sim \mathsf{y} \rightarrow (\mathsf{x} = \mathsf{y} \vee \neg(\mathsf{x} = \mathsf{y}))]$.
(ii) $(\mathcal{N}, =, \sim_V^\omega) \models \forall \mathsf{x} \neg\forall \mathsf{y}[\mathsf{x} \sim \mathsf{y} \rightarrow (\mathsf{x} = \mathsf{y} \vee \neg(\mathsf{x} = \mathsf{y}))]$.

Proof (i) Let γ, α be given such that $\gamma \sim_V \alpha$.

Find n such that $\forall m > n[\gamma(m) = \alpha(m)]$ and distinguish two cases.

Either $\overline{\gamma}(m+1) = \overline{\alpha}(m+1)$ and $\gamma = \alpha$, *or* $\overline{\gamma}(m+1) \neq \overline{\alpha}(m+1)$ and $\neg(\gamma = \alpha)$.

Conclude: $\forall \gamma \forall \alpha[\gamma \sim_V \alpha \rightarrow (\gamma = \alpha \vee \neg(\gamma = \alpha))]$.

(ii) Let γ be given.

Consider $\mathcal{F}_1^\gamma := \{\alpha \mid \forall m \forall n[(\alpha(m) \neq \gamma(m) \wedge \alpha(n) \neq \gamma(n)) \rightarrow m = n]\}$.

Note: $\mathcal{F}_1^\gamma$ is a spread. Also: $\forall \alpha \in \mathcal{F}_1^\gamma[\gamma \sim_V^1 \alpha]$ [29] and, therefore, $\forall \alpha \in \mathcal{F}_1^\gamma[\gamma \sim_V^\omega \alpha]$. Assume $\forall \alpha \in \mathcal{F}_1^\gamma[\gamma = \alpha \ \vee \ \neg(\gamma = \alpha)]$. Applying Lemma 1, find p such that *either* $\forall \alpha \in \mathcal{F}_1^\gamma[\overline{\gamma}p \sqsubset \alpha \to \gamma = \alpha]$ *or* $\forall \alpha[\overline{\gamma}p \sqsubset \alpha \to \neg(\gamma = \alpha)]$, and note that both alternatives are false.

Conclude: $\forall \gamma \neg \forall \alpha[\gamma \sim_V^\omega \alpha \ \vee \neg(\gamma = \alpha)]$.

□

Lemma 9 $(\sim_V^{\neg\neg})^+ \subseteq \sim_V^{\neg\neg}$ *and* $(\sim_V^{almost})^+ \subseteq \sim_V^{almost}$.[30]

Proof Assume $\alpha(\sim_V^{\neg\neg})^+\beta$.

Find n such that $\forall m > n[\alpha(m) \neq \beta(m) \to \alpha \sim_V^{\neg\neg} \beta]$.

Note: if $\exists m > n[\alpha(m) \neq \beta(m)]$, then $\alpha \sim_V^{\neg\neg} \beta$, and if $\neg\exists m > n[\alpha(m) \neq \beta(m)]$, then $\forall m > n[\alpha(m) = \beta(m)]$ and $\alpha \sim_V \beta$ and also $\alpha \sim_V^{\neg\neg} \beta$.

Conclude: $\neg\neg(\alpha \sim_V^{\neg\neg} \beta)$, and, therefore, $\alpha \sim_V^{\neg\neg} \beta$.

Assume $\alpha(\sim_V^{almost})^+\beta$.

Find n such that $\forall m > n[\alpha(m) \neq \beta(m) \to \alpha \sim_V^{almost} \beta]$.

Let ζ in $[\omega]^\omega$ be given. Note: $\zeta(n+1) > n$.

Either: $\alpha \circ \zeta(n+1) = \beta \circ \zeta(n+1)$

or: $\alpha \sim_V^{almost} \beta$ and $\exists p[\alpha \circ \zeta(p) = \beta \circ \zeta(p)]$.

We thus see: $\forall \zeta \in [\omega]^\omega \exists n[\alpha \circ \zeta(n) = \beta \circ \zeta(n)]$, that is: $\alpha \sim_V^{almost} \beta$. □

Lemma 10 *For every shift-invariant binary relation R on $\mathcal{N}$,*
$R^+ \subseteq R$ *if and only if* $(\mathcal{N}, =, R) \models \forall x \forall y[(AP(x, y) \to x \sim y) \to x \sim y]$.

Proof First assume $R^+ \subseteq R$.

Assume $\alpha \# \beta \to \alpha R \beta$.

Then: $\forall m > 0[\alpha(m) \neq \beta(m) \to \alpha R \beta]$, so: $\alpha R^+ \beta$, and, therefore: $\alpha R \beta$.

We thus see: $(\mathcal{N}, =, R) \models \forall x \forall y[(AP(x, y) \to x \sim y) \to x \sim y]$.

Now assume $(\mathcal{N}, =, R) \models \forall x \forall y[(AP(x, y) \to x \sim y) \to x \sim y]$.

Assume $\alpha R^+ \beta$. Find n such that $\forall m > n[\alpha(m) \neq \beta(m) \to \alpha R \beta]$.

Define γ, δ such that $\forall m[\gamma(m) = \alpha(n+1+m) \ \wedge \ \delta(m) = \beta(n+1+m)]$.

Note: $\gamma \# \delta \to \alpha R \beta$, and, as R is shift-invariant, also: $\gamma \# \delta \to \gamma R \delta$, and, therefore: $\gamma R \delta$, and also: $\alpha R \beta$.

We thus see: $R^+ \subseteq R$. □

Corollary 6 (i) $(\mathcal{N}, =, \sim_V^{\neg\neg}) \models \forall x \forall y[(AP(x, y) \to x \sim y) \to x \sim y]$.
(ii) $(\mathcal{N}, =, \sim_V^{almost}) \models \forall x \forall y[(AP(x, y) \to x \sim y) \to x \sim y]$.
(iii) *For each R in $\mathcal{E}$,* $(\mathcal{N}, =, R) \models \neg\forall x \forall y[(AP(x, y) \to x \sim y) \to x \sim y]$.

Proof Use Lemmas 9 and 10 and Corollary 3. □

[29] See the proof of Theorem 18 (iii).

[30] Following the terminology in Veldman (1995), a binary relation R on $\mathcal{N}$ should be called *perhapsive* if $R^+ \subseteq R$.

1.13 Notations and Conventions

We use $m, n, \ldots$ as variables over the set $\omega = \mathbb{N}$ of the natural numbers.

For every $P \subseteq \mathbb{N}$ such that $\forall n[P(n) \vee \neg P(n)]$, for all m,

$m = \mu n[P(n)] \leftrightarrow \big(P(m) \wedge \forall n < m[\neg P(n)]\big)$.

$(m, n) \mapsto J(m, n)$ is a one-to-one surjective mapping from $\omega \times \omega$ onto ω.

$K, L : \omega \to \omega$ are its inverse functions, so $\forall n[J\big(K(n), L(n)\big) = n]$.

For each n, $n' := K(n)$ and $n'' := L(n)$.

$(n_0, n_1, \ldots, n_{k-1}) \mapsto \langle n_0, n_1, \ldots, n_{k-1}\rangle$ is a one-to-one surjective mapping from the set of finite sequences of natural numbers to the set of the natural numbers.

$\langle n_0, n_1, \ldots, n_{k-1}\rangle$ is the *code* of the finite sequence $(n_0, n_1, \ldots, n_{k-1})$.

$s \mapsto length(s)$ is is the function that, for each s, gives the length of the finite sequence coded by s.

$s, n \mapsto s(n)$ is the function that, for all s, n, gives the value of the finite sequence coded by s at n. If $n \geq length(s)$, then $s(n) = 0$.

For all s, k, if $length(s) = k$, then $s = \langle s(0), s(1), \ldots s(k-1)\rangle$.

$0 = \langle\ \rangle$ codes the empty sequence of natural numbers,
the unique finite sequence s such that $length(s) = 0$.

$\omega^k := \{s \mid length(s) = k\}$.

$[\omega]^k := \{s \in \omega^k \mid \forall i[i + 1 < k \to s(i) < s(i+1)]\}$.

$[\omega]^{<\omega} := \bigcup_k [\omega]^k$.

For all s, k, t, l, if $s \in \omega^k$ and $t \in \omega^l$, then $s * t$ is the element u of ω^{k+l} such that $\forall i < k[u(i) = s(i)]$ and $\forall j < l[u(k + j) = t(j)]$.

$s \sqsubseteq t \leftrightarrow \exists u[s * u = t]$.

$s \sqsubset t \leftrightarrow (s \sqsubseteq t \wedge s \neq t)$.

We use $\alpha, \beta, \ldots$ as variables over *Baire space*, the set $\omega^\omega := \mathcal{N}$ of functions from $\mathbb{N}$ to $\mathbb{N}$.

$(\alpha, n) \mapsto \alpha(n)$ is the function that associates to all α, n, the value of α at n.

For all α, β, $\alpha \circ \beta$ is the element γ of $\mathcal{N}$ such that $\forall n[\gamma(n) = \alpha\big(\beta(n)\big)]$.

$2^\omega := \mathcal{C} := \{\alpha \mid \forall n[\alpha(n) < 2]\}$ is *Cantor space*.

For all α, for all k, for all s in ω^k, $\alpha \circ s$ is the element t of ω^k satisfying $\forall n < k[t(k) = \alpha\big(s(k)\big)]$.

For each s, k, if $s \in \omega^k$, then, for each α, $s * \alpha$ is the element β of $\mathcal{N}$ such that $\forall i < k[\beta(i) = s(i)]$ and $\forall i[\beta(k + i) = \alpha(i)]$.

For each s, for each $\mathcal{X} \subseteq \mathcal{N}$, $s * \mathcal{X} := \{s * \alpha \mid \alpha \in \mathcal{X}\}$.

For each α, for each n, α^n is the element of $\mathcal{N}$ satisfying $\forall m[\alpha^n(m) = \alpha\big(J(n, m)\big)]$.

For each m, $\underline{m} \in \mathcal{N}$ is the element of $\mathcal{N}$ satisfying $\forall n[\underline{m}(n) = m]$.

S is the element of $\mathcal{N}$ satisfying $\forall n[S(n) = n + 1]$.

$\forall n[\alpha'(n) = \big(\alpha(n)\big)' \wedge \alpha''(n) = \big(\alpha(n)\big)'']$.

$\overline{\alpha}n := \langle\alpha(0), \alpha(1), \ldots \alpha(n-1)\rangle$.

$s \sqsubset \alpha \leftrightarrow \exists n[\overline{\alpha}n = s]$.

$\alpha \perp \beta \leftrightarrow \alpha \# \beta \leftrightarrow \exists n[\alpha(n) \neq \beta(n)]$.

$[\omega]^\omega := \{\zeta \in \mathcal{N} \mid \forall i[\zeta(i) < \zeta(i+1)]\}$.

$\mathbb{Q}$, the set of the rationals, may be defined as a subset of ω, with accompanying relations $=_{\mathbb{Q}}$, $<_{\mathbb{Q}}$, $\leq_{\mathbb{Q}}$ and operations $+_{\mathbb{Q}}$, $-_{\mathbb{Q}}$, $\cdot_{\mathbb{Q}}$.

$\mathcal{R} := \{\alpha \mid \forall n[\alpha'(n) \in \mathbb{Q} \;\wedge\; \alpha''(n) \in \mathbb{Q}] \;\wedge \forall n[\alpha'(n) \leq_{\mathbb{Q}} \alpha'(n+1) \leq_{\mathbb{Q}} \alpha''(n+1) \leq_{\mathbb{Q}} \alpha''(n)] \;\wedge\; \forall m \exists n[\alpha''(n) -_{\mathbb{Q}} \alpha'(n) <_{\mathbb{Q}} \frac{1}{2^m}]\}$.

For all α, β in $\mathcal{R}$,

$\alpha <_{\mathcal{R}} \beta \leftrightarrow \exists n[\alpha''(n) <_{\mathbb{Q}} \beta'(n)]$ and $\alpha =_{\mathcal{R}} \beta \leftrightarrow \big(\neg(\alpha <_{\mathcal{R}} \beta) \;\wedge\; \neg(\beta <_{\mathcal{R}} \alpha)\big)$.

Operations $+_{\mathcal{R}}$, $-_{\mathcal{R}}$ are defined straightforwardly.

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Chapter 2
Binary Modal Companions for Subintuitionistic Logics

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Abstract The weak subintuitionistic logic WF, for which no standard unary modal companion is known, is found to have a strict implication logic as its binary modal companion. It is also shown that for all modal logics extending the weak logic EN, classical modal logic with necessitation, a strict implication logic exists which is essentially equivalent to it. This logic extends a basic strict implication logic plus an axiom U, and conversely each such logic corresponds to a modal logic extending EN. Among other things this means that any subintuitionistic logic which has a modal companion has a strict implication companion as well.

Keywords Subintuitionistic logic · Modal logic · Classical modal logic · Intermediate logic · Modal companion · Strict implication

2.1 Introduction

Subintuitionistic logics as a theme were first studied by Corsi (1987), who introduced a basic system F. The system F, which cannot prove formulas like $A \to (B \to A)$ and $A \to (B \to A \wedge B)$, has Kripke frames in which no assumption of preservation of truth is made and which are neither reflexive nor transitive. She also introduced Gödel-type translations of these systems into modal logic. Restall (1994) defined

The original version of this chapter was revised: The chapter author's (Dick de Jongh) incorrect family name information have been updated. The correction to this chapter is available at https://doi.org/10.1007/978-3-030-53654-1_19

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M. Mojtahedi et al. (eds.), *Mathematics, Logic, and their Philosophies*, Logic, Epistemology, and the Unity of Science 49,
https://doi.org/10.1007/978-3-030-53654-1_2

a similar system SJ (see also de Jongh and Shirmohammadzadeh Maleki (2017)). Basic logic BPC, a much studied extension of F, had already been introduced before by Visser (1981) in a study mainly focussed on a further extension FPC of BPC with a provability interpretation. The system BPC has irreflexive Kripke frames with transitivity and preservation. A considerable amount work in the area, especially on BPC, has been done by Ardeshir in cooperation with members of his school and with W. Ruitenburg (see e.g. Ardeshir (1995), Ardeshir and Ruitenburg (1998)).

In our papers de Jongh and Shirmohammadzadeh Maleki (2018, 2019), Shirmohammadzadeh Maleki and de Jongh (2016) we introduced a basic logic WF much weaker than F, and we developed two types of neighborhood semantics for this logic and its extensions. In de Jongh and Shirmohammadzadeh Maleki (2018) we discussed the strength of the various subintuitionistic logics by investigating which part of intuitionistic logic IPC they are able to prove. A translation from IPC into BPC discovered by Ardeshir (1999) played an important role.

Furthermore, we discovered modal companions for a number of logics extending $\mathsf{WF_N}$, an extension of WF by a rule. The logic WF did not lend itself to our treatment because its semantics is too different from the usual neighborhood semantics for modal logic. In the present paper we looked for a binary modal companion for WF instead of the usual unary one. This exploration was successful as we will show. It also lead us to investigate the notion of binary modal logic, its neighborhood semantics and its relation to ordinary unary modal logic, and more specifically what we call classical strict implication logic, for which we give a complete basic system $\mathsf{E}^2_{\mathsf{Imp}}$. In fact, we show that all extensions L of the weak logic EN (classical modal logic with necessitation) have a unique counterpart logic L^* with a strict implication. All logics extending $\mathsf{E}^2_{\mathsf{Imp}}$ plus an axiom U are such a counterpart L^*, each is mutually interpretable with L, shares with it the usual logical properties and functions as a modal companion to the same subintuitionistic logics. This result exhibits which conditional logics can be represented in ordinary unary modal logic, at least if one restricts one's attention to modal logics extending EN, which indeed does seem to be a bare minimum.

2.2 Neighborhood Semantics for Modal and Subintuitionistic Logics

In this section we will give in Sect. 2.2.1 an introduction to the usual neighborhood semantics for modal logic followed in Sect. 2.2.2 by a quick survey of our neighborhood semantics for subintuitionistic logics and a summary of the results previously obtained by us.

2.2.1 Neighborhood Semantics for Modal Logic

Definition 2.1 The **modal language** $\mathcal{L}^{\Box}(At)$ is the smallest set of formulas generated by the following grammar, where $p \in At$:

$$p \mid \neg A \mid A \wedge B \mid \Box A.$$

The sublanguage $\mathcal{L}_c(At)$ of $\mathcal{L}^{\Box}(At)$ containing its formulas without $\Box$ is the language of (classical) propositional logic. We add to $\mathcal{L}_c(At)$ the symbols $\rightarrow$, $\leftrightarrow$, $\top$ and $\bot$ as symbols defined in the usual way.

Definition 2.2 A pair $\mathfrak{F} = \langle W, N\rangle$ is a ***Neighborhood Frame*** of modal logic if W is a non-empty set and N is a function from W into $\mathcal{P}(\mathcal{P}(W))$.
In a ***Neighborhood Model*** $\mathfrak{M} = \langle W, N, V\rangle$, $V : At \to \mathcal{P}(W)$ is a valuation function on the set of propositional variables At.

Definition 2.3 Let $\mathfrak{M} = \langle W, N, V\rangle$ be a neighborhood model and $w \in W$.
Truth of a propositional formula in a world w is defined inductively as follows.

1. $\mathfrak{M}, w \models p \quad \Leftrightarrow \quad w \in V(p)$,
2. $\mathfrak{M}, w \models \neg A \quad \Leftrightarrow \quad \mathfrak{M}, w \not\models A$,
3. $\mathfrak{M}, w \models A \wedge B \quad \Leftrightarrow \quad \mathfrak{M}, w \models A$ and $\mathfrak{M}, w \models B$,
4. $\mathfrak{M}, w \models \Box A \quad \Leftrightarrow \quad A^{\mathfrak{M}} \in N(w)$,

where $A^{\mathfrak{M}}$ denotes the truth set of A.

We consider the following axiom schemas and rules.

PC Any axiomatization of propositional calculus

N $\Box\top$

RE $\dfrac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$

MP $\dfrac{A \quad A \rightarrow B}{B}$

Nec $\dfrac{A}{\Box A}$

E is the smallest classical modal logic containing all instances of PC which is closed under the rules MP and RE. The logic EN extends E by adding the axiom scheme N, or by adding the rule Nec (Pacuit 2017).

Theorem 2.1

1. *The logic* E *is sound and strongly complete with respect to the class of all neighborhood frames* (Pacuit 2017).
2. *The logic* EN *is sound and strongly complete with respect to the class of neighborhood frames that contain the unit, i.e. for all* $w \in W$, $W \in N(w)$ (Pacuit 2017).

2.2.2 Neighborhood Semantics for Subintuitionistic Logics

Definition 2.4 The **language of intuitionistic propositional logic** $\mathcal{L}(At)$ is the smallest set of formulas generated by the following grammar, where $p \in At$:

$$p \mid A \wedge B \mid A \vee B \mid A \rightarrow B \mid \bot$$

As usual we consider $\mathcal{L}(At)$ to be an extension of $\mathcal{L}_c(At)$, so we will write $\rightarrow$ for both intuitionistic and classical implication. From the context it should be clear which is meant. To $\mathcal{L}(At)$ the the symbols $\neg$ and $\leftrightarrow$ are added as defined symbols in the usual manner. Again this should not create confusion with the symbols of classical propositional logic.

Definition 2.5 An **NB-Neighborhood Frame** $\mathfrak{F} = \langle W, NB \rangle$ for subintuitionistic logic consists of a non-empty set W, and a function NB from W into $\mathcal{P}((\mathcal{P}(W))^2)$ such that:

$$\forall w \in W,\ \forall X, Y \in \mathcal{P}(W)\ (X \subseteq Y \ \Rightarrow\ (X, Y) \in NB(w)).$$

In an **NB-Neighborhood Model** $\mathfrak{M} = \langle W, NB, V \rangle$, $V : At \rightarrow \mathcal{P}(W)$ is a valuation function on the set of propositional variables At.

Definition 2.6 Let $\mathfrak{M} = \langle W, NB, V \rangle$ be an NB-neighborhood model.
Truth of a propositional formula in a world w is defined inductively as follows.

1. $\mathfrak{M}, w \Vdash p \quad \Leftrightarrow \quad w \in V(p)$;
2. $\mathfrak{M}, w \Vdash A \wedge B \quad \Leftrightarrow \quad \mathfrak{M}, w \Vdash A$ and $\mathfrak{M}, w \Vdash B$;
3. $\mathfrak{M}, w \Vdash A \vee B \quad \Leftrightarrow \quad \mathfrak{M}, w \Vdash A$ or $\mathfrak{M}, w \Vdash B$;
4. $\mathfrak{M}, w \Vdash A \rightarrow B \quad \Leftrightarrow \quad \left(A^{\mathfrak{M}}, B^{\mathfrak{M}}\right) \in NB(w)$;
5. $\mathfrak{M}, w \nVdash \bot$.

A is **valid** in $\mathfrak{M}$, $\mathfrak{M} \Vdash A$, if for all $w \in W$, $\mathfrak{M}, w \Vdash A$, and A is valid in $\mathfrak{F}$, $\mathfrak{F} \Vdash A$ if for all $\mathfrak{M}$ on $\mathfrak{F}$, $\mathfrak{M} \Vdash A$. We write $\Vdash A$ if $\mathfrak{M} \Vdash A$ for all $\mathfrak{M}$. Also we define $\Gamma \Vdash A$ iff for all $\mathfrak{M}$, $w \in \mathfrak{M}$, if $\mathfrak{M}, w \Vdash \Gamma$ then $\mathfrak{M}, w \Vdash A$.

Definition 2.7 WF is the logic given by the following axiom schemas and rules,

1. $A \rightarrow A \vee B$	2. $B \rightarrow A \vee B$	3. $A \rightarrow A$
4. $A \wedge B \rightarrow A$	5. $A \wedge B \rightarrow B$	6. $\frac{A \quad A \rightarrow B}{B}$
7. $\frac{A \rightarrow B \quad A \rightarrow C}{A \rightarrow B \wedge C}$	8. $\frac{A \rightarrow C \quad B \rightarrow C}{A \vee B \rightarrow C}$	9. $\frac{A \rightarrow B \quad B \rightarrow C}{A \rightarrow C}$
10. $\frac{A}{B \rightarrow A}$	11. $\frac{A \leftrightarrow B \quad C \leftrightarrow D}{(A \rightarrow C) \leftrightarrow (B \rightarrow D)}$	12. $\frac{A \quad B}{A \wedge B}$
13. $A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$		14. $\bot \rightarrow A$

$\Gamma \vdash_{\mathsf{WF}} A$ iff there is a derivation of A from Γ using the rules 7, 8, 9, 10, 11 only when there are no assumptions, and the rule 6, MP, only when the derivation of $A \rightarrow B$ contains no assumptions.

For a discussion of the definition of $\Gamma \vdash_{\mathsf{WF}} A$ see Definition 4 of de Jongh and Shirmohammadzadeh Maleki (2019) and its introduction.

Theorem 2.2 (Weak Deduction Theorem, Shirmohammadzadeh Maleki and de Jongh (2016) Theorem 2.19)
$A \vdash_{\mathsf{WF}} B$ *iff* $\vdash_{\mathsf{WF}} A \to B$.
$A_1, \ldots, A_n \vdash_{\mathsf{WF}} B$ *iff* $\vdash_{\mathsf{WF}} A_1 \wedge \cdots \wedge A_n \to B$.

Theorem 2.3 *The logic* WF *is sound and strongly complete with respect to the class of* NB-*neighborhood frames.*

We now define a second type of neighborhood semantics for subintuitionistic logics, N-neighborhood frames and models. In fact these are exactly the same frames and models as for modal logic, except of course for the truth definition. This may be confusing but it enables us to compare the logics very comfortably.

Definition 2.8 $\mathfrak{F} = \langle W, N \rangle$ is an **N-Neighborhood Frame** of subintuitionistic logic if W is a non-empty set, N is a function from W into $\mathcal{P}(\mathcal{P}(W))$, and for each $w \in W$, $W \in N(w)$.
Valuation $V : At \to \mathcal{P}(W)$ makes $\mathfrak{M} = \langle W, N, V \rangle$ an **N-Neighborhood Model**. **Truth** of a propositional formula in a world w is defined inductively as follows.

1. $\mathfrak{M}, w \Vdash p \quad \Leftrightarrow \quad w \in V(p)$;
2. $\mathfrak{M}, w \Vdash A \wedge B \quad \Leftrightarrow \quad \mathfrak{M}, w \Vdash A$ and $\mathfrak{M}, w \Vdash B$;
3. $\mathfrak{M}, w \Vdash A \vee B \quad \Leftrightarrow \quad \mathfrak{M}, w \Vdash A$ or $\mathfrak{M}, w \Vdash B$;
4. $\mathfrak{M}, w \Vdash A \to B \quad \Leftrightarrow \quad \{v \mid v \Vdash A \;\Rightarrow\; v \Vdash B\} = \overline{A^{\mathfrak{M}}} \cup B^{\mathfrak{M}} \in N(w)$;
5. $\mathfrak{M}, w \nVdash \bot$.

A formula A is **valid** in $\mathfrak{M}$, $\mathfrak{M} \Vdash A$, if for all $w \in W$, $\mathfrak{M}, w \Vdash A$, and A is valid in $\mathfrak{F}$, $\mathfrak{F} \Vdash A$ if for all $\mathfrak{M}$ on $\mathfrak{F}$, $\mathfrak{M} \Vdash A$. We write $\Vdash A$ if $\mathfrak{M} \Vdash A$ for all $\mathfrak{M}$. Also we define $\Gamma \Vdash A$ iff for all $\mathfrak{M}$, $w \in \mathfrak{M}$, if $\mathfrak{M}, w \Vdash \Gamma$ then $\mathfrak{M}, w \Vdash A$.

The question whether validity in NB-neighborhood frames and N-neighborhood frames is the same was resolved in de Jongh and Shirmohammadzadeh Maleki (2019). The difference resides in the rule N. To the system WF we add this rule to obtain the logic $\mathsf{WF_N}$:

$$\frac{A \to B \vee C \quad C \to A \vee D \quad A \wedge C \wedge D \to B \quad A \wedge C \wedge B \to D}{(A \to B) \leftrightarrow (C \to D)} \qquad (\mathsf{N})$$

Theorem 2.4 (Weak Deduction Theorem, de Jongh and Shirmohammadzadeh Maleki (2019) Theorem 8)
$A \vdash_{\mathsf{WF_N}} B$ *iff* $\vdash_{\mathsf{WF_N}} A \to B$.
$A_1, \ldots, A_n \vdash_{\mathsf{WF_N}} B$ *iff* $\vdash_{\mathsf{WF_N}} A_1 \wedge \cdots \wedge A_n \to B$.

Theorem 2.5 (Completeness of $\mathsf{WF_N}$, de Jongh and Shirmohammadzadeh Maleki (2019) Theorem 12) *The logic* $\mathsf{WF_N}$ *is sound and strongly complete with respect to the class of* N-*neighborhood frames.*

We consider the translation $\Box$ from $\mathcal{L}(At)$, the language of intuitionistic propositional logic, to $\mathcal{L}^{\Box}(At)$, the language of modal propositional logic (see Corsi (1987), de Jongh and Shirmohammadzadeh Maleki (2018)). It is given by:

1. $p^{\Box} = p$;
2. $\bot^{\Box} = \bot$;
3. $(A \wedge B)^{\Box} = A^{\Box} \wedge B^{\Box}$;
4. $(A \vee B)^{\Box} = A^{\Box} \vee B^{\Box}$;
5. $(A \to B)^{\Box} = \Box(A^{\Box} \to B^{\Box})$.

Note that in (5.) above the first $\to$ is a symbol from $\mathcal{L}$ whereas the second $\to$ is a defined symbol of $\mathcal{L}_c$. This need not cause confusion since from the context in which $\to$ occurs it will always be clear in which language it occurs.

Theorem 2.6 (de Jongh and Shirmohammadzadeh Maleki (2018), Theorem 5.17) *For all formulas A,*

$$\vdash_{\mathsf{WF_N}} A \ \text{ iff } \ \vdash_{\mathsf{EN}} A^{\Box}.$$

As one says, EN is a *modal companion* of $\mathsf{WF_N}$. For WF the question how to provide it with a modal companion was left open in de Jongh and Shirmohammadzadeh Maleki (2018). It is not easy to imagine a modal logic which weakens EN but leaves N in.

2.3 A Complete Basic System for Strict Implication

In this section we define a neighborhood semantics for modal logic with a binary operator and we introduce a basic system which is sound and complete for this semantics. One might not consider it to be quite proper to call this basic system a system of strict implication since it allows extensions to systems for counterfactuals but it is the best we have come up with.

Definition 2.9 The **strict implication language** $\mathcal{L}^{\Rightarrow}(At)$ is the smallest set of formulas generated by the following grammar, where $p \in At$:

$$p \mid \neg A \mid A \wedge B \mid A \Rightarrow B.$$

As in the case the modal language the language $\mathcal{L}_c(At)$ is a sublanguage of $\mathcal{L}^{\Rightarrow}(At)$, and we again have the usual defined symbols. The *NB*-neighborhood frames and models of subintuitionistic logic can be used as frames and models for strict implication logic, again with a different truth definition.

Definition 2.10 A pair $\mathfrak{F} = \langle W, NB \rangle$ is called a **Neighborhood Frame** of strict implication logic if W is a non-empty set and NB is a neighborhood function from W into $\mathcal{P}((\mathcal{P}(W))^2)$ such that

$$\forall w \in W,\ \forall X, Y \in \mathcal{P}(W),\ (X \subseteq Y\ \Rightarrow\ (X, Y) \in NB(w)).$$

If we delete the final requirement on the neighborhood function we obtain a more general semantics for binary modal logic, but in this article we focus on implication because this is all we are interested in at this point. Generally, our results will stand when we delete this condition. The results will then concern not EN but E.

Definition 2.11 A **Neighborhood Model** of strict implication logic is a tuple $\mathfrak{M} = \langle W, NB, V\rangle$, where $\langle W, NB\rangle$ is a neighborhood frame of strict implication logic and $V : At \to \mathcal{P}(W)$ a valuation function.

Definition 2.12 Let $\mathfrak{M} = \langle W, NB, V\rangle$ be a neighborhood model for strict implication logic and $w \in W$. **Truth** of a propositional formula in a world w is defined inductively as follows.

1. $\mathfrak{M}, w \models p \quad \Leftrightarrow \quad w \in V(p)$,
2. $\mathfrak{M}, w \models \neg A \quad \Leftrightarrow \quad \mathfrak{M}, w \not\models A$,
3. $\mathfrak{M}, w \models A \wedge B \quad \Leftrightarrow \quad \mathfrak{M}, w \models A$ and $\mathfrak{M}, w \models B$,
4. $\mathfrak{M}, w \models A \Rightarrow B \quad \Leftrightarrow \quad \left(A^{\mathfrak{M}}, B^{\mathfrak{M}}\right) \in NB(w)$,

where $A^{\mathfrak{M}}$ denotes the truth set of A.

Definition 2.13 A formula A is **valid in a model** $\mathfrak{M} = \langle W, NB, V\rangle$, $\mathfrak{M} \models A$, if for all $w \in W$, $\mathfrak{M}, w \models A$. If all models force A, we write $\models A$ and call A **valid**. A formula A is **valid on a frame** $\mathfrak{F} = \langle W, NB\rangle$, $\mathfrak{F} \models A$ if A is valid in every model based on that frame. We write $\Gamma \models A$, A is a **valid consequence of** Γ, if, for each model $\mathfrak{M} = \langle W, NB, V\rangle$ and $w \in W$, if $\mathfrak{M}, w \models \Gamma$, then $\mathfrak{M}, w \models A$.

The definitions above mean that a model $\mathfrak{M} = \langle W, NB, V\rangle$ will simultaneously be a model for the subintuitionistic language and for the strict implication language. This will enable us to compare the languages and the systems formulated in them directly in Sect. 2.4. We will then, to avoid confusion, use different symbols for the two notions of $\models$.

In this section we will be interested in the following axiom schemas and rules.

$$E^2\quad \frac{A \leftrightarrow B \quad C \leftrightarrow D}{(A \Rightarrow C) \leftrightarrow (B \Rightarrow D)}$$

$$Imp\quad \frac{A \to B}{A \Rightarrow B}$$

Definition 2.14 $\mathsf{E}^2_{\mathsf{Imp}}$ is the smallest set of formulas containing all instances of PC closed under the rules E^2, Imp and MP. We call it **Classical Strict Implication Logic**.

If one leaves out the rule Imp, then one obtains what one might call *Classical Binary Modal Logic*. In fact, this logic occurs as CK in Chellas (1975). We won't discuss it here, but as said, basically our results will extend to that more general case. We will now prove the completeness of $\mathsf{E}^2_{\mathsf{Imp}}$ in a rather standard way (compare Shirmohammadzadeh Maleki and de Jongh (2016)).

Definition 2.15 Let $W_{\mathsf{E}^2_{\mathsf{Imp}}}$ be the set of all $\mathsf{E}^2_{\mathsf{Imp}}$-maximally consistent sets of formulas. Given a formula A, we define the set $[\![A]\!]$ as follows,

$$[\![A]\!] = \left\{\Delta \mid \Delta \in W_{\mathsf{E}^2_{\mathsf{Imp}}},\ A \in \Delta\right\}.$$

Lemma 2.1 *Let C and D are formulas. Then*

(a) $[\![C \wedge D]\!] = [\![C]\!] \cap [\![D]\!]$.
(b) $[\![C \vee D]\!] = [\![C]\!] \cup [\![D]\!]$.
(c) *If* $[\![C]\!] \subseteq [\![D]\!]$ then $\vdash C \to D$.
(d) $[\![C]\!] = [\![D]\!]$ iff $\vdash C \leftrightarrow D$.

Proof The proofs are easy. □

Definition 2.16 The **Canonical model** $\mathfrak{M}^{\mathsf{E}^2_{\mathsf{Imp}}} = \langle W_{\mathsf{E}^2_{\mathsf{Imp}}}, NB_{\mathsf{E}^2_{\mathsf{Imp}}}, V\rangle$ of $\mathsf{E}^2_{\mathsf{Imp}}$ is defined by:

1. For each $\Gamma \in W_{\mathsf{E}^2_{\mathsf{Imp}}}$ and all formulas A and B,

$$NB_{\mathsf{E}^2_{\mathsf{Imp}}}(\Gamma) = \{([\![A]\!], [\![B]\!]) \mid A \Rightarrow B \in \Gamma\} \cup \{(X, Y) \mid X \subseteq Y\}.$$

2. If $p \in At$, then $V(p) = [\![p]\!] = \left\{\Gamma \mid \Gamma \in W_{\mathsf{E}^2_{\mathsf{Imp}}} \textit{ and } p \in \Gamma\right\}$.

In the completeness proof we need to be sure that, if $([\![A]\!], [\![B]\!]) \in NB_{\mathsf{E}^2_{\mathsf{Imp}}}(\Gamma)$, then $A \Rightarrow B \in \Gamma$.

Lemma 2.2 *If* $NB_{\mathsf{E}^2_{\mathsf{Imp}}} : W_{\mathsf{E}^2_{\mathsf{Imp}}} \to \mathcal{P}((\mathcal{P}(W_{\mathsf{E}^2_{\mathsf{Imp}}}))^2)$ *is a function such that for each* $\Gamma \in W_{\mathsf{E}^2_{\mathsf{Imp}}}$, $NB_{\mathsf{E}^2_{\mathsf{Imp}}}(\Gamma)=\{([\![A]\!], [\![B]\!]) \mid A \Rightarrow B \in \Gamma\} \cup \{(X, Y) \mid X \subseteq Y\}$. *Then* $([\![A]\!], [\![B]\!])\in NB_{\mathsf{E}^2_{\mathsf{Imp}}}(\Gamma)$ *implies* $A \Rightarrow B \in \Gamma$.

Proof Assume $([\![A]\!], [\![B]\!]) \in NB_{\mathsf{E}^2_{\mathsf{Imp}}}(\Gamma)$. This gives us two possibilities:

1. For some C, D, $[\![A]\!] = [\![C]\!]$, $[\![B]\!] = [\![D]\!]$, $C \Rightarrow D \in \Gamma$,
2. $[\![A]\!] \subseteq [\![B]\!]$.

If (1), then by Lemma 2.1, we have $\vdash A \leftrightarrow C$ and $\vdash B \leftrightarrow D$. Hence by rule E^2 we will have $\vdash (A \Rightarrow B) \leftrightarrow (C \Rightarrow D)$. By assumption, $C \Rightarrow D \in \Gamma$. Hence, $A \Rightarrow B \in \Gamma$.

If (2), then by Lemma 2.1, we have $\vdash A \to B$. Then by rule Imp we will have $\vdash A \Rightarrow B$. Hence, $A \Rightarrow B \in \Gamma$. □

Theorem 2.7 (***Truth Lemma***) *For any consistent formula D, if $\mathfrak{M}$ is the canonical model of* $\mathsf{E}^2_{\mathsf{Imp}}$, *then* $D^{\mathfrak{M}} = [\![D]\!]$.

Proof We only consider the $D := A \Rightarrow B$ case, the other cases are as usual. Let $\Gamma \in W_{\mathsf{E}^2_{\mathsf{Imp}}}$, then,

$$\begin{array}{rcl} \Gamma \models A \Rightarrow B & \iff & (A^{\mathfrak{M}}, B^{\mathfrak{M}}) \in NB_{\mathsf{E}^2_{\mathsf{Imp}}}(\Gamma) \\ \text{(by induction hypothesis)} & \iff & ([\![A]\!], [\![B]\!]) \in NB_{\mathsf{E}^2_{\mathsf{Imp}}}(\Gamma) \\ \text{(by Lemma 2.2)} & \iff & A \Rightarrow B \in \Gamma. \end{array}$$

□

Theorem 2.8 *The classical strict implication logic* $\mathsf{E}^2_{\mathsf{Imp}}$ *is sound and strongly complete with respect to the class of neighborhood frames.*

Proof Soundness is straightforward. For strong completeness, suppose Σ is a consistent set of the classical strict implication logic $\mathsf{E}^2_{\mathsf{Imp}}$. By Lindenbaum's Lemma there is a maximal consistent set Σ^* extending Σ. Then by Lemma 2.7, $\mathfrak{M}^{\mathsf{E}^2_{\mathsf{Imp}}}, \Sigma^* \models \Sigma$, and we have shown that each consistent set has a model. □

2.4 Modal Companions

We consider the translation $\Rightarrow$ from $\mathcal{L}$, the language of intuitionistic propositional logic, to $\mathcal{L}^{\Rightarrow}$, the language of classical strict implication logic. It is given by:

1. $p^{\Rightarrow} := p$;
2. $\bot^{\Rightarrow} := \bot$;
3. $(A \wedge B)^{\Rightarrow} := A^{\Rightarrow} \wedge B^{\Rightarrow}$;
4. $(A \vee B)^{\Rightarrow} := A^{\Rightarrow} \vee B^{\Rightarrow}$;
5. $(A \to B)^{\Rightarrow} := (A^{\Rightarrow} \Rightarrow B^{\Rightarrow})$.

As said above we can use neighborhood models to interpret subintuitionistic formulas and modal or strict implication formulas simultaneously. We distinguish these uses by writing $\Vdash$ for truth for subintuitionistic formulas and $\models$ for truth for classical strict implication formulas.

Lemma 2.3 *Let* $\mathfrak{M} = \langle W, NB, V\rangle$ *be a neighborhood model. Then for all* $w \in W$,

$$\mathfrak{M}, w \Vdash A \quad \text{iff} \quad \mathfrak{M}, w \models A^{\Rightarrow}.$$

Proof The proof is by induction on A. The atomic case holds by induction and the conjunction and disjunction cases are easy. We only check the implication case. So let $A = C \to D$, then

$$\begin{array}{rcl} \mathfrak{M}, w \Vdash C \to D & \iff & (C^{\mathfrak{M}}, D^{\mathfrak{M}}) \in NB(w) \\ \text{(by induction hypothesis)} & \iff & ((C^{\Rightarrow})^{\mathfrak{M}}, (D^{\Rightarrow})^{\mathfrak{M}}) \in NB(w) \\ & \iff & \mathfrak{M}, w \models C^{\Rightarrow} \Rightarrow D^{\Rightarrow} \\ & \iff & \mathfrak{M}, w \models (C \to D)^{\Rightarrow}. \end{array}$$

□

Theorem 2.9 *For all formulas* A,

$$\vdash_{\mathsf{WF}} A \ \text{ iff } \ \vdash_{\mathsf{E}^2_{\mathsf{Imp}}} A^{\Rightarrow}.$$

Proof By Theorem 2.8 and Lemma 2.3. □

Lemma 2.4 *If* $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} A \Rightarrow B$ *then* $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} A \to B$.

Proof Suppose that there is a model $\mathfrak{M} = \langle W, NB, V\rangle$ and a point $w \in W$ such that $\mathfrak{M}, w \nvDash_{\mathsf{E}^2_{\mathsf{Imp}}} A \to B$. Then, $\mathfrak{M}, w \vDash_{\mathsf{E}^2_{\mathsf{Imp}}} A$ and $\mathfrak{M}, w \nvDash_{\mathsf{E}^2_{\mathsf{Imp}}} B$, therefore $A^{\mathfrak{M}} \nsubseteq B^{\mathfrak{M}}$. Let $\mathfrak{F}'$ be $\mathfrak{F}$ augmented by a g such that $NB(g) = \{(X, Y) \mid X \subseteq Y\}$ and $\mathfrak{M}' = \langle \mathfrak{F}', V\rangle$. Since $A^{\mathfrak{M}} \nsubseteq B^{\mathfrak{M}}$ and hence $\mathfrak{M}', g \nvDash_{\mathsf{E}^2_{\mathsf{Imp}}} A \Rightarrow B$, we have $\nvDash_{\mathsf{E}^2_{\mathsf{Imp}}} A \Rightarrow B$. □

The following theorem, proved by using the Weak Deduction Theorem (2.2) and Lemmas 2.9 and 2.4, shows that the translation works under assumptions.

Theorem 2.10 $\Gamma \vdash_{\mathsf{WF}} A$ iff $\Gamma^{\Rightarrow} \vdash_{\mathsf{E}^2_{\mathsf{Imp}}} A^{\Rightarrow}$.

Proof $B_1, \ldots, B_k \vdash_{\mathsf{WF}} A \;\Leftrightarrow\; \vdash_{\mathsf{WF}} B_1 \wedge \cdots \wedge B_k \to A \;\Leftrightarrow\; \vdash_{\mathsf{E}^2_{\mathsf{Imp}}} (B_1 \wedge \cdots \wedge B_k \to A)^{\Rightarrow} \;\Leftrightarrow\; \vdash_{\mathsf{E}^2_{\mathsf{Imp}}} (B_1^{\Rightarrow} \wedge \cdots \wedge B_k^{\Rightarrow} \Rightarrow A^{\Rightarrow}) \;\Leftrightarrow\; \vdash_{\mathsf{E}^2_{\mathsf{Imp}}} B_1^{\Rightarrow} \wedge \cdots \wedge B_k^{\Rightarrow} \to A^{\Rightarrow} \;\Leftrightarrow\; B_1^{\Rightarrow}, \ldots, B_k^{\Rightarrow} \vdash_{\mathsf{E}^2_{\mathsf{Imp}}} A^{\Rightarrow}$. □

2.5 Translations

In this section we will show that $\mathsf{E}^2_{\mathsf{Imp}}$ and EN are very closely related by translations. The first section will treat formulas, the second will extend this to logics, and in the third we will show what happens to axiomatizations.

2.5.1 *Translations Between* $\mathsf{E}^2_{\mathsf{Imp}}$ *and* EN

Definition 2.17 The mapping * from $\mathcal{L}^{\Box}$ to $\mathcal{L}^{\Rightarrow}$ is defined by

1. $(p)^* := p$,
2. $(\neg A)^* := \neg A^*$,
3. $(A \wedge B)^* := A^* \wedge B^*$,
4. $(\Box A)^* := \top \Rightarrow A^*$.

Theorem 2.11 *If* $\vdash_{\mathsf{EN}} A$, *then* $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} A^*$.

Proof We use induction on the derivation of A. We only consider the rules Nec and E. First rule $\frac{A}{\Box A}$:

1. $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} A^*$ by induction hypothesis
2. $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} \top \to A^*$ by 1
3. $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} \top \Rightarrow A^*$ by 2 and rule Imp
4. $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} (\Box A)^*$ by 3

Rule $\frac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$:

1. $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} A^* \leftrightarrow B^*$ by induction hypothesis
2. $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} \top \leftrightarrow \top$
3. $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} (\top \Rightarrow A^*) \leftrightarrow (\top \Rightarrow B^*)$ by 1, 2 and rule E^2
4. $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} (\Box A)^* \leftrightarrow (\Box B)^*$ by 3

□

Definition 2.18 The mapping $\sharp$ from $\mathcal{L}^{\Rightarrow}$ to $\mathcal{L}^{\Box}$ is defined by

1. $(p)^\sharp := p$,
2. $(\neg A)^\sharp := \neg A^\sharp$,
3. $(A \wedge B)^\sharp := A^\sharp \wedge B^\sharp$,
4. $(A \Rightarrow B)^\sharp := \Box(A^\sharp \rightarrow B^\sharp)$.

Theorem 2.12 *If* $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} A$, *then* $\vdash_{\mathsf{EN}} A^\sharp$.

Proof We use induction on the derivation of A. We only consider the rules Imp and E^2. First, rule $\frac{A \rightarrow B}{A \Rightarrow B}$:

1. $\vdash_{\mathsf{EN}} (A \rightarrow B)^\sharp$ by induction hypothesis
2. $\vdash_{\mathsf{EN}} A^\sharp \rightarrow B^\sharp$ by 1
3. $\vdash_{\mathsf{EN}} \Box(A^\sharp \rightarrow B^\sharp)$ by 2 and rule Nec
4. $\vdash_{\mathsf{EN}} (A \Rightarrow B)^\sharp$ by 3

Rule $\dfrac{A \leftrightarrow B \quad C \leftrightarrow D}{(A \Rightarrow C) \leftrightarrow (B \Rightarrow D)}$:

1. $\vdash_{\mathsf{EN}} A^\sharp \leftrightarrow B^\sharp$ by induction hypothesis
2. $\vdash_{\mathsf{EN}} C^\sharp \leftrightarrow D^\sharp$ by induction hypothesis
3. $\vdash_{\mathsf{EN}} (A^\sharp \rightarrow C^\sharp) \leftrightarrow (B^\sharp \rightarrow D^\sharp)$ by 1, 2
4. $\vdash_{\mathsf{EN}} \Box(A^\sharp \rightarrow C^\sharp) \leftrightarrow \Box(B^\sharp \rightarrow D^\sharp)$ by 3 and rule RE
5. $\vdash_{\mathsf{EN}} (A \Rightarrow C)^\sharp \leftrightarrow (B \Rightarrow D)^\sharp$ by 4
6. $\vdash_{\mathsf{EN}} ((A \Rightarrow C) \leftrightarrow (B \Rightarrow D))^\sharp$

□

We can combine the * and $\sharp$-translations:

Lemma 2.5 $\vdash_{\mathsf{EN}} A \leftrightarrow A^{*\sharp}$.

Proof By induction on A. The atomic case holds by definition and the conjunction and disjunction cases are trivial.

Assume $A = \Box B$, we need to show that $\vdash \Box B \leftrightarrow (\Box B)^{*\sharp}$. By definition, $(\Box B)^{*\sharp}$ is equal to $(\top \Rightarrow B^*)^\sharp$, which is equal to $\Box(\top \rightarrow B^{*\sharp})$, which is $\Box(B^{*\sharp})$. Then this is equal to $\Box B$, by the induction hypothesis. □

Theorem 2.13 *If* $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} A^*$ *then* $\vdash_{\mathsf{EN}} A$.

Proof Assume $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} A^*$, then by Lemma 2.12, $\vdash_{\mathsf{EN}} A^{*\sharp}$. Again, by Lemma 2.5, we conclude that $\vdash_{\mathsf{EN}} A$. □

Corollary 2.1

1. $\vdash_{\mathsf{EN}} A$ *iff* $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} A^*$

2. $\Gamma \vdash_{\mathsf{EN}} A$ *iff* $\Gamma^* \vdash_{\mathsf{E}^2_{\mathsf{Imp}}} A^*$.

Proof (1) By combining Theorem 2.13 with Theorem 2.11.
(2) By applying the weak deduction theorem to (1). □

We call a translation a *faithful interpretation* if provability is preserved in both directions. So, with this terminology we can say that Corollary 2.1 states that $*$ is a faithful interpretation of EN into $\mathsf{E}^2_{\mathsf{Imp}}$.

Contrary to this result about $*$ it is not so that $\sharp$ is a faithful interpretation of $\mathsf{E}^2_{\mathsf{Imp}}$ into EN. Clearly $\vdash_{\mathsf{EN}} ((\top \Rightarrow (p \to q)) \leftrightarrow (p \Rightarrow q))^\sharp$, but if we consider the neighborhood frame $\mathfrak{F} = \langle W, NB \rangle$ with

$$W = \{w, v\},\ NB(w) = \{(\{v\}, \{w\})\} \cup \{(X, Y) \mid X \subseteq Y\},$$
$$NB(v) = \{(X, Y) \mid X \subseteq Y\},$$

and the valuation $V(p) = \{v\}$, $V(q) = \{w\}$, then it is easy to show that $w \nvDash (\top \Rightarrow (p \to q)) \leftrightarrow (p \Rightarrow q)$, that is $\nvdash_{\mathsf{E}^2_{\mathsf{Imp}}} (\top \Rightarrow (p \to q)) \leftrightarrow (p \Rightarrow q)$.

2.5.2 Translations Between Extensions of $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}$ *and* EN

To make $\sharp$ a faithful interpretation we have to extend $\mathsf{E}^2_{\mathsf{Imp}}$ by an axiom. Let us introduce the axiom U: $(\top \Rightarrow (A \to B)) \leftrightarrow (A \Rightarrow B)$. It characterizes the class of frames closed under equivalence. $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}$ is the system $\mathsf{E}^2_{\mathsf{Imp}}$ with the axiom U.

Definition 2.19 Neighborhood frame $\mathfrak{F} = \langle W, NB \rangle$ is closed under **equivalence** if for all $w \in W$, $(X, Y) \in NB(w)$ if and only if $(W, \overline{X} \cup Y) \in NB(w)$.

Lemma 2.6 *The formula* $(\top \Rightarrow (p \to q)) \leftrightarrow (p \Rightarrow q)$ *characterizes the class of neighborhood frames* $\mathfrak{F} = \langle W, NB \rangle$ *satisfying closure under equivalence.*

Proof Let $\mathfrak{F}$ be closed under equivalence and $\mathfrak{M} = \langle W, NB, V \rangle$ be any model based on $\mathfrak{F}$. We have to prove for all $w \in W$, $w \models (\top \Rightarrow (p \to q)) \leftrightarrow (p \Rightarrow q)$. This is easy, because:

$w \models \top \Rightarrow (p \to q)$	iff	$(W, \overline{V(p)} \cup V(q)) \in NB(w)$
by the equivalence condition	iff	$(V(p), V(q)) \in NB(w)$
	iff	$w \models p \Rightarrow q$.

For the other direction, we use contraposition. Suppose that the class is not closed under equivalence. Then there is a frame $\mathfrak{F}$ and $w \in \mathfrak{F}$ such that $(X, Y) \in NB(w)$ but

$(W, \overline{X} \cup Y) \notin NB(w)$. Consider the valuation V such that, $V(p) = X$ and $V(q) = Y$. Then, $w \models p \Rightarrow q$ and $w \not\models \top \Rightarrow (p \rightarrow q)$. Therefore $\mathfrak{F} \not\models (p \Rightarrow q) \rightarrow (\top \Rightarrow (p \rightarrow q))$. Similarly to this we can show that if $(W, \overline{X} \cup Y) \in NB(w)$ and $(X, Y) \notin NB(w)$ then $\mathfrak{F} \not\models (\top \Rightarrow (p \rightarrow q)) \rightarrow (p \Rightarrow q)$. □

The translations $*$ and $\sharp$ have semantical meaning as well of course. This is especially useful in the case of extensions of $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}$. That is because NB-neighborhood models satisfying closure under equivalence are essentially equivalent to N-neighborhood models (see de Jongh and Shirmohammadzadeh Maleki (2019)). We state the crucial lemmas from that paper.

Lemma 2.7 *Let $\langle W, N\rangle$ be an* N*-neighborhood frame. Then there exists an equivalent* NB*-neighborhood frame $\langle W, NB\rangle$. This* NB*-frame is closed under* N*-equivalence, i.e., if $(X, Y) \in NB(w)$ and $(X, Y) \equiv (X', Y')$, then $(X', Y') \in NB(w)$. In addition, for all X, Y, w, if $X \subseteq Y$, then $(X, Y) \in NB(w)$.*

Proof The proof is straightforward by considering, for each $w \in W$,

$$NB(w) = \left\{(X, Y) \mid \overline{X} \cup Y \in N(w)\right\}.$$ □

Lemma 2.8 *Let $\langle W, NB\rangle$ be an* NB*-neighborhood frame closed under* N*-equivalence. Then there exists an equivalent* N*-neighborhood frame $\langle W, N\rangle$.*

Proof The proof is straightforward by considering, for each $w \in W$,

$$N(w) = \left\{\overline{X} \cup Y \mid (X, Y) \in NB(w)\right\}.$$ □

This allows us to interpret strict implication formulas in N-neighborhood models and modal formulas in NB-neighborhood models for $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}$. We just state the consequences here without working out the details completely.

Lemma 2.9

1. *For any N-neighborhood model $\mathfrak{M}$ for modal logic and any modal formula $A(p_1, \ldots, p_n)$, $A^{\mathfrak{M}} = (A^*)^{\mathfrak{M}}$.*
2. *For any neighborhood model for strict implication logic which is closed under* N*-equivalence and any strict implication formula $A(p_1, \ldots, p_n)$, $A^{\mathfrak{M}} = (A^\sharp)^{\mathfrak{M}}$.*

This lemma extends to the Kripke model case when we define $w \models A \Rightarrow B$ as, for all v such that wRv, if $w \models A$, then $w \models B$ (see Definition 2.22).

Lemma 2.10

1. *For any Kripke model $\mathfrak{M}$ for modal logic and any modal formula $A(p_1, \ldots, p_n)$, $A^{\mathfrak{M}} = (A^*)^{\mathfrak{M}}$.*
2. *For any Kripke model $\mathfrak{M}$ for strict implication logic and any strict implication formula $A(p_1, \ldots, p_n)$, $A^{\mathfrak{M}} = (A^\sharp)^{\mathfrak{M}}$.*

Theorem 2.14 *If $\vdash_{\mathsf{E}^2_{Imp}U} A$, then $\vdash_{\mathsf{EN}} A^\sharp$.*

Proof By Theorem 2.12, we just need to show that $\vdash_{\mathsf{EN}} \mathsf{U}^\sharp$ and this is easy. Because $(\top \Rightarrow (A \to B))^\sharp \leftrightarrow (A \Rightarrow B))^\sharp$ is equal to $\Box(\top \to (A^\sharp \to B^\sharp) \leftrightarrow \Box(A^\sharp \to B^\sharp)$, which is provable in EN. □

Lemma 2.11 $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}} A \leftrightarrow A^{\sharp *}$.

Proof By induction on A. The atomic case holds by definition and the conjunction and disjunction cases are trivial.

Assume $A = C \Rightarrow D$, we need to show that $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}} (C \Rightarrow D) \leftrightarrow (C \Rightarrow D)^{\sharp *}$. By definition, $(C \Rightarrow D)^{\sharp *}$ is equal to $(\Box(C^\sharp \to D^\sharp))^*$, which is equal to $(\top \Rightarrow (C^{\sharp *} \to D^{\sharp *}))$, and by axiom U is equal to $(C^{\sharp *} \Rightarrow D^{\sharp *})$. Then this is equal to $(C \Rightarrow D)$, by the induction hypothesis. □

Theorem 2.15 *If* $\vdash_{\mathsf{EN}} A^\sharp$ *then* $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}} A$.

Proof Assume $\vdash_{\mathsf{EN}} A^\sharp$, then by Theorem 2.11 $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}} A^{\sharp *}$. Again, by Lemma 2.11 we conclude that $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}} A$. □

Corollary 2.2 $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}} A$ *iff* $\vdash_{\mathsf{EN}} A^\sharp$.

Proof By combining Theorem 2.15 with Theorem 2.14. □

So, we have that $\sharp$ is a faithful translation of $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}$ into EN. We will now see that the classes of logics extending EN and $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}$ are closely related as well. A logic extending EN will be a set of formulas containing EN closed under its rules and uniform substitution. A logic extending $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}$ is similarly defined.

Definition 2.20

1. Suppose that L is a logic extending EN. We define L^* as the closure of $\{A^* | A \in L\} \cup \{\mathsf{U}\}$ under the rules of $\mathsf{E}^2_{\mathsf{Imp}}$.
2. Suppose that L is a logic extending $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}$. We define $L^\sharp$ as the closure of $\{A^\sharp | A \in L\}$ under the rules of EN.

Lemma 2.12 *If L is a logic extending* EN, *and $A \in L^*$, then $A^\sharp \in L$.*

Proof Suppose $A \in L^*$, then there is a finite number of $B_1^*, ..., B_n^*$, with $B_i \in L$, $1 \leq i \leq n$, such that $B_1^* \wedge \cdots \wedge B_n^* \vdash_{\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}} A$ and so $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}} B_1^* \wedge \cdots \wedge B_n^* \to A$. By Lemma 2.14 we have $\vdash_{\mathsf{EN}} B_1^{*\sharp} \wedge \cdots \wedge B_n^{*\sharp} \to A^\sharp$. Again, by Lemma 2.5, we conclude that $\vdash_{\mathsf{EN}} B_1 \wedge \cdots \wedge B_n \to A^\sharp$. Since $B_1 \wedge \cdots \wedge B_n \in L$, we have $A^\sharp \in L$. □

Theorem 2.16 *If L is a logic extending* EN, *then $L = L^{*\sharp}$.*

Proof First we prove $L \subseteq L^{*\sharp}$. Assume $A \in L$ then $A^* \in L^*$ and $A^{*\sharp} \in L^{*\sharp}$. By Lemma 2.5 $\vdash_{\mathsf{EN}} A \leftrightarrow A^{*\sharp}$. Hence $A \in L^{*\sharp}$.

For the opposite direction assume $A \in L^{*\sharp}$. Then there exist $B_1^\sharp, ..., B_n^\sharp$, with $B_i \in L^*$, $1 \leq i \leq n$, such that $B_1^\sharp \wedge \cdots \wedge B_n^\sharp \vdash_{\mathsf{EN}} A$. By Lemma 2.12 each $B_i^\sharp$ is in L. Therefore $A \in L$. □

This theorem basically means that each logic extending EN is represented by a logic extending $\mathsf{E}^2_{Imp}U$, by L^*. We can now directly see by an analogous proof that for extensions of $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}$ we can reverse the order of the translations in Theorem 2.16.

Theorem 2.17 *If L is a logic extending $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}$, then $L = L^{\sharp *}$.*

The two theorems together mean that there is a 1-1-correspondence between the logics extending EN and extending $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}$. To find the corresponding logic on the opposite side one only has to check the derivability via the translations on both directions. By the semantic meaning of the translations completeness of the corresponding logic then immediately follows for the same semantics. In fact, this holds for all the usual logical properties since the logics are essentially the same. Also, if one has a unary modal companion one finds in that manner a binary one and vice versa. Of course, this is restricted to logics extending EN or extending $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}$ respectively.

As an illustration we show directly that the new system $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}$ is a modal companion of $\mathsf{W}F_N$. First a very straightforward proposition.

Proposition 2.1 *For all subintuitionistic formulas A, $A^{\Box}$ is identical to $A^{\Rightarrow\sharp}$.*

Theorem 2.18 $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}$ *is a modal companion of* $\mathsf{WF_N}$.

Proof We can reason completely syntactically in this case. From Theorem 2.6 we know that EN is a modal companion of $\mathsf{WF_N}$: $\mathsf{WF_N} \vdash A$ iff $\mathsf{EN} \vdash A^{\Box}$. Thus, by Proposition 2.1, $\mathsf{WF_N} \vdash A$ iff $\mathsf{EN} \vdash A^{\Rightarrow\sharp}$. Applying Corollary 2.2 we then immediately get the desired conclusion: $\mathsf{WF_N} \vdash A$ iff $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U} \vdash A^{\Rightarrow}$. □

2.5.3 *Translations, Axiomatizations and Standard Modal Logics*

In this subsection we consider what happens if a logic extending EN is axiomatized by an axiom A. Then A does not function as a single sentence but it represents all its uniform substitution instances.

Theorem 2.19 $(\mathsf{EN} + A)^* = \mathsf{E}^2_{\mathsf{Imp}}\mathsf{U} + A^*$.

Proof Obviously $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U} + \mathsf{A}^* \subseteq (\mathsf{EN} + A)^*$. So, we just show the opposite inclusion. Assume $(\mathsf{EN} + A)^* \vdash B$. Then there are substitution instances $A_1, \ldots A_n$ of A such that EN proves $A_1 \wedge \cdots \wedge A_n \rightarrow B$. It is a trivial fact of translations and substitution that $(A_1)^*, \ldots (A_n)^*$ are substitution instances of A^*. So, $\mathsf{E}^2_{\mathsf{Imp}}U + A^*$ proves B^*. So, also $(\mathsf{EN} + A)^* \subseteq \mathsf{E}^2_{\mathsf{Imp}}U + A^*$. □

In other words, if L is a logic extending EN axiomatized over EN by A, then L^* is the logic axiomatized over $\mathsf{E}^2_{\mathsf{Imp}}U$ by A^*.

We now apply the results we have obtained to logics having Kripke models. We will find the strict implication variants $\overrightarrow{\mathsf{K}}$, $\overrightarrow{\mathsf{KT}}$, $\overrightarrow{\mathsf{K4}}$ and $\overrightarrow{\mathsf{S4}}$ as the unique correspondents of the logics K, KT, K4 and S4 obtained from the following schemas.

K	$\Box(A \to B) \to (\Box A \to \Box B)$
T	$\Box A \to A$
4	$\Box A \to \Box\Box A$

Definition 2.21 A **Kripke frame** $\mathfrak{F}$ is a pair $\langle W, R \rangle$, where W is a nonempty set and R is a binary relation on W. A **Kripke Model** $\mathfrak{M}$ based on a frame $\mathfrak{F}$ is a tuple $\langle W, R, V \rangle$ where $V : At \to 2^W$ is called a valuation function.

Definition 2.22 (**Truth in Kripke Models**) Let $\mathfrak{M} = \langle W, R, V \rangle$ be a Kripke model and $w \in W$. Truth of a propositional formula in a world w is defined inductively as follows.

1. $\mathfrak{M}, w \models p \quad \Leftrightarrow \quad w \in V(p)$,
2. $\mathfrak{M}, w \models \neg A \quad \Leftrightarrow \quad \mathfrak{M}, w \not\models A$,
3. $\mathfrak{M}, w \models A \wedge B \quad \Leftrightarrow \quad \mathfrak{M}, w \models A$ and $\mathfrak{M}, w \Vdash B$,
4. $\mathfrak{M}, w \models A \Rightarrow B \;\Leftrightarrow\;$ for each $w' \in W$ with wRw', if $\mathfrak{M}, w' \models A$, then $\mathfrak{M}, w' \models B$.

By Theorem 2.19 it is almost immediate that:

Theorem 2.20

1. $\overrightarrow{\mathsf{K}} = \mathsf{E}^2_{\mathsf{Imp}}\mathsf{UK}^*$,
2. $\overrightarrow{\mathsf{KT}} = \mathsf{E}^2_{\mathsf{Imp}}\mathsf{UK}^*\mathsf{T}^*$,
3. $\overrightarrow{\mathsf{K4}} = \mathsf{E}^2_{\mathsf{Imp}}\mathsf{UK}^*4^*$,
4. $\overrightarrow{\mathsf{S4}} = \mathsf{E}^2_{\mathsf{Imp}}\mathsf{UK}^*\mathsf{T}^*4^*$.

Proof We only need to note that EN follows from K. □

Let us just list the ∗-translations here:
$\mathsf{K}^* = (\top \Rightarrow (p \to q)) \to ((\top \Rightarrow p) \to (\top \Rightarrow q))$
$\mathsf{T}^* = (\top \Rightarrow p) \to p$
$4^* = (\top \Rightarrow p) \to ((\top \Rightarrow (\top \Rightarrow p))$

We do immediately get completeness of each of the systems $\overrightarrow{\mathsf{K}}$, $\overrightarrow{\mathsf{KT}}$, $\overrightarrow{\mathsf{K4}}$, $\overrightarrow{\mathsf{S4}}$ for their Kripke frames and all the regular properties of their correspondents. Surely, these logics can be given more elegant axiomatizations. For example, $\overrightarrow{\mathsf{K}}$ can also be axiomatized as $\mathsf{E}^2_{\mathsf{Imp}} + ((A \to B) \Rightarrow (C \to D)) \to ((A \Rightarrow B) \to (C \Rightarrow D))$.

Also, we immediately get

Theorem 2.21

1. $\overrightarrow{\mathsf{K}}$ *is a strict implication companion of* F*,*
2. $\overrightarrow{\mathsf{K4}}$ *is a strict implication companion of* BPC*,*
3. $\overrightarrow{\mathsf{S4}}$ *is a strict implication companion of* IPC*.*

Similarly we obtain that also the correspondent $\overrightarrow{\mathsf{wK4}}$ of wK4 is a strict implication companion of BPC because wK4 is a modal companion of BPC (see Sano and Ma (2015)).

2.6 Conclusion

We looked for a binary modal companion of the weak subintuitionistic logic WF and found it in the strict implication logic E^2_{Imp}. During this search we established also that any extension of the weak modal logic EN can just as well be represented as an equivalent strict implication logic, satisfying a new axiom U and conversely. Among other things this implies that any sub- or superintuitionistic logic which has a standard modal companion has a strict implication companion as well. This is grounded in the fact that $E^2_{Imp}U$ is a strict implication companion of WF_N. A next research goal would be the opposite direction: to find sub- and superintuitionistic logics corresponding to strict implication logics. This of course can only work if the strict implication logics satisfy the rules E^2 and Imp and the axiom U. Most of them do satisfy the rules E^2 and Imp (see Nute (1984)). Whether such logics satisfy the axiom U is another matter. Logics with Kripke models do satisfy U, but certainly the interpretability logics IL and its extensions (see e.g. Japaridze and de Jongh (1998)) do not qualify, since $\Box A$ is not definable as $\top \Rightarrow A$, but as $\neg A \Rightarrow \bot$. Also, logics for counterfactuals (see Lewis (1973), Veltman (1973)) do not satisfy axiom U. These may be approached differently.

Acknowledgements We thank Wesley Holliday for making the fruitful suggestion to us to look for a binary modal companion for WF instead of the usual unary modal companions. We thank Frank Veltman and Johannes Marti for helpful suggestions concerning conditional logic. Finally, we are greatly thankful to an unknown referee for coming up with many small points and some larger ones to improve the readability and content of the paper and for heeding us from some mistakes. The second author was as a postdoc researcher at school of Mathematics, Statistics and Computer Science of university of Tehran during year 2018.

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Chapter 3
Extension and Interpretability

Albert Visser

Abstract In this paper we study the combined structure of the relations of *theory-extension* and *interpretability between theories* for the case of finitely axiomatised theories. We focus on two main questions. The first is definability of salient notions in terms of the structure. We show, for example, that local tolerance, locally faithful interpretability and the finite model property are definable over the structure. The second question is how to think about 'good' properties of theories that are independent of implementation details and of 'bad' properties that do depend on implementation details. Our degree structure is suitable to study this contrast, since one of our basic relations, to wit theory-extension, is dependent on implementation details and the other relation, interpretability, is not. Nevertheless, we can define new good properties using bad ones. We introduce a new notion of sameness of theories *i-bisimilarity* that is second-order definable over our structure. We define a notion of *goodness* in terms of this relation. We call this notion *being capital*. We illustrate that some intuitively good properties, like being a complete theory, are not capital.

Keywords Degrees of interpretability · Lindenbaum algebras · Arithmetic · Sequentiality

Dedicated to Mohammad Ardeshir.

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M. Mojtahedi et al. (eds.), *Mathematics, Logic, and their Philosophies*,
Logic, Epistemology, and the Unity of Science 49,
https://doi.org/10.1007/978-3-030-53654-1_3

3.1 Introduction

The notion of first-order theory is surprisingly successful in logic. It is central in both proof theory and model theory. However, it clearly has its disadvantages. For example, we cannot define the notion of finiteness in a first-order way.[1] Moreover, the usual representations of first-order theories do not reflect mathematical practice well. For one thing, they are insufficiently abstract. In practice, we switch, as a matter of course, from one representation of the basic concepts to another one. Predicate logic however insists on a fixed signature. This rigidity could be called *the tyranny of signature*.

The present paper is a study within one paradigm of abstracting away from signature, the degrees of interpretability. Here we only think of theories modulo interpretability. This approach, however, has a striking disadvantage—at least as far as current knowledge goes. It is difficult to express intuitive salient notions in terms of the degree structure. We can do somewhat better, e.g., by switching to categories of theories and interpretations. See e.g. Visser (2006).

In the present paper, we follow a different strategy to enhance the expressive power of degree structures. We enrich our structures with the (signature dependent) notion of *theory extension*. As we will see, even if theory extension is bad from the standpoint of the need to abstract away from signature, it still can help us to define good notions.

We study degrees of interpretability enriched with theory extension for finitely axiomatised theories. Thus, our study has some connection with the study of sentence algebras. Sentence algebras are studied, e.g., in Hanf (1975) and de Myers (1989). The degrees of interpretability for finitely axiomatised theories, are studied in, e.g., Švejdar (1978), Lindström (1979, 1984a, b, 2003), Bennet (1986), Friedman (2007), Visser (2012, 2014).[2]

We consider the structure of consistent, finitely axiomatised theories with as relations *extension-in-the-same-language* and *parameter-free, multi-dimensional, piecewise interpretability*. As will become apparent from the paper, this structure seems to be best suited to the treatment of *local* notions. We show, for example, that locally faithful interpretability, local tolerance and the finite model property are definable in the structure. With respect to faithful interpretability and sequentiality, we have the partial negative result that they are not definable in a certain fragment of the first-order language over the structure. The question of definability in the full language remains open.

The best notion of sameness between theories is, in my opinion, *bi-interpretability*. This notion preserves many good mathematical properties of theories that are as diverse as finite axiomatisability, decidability and κ-categoricity. Bi-interpretability is however not definable in the framework of our paper. We replace bi-interpretability

[1] This weakness is also a strength, since it gives us e.g. non-standard models of arithmetic.

[2] In a sense, the fundamental paper Mycielski et al. (1990) also belongs to this list, since chapters, i.e. degrees of local interpretability, can be viewed as ideals of the degrees of global interpretability for finitely axiomatised theories.

by a cruder notion, to wit (*local*) *i-bisimilarity*. We study the concept of 'good property' derived from i-bisimilarity. We call this idea of goodness *being capital*. We illustrate what i-bisimilarity in the context of our degree structure can and cannot do. E.g., we show that the notions of complete theory, of sequential theory and of faithful interpretability are not capital, where they are definitely good when measured by our best notion of sameness: bi-interpretability.

3.2 Preliminaries

In this section, we give basic definitions and discuss some basic facts.

3.2.1 Signatures, Formulas, Theories

Signatures will be finite in this paper. A signature is given by a finite set of predicate symbols and an arity function that assigns an arity in ω to each symbol. To avoid having to deal with class-many signatures, we restrict the predicate symbols to elements of a fixed countable list $P_0, P_1, \ldots$.

For any signature Σ, we take $\mathcal{L}_\Sigma$ to be the set of sentences of that signature. We assign a finitely axiomatised theory $\mathcal{I}_\Sigma$ to Σ. This is the theory of identity for Σ including the axiom $\exists x\, x = x$.

We will use $A, B, C, \ldots$ as variables ranging over consistent finitely axiomatised theories of finite signature. We take the theory of identity $\mathcal{I}_\Sigma$ as part of predicate logic CQC_Σ of signature Σ. Still we insist that $\mathcal{I}_\Sigma$ is *also* syntactically present in a theory. The reason for this somewhat strange stipulation is that we allow identity to be translated to some other relation than identity. Hence, we need the axioms of identity to be interpreted.[3]

We use φ, ψ as ranging over sentences. We suppose that theories and sentences have built-in signature, so that we can read out the signature from the sentence. So, e.g. $\exists x\, P(x)$ is a different sentence depending on whether it occurs in a language of signature with only the predicate symbol P or whether there is, e.g., a second predicate symbol Q.

A basic relation between theories is extension. By this we mean: extension-in-the-same-language. More precisely B is an extension A when $\Sigma_A = \Sigma_B$ and $B \vdash A$. We write $A \subseteq B$ for: B extends A in the same language.

A second basic relation is interpretability or $\lhd$. We will explain this in Sect. 3.2.2 and in Appendix 3.8.

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[3]The unnaturality is caused by the fact that, on the one hand, we treat identity as *a logical relation*, and, on the other hand, unlike other logical relations, we do not translate it to itself. The truly coherent way of proceeding would be to work without identity as part of the logic and with a free logic. This way would greatly simplify things. In this paper, however, we stick to the traditional framework and do the sometimes boring homework to treat identity.

A special theory we will consider is the arithmetical theory S^1_2. See, e.g., Buss (1986) or Hájek and Pudlák (1993, Chap. V). The theory S^1_2 is finitely axiomatisable. See Hájek and Pudlák (1993, Chap. V(e)). Let the signature of the language of S^1_2 be $\mathcal{A}$.

3.2.2 Translations and Interpretations

In this subsection, we give a brief informal explanation of what translations and interpretations are. For all details, the reader is referred to Appendix 3.8.

A translation maps a language to a language. The basic idea is very simple: the translation commutes with the logical connectives. In spite of the apparent simplicity, there is some work to do to give a full and correct definition of a translation. There are two reasons. The first is that we have to get nasty details concerning renaming of variables out of the way. The second is that we want to add a number of features. The idea in this paper is to ignore the problem of the variables. It is clear that it can be taken care of in some appropriate way. We will however have a lot to say in Appendix 3.8 about the extra features. The features are these:

- Our translations do not necessarily translate identity to identity. Identity may go to a formula that is intended to represent an equivalence relation.
- Our translations are relativised. We will relativise the translations of the quantifiers to prespecified domains.
- Our translations are more dimensional. This means that one object of the domain of quantification of the translated language may be represented as a sequence of objects in the translating language. In a more syntactic formulation: one variable of the translated language is associated with a sequence of variables in the translating language.
- We allow piecewise translations. This means that our domains can be built up from various pieces. These pieces may be of different dimensions. Moreover, even if they are of the same dimension, a sequence of elements shared by two pieces may, in the context of the first piece, represent a different object of the translated language than the object represented by the sequence in the context of the second piece.

! Nota Bene
In general, translations may have parameters. In this paper, however, we only consider the parameter-free case.

We use τ, τ', μ, ν, ... to range over translations. We write φ^τ for the τ-translation of φ.

The notion of translation yields the notion of interpretation. If we have theories U and V, then U *interprets* V or $U \rhd V$ or $V \lhd U$ iff $U \vdash \varphi^{\tau}$, for all theorems φ of V. We say that U and V are *mutually interpretable* or $U \bowtie V$ iff $U \rhd V$ and $V \lhd U$.

3.2.3 The Structure

We will study the structure $\mathbb{EI}$ of extension and interpretability. Its objects are finitely axiomatised consistent theories A, B, ... The structure has two relations $\subseteq$ and $\lhd$. Here $\subseteq$ is a partial ordering, $\lhd$ is a partial pre-ordering and $\subseteq$ is a subrelation of $\lhd$.

The structure $\mathbb{EI}$ is countable since we restricted the symbols from our signatures to symbols from a fixed countable list.

? Question
How does $\mathbb{EI}$ as we defined it relate to $\mathbb{EI}^+$, the analogous structure where we allow class-many signatures? In some sense nothing new should happen, but, since we are interested both in first- and in second-order properties, a careful analysis of the relevant notion of sameness is needed.

We will be interested in properties $P(A_0, \ldots, A_{n-1})$ of finitely axiomatised consistent theories that can be formulated in terms of the structure. In various ways these properties can be divided in good and bad properties. Given a notion of sameness for theories $\sim$ we say that P is $\sim$*-invariant* if P preserves $\sim$, in other words, if $A_0 \sim B_0$, ..., $A_{n-1} \sim B_{n-1}$ and $P(A_0, \ldots, A_{n-1})$, then $P(B_0, \ldots, B_{n-1})$.

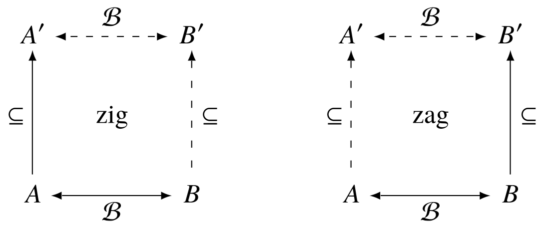
I believe in the intuitive thesis that the right notion of sameness between theories is *bi-interpretability*. In other words, preservation of $\sim_{\mathsf{bii}}$ is true goodness. Bi-interpretability is more flexible than the very strict notion of synonymy, but still preserves many mathematically interesting properties like categoricity and decidability. There are two more salient notions iso-congruence and sentential or elementary congruence. See Appendix 3.8.6. All of these notions have an important property that mutual interpretability and mutual faithful interpretability lack. Suppose $A \sim B$, where $\sim$ is synonymy or bi-interpretability or iso-congruence or sentential congruence. Suppose we add a new axiom φ to A. Then, we can add a matching axiom ψ to B such that $(A + \varphi) \sim (B + \psi)$.[4] Of course, since our relations $\sim$ are equivalence relations, we also have the converse: if we add a ψ to B, there is a matching φ for A. Thus $\sim$ is a bisimulation with respect to $\subseteq$. There is a crudest notion below mutual interpretability or $\bowtie$ that has this property, to wit *i-bisimilarity*.[5]

[4]For a proof of this claim, see Appendix 3.8.6.6.

[5]The notion should really be called: *local i-bisimilarity*. However, since this paper is about local notions, we omit the 'local'.

An *i-bisimulation* is a relation $\mathcal{B}$ on $\mathbb{EI}$ such that (i) $\mathcal{B}$ is a subrelation of $\bowtie$ and (ii) $\mathcal{B}$ is a bisimulation w.r.t. $\subseteq$. We remind the reader that $\mathcal{B}$ is a bisimulation w.r.t. $\subseteq$ iff, it has both the forward or zig property and the backward or zag property: if $A \ \mathcal{B} \ B$, then

zig for all $A' \supseteq A$, there is a $B' \supseteq B$, such that $A' \ \mathcal{B} \ B'$;
zag for all $B' \supseteq B$, there is an $A' \supseteq A$, such that $A' \ \mathcal{B} \ B'$.



The theories A and B are *i-bisimilar* or $A \approx B$ iff, there an i-bisimulation $\mathcal{B}$, such that $A \ \mathcal{B} \ B$. We note that $\approx$ is the maximal i-bisimulation. Since i-bisimulations contain the identity relation and are closed under converse and composition, we find that $\approx$ is an equivalence relation.

We note that each of synonymy, bi-interpretability, iso-congruence and sentential congruence is both a bisimulation and a sub-relation of $\bowtie$. Hence each of these relations is a sub-relation of $\approx$.

Example 3.1 We consider the theory EQ of pure equality and the theory $\mathbb{1}$ in the language of pure equality with an axiom that states that there is precisely one object. Let INF be the theory in the language of pure equality with axioms that state 'there are at least n objects' for each n. We note that:

- If both T and U have a finite model, then T and U are mutually interpretable. (See Remark 3.5.)
- Any finite extension of EQ has a finite model.
- $\mathbb{1}$ and INF are not mutually interpretable.

It follows that $\mathbb{1} \approx \mathsf{EQ}$. So,

a. $A \approx B$ does not imply that A and B are mutually faithfully interpretable.
b. If we also allow infinitely axiomatised theories, then $\mathbb{1}$ and EQ are not i-bisimilar in the modified sense, since $\mathbb{1}$ and INF are not mutually interpretable.
c. If we replace $\subseteq$ by $\subsetneq$, then $\mathbb{1}$ and EQ are not i-bisimilar in the modified sense, since we can not strictly extend $\mathbb{1}$.

The example is somewhat trifling. The reader may well feel the wish to tweak the framework a bit to exclude the example. We note however that, if we omit piecewise interpretations, then a slightly modified version of the example still works; similarly, for the case where we exclude theories with finite models. What happens when we restrict ourselves to one-dimensional interpretations? We take this up in Example 3.8.

? Question
We define $A \approx_0 B$ iff $A \bowtie B$ and $A \approx_{n+1} B$ iff $A \bowtie B$ and, for all $A' \supseteq A$, there is a $B' \supseteq B$, such that $A' \approx_n B'$, and for all $B' \supseteq B$, there is an $A' \supseteq A$, such that $A' \approx_n B'$. Clearly, $\approx_{n+1}$ is a subrelation of $\approx_n$, and $\approx$ is a subrelation of the intersection $\approx_\omega$ of the $\approx_n$. Are all these inclusions strict?

We see that the notion of i-bisimularity has a Σ^1_1-description over $\mathbb{EI}$. We call a property *capital* if it is $\approx$-invariant.

We note that all capital properties are also $\sim$-invariant where $\sim$ is synonymy, bi-interpretability, iso-congruence and sentential equivalence, since each of these relations is a sub-relation of $\approx$. The converse does not hold: as we will see *being a complete theory* is not capital but it is $\sim$-invariant for sentential congruence and, *ipso facto*, for all more refined equivalence relations.

We introduce a fragment $\mathcal{F}$ of first-order formulas over $\mathbb{EI}$ such that every formula φ in $\mathcal{F}$ defines a capital property. We define $\exists A' \trianglerighteq A \ldots$ by $\exists A'\,(A \supseteq A' \wedge \ldots)$ and we define $\forall A' \trianglerighteq A \ldots$ by the formula $\forall A'\,(A \supseteq A' \to \ldots)$. Here we assume that A' and A are distinct variables. The fragment $\mathcal{F}$ is defined as the smallest set of formulas such that $\bot$, $\top$ and $A \lhd B$ are in $\mathcal{F}$, and, if φ and ψ are in $\mathcal{F}$, then so are $\neg\varphi$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \to \psi)$, $\forall A\,\varphi$, $\exists A\,\varphi$, $\forall A \trianglerighteq B\,\varphi$, and $\exists A \trianglerighteq B\,\varphi$, where, in the last two cases, A and B are distinct.

We claim that every formula in $\mathcal{F}$ defines a capital relation. The proof is by induction of the formulas in $\mathcal{F}$. We treat the case of the reverse bounded existential quantifier. Suppose $\varphi(A, \mathbf{C})$ defines a capital relation. Here the $\mathbf{C}$ cover all occurrences of free variables in φ except A and A represents all occurrences of the free variable A. We claim that the relation given by $\exists A \trianglerighteq B\,\varphi(A, \mathbf{C})$ is again capital. Here B may be one of the $\mathbf{C}$. We assume that A and B are distinct. It is clearly sufficient to consider the case where we replace one occurrence of D by D' with $D \approx D'$, since more complicated replacements can be constructed as a number of such simpler replacements. Each such replacement is unproblematic as long as we do not replace the first occurrence of B. Suppose we replace the first occurrence of B by B', where $B \approx B'$. For any $A \supseteq B$, we can find $A' \supseteq B'$ with $A \approx A'$. Hence, by the induction hypothesis, $\varphi(A, \mathbf{C})$ iff $\varphi(A', \mathbf{C})$. It follows that $\exists A \trianglerighteq B\,\varphi(A, \mathbf{C})$ iff $\exists A' \trianglerighteq B'\,\varphi(A', \mathbf{C})$ and, by renaming of the bound variable A' to A, we have $\exists A \trianglerighteq B\,\varphi(A, \mathbf{C})$ iff $\exists A \trianglerighteq B'\,\varphi(A, \mathbf{C})$.

? Question
Is every $\mathbb{EI}$-definable capital property definable by a formula from $\mathcal{F}$? See also Question 5.

3.2.4 Some Salient Notions

We introduce a number of salient notions relevant to our framework. These notions are generalisations and localisations of the notions introduced by Per Lindström and Giorgi Japaridze for extensions of PA. See Japaridze and de Jongh (1998, Sect. 11). Because of our restriction to finitely axiomatised theories our framework seems to be primarily suitable for the study of *local* versions of the various notions. The first two concepts, to wit the logic Λ_A^Θ of A and admissible inference over A, will play no essential role in this paper. We just add them to make our list reasonably complete.

We remind the reader of the fact that $\mathcal{I}_{\Sigma_A}$ is built in in A. This is the reason that $\mathcal{I}$ only occurs in the first item.

- $\Lambda_A^\Theta := \{\varphi \in \mathcal{L}_\Theta \mid \text{for all } \tau : \Theta \to \Sigma_A \text{ such that } A \vdash \mathcal{I}_\Theta^\tau \text{ we have } A \vdash \varphi^\tau\}$. Λ_A^Θ is the logic of A for Θ.
- $B \mathrel{|\!\!\sim}_A \varphi$ iff, $\Sigma_B = \Sigma_\varphi$ and, for all $\tau : \Sigma_B \to \Sigma_A$, if $A \vdash B^\tau$, then $A \vdash \varphi^\tau$. The relation $\mathrel{|\!\!\sim}_A$ represents admissible inference over A.
- *A interprets B* or $A \rhd B$ iff, for some τ, $A \vdash B^\tau$.
- *A weakly interprets B* or *A tolerates B* or $A \Rsh B$ iff, for some translation τ, we have that $A + B^\tau$ is consistent.
- A is *locally tolerant* iff $A \Rsh B$, for each B.
- A is *essentially locally tolerant* iff, for each $A' \supseteq A$, we have $A' \Rsh B$, for each B.
- *A co-interprets B* or $A \blacktriangleright B$ if, for some τ, for all φ, if $B \vdash \varphi^\tau$, then $A \vdash \varphi$.[6]
- *A locally co-interprets B* or $A \blacktriangleright_{\mathsf{loc}} B$ if, for all φ, there is a τ such that, if $B \vdash \varphi^\tau$, then $A \vdash \varphi$.
- *A faithfully interprets B* or $A \rhd_{\mathsf{faith}} B$ iff, there is a τ, such that, for all φ, we have: $B \vdash \varphi$ iff $A \vdash \varphi^\tau$.
- *A locally faithfully interprets B* or $A \rhd_{\mathsf{lofa}} B$ iff, for all φ with $B \nvdash \varphi$, there is a τ such that $A \vdash B^\tau$ and $A \nvdash \varphi^\tau$.

3.3 Characterisations

Our salient local notions have various characterisations. In this section we collect those characterisations that are 'theory-free', i.e. for which we do not need results like the Interpretation Existence Lemma (see e.g. Visser (2018)). More theoretically involved characterisations will be treated in Sect. 3.6.

[6]In our formulations we will always implicitly assume that the φ we quantify over are of the right signature. In this case the hidden assumption is that $\Sigma_\varphi = \Sigma_A$.

3.3.1 The Logic of a Theory, Admissibility and Interpretability

We give various characterisations of the logic of a theory, of admissible rules and of interpretability. We repeat the relevant definitions.

- $\Lambda_A^\Theta := \{\varphi \in \mathcal{L}_\Theta \mid \text{for all } \tau : \Theta \to \Sigma_A \text{ such that } A \vdash \mathcal{I}_\Theta^\tau \text{ we have } A \vdash \varphi^\tau\}$.
- $B \mathrel{|\!\sim}_A \varphi$ iff, $\Sigma_B = \Sigma_\varphi$ and, for all $\tau : \Sigma_B \to \Sigma_A$, if $A \vdash B^\tau$, then $A \vdash \varphi^\tau$.
- *A interprets B* iff, for some τ, $A \vdash B^\tau$.

Theorem 3.1

1. $\Lambda_A^\Theta = \{\varphi \in \mathcal{L}_\Theta \mid \mathcal{I}_\Theta \mathrel{|\!\sim} \varphi\}$.
2. $\varphi \in \Lambda_A^\Theta$ *iff,* $\varphi \in \mathcal{L}_\Theta$, *and, for all* $\tau\colon \Theta \to \Sigma_A$, *we have* $A \vdash (\bigwedge \mathcal{I}_\Theta \to \varphi)^\tau$.
3. $\varphi \notin \Lambda_A^\Theta$ *iff* $A \Rsh (\mathcal{I}_\Theta \wedge \neg\, \varphi)$.

Proof Claim (1) is trivial.

We prove Claim (2). Suppose $\varphi \in \Lambda_A^\Theta$ and $\tau\colon \Theta \to \Sigma_A$. Let $\tau_0\colon \Theta \to \Sigma_A$ be defined as follows. The translation τ_0 is 1-dimensional, $\delta_\tau(x) := \top$, and τ_0 sends any predicate symbol including identity to $\top$.[7] We define $\tau^\star := \tau\langle \mathcal{I}_\Theta^\tau \rangle \tau_0$.[8] Clearly, $A \vdash \mathcal{I}_\Theta^{\tau^\star}$. Hence, $A \vdash \varphi^{\tau^\star}$. We may conclude that $A \vdash (\bigwedge \mathcal{I}_\Theta \to \varphi)^\tau$. The converse is immediate.

Claim (3) is immediate from Claim (2). □

The characterisation of Theorem 3.1(2) is very useful. We will employ it without mentioning that we apply the theorem.

Theorem 3.2 *$A \rhd B$ iff $B \mathrel{|\!\not\sim}_A \bot$.*

Proof We remind the reader that 'A' ranges over *consistent* finitely axiomatised theories. We have:

$$\begin{aligned} B \mathrel{|\!\not\sim}_A \bot &\Leftrightarrow \exists \tau\, (A \vdash B^\tau \text{ and } A \nvdash \bot^\tau) \\ &\Leftrightarrow A \rhd B \end{aligned}$$ □

Theorem 3.3 *Suppose B and φ have signature Θ. We have:*

$$B \mathrel{|\!\sim}_A^\Theta \varphi \quad \Leftrightarrow \quad (B \mathrel{|\!\sim}_A^\Theta \bot \text{ or } B \vdash_{\Lambda_A^\Theta} \varphi).$$

Proof The right-to-left direction is trivial. We prove left-to-right. Suppose we have $B \mathrel{|\!\sim}_A^\Theta \varphi$. In case we have $B \mathrel{|\!\sim}_A^\Theta \bot$, we are done. Otherwise there is a τ_0 such that $A \vdash B^{\tau_0}$. Consider any translation $\tau : \Theta \to \Sigma_A$. Let $\tau^\star := \tau\langle B^\tau \rangle \tau_0$. We clearly have $A \vdash B^{\tau^*}$ and, hence, $A \vdash \varphi^{\tau^*}$. It follows that $A + B^\tau \vdash \varphi^\tau$. □

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[7] It would be more natural to use a 0-dimensional translation, but I wanted to avoid any suspicion that our argument depends on some special etheric feature of the notion of translation.

[8] $\tau\langle \mathcal{I}_\Theta^\tau \rangle \tau_0$ is the translation that is τ if $\mathcal{I}_\Theta^\tau$ and that is τ_0 otherwise. See Appendices 3.8.2 and 3.8.5 for the official definitions.

Example 3.2 A general version of the second incompleteness theorem tells us that $A \nvdash (\mathsf{S}^1_2 + \mathsf{con}(A))$. (See, e.g., Visser (2011).) Thus, $(\mathsf{S}^1_2 + \mathsf{con}(A)) \vdash^{\mathcal{A}}_{A} \bot$. On the other hand, all sequential A interpret S^1_2 faithfully. (See. e.g., Visser (2005).) For these theories we have $\mathsf{S}^1_2 + \mathsf{con}(A) \nvdash_{\Lambda^{\mathcal{A}}_{A}} \bot$. So, the first disjunct of Theorem 3.3 is essential.

3.3.2 *Weak Interpretability and Local Cointerpretability*

We first repeat the definitions of weak interpretability and of local cointerpretability.

- $A \upharpoonright B$ iff, for some translation τ, we have that $A + B^\tau$ is consistent.
- $A \blacktriangleright_{\mathsf{loc}} B$ iff, for all φ, there is a τ such that, if $B \vdash \varphi^\tau$, then $A \vdash \varphi$.

Theorem 3.4 *$A \upharpoonright B$ iff $\exists C \supseteq A\ C \rhd B$.*

Proof Suppose $A \upharpoonright B$. Then for some τ, the theory $A + B^\tau$ is consistent. Hence $A \subseteq C := A + B^\tau$ and $C \rhd B$.

Conversely, suppose $A \subseteq C$ and $C \rhd B$. Let τ witness the interpretability of B in C. Then certainly $A + B^\tau$ is consistent. □

We note that the formula $\exists C \supseteq A\ C \rhd B$ is in $\mathcal{F}$ and hence weak interpretability is a capital relation. Our next order of business is to show that local cointerpretability can also be defined in the fragment $\mathcal{F}$. Local cointerpretability is preservation of the tolerated. Thus, local cointerpretability is a capital relation.

Theorem 3.5 *$A \blacktriangleright_{\mathsf{loc}} B$ iff, for all C, if $A \upharpoonright C$, then $B \upharpoonright C$.*

Proof Suppose $A \blacktriangleright_{\mathsf{loc}} B$. Consider any C and suppose $A \upharpoonright C$. Then, for some σ, we have $A \nvdash (\neg C)^\sigma$. Taking $\varphi := \neg C^\sigma$ in the definition of $A \blacktriangleright_{\mathsf{loc}} B$, we find, for some τ, that $B \nvdash (\neg C)^{\sigma\tau}$. It follows that $B \upharpoonright C$.

Suppose that, for all C, if $A \upharpoonright C$, then $B \upharpoonright C$. Consider any φ. We take $C := (\mathcal{I}_{\Sigma_A} \wedge \neg\varphi)$. Suppose $A \nvdash \varphi$, then $A \nvdash (\mathcal{I}_{\Sigma_A} \to \varphi)$, since $A \vdash I_{\Sigma_A}$. Hence, $A \upharpoonright C$. It follows that $B \upharpoonright C$, so, for some τ, we have $B \nvdash (\mathcal{I}_{\Sigma_A} \to \varphi)^\tau$, and, *a fortiori*, $B \nvdash \varphi^\tau$. □

We can write Theorem 3.5 a bit differently. If we define $[\![A]\!] := \{C \mid A \upharpoonright C\}$. Then, $A \blacktriangleright_{\mathsf{loc}} B$ iff $[\![A]\!] \subseteq [\![B]\!]$.

Here is another characterisation in $\mathcal{F}$ that is easy to remember.

Theorem 3.6 *$A \blacktriangleright_{\mathsf{loc}} B$ iff, for all $A' \supseteq A$, there is a $B' \supseteq B$, such that $B' \rhd A'$.*

Proof We use Theorem 3.5. Suppose that, for all C, if $A \upharpoonright C$, then $B \upharpoonright C$. Suppose $A' \supseteq A$. Then, the identity interpretation witnesses that $A \upharpoonright A'$. It follows that $B \upharpoonright A'$. Hence, for some τ, we have $B' := B + (A')^\tau$ is consistent. Clearly $B' \supseteq B$ and $B' \rhd A'$.

Conversely suppose that, for all $A' \supseteq A$, there is a $B' \supseteq B$, such that $B' \rhd A'$. Suppose ρ witnesses that $A \upharpoonleft C$. Then $A' := A + C^{\rho}$ is consistent. It follows that there is a $B' \supseteq B$ such that $B' \rhd A'$. Suppose $B' \rhd A'$ is witnessed by σ. We may conclude that $B + C^{\rho\sigma}$ is consistent, so $B \upharpoonleft C$. □

We prove a number of further equivalents of cointerpretability.

Theorem 3.7 *The following are equivalent:*

1. $A \blacktriangleright_{\mathsf{loc}} B$.
2. *For all* Θ, *we have:* $\Lambda_B^{\Theta} \subseteq \Lambda_A^{\Theta}$.
3. $\Lambda_B^{\Sigma_A} \subseteq \Lambda_A^{\Sigma_A}$.
4. $\Lambda_B^{\Sigma_A} \subseteq A$.

Proof We prove: (1) $\Rightarrow$ (2). Suppose $A \blacktriangleright_{\mathsf{loc}} B$. Let Θ be any signature. Consider any $\varphi \in \Lambda_B^{\Theta}$. So, for all $\sigma : \Theta \to \Sigma_B$, $B \vdash (\mathcal{I}_{\Theta} \to \varphi)^{\sigma}$. Consider any $\rho : \Theta \to \Sigma_A$. Let τ be the witness of $A \blacktriangleright_{\mathsf{loc}} B$ for $(\mathcal{I}_{\Theta} \to \varphi)^{\rho}$. We have $(\tau \circ \rho) : \Theta \to \Sigma_B$ and hence $B \vdash (\mathcal{I}_{\Theta} \to \varphi)^{\rho\tau}$. It follows that $A \vdash (\mathcal{I}_{\Theta} \to \varphi)^{\rho}$. Thus, φ in Λ_A^{Θ}.

We get (2) $\Rightarrow$ (3) by universal instantiation.

We get (3) to (4), since, trivially $\Lambda_A^{\Sigma_A} \subseteq A$.

We prove (4) $\Rightarrow$ (1). Suppose $\Lambda_B^{\Sigma_A} \subseteq A$. We note that $A \blacktriangleright_{\mathsf{loc}} B$ iff, for all φ, if (for all $\tau : \Sigma_B \to \Sigma_A$, we have $B \vdash \varphi^{\tau}$), then $A \vdash \varphi$.

Suppose, for all $\tau : \Sigma_B \to \Sigma_A$, we have $B \vdash \varphi^{\tau}$. Then, *a fortiori*, for all $\tau : \Sigma_B \to \Sigma_A$, we have $B \vdash \mathcal{I}_{\Sigma_B} \to \varphi^{\tau}$. Thus, $\varphi \in \Lambda_B^{\Sigma_A}$. Hence, $A \vdash \varphi$. □

Remark 3.1 We consider the following to notions.

- $A \blacktriangleright_{\mathsf{mod}} B$ iff, there is a τ such that, for every model $\mathcal{M}$ of A, there is a model $\mathcal{N}$ of B, such that $\mathcal{M}$ is elementarily equivalent to $\widetilde{\tau}(\mathcal{N})$.
- $A \blacktriangleright_{\mathsf{lomo}} B$ iff, for every model $\mathcal{M}$ of A, there is a τ and there is a model $\mathcal{N}$ of B, such that $\mathcal{M}$ is elementarily equivalent to $\widetilde{\tau}(\mathcal{N})$.

It is easily seen that $\blacktriangleright_{\mathsf{mod}}$ coincides with $\blacktriangleright$. Also, if $A \blacktriangleright_{\mathsf{lomo}} B$, then $A \blacktriangleright_{\mathsf{loc}} B$. The converse direction, however, fails. We have, for example, $\mathsf{EQ} \blacktriangleright_{\mathsf{loc}} \mathbb{1}$. On the other hand, $\mathsf{EQ} \not\blacktriangleright_{\mathsf{lomo}} \mathbb{1}$.

Finally, note that, if we replace elementary equivalence by isomorphism in the above definitions we obtain two further notions that are *prima facie* different.

3.3.3 (Locally) Faithful Interpretability

We remind the reader that:

- $A \rhd_{\mathsf{lofa}} B$ iff,
  for all φ such that $B \nvdash \varphi$, there is a τ, such that $A \vdash B^{\tau}$ and $A \nvdash \varphi^{\tau}$.

We start with an immediate observation.

Theorem 3.8 *$A \rhd_{\mathsf{lofa}} B$ iff, for all φ, if $B \;\, \mathrel{\rotatebox{0}{$\vdash$}}_A \varphi$, then $B \vdash \varphi$.*

Proof We have:

$$\begin{aligned} A \rhd_{\mathsf{lofa}} B &\Leftrightarrow \forall \varphi\, (B \nvdash \varphi \Rightarrow \exists \tau\, (A \vdash B^\tau \text{ and } A \nvdash \varphi^\tau)) \\ &\Leftrightarrow \forall \varphi\, (\forall \tau\, (A \vdash B^\tau \Rightarrow A \vdash \varphi^\tau) \Rightarrow B \vdash \varphi) \\ &\Leftrightarrow \forall \varphi\, (B \;\, \vdash_A \varphi \Rightarrow B \vdash \varphi) \end{aligned}$$ □

The following theorem shows that $A \rhd_{\mathsf{lofa}} B$ is definable in $\mathcal{F}$ and, hence, capital.

Theorem 3.9 *$A \rhd_{\mathsf{lofa}} B$ iff $(A \rhd B$ and $B \blacktriangleright_{\mathsf{loc}} A)$.*

Proof We use the characterisation of Theorem 3.6.

Suppose $A \rhd_{\mathsf{lofa}} B$. By putting $\varphi := \bot$ in the definition of $\rhd_{\mathsf{lofa}}$, we see that $A \rhd B$.

Consider any $B' \supseteq B$. Suppose B' is axiomatised by φ. Since B' is consistent, it follows that $B \nvdash \neg \varphi$. So there is a τ such that $A \vdash B^\tau$ and $A \nvdash (\neg \varphi)^\tau$. We may conclude that $A \rhd B$ and that $A' := A + \varphi^\tau$ is consistent. Clearly, $A' \rhd B'$. We may conclude $B \blacktriangleright_{\mathsf{loc}} A$.

Suppose $A \rhd B$ and $B \blacktriangleright_{\mathsf{loc}} A$. Let τ_0 witness $A \rhd B$. Suppose $B \nvdash \varphi$. If follows that $B' := B + \neg \varphi$ is consistent. Then, for some $A' \supseteq A$ we have $A' \rhd B'$. Let τ be the witness of $A' \rhd B'$. We take $\tau^\star := \tau\langle (B')^\tau \rangle \tau_0$. Clearly, $A \vdash B^{\tau^\star}$. Suppose $A \vdash \varphi^{\tau^\star}$. Then, $A + (B')^\tau \vdash \varphi^\tau$, since under the assumption $(B')^\tau$, the translations τ and $\tau^\star$ coincide. On the other hand, $A + (B')^\tau \vdash \neg \varphi^\tau$, so $A + (B')^\tau \vdash \bot$. But $A + (B')^\tau \subseteq A'$ and A' is consistent. So, we have a contradiction. Ergo, $A \nvdash \varphi^{\tau^\star}$. □

We note that Theorem 3.9 in combination with the various characterisations of $\blacktriangleright_{\mathsf{loc}}$ gives us many alternate characterisations for $\rhd_{\mathsf{lofa}}$.

The nice decoupling into interpretability and cointerpretability also works in the non-local case—at least as long as the theories we consider are finite.

Theorem 3.10 *$A \rhd_{\mathsf{faith}} B$ iff $(A \rhd B$ and $B \blacktriangleright A)$.*

Proof From-left-to-right is easy. We treat from-right-to-left. Suppose τ_0 witnesses $A \rhd B$ and τ_1 witnesses $B \blacktriangleright A$. Let $\tau^\star := \tau_1\langle B^{\tau_1} \rangle \tau_0$. Clearly $A \vdash B^{\tau^\star}$. Suppose $A \vdash \varphi^{\tau^\star}$. Then $A + B^{\tau_1} \vdash \varphi^{\tau_1}$. It follows that $B \vdash (B \to \varphi)$, so $B \vdash \varphi$. □

We note that faithful interpretability is not capital since $\mathbb{1} \approx \mathsf{EQ}$ but $\mathbb{1}$ does not faithfully interpret EQ. Since $\rhd$ is capital, we find that $\blacktriangleright$ is not capital.

Remark 3.2 We pick up the thread of Remark 3.1. We define $A \rhd_{\mathsf{lomo}} B$ iff $A \rhd B$ and $B \blacktriangleright_{\mathsf{lomo}} A$. We note that $\lhd_{\mathsf{faith}}$ is a sub-relation of $\lhd_{\mathsf{lomo}}$ and $\lhd_{\mathsf{lomo}}$ is a sub-relation of $\lhd_{\mathsf{lofa}}$.

On the other hand, we have $\mathsf{EQ} \lhd_{\mathsf{lofa}} \mathbb{1}$, but $\mathsf{EQ} \not\lhd_{\mathsf{lomo}} \mathbb{1}$. Let C be the theory of linear dense orderings without endpoints. It is easy to see that $\mathsf{EQ} \lhd_{\mathsf{lomo}} C$. On the other hand, since C is a complete theory, $\mathsf{EQ} \not\lhd_{\mathsf{faith}} C$.[9] So both inclusions are strict.

[9]This insight is trivial since we do not work with parameters: parameter-free faithful interpretability preserves completeness from the interpreting theory to the interpreted theory. The insight still holds when we do allow parameters. In this case one uses that, in the unique countable model of C (modulo isomorphism), the parameters only have finitely many constellations modulo automorphism.

The notion of faithfulness one obtains by combining interpretability with model theoretic cointerpretability with isomorphism was studied by Szczerba (1976).

3.3.4 Local Tolerance

We remind the reader of two definitions.

- A is locally tolerant iff $\forall B\ A \Rsh B$.
- A is essentially locally tolerant iff $\forall B \supseteq A\, \forall C\ B \Rsh C$.

We note that both notions are capital. We give two important characterisations of local tolerance.

Theorem 3.11 *The following are equivalent:*

i. *A is locally tolerant.*
ii. *For all B, if $A \rhd B$, then $A \rhd_{\mathsf{lofa}} B$.*
iii. *$A \rhd_{\mathsf{lofa}} \mathsf{CQC}_2$, where CQC_2 is predicate logic for a binary relation symbol.*

These characterisations are the local versions of well-known characterisations of tolerance. See e.g. Visser (2005). In Sect. 3.6, we will add a further characterisation.

Proof (i) $\Rightarrow$ (ii). Suppose A is locally tolerant and $A \rhd B$. Let τ_0 witness $A \rhd B$. Suppose $B \nvdash \varphi$. Let $B' := B + \neg\varphi$. We have $A \Rsh B'$. Let τ_1 be the witness of $A \Rsh B'$. We define $\tau^\star := \tau_1 \langle (B')^{\tau_1} \rangle \tau_0$. Clearly, $A \vdash B^{\tau^\star}$. Suppose $A \vdash \varphi^{\tau^\star}$. In that case, we must have $A + (B')^{\tau_1} \vdash \bot$. Quod non.

(ii) $\Rightarrow$ (iii). This is immediate from the fact that $A \rhd \mathsf{CQC}_2$.

(iii) $\Rightarrow$ (i). We can do this in two ways. First, we can use a theorem from Hodges (1993) that any theory B is bi-interpretable with a theory $\widetilde{B}$ in the signature of one binary relation. Since $\mathsf{CQC}^2 \nvdash \neg\widetilde{B}$, it follows that $A + (\widetilde{B})^\tau$ is consistent, for some τ. So, $A \Rsh \widetilde{B}$, and, hence, $A \Rsh B$.

Alternatively, we can use the fact that we have an interpretation of S^1_2 in adjunctive set theory AS, which is a theory in one binary relation. Say this is witnessed by ν. We want a Σ_1-sound translation here: say, when we interpret AS in the hereditarily finite sets, then ν gives us an isomorphic copy of the natural numbers. Consider any theory B. We easily show that $A \Rsh (\mathsf{AS} + (\mathsf{S}^1_2 + \mathsf{con}(B))^\nu)$ —noting that the theory that is tolerated is true in the hereditarily finite sets. Since, $(\mathsf{AS} + (\mathsf{S}^1_2 + \mathsf{con}(B))^\nu) \rhd (\mathsf{S}^1_2 + \mathsf{con}(B))$. and $(\mathsf{S}^1_2 + \mathsf{con}(B)) \rhd B$, it follows that $A \Rsh B$. □

Remark 3.3 The above result matches precisely with what we know about tolerance. A theory A is *tolerant* iff, for all (possibly not finitely axiomatised) theories X, we have $A \Rsh X$. Alternatively, A is tolerant iff it tolerates S^1_2 plus all true Π_1-sentences. Then, the following are equivalent:

i. A is tolerant.
ii. For all B, if $A \rhd B$, then $A \rhd_{\mathsf{faith}} B$.
iii. $A \rhd_{\mathsf{faith}} \mathsf{CQC}_2$.

The reader is referred to Visser (2005) for explanation and proofs.

Finally, we remind the reader that $A \lhd_{\mathsf{loc}} B$ iff $[\![A]\!] \supseteq [\![B]\!]$, where we define $[\![C]\!] := \{D \mid C \Rsh D\}$. Hence, a theory is locally tolerant iff it is minimal w.r.t. $\lhd_{\mathsf{loc}}$. Thus, we have:

Theorem 3.12 *A is locally tolerant iff, for all B, we have* $A \lhd_{\mathsf{loc}} B$.

In Sect. 3.5.2, we will see a characterisation of the maximal elements of $\lhd_{\mathsf{loc}}$: these are precisely the theories with the finite model property.

3.4 Disjoint Sum is a Capital Operation

The disjoint sum of two theories is more or less what you expect it to be: make the signatures disjoint, relativise both theories to newly introduced domains and take the union. The disjoint sum is introduced and discussed in Appendix 3.8.7. In this section, we show that $\boxplus$ preserves $\approx$.

Theorem 3.13 $\boxplus$ *is a capital operation.*

Proof Suppose $A_0 \approx B_0$ and $A_1 \approx B_1$. We want to show that $(A_0 \boxplus A_1) \approx (B_0 \boxplus B_1)$.

We define $C \mathcal{B} D$ iff, for some $k > 0$, and, for some $\varphi_{00}, \ldots, \varphi_{0(k-1)}$ in the language of A_0, and for some $\varphi_{10}, \ldots, \varphi_{1(k-1)}$ in the language of A_1, and for some $\psi_{00}, \ldots, \psi_{0(k-1)}$ in the language of B_0, and some $\psi_{10}, \ldots, \psi_{1(k-1)}$ in the language of B_1, we have:

- $C = ((A_0 \boxplus A_1) + \bigvee_{i<k}(\varphi_{0i} \wedge \varphi_{1i}))$,
- $D = ((B_0 \boxplus B_1) + \bigvee_{i<k}(\psi_{0i} \wedge \psi_{1i}))$,
- for all $j < 2$ and $i < k$, we have $(A_j + \varphi_{ji})$ is consistent and $(B_j + \psi_{ji})$ is consistent and $(A_j + \varphi_{ji}) \approx (B_j + \psi_{ji})$.

We first show that $(A_0 \boxplus B_0) \mathcal{B} (A_1 \boxplus B_1)$. This is immediate from $A_0 \approx B_0$ and $A_1 \approx B_1$, by taking $k := 1$ and $\varphi_{00} := \top$, $\varphi_{10} := \top$, $\psi_{00} := \top$ and $\psi_{10} := \top$.

We check that $C \mathcal{B} D$ implies $C \bowtie D$. Suppose $C \mathcal{B} D$. We have witnessing φ_{ji} and ψ_{ji}. with $(A_j + \varphi_{ji}) \approx (B_j + \psi_{ji})$, and, hence, $(A_j + \varphi_{ji}) \bowtie (B_j + \psi_{ji})$. Suppose τ_{ji} witnesses $(A_j + \varphi_{ji}) \rhd (B_j + \psi_{ji})$. Then $\tau_i^\star := \tau_{0i} \boxplus \tau_{1i}$ witnesses

$$\begin{aligned} ((A_0 \boxplus A_1) + (\varphi_{0i} \wedge \varphi_{1i})) &= ((A_0 + \varphi_{0i}) \boxplus (A_1 + \varphi_{1i})) \\ &\rhd ((B_0 + \psi_{0i}) \boxplus (B_1 + \varphi_{1i})) \\ &= ((B_0 \boxplus B_1) + (\psi_{0i} \wedge \psi_{1i})) \end{aligned}$$

Let:

$$\nu := \tau_0^\star \langle \varphi_{00} \wedge \varphi_{10} \rangle (\tau_1^\star \langle \varphi_{01} \wedge \varphi_{11} \rangle (\cdots \langle \varphi_{0(k-2)} \wedge \varphi_{1(k-2)} \rangle \tau_{k-1}^\star) \cdots).$$

Then ν witnesses:

$$C = ((A_0 \boxplus A_1) + \bigvee_{i<k}(\varphi_{0i} \wedge \varphi_{1i})) \rhd ((B_0 \boxplus B_1) + \bigvee_{i<k}(\psi_{0i} \wedge \psi_{1i})) = D.$$

Similarly, in the other direction.

We check the forward property. The backward property is, of course, analogous. Suppose $C \mathcal{B} D$ and let, as before φ_{ji} and ψ_{ji} be the witnessing formulas. Suppose $C' \supseteq C$. Since $C' \supseteq (A_0 \boxplus A_1)$, we have by Theorem 3.34, that $C' = (A_0 \boxplus A_1) + \bigvee_{\ell<n}(\chi_{0\ell} \wedge \chi_{1\ell})$, where the $\chi_{0\ell}$ are in the A_0-language and the $\chi_{1\ell}$ are in the A_1-language. Since, $\bigvee_{i<k}(\varphi_{0i} \wedge \varphi_{1i})$ is also in C, we have by propositional logic:

$$C' = (A_0 \boxplus A_1) + \bigvee_{i<k,\ell<n} ((\varphi_{0i} \wedge \chi_{0\ell}) \wedge (\varphi_{1i} \wedge \chi_{1\ell})).$$

By our assumption, we have $(A_j + \varphi_{ji}) \approx (B_j + \psi_{ji})$. Moreover, we have $(A_j + \varphi_{ji}) \subseteq (A_j + (\varphi_{ji} \wedge \chi_{j\ell}))$. Hence, by the forward property for $\approx$, there is a $\nu_{1i\ell}$ such that:

$$(B_j + \psi_{ji}) \subseteq (B_j + \nu_{ji\ell}) \text{ and } (A_j + (\varphi_{ji} \wedge \chi_{j\ell})) \approx (B_j + \nu_{1i\ell}).$$

Thus, we can take $D' := (B_0 \boxplus B_1) + \bigvee_{i<k,\ell<n}(\nu_{0i\ell} \wedge \nu_{1i\ell})$. Clearly, we have $C' \mathcal{B} D'$. $\square$

? Question

Let $\mathfrak{S}(A, B, C)$ be the relation $A \boxplus B \approx C$. Is $\mathfrak{S}$ $\mathbb{EI}$-definable? Is $\mathfrak{S}$ $\mathcal{F}$-definable?

3.5 Faithful Interpretability and Locally Faithful Interpretability

Locally faithful interpretability looks to me like a fairly natural notion. Still I have not seen it formulated before. In the present section, we provide a few basic insights concerning the new notion. Finally, we briefly indicate the connection of faithful interpretability and locally faithful interpretability with the forward property.

We have seen that local faithful interpretability is capital. On the other hand, Example 3.1 shows that faithful interpretability is not capital. So, faithful interpretability and locally faithful interpretability are extensionally distinct notions.

We note that we have $A \rhd_{\mathsf{lofa}} A$. Suppose $A \approx B$, then, since $\rhd_{\mathsf{lofa}}$ is capital, we find $A \rhd_{\mathsf{lofa}} B$. Thus, we may conclude that i-bisimilarity implies mutual local faithful interpretability.

3.5.1 Decidability

We have the following theorem.

Theorem 3.14 *The relation $\rhd_{\mathsf{lofa}}$ preserves decidability, i.e., if $A \rhd_{\mathsf{lofa}} B$ and A is decidable, then B is decidable.*

Proof Suppose $A \rhd_{\mathsf{lofa}} B$ and A is decidable. Is φ a theorem of B? On the positive side we enumerate the theorems of B until we find a proof of φ; on the negative side we run through translations $\tau_i : \Sigma_B \to \Sigma_A$ and decide whether $A \vdash B^{\tau_i}$ and $A \nvdash \varphi^{\tau_i}$. Eventually, one of the parallel processes must yield an answer. □

Unfortunately, our argument fails when we switch to infinitely axiomatised theories. So, in the infinite case, ordinary faithful interpretability seems to be in better shape.

? Question
Is there an example of $U \rhd_{\mathsf{lofa}} V$ (or even $A \rhd_{\mathsf{lofa}} V$), where U is decidable and V is not?

We note that it follows that decidability is a capital property.

? Question
Is decidability first-order or even second-order definable in $\mathbb{EI}$?

Example 3.3 Let $A := (\forall x\, \forall y\, x = y \vee \bigwedge \mathsf{S}^1_2)$. We have $\mathbb{1} \rhd A$, but not $\mathbb{1} \rhd_{\mathsf{lofa}} A$, since A is not decidable. This gives us a separating example between $\rhd_{\mathsf{lofa}}$ and $\rhd$.

3.5.2 The Finite Model Property

A basic property of theories is the finite model property or FMP. A theory A has the finite model property iff, for every φ with $A \nvdash \varphi$, there is a finite model $\mathcal{M}$ such that $\mathcal{M} \models A$ and $\mathcal{M} \models \neg\,\varphi$.

There are many theories with the finite model property that also have an infinite model. The theory of pure identity EQ or CQC_0 is one example of these. Another example is the theory of discrete linear orderings with endpoints.

Theorem 3.15 *The theory of discrete linear orderings with endpoints has the finite model property.*

Proof All infinite models of this theory are of the form $\omega + \zeta \cdot \alpha + \omega^*$, where ζ is the order type of the integers and α is an arbitrary order type. By a Fraïssé style argument all these models are elementarily equivalent.[10]

If a sentence φ is true in all models, then certainly it is true in all finite models. Conversely, if φ is true in all finite models, by compactness, it is true in a least one infinite model, and, hence, in all models. □

Theorem 3.16 *The following are equivalent.*

i. *A has the finite model property.*
ii. $\mathbb{1} \approx A$.
iii. $\mathbb{1} \bowtie_{\mathsf{lofa}} A$.
iv. $\mathbb{1}$ *is mutually locally cointerpretable with* A.

Proof We need two simple observations. First, any A with the finite model property is mutually interpretable with $\mathbb{1}$. Any theory interprets $\mathbb{1}$. Suppose $\mathcal{M}$ is a finite model of A. Since, we allow piecewise interpretations, we can transform our finite model into an interpretation of A in $\mathbb{1}$.

Secondly, the finite model property is preserved over $\subseteq$. Suppose A has the finite model property. Consider any $B \supseteq A$ and suppose $B \nvdash \varphi$. Then, $A \nvdash B \to \varphi$. So there is a finite model $\mathcal{M}$ of A such that $\mathcal{M} \models \neg(B \to C)$, so $\mathcal{M} \models B$ and $\mathcal{M} \models \neg\varphi$. We may conclude that B has the finite model property.

(i) $\Rightarrow$ (ii). We define $D\,\mathcal{B}\,E$ iff $D = \mathbb{1}$ and $E \supseteq A$. By the above observations, $\mathcal{B}$ is a i-bisimulation. Hence $\mathbb{1} \approx A$.

(ii) $\Rightarrow$ (iii). We have already seen that i-bisimilarity implies mutual locally faithful interpretability.

(iii) $\Rightarrow$ (iv). From $A \bowtie_{\mathsf{lofa}} \mathbb{1}$, we have that A and $\mathbb{1}$ are mutually locally cointerpretable.

(iv) $\Rightarrow$ (i). Suppose $A \blacktriangleright_{\mathsf{loc}} \mathbb{1}$. Suppose $A \nvdash \varphi$. Then $A \upharpoonright (A + \neg\varphi)$. It follows that $\mathbb{1} \upharpoonright (A + \neg\varphi)$. Hence $\mathbb{1} + (A + \neg\varphi)^\tau$ is consistent, for some τ. Since, $\mathbb{1}$ is complete it follows that $\mathbb{1} \vdash (A + \neg\varphi)^\tau$. Thus, τ defines a finite internal model of $A + \neg\varphi$ in the unique model of $\mathbb{1}$. Thus, A has the finite model property. □

It follows that the finite model property is a capital, $\mathcal{F}$-definable, property.

We end this subsection with the observation that the theories with the finite model property are precisely the theories that are maximal with respect to $\blacktriangleleft_{\mathsf{loc}}$. So:

Theorem 3.17 *A has the finite model property iff, for all B, we have* $A \blacktriangleright_{\mathsf{loc}} B$.

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[10] I am grateful to the anonymous referee for this very short proof.

3.5.3 Separating Examples

In this section, we provide separating examples between a number of salient notions. Some of the examples we already covered before but it is pleasant to repeat them for the sake of overview.

Example 3.4 *i-Bisimilarity does not imply mutual faithful interpretability.* We have seen that $\mathbb{1} \approx \mathsf{EQ}$. However, it is impossible that $\mathbb{1} \rhd_{\mathsf{faith}} \mathsf{EQ}$, since for any τ such that $\mathbb{1} \vdash \mathcal{I}_0^\tau$, necessarily there is an n such that $\mathbb{1}$ proves the τ-translation of *there are at most n elements*. So, faithful interpretability is not capital.

> **? Question**
> Is faithful interpretability first-order or even second-order definable over $\mathbb{EI}$?

Example 3.5 *Mutual faithful interpretability does not imply i-Bisimilarity.* Let A be sequential. It is easy to see that the theories A and $A \boxplus A$ are mutually faithfully interpretable and, *a fortiori*, mutually locally faithfully interpretable. By Theorem 3.19, there are $A_0 \supseteq A$ and $A_1 \supseteq A$ such that $A_0 \not\rhd A_1$ and $A_1 \not\rhd A_0$. We have: $(A \boxplus A) \subseteq (A_0 \boxplus A_1)$. Suppose there were an $A' \supseteq A$ with $A' \bowtie (A_0 \boxplus A_1)$. Then, since A' is sequential and, hence, connected,[11] it follows that $A_i \rhd A'$, for some i. But this would give us: $A_i \rhd A' \rhd (A_0 \boxplus A_1) \rhd A_{1-i}$. *Quod non.* We may conclude that $A \not\approx (A \boxplus A)$.

We show that mutual faithful interpretability is incomparable to $\approx$.

Example 3.6 *Mutual faithful interpretability is not mutual locally faithful interpretability and mutual locally faithful interpretability is not i-bisimilarity.* We note that both $\bowtie_{\mathsf{faith}}$ and $\approx$ are subrelations of $\bowtie_{\mathsf{lofa}}$ Hence, Example 3.4 also separates $\bowtie_{\mathsf{lofa}}$ and $\bowtie_{\mathsf{faith}}$. Example 3.5 separates $\bowtie_{\mathsf{lofa}}$ and $\approx$.

Example 3.7 *Interpretability does not imply locally faithful interpretability.* Let A be the theory of dense linear orderings without end-points and let B be $\mathsf{CQC}_{\mathcal{A}}$, i.e. predicate logic for the signature of arithmetic. Clearly, we have $A \rhd B$. On the other hand, for any interpretation τ such that $A \vdash B^\tau$, we have $A \vdash (\neg \bigwedge \mathsf{Q})^\tau$, where Q is Robinson's Arithmetic, since A is decidable and Q is essentially undecidable. Thus, $A \not\rhd_{\mathsf{lofa}} B$.

Example 3.8 *Local faithful interpretability does not imply faithful interpretability.* We already showed that $\mathbb{1} \rhd_{\mathsf{lofa}} \mathsf{EQ}$ but $\mathbb{1} \not\rhd_{\mathsf{faith}} \mathsf{EQ}$.

It might be thought that our example leans on very specific features of piecewise or many-dimensional interpretations. However, we can improve our example in order to

[11] See Sect. 3.7.

get the witnessing interpretations for $A \rhd_{\mathsf{lofa}} B$ one-dimensional and without pieces. We take A the theory of a linear discrete ordering with an initial point without endpoint, i.e., the theory of the ordering of the natural numbers. This is a complete finitely axiomatisable theory. See Enderton (2001, Sect. 3.2). We take B the theory of discrete linear order with both initial point and endpoint. By Theorem 3.15, the theory B has the finite model property. Clearly, the theory of any finite model of B can be interpreted in A by a one-dimensional interpretation. It follows that, $A \rhd_{\mathsf{lofa}} B$ using only one-dimensional interpretations. However, since A is complete, there can be no faithful interpretation of A in B.

3.5.4 The Forward Property

We have, by Theorems 3.6 and 3.10, the following characterisation of $\lhd_{\mathsf{lofa}}$.

- $A \lhd_{\mathsf{lofa}} B$ iff $A \lhd B$ and, for all $A' \supseteq A$, there is a $B' \supseteq B$, such that $A' \lhd B'$.

We note that this characterisation is reminiscent of the forward property for $\lhd_{\mathsf{lofa}}$. It is, so to speak, the first step towards the forward property. It turns out that faithful interpretability does have the forward property with respect to $\lhd$.

Theorem 3.18 *The relation $\lhd_{\mathsf{faith}}$ has the forward property.*

Proof Suppose $K : A \lhd_{\mathsf{faith}} B$ and $A \subseteq A'$. We claim that $A' \lhd_{\mathsf{faith}} (B + (A')^{\tau_K})$. Clearly, $K' : A' \lhd (B + (A')^{\tau_K})$, where K' is based on τ_K. Suppose $B + (A')^{\tau_K} \vdash \varphi^{\tau_K}$. Then by faithfulness: $A' = A + A' \vdash \varphi$. Hence, K' is faithful. We note that it also follows that $B + (A')^{\tau_K}$ is consistent. So, we can take $B' := (B + (A')^{\tau_K})$. □

We define the obvious analogue of $\approx$ for the forward property.

- A relation $\mathcal{S}$ is an *i-simulation* iff (i) $\mathcal{S}$ is a subrelation of $\lhd$ and (ii) $\mathcal{S}$ has the forward or zig property: if $A \mathcal{S} B$ and $A' \supseteq A$, then, there is a $B' \supseteq B$, such that $A \mathcal{S} B$.
- B simulates A, or A is simulated by B, or $A \precsim B$ iff, there is a simulation $\mathcal{S}$ such that $A \mathcal{S} B$.
- $A \cong B$ iff $A \precsim B$ and $A \succsim B$.

Trivially. $\precsim$ and $\cong$ are capital.

We note that $\approx$ is a subrelation of $\cong$. In Sect. 3.7, we will provide an example that $\cong$ and $\approx$ do not coincide.

We note that $\mathsf{EQ} \precsim \mathbb{1}$, but not $\mathsf{EQ} \lhd_{\mathsf{faith}} \mathbb{1}$. Hence, $\lhd_{\mathsf{faith}}$ is strictly contained in $\precsim$.

In Appendix 3.8.6.6 we show that faithful retractions in the categories INT_i, for $i \leq 3$, have the forward property. So we could define $A \sqsubseteq B$ as the maximal simulation that is a subrelation of $\bowtie$. This relation would contain the faithful retractions of INT_3 (and, *a fortiori*, the faithful retractions of INT_i for $i \leq 3$). We note that the induced equivalence relation of $\sqsubseteq$ is precisely $\approx$.

3.6 Arithmetic

In the present section, we collect a number of characterisations and results connected with the arithmetical theory S^1_2.

3.6.1 Incomparable Theories

We need a sufficient store of incomparable extensions of given finitely axiomatised theories. The following theorem provides these.

Theorem 3.19 *Suppose A and B both tolerate S^1_2. Then, there are $A^\star \supseteq A$ and $B^\star \supseteq B$, that are incomparable w.r.t. $\lhd$, i.e., $A^\star \not\rhd B^\star$ and $B^\star \not\rhd A^\star$.*

Proof Suppose τ witnesses that A tolerates S^1_2 and ν witnesses that B tolerates S^1_2. We take $A' := A + (\mathsf{S}^1_2)^\tau$ and $B' := B + (\mathsf{S}^1_2)^\nu$. By the Gödel Fixed Point Lemma, we find R such that:

$$\mathsf{S}^1_2 \vdash R \leftrightarrow ((A' + R^\tau) \rhd (B' + \neg R^\nu)) \leq ((B' + \neg R^\nu) \rhd (A' + R^\tau)).$$

We take $A^\star := A' + R^\tau$ and $B^\star := B' + \neg R^\nu$. Suppose $A^\star \rhd B^\star$. It follows that R or $R^\perp$. In case we have R, we find, by Σ_1-completeness, that $A' \rhd \perp$. Quod non. If we have $R^\perp$, it follows by the fixed point equation that $(B' + \neg R^\tau) \rhd (A' + R^\nu)$. By Σ_1-completeness, we have $B' \rhd \perp$. Quod non. We may conclude that $A^\star \not\rhd B^\star$.

The proof that $B^\star \not\rhd A^\star$ is similar. □

3.6.2 Characterisations

We can connect our previous characterisations to arithmetical ones using two basic insights. We employ complexity measure ρ which is *depth of quantifier alternations*. The formula $\mathsf{con}_n(A)$ refers to consistency for n-provability. Here we only allow n-proofs, i.e., proofs involving formulas of complexity $\leq n$. See also Appendix 3.8.1.

For the notion of sequentiality, see Appendix 3.9.

I. $(\mathsf{S}^1_2 + \mathsf{con}_n(A)) \rhd A$, where $n \geq \rho(A)$.
II. If A is sequential, then $A \rhd (\mathsf{S}^1_2 + \mathsf{con}_n(A))$, where $n \geq \rho(A)$.

In (I), the translation that realises the interpretation is the Henkin translation η. The proof of (I) is described in great detail in Visser (2018). The proof of (II) is described in great detail in Visser (2019).

We have the following basic insight.

Theorem 3.20 *Suppose A is sequential. Then, $A \upharpoonright B$ iff $A \upharpoonright (\mathsf{S}^1_2 + \mathsf{con}_{\rho(B)}(B))$.*

Proof Suppose A is sequential and $A \upharpoonright B$. Then $A + B^\tau$ is consistent, for some τ. But $A + B^\tau$ is sequential, hence it interprets $\mathsf{S}^1_2 + \mathsf{con}_{\rho(B)}(B)$. We can see this by noting that for any $n \geq \rho(A + B^\tau)$, we have $(A + B^\tau) \rhd (\mathsf{S}^1_2 + \mathsf{con}_n(A + B^\tau))$. On the other hand, for sufficiently large n, we have $\mathsf{S}^1_2 \vdash \Box_{B,\rho(B)}\bot \to \Box_{A+B^\tau,n}\bot$. So it follows that $(A + B^\tau) \rhd (\mathsf{S}^1_2 + \mathsf{con}_{\rho(B)}B)$. Thus, we may conclude that $A \upharpoonright (\mathsf{S}^1_2 + \mathsf{con}_{\rho(B)}B)$.

The other direction is immediate by Basic Insight (I). □

If we are interested in local tolerance, we do not need the assumption of sequentiality.

Theorem 3.21 *A is locally tolerant iff, for all true Π_1-sentences P, we have $A \upharpoonright (\mathsf{S}^1_2 + P)$.*

Proof From-left-to-right is just specialisation. From-right-to-left, we may conclude $A \upharpoonright B$, from the fact that $A \upharpoonright (\mathsf{S}^1_2 + \mathsf{con}_n(B))$. □

We can characterise interpretability as local Π_1-conservativity in case either the target theory or the source theory is sequential.

Theorem 3.22 *Suppose B is sequential. Then, $A \rhd B$ iff, for all Π_1-sentences P, iff $B \rhd (\mathsf{S}^1_2 + P)$, then $A \rhd (\mathsf{S}^1_2 + P)$.*

Proof Suppose $A \rhd B$. Then, trivially, for all Π_1-sentences P, iff $B \rhd (\mathsf{S}^1_2 + P)$, then $A \rhd (\mathsf{S}^1_2 + P)$.

Conversely, suppose, for all Π_1-sentences P, iff $B \rhd (\mathsf{S}^1_2 + P)$, then $A \rhd (\mathsf{S}^1_2 + P)$.

Suppose B is sequential. Then, we have $B \rhd (\mathsf{S}^1_2 + \mathsf{con}_{\rho(B)}(B))$ and, hence, $A \rhd (\mathsf{S}^1_2 + \mathsf{con}_{\rho(B)}(B)) \rhd B$. □

We remind the reader of the Friedman Characterisation.

Theorem 3.23 *Suppose A is sequential. Then,*
$A \rhd B$ iff $(\mathsf{EA} + \mathsf{con}_{\rho(A)}(A)) \supseteq (\mathsf{EA} + \mathsf{con}_{\rho(B)}(B))$.

Here EA is Elementary Arithmetic or $\mathrm{I}\Delta_0 + \mathsf{Exp}$. In the context of this paper the following characterisation is relevant.

Theorem 3.24 *Suppose A is sequential. The following are equivalent:*

a. *$A \rhd B$.*
b. *For all $n \geq \rho(B)$, there is an $m \geq \rho(A)$, such that $(\mathsf{S}^1_2 + \mathsf{con}_m(A)) \supseteq (\mathsf{S}^1_2 + \mathsf{con}_n(B))$.*
c. *There is an $m \geq \rho(A)$, such that $(\mathsf{S}^1_2 + \mathsf{con}_m(A)) \supseteq (\mathsf{S}^1_2 + \mathsf{con}_{\rho(B)}(B))$.*

Proof (a) $\Rightarrow$ (b). Suppose τ witnesses $A \rhd B$. We can use τ to transform, in the context of S^1_2, an n-inconsistency proof of B into an m-inconsistency proof of A, where m is roughly $n + \rho(\tau)$. Here $\rho(\tau)$ is the maximum of $\rho(\delta_\tau)$ and the $\rho(P_\tau)$. The main point is that $\rho(\varphi^\tau)$ will be $\rho(\varphi) + \rho(\tau)$ plus some fixed standard overhead.

(b) $\Rightarrow$ (c). This is just specialisation.

(c) $\Rightarrow$ (a). We have: $A \rhd (\mathsf{S}^1_2 + \mathsf{con}_m(A)) \supseteq (\mathsf{S}^1_2 + \mathsf{con}_{\rho(B)}(B)) \rhd B$. □

3.6.3 Local (In)tolerance

We start with a question.

? Question
Let's say that a theory is *self-confident* iff it is mutually interpretable with a sequential theory. Does every locally tolerant theory have a self-confident extension? If not, does every tolerant theory have a self-confident extension?

It is easy to give examples of theories that are locally intolerant. For example, every decidable theory is locally intolerant since it does not tolerate S^1_2. It is unknown whether there is a theory that tolerates S^1_2 but still is locally intolerant.

? Question
Is there a locally intolerant theory A with $A \upharpoonright \mathsf{S}^1_2$?

In this subsection, we take a small step in thinking about this question by proving the following theorem.

Theorem 3.25 *Suppose A is locally intolerant. Then, it has an extension B such that* $\mathsf{S}^1_2 \vdash_{\Lambda^{\mathcal{A}}_B} \Box_{B,\rho(B)}\bot$. *So, B believes in its own restricted inconsistency in the strongest possible way.*

Proof Suppose, A is locally intolerant. This tells us that there exists a false Σ_1-sentence S, such that, for all $\tau : \mathcal{A} \to \Sigma_A$, we have $A + (\mathsf{S}^1_2)^\tau \vdash S^\tau$. In other words, $\mathsf{S}^1_2 \vdash_{\Lambda^{\mathcal{A}}_A} S$.

We show that there is a $B \supseteq A$, such that, for all $\tau : \mathcal{A} \to \Sigma_A$, we have $B + (\mathsf{S}^1_2)^\tau \vdash \Box^\tau_{B,\rho(B)}\bot$. In other words, $\mathsf{S}^1_2 \vdash_{\Lambda_B} \Box_{B,\rho(B)}\bot$.

In case S^1_2 is not interpretable in A, we are easily done, taking $B := A$. Suppose ν witnesses $\mathsf{S}^1_2 \lhd A$. We find R such that $\mathsf{S}^1_2 \vdash R \leftrightarrow S \leq \Box_{A,m}R^\nu$. Here we take m to be $\mathsf{max}(\rho(A), \rho(R)) + 1$. We note that the complexity of R only depends on the complexity of S plus some fixed constant derived from the complexity of the provability predicate and the overhead of the fixed point construction.

Consider any $\tau : \mathcal{A} \to \Sigma_A$. We work in $\alpha := A + (\mathsf{S}^1_2)^\tau$. We allow α to be inconsistent. We take a cut I of τ, such that $\alpha \vdash \forall x \in I\ 2^{2^x} \in \delta_\tau$ and $\alpha \vdash (\mathsf{T}^1_2)^I$. We have, *ex hypothesis*, $\alpha \vdash S^I$. Hence, $\alpha \vdash (R \vee R^\perp)^I$. We have, by verifiable Σ_1-completeness, using that I is sufficiently 'deep', $\alpha \vdash R^I \to \Box^\tau_{A,m}R^\nu$ and, by the fixed-point equation, $\alpha \vdash (R^\perp)^I \to \Box^I_{A,m}R^\nu$. Hence, $A + (\mathsf{S}^1_2)^\tau \vdash \Box^\tau_{A,m}R^\nu$.

We take $B := A + \neg R^\nu$. If B would be inconsistent, we would have $A \vdash R^\nu$. Hence, by cut-elimination, $A \vdash_m R^\nu$. So, for some standard p, we may conclude that $A \vdash \underline{p}\ \mathsf{wit}\ \Box^\nu_{A,m}R^\nu$. It follows that $A \vdash \bigvee_{q \leq p}(\underline{q}\ \mathsf{wit}\ S)^\nu$. Since $\neg S$, we have

$\bigwedge_{q \leq p} \neg\, q$ wit S, and, hence, by Σ_1-completeness, $A \vdash \bigwedge_{q \leq p}(\neg\, \underline{q}$ wit $S)^{\nu}$. So, A is inconsistent. *Quod non.*

Clearly, for any τ, we have $B + (\mathsf{S}^1_2)^{\tau} \vdash \Box^{\tau}_{B,\rho(B)}\bot$, noting that $\rho(B) \geq m$. □

Our result shows that a sequential A must be locally tolerant. If a sequential A were locally intolerant, then there would be a sequential B that proves $\Box_{B,\rho(B)}\bot$ for all interpretations of S^1_2 in B. But we know there is an interpretation of S^1_2 on which we have $\mathsf{con}_{\rho(B)}(B)$.[12] This is a weaker result than the result proved by Harvey Friedman and, independently, by Jan Krajíček. See, e.g., Smoryński (1985), Krajíček (1987) and Visser (2005). However, the aim of this subsection was more than just reproving that earlier theorem.

3.7 Sequential Theories

In the present section we discuss a number of properties of sequential theories visible over our framework. We have the following *Basic Insights*.

Theorem 3.26 *Suppose A is sequential.*

a. For every B there is a sequential C with $C \rhd B$.
b. If $A \subseteq B$*, then B is sequential.*
c. A is connected, i.e., suppose $A \lhd B \boxplus C$*, then* $A \lhd B$ *or* $A \lhd C$.
d. If $A \bowtie B$*, then B is tolerant, and,* a fortiori*, locally tolerant.*
e. If B is sequential and $A \lhd B$*, then there is an* $A' \supseteq A$ *such that* $A' \bowtie B$.

Proof

Ad (a) This can be seen in two ways. We can form $C := \mathsf{seq}(B)$ as follows. We add a unary predicate $\triangle$ and a binary predicate $\in$ to the language of B. We relativise B to $\triangle$ and we add AS. Alternatively, we can take $C := (\mathsf{S}^1_2 + \mathsf{con}_{\rho(B)}(B))$.

Ad (b) This is a triviality.

Ad (c) This is a non-trivial result. It was first proved by Pudlák (1983). An essentially different proof was given by Stern (1989). For a discussion of the significance of this result: see Mycielski et al. (1990).

Ad (d) Tolerance simply means that for any possible infinitely axiomatised theory X, there is a τ such that $B + X^{\tau}$ is consistent. Alternatively, tolerance means that B faithfully interprets predicate logic in the language with one binary predicate.
The fact that any sequential theory is tolerant was proved by Harvey Friedman (see Smoryński (1985)) and, independently, Jan Krajíček (see Krajíček (1987)). The strengthening involving mutual interpretablity was noted in Visser (2005).

Ad (e) This insight is one of the central results of Visser (2014). □

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[12]This argument does not work for self-confident theories, where a theory C is self-confident if $C \rhd (\mathsf{S}^1_2 + \mathsf{con}_{\rho(C)}(C))$. So, one may wonder whether it can be adapted to accommodate these?

The property of being mutually interpretable with a sequential theory can be given the following form. A theory A is *self-confident* if $A \rhd (\mathsf{S}^1_2 + \mathsf{con}_{\rho(A)}(A))$. It is easy to see that the self-confident theories are precisely the theories that are mutually interpretable with a sequential theory. (*Warning:* This last result holds only in the finitely axiomatised case.) Thus, Basic Insight (d) tells us that all self-confident theories are locally tolerant, and we even know that all such theories are tolerant. We note that self-confidence is definitely a capital notion. Example 3.5 illustrates that, while sequentiality is upwards closed under $\subseteq$, self-confidence is not upwards closed under $\subseteq$.

Basic Insight (e) can be connected to our framework in an interesting way.

Theorem 3.27 *Suppose A and B are sequential, then $A \approx B$ iff $A \bowtie B$.*

Proof From-left-to-right, is immediate, since $\bowtie$ is capital. We prove the right-to-left direction. Let $\mathcal{B}$ be $\bowtie$ restricted to the sequential theories. Suppose A and B are sequential and $A \bowtie B$ and $A \subseteq A'$. Then, $A' \rhd B$. Then, by (d), there is a $B' \supseteq B$ such that $A' \bowtie B'$. Thus, we have the forward property. Similarly, we have the backward property. Trivially, $\mathcal{B}$ is a subrelation of $\bowtie$. We may conclude that $\mathcal{B}$ is a i-bisimulation. Hence $\mathcal{B}$ is a subrelation of $\approx$. □

Example 3.9 It is easily seen that, for any theory A, we have $A \approx (A \boxplus \mathbb{1})$. Since, $\mathbb{1} \approx \mathsf{EQ}$, it follows that $A \approx (A \boxplus \mathsf{EQ})$. It is easy to see that $A \boxplus \mathsf{EQ}$ cannot be sequential. Thus, we have, for sequential A, that $A \approx (A \boxplus \mathsf{EQ})$ and $A \boxplus \mathsf{EQ}$ is not sequential. Hence, sequentiality is not capital.

Example 3.10 We note that, for sequential A, we have $A \bowtie (A \boxplus A)$. It follows that $A \bowtie_{\mathsf{faith}} (A \boxplus A)$.[13] By Theorem 3.19, there are A_0 and A_1 extending A such that $A_0 \not\rhd A_1$ and $A_1 \not\rhd A_0$. So, $(A_0 \boxplus A_1) \supseteq (A \boxplus A)$, Since any A' extending A is sequential by Basic Insight (b), no such A' can be mutually interpretable with $(A_0 \boxplus A_1)$ by Basic Insight (c). So, $A \not\approx (A \boxplus A)$. Thus, there are theories B that are mutually faithfully interpretable with a sequential theory A, but that are not i-bisimilar to it.

? Question
Are the sequential theories closed under sentential congruence? And, if not, are they closed under iso-congruence? We already know that the sequential theories are closed under bi-interpretability.

We have the following theorem.

Theorem 3.28 *Suppose B is sequential. Then, $A \precsim B$ iff $A \lhd B$.*

[13]This is immediate by the Friedman-Krajíček result (see Smoryński (1985), Krajíček (1987), Visser (2005)).

Proof We define $C\ \mathcal{S}\ D$ iff $C \lhd D$ and D is sequential. We show that $\mathcal{S}$ has the forward property. Suppose $C\ \mathcal{S}\ D$. Then $C \lhd D$ and D is sequential. Let $C' \supseteq C$. Since D is locally tolerant, there is a $D' \supseteq D$ such that $D' \rhd C'$.

Alternatively, we know by the results of Visser (2005), that $A \lhd B$ iff $A \lhd_{\mathsf{faith}} B$. Theorem 3.18 tells us that $\lhd_{\mathsf{faith}}$ is a subrelation of $\precsim$, which is in its turn a subrelation of $\lhd$. □

We note that in the first proof of Theorem 3.28 we just used the fact that sequential theories are essentially locally tolerant. So, in fact, we have proved: if B is essentially locally tolerant, then $A \precsim B$ iff $A \lhd B$.

Example 3.11 Suppose A is sequential. We have already seen that $A \not\approx (A \boxplus A)$. On the other hand, we have $A \rhd (A \boxplus A)$, and, hence $A \succsim (A \boxplus A)$. It is clear that the mapping $A \mapsto (A \boxplus A)$ has the forward property. Hence $A \cong (A \boxplus A)$.

Alternatively, we could simply note that, by Theorem 3.28, $A \cong B$ iff $A \bowtie B$, for sequential A. Moreover, by the results of Visser (2005), we have that $A \bowtie B$ iff $A \bowtie_{\mathsf{faith}} B$, for sequential A. So we can apply Example 3.10.

Let $[A]_\sim$ be the equivalence class of $\sim$. Let SEQ be the class of sequential theories. Our knowledge at this point is summarised by the following theorem.

Theorem 3.29 *Suppose A is sequential Then:*

$$([A]_{\bowtie} \cap \mathsf{SEQ}) = ([A]_{\approx} \cap \mathsf{SEQ}) \subsetneq [A]_{\approx} \subsetneq [A]_{\cong} = [A]_{\bowtie_{\mathsf{faith}}} = [A]_{\bowtie_{\mathsf{lofa}}} = [A]_{\bowtie}.$$

Proof Suppose A is sequential. Let's number the claims of the theorem.

$$([A]_{\bowtie} \cap \mathsf{SEQ}) \overset{(1)}{=} ([A]_{\approx} \cap \mathsf{SEQ}) \overset{(2)}{\subsetneq} [A]_{\approx} \overset{(3)}{\subsetneq} [A]_{\cong} \overset{(4)}{=} [A]_{\bowtie_{\mathsf{faith}}} \overset{(5)}{=} [A]_{\bowtie_{\mathsf{lofa}}} \overset{(6)}{=} [A]_{\bowtie}.$$

Claim 1 is Theorem 3.27. The non-identity in Claim 2 is Example 3.9. The non-identity in Claim 3 is by Example 3.11. The identities 5 and 6 follow since by the results of Visser (2005), we have $A \bowtie B$ iff $A \bowtie_{\mathsf{faith}} B$ in case A is sequential. Moreover, $\bowtie_{\mathsf{lofa}}$ is between $\bowtie_{\mathsf{faith}}$ and $\bowtie$. Finally, by Theorem 3.28 in combination with identities 5 and 6, we have identity 4. □

Remark 3.4 We can do a bit of reverse meta-mathematics and rederive earlier insights from Theorem 3.27.

We note that the fact that sequential theories are locally tolerant is immediate from Theorem 3.27. Consider any sequential A. Our theory is mutually interpretable with $\mathsf{S}^1_2 + \mathsf{con}_{\rho(A)}(A)$. Hence, $A \approx (\mathsf{S}^1_2 + \mathsf{con}_{\rho(A)}(A))$. By Theorem 3.21, it is clear that $\mathsf{S}^1_2 + \mathsf{con}_{\rho(A)}(A)$ is locally tolerant. Since local tolerance is a capital property, we find that A is locally tolerant. I do not see how we to derive Basic Insight (d) in full from Theorem 3.27 without self-referential arguments.

We show that Basic Insight (e) follows from Theorem 3.27. Suppose A and B are sequential and $A \lhd B$. By Theorem 3.24, there is an $m \geq \rho(B)$, such that

$$(\mathsf{S}^1_2 + \mathsf{con}_m(B)) \supseteq (\mathsf{S}^1_2 + \mathsf{con}_{\rho(A)}(A)).$$

Since $A \approx (\mathsf{S}^1_2 + \mathsf{con}_{\rho(A)}(A))$ and $B \approx (\mathsf{S}^1_2 + \mathsf{con}_m(B))$, we are easily done. □

Acknowledgements I thank Johan van Benthem for his comments and questions. I am grateful to Volodya Shavrukov for his comments and help with the references. I thank Victor Pambuccian for pointing me to Szczerba's work. I thank Clemens Grabmayer, Vincent van Oostrom and Freek Wiedijk for their help with a LATEX problem. I am grateful to the anonymous referee for his/her careful reading of the paper. I thank the referee especially for a nice short proof of Theorem 3.15.

3.8 Appendix: Basics

In this appendix, we provide detailed definitions of translations, interpretations and morphisms between interpretations.

3.8.1 Theories and Provability

Theories in this paper are one-sorted theories of first order predicate logic of finite relational signature. We take identity to be a logical constant. Our official signatures are relational, however, via the term-unwinding algorithm, we can also accommodate signatures with functions. For most purposes in the present paper a theory can be identified with a deductively closed set of sentences of the given language. The exception is the few places where we use Rosser style arguments.

Our main focus will be on finitely axiomatised theories, but in this appendix, we will develop the material also for the infinitely axiomatised case.

We will sometimes use the modal notation $\Box_A\varphi$ for $\mathsf{prov}_A(\underline{\ulcorner\varphi\urcorner})$. We will also consider *restricted provability*. This is provability where we restrict the formulas occurring in the proof to formulas of complexity n, for some given n. Our measure of complexity is *depth of quantifier alternations*. This measure is defined officially as follows: $\rho := \rho_\exists$, where:

- $\rho_\exists(A) := \rho_\forall(A) = 1$, if A is atomic.
- $\rho_\exists(\neg B) := \rho_\forall(B)$, $\rho_\forall(\neg B) := \rho_\exists(B)$.
- $\rho_\exists(B \wedge C) := \mathsf{max}(\rho_\exists(B), \rho_\exists(C))$, $\rho_\forall(B \wedge C) := \mathsf{max}(\rho_\forall(B), \rho_\forall(C))$.
- $\rho_\exists(B \vee C) := \mathsf{max}(\rho_\exists(B), \rho_\exists(C))$, $\rho_\forall(B \vee C) := \mathsf{max}(\rho_\forall(B), \rho_\forall(C))$.
- $\rho_\exists(B \to C) := \mathsf{max}(\rho_\forall(B), \rho_\exists(C))$, $\rho_\forall(B \to C) := \mathsf{max}(\rho_\exists(B), \rho_\forall(C))$.
- $\rho_\exists(\exists v\, B) := \rho_\exists(B)$, $\rho_\forall(\exists v\, B) := \rho_\exists(B) + 1$.
- $\rho_\exists(\forall v\, B) := \rho_\forall(B) + 1$, $\rho_\forall(\forall v\, B) := \rho_\forall(B)$.

We write $\Box_{A,n}\varphi$ for provability restricted to formulas ψ with $\rho(\psi) \leq n$.

3.8.2 Translations

Translations are the heart of our interpretations. In fact, they are often confused with interpretations, but we will not do that officially. In practice it is often convenient to conflate an interpretation and its underlying translation. The distinction is essential when we consider categories of theories and interpretations.

To formulate the notion of translation it is pleasant to allow in the target language lambda terms of the form $\lambda x_0 \dots x_{n-1}.\varphi(x_0, \dots, x_{n-1})$, where A is a formula. We will call a term of this form an n-term. We think of such terms modulo α-conversion (renaming of bound variables) as is usual in λ-calculus.

As a start. we define more-dimensional, one-sorted, one-piece relative translations without parameters. We will later indicate how to modify the definition to get piecewise translations and interpretations. We do not treat the case where we add parameters.

Let Σ and Θ be one-sorted signatures. A translation $\tau : \Sigma \to \Theta$ is given by a triple $\langle m, \delta, F\rangle$. Here δ will be a closed m-term. The mapping F associates to each relation symbol R of Σ with arity n a closed $m \times n$-term of signature Θ.

We demand that predicate logic proves $F(R)(\mathbf{x}_0, \dots, \mathbf{x}_{n-1}) \to (\delta(\mathbf{x}_0) \wedge \dots \delta(\mathbf{x}_{n-1}))$. Of course, given any candidate $m \times n$-term $F(R)$ not satisfying the restriction, we can obviously modify it to satisfy the restriction.

We translate Σ-formulas to Θ-formulas as follows.

- $(R(x_0, \dots, x_{n-1}))^\tau := F(R)(\mathbf{x}_0, \dots, \mathbf{x}_{n-1})$.
  Here we demand that the sequences $\mathbf{x}_i$ are fully disjoint if the original variables are x_i are different.
  The single variable x_i of the source language needs to have no obvious connection with the sequence of variables $\mathbf{x}_i$ of the target language that represents it. We need some conventions to properly handle the association $x_i \mapsto \mathbf{x}_i$.[14]
- $(\cdot)^\tau$ commutes with the propositional connectives;
- $(\forall x\, \varphi)^\tau := \forall \mathbf{x}\, (\delta(\mathbf{x}) \to \varphi^\tau)$;
- $(\exists x\, \varphi)^\tau := \exists \mathbf{x}\, (\delta(\mathbf{x}) \wedge \varphi^\tau)$.

Here are some convenient conventions and notations.

- We write δ_τ for 'the δ of τ' and F_τ for 'the F of τ'.
- We write R_τ for $F_\tau(R)$.
- We write $\mathbf{x} \in \delta$ for: $\delta(\mathbf{x})$.

[14]There are several ways of handling such conventions. First we can work with a fixed global association between the x_i and the $\mathbf{x}_i$. Secondly, we can make such an association local and carry it around as an extra argument of the translation. Thirdly, we can throw away the mechanism of using variable-names and work in a language that works with explicit links between places. Fourthly, we can sidestep the problem by working in many-sorted languages and, for every k, adding sequences of length k (of various sorts). This construction can be viewed as a representation of more dimensional interpretations as arrows in a Kleisli category. Regrettably, each way of proceeding needs some work and produces some awkwardness somewhere. In this paper, we will assume that these details are taken care of by one strategy or by another.

There are some natural operations on translations. The identity translation $\mathsf{id} := \mathsf{id}_\Theta$ is one-dimensional and it is defined by:

- $\delta_{\mathsf{id}} := \lambda x.(x = x)$,
- $R_{\mathsf{id}} := \lambda \mathbf{x}.R\mathbf{x}$.

We can compose relative translations as follows. Suppose τ is an m-dimensional translation from Σ to Θ, and ν is a k-dimensional translation from Θ to Ξ. We define the $m \times k$-dimensional interpretation $\tau\nu$ or $\nu \circ \tau$ as follows.

- We suppose that with the variable x we associate under τ the sequence $x_0, \ldots, x_{m-1}$ and under ν we send x_i to $\mathbf{x}_i$.
  $\delta_{\tau\nu}(\mathbf{x}_0, \ldots, \mathbf{x}_{m-1}) := \delta_\nu(\mathbf{x}_0) \wedge \ldots \wedge \delta_\nu(\mathbf{x}_{m-1}) \wedge (\delta_\tau(x_0, \ldots, x_{m-1}))^\nu$,
- Let R be n-ary. Suppose that under τ we associate with x_i the sequence $x_{i,0}, \ldots, x_{i,m-1}$ and that under ν we associate with $x_{i,j}$ the sequence $\mathbf{x}_{i,j}$. We take:
  $R_{\tau\nu}(\mathbf{x}_{0,0}, \ldots \mathbf{x}_{n-1,m-1}) =$
  $$\delta_\tau(\mathbf{x}_{0,0}) \wedge \ldots \wedge \delta_\tau(\mathbf{x}_{n-1,m-1}) \ \wedge (R_\tau(x_{0,0}, \ldots x_{n-1,m-1}))^\nu.$$

We can make a disjunctive interpretation as follows. Suppose τ and ν are translations from Σ to Θ. We assume that τ is k-dimensional and ν is m-dimensional. Let φ be a Θ-sentence. We introduce a $\mathsf{max}(k, m)$-dimensional interpretation $\tau\langle\varphi\rangle\nu$.

We first 'lift' one of the interpretations by padding to get the dimensions equal.[15] Suppose, e.g., that $k < m$. Then we define the auxiliary translation τ' as follows:

- $\delta_{\tau'}(\mathbf{x}\mathbf{z}) :\leftrightarrow \delta_\tau(\mathbf{x})$,
- $P_{\tau'}(\mathbf{x}_0\mathbf{z}_0, \ldots, \mathbf{x}_{n-1}\mathbf{z}_{n-1}) := P_\tau(\mathbf{x}_0, \ldots, \mathbf{x}_{n-1})$.

Here the dimension of the $\mathbf{z}$ is $m - k$. In this case $\nu' := \nu$. If $m < k$ the construction is similar.

Suppose the results of the padding operation are τ' and ν'. We define $\tau\langle\varphi\rangle\nu$ as follows:

- $\delta_{\tau\langle\varphi\rangle\nu}(\mathbf{x}) := ((\varphi \wedge \delta_{\tau'}(\mathbf{x})) \vee (\neg\, \varphi \wedge \delta_{\nu'}(\mathbf{x})))$.
- $R_{\tau\langle\varphi\rangle\nu}(\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}) :=$
  $$((\varphi \wedge R_{\tau'}(\mathbf{x}_0, \ldots, \mathbf{x}_{n-1})) \vee (\neg\, \varphi \wedge R_{\nu'}(\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}))).$$
  Here the $\mathbf{x}$ are $\mathsf{max}(k, m)$-dimensional.

An m-dimensional translation τ *preserves identity* if

$$\mathbf{x} =_\tau \mathbf{y} := \bigwedge_{i<m} (\delta_\tau(x_i) \wedge \delta_\tau(y_i) \wedge x_i = y_i).$$

An m-dimensional translation τ *is unrelativized* if $\delta_\tau(\mathbf{x}) := \top$. An m-dimensional translation τ is *direct* if it is unrelativised and preserves identity. Note that all these properties are preserved by composition (modulo provable equivalence in predicate logic).

Consider a model $\mathcal{M}$ with domain M of signature Σ and k-dimensional translation $\tau : \Sigma \to \Theta$. Let $N := \delta_\tau^{\mathcal{M}} := \{\mathbf{m} \in M^k \mid \mathcal{M} \models \delta_\tau \mathbf{m}\}$. Suppose N is not empty. Let

[15] As we will see the padding can be avoided by using piecewise interpretations.

E be the equivalence relation on N defined in $\mathcal{M}$ by $=_\tau$. Then τ specifies an internal model $\mathcal{N}$ of $\mathcal{M}$ with domain N/E and with $\mathcal{N} \models R([\mathbf{m}_0]_E, \ldots, [\mathbf{m}_{n-1}]_E)$ iff $\mathcal{M} \models R_\tau(\mathbf{m}_0, \ldots, \mathbf{m}_{n-1})$. We will write $\widetilde{\tau}(\mathcal{M})$ for the internal model of $\mathcal{M}$ given by τ. We treat the mapping $\tau, \mathcal{M} \mapsto \widetilde{\tau}\mathcal{M}$ as a partial function that is defined precisely if $\delta_\tau^{\mathcal{M}}$ is non-empty. Let Mod or $\widetilde{(\cdot)}$ be the function that maps τ to $\widetilde{\tau}$. We have:

$$\mathsf{Mod}(\tau \circ \rho)(\mathcal{M}) = (\mathsf{Mod}(\rho) \circ \mathsf{Mod}(\tau))(\mathcal{M}).$$

So, Mod behaves contravariantly.

3.8.3 Relative Interpretations

A translation τ supports a *relative interpretation* of a theory U in a theory V, if, for all U-sentences φ, we have $U \vdash \varphi \Rightarrow V \vdash \varphi^\tau$. Note that this automatically takes care of the theory of identity and assures us that δ_τ is inhabited. We will write $K = \langle U, \tau, V \rangle$ for the interpretation supported by τ. We write $K : U \to V$ for: K is an interpretation of the form $\langle U, \tau, V \rangle$. If M is an interpretation, τ_M will be its second component, so $M = \langle U, \tau_M, V \rangle$, for some U and V.

Par abus de langage, we write 'δ_K' for: δ_{τ_K}; 'R_K' for: R_{τ_K}; 'A^K' for: A^{τ_K}, etc. Here are the definitions of three central operations on interpretations.

- Suppose U has signature Σ. We define:
  $\mathsf{ID}_U : U \to U$ is $\langle U, \mathsf{id}_\Sigma, U \rangle$.
- Suppose $K : U \to V$ and $M : V \to W$. We define:
  $M \circ K : U \to W$ is $\langle U, \tau_M \circ \tau_K, W \rangle$.
- Suppose $K : U \to (V + \varphi)$ and $M : U \to (V + \neg\, \varphi)$. We define:
  $K\langle\varphi\rangle M : U \to V$ is $\langle U, \tau_K\langle\varphi\rangle\tau_M, V \rangle$.

It is easy to see that we indeed correctly defined interpretations between the theories specified.

3.8.4 Global and Local Interpretability

We can view interpretability as a generalisation of provability. When we take this standpoint, we write:

- $U \rhd V$ (or $V \lhd U$) for: $\exists K\ K : V \to U$, or: U interprets V (or: V is interpretable in U).
- $U \bowtie V$ for: ($U \rhd V$ and $V \rhd U$), or: U and V are mutually interpretable.

A closely related notion is local interpretability. We define

- U locally interprets V or $U \rhd_{\mathsf{loc}} V$ iff, for every finitely axiomatised subtheory V_0 of V we have $U \rhd V_0$.

- We write $V \lhd_{\mathsf{loc}} U$ and $U \bowtie_{\mathsf{loc}} V$ with the obvious meanings.

If we want to stress the contrast between local and ordinary interpretability, we call ordinary interpretability *global interpretability*. We will write $\rhd_{\mathsf{glob}}$, etcetera.

The degrees of of global interpretability are $\mathsf{DEG}_{\mathsf{glob}}$ and the degrees of local interpretability are $\mathsf{DEG}_{\mathsf{loc}}$.

Example 3.12 Let $\mathbb{2}$ be the theory in the language of identity that says that there are precisely two elements. Let INF be the theory in the language of identity that has for every n and axiom saying 'there are at least n elements'. Then, $\mathbb{2} \rhd_{\mathsf{loc}} \mathsf{INF}$ but $\mathbb{2} \not\rhd_{\mathsf{glob}} \mathsf{INF}$. We will see that when we admit piecewise interpretability, we may replace $\mathbb{2}$ by $\mathbb{1}$ in this observation.

3.8.5 Piecewise Translations and Interpretations

In this subsection we introduce piecewise translations and interpretations. For some further information on piecewise translations and interpretations see Visser (2012).

Before explaining what piecewise translations and interpretations are, we state some of their advantages.

- Many constructions are conceptually cleaner when we use piecewise translations and interpretations. Specifically, we avoid a lot of padding. As a consequence the heuristics for a construction is usually easier to grasp.
- The unnatural difference between one-element and at-least-two-element domains disappears.

We will show, in Appendix 3.8.8, that, in case our interpreting theory proves that there are at least two elements, piecewise translations can be simulated by translations without pieces.

The idea of piecewise translations is that we can build up the domain from a number of pieces that may or may not be of the same dimension and that may or may not overlap.

A piecewise translation is a tuple $\langle X, f, \delta, F \rangle$. Here X is a non-empty set of pieces and f is a function from X to ω. The function f gives us the arity of the domain associated to each piece. We use $\mathfrak{a}$, $\mathfrak{b}$, ... to range over pieces.

The term $\delta^{\mathfrak{a}}$ is a $f\mathfrak{a}$-term. Suppose P is n-ary. Let g be a function from $\{0, \ldots, n-1\}$ to X. Then $F^g(P)$ is a $(fg0 + fg1 + \cdots + fg(n-1))$-term.

Here are the clauses to lift our translation to the full language.

- Consider an n-ary predicate symbol P. Let j be a function from the set $\{0, \ldots, n-1\}$ to variables. Say $j(i) = x_i$. Suppose h is a function from $\{x_0, \ldots, x_{n-1}\}$ to X. (Here we allow that, for some i and j, the variables x_i and x_j are the same.) We define:
$$(R(x_0, \ldots, x_{n-1}))^{\tau,h} := F^{h \circ j}(R)(\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}).$$

Here $\mathbf{x}_i$ has length $fh(x_i)$. We demand that the sequences $\mathbf{x}_i$ are fully disjoint if the original variables are x_i are different.[16]

- Suppose h is a function from the free variables of $(\varphi \wedge \psi)$ to pieces. Then, $(\varphi \wedge \psi)^{\tau,h} = (\varphi^{\tau,h\restriction \mathsf{FV}(\varphi)} \wedge \psi^{\tau,h\restriction \mathsf{FV}(\psi)})$.
  Similarly, for the other propositional connectives.
- $(\forall x\, \varphi)^{\tau,h} := \bigwedge_{\mathfrak{a}\in X} \forall \mathbf{x}_{\mathfrak{a}}\, (\delta^{\mathfrak{a}}(\mathbf{x}_{\mathfrak{a}}) \to \varphi^{\tau,h[x:=\mathfrak{a}]})$.
  Here $\mathbf{x}_{\mathfrak{a}}$ is the sequence of variables we associate to x for the piece $\mathfrak{a}$. Similarly, for the existential quantifier.

We can now define the notion of piecewise interpretation in the usual way using piecewise translations.

We give three important examples of how piecewise interpretation works. A fourth example is given in Appendix 3.8.7.

Example 3.13 Let U be any theory and let $\mathbb{2}^+$ be the theory of two *named* elements, say c and d. We show that $U \rhd \mathbb{2}^+$. We write ε for the empty sequence. We define:

- $X := \{0, 1\}$,
- $f(i) := 0$,
- $\delta^i(\varepsilon) := \top$,
- $F^{ij}(=)(\varepsilon, \varepsilon) := \top$, if $i = j$
  $F^{ij}(=)(\varepsilon, \varepsilon) := \bot$, otherwise,
- $F^0(C)(\varepsilon) := \top$, $F^1(C)(\varepsilon) := \bot$,
- $F^0(D)(\varepsilon) := \bot$, $F^1(D)(\varepsilon) := \top$,

We leave the easy verification that we did indeed define an interpretation of $\mathbb{2}^+$ to the reader. We see that our interpretation is, in a sense, entirely independent of (the language of) U. We note that the interpretation would have worked even if we had started from a free logic. Thus, piecewise interpretation truly makes *creatio ex nihilo* possible.

Remark 3.5 We can use a similar construction to show that the theory of any finite model is interpretable in any theory. It follows that any two theories U and V such that both U and V have a finite model are mutually interpretable.

Example 3.14 We define the operation $\boxplus$ on theories as follows. The signature of $U_0 \boxplus U_1$ is the disjoint union of the signatures of U_0 and U_1, plus two new unary predicate $\triangle_0$ and $\triangle_1$. We keep identity out of this construction. Identity simple remains identity. The axioms of $U_0 \boxplus U_1$ are:

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[16]We treat identity as any relation. Note that, in this way, we allow identity across pieces. We could also opt to stipulate that the pieces always be disjoint. For an appropriate notion of definable isomorphism, an interpretation with non-disjoint pieces is always definably isomorphic to an interpretation with only disjoint pieces.

- The theory of identity for the extended signature,
- $P(x_0, \ldots, x_{n-1}) \to \bigwedge_{j<n} \triangle_i(x_j)$, if P is derived from the signature of U_i,
- the axioms of U_i relativised to $\triangle_i$,
- $\forall x\, (\triangle_0(x) \vee \triangle_1(x))$,
- $\forall x\, \neg\, (\triangle_0(x) \wedge \triangle_1(x))$.

Suppose τ_0 witnesses $V \rhd U_0$ and τ_1 witnesses $V \rhd U_1$. We construct a translation $\nu := [\tau_0, \tau_1]$ that witnesses $V \rhd (U_0 \boxplus U_1)$.

- $X^{\nu} := X^{\tau_0} \oplus X^{\tau_1} := (\{0\} \times X^{\tau_0}) \cup (\{1\} \times X^{\tau_1})$,
- $\delta^{\nu,(i,\mathfrak{a})} := \delta^{\tau_i,\mathfrak{a}}$,
- Suppose P is derived from an n-ary predicate of U_i. Then:
  $F^{\nu,h}(P) := F^{\tau_i,\pi_1\circ h}(P)$ if $(\pi_0 \circ h)(k) = i$, for all $k < n$,
  $F^{\nu,h}(P) := \lambda \mathbf{x}_0 \cdots \mathbf{x}_{n-1} \cdot \bot$, otherwise.
  Here $\mathbf{x}_j$ is a sequence of variables corresponding to the piece $\pi_1(h(j))$.

We note that we can redo Example 3.14, by noting that, for any U, we have $U \rhd \mathbb{1}$. Hence, $U \rhd \mathbb{1} \boxplus \mathbb{1}$.

Example 3.15 We define the operation $\nu := \tau_0\langle\varphi\rangle\tau_1$ using pieces.

- $X^{\nu} := X^{\tau_0} \oplus X^{\tau_1}$,
- $\delta^{\nu,(0,\mathfrak{a})}(\mathbf{x}) := (\varphi \wedge \delta^{\tau_0,\mathfrak{a}}(\mathbf{x}))$,
  $\delta^{\nu,(1,\mathfrak{a})}(\mathbf{x}) := (\neg\,\varphi \wedge \delta^{\tau_1,\mathfrak{a}}(\mathbf{x}))$,
- Suppose P is derived from an n-ary predicate of U_i. Then:
  $F^{\nu,h}(P)(\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}) := (\varphi \wedge P^{\pi_1\circ h}_{\tau_0}(\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}))$,
  if $(\pi_0 \circ h)(k) = 0$, for all $k < n$,
  $F^{\nu,h}(P)(\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}) := (\neg\,\varphi \wedge P^{\pi_1\circ h}_{\tau_1}(\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}))$,
  if $(\pi_0 \circ h)(k) = 1$, for all $k < n$,
  $F^{\nu,h}(P)(\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}) := \bot$, otherwise.

The definition of *direct* for piecewise translations is simply that every piece has an unrelativised domain and as identity the pointwise identity of the components of the sequences representing the elements.

Regrettably there is no worked out treatment of piecewise interpretations with parameters.

We discuss the connection between piecewise and piece-free interpretations more closely in Appendix 3.8.8. Appendix 3.8.6 is needed for the proper perspective on Appendix 3.8.8.

3.8.6 Five Categories

We do not automatically get a category of theories and interpretations from the machinery we built up until now. For example, $\mathsf{ID}_U \circ \mathsf{ID}_U$ will not be strictly speaking identical with ID_U. We will obtain a category, when we divide out a suitable equivalence among interpretations. Below we will consider five kinds of equivalence that will give us five different categories. One important point of the categories is that isomorphism in each of them defines a salient notion of sameness of theories.

We treat our categories in the case where we do not have parameters, nor pieces. To add these features, we have to adapt our definitions a bit. We sketch the addition of pieces via the Kleisli construction in Appendix 3.8.8.

3.8.6.1 Provable Equivalence of Interpretations

Two interpretations are *provably equivalent* when the *target theory thinks they are the same*. Specifically, two interpretations $K, M : U \to V$ are provably equivalent if they have the same dimension, say m, and:

- $V \vdash \forall \mathbf{x}\ (\delta_K(\mathbf{x}) \leftrightarrow \delta_M(\mathbf{x}))$,
- $V \vdash \forall \mathbf{x}_0, \ldots, \mathbf{x}_{n-1} \in \delta_K\ (R_K(\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}) \leftrightarrow R_M(\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}))$.

Modulo this identification, the operations identity and composition give rise to a category INT_0, where the theories are objects and the interpretations arrows. Isomorphism in this category is *synonymy* or *definitional equivalence*. This is the strictest notion of identity between theories in the literature. It was first introduced by de Bouvère (1965a, b).

Let MOD be the category with as objects classes of models, where the models in each class have the same signature, and as morphisms all functions between these classes. We define $\mathsf{Mod}(U)$ as the class of all models of U. Suppose $K : U \to V$. Then, $\mathsf{Mod}(K)$ is the function from $\mathsf{Mod}(V)$ to $\mathsf{Mod}(U)$ given by: $\mathcal{M} \mapsto \widetilde{K}(\mathcal{M}) := \widetilde{\tau_K}(\mathcal{M})$. It is clear that Mod is a *contravariant functor* from INT_0 to MOD.

3.8.6.2 Definable Isomorphism of Interpretations

For many applications provable equivalence is too strict. A better notions is provable isomorphism or *i-isomorphism*.

Consider $K, M : U \to V$. Suppose K is m-dimensional and M is k-dimensional. An i-isomorphism between interpretations $K, M : U \to V$ is given by an $m + k$-term F in the language of V. We demand that V verifies that "F is an isomorphism between K and M", or, equivalently, that, for each model $\mathcal{M}$ of V, the function $F^{\mathcal{M}}$ is an isomorphism between $\widetilde{K}(\mathcal{M})$ and $\widetilde{M}(\mathcal{M})$.

We spell out the syntactical definition of an i-isomorphism $F : K \Rightarrow M$.

- $V \vdash \mathbf{x}\ F\ \mathbf{y} \to (\mathbf{x} \in \delta_K \wedge \mathbf{y} \in \delta_M)$.
- $V \vdash (\mathbf{x} =_K \mathbf{u} \wedge \mathbf{u}\ F\ \mathbf{v} \wedge \mathbf{v} =_M \mathbf{y}) \to \mathbf{x}\ F\ \mathbf{y}$.
- $V \vdash \forall \mathbf{x} \in \delta_K\ \exists \mathbf{y} \in \delta_M\ \ \mathbf{x}\ F\ \mathbf{y}$.
- $V \vdash (\mathbf{x}_0 F \mathbf{y}_0 \wedge \ldots \mathbf{x}_{n-1} F \mathbf{y}_{n-1}) \to (R_K(\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}) \leftrightarrow R_M(\mathbf{y}_0, \ldots, \mathbf{y}_{n-1}))$.

Here the last item includes identity in the role of R !

Two interpretations $K, M : U \to V$, are *i-isomorphic* iff there is an i-isomorphism between K and M. Wilfrid Hodges calls this notion: *homotopy*. See Hodges (1993), p. 222.

We can also define the notion of being i-isomorphic semantically. The interpretations $K, M : U \to V$, are *i-isomorphic* iff there is an F such that, for all V-models, $\mathcal{M}$, the relation $F^{\mathcal{M}}$ is an isomorphism between $\widetilde{K}(\mathcal{M})$ and $\widetilde{M}(\mathcal{M})$.

The default in this paper is that theories have finite signature: In this case we have a third characterisation. The interpretations $K, M : U \to V$, are *i-isomorphic* iff, for every V-model $\mathcal{M}$, there is an $\mathcal{M}$-definable isomorphism between $\widetilde{K}(\mathcal{M})$ and $\widetilde{M}(\mathcal{M})$. This characterisation follows by a simple compactness argument.

Clearly, if K and M are provably equivalent in the sense of the previous sub-subsection, they will be i-isomorphic. The notion of i-isomorphism give rise to a category of interpretations modulo i-isomorphism. We call this category INT_1.

Isomorphism in INT_1 is *bi-interpretability*. Bi-interpretability is a very good notion of sameness that preserves such diverse properties as finite axiomatisability and κ-categoricity.

3.8.6.3 Isomorphism

Our third notion of sameness of the basic list is that K and M are the same if, for all models $\mathcal{M}$ of V, the internal models $\widetilde{K}(\mathcal{M})$ and $\widetilde{M}(\mathcal{M})$ are isomorphic. We will simply say that K and M are isomorphic. Clearly, i-isomorphism implies isomorphism. We call the associated category INT_2. Isomorphism in INT_2 is *iso-congruence*.

3.8.6.4 Elementary Equivalence

The fourth notion is to say that two interpretations K and M are the same if, for each $\mathcal{M}$, the internal models $\widetilde{K}(\mathcal{M})$ and $\widetilde{M}(\mathcal{M})$ are elementary equivalent. We will say that K and M are elementary equivalent.

By the Completeness Theorem, we easily see that this notion can be alternatively defined by saying that K is elementary equivalent to M iff, for all U-sentences A, we have $V \vdash A^K \leftrightarrow A^M$. It is easy to see that isomorphism implies elementary equivalence. We call the associated category INT_3. Isomorphism in INT_3 is *elementary congruence* or *sentential congruence*.

3.8.6.5 Identity of Source and Target

Finally, we have the option of abstracting away from the specific identity of interpretations completely, simply counting any two interpretations $K, M : U \to V$ the same. The associated category is INT_4. This is the structure of degrees of (global) interpretability $\mathsf{DEG}_{\mathsf{glob}}$. Isomorphism in INT_4 is *mutual interpretability*.

3.8.6.6 Sections, Faithful Retractions and Isomorphisms

We remind the reader of the following. Consider a category C. Suppose $f : x \to y$ and $g : y \to x$ and $g \circ f = \mathsf{id}_x$. In this case, we call f a *section* or *split monomorphism*. We call g a *retraction* or *split epimorphism*. The object x is in this situation a *retract* of y.

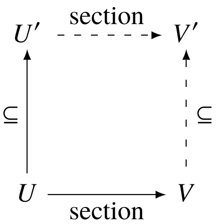
We have:

Theorem 3.30 *Sections in* INT_i, *for* $i = 0, 1, 2, 3$ *are faithful interpretations.*

Proof Since a section in INT_i for $i \leq 3$, is automatically a section in INT_3. It is sufficient to prove out claim for INT_3. Suppose $K : U \to V$ is a section with inverse $M : V \to U$. We have:

$$\begin{aligned} V \vdash \varphi^{\tau_K} &\Rightarrow U \vdash \varphi^{\tau_K \tau_M} \\ &\Rightarrow U \vdash \varphi \end{aligned}$$ □

The section relation has the forward or zig property w.r.t. theory-extension in INT_i, for $i = 0, 1, 2, 3$. This is illustrated by the following diagram.



Theorem 3.31 *Let* $i \in \{0, 1, 2, 3\}$. *The section relation in* INT_i *has the forward or zig property with respect to theory extension.*

Proof Suppose $K : U \to V$ is a section in INT_i. Let M be an inverse of K, so $M : V \to U$ and $M \circ K = \mathsf{ID}_U$ in INT_i. Suppose $U \subseteq U'$. We define

$$V' := \{\varphi \in \mathsf{sent}_{\Sigma_V} \mid U' \vdash \varphi^{\tau_M}\}.$$

Clearly, we have an interpretation $M' : V' \to U'$ based on τ_M. We have:

$$\begin{aligned} U' \vdash \psi &\Rightarrow U' \vdash \psi^{\tau_K \tau_M} \\ &\Rightarrow V' \vdash \psi^{\tau_K} \end{aligned}$$

Hence there is an interpretation K' based on τ_K such that $K' : U' \to V'$.

We note that the further properties needed to be a retraction in one of our categories are trivially upwards preserved from U, K, M, V to U', K', M', V', since the corresponding interpretations are based on the same translations. E.g., in the case of INT_2, suppose $\mathcal{M}$ is a model of U'. Consider the inner model $\mathcal{M}^* := \widetilde{\tau}_K \widetilde{\tau}_M(\mathcal{M})$ Since $\mathcal{M}$ is a model of U, it follows that $\mathcal{M}^*$ is isomorphic to $\mathcal{M}$. □

The above construction has an important disadvantage: it does not *prima facie* preserve finite axiomatisation. Even if U, V and U' are finitely axiomatised, why should V' be finitely axiomatised? The next result fares better in this respect: faithful retractions also have the forward property and here finiteness is preserved.

Theorem 3.32 *Let* $i \in \{0, 1, 2, 3\}$. *The relation of being a faithful retraction in* INT_i *has the forward or zig property with respect to theory extension. Moreover, the result is preserved when we restrict ourselves to finitely axiomatised theories.*

Proof Suppose $K : U \to V$ is a faithful retraction in INT_i. Let M be an inverse of K, so $M : V \to U$ and $K \circ M = \mathsf{ID}_V$ in INT_i. Suppose $U \subseteq U'$. We define:

$$V' := V + \{\varphi^{\tau_K} \in \mathsf{sent}_{\Sigma_V} \mid U' \vdash \varphi\}.$$

Clearly, we have an interpretation $K' : U' \to V'$ based on τ_K.

Suppose $V' \vdash \varphi^{\tau_K}$. Then, for some χ, we have $U' \vdash \chi$ and $V + \chi^{\tau_K} \vdash \varphi^{\tau_K}$. By the faithfulness of K, we find: $U + \chi \vdash \varphi$. Ergo $U' \vdash \varphi$. So K' is faithful.

Suppose $V' \vdash \psi$. Then, $V + \varphi^{\tau_K} \vdash \psi$, for some φ such that $U' \vdash \varphi$. It follows that $V + \varphi^{\tau_K} \vdash \psi^{\tau_M \tau_K}$, since we are in INT_i with $i \leq 3$. By the faithfulness of K, we have $U + \varphi \vdash \psi^{\tau_M}$. Hence, $U' \vdash \psi^{\tau_M}$. So there is an interpretation M' based on τ_M such that $M' : V' \to U'$.

We note that the further properties needed to be a retraction in one of our categories are upwards preserved from U, K, M, V to U', K', M', V', since the corresponding interpretations are based on the same translations.

Finally, if V and U' are finite, then, so is V'. Specifically, if $B := V$ and $A' := U'$, then $B' := V' := B + (A')^{\tau_K}$. □

Theorem 3.33 *Let* $i \in \{0, 1, 2, 3\}$. *The relation of being isomorphic in* INT_i *is a bisimulation with respect to theory extension. Moreover, the result is preserved when we restrict ourselves to finitely axiomatised theories.*

Proof We note that all relevant arrows are sections in INT_3 and, hence, faithful. We can use the proof of the previous theorem to prove the zig and the zag property. To see that the pairs of interpretations we found do indeed form isomorphisms, we note that the further properties needed to be an isomorphism in one of our categories are upwards preserved from U, K, M, V to U', K', M', V', since the corresponding interpretations are based on the same translations. □

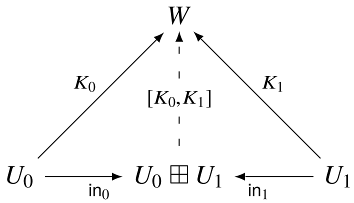
3.8.7 Sums

We show how to treat the notion of sum in the case where we allow piecewise interpretations.

The sum $U_0 \boxplus U_1$ of theories U_0 and U_1 is defined in Example 3.14. Suppose $K_i : U_i \to W$, for $i < 2$. In Example 3.14, we defined the interpretation $[K_0, K_1] : U_0 \boxplus U_1 \to W$.

We note that $U_0 \boxplus U_1$ is synonymous with $U_1 \boxplus U_0$ and $(U_0 \boxplus U_1) \boxplus U_2$ is synonymous with $U_0 \boxplus (U_1 \boxplus U_2)$ and that both are synonymous with the ternary sum $\boxplus(U_0, U_1, U_2)$ which is defined in the obvious way using $\triangle_0$, $\triangle_1$ and $\triangle_2$.

We show that $\boxplus$ is the sum in the categories INT_i for $1 \leq i \leq 4$ where we also include piecewise interpretability. We remind the reader of the sum diagram.



The arrows in_j interprets U_j in $U_0 \boxplus U_1$ by relativisation to $\triangle_j$. We note that, by our conventions we should take $x =_{\mathsf{in}_j} y$ iff $\triangle_j(x) \wedge \triangle_j(y) \wedge x = y$. The other predicate symbols do not need this addition. We leave it to the reader to verify that $[K_0, K_1]$ is indeed the unique arrow satisfying the diagram.

In the case that we do not allow piecewise interpretations, we can simulate our construction for the case that we restrict ourselves to theories W that prove that there are at least two elements. See also Appendix 3.8.8.

There is an alternative construction of a sum $U_0 \oplus U_1$ in Mycielski et al. (1990) or Stern (1989). This alternative construction is for many purposes more convenient.

We have the following basic theorem.

Theorem 3.34 *Consider the theory $W := U \boxplus V$. Consider any formula $A\mathbf{xy}$ in the language of W. Then, there are formulas $B_i\mathbf{x}$ in the language of U and formulas $C_j\mathbf{y}$ in the language of V, such that $A\mathbf{xy}$ is equivalent to a Boolean combination of $B_i^{\mathsf{in}_0}\mathbf{x}$ and $C_j^{\mathsf{in}_1}\mathbf{y}$ in the theory $W + \bigwedge_k \; x_k \in \triangle_0 + \bigwedge_\ell \; y_\ell \in \triangle_1$.*

Proof The proof of the theorem is by a simple induction on A. □

An important notion that is defined in terms of the notion of *sum* is *connectedness*. We say that a theory W is *connected* if, for any theories U and V, if $(U \boxplus V) \rhd_{\mathsf{loc}} W$, then $U \rhd_{\mathsf{loc}} W$ or $V \rhd_{\mathsf{loc}} W$. The following fundamental theorem is due to Pudlák (1983). It was reproved with a markedly different proof by Stern (1989). For more context, see also Mycielski et al. (1990).

Theorem 3.35 *Every sequential theory is connected.*

The notion of *sequentiality* is introduced in Appendix 3.9.

3.8.8 Adding Pieces

Many features of interpretations can be added via the Kleisli and the co-Kleisli construction. See MacLane (1971). We could, for example, start with a category for 1-dimensional direct interpretations and use the co-Kleisli construction to add domains, non-trivial identity and parameters. We can use the Kleisli construction to add more-dimensionality for each dimension m and to add piecewise interpretations. We sketch the addition of pieces using this road using the Kleisli construction. Regrettably, there is no place where the full result is worked out in detail.

Let $\mathbb{2}^+$ be the theory of two named elements. Let's say the named elements are 0 and 1. We define a functor $\Psi : \mathsf{INT}_1^{\mathsf{piece}} \to \mathsf{INT}_1^{\mathsf{npiece}}$. The first category is with and the second without pieces.[17] We take $\Psi(U) := U \boxplus \mathbb{2}^+$. Suppose K is a piecewise interpretation based on $\tau = \langle X, f, \delta, F \rangle$. We define $\Psi(K)$ as follows.

- Let s be the minimum number such that $2^s \geq |X|$. We take $m_{\Psi(K)} := \mathsf{max}(\{f(\mathfrak{a}) \mid \mathfrak{a} \in X\}) + s$.
- The domain $\delta_{\Psi(K)}$ is specified as follows. We first number the pieces in X starting the count with 0. We represent each piece by a binary string of length s corresponding to its associated number. Say this string is $\sigma(\mathfrak{a})$. Let $\zeta(\mathfrak{a})$ be a sequence of $m_{\Psi(K)} - f(\mathfrak{a}) - s - 1$ zero's. For each piece $\mathfrak{a}$ in X, we add elements to the domain of the following form:

$$(\overbrace{0, 1, \ldots, 1, 0}^{\sigma(\mathfrak{a})}, \overbrace{d_0, \ldots, d_{f_K(\mathfrak{a})-1}}^{\text{element of } \delta_K^{\mathfrak{a}}}, \overbrace{0, 0 \ldots, 0, 0}^{\zeta(\mathfrak{a})}).$$

- $P_{\Psi(K)}(\mathbf{x}_0, \ldots, \mathbf{x}_{n-1})$, whenever each $\mathbf{x}_i$ is of the form $\sigma(\mathfrak{a}_i)\mathbf{d}_i\mathbf{\imath}(\mathfrak{a}_i)$, where $\mathbf{d}_i$ is in $\delta_K^{\mathfrak{a}_i}$ and $P_K^h(\mathbf{d}_0, \ldots, \mathbf{d}_{n-1})$, where $h(i) := \mathfrak{a}_i$.

We note that our translation is not uniquely specified since it depends on the choice of the numbering of X. However, on the level of interpretations in $\mathsf{INT}_1^{\mathsf{npiece}}$, this choice is 'erased', since all choices give i-isomorphic interpretations.

In the other direction we define the functor π. We take $\pi(V) := V$. Moreover, $\pi(M)$ will be given by: $X_{\pi(M)} := \{0\}$, $f_{\pi(M)}(0) := m_M$, $\delta^0_{\pi(M)}(\mathbf{x}) := \delta_M(\mathbf{x})$, $P^h_{\pi(M)}(\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}) := P_M(\mathbf{x}_0, \ldots, \mathbf{x}_{n-1})$. So, π is simply representing a non-piecewise interpretation as a one-piece interpretation.

We find that Ψ is a right adjoint of π. The Kleisli construction allows us to define an isomorphic copy of $\mathsf{INT}_1^{\mathsf{piece}}$ inside $\mathsf{INT}_1^{\mathsf{nopiece}}$ by taking as arrows piece-free interpretations $K : U \to \Psi(V)$.

One can show that $\pi \circ \Psi$ is the identity endofunctor of $\mathsf{INT}_1^{\mathsf{piece}}$. This tells us approximately that $\Psi(K)$ and K are i-isomorphic.

We note that if U proves that there are at least two elements, then U is bi-interpretable with $U \boxplus \mathbb{2}^+$. This means that as soon as the interpreting theory proves that there are at least two elements, piecewise interpretations contribute, in a sense, nothing new.

[17] We have to adapt the notion of definable isomorphism to handle the pieces. For piecewise interpretations K and M. An i-isomorphism F will be built up from partial isomorphisms between pieces $\mathfrak{a}$ of K and $\mathfrak{b}$ of M.

3.9 Appendix: Sequential Theories

Sequential theories are theories of sequences where the possible length of the sequence is internally determined. The presence of sequences provides many good properties for such theories. For example, sequential theories are locally reflexive due to the presence of partial satisfaction predicates. We refer the reader to Visser (2013) for more information about sequential and poly-sequential theories.

Even if the basic idea of sequentiality involves sequences and *ipso facto* numbers, sequentiality has a surprisingly simple definition. The theory AS is given by:

$\mathsf{AS}1.\ \vdash \exists x\, \forall y\, y \notin x$
$\mathsf{AS}2.\ \vdash \exists z\, \forall u\, (u \in z \leftrightarrow (u \in x \vee u = y))$

A theory is poly-sequential if it directly interprets AS. A theory is a sequential if it directly interprets AS via a 1-dimensional interpretation. Using these definitions, one may obtain the desired numbers and sequences by a substantial bootstrap.

Since direct interpretations are closed under composition, each theory that directly interprets a (poly-)sequential theory is itself a poly-sequential theory. Obviously, the identical embedding of a theory in an extension-in-the-same-language is direct. Ergo, being a (poly-)sequential theory is preserved under extension-in-the-same-language. Poly-sequentiality is also preserved under INT_1-retractions.

Theorem 3.36 *Let U be a poly-sequential theory and suppose that V is a retraction in INT_1 of U. Then, V is a poly-sequential theory.*

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Chapter 4
Residuated Expansions of Lattice-Ordered Structures

Majid Alizadeh and Hiroakira Ono

Abstract In this paper, residuated expansions of lattice-ordered structures are explored, in particular, of both lattice-ordered groupoids and lattices with implication. Here, a residuated expansion is an expansion in which the law of (left) residuation between fusion and implication holds. Thus, residuated expansions discussed here take the form of (left) residuated lattice-ordered groupoids. A necessary and sufficient condition is given for a lattice-ordered groupoid and also for a lattices with implication to be expandable to a (left) residuated one. Then, our attention is focused to the case where these lattice-ordered structures are bounded and distributive. Each of these structures is shown to be embedded into a residuated one in most cases. Weak Heyting algebras are algebras for subintuitionistic logics, which are special bounded distributive lattices with implication. By applying the above result to them, it is shown that every weak Heyting algebra can be embedded into the *canonical* residuated expansion. This establishes a close link between weak Heyting algebras and the residuated ones, which is examined in more detail for the finite embeddability property and the amalgamation property.

Keywords Residuated expansion · The law of residuation · Lattice-ordered groupoid · Lattice with implication · Weak Heyting algebra · Visser algebra

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M. Mojtahedi et al. (eds.), *Mathematics, Logic, and their Philosophies*, Logic, Epistemology, and the Unity of Science 49,
https://doi.org/10.1007/978-3-030-53654-1_4

4.1 Introduction

In the present paper, we discuss *residuated expansions* of lattice-ordered groupoids and lattices with implication. A residuated expansion of either of these lattice-ordered structures is an expansion in which the law of residuation between fusion and implication holds always. Thus, a residuated expansion of a lattice-ordered groupoid has an additional operation, called an implication, for which basic conditions for lattices with implication hold. On the other hand, a residuated expansion of a lattice with implication has an additional operation, called a fusion, for which basic conditions for lattice-ordered groupoids hold. In this way, residuated expansions can combine these two classes of lattice-ordered structures. We do not assume that a given fusion is commutative. This means that we need to consider the laws of both left and right residuation which will induce two implications, and they must be distinguished in general. On the other hand, in each of our lattices with implication, we assume the existence of a single implication. As there is an apparent symmetry between two implications induced by the law of residuation, we can choose either of them without loss of generality. In the present paper we are mostly concerned with *left residuations* and hence with *left residuated expansions* in our paper.[1] The main aim of our paper is to present and discuss properties of lattice-ordered structures, including some known results, from a perspective of residuated expansions.

A basic question on residuated expansions is obviously whether or not and how an implication can be introduced into a given lattice-ordered groupoid, and also whether or not and how a fusion can be introduced into a given lattice with implication so that the law of residuation holds in the expansion. We make a survey and examine known results related to these questions in Sects. 4.2 and 4.3. By considering residuated expansions, we can clarify close connections between lattice-ordered groupoids and lattices with implication. For example, *integrality* and *contractivity* of lattice-ordered groupoids are known as key notions when we study *substructural features* of monoids and groupoids (for non-associative substructural logics). (See e.g. Galatos et al. 2007). Then with the help of the law of residuation, these notions can be alternatively expressed as familiar conditions in the language of lattices with implication (see Sect. 4.4). Such connections will play an important role in later sections.

From Sect. 4.5 we will concentrate on residuated expansions of *bounded distributive* lattice-ordered structures. Section 4.5 will be devoted to representation theorems of both lattice-ordered groupoids and lattices with implication, assuming that they are bounded and distributive. Our representation theorems are obtained by using *frames with ternary relations*. We owe the idea to Celani (2004) though we take a slightly different approach. These representations together with results obtained in Sects. 4.2 and 4.3 will give us a general embedding theorem of these lattice-ordered structures into residuated ones. This embedding result will be effectively applied in Sects. 4.6 and 4.7. We will take up *weak Heyting algebras* in Sect. 4.6 as special bounded distributive lattices with implication. As a consequence of our embedding theorem, we can show that every weak Heyting algebra can be embedded into its

[1] Sometimes, we call it simply *residuated expansions* when no confusions will occur.

canonical extension, which in turn is expanded to a residuated one, which we call *canonical residuated expansion*. The result holds also for Visser algebras.

In Sect. 4.7, we consider some subvarieties of the variety WH of all weak Heyting algebras and also subvarieties of the variety reWH of all left residuated weak Heyting algebras. First we show that reWH is *conservative* over WH. That is, no new valid equation can be produced by residuated expansions as long as the equation contains no fusion.[2] This suggests that these two varieties may share common algebraic properties. We will see this for the finite embeddability property and the amalgamation property. It is shown that all of these varieties under consideration in this section have the finite embeddability property, and therefore their decision problems are decidable. Also, it is shown that all of these varieties have the amalgamation property, by using the techniques developed in Sect. 4.5.

4.2 Expansions of Lattice-Ordered Groupoids into Residuated Ones

In the following, we assume familiarity with basic notions and results on lattice-ordered structures. (See Davey and Priestley 2002 for general information.) The first class of lattice-ordered structures which we are going to discuss is the class of lattice-ordered groupoids. Residuated lattice-ordered groupoids have been discussed already in connection with non-associative substructural logics. (See e.g. Celani 2004; Galatos and Ono 2010.)

An algebra $\mathbf{A} = \langle A, \cdot, \leq \rangle$ is a *partially ordered groupoid* if it satisfies:

1. $\langle A, \leq \rangle$ is a poset,
2. $\cdot$ is a binary operation on A which is compatible with $\leq$, i.e.,

$$\mathrm{x} \leq \mathrm{z} \text{ and } \mathrm{y} \leq \mathrm{w} \text{ imply } \mathrm{x} \cdot \mathrm{y} \leq \mathrm{z} \cdot \mathrm{w}.$$

Such an operation $\cdot$ is called a *fusion*. The second condition in the above says that a fusion is *monotone* in both coodinates with respect to $\leq$. An algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot \rangle$ is a *lattice-ordered groupoid* if it satisfies:

1. $\langle A, \wedge, \vee \rangle$ is a lattice,
2. $\cdot$ is a binary operation on A such that $\langle A, \cdot, \leq \rangle$ is a partially ordered groupoid, where $\leq$ is the order which is associated with the lattice $\langle A, \wedge, \vee \rangle$.

We notice that the compatibility of fusion with the order $\leq$ in any lattice-ordered groupoid can be expressed also as follows. Here, (wld) and (wrd) mean *weak left* and *weak right distributivity of join over fusion*, respectively. For all $x, y, z \in A$,

(wld) $\quad (x \cdot z) \vee (y \cdot z) \leq (x \vee y) \cdot z,$
(wrd) $\quad (z \cdot x) \vee (z \cdot y) \leq z \cdot (x \vee y).$

[2] Similar topics are discussed in §3.1 of Restall (2006).

Definition 1 An algebra $\mathbf{A} = \langle A, \wedge, \vee, \rightarrow, \cdot \rangle$ is a *left residuated lattice-ordered groupoid* (abbreviated as a *left Relog*) if

1. $\langle A, \wedge, \vee, \cdot \rangle$ is a lattice-ordered groupoid,
2. $\rightarrow$ is a binary operation on A which satisfies that $x \cdot y \le z$ iff $y \le x \rightarrow z$ for every $x, y, z \in A$. (the law of left residuation).

An algebra $\mathbf{A} = \langle A, \wedge, \vee, \rightsquigarrow, \cdot, \rangle$ is a *right residuated lattice-ordered groupoid* (abbreviated as a *right Relog*) if

1. $\langle A, \wedge, \vee, \cdot \rangle$ is a lattice-ordered groupoid,
2. the binary operation $\rightsquigarrow$ satisfies that $x \cdot y \le z$ iff $x \le y \rightsquigarrow z$ for every $x, y, z \in A$. (the law of right residuation).

It is easy to see that the following *right distributivity* (rd), a stronger form of (wrd), holds in every left Relog. For all x, y, z,

(rd) $\quad z \cdot (x \vee y) = (z \cdot x) \vee (z \cdot y).$

Similarly, the following *left distributivity* (ld), a stronger form of (wld), holds in every right Relog. For all x, y, z,

(ld) $\quad (x \vee y) \cdot z = (x \cdot z) \vee (y \cdot z).$

Operations $\rightarrow$ and $\rightsquigarrow$ are sometimes called *left implication* and *right implication* (*induced by* a fusion $\cdot$), respectively.[3] We note that the law of left (right) residuation determines left (right) implication uniquely. It is easy to see that right (left) monotonicity of fusion follows from the law of left (right) residuation, respectively.

Definition 2 An algebra $\mathbf{A} = \langle A, \wedge, \vee, \rightarrow, \rightsquigarrow, \cdot \rangle$ is a *residuated lattice-ordered groupoid* (abbreviated as a *Relog*) when both the law of left residuation for $\rightarrow$ and the law of right residuation for $\rightsquigarrow$ hold.

A left (or, right) Relog is *bounded* when the lattice $\langle A, \wedge, \vee \rangle$ has the smallest element $\bot$ and the greatest $\top$, *distributive* when the lattice $\langle A, \wedge, \vee \rangle$ is distributive, and is *complete* if the lattice $\langle A, \wedge, \vee \rangle$ is complete. As there is an apparent symmetry between left Relogs and right Relogs, we will mostly discuss left Relogs in the following, not repeating the same discussions on right Relogs. So, the term 'implication' refers the left implication in the following when no confusions will occur.

In every left Relog, the monotonicity of $\rightarrow$ in the consequent and the antimonotonicity of $\rightarrow$ in the antecedent hold. That is, for all x, y, z,

- if $x \le y$ then $z \rightarrow x \le z \rightarrow y$,
- if $x \le y$ then $y \rightarrow z \le x \rightarrow z$.

They can be expressed also as the following inequalities, respectively. For all x, y, z,

[3] $x \rightarrow z$ and $y \rightsquigarrow z$ are sometimes expressed as $x \backslash z$ and z / y, and are read as left and right divisions, respectively.

(im) $\quad x \to (y \wedge z) \leq (x \to y) \wedge (x \to z),$
(ij) $\quad (y \vee z) \to x \leq (y \to x) \wedge (z \to x).$

As a matter of fact, the above first inequality can be replaced by the following equality in every left Relog;

(eim) $\quad x \to (y \wedge z) = (x \to y) \wedge (x \to z).$

For, if $w \leq (x \to y) \wedge (x \to z)$ then both $x \cdot w \leq y$ and $x \cdot w \leq z$ hold. Hence $x \cdot w \leq y \wedge z$. Thus $w \leq x \to (y \wedge z)$. This with (im) implies that $x \to (y \wedge z)$ is the greatest lower bound of $\{x \to y, x \to z\}$. Therefore (eim) holds. On the other hand, this is not the case for the second inequality. We can show that in any left Relog the equality

(eij) $\quad (y \vee z) \to x = (y \to x) \wedge (z \to x)$

holds always iff the following left distributivity of join over fusion

(ld) $\quad (y \vee z) \cdot w = (y \cdot w) \vee (z \cdot w)$

holds always. We notice first that $(y \vee z) \cdot w$ is an upper bound of $\{y \cdot w, z \cdot w\}$, by the monotonicity of fusion. Now we assume that (eij) holds and moreover that v is any upper bound of $\{y \cdot w, z \cdot w\}$. By the law of left residuation, w is a lower bound of $\{y \to v, z \to v\}$. Then, $w \leq (y \vee z) \to v$ follows from (eij), as it says that $(y \vee z) \to v$ is the greatest lower bound of $\{y \to v, z \to v\}$, Therefore, $(y \vee z) \cdot w \leq v$ holds. This means that $(y \vee z) \cdot w$ is the least upper bound of $\{y \cdot w, z \cdot w\}$. Hence, (ld) holds. In the same way, we can show that (ld) implies (eij). In general, we have the following lemma. (See e.g. Chap. 3 of Galatos et al. (2007) for the proof.)

Lemma 1 *The following statements hold for every left Relog* **A**.

1. *For each $c \in A$ and each $\{x_i\}_i \subseteq A$, if $\bigvee_i x_i$ exists then $\bigvee_i (c \cdot x_i)$ exists and is equal to $c \cdot (\bigvee_i x_i)$.*
2. *For each $c \in A$ and each $\{x_i\}_i \subseteq A$, if $\bigwedge_i x_i$ exists then $\bigwedge_i (c \to x_i)$ exists and is equal to $c \to (\bigwedge_i x_i)$.*
3. *Suppose that $\bigvee_j y_j$ exists for $\{y_j\}_j \subseteq A$. Then, $\bigwedge_j (y_j \to d)$ exists and is equal to $(\bigvee_j y_j) \to d$ for all $d \in A$ iff $\bigvee_j (y_j \cdot e)$ exists and is equal to $(\bigvee_j y_j) \cdot e$ for all $e \in A$.*

The similar situation happens for right Relogs, in which the law of right residuation holds. Inequalities (im) and (ij) for $\rightsquigarrow$ hold always, and (im) for $\rightsquigarrow$ can be replaced by (eim) for $\rightsquigarrow$. On the other hand, the equality (eij) for $\rightsquigarrow$, i.e., $(y \vee z) \rightsquigarrow x = (y \rightsquigarrow x) \wedge (z \rightsquigarrow x)$ is shown to be equivalent to the right distributivity (rd) of join over fusion. Obviously, for every right Relog, these facts can be stated in general like Lemma 1, by replacing $\to$ by $\rightsquigarrow$ and also by replacing distribution of join over fusion with respect to the first (second) argument by the second (first) argument, respectively, anywhere in the above lemma.

We will discuss next when a given lattice-ordered groupoid can be expanded to a left residuated one. The following lemma gives us a necessary and sufficient condition for a given lattice-ordered groupoid to be expandable. This was discussed e.g. in Sect. 3.1 of Restall (2006) and also in Kowalski and Ono (2010).

Theorem 2 *Let* $\mathbf{A} = \langle A, \wedge, \vee, \cdot \rangle$ *be any lattice-ordered groupoid. Then, there exists a binary operation* $\to$ *on* A *such that the algebra* $\mathbf{A}' = \langle A, \wedge, \vee, \to, \cdot \rangle$ *forms a left Relog if and only if the following two conditions (i) and (ii) hold in* $\mathbf{A}$.

(i) Whenever $\bigvee x_i$ *exists,* $\bigvee (c \cdot x_i)$ *exists always and is equal to* $c \cdot (\bigvee x_i)$ *for all* c.
(ii) For all $a, b \in A$, $\bigvee \{x \in A \mid a \cdot x \leq b\}$ *exists.*

Proof Let us suppose first that the law of left residuation holds between $\cdot$ and $\to$ in our left Relog $\mathbf{A}'$. To show (i), let y be $\bigvee x_i$ for any given x_i's. Then for each c we have $c \cdot x_i \leq c \cdot y$ for every i by using the monotonicity of $\cdot$. This means that $c \cdot y$ is an upper bound of $\{c \cdot x_i\}_i$. Let d be an arbitrary upper bound of $\{c \cdot x_i\}_i$. Then $x_i \leq c \to d$ for each i, and hence $y = \bigvee x_i \leq c \to d$ holds, which implies $c \cdot y \leq d$. Therefore, $c \cdot y$ $(= c \cdot (\bigvee x_i))$ is the least upper bound of $\{c \cdot x_i\}_i$. Thus the first condition (i) holds. For the second condition (ii), let $D = \{x \in A \mid a \cdot x \leq b\}$ for given a, b. Clearly, $a \to b \in D$ and moreover $x \leq a \to b$ for any $x \in D$. This means that $a \to b$ is the least upper bound of D. That is, $\bigvee \{x \in A \mid a \cdot x \leq b\}$ exists, and is equal to $a \to b$.

To show the converse direction, let us define a binary operation $\to$ on $\mathbf{A}$ by the condition that for all $a, b \in A$, $a \to b = \bigvee D$ with $D = \{x \in A \mid a \cdot x \leq b\}$. Here, the existence of $\bigvee D$ is guaranteed by our assumption (ii). It remains to show that the law of left residuation holds between $\cdot$ and $\to$. Suppose that $a \cdot c \leq b$. Then $c \in D$ and hence $c \leq \bigvee D = a \to b$. Conversely suppose that $c \leq a \to b = \bigvee D$. By using the monotonicity of $\cdot$, and the condition (i), we have $a \cdot c \leq a \cdot \bigvee D = \bigvee_{w \in D} (a \cdot w) \leq b$. We note that the above conditions (i) and (ii) imply that the element $\bigvee \{x \in A \mid a \cdot x \leq b\}$ in (ii) is in fact equal to $\max\{x \in A \mid a \cdot x \leq b\}$ for all a, b. □

Because of the law of left residuation, such a left Relog $\mathbf{A}'$ is uniquely determined by a given lattice-ordered groupoid $\mathbf{A}$ as long as it satisfies these two conditions. We call $\mathbf{A}'$, the *left residuated expansion* (or, simply *residuated expansion*, when no confusions will occur) of the lattice-ordered groupoid $\mathbf{A}$. Clearly, the next result immediately follows from Theorem 2.

Corollary 3 *A complete lattice with implication* $\mathbf{A} = \langle A, \wedge, \vee, \to \rangle$, *has the left residuated expansion if and only if* $\bigvee_{v \in D} (c \cdot v) = c \cdot (\bigvee D)$ *for all* $c \in A$ *and all* $D \subseteq A$. *In particular, a finite lattice-ordered groupoid has the left residuated expansion if and only if the equality (rd) holds in it.*

Obviously, any lattice can be identified with a lattice-ordered groupoid in which the fusion is equal to the meet. In such a case, the equality (rd) expresses the distributive law. Then the second statement of Corollary 3 means a well-known fact, which says that for any finite lattice, it is distributive if and only if it can be expanded to an algebra in which the law of left residuation with respect to meet holds, i.e., it can be expanded to a Heyting algebra. Note that since the operation 'meet' is commutative, the law of left residuation is the same as the law of residuation.

4.3 Left Residuated Expansions of Lattices with Implication

The other class of lattice-ordered structures we discuss in the present paper is the class of *lattices with implication*. It is necessary to clarify what kind of lattices with implication we will consider in the present paper, as similar structures are already introduced and discussed, e.g., in Celani (2004).

Definition 3 An algebra $\mathbf{A} = \langle A, \wedge, \vee, \rightarrow\rangle$ is a *lattice with implication*, if $\langle A, \wedge, \vee\rangle$ is a lattice and $\rightarrow$ is a binary operation on A which satisfies

1. if $x \leq y$ then $z \rightarrow x \leq z \rightarrow y$ (the monotonicity in the consequent),
2. if $x \leq y$ then $y \rightarrow z \leq x \rightarrow z$ (the antimonotonicity in the antecedent).

Such an operation $\rightarrow$ is called, an *implication* on $\mathbf{A}$.

As mentioned in the previous section, the two conditions for an implication in the above definition can be expressed also by the following inequalities.

(im) $x \rightarrow (y \wedge z) \leq (x \rightarrow y) \wedge (x \rightarrow z)$,
(ij) $(y \vee z) \rightarrow x \leq (y \rightarrow x) \wedge (z \rightarrow x)$.

It is clear that the fusion-free reduct of the left residuated expansion of a given lattice-ordered groupoid forms a lattice with implication. Now suppose that a lattices with an implication $\rightarrow$ is given. Similarly as in the previous section, we consider a necessary and sufficient condition, under which a fusion can be introduced so that the law of left residuation holds for it with this $\rightarrow$.[4]

The proof of the following lemma goes mostly in parallel with the proof of Theorem 2, as shown below.

Theorem 4 *Let* $\mathbf{A} = \langle A, \wedge, \vee, \rightarrow\rangle$ *be any lattice with implication* $\rightarrow$*. Then, there exists a binary operation* $\cdot$ *on* A *such that* $\langle A, \wedge, \vee, \rightarrow, \cdot\rangle$ *forms a left Relog if and only if the following two conditions (i) and (ii) hold in* $\mathbf{A}$*:*

(i) Whenever $\bigwedge x_i$ *exists,* $\bigwedge(c \rightarrow x_i)$ *exists always and is equal to* $c \rightarrow \bigwedge x_i$ *for all* $c \in A$.
(ii) For all $a, b \in A$, $\bigwedge\{x \in A \mid a \leq b \rightarrow x\}$ *exists.*

Proof We suppose first that the law of left residuation holds between $\cdot$ and $\rightarrow$. To show (i), let $y = \bigwedge x_i$ for any given x_i's. Then for each c we have $c \rightarrow y \leq c \rightarrow x_i$ for every i by the monotonicity of $\rightarrow$ in the consequent. Thus, $c \rightarrow y$ is a lower bound of $\{c \rightarrow x_i\}_i$. Let d be an arbitrary lower bound of $\{c \rightarrow x_i\}_i$. Then $c \cdot d \leq x_i$ for each i, and hence $c \cdot d \leq y$ holds, which implies $d \leq c \rightarrow y$. Therefore, $c \rightarrow y$ is the greatest lower bound of $\{c \rightarrow x_i\}_i$. So the first condition (i) holds. For the second

[4] Alternatively, it is possible to discuss conditions when the law of right residuation holds. But, this is a matter of choice, and it is easy to transfer results on the former to those on the latter, and vice versa.

condition (ii), let $E = \{x \in A \mid a \leq b \to x\}$ for given a, b. Clearly, $b \cdot a \in E$ and moreover $b \cdot a \leq z$ for any $z \in E$. This means that $b \cdot a$ is the greatest lower bound of E. Hence, $\bigwedge\{x \in A \mid a \leq b \to x\}$ exists, and is equal to $b \cdot a$.

To show the converse direction, we introduce a binary operation $\cdot$ by the condition that for any $a, b \in A$, the element $b \cdot a$ is defined to be $\bigwedge E$, where $E = \{x \in A \mid a \leq b \to x\}$. Our assumption (ii) guarantees the existence of $\bigwedge E$. We show now that the law of left residuation holds between $\cdot$ and $\to$. Suppose that $b \cdot a \leq c$. Since $\bigwedge E$ exists, by our assumption (i) $\bigwedge_{w \in E}(b \to w)$ exists and is equal to $b \to \bigwedge E$. Thus $a \leq \bigwedge_{w \in E}(b \to w) = b \to \bigwedge E = b \to (b \cdot a) \leq b \to c$, by the monotonicity of $\to$ in the consequent. Conversely, if $a \leq b \to c$ then $c \in E$. As $b \cdot a$ is a lower bound of E, obviously the inequality $b \cdot a \leq c$ holds. We note here that the conditions (i) and (ii) imply that the element $\bigwedge\{x \in A \mid a \leq b \to x\}$ in (ii) is in fact equal to $\min\{x \in A \mid a \leq b \to x\}$ for all a, b.

It remains to show that the fusion $\cdot$ which we introduced here satisfies the monotonicity, so as for $\langle A, \wedge, \vee, \cdot\rangle$ to be a lattice-ordered groupoid. The left residuation obviously implies the right monotonicity of fusion. To show the left monotonicity of fusion, suppose that $b \leq b'$. For a given a, take an arbitrary $x \in A$ such that $a \leq b' \to x$ holds. Since we assume the antimonotonicity of $\to$ in the antecedent, $b' \to x \leq b \to x$, which implies $a \leq b \to x$ holds. This shows the set inclusion $\{x \in A \mid a \leq b' \to x\} \subseteq \{x \in A \mid a \leq b \to x\}$. Therefore, $b \cdot a = \bigwedge\{x \in A \mid a \leq b \to x\} \leq \bigwedge\{x \in A \mid a \leq b' \to x\} = b' \cdot a$. Thus we have the left monotonicity of fusion. □

It is easily shown that the left Relog $\mathbf{A}^* = \langle A, \wedge, \vee, \to, \cdot\rangle$ is determined uniquely by a given lattice with implication $\mathbf{A} = \langle A, \wedge, \vee, \to\rangle$, if exists, which is called the *left residuated expansion* of a lattice with implication $\mathbf{A}$.

Corollary 5 *A complete lattice with implication $\mathbf{A} = \langle A, \wedge, \vee, \to\rangle$, has the left residuated expansion if and only if $\bigwedge_{w \in E}(b \to w) = b \to \bigwedge E$ holds for all $b \in A$ and all $E \subseteq A$. In particular, a finite lattice with implication has the left residuated expansion if and only if the equality (eim) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$ holds in it.*

It is easy to see that as long as a finite lattice with implication is linearly ordered, the equality (eim) holds always. Thus, we can conclude that any finite linearly ordered lattice with implication has always the left residuated expansion.

4.4 Links Between Lattices with Implication and Lattice-Ordered Groupoids

The law of left (and right) residuation can connect lattices with implication with lattice-ordered groupoids, if it holds. We can observe *substructural features* of implication through this link. Here, we will consider some typical cases which will play an important role in later sections.[5] We consider first *integrality* which is usually expressed by two inequalities $x \cdot y \leq x$ and $x \cdot y \leq y$ (see e.g. Galatos et al. 2007). We have the following lemma immediately.

Lemma 6 *The following statements hold in every Relog* **A***.*

1. *For any given $x \in A$, $x \cdot y \leq x$ for all $y \in A$* (left integral) *iff $y \leq x \to x$ for all $y \in A$ iff $x \leq y \rightsquigarrow x$ for all $y \in A$. Thus, when the greatest element $\top$ exists in A, the two conditions in the if-clause can be replaced by $x \to x = \top$ and $x \leq \top \rightsquigarrow x$, respectively.*
2. *For any given $y \in A$, $x \cdot y \leq y$ for all $x \in A$* (right integral) *iff $y \leq x \to y$ for all $x \in A$ iff $x \leq y \rightsquigarrow x$ for all $x \in A$. Thus, when the greatest element $\top$ exists in A, the two conditions in the if-clause can be replaced by $y \leq \top \to y$ and $y \rightsquigarrow y = \top$, respectively.*

We note here that the condition $x \to x = \top$ can be expressed alternatively as the condition $x \leq y \Rightarrow x \to y = \top$. The next result follows immediately from Lemma 6.

Corollary 7 *In every left Relog with the greatest element $\top$, $x \cdot y \leq x \wedge y$ holds for all x, y* (integrality) *iff both $x \to x = \top$ and $x \leq \top \to x$ holds for all x. A similar statement holds also for every right Relog, if $\to$ is replaced by $\rightsquigarrow$.*

For weaker forms of integrality, we can show the following.

Lemma 8 *In each left Relog* **A***, the following two statements hold.*

1. *$z \cdot (x \cdot y) \leq z \cdot x$ for all $x, y, z \in A$* (weak left integral (1)) *iff $y \leq (z \to x) \to (z \to x)$ for all $x, y, z \in A$,*
2. *$z \cdot (x \cdot y) \leq z \cdot y$ for all $x, y, z \in A$* (weak right integral (1)) *iff $z \to x \leq y \to (z \to x)$ for all $x, y, z \in A$.*

Similarly, in each right Relog **A***, the following two statements hold.*

3. *$(x \cdot y) \cdot z \leq x \cdot z$ for all $x, y, z \in A$* (weak left integral (2)) *iff $z \rightsquigarrow x \leq y \rightsquigarrow (z \rightsquigarrow x)$ for all $x, y, z \in A$.*
4. *$(x \cdot y) \cdot z \leq y \cdot z$ for all $x, y, z \in A$* (weak right integral (2)) *iff $y \leq (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow x)$ for all $x, y, z \in A$,*

Proof We will give a proof of the first statement, since the remaining cases can be proved similarly. Assume that $z \cdot (x \cdot y) \leq z \cdot x$ for all $x, y, z \in A$. For any $u, v, w \in A$, $w \cdot ((w \to u) \cdot v) \leq w \cdot (w \to u) \leq u$ holds by using our assumption. From this, it follows immediately that $v \leq (w \to u) \to (w \to u)$, by using the law of left residuation. Conversely, suppose that $y \leq (z \to s) \to (z \to s)$. Assume moreover that $z \cdot x \leq s$. Then $x \leq z \to s$. By using the antimonotonicity of $\to$ in the antecedent, we have $y \leq x \to (z \to s)$. Thus, $z \cdot (x \cdot y) \leq s$ holds. By taking $z \cdot x$ for s, we get the required inequality. □

[5] A general discussion on related topics in this section is developed in Sect. 2.4 of Ma and Zhao (2017) for distributive cases.

Next, consider *contractivity*, which sometimes is called *square-increasingness*,

Lemma 9 *In each Relog* **A***, the following three statements are mutually equivalent.*

1. $x \wedge (x \to y) \leq y$ *for all* $x, y \in A$,
2. $x \leq x \cdot x$ *for all* $x \in A$ (contractive),
3. $x \wedge (x \rightsquigarrow y) \leq y$ *for all* $x, y \in A$.

Proof We give a proof of the equivalence of the first two statements. The proof of the equivalence of the third to the second goes in the same way. Suppose that the first statement holds. If $x \cdot x \leq u$ then $x \leq x \to u$. Therefore $x = x \wedge x \leq x \wedge (x \to u) \leq u$ by our assumption. Thus $x \leq u$. By taking $x \cdot x$ for u, we have the second. Conversely, suppose that the second statement holds. If $w \leq x \wedge (x \to y)$, then $w \leq x$ and $x \cdot w \leq y$ hold. Using the assumption and the monotonicity, $w \leq w \cdot w \leq x \cdot w \leq y$. Thus, we have $w \leq y$, from which the first statement follows. □

We note that in any lattice-ordered groupoid, the second condition in the above lemma can be replaced by the condition that $x \wedge y \leq x \cdot y$ for all y, z.

Lemma 10 *In each left Relog* **A** *the following two statements are mutually equivalent.*

1. $x \cdot y \leq (x \cdot y) \cdot y$ *for all* $x, y \in A$ (weak right contractive),
2. $(x \to y) \wedge (y \to z) \leq x \to z$ *for all* $x, y, z \in A$.

Similarly, in each right Relog, the following two statements are mutually equivalent.

3. $y \cdot x \leq y \cdot (y \cdot x)$ *for all* $x, y \in A$ (weak left contractive),
4. $(x \rightsquigarrow y) \wedge (y \rightsquigarrow z) \leq x \rightsquigarrow z$ *for all* $x, y, z \in A$.

Proof Suppose that the first statement holds. If $w \leq (x \to y) \wedge (y \to z)$, then both $x \cdot w \leq y$ and $y \cdot w \leq z$. Therefore, $x \cdot w \leq (x \cdot w) \cdot w \leq y \cdot w \leq z$ by using our assumption and the monotonicity for fusion. Hence, $w \leq x \to z$. Thus we have the second statement. Conversely, suppose that the second statement holds. If $(x \cdot y) \cdot y \leq z$ then $y \leq (x \cdot y) \to z$. As $y \leq x \to (x \cdot y)$ holds always by the law of left residuation, $y = y \wedge y \leq (x \to (x \cdot y)) \wedge ((x \cdot y) \to z) \leq (x \to z)$ by 2. Hence, $x \cdot y \leq z$ follows from $y \leq x \to z$. Thus we have the first. The equivalence between the third and the fourth statements can be proved similarly. □

The following lemma shows when the law of residuation with respect to meet holds in a bounded lattice with implication. From this, we can derive related results for weak Heyting algebras in Celani and Jansana (2005). (Weak Heyting algebras are discussed in Sect. 4.6.)

Lemma 11 *The following equivalences hold in any bounded lattice with implication.*

1. *Suppose that the equation (eim):* $(x \to y) \wedge (x \to z) = x \to (y \wedge z)$ *holds for all* x, y, z *in a given bounded lattice with implication. Then,* $x \wedge y \leq z$ *implies* $x \leq y \to z$ *for all* x, y, z *iff both* $x \leq \top \to x$ *and* $x \to x = \top$ *for all* x,

2. *$x \le y \to z$ implies $x \wedge y \le z$ for all x, y, z iff $x \wedge (x \to y) \le y$ for all x, y.*

Proof We give a proof only of the if-part of the first statement. Suppose that $x \wedge y \le z$ for given x, y, z. Then, $x \le \top \to x \le y \to x = (y \to x) \wedge (y \to y) = y \to (x \wedge y) \le y \to z$. □

Corollary 12 *A bounded lattice with implication is a Heyting algebra iff it satisfies all conditions (eim), $x \le \top \to x$, $x \to x = \top$ and $x \wedge (x \to y) \le y$, where $\top$ is the greatest element.*

Recall that a lattice with implication having the smallest element is a Heyting algebra iff the law of residuation holds in it. Also we notice that the condition (eim) follows from the law of residuation. We can see that (eim) is a necessary condition for a lattice with implication to have the left residuated expansion (Theorem 4), conditions $x \le \top \to x$ and $x \to x = \top$ (integrality) implies $x \cdot y \le x \wedge y$ (Corollary 7), and the condition $x \wedge (x \to y) \le y$ (contractivity) implies $x \wedge y \le x \cdot y$ (a remark just below Lemma 9).

4.5 Representation and Left Residuated Expansion of Bounded Distributive lattice-Ordered Structures

To develop the study of residuated expansions of lattice-ordered structures further, we focus our attention particularly on bounded, distributive case. For this purpose, we discuss representation theorem of these structures, using *frames with ternary relation*. They have been studied already in e.g. Dunn and Hardegree (2001), Celani (2004), Ma and Zhao (2017), Shkatov and van Alten (2019). Here we will follow above all (Celani 2004) by S. Celani but in a slightly modified form.[6]

Definition 4 A *frame* is a structure $\langle W, \le, U \rangle$, where $\langle W, \le \rangle$ is a nonempty poset and U is a ternary relation on W satisfying that

(♯) if $U(x, y, z)$, $x' \le x$, $y' \le y$ and $z \le z'$, then $U(x', y', z')$.

Suppose that a frame $\langle W, \le, U \rangle$ is given. Let $Up(W)$ be the collection of upward-closed subsets of W (with respect to $\le$). Obviously, both $X \cap Y$ and $X \cup Y$ belong to $Up(W)$ if $X, Y \in Up(W)$. Moreover, $\langle Up(W), \cap, \cup \rangle$ is shown to be a bounded distributive lattice. For $X, Y \in Up(W)$, we introduce two operations $\circ_U$ and $\Rightarrow_U$.

- $X \circ_U Y = \{w \in W \mid$ there exists $x \in X, y \in Y$ such that $U(x, y, w)\}$,
- $X \Rightarrow_U Y = \{w \in W \mid$ for all $x, y \in W$ such that $U(x, w, y)$, if $x \in X$ then $y \in Y\}$.

[6] For general information on ternary frames, see e.g. Restall (2006).

Due to our assumption (♯) on ternary relations of frames, we can show that both $X \circ_U Y$ and $X \Rightarrow_U Y$ belong to $Up(W)$ whenever $X, Y \in Up(W)$. We can omit the subscript U of $\circ$ or $\Rightarrow$, when a ternary relation U under consideration is clear from the context.

Lemma 13 *For a given frame $\mathbf{M} = \langle W, \leq, U \rangle$, define $\mathbf{M}^G$ to be an algebra $\langle Up(W), \cap, \cup, \circ \rangle$. Then, $\mathbf{M}^G$ is a bounded distributive lattice-ordered groupoid satisfying both (ld) and (rd). In fact, it is complete (as a lattice) which satisfies that for any $Y \in Up(W)$ and any $\{X_i\}_i \subseteq Up(W)$,*

- $\bigcup_i (Y \circ X_i) = Y \circ (\bigcup_i X_i)$,
- $\bigcup_i (X_i \circ Y) = (\bigcup_i X_i) \circ Y$,
- $Y \circ \emptyset = \emptyset \circ Y = \emptyset$.

Lemma 14 *For a given frame $\mathbf{M} = \langle W, \leq, U \rangle$, define $\mathbf{M}^I$ to be an algebra $\langle Up(W), \cap, \cup, \Rightarrow \rangle$. Then, $\mathbf{M}^I$ is a bounded distributive lattice with weak implication satisfying both (eim) and (eij). In fact, it is complete (as a lattice) which satisfies that for any $Y \in Up(W)$ and any $\{X_i\}_i \subseteq Up(W)$,*

- $Y \Rightarrow (\bigcap_i X_i) = \bigcap_i (Y \Rightarrow X_i)$,
- $(\bigcup_i X_i) \Rightarrow Y = \bigcap_i (X_i \Rightarrow Y)$,
- $W \Rightarrow W = W$.

We note that both $\mathbf{M}^G$ and $\mathbf{M}^I$ are complete and satisfy the conditions in Theorems 2 and 4, respectively. Therefore, either of them can be extended to left Relogs. This fact can be confirmed also in a way as Celani did (see Lemma 2.7 of Celani 2004), as shown below. For the time being, we consider any frame $\mathbf{M}_{V,U}$ of the form $\langle W, \leq, V, U \rangle$ with two ternary relations V and U, both of which satisfies (♯). Consider an algebra $(\mathbf{M}_{V,U})^R = \langle Up(W), \cap, \cup, \Rightarrow_V, \circ_U \rangle$, which can be regarded as a combination of two algebras $\mathbf{M}^G$ and $\mathbf{M}^I$.

Lemma 15 *The law of left residuation holds between $\circ_U$ and $\Rightarrow_V$ in $\langle Up(W), \cap, \cup, \Rightarrow_V, \circ_U \rangle$, i.e., for any $X, Y, Z \in Up(W)$, $X \circ_U Y \subseteq Z \Leftrightarrow Y \subseteq X \Rightarrow_V Z$ if and only if ternary relations U and V are equal.*

Proof By our definition, $X \circ_U Y \subseteq Z$ holds iff for all $w \in W$ ($U(x, y, w)$ for some $x \in X$ and some $y \in Y$ implies $w \in Z$), iff

$(*)$ for all $w \in W$, all $x \in X$ and all $y \in Y$, ($U(x, y, w)$ implies $w \in Z$).

On the other hand, $Y \subseteq X \Rightarrow_V Z$ holds iff for all $y \in Y$ and all $x, w \in W$ such that $V(x, y, w)$, if $x \in X$ then $w \in Z$, iff

$(**)$ for all $w \in W$, all $x \in X$ and all $y \in Y$, ($V(x, y, w)$ implies $w \in Z$).

Clearly, when $U = V$, $(*)$ is equivalent to $(**)$. Conversely, suppose that for given $X, Y \in Up(W)$, $(*)$ is equivalent to $(**)$ for any $Z \in Up(W)$. Clearly, both $\{z \mid U(x, y, z)\}$ and $\{z \mid V(x, y, z)\}$ are upward-closed subsets of W for any fixed $x \in X, y \in Y$. By taking each of them for Z in these statements $(*)$ and $(**)$, the equality $U = V$ can be derived. □

Corollary 16 *For a given frame* $\mathbf{M} = \langle W, \leq, U \rangle$*, the algebra* $\mathbf{M}^R$ *defined by* $\langle Up(W), \cap, \cup, \Rightarrow_U, \circ_U \rangle$ *is a bounded distributive left Relog which is the left residuated expansion of both* $\mathbf{M}^G$ *and* $\mathbf{M}^I$*.*

Now we discuss representation of bounded distributive lattice-ordered structures. Let us suppose that a bounded distributive lattice $\mathbf{A} = \langle A.\wedge, \vee \rangle$ is given. To simplify our presentation, we assume in the following that;

- $\mathbf{A}_g$ is any lattice-ordered groupoid which satisfies (ld), (rd) and $x \cdot \bot = \bot \cdot x = \bot$,
- $\mathbf{A}_i$ is any lattice with implication which satisfies (eim), (eij) and $\top = \top \to \top$,
- the lattice reduct of each of $\mathbf{A}_g$ and $\mathbf{A}_i$ is $\mathbf{A}$.

Conditions on $\mathbf{A}_g$ and $\mathbf{A}_i$ in the above will be necessary. In fact, in our representation theorems (Theorems 20 and 21), we use algebras induced by frames with ternary relations in which infinite analogues of these conditions are satisfied, as shown in Lemmas 13 and 14.

Let us proceed to our proof by following mostly the arguments of Sect. 4.2 of Celani (2004), but with a more detailed examination.[7] In the following, $\mathcal{F}_p(\mathbf{A})$ denotes the set of prime filters of the lattice $\mathbf{A}$. Clearly, $\langle \mathcal{F}_p(\mathbf{A}), \subseteq \rangle$ forms a poset. We will introduce two ternary relations U^A and V^A on $\mathcal{F}_p(\mathbf{A})$, which are defined as follows. For all $P, F, Q \in \mathcal{F}_p(\mathbf{A})$,

- $U^A(P, F, Q) \Leftrightarrow$ for all x, y, if $x \in P$ and $y \in F$ then $x \cdot y \in Q$,
- $V^A(P, F, Q) \Leftrightarrow$ for all x, y, if $x \in P$ and $x \to y \in F$ then $y \in Q$.

For a given bounded distributive lattice-ordered groupoid $\mathbf{A}_g$ satisfying that (ld), (rd) and $x \cdot \bot = \bot \cdot x = \bot$, define $(\mathbf{A}_g)_*$ to be the frame $\langle \mathcal{F}_p(\mathbf{A}), \subseteq, U^A \rangle$. Also, for a given bounded distributive lattice with implication $\mathbf{A}_i$ satisfying that (eim), (eij) and $\top = \top \to \top$, define $(\mathbf{A}_i)_*$ to be the frame $\langle \mathcal{F}_p(\mathbf{A}), \subseteq, V^A \rangle$. Obviously, both U^A and V^A satisfy the condition ($\sharp$) of frames. The following is easily seen.

Lemma 17 *The equality* $U^A = V^A$ *holds, when* $\mathbf{A}$ *is a lattice reduct of a left Relog, That is, for all prime filters* P, F, Q *of* $\mathbf{A}$*,* $x \in P$ *and* $y \in F$ *implies* $x \cdot y \in Q$ *for all* $x, y \in A$ *if and only if* $x \in P$ *and* $x \to y \in F$ *implies* $y \in Q$ *for all* $x, y \in A$*.*

The following two lemmas can be shown by using the prime filter theorem of distributive lattices. In the proof of Lemma 18, the conditions (ld) and (rd) are used in showing the only-if part.

Lemma 18 *Suppose that* $\mathbf{A}_g$ *is a lattice-ordered groupoid satisfying (ld) and (rd). For any prime filter* Q *of* $\mathbf{A}$*, and any* $a, b \in A$*,*

- $a \cdot b \in Q \Leftrightarrow$ *there exist prime filters* $a \in P$ *and* $b \in F$ *such that* $U^A(P, F, Q)$*.*

Lemma 19 *Suppose that* $\mathbf{A}_i$ *is a lattice with implication satisfying (eim), (eij) and* $\top = \top \to \top$*. For any prime filter* F *of* $\mathbf{A}$*, and any* $a, b \in A$*,*

[7] There are some, but inessential, differences between algebras discussed in the present section and Celani's lattices with fusion and with implication in Celani (2004).

- *$a \to b \in F \;\Leftrightarrow$ for all prime filters P and Q such that $V^A(P, F, Q)$, if $a \in P$ then $b \in Q$.*

Proof The only-if part is trivial by the definition of V^A. Here, we give a brief outline of a proof of the if part. By taking the contraposition, we suppose that $a \to b \notin F$. Let $K^a = \{x \in A \mid a \to x \in F\}$ and $I_b = \{x \in A \mid x \leq b\}$. We can show that K^a is a nonempty filter. For, $\top = \top \to \top \leq a \to T$ and thus $a \to T \in F$. Therefore, $\top \in K^a$. The set K^a is shown to be a filter, by using (eim). Also, $K^a \cap I_b = \emptyset$, by our assumption $a \to b \notin F$. Now by using the prime filter theorem of distributive lattices, we have that there exists a prime filter Q such that $K^a \subseteq Q$ and $b \notin Q$.

For a given such a prime filter Q, let $\mathcal{H}$ be the set of all filters H such that (i) $a \in H$ and (ii) for all $y \in A$ and all $z \in H$, $z \to y \in F$ implies $y \in Q$. Then $\mathcal{H}$ is nonempty as the principal filter F_a generated by a belongs to it. As $\mathcal{H}$ is an inductive set, there exists a maximal element $P \in \mathcal{H}$ by Zorn's lemma. To show that P is a prime filter, we assume that there are $c, d \in A$ such that $c \vee d \in P$ but $c, d \notin P$. Let $P(c)$ and $P(d)$ are filters generated by $P \cup \{c\}$ and $P \cup \{d\}$, respectively. Since P is maximal in $\mathcal{H}$ and $c, d \notin P$, there exist

- $y_1 \in A$ and $z_1 \in P$ such that $(z_1 \wedge c) \to y_1 \in F$ and $y_1 \notin Q$,
- $y_2 \in A$ and $z_2 \in P$ such that $(z_2 \wedge d) \to y_2 \in F$ and $y_2 \notin Q$.

Let $z = z_1 \wedge z_2$ and $y = y_1 \vee y_2$. As P is a filter and Q is a prime filter, both $y \in A$ and $z \in P$ hold, and moreover they satisfy that $(z \wedge c) \to y$, $(z \wedge d) \to y \in F$ and $y \notin Q$. By (eij) and the law of distributivity, $(z \wedge (c \vee d)) \to y \in F$. On the other hand, $z \wedge (c \vee d) \in P$ by our assumption. This contradicts the assumption that P belongs to $\mathcal{H}$. Thus, P must be prime. It means that $V^A(P, F, Q)$ holds for prime filters P and Q such that $a \in P$ but $b \notin Q$. This completes our proof. □

We have now representation theorems for lattice-ordered groupoids and also for lattices with implication. Let us take a lattice-ordered groupoid $\mathbf{A}_g$ and a lattice with implication $\mathbf{A}_i$ such that each of them has the lattice reduct $\mathbf{A}$ and satisfies either of conditions mentioned above. We construct two frames $(\mathbf{A}_g)_*$ and $(\mathbf{A}_i)_*$, respectively, each of whose underlying set is $\mathcal{F}_p(\mathbf{A})$. Let us consider two complete, bounded distributive lattices $((\mathbf{A}_g)_*)^G$ and $((\mathbf{A}_i)_*)^I$, obtained from these frames. Taking either $\mathbf{A}_g$ or $\mathbf{A}_i$ for $\mathbf{B}$, define a mapping h from $\mathbf{B}$ to $Up(\mathcal{F}_p(\mathbf{B}))$ by $h(a) = \{F \in \mathcal{F}_p(\mathbf{B}) \mid a \in F\}$ for each $a \in A$. We note that $Up(\mathcal{F}_p(\mathbf{B}))$ is the underlying set of the algebra $(\mathbf{B}_*)^K$ where K is either G or I. From Lemmas 18 and 19, it follows that h is a homomorphism in either case. Moreover, by prime filter theorem, we have that h is in fact an injective isomorphism. This completes our proof of the representation theorems. Furthermore, by combining Theorems 2 and 4 together with Lemmas 13 and 14, respectively, we have the following results on left residuated expansions.

Theorem 20 *Every bounded distributive lattice-ordered groupoid* $\mathbf{C}$ *satisfying (ld), (rd) and* $x \cdot \bot = \bot \cdot x = \bot$ *can be embedded into a complete, bounded distributive lattice-ordered groupoid* $(\mathbf{C}_*)^G$*, which satisfies both (ld) and (rd) and moreover has the left residuated expansion. Thus, every bounded distributive lattice-ordered groupoid satisfying (ld) and (rd) can be embedded into a left Relog.*

Theorem 21 *Every bounded distributive lattice with implication* **D** *satisfying (eim), (eij) and* $\top = \top \to \top$ *can be embedded into a complete, bounded distributive lattice with implication* $(\mathbf{D}_*)^I$*, which satisfies (eim), (eij) and* $\top = \top \to \top$ *and moreover has the left residuated expansion. Thus, every bounded distributive lattice with implication satisfying (eim), (eij) and* $\top = \top \to \top$ *can be embedded into a left Relog.*

In particular, we have also the following representation theorem for left Relogs.

Theorem 22 *Every bounded distributive left Relog* **E** *satisfying (eij) and* $\top = \top \to \top$ *can be embedded into a complete, bounded distributive left Relog* $(\mathbf{E}_*)^R$ *which satisfies (eij) and* $\top = \top \to \top$*.*

Proof In fact, $U^E = V^E$ holds by Lemma 17, and therefore $(\mathbf{E}_*)^R$ is a complete, bounded distributive left Relog by Corollary 16. The mapping h defined as above is no other than an embedding **E** into $(\mathbf{E}_*)^R$. Here, we remind you of the fact that (eim) holds in any left Relog. □

We say that $(\mathbf{C}_*)^G$ in Theorem 20 is the *canonical extension* of a bounded distributive lattice-ordered groupoid **C**. Similarly, $(\mathbf{D}_*)^I$ in Theorem 21 (and $(\mathbf{E}_*)^R$ in Theorem 22) is the *canonical extension* of a bounded distributive lattice with implication **D** (and of a bounded distributive left Relog **E**, respectively).

4.6 Left Residuated Weak Heyting Algebras

The class of all lattices with implication includes many important subclasses of algebras for subintuitionisitc logics. We take some examples from Celani and Jansana (2005) and apply our Theorem 21. An algebra **A** of the form $\langle A, \wedge, \vee, \to, \bot, \top\rangle$ is called a *weak Heyting algebra* (abbreviated as a *WH-algebra*), when $\langle A, \wedge, \vee, \to, \bot, \top\rangle$ is a *bounded distributive* lattice with implication $\to$ satisfying (eim), (eij) and moreover the following two equations:

1. $(x \to y) \wedge (y \to z) \le x \to z$,
2. $x \to x = \top$.

An *RWH-algebra* is a WH-algebra which satisfies the following inequality.[8]

3. $x \wedge (x \to y) \le y$.

A *Visser algebra* is a WH-algebra satisfying the following inequality[9]:

4. $x \le \top \to x$.

[8]Due to Celani and Jansana (2005), where ‘R’ comes from ‘reflexive’.

[9]Sometimes, this algebra is called a basic algebra. The present name came from Visser (1981).

When a given algebra $\mathbf{A}$ is a left residuated expansion of a WH-algebra, $\mathbf{A}$ is called a *left residuated WH-algebra*. Similarly, *left residuated RWH-algebras* and *left residuated Visser algebras* can be defined. We notice that in every left residuated WH-algebra, the equality (eim) holds always, and also (eij) is equivalent to (ld).

As an immediate consequence of Corollary 5, we have that every finite WH-algebra can expanded to a left residuated WH-algebra. On the other hand, this is not always the case for infinite WH-algebras, even if they are linearly ordered and complete (see also a remark just below Corollary 5).

Theorem 23 *There is a complete, linearly ordered Visser algebra that cannot be expanded to a left residuated Visser algebra.*

Proof Let $C = [0, 1] \cup \{\omega, \top\}$, where $[0, 1]$ is the unit interval of the reals with the natural order. We assume moreover that $r < \omega < \top$ for every r such that $0 \leq r \leq 1$. Clearly, C forms a complete distributive lattice. Define an operation $\rightarrow$ on C by

$$u \rightarrow v = \begin{cases} \top & \text{if } u \leq v \text{ or } v = \omega \\ \omega & \text{if } u > v \text{ and } 0 < v \leq 1 \\ 1 & \text{if } u > v \text{ and } v = 0. \end{cases}$$

One can show that $\mathbf{C} = \langle C, \min, \max, \rightarrow, 0, \top\rangle$ is a Visser algebra. Let $D = C - \{0\}$. It is clear that $\bigwedge D = 0$. Hence $\top \rightarrow \bigwedge D = \top \rightarrow 0 = 1$. On the other hand, $\bigwedge_{u \in D}(\top \rightarrow u) = \omega$. It says that the condition (i) in Theorem 4 fails in $\mathbf{C}$. Thus, $\mathbf{C}$ cannot be extended to a left residuated Visser algebra. □

On the other hand, we can show that every WH-algebra can be embedded into a left residuated WH-algebra, as a corollary of Theorem 21. We note that the first half of the statement of Theorem 21 is shown already in Celani and Jansana (2005), in which the notion of *WH-frames* was introduced. Here, a WH-frame is a structure $\langle W, \leq, R\rangle$, where $\langle W, \leq\rangle$ is a poset and R is a binary relation on W such that $(\leq \circ R) \subseteq R$. Let $Up(W)$ be the collection of upward-closed subsets of W (with respect to $\leq$). For $X, Y \in Up(W)$, let

$$X \Rightarrow Y = \{w \in W \mid \forall u \in W(wRu \textit{ and } u \in X \Longrightarrow u \in Y)\}.$$

In fact, every WH-frame $\langle W, \leq, R\rangle$ is a special case of a frame (for lattices with implication) $\langle W, \leq, V\rangle$ in our sense, in which $V(u, w, v)$ holds only when $v = u$ for every $u, w, v \in W$. In such a case, wRu is defined by $V(u, w, u)$. The condition $(\leq \circ R) \subseteq R$ follows from the condition ($\sharp$) for the ternary relation V. An *RWH-frame* is a WH-frame $\langle W, \leq, R\rangle$ such that R is reflexive. A *V-frame* is a WH-frame $\langle W, \leq, R\rangle$ such that $R \subseteq \leq$. Notice that $(R \circ R) \subseteq R$ holds in each V-frame, and hence R must be transitive. When R is equal to $\leq$, the frame $\langle W, \leq, R\rangle$ is essentially the same as the intuitionistic frame $\langle W, \leq\rangle$. Like Lemma 14, we can show the following.

Lemma 24 *For every WH-frame* $\mathbf{M} = \langle W, \leq, R\rangle$, $\mathbf{M}^I = \langle Up(W), \cap, \cup, \Rightarrow, \emptyset, W\rangle$ *is a complete WH-algebra.*

When R is reflexive, it is easy to see that for all $X, Y \in Up(W)$, the inclusion $X \cap (X \Rightarrow Y) \subseteq Y$ holds. For, if $w \in X$ and $w \in X \Rightarrow Y$ then $w \in Y$ as wRw holds. Hence, $\mathbf{M}^I$ becomes a RWH-algebra. On the other hand, when $\mathbf{M} = \langle W, \leq, R\rangle$ is a V-frame, the inclusion $X \subseteq W \Rightarrow X$ holds for any $X \in Up(W)$. For, if $u \in X$ and uRz then $u \leq z$. Since $X \in Up(W)$, $z \in X$. As $z \in W$ holds always, this means that $X \subseteq W \Rightarrow X$ holds. Thus, $\mathbf{M}^I$ becomes a Visser algebra. Hence,

Corollary 25 *For every RWH-frame (V-frame)* $\mathbf{M} = \langle W, \leq, R\rangle$, $\mathbf{M}^I = \langle Up(W), \cap, \cup, \Rightarrow, \emptyset, W\rangle$ *is a complete RWH-algebra (Visser algebra, respectively).*

Next, suppose that a WH-algebra $\mathbf{A}$ is given. Define a binary relation R^A on $\mathcal{F}_p(\mathbf{A})$ by $FR^AP \Leftrightarrow V^A(P, F, P)$ for all $F, P \in \mathcal{F}_p(\mathbf{A})$. That is;

$$FR^AP \Leftrightarrow \text{ for all } x, y \text{ such that } x \to y \in F, \text{ if } x \in P \text{ then } y \in P.$$

Lemma 26 *For every WH-algebra* $\mathbf{A}$*, the algebra* $\mathbf{A}_* = \langle \mathcal{F}_p(\mathbf{A}), \subseteq, R^A\rangle$ *is a WH-frame. If* $\mathbf{A}$ *is moreover an RWH-algebra (a Visser algebra), then* $\mathbf{A}_*$ *is an RWH-frame (a V-frame, respectively).*

Proof We will show the second statement of this proposition. Suppose first that $\mathbf{A}$ is an RWH-algebra. Let F be any prime filter. If $x \to y \in F$ and $x \in F$ then $x \wedge (x \to y) \in F$. As $x \wedge (x \to y) \leq y$ holds always in $\mathbf{A}$, we have $y \in F$. Thus, FR^AF holds for any prime filter F, and hence $\mathbf{A}_*$ is an RWH-frame.

Next suppose that $\mathbf{A}$ is a Visser algebra and that FR^AP holds for prime filters F and P. If $a \in F$ then $\top \to a \in F$, since $a \leq \top \to a$ holds. As $\top \in P$ and FR^AP, the element a must be in P. Thus FR^AP implies $F \subseteq P$. Hence, $\mathbf{A}_*$ is a V-frame. □

From Lemmas 26, 24, Corollaries 25 and 16, we can derive the following.

Theorem 27 *Every WH-algebra* $\mathbf{A}$ *can be embedded into a complete WH-algebra* $(\mathbf{A}_*)^I$ *by a mapping* h *defined by* $h(a) = \{F \in \mathcal{F}_p(\mathbf{A}) \mid a \in F\}$ *for each* $a \in A$*. Moreover* $(\mathbf{A}_*)^I$ *can be expanded to the left residuated WH-algebra* $(\mathbf{A}_*)^R$*. The same statement holds also for RWH-algebras and for Visser algebras.*

The left residuated WH-algebra $(\mathbf{A}_*)^R$ in Theorem 27 is called the *canonical residuated expansion* of a WH-algebra $\mathbf{A}$. From results in Sect. 4.4, we can get a substructural characterization of some subclasses of left residuated WH-algebras.

Corollary 28 *1. An algebra* $\mathbf{A}$ *is a left residuated WH-algebra iff it is a bounded distributive weak right contractive and left integral left Relog satisfying (ld):* $(x \vee y) \cdot z = (x \cdot z) \vee (y \cdot z)$ *for all* x, y, z*, i.e., a bounded distributive left Relog satisfying* $x \cdot y \leq (x \cdot y) \cdot y$ *and* $x \cdot y \leq x$ *for all* x, y*, in addition to (ld).*

2. *An algebra* $\mathbf{A}$ *is a left residuated RWH-algebra iff it is a bounded distributive contractive left Relog satisfying (ld).*
3. *An algebra* $\mathbf{A}$ *is a left residuated Visser algebra iff it is a bounded distributive weak right contractive and integral left Relog satisfying (ld).*

The third statement of the above corollary was already shown in Ma and Lin (2014), though their approach is quite different from ours. Since left and right residuated Visser algebras are considered in it, our condition (ld) becomes redundant as it holds always in any right Relog (see Sect. 4.1).

4.7 Conservative Extensions, Finite Embeddability Property and Amalgamation Property

A class $\mathcal{V}$ of algebras of the same type is called a *variety* if $\mathcal{V}$ is closed under homomorphic images, subalgebras and direct products. By Birkhoff's theorem on the equivalence between varieties and equational classes, every variety can be regarded as the class of all algebras which are models of some set E of equations.[10] Note that any inequality $s \leq t$ in the language of lattices can be expressed as an equation $s \wedge t = s$. (For the simplicity's sake, here we use same symbols for syntactic objects like terms and equality symbols as those for mathematical objects.) Hence, the class of all lattice-ordered groupoids, all left Relogs, all lattices with implication, all WH-algebras, all RWH-algebras and all Visser algebras are varieties by their definition. It is easy to see that the law of left resiuation can be expressed by two inequalities $y \leq x \to ((x \cdot y) \vee z)$ and $x \cdot (y \wedge (x \to z)) \leq z$. Therefore, the classes of all left residuated WH-algebras, all left residuated RWH-algebras and all left residuated Visser algebras are also varieties. As a corollary of Theorem 27, we have the following result on conservative extensions, which says that the law of residuation does not produce any new equality in the original language.

Corollary 29 *Suppose that s and t are arbitrary terms in the language $\mathcal{L}$ consisting of $\wedge, \vee, \to, \bot, \top$ (but without fusion). Then the equality $s = t$ is satisfied in the variety* WH *of all WH-algebras iff it is satisfied in the variety* reWH *of all left residuated WH-algebras. This equivalence holds also between the variety of all RWH-algebras* (*of all Visser algebras*) *and the variety of all left residuated RWH-algebras* (*of all left residuated Visser algebras, respectively*).

Proof The only-if part is trivial, as every left residuated WH-algebra is also a WH-algebra. For the if part, suppose that $s = t$ is not valid in a WH-algebra $\mathbf{A}$ for given terms s and t. Then there exists an assignment f on A such that $f(s)$ is not equal to $f(t)$ in $\mathbf{A}$. By Theorem 27, $\mathbf{A}$ is embedded into its canonical residuated expansion $\mathbf{B}$ by an injective isomorphism h. Let g is an assignment on B defined by $g(x) = h(f(x))$ for each variable x. Then $g(s) = h(f(s))$ and $g(t) = h(f(t))$. As h is an isomorphism, $g(s)$ cannot be equal to $g(t)$ in $\mathbf{B}$. Hence, $s = t$ is not satisfied in the variety of all left residuated WH-algebras. □

For the case of the variety of Visser algebras, the above result was essentially obtained in Ma and Lin (2014). On the other hand, Theorem 27 makes the proof

[10] For general information on varieties and equational classes, consult (Burris and Sankappanavar 1981).

much simpler. The above result will suggest that each subvariety of the variety reWH shares various properties with the subvariety of the variety WH corresponding to it. Results on finite embeddability property and amalgamation property in the following may affirm such observations.

We will show next the finite embeddability property of some subvarieties of the variety reWH, including reWH itself.

Definition 5 A given class $\mathcal{K}$ of algebras (of the same finite type) has the *finite embeddability property (FEP)* if for every finite partial subalgebra $\mathbf{B}$ of an algebra $\mathbf{A}$ in $\mathcal{K}$ there exists a finite algebra $\mathbf{D}$ in $\mathcal{K}$ into which $\mathbf{B}$ is embedded.

Our results below can be obtained by applying the method developed in Haniková and Horčík (2014), in which the FEP of the variety BRDG of all bounded distributive Relogs is shown.[11] So, first we will give a brief sketch of the proof in the paper Haniková and Horčík (2014) of the FEP. But for our purpose, we will modify it and show the FEP of the variety of all bounded distributive left Relogs, instead. Let $\mathbf{A} = \langle A, \wedge^A, \vee^A, \rightarrow^A, \cdot^A, \top^A, \bot^A \rangle$ be a bounded distributive left Relog and $\mathbf{B}$ be a finite partial subalgebra of $\mathbf{A}$. To get a required algebra $\mathbf{D}(\mathbf{A}, \mathbf{B})$ into which $\mathbf{B}$ is embedded, first we construct the bounded sublattice $\mathbf{D} = \langle D, \wedge^D, \vee^D, \top^D, \bot^D \rangle$, generated by the set $B \cup \{\top^A, \bot^A\}$ in $\mathbf{A}$. It is easy to see that $\mathbf{D}$ is a finite bounded distributive lattice, satisfying that $\top^D = \top^A$ and $\bot^D = \bot^A$. Next define a closure operator γ and an interior operator σ by

$$\gamma(a) = \bigwedge\{b \in D \mid a \leq^A b\} \text{ and } \sigma(a) = \bigvee\{b \in D \mid b \leq^A a\}.$$

We notice that, since both $\top^A$ and $\bot^A$ belong to $\mathbf{D}$, $\gamma(a)$ and $\sigma(a)$ are always exist. It is obvious that for every $a \in D$ we have $a = \gamma(a) = \sigma(a)$. Now, we define operations $\rightarrow^D$ and $\cdot^D$ on D by $a \rightarrow^D b = \sigma(a \rightarrow^A b)$ and $a \cdot^D b = \gamma(a \cdot^A b)$ for all $a, b \in D$. Let us define the algebra $\mathbf{D}(\mathbf{A}, \mathbf{B})$ by $\langle D, \wedge^D, \vee^D, \rightarrow^D, \cdot^D, \top^D, \bot^D \rangle$. Then, we can show that the law of left residuation holds in $\mathbf{D}(\mathbf{A}, \mathbf{B})$, by using the fact that $\mathbf{A}$ is a left Relog. (See Sect. 4.3 of Haniková and Horčík (2014) for the details.) This means that the algebra $\mathbf{D}(\mathbf{A}, \mathbf{B})$ is in fact a finite bounded distributive left Relog. Obviously, the identity map is the required embedding. In this way, the FEP of the variety of all bounded distributive left Relogs can be shown.

As for the FEP of the variety reWH, we take an arbitrary left residuated WH-algebra for $\mathbf{A}$, instead. The remaining argument goes in the same way. But in this case, we need to show that $\mathbf{D}(\mathbf{A}, \mathbf{B})$ is a left residuated WH-algebra. By Corollary 28, it is enough to show that $\mathbf{D}(\mathbf{A}, \mathbf{B})$ satisfies weak right contractivity, left integrality and (ld). First we show that $a \cdot^D b \leq^D (a \cdot^D b) \cdot^D b$, for all $a, b \in D$. As $\mathbf{A}$ is weak right contractive, $a \cdot^A b \leq^A (a \cdot^A b) \cdot^A b$ holds. Since γ is a closure operator, $a \cdot^A b \leq \gamma(a \cdot^A b)$ and hence $a \cdot^A b \leq^A \gamma(a \cdot^A b) \cdot^A b$. Thus, $a \cdot^D b = \gamma(a \cdot^A b) \leq^D \gamma(\gamma(a \cdot^A b) \cdot^A b) = (a \cdot^D b) \cdot^D b$. Similarly, as $a \cdot^A b \leq a$ holds in A for

[11] See also Farulewski (2008) for the FEP of BRDG.

$a, b \in D$, we can show that $a \cdot^D b = \gamma(a \cdot^A b) \leq^D \gamma^A(a) = a$. Thus, $\mathbf{D}(\mathbf{A}, \mathbf{B})$ is left integral. Lastly, suppose that $\mathbf{A}$ satisfies (ld). Then, for $a, b, c \in D$, $(a \vee^D b) \cdot^D c = \gamma((a \vee^A b) \cdot^A c) = \gamma((a \cdot^A c) \vee^A (b \cdot^A c)) \geq \gamma(a \cdot^A c) \vee^D \gamma(b \cdot^A c)$. On the other hand, by the definition, $\gamma(a \cdot^A c) = \bigwedge\{d \in D \mid a \cdot^A c \leq^A d\}$ and $\gamma(b \cdot^A c) = \bigwedge\{e \in D \mid b \cdot^A c \leq^A e\}$. Therefore, by using the distributivity,

- $\gamma((a \cdot^A c) \vee^A (b \cdot^A c)) = \bigwedge\{g \in D \mid (a \cdot^A c) \vee^A (b \cdot^A c) \leq^A g\} \leq \bigwedge\{d \vee^D e \in D \mid a \cdot^A c \leq^A d \text{ and } b \cdot^A c \leq^A e\} = \gamma(a \cdot^A c) \vee^D \gamma(b \cdot^A c) = (a \cdot^D c) \vee^D (b \cdot^D c)$.

Consequently, $\mathbf{D}(\mathbf{A}, \mathbf{B})$ is a left residuated WH-algebra.

Theorem 30 *The variety* reWH *has the finite embeddability property.*

Similarly, we can show that $\mathbf{D}(\mathbf{A}, \mathbf{B})$ is contractive (integral) when $\mathbf{A}$ is contractive (integral, respectively). Thus, we have also the following with the help of Corollary 28.[12]

Theorem 31 *Both the variety of all left residuated RWH-algebras and the variety of all left residuated Visser algebras have the finite embeddability property.*

From these two theorems together with Theorem 27 we can derive also the following. (See Sect. 4.5 of Celani and Jansana (2005) for a related result.)

Theorem 32 *The variety* WH*, and also the varieties of all RWH-algebras and all Visser algebras have the finite embeddability property.*

Proof We will give a proof for the case of WH. Let $\mathbf{A}$ be a WH-algebra and $\mathbf{B}$ be a finite partial subalgebra of $\mathbf{A}$. By Theorem 27, $\mathbf{A}$ can be embedded into its canonical residuated expansion $\mathbf{A}^\dagger$. Then, $\mathbf{B}$ can be regarded as a finite partial subalgebra of $\mathbf{A}^\dagger$. By the above proof of Theorem 30, we can infer that $\mathbf{D}(\mathbf{A}^\dagger, \mathbf{B})$ is a finite (left residuated) WH-algebra into which $\mathbf{B}$ is embedded. □

Our last topic is the amalgamation property. We will explain below how the amalgamation property of the variety reWH can be obtained from the amalgamation property of the variety WH. The amalgamation property of the latter is already shown in Celani and Jansana (2005). To clarify the connection between the amalgamation property of WH and that of reWH, we will outline here an alternative proof of them, which is based on the idea developed in Maksimova (1977) for the variety of all Heyting algebras, and also in Alizadeh and Ardeshir (2006) for the variety of all Visser algebras.

Definition 6 A given class $\mathcal{K}$ of algebras has the *amalgamation property (AP)* if for all algebras $\mathbf{A}_0$, $\mathbf{A}_1$ and $\mathbf{A}_2$ in $\mathcal{K}$ and for all embeddings $f_1 : \mathbf{A}_0 \longrightarrow \mathbf{A}_1$ and $f_2 : \mathbf{A}_0 \longrightarrow \mathbf{A}_2$, there exist an algebra $\mathbf{B}$ in $\mathcal{K}$, embeddings $g_1 : \mathbf{A}_1 \longrightarrow \mathbf{B}$ and $g_2 : \mathbf{A}_2 \longrightarrow \mathbf{B}$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

[12] A related result on the variety of all residuated Visser algebras is shown in Ma and Lin (2014).

We give here a brief sketch of a proof of AP of WH. As a matter of fact, our argument has many parallels with discussions in §5 on the representation theorem of bounded distributive lattices with implication. Compare the following proof also with the proofs of Lemma 26 and Theorem 27. For further details, consult Maksimova (1977) and Alizadeh and Ardeshir (2006).

Without loss of generality we can assume that the WH-algebra $\mathbf{A}_0$ is a common subalgebra of both WH-algebras $\mathbf{A}_1$ and $\mathbf{A}_2$, and also that both f_1 and f_2 are inclusion maps. Using $\mathbf{A}_1$ and $\mathbf{A}_2$, we define a WH-frame $\mathbf{M}$ below, from which the required WH-algebra $\mathbf{B}$ is obtained. Let us define $\mathbf{M} = \langle W, \leq, V \rangle$ by

- $W = \{(F_1, F_2) | F_i \in \mathcal{F}_p(\mathbf{A}_i)$ for $i = 1, 2$ such that $F_1 \cap A_0 = F_2 \cap A_0\}$,
- $(F_1, F_2) \leq (P_1, P_2)$ iff $F_1 \subseteq P_1$ and $F_2 \subseteq P_2$,
- $V((P_1, P_2), (F_1, F_2), (Q_1, Q_2))$ iff $V_1(P_1, F_1, Q_1)$ and $V_2(P_2, F_2, Q_2)$,

where $V_i(P_i, F_i, Q_i)$ is defined by

$$V_i(P_i, F_i, Q_i) \;\Leftrightarrow\; \text{for all } x, y \in A_i \text{ if } x \in P_i \text{ and } x \to y \in F_i \text{ then } y \in Q_i$$

for $i = 1, 2$ (see §5). Then, it can be shown that $\mathbf{M}$ is in fact a WH-frame. Define the algebra $\mathbf{B}$ as $\mathbf{M}^I$. Then, $\mathbf{B}$ is a WH-algebra by Lemma 24. Now, let g_i be the mapping from $\mathbf{A}_i$ to $\mathbf{B}$, defined by $g_i(a) = \{(F_1, F_2) \in W; a \in F_i\}$ for each $a \in A_i$, for $i = 1, 2$. Then, we can show that both g_1 and g_2 are (WH-)embeddings, and moreover that $g_1 \circ f_1 = g_2 \circ f_2$ holds. (We note that for each $a \in A_0$, $a \in F_1$ holds if and only if $a \in F_2$, whenever $(F_1, F_2) \in W$.) In this way, the amalgamation property of the variety WH is shown.

The proof of AP of reWH goes almost the same as above. The only exception is that we take $\mathbf{M}^R$ instead of $\mathbf{M}^I$ for $\mathbf{B}$. Since both $\mathbf{A}_1$ and $\mathbf{A}_2$ are left residuated, by Lemma 17 $V_i(P, F, Q)$ is equivalent to $U_i(P, F, Q)$, where $U_i(P, F, Q)$ is the condition defined by

$$U_i(P_i, F_i, Q_i) \;\Leftrightarrow\; \text{if } x \in P \text{ and } y \in F \text{ then } x \cdot y \in Q \text{ for all } x, y \in A_i.$$

(We use the letter 'V' (possibly, with subscripts) to express conditions on implication, while the letter 'U' (possibly, with subscripts) to express conditions on fusion.) What we need to add is to show that the mapping g_i preserves also the fusion, i.e., $g_i(x \cdot y) = g_i(x) \circ g_i(y)$ for all $x, y \in A_i$ for $i = 1, 2$. We will show this for $i = 1$ in the following. First we show that $g_1(x) \circ g_1(y) \subseteq g_1(x \cdot y)$ for all $x, y \in A_1$. Take any $G = (P, Q) \in g_1(x) \circ g_1(y)$. Then there exist $G_1 = (P_1, P_1') \in g_1(x)$ and $G_2 = (P_2, P_2') \in g_2(x)$ such that $V(G_1, G_2, G)$. This implies that $U_1(P_1, P_2, P)$. Since $x \in P_1$ and $y \in P_2$, we have $x \cdot y \in P$. This means that $G \in g_1(x \cdot y)$.

Next we show that $g_1(x \cdot y) \subseteq g_1(x) \circ g_1(y)$ for all $x, y \in A_1$. Let $G = (P, Q) \in g_1(x \cdot y)$, i.e., $x \cdot y \in P$. Then there exist prime filters P_1 and P_2 of $\mathbf{A}_1$ such that $x \in P_1$, $y \in P_2$ and $U_1(P_1, P_2, P)$. Put $Q_j = [P_i \cap A_0)_{A_2}$, that is, the filter generated by $P_j \cap A_0$ in $\mathbf{A}_2$ for $j = 1, 2$. It is clear that for each j, $P_j \cap A_0 = Q_j \cap A_0$. To show that $U_2(Q_1, Q_2, Q)$ holds, we need to show that $c \cdot d \in Q$ under the assumption that $c \in Q_1$ and $d \in Q_2$ where $c, d \in A_2$. By the definition of Q_1 and Q_2, there exist $a_1, \ldots, a_m \in P_1 \cap A_0$ and $b_1, \ldots, b_n \in P_2 \cap A_0$ such that $\bigwedge a_i \leq c$ and $\bigwedge b_j \leq d$. Let $c' = \bigwedge a_i$ and $d' = \bigwedge b_j$. Since both P_1 and P_2 are filters and $\mathbf{A}_0$ is a subalgebra of both $\mathbf{A}_1$ and $\mathbf{A}_2$, c' itself belongs to $P_1 \cap A_0$ and also d' belongs to $P_2 \cap A_0$. From $U_1(P_1, P_2, P)$, it follows that $c' \cdot d' \in P$. As $\mathbf{A}_0$ is a left residuated WH-algebra in the present case, $c' \cdot d' \in A_0$. Hence $c' \cdot d' \in P \cap A_0 = Q \cap A_0 \subseteq Q$. On the other hand, as $c' \cdot d' \leq c \cdot d$, also $c \cdot d$ belongs to Q. Thus, $U_2(Q_1, Q_2, Q)$ holds. But, this is still not enough since it is uncertain that both Q_1 and Q_2 are *prime filters*. So, let us define the set Σ by

- $\{(F_1, F_2) |$ both F_1 and F_2 are filters of $\mathbf{A}_2$, $P_1 \cap A_0 = F_1 \cap A_0$, $P_2 \cap A_0 = F_2 \cap A_0$ and $U_2(F_1, F_2, Q)\}$.

Then, Σ is non-empty since $(Q_1, Q_2) \in \Sigma$. By Zorn's lemma, Σ has a maximal element, which we denote (F_1^*, F_2^*). We can show that both F_1^* and F_2^* are prime filters of A_2 as follows. To the contrary, suppose that F_1^* is not prime. Then there exist u and v such that $u \vee v \in F_1^*$ but $u, v \notin F_1^*$. Let G_u (and G_v) be the filter generated by the set $F_1^* \cup \{u\}$ (and $F_1^* \cup \{v\}$, respectively). From $U_2(F_1^*, F_2^*, Q)$, by the condition (♯) in Definition 4 it follows that both $U_2(G_u, F_2^*, Q)$ and $U_2(G_v, F_2^*, Q)$ hold. Because of the maximality of (F_1^*, F_2^*) in Σ, each of $G_u \cap A_0$ and $G_v \cap A_0$ includes $F_1^* \cap A_0$ as a proper subset. Thus, there exist t and s such that $t \in (G_u \cap A_0) \setminus (F_1^* \cap A_0)$ and $s \in (G_v \cap A_0) \setminus (F_1^* \cap A_0)$. It is clear that $t \vee s \in A_0$. Since $t \in G_u$ and $s \in G_v$, there exist $a, b \in F_1^*$ such that $a \wedge u \leq t$ and $b \wedge v \leq s$. Then $(a \wedge u) \vee (b \wedge v) \leq t \vee s$. Meanwhile, $(a \wedge u) \vee (b \wedge v)$ is equal to $(a \vee b) \wedge (a \vee v) \wedge (u \vee b) \wedge (u \vee v)$, which is a conjunction of elements of F_1^*, and hence $t \vee s$ must be in F_1^* and hence in $F_1^* \cap A_0$, which is equal to $P_1 \cap A_0$. As P_1 is a prime filter, either of t and s must be in $P_1 \cap A_0$, and hence in $F_1^* \cap A_0$. This contradicts the choice of elements t and s. Thus we can conclude that F_1^* is a prime filter. Similarly, we can show that F_2^* is a prime filter. Now we have that $(P_1, F_1^*) \in g_1(x)$, $(P_2, F_2^*) \in g_1(y)$, $U_1(P_1, P_2, P)$ and $U_2(F_1^*, F_2^*, Q)$, Hence, $U((P_1, F_1^*), (P_2, F_2^*), (P, Q))$ holds. Thus, $G = (P, Q) \in g_1(x) \circ g_1(y)$.

Theorem 33 *The varieties* WH *and* reWH *have the amalgamation property.*

By using the same argument as above, and in addition using also Lemma 26, Corollary 25 and Theorem 27, we have also the following. We note that the first and the third results are shown already in Celani and Jansana (2005) and Alizadeh and Ardeshir (2006), respectively.

Theorem 34 *The varieties of all RWH-algebras, all left residuated RWH-algebras, all Visser algebras and all left residuated Visser algebras have the amalgamation property.*

4.8 Concluding Remarks

In the present paper, we have introduced a framework of discussing lattice-ordered groupoids together with lattices with implication. Our key notion is *residuated expansions* of both classes of lattice-ordered structures. As shown in Sect. 4.5, this framework runs particularly well for bounded distributive case. Since fusion induced by a residuated expansion of important classes of lattices with implication is often non-associative and non-commutative, we have focused our attention mostly to only *left residuated expansions* but it is not difficult to extend our framework to left and right residuated expansions. In Sects. 4.6 and 4.7, we took up and discussed WH-algebras, RWH-algebras, Visser algebras and their left residuated expansions. It will not be hard to extend results in these sections to any other class of bounded distributive lattices with implication, as long as it is closed under canonical extensions and moreover canonical residuated expansions.

Results on algebraic properties of varieties of (residuated) lattices with implication shown in Sect. 4.7 will suggest to introduce the notion of the *residuated expansion* of a given logic. We are now developing such a logical study in connection with subintuitionistic logics and logics of strict implication, which will be announced in a separate paper.

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Chapter 5
Everyone Knows that Everyone Knows

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Abstract A gossip protocol is a procedure for sharing secrets in a network. The basic action in a gossip protocol is a telephone call wherein the caller and the callee exchange all the secrets they know. An agent who knows all secrets is an expert. The usual termination condition is that all agents are experts. Instead, we explore some protocols wherein the termination condition is that all agents know that all agents are experts. We call such agents super experts. Additionally, we model that agents who already know that all agents are experts, do not make and do not answer calls. We also model that such protocols are common knowledge among the agents. We investigate conditions under which such gossip protocols terminate, both in the synchronous case, where there is a global clock, and in the asynchronous case, where there is not. We show that a protocol with missed calls can terminate faster than the same protocol without missed calls.

Keywords Modal logic · Gossip protocol · Multi-agent knowledge · Dynamics · Common knowledge · Higher-order epistemic goals · Telephone problem

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M. Mojtahedi et al. (eds.), *Mathematics, Logic, and their Philosophies*, Logic, Epistemology, and the Unity of Science 49,
https://doi.org/10.1007/978-3-030-53654-1_5

5.1 Introduction

The great 7th century lexicographer Al Khalil Ibn Ahmad wrote this famous epigraph:

رجل یدری و یدری أنه یدری، فذلک عالم فاتبعوه،
و رجل یدری و لا یدری أنه یدری، فذلک نائم فأیقظوه،
و رجل لا یدری و یدری أنه لا یدری، فذلک مسترشد فأرشدوه
و رجل لا یدری و لا یدری أنه لا یدری، فذلک جاهل فارفضوه

—*Men who know and know that they know, they are expert, follow them*;
—*Men who know and do not know that they know, they are asleep, wake them*;
—*Men who do not know and know that they do not know, they search for guidance, lead them*;
—*Men who do not know and do not know that they do not know, they are ignorant, shun them.*

This contribution is on gossip protocols that terminate when everyone knows that everyone knows all secrets. It is dedicated to a man who knows that he knows.

The gossip problem addresses how to spread secrets among a group of agents by pairwise message exchanges, in other words: telephone calls. We assume that each agent holds a single secret, that when calling each other, the agents exchange all the secrets they know, and that the goal of the information dissemination is that all agents know all secrets. In gossip terminology, an agent who knows all secrets is an *expert* (unlike in the citation above, where an expert is a man who knows that he knows). The situation can be represented by a graph or network where the nodes are the agents and where, when two nodes are linked, the agents can call each other. We say that they know their (telephone) numbers. The linked nodes are pairs in the binary number relation. Similarly, we can represent the secrets that the agents know in a binary secret relation.

There are many variations of the problem. It goes back to the early 1970s (Baker and Shostak 1972; Tijdeman 1971). In this 'classical' setting (for an overview, see Hedetniemi et al. (1988)) only secrets are exchanged, and the focus is on minimum execution length of protocols executed by a central scheduler. Later publications assume that the scheduling is *distributed* (Kermarrec and Steen 2007). Fairly recent developments focus on gossip protocols with *epistemic* preconditions for calls (Apt et al. 2015; Attamah et al. 2014). For example, agents may only call another agent once, or only if they do not know the other agent's secret, etc.

In the problem of *dynamic* gossip (van Ditmarsch et al. 2017, 2019) the agents do not only exchange all the secrets they known but also all the telephone numbers they know. This results in network expansion: not only the secret relation but also the number relation is expanded after a call. The network is then dynamic, which explains the term. However, if the number relation is a complete digraph (the universal relation), i.e., when all agents know all telephone numbers, then the dynamic and classical gossip problem coincide. Here we will assume complete digraphs and thus not investigate dynamic gossip.

Another way to load the messages beyond merely exchanging secrets is to exchange *knowledge about secrets*. This approach is taken in Herzig and Maffre

(2017): in a call the two agents may exchange all the secrets they know. But once this is done (and more), they may also exchange the information 'everyone knows all the secrets'. This requires that the number of agents is known. And once *that* is done, they may exchange the information 'everyone knows that everyone knows all the secrets', and so on. They thus achieve higher-order shared knowledge of all secrets (all the agents know that all the agents know, ...).

In this contribution we also investigate gossip protocols that achieve such higher-order knowledge. However, unlike Herzig and Maffre (2017) we do not achieve this by loading the messages with epistemic features. We continue to exchange the same basic information as in the classical gossip problem, i.e. only secrets. Instead we make three modifications:

- Agents who know that everyone is an expert *no longer make calls*.
- Agents who know that everyone is an expert *no longer answer calls*.
- The protocol terminates when *everyone knows that everyone is an expert*.

An agent who knows that everyone is an expert is a *super expert*. Hence in our new setting, super experts do not make and do not answer calls and the goal is to turn all agents into super experts. Finally, we assume that all these new conditions are common knowledge among the agents.

In the remainder of this introductory section we gradually develop detailed running examples to motivate our approach and discuss first results.

Let there be four agents a, b, c, d. Each agent holds a single secret to share. Consider the call sequence ab; cd; ac; bd. In a call, agents exchange all secrets they know. After the call ab, agents a and b both know two secrets, and similarly after the call cd, agents c and d both know two secrets. Therefore, after the subsequent call ac, agents a and c both know all four secrets: they are experts. Similarly, after the final call bd, b and d are experts. So, after ab; cd; ac; bd, all agents are experts.

In fact, the agents know a bit more than that. After call ac agent a is not only herself an expert but she also knows that agent c is an expert, and agent c also knows that agent a is an expert. (We typically use alternating pronouns: a is female, b is male, c is female, and so on.) Similarly, after call bd, agent b also knows that d is an expert, and d also knows that b is an expert. Can the agents continue calling each other until they all know that they are all experts, i.e., until they all know that they all know all secrets? Yes, they can. Let us first consider agent a. In order to get to know that everyone knows all secrets, a has to make two further calls: ab and ad. Let us suppose these calls are made, and in that order. First, note that before and after those calls the agents involved are already experts, so no factual information is exchanged. However, the agents still learn about each other that they are experts. Hence, after ab, agent a knows that b is an expert and after ad she knows that d is an expert. As she also knows this from herself, a therefore now knows that everyone is an expert. Let us now consider agent b. In call bd he learnt that d is an expert, and in the additional call ab he learnt that a is an expert. And again he obviously knows from himself that he is an expert. Therefore, in order to get to know that everyone is an expert, b only needs to make one additional call, bc, and b then knows that everyone is an expert. Similarly, after yet another call cd, c knows that everyone is an expert, which can be observed by highlighting the calls wherein *c* learns that another agent is an expert, as follows: *ab*; *cd*; ***ac***; *bd*; *ab*; *ad*; ***bc***; ***cd***. We caught two birds in one throw, because after that final call *cd* also agent *d* knows that all agents are experts: *ab*; *cd*; *ac*; ***bd***; *ab*; ***ad***; *bc*; ***cd***.

This contribution is about gossip protocols with the termination condition that everyone knows that everyone knows all secrets. To our knowledge this setting has not been studied in detail before. In particular it differs from Herzig and Maffre (2017) because we do not allow agents to exchange more information than merely their secrets.

As a first idea to motivate our new call rules, suppose any agent who is an expert no longer makes calls and no longer answers calls. A call that is not answered we name a *missed call* (even though in this case not answering is intentional). Given this new rule, can everyone still become an expert? Yes. For example, after the already mentioned call sequence *ab*; *cd*; *ac*; *bd* all agents are experts, and all calls were answered. However, now consider the sequence *ab*; *ac*; *ad*. After this, agents *a* and *d* are experts. Agents *b* and *c* can now no longer become experts: if either were to call *a* or *d*, this would be a missed call. Note that agents do not learn any secrets from a missed call. Hence in this case *b* and *c* can never learn the secret of *d*: they can still call each other, and after additional call *bc* or *cb* agents *b* and *c* would both know three secrets but not all four secrets, they are not experts. The protocol cannot terminate.

We could additionally assume common knowledge among the agents that a missed call means that the agent not answering the call is an expert. But that does not make a big difference. After a missed call as above agents *b* and *c* would thus know that *a* and *d* are experts. But, for example, that agent *b* knows that *a* knows the secret of *d*, does not make *b* himself know the secret of *d*. They cannot use that knowledge to become experts themselves. We conclude that this first idea of a condition for missed calls is not very satisfactory.

In this contribution we therefore employ the idea of missed calls in a different way. Let us suppose that an agent *who knows that everyone is an expert* no longer makes calls and no longer answers calls. This requirement is harder to fulfil than the previous requirement that an agent *who is an expert* stops making and answering calls.

We can already satisfy the stronger termination requirement that everyone knows that everyone is an expert without such missed calls, for example, with the already given sequence *ab*; *cd*; *ac*; *bd*; *ab*; *ad*; *bc*; *cd*. Admittedly, this is not entirely obvious. Please consider the above explanation again, and observe it is also the case that after the subsequence *ab*; *cd*; *ac*; *bd*; *ab*; *ad only* agent *a* knows that everyone is an expert, and in the subsequent call *bc only* agent *b* learns that, and *only* in the final call *cd* agents *c* and *d* simultaneously learn that. No call is made to an agent who knows that all agents know all secrets. Therefore, there are no missed calls.

However, now consider the call sequence *ab*; *cd*; *ac*; *bd*; *ab*; *ad*; *ba*; *ca*; *da*. All final three calls are missed calls, because *a* already knows that everyone is an expert. What do *b*, *c*, and *d* respectively learn from these calls? Well, nothing whatsoever, as just like above we did not make any assumptions so far about the meaning of a missed call in this new context. Therefore, after those calls we can still make the additional calls *bc*; *cd* in order to satisfy that everyone knows that everyone is an expert.

Let us now, as above, additionally assume that it is common knowledge among the agents that a missed call means that the agent not answering the call knows that all agents are experts. Now, unlike above, that makes a big difference. Given the sequence *ab*; *cd*; *ac*; *bd*; *ab*; *ad*; *ba*; *ca*; *da*, in the three final missed calls *ba*, *ca*, and *da*, respectively, agents *b*,*c*,*d* then learn from *a* that all agents are experts, so that after the entire sequence all agents know that all agents are experts. Again, we are done. Before we continue, let us make two more observations. Firstly, if the three missed calls had been ordinary calls, the termination condition would not yet have been met. For example, agent *d* would then not know that agent *c* knows all secrets. Additional calls would have been needed. Secondly, although the sequence with three missed calls is one call longer than the previous sequence that also realizes the knowledge objective, in general there are terminating sequences with missed calls that are shorter than any other terminating sequence without missed calls, as we will prove later.

The modelling solution for missed calls, that is novel, is similar to a modelling solution for making protocols common knowledge, presented in van Ditmarsch et al. (2019). We incorporated both together in this contribution. This also allows us to investigate how we can achieve that everyone knows that everyone is an expert with the constraints of some protocols known from the literature, such as the protocol wherein you are only allowed once to be involved in a call (as the agent making or receiving the call) (van Ditmarsch et al. 2019). For example, consider again the sequence *ab*; *ac*; *ad* after which agents *a* and *d* are experts. As agent *a* may no longer be involved in any subsequent call, it is impossible for her to get to know that everyone is an expert. So, common knowledge of a protocol comes with additional constraints. It may also come with additional advantages: in this case we can sometimes achieve common knowledge of termination under synchronous conditions, i.e., if all agents know how many calls have been made, even if they were not involved themselves in all those calls. We will report some such cases.

Outline In Sect. 5.2 we define gossip protocols and different frameworks for executing gossip protocols. Section 5.3 presents some detailed examples for the protocol wherein any call is allowed, not surprising called ANY. In Sect. 5.4 we provide more general results involving other protocols, namely the CMO protocol (for 'Call Me Once': a call between two agents may only occur once) and the PIG protocol (for 'Possible Information Growth': a call is allowed if the caller considers it possible that the caller or callee will learn a new secret in that call).

5.2 Definitions

Suppose a finite set of agents $A = \{a, b, c, \dots\}$ is given. We assume that two agents can always call each other, i.e., a complete network connects all the agents. Let

$S \subseteq A^2$ be a binary relation such that we read Sxy (for $(x, y) \in S$) as "agent x knows the secret of agent y." For the identity relation $S = \{(x, x) \mid x \in A\}$ we write I.

The agents communicate with each other through telephone calls. During a call between two agent x and y, they exchange all the secrets that they knew before the call. So if a call takes place the binary relation S will be changed.

A *call* or telephone call is a pair (x, y) of agents $x, y \in A$ for which we write xy; agent x is the *caller* and agent y is the *callee*. An agent x is *involved* in a call yz iff $y = x$ or $z = x$. A *call sequence* is defined by induction: the empty sequence ϵ is a call sequence. If σ is a call sequence and xy is a call, then $\sigma; xy$ is a call sequence. Informally, we occasionally consider infinite call sequences. Let S be the secret relation between agents and σ a call sequence. The result of applying σ to S is defined recursively as:

- $S^\epsilon = S$;
- $S^{\sigma;xy} = S^\sigma \cup \{(x, y), (y, x)\} \circ S^\sigma$.

We use $|\sigma|$ for the length of a call sequence, $\sigma[i]$ for the ith call of the sequence, $\sigma|i$ for the first i calls of the sequence, and σ_x for the subsequence of σ that only contains calls involving x.

For a given set of agents A a gossip state is a pair (S, σ), where S is a secret relation and σ a call sequence. A gossip state is initial if $S = I$ and $\sigma = \epsilon$. In this contribution we only consider gossip states (I, σ), in which case we often omit I (we recall that I is the identity secret relation where every agent only knows its own secret).

The following two Definitions 1 and 2 are given by simultaneous induction (double recursion). This is because the language contains a primitive formula $K_a^{\mathsf{P}}\varphi$ that assumes a protocol P, where the protocol conditions P_{xy} that occur in a protocol P are formulas.[1]

Definition 1 (*Language*) For a given finite set of agents A the language $\mathcal{L}$ of protocol conditions is given by the following BNF:

$$\varphi := \top \mid Sab \mid Cab \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_a^{\mathsf{P}}\varphi \mid [\pi]\varphi$$
$$\pi := ?\varphi \mid ab \mid (\pi; \pi) \mid (\pi \cup \pi) \mid \pi^*$$

where a, b range over A. We will use the usual abbreviations for implication, disjunction and for dual modalities, and often omit parentheses.

The atomic formula Sab reads as 'agent a has the secret of b', whereas Cab reads 'a call from a to b has been executed in the past'. The formula $K_a^{\mathsf{P}}\varphi$ reads 'agent a knows that φ is true given protocol P', where 'given protocol P' means 'given that protocol P is common knowledge among the agents'. For the 'make any call' protocol ANY (see below) with all protocol conditions $\mathsf{P}_{ab} = \top$, we write $K_a\varphi$ instead of $K_a^{\mathsf{ANY}}\varphi$. Expression $[\pi]\varphi$ reads as 'after executing the program π, φ is true'.

[1]This is well-defined, see van Ditmarsch et al. (2019) for a justification of a similar semantics.

We also defined the abbreviation $E\varphi := \bigwedge_{a \in A} K_a^{\mathsf{P}} \varphi$ and read it as 'everyone knows φ' (given protocol P — this is left implicit); $E\varphi$ is also known as *shared knowledge* of φ. Program iteration is defined as: $\pi^1 := \pi$, and for $n > 1$, $\pi^{n+1} := \pi^n; \pi$.

Agent a is *expert* if she knows all the secrets, formally $\bigwedge_{b \in A} Sab$. *Everyone is an expert* is represented by the formula $E_{exp} := \bigwedge_{a \in A} \bigwedge_{b \in A} Sab$. Agent a is a *super expert* if she knows that everyone is an expert, formally $K_a^{\mathsf{P}} E_{exp}$. For 'everyone knows that everyone is an expert' (i.e., everyone is a super expert), EE_{exp}, we write E^2_{exp}.

This usage of the E operator can be informally justified as follows. Instead of 'agent a has the secret of b', to represent Sab, it is also often thought of as 'agent a knows the secret of b'. Assuming for this paragraph that a secret of b were a propositional variable p_b, agent a knowing the secret of b then means that agent a knows whether p_b, in a formula, $K_a p_b \vee K_a \neg p_b$. The semantics we present later enforces that Sab implies $K_a Sab$ (if a knows the secret of b, she knows that she knows the secret of b), and that $\neg Sab$ implies $K_a \neg Sab$ (if a does not know the secret of b, she knows that she does not know the secret of b), which corresponds to the positive and negative introspection properties of knowledge. And indeed, these also apply to the description $K_a p_b \vee K_a \neg p_b$ and its negation. Also, E_{exp} for 'everyone knows all the secrets' is equivalent to $\bigwedge_{a \in A} K_a \bigwedge_{b \in A} Sab$, which justifies the usage of 'everyone knows' in its intuitive description. This therefore also justifies the 'everyone knows that everyone knows' and the upper index 2 in E^2_{exp}.

Definition 2 (*Protocol*) A protocol P is a program defined by

$$\mathsf{P} := \Big(\bigcup_{a \neq b \in A} (?(\neg K_a^{\mathsf{P}} E_{exp} \wedge \mathsf{P}_{ab}); ab)\Big)^*; ?E^2_{exp}$$

where for all $a \neq b \in A$, $\mathsf{P}_{ab} \in \mathcal{L}$ is the *protocol condition* for call ab of protocol P.

Some informal explanations are in order to explain this protocol definition. The usual, expected, definition for gossip protocols would have been

$$(?\neg E_{exp}; \bigcup_{a \neq b \in A} (?\mathsf{P}_{ab}; ab))^*; ?E_{exp}$$

where we note that this corresponds to "while $\neg E_{exp}$ do $\bigcup_{a \neq b \in A} (?\mathsf{P}_{ab}; ab)$," in words: "while not everyone is an expert, choose different agents a and b that satisfy the condition P_{ab} and let a call b."

As our termination condition is stronger, this should then become

$$(?\neg E^2_{exp}; \bigcup_{a \neq b \in A} (?\mathsf{P}_{ab}; ab))^*; ?E^2_{exp}$$

Then, as we do not want super experts to make calls, we need to strengthen the protocol condition as in

$$(?\neg E^2_{exp}; \bigcup_{a \neq b \in A} (?(\neg K^{\mathsf{P}}_a E_{exp} \wedge \mathsf{P}_{ab}); ab))^*; ?E^2_{exp}$$

Finally, as $\bigwedge_{a \in A} K^{\mathsf{P}}_a E_{exp}$ is E^2_{exp}, it is not hard to see that the same call sequences are allowed when we remove the first test on $\neg E^2_{exp}$, which leads to the above Definition 2.

The following two Definitions 3 and 4 are also given by simultaneous induction. To define $\approx^{\mathsf{P}}_a$ (or $\sim^{\mathsf{P}}_a$), we need $\models$, and to define $\models$, we need $\approx^{\mathsf{P}}_a$.[2]

Definition 3 (*Synchronous and asynchonous epistemic relation*) Let A be a set of agents and $a \in A$. We define the synchronous accessibility relation $\approx^{\mathsf{P}}_a$ between gossip states inductively as follows:

- $\epsilon \approx^{\mathsf{P}}_a \epsilon$,
- if $\sigma \approx^{\mathsf{P}}_a \tau$, $a \notin \{b, c, d, e\}$, $\sigma \models \neg K^{\mathsf{P}}_b E_{exp} \wedge \mathsf{P}_{bc}$ and $\tau \models \neg K^{\mathsf{P}}_d E_{exp} \wedge \mathsf{P}_{de}$ then $\sigma; bc \approx^{\mathsf{P}}_a \tau; de$
- if $\sigma \approx^{\mathsf{P}}_a \tau$, $S^{\sigma}_b = S^{\tau}_b$, $\sigma \models \neg K^{\mathsf{P}}_a E_{exp} \wedge \mathsf{P}_{ab}$, $\tau \models \neg K^{\mathsf{P}}_a E_{exp} \wedge \mathsf{P}_{ab}$, and ($\sigma \models K^{\mathsf{P}}_b E_{exp}$ iff $\tau \models K^{\mathsf{P}}_b E_{exp}$), then $\sigma; ab \approx^{\mathsf{P}}_a \tau; ab$
- if $\sigma \approx^{\mathsf{P}}_a \tau$, $S^{\sigma}_b = S^{\tau}_b$, $\sigma \models \neg K^{\mathsf{P}}_b E_{exp} \wedge \mathsf{P}_{ba}$, $\tau \models \neg K^{\mathsf{P}}_b E_{exp} \wedge \mathsf{P}_{ba}$, and ($\sigma \models K^{\mathsf{P}}_a E_{exp}$ iff $\tau \models K^{\mathsf{P}}_a E_{exp}$), then $\sigma; ba \approx^{\mathsf{P}}_a \tau; ba$

The asynchronous accessibility relation $\sim^{\mathsf{P}}_a$ between gossip states is the same as the relation $\approx^{\mathsf{P}}_a$ except that the second clause is replaced by

- if $\sigma \sim^{\mathsf{P}}_a \tau$, $a \notin \{b, c\}$, and $\sigma \models \neg K^{\mathsf{P}}_b E_{exp} \wedge \mathsf{P}_{bc}$, then $\sigma; bc \sim^{\mathsf{P}}_a \tau$ and $\sigma \sim^{\mathsf{P}}_a \tau; bc$

Note that $\approx^{\mathsf{P}}_a$ and $\sim^{\mathsf{P}}_a$ are equivalence relations. For $\sim^{\mathsf{ANY}}_a$ we write $\sim_a$ and for $\approx^{\mathsf{ANY}}_a$ we write $\approx_a$.

We recall that we identify initial gossip states (I, σ) with call sequences σ. The above definitions incorporate common knowledge that (i) protocol P is used *and* (ii) agents stop making calls once they know all agents are experts (once they are super experts). This is as in van Ditmarsch et al. (2019).

However, our semantics also has the additional feature "$\sigma \models K^{\mathsf{P}}_b E_{exp}$ iff $\tau \models K^{\mathsf{P}}_b E_{exp}$" in the relations. This extra condition models that agents no longer answer calls if they know that all agents are experts (if they are super experts). It has the effect that after a missed call ab any state τ wherein agent b does *not* know that all agents are experts is no longer considered possible by agent a. In other words, in that case we have $\sigma; ab \not\approx^{\mathsf{P}}_a \tau; ab$. Given the subsequent semantics, this means that $K^{\mathsf{P}}_a K^{\mathsf{P}}_b E_{exp}$, and therefore also $K^{\mathsf{P}}_a E_{exp}$. The dual effect of this semantics for missed calls is that when b answers a call ab by a, any state τ wherein agent b would have been a super expert is no longer considered possible by a. In particular, when a learns that b already knew all secrets before the call, she learns that b was not yet a super expert at the time. Of course, b may have become a super expert in the call ab.

The clause for the missed call ba to agent a is somewhat (but not entirely) similar to the clause for the missed call ab from agent a.

[2]This is also well-defined, see again (van Ditmarsch et al. 2019) for a justification of a similar semantics.

Informally, the synchronous accessibility relation encodes that agents not involved in a call are still aware that a call has taken place, as considered in Apt et al. (2015), Attamah et al. (2014). This also implies that all agents know how many calls have taken place, i.e., there is a global clock. The asynchronous accessibility relation does not make any such assumption. Then, agents are only aware of the calls in which they are involved. Any information on other calls has to be deduced from the secrets they obtain from their calling partners.

Definition 4 (*Semantics*) Let call sequence σ and formula $\varphi \in \mathcal{L}$ be given. We define $\sigma \models \varphi$ by induction on φ.

$$
\begin{array}{ll}
\sigma \models \top & \text{iff } always \\
\sigma \models Sab & \text{iff } I^{\sigma}ab \\
\sigma \models Cab & \text{iff } ab \in \sigma \\
\sigma \models \neg\varphi & \text{iff } \sigma \not\models \varphi \\
\sigma \models \varphi \wedge \psi & \text{iff } \sigma \models \varphi \text{ and } \sigma \models \psi \\
\sigma \models K_a^{\mathsf{P}}\varphi & \text{iff } \tau \models \varphi \text{ for all } \tau \text{ such that } \sigma \approx_a^{\mathsf{P}} \tau \\
\sigma \models [\pi]\varphi & \text{iff } \tau \models \varphi \text{ for all } \tau \text{ such that } \sigma[\![\pi]\!]\tau
\end{array}
$$

where

$$
\begin{array}{ll}
\sigma[\![?\varphi]\!]\tau & \text{iff } \sigma \models \varphi \text{ and } \tau = \sigma \\
\sigma[\![ab]\!]\tau & \text{iff } \tau = \sigma; ab \\
\sigma[\![\pi;\pi']\!]\tau & \text{iff there is } \rho \text{ such that } \sigma[\![\pi]\!]\rho \text{ and } \rho[\![\pi']\!]\tau \\
\sigma[\![\pi \cup \pi']\!]\tau & \text{iff } \sigma[\![\pi]\!]\tau \text{ or } \sigma[\![\pi']\!]\tau \\
\sigma[\![\pi^*]\!]\tau & \text{iff there is } n \in \mathbb{N} \text{ such that } \sigma[\![\pi^n]\!]\tau
\end{array}
$$

The inductive clause for $K_a^{\mathsf{P}}\varphi$ above is for the synchronous setting. For the asynchronous setting we replace $\sigma \approx_a^{\mathsf{P}} \tau$ by $\sigma \sim_a^{\mathsf{P}} \tau$ in that clause. For simplicity we do not use a separate symbol for the asynchronous semantics — it will always be clear from the context what '$\models$' stands for. A formula is *valid* iff for all call sequences σ, $\sigma \models \varphi$, in which case we also write $\models \varphi$.

We assume that all our protocols are *symmetric*, which means that for all $a \neq b \in A$ and $c \neq d \in A$, simultaneously replacing a by c and b by d in the protocol condition P_{ab} yields P_{cd}. Moreover, we only consider protocols that are *epistemic*, which means that $\models \mathsf{P}_{ab} \leftrightarrow K_a^{\mathsf{P}}\mathsf{P}_{ab}$.

Some elementary properties of the semantics that we list without proof, are as follows. Here $a, b \in A$, protocols P, P', and $\varphi \in \mathcal{L}$ are all arbitrary.

- $\models K_a^{\mathsf{P}}\varphi \to K_a^{\mathsf{P}}K_a^{\mathsf{P}}\varphi$, and $\models \neg K_a^{\mathsf{P}}\varphi \to K_a^{\mathsf{P}}\neg K_a^{\mathsf{P}}\varphi$. Intuitively, K_a^{P} has two of the standard properties of knowledge, namely positive and negative introspection.
- $K_a^{\mathsf{P}}\varphi \to \varphi$ is not valid. In fact, whenever σ is not P-permitted as defined below, then $\sigma \models K_a^{\mathsf{P}}\bot$.
- $\models \mathsf{P}_{ab} \to \mathsf{P}'_{ab}$ implies $\models K_a^{\mathsf{P}'}\varphi \to K_a^{\mathsf{P}}\varphi$; a corollary is that $\models K_a\varphi \to K_a^{\mathsf{P}}\varphi$;
- $\models Sab \leftrightarrow K_aSab$ and $\models \neg Sab \leftrightarrow K_a\neg Sab$;
- $\sigma \models K_b^{\mathsf{P}}E_{exp}$ implies $\sigma \models [ab]K_a^{\mathsf{P}}K_b^{\mathsf{P}}E_{exp}$. That is, if b is a super expert and a calls b, then a learns that b is a super expert from the missed call ab.

We continue with some terminology on protocol termination. In some of this subsequent terminology we need to consider infinite call sequences. We denote potentially infinite call as σ_∞.

If $\sigma \models \mathsf{P}_{ab} \wedge \neg K_a^{\mathsf{P}} E_{exp}$ we say that call ab is P-*permitted* after σ. A P-permitted call sequence is a call sequence consisting of P-permitted calls. An infinite call sequence σ_∞ is P-permitted if any prefix $\sigma_\infty|i$ (where $i \in \mathbb{N}$) is P-permitted.

A P-permitted sequence σ_∞ is P-*fair* iff either σ_∞ is finite or for all $x \neq y \in A$, if for all i there is $j > i$ such that xy is P-permitted in $\sigma_\infty|j$ then for all i there is $j > i$ such that $\sigma_\infty[j] = xy$.

A call sequence σ is *super successful* if after σ all the agents are super experts. (For the termination condition E_{exp}, the corresponding terminology is that call sequence σ is successful if after σ all the agents are experts, see van Ditmarsch et al. (2019).) A sequence σ is P-*maximal* (or P-*terminal*, or *terminating*) iff it is P-permitted and if for any $x, y \in A$, $\sigma; xy$ is not P-permitted. An infinite call sequence σ_∞ is P-maximal if any prefix $\sigma_\infty|i$ is P-permitted. A protocol P is super successful iff all P-maximal sequences are super successful (and thus finite). A protocol P is *fairly super successful* iff all fair P-maximal sequences are super successful.

Let us give some further intuitions to explain these definitions. The notion of fairness is needed because already very simply protocols allow infinite execution sequences. For example, the protocol ANY (wherein you can make any call) has execution sequences $ab; ab; \ldots$. We call this unfair because only call ab occurs in this sequence, but not, for example, ba, or if there is another agent c, any of ca, ac, bc, cb. All of these are permitted at any stage. Therefore, in any *fair* ANY-sequence all of these should occur arbitrarily often. Indeed, the protocol ANY is not super successful, but it is fairly super successful (van Ditmarsch et al. 2019).

After these technical requirements, we continue with some detailed examples of the semantics.

5.3 Examples

All examples in this section are for the protocol ANY. We recall that for all $x \neq y \in A$, $\mathsf{ANY}_{xy} = \top$. Therefore, we can write K_a ($= K_a^{\mathsf{ANY}}$) everywhere instead of K_a^{P}.

Three agents, make any call, asynchronous

Let $A = \{a, b, c\}$, let the protocol be ANY, and let the environment be asynchronous. Consider the call sequence $ab; ac; ab; cb$.

- After the prefix $ab; ac$, agents a and c are experts.
- After the prefix $ab; ac; ab$, agents a and b know that all agents are experts. Agent a already knew that c is an expert and in call ab also learns that b now is an expert. Therefore, she knows that all agents are experts: $ab; ac; ab \models K_a E_{exp}$. In the call ab, agent b learns that a is an expert. Because in the first call ab agent a did not know the secret of c yet, but now gives it to b, agent b can infer that the call ac

must have taken place before the second call ab. As in that call ac agent c became an expert, agent b also knows that agent c is an expert. Therefore also agent b knows that all agents are experts: $ab; ac; ab \models K_b E_{exp}$.

- Now consider the entire sequence $ab; ac; ab; cb$. Call cb is a missed call: agent b will not answer agent c's call, because b already knows that all agents are experts. Therefore, agent c learns in that call that all agents are experts: $ab; ac; ab; cb \models K_c E_{exp}$. We now have that $ab; ac; ab; db \models E^2_{exp}$.

Possibly of interest is that if the last call had been answered by b, then c would also have learnt that all agents are experts. This is because in agent c's prior call ac, agent a did not know the secret of b yet. Therefore agent c can infer that call ab must have taken place before the final call cb. As in that call ab agent a became an expert, agent c also knows that agent a is an expert. Therefore agent c knows that all agents are experts. A crucial assumption here is that caller and callee not only learn what the union is of the set of secrets they each held before the call, but that they also learn what set of secrets the other agent held before the call. This is known as the "inspect-then-merge" form of observation (Apt et al. 2018).

It is remarkable that we can come to this conclusion for the asynchronous situation. This is because there are three agents only. Such scenario's for small numbers of agents (although not for the alternative termination condition E^2_{exp}, or involving missed calls) are also discussed in Apt et al. (2018).

Four agents, make any call, asynchronous Let now $A = \{a, b, c, d\}$, let the protocol be ANY, and let the environment be asynchronous. A terminating sequence $ab; cd; ac; bd; ab; ad; bc; cd$ consisting of eight calls was already given in the introductory Sect. 5.1. In that section it was already justified that we obtain E^2_{exp} by this sequence without any missed call. Asynchronously, we cannot realize E^2_{exp} for four agents in less than eight calls. In the next examples we show that for larger numbers of agents there are shorter terminating sequences in the synchronous situation.

Four agents, make any call, synchronous Let again $A = \{a, b, c, d\}$, let the protocol be ANY, but let now the environment be synchronous. We claim that after the five call sequence $ab; cd; ac; bd; ba$ agent b already knows that all agents are experts, which can only be obtained after a six call sequence in the asynchronous case. This illustrates the difference between the synchronous and the asynchronous situation.

Let us consider agent b. Clearly, after prefix $ab; cd; ac; bd$ agents b is an expert. He does not know between who the second and third calls were, but he knows that no call between a and d took place. However, he is uncertain whether agent a already is an expert. For example, an alternative sequence considered possible by b is $ab; cd; cd; bd$. This uncertainty is resolved in the fifth call. Consider the sequence $ab; cd; ac; bd; ab$. This reveals to b that a must have been involved in the second or third call of the sequence, and, given b's call bd, that this cannot be the second call. As a is already an expert in the call ab, this reveals that the third call must have been between a and c. Agent b now only consider possible the sequence $ab; cd; ac; bd; ab$ (where the calls not involving him could also have been in the other direction). Therefore, agent b knows that all agents are experts.

We were not able to reach E^2_{exp} for four agents with less than eight calls, for example, as an extension of the above sequence.

Six agents, make any call, asynchronous In this example we show that with missed calls we can obtain shorter termination sequences than without.

Let there now be six agents a, b, c, d, e, f. Let us first assume a semantics without missed calls, but still with termination condition E^2_{exp}. A standard solution to obtain E_{exp} is $ae; af; ab; cd; ac; bd; ae; af$. It consists of eight calls. This is the minimum (for $n \geq 4$ agents, $2n - 4$ calls are needed to reach E_{exp} (Tijdeman 1971)). Note that after any of the final four calls $ac; bd; ae; af$, the involved agents are experts. For n agents, to obtain E^2_{exp} we need $\binom{n}{2}$ calls after which the calling agents are experts. This is because each agent has to verify for all other agents that they know all secrets, hence all pairs need to occur as calls. This gives us $\binom{n}{2}$. But, of course, four of those already occurred at the end of the sequence obtaining E_{exp}, as above. In the case of 6 agents, where $\binom{6}{2} = 15$, we therefore need $8 + 15 - 4 = 19$ calls. For example, one possible terminating sequence is this:

$$ae; af; ab; cd; ac; bd; ae; af; ab; ad; \boldsymbol{ba}; bc; bd; bf; cd; ce; cf; ed; ef$$

Let us now consider our new semantics with missed calls. The above sequence is no longer permitted. For example, before the call ba in bold a is already a super expert and thus will not answer the call. After the call ba also b is a super expert and will stop making calls, hence $bc; bd; bf$ are no longer permitted. Thus, in our new setting, a simpler sequence with 15 instead of 19 calls is terminating:

$$ae; af; ab; cd; ac; bd; ae; af; ab; ad; ba; ca; da; ea; fa$$

In this sequence first a becomes a super expert and then all other agents call agent a. The final five calls $ba; ca; da; ea; fa$ are all missed calls in which b to f become super experts.

We conjecture that not only in this example but for any $n \geq 4$ agents at least $n - 2 + \binom{n}{2}$ calls are needed.

Firstly, note that given a graph with n nodes without any connections, such as the graph for the initial secrets relation S, at least $n - 1$ links are needed to connect n nodes (this may be by way of one agent calling all $n - 1$ other agents, but there are many other ways, such as in the above sequence). Therefore, after $n - 2$ calls no agent will be an expert. Call $n - 1$ is the first call wherein two agents may become experts.

Secondly, under asynchronous conditions and without any knowledge of the protocol, and given that there are at least 4 agents, whenever two agents become expert in a call, they are uncertain whether the agents not involved in that call are already experts. It therefore seems (which we wish to prove in future research) that they need to call all other agents to confirm that. As this holds for all agents, on this assumption all calls ab for $a, b \in A$ need therefore to occur: this is the number $\binom{n}{2}$ already mentioned above. This would also mean that in general, for n agents, using the new semantics with missed calls at least $\mathcal{O}(n)$ additional calls are needed to reach E^2_{exp}, whereas without missed calls at least $\mathcal{O}(n^2)$ calls are needed.

Yet more examples are found in the introductory section. The reader may wish to go through these examples again, and validate them with the semantics of the previous section.

5.4 The Protocols CMO and PIG

5.4.1 The Protocol CMO

The *Call Me Once* (CMO) protocol was introduced in van Ditmarsch et al. (2019) (and is reminiscent of Doerr et al. (2014)), wherein it was shown that for the termination condition E_{exp} the protocol CMO is successful in weakly connected graphs, and therefore also in the fully connected graphs that are implicit in our contribution. The protocol condition for CMO is that you are allowed to call an agent if you have not yet been involved in a call with that agent. Formally we define it as follows.

Protocol (CMO—Call Me Once) *The* Call Me Once *protocol is defined by the condition* $\mathsf{P}_{xy} := \neg Cxy \wedge \neg Cyx$. □

As any two agents x, y out of $|A| = n$ agents are only allowed to call each other once (either as xy or as yx), the maximum number of calls before termination is therefore $\binom{n}{2} = \frac{n(n-1)}{2}$.

In this section we show that CMO is not (always) super successful if the protocol is not common knowledge among the agents or if the conditions are asynchronous.

Theorem 1 *Let* $|A| \geq 4$. *The protocol* CMO *is not super successful if the protocol is not common knowledge or if the setting is asynchronous.*

Proof We need to consider two cases: (i) the protocol is not common knowledge, and (ii) the setting is asynchronous. For both cases we have to show that CMO is not (always) super successful.

Case (i). Let the protocol not be common knowledge. Given $A = \{a_1, a_2, \dots, a_n\}$, let ρ be a maximal CMO sequence between agents $\{a_1, a_2, \dots, a_{n-1}\}$. We may assume (from van Ditmarsch et al. (2019)) that after ρ all agents $a_1, a_2, \dots, a_{n-1}$ know all their secrets (so they are experts except that none of them knows the secret of a_n). Now define the call sequence σ by having agent a_n call all other agents after ρ:

$$\sigma := \rho; a_n a_1; a_n a_2; \dots; a_n a_{n-1}$$

We note that σ is again a maximal CMO sequence. After σ, all agents are experts, and agent a_n is the only agent who knows that all the agents are expert. Let $i, j < n$ and $i \neq j$. Now consider the following call sequence τ where a_n only calls a_j (many times) and q_i (once, at the same moment as in σ:

$$\tau := \rho; \overbrace{a_n a_j; a_n a_j; \ldots a_n a_j}^{i-1 \text{ times}}; a_n a_i; \overbrace{a_n a_j; a_n a_j \ldots a_n a_j}^{n-i-1 \text{ times}}$$

We then have that $\sigma \approx_{a_i} \tau$ and that $\tau \not\models E_{exp}$. Therefore, $\sigma \models \neg K_{a_i} E_{exp}$. As σ is maximal and not super successful, CMO is not super successful.

Case (ii). Let now the setting be asynchronous. Consider again the call sequence ρ and σ from the case (i). The sequence $\rho; a_n a_i$ is CMO permitted, and $\sigma \sim_{a_i}^{\mathsf{CMO}} \rho; a_n a_i$. After $\rho; a_n a_i$, only agents a_n and a_i are experts but none of the remaining agents. Therefore, $\sigma \models \neg K_{a_i} E_{exp}$. As σ is maximal and not super successful, CMO is not super successful.

Let us illustrate Theorem 1 with an example. Let $A = \{a, b, c, d\}$ and consider the sequence $\sigma := ab; ac; bc; da; db; dc$. This sequence is CMO-permitted, CMO-terminal, and satisfies E_{exp}.

First, assume that the setting be asynchronous but that the protocol CMO is common knowledge. Consider the prefix $ab; ac; bc; da$ of σ. Note that $\sigma \sim_a^{\mathsf{CMO}} ab; ac; bc; da$, as a is not involved in the final two calls. Now observe that after $ab; ac; bc; da$ agents b and c do not know the secret of d ($ab; ac; bc; da \models \neg Sbd \wedge \neg Scd$), so that $ab; ac; bc; da \not\models E_{exp}$. From that and $ab; ac; bc; da; db; dc \sim_a^{\mathsf{CMO}} ab; ac; bc; da$ it follows that $\sigma \not\models K_a E_{exp}$, which implies $\sigma \not\models E_{exp}^2$, so that σ is not super successful.

Next, assume that the protocol is not common knowledge (but still assume the more informative synchronous setting). Then $\sigma \approx_b ab; ac; bc; da; db; da$, where in the call sequence on the right side we replaced the final call dc in σ by da. This sequence is not CMO-permitted, as call da occurs twice. After $ab; ac; bc; da; db; da$, agent c does not know the secret of d, therefore $ab; ac; bc; da; db; da \not\models E_{exp}$. From that and $\sigma \approx_b ab; ac; bc; da; db; da$ then follows that $\sigma \not\models K_b E_{exp}$, and therefore $\sigma \not\models E_{exp}^2$, so that again σ is not super successful.

The other direction of Theorem 1 is false, which can be shown by yet another example for the four-agent case. However, under certain additional conditions, which would define a slightly different gossip protocol, we conjecture that the other direction is true:

> The (changed) protocol CMO is super successful if the protocol is common knowledge and the setting is synchronous.

For four agents, this seems to be the case if the first two calls are between pairs of disjoint agents. Such results seem of interest, as we may then even be able to obtain that the agents have *common knowledge that everybody is expert*, a stronger condition than the E_{exp}^2 of our investigation. Such common knowledge that everybody is expert may, possibly, also be obtainable by allowing 'skip' calls after termination in the synchronous case: even when the goal E_{exp}^2 has been achieved, it may be that some agents do not know that this has been achieved, and consider it possible that other agents can only become super experts after yet another call (or more calls). Such a 'skip' call would then function as a 'clock tick': it may be informative to agents that one more unit of time has passed. Without a skip, no such information growth can take place, as the protocol has already terminated, and as (also in the synchronous case) the unit of time between calls is unknown. We intend to report this in future research.

5.4.2 The Protocol PIG

The Possible Information Growth (PIG) protocol has been investigated in Attamah et al. (2014), van Ditmarsch et al. (2017). In this protocol agent x will call agent y if she considers it possible that either herself or the agent y being called will learn at least one new secret in the call. As usual, the termination condition studied in Attamah et al. (2014), van Ditmarsch et al. (2017) is E_{exp}, whereas we are now interested again in E^2_{exp}. Do we have to adjust the protocol to also achieve this stronger termination condition? Interestingly enough, we do not have to change anything, as we will see in Theorem 2 below. We thus define the PIG protocol as follows.

Protocol (PIG — Possible Information Growth) *The* Possible Information Growth *protocol is defined by the condition*

$$\mathsf{P}_{xy} := \hat{K}_x \bigvee_{z \in A} ((Sxz \land \neg Syz) \lor (\neg Sxz \land Syz))$$

Intuitively, the call xy is permitted if x considers it possible that there is a secret z that x but not y knows, or vice versa. □

The PIG protocol has infinite executions for four or more agents. Under asynchronous conditions for example, $ab; ab; ab; \ldots$ is PIG-permitted, because ab is indistinguishable for agent a from $ab; ac$, thus ab is again PIG-permitted. Then, $ab; cd; ab \sim_a^{\mathsf{PIG}} ab; cd; ab; ac$, thus ab is again PIG-permitted, and so on. Somewhat similarly, under synchronous conditions, the sequence $ab; cd; ab; cd; ab; cd; \ldots$ is PIG-permitted, as after any even number of calls agent a considers it possible that agent b was involved in the last call and would thus have learnt a new secret in that call. Therefore, each odd call can again be call ab. Termination results for the PIG protocol are therefore restricted to fair call sequences.

Theorem 2 *The protocol* PIG *is fairly super successful.*

Proof If agent a considers it possible that she or another agent know different secrets, then clearly agent a considers it possible that not all agents know all secrets. This implies that agent a does not know that all agents know all secrets, $\neg K_a E_{exp}$ (strictly, $\neg K_a^{\mathsf{PIG}} E_{exp}$, but with the validity $K_a \varphi \to K_a^{\mathsf{PIG}} \varphi$ and contraposition this implies $\neg K_a E_{exp}$). Which is, as we may recall, the strengthened protocol condition in the protocols in our contribution. We then recall that E^2_{exp} is satisfied if no agent satisfies $\neg K_a E_{exp}$ anymore.

The remainder of the argument is now as usual for fair executions, where we will also see that missed calls play no role (so that it applies to our setting).

Let σ_∞ be a (possibly infinite) fair maximal PIG sequence. Towards a contradiction suppose the protocol does not terminate after each finite prefix of σ_∞. Consider the following two cases.

- The sequence σ_∞ is finite. Let x be an agent who is not a super expert. If x is not even an expert and $y \in A \setminus S_x^{\sigma_\infty}$ then the call xy is PIG permitted after σ_∞, this contradicts with the maximality of σ_∞. If x is an expert and she does not know that y is an expert, then again the call xy is PIG permitted after σ_∞, which contradicts the maximality of σ_∞.
- The sequence σ_∞ has infinite length. Then there is a finite prefix $\tau \sqsubset \sigma_\infty$, such that for all sequences $\tau \sqsubseteq \rho \sqsubset \sigma_\infty$ we have $S^\tau = S^\rho$. Consider the following two cases.
  - $S^\tau \neq A^2$. Then there are $x, y \in A$ such that $y \in A \setminus S_x^\tau$. So the call xy is permitted after τ but it is not executed. This contradicts the fairness assumption.
  - $S^\tau = A^2$. Then there are $x, y \in A$ such that after every prefix of σ_∞, agent x does not know that y is expert. This means for any sequence ρ with $\tau \sqsubseteq \rho \sqsubset \sigma_\infty$ there is a sequence π such that $\rho \sim_x^{\mathsf{PIG}} \pi$ (or, for the less general synchronous situation, $\rho \approx_x^{\mathsf{PIG}} \pi$) and $A = S_x^\pi \neq S_y^\pi$. Therefore the call xy is permitted after all such ρ but it is never executed, again this contradicts the fairness assumption. □

We close this part on the PIG protocol with an example. Let $A = \{a, b, c, d\}$, assume that the agents execute the PIG protocol but that this is not common knowledge, and let the setting be asynchronous. Consider the call sequence $ab; cd; ac; bd; ab; ad; cb; cd$. After the prefix $ab; cd; ac; bd$ all the agents are already experts. However, as they do not know that the others are experts, they will now make additional calls. After the next four calls $ab; ad; cb; cd$ all agents know that all the agents are experts. Thus, the PIG protocol terminates after σ.

In fact, we recall this example from the introduction, but for the protocol ANY wherein we can make any call, as the condition $\neg K_a E_{exp} \wedge \top$, i.e., $\neg K_a E_{exp}$, and which is here equivalent to $\bigvee_{c \neq a \in A} \mathsf{PIG}_{ac}$, is satisfied.

Acknowledgements Rasoul Ramezanian and Rahim Ramezanian would like to thank Prof. Ardeshir for his support and encouragement during their Ph.D. research. Hans van Ditmarsch very kindly remembers the fabulous *Workshop on Modal Logic and its Application in Computer Science*, organized at the University of Tehran in 2016, of which he and Mohammad Ardeshir were co-organizers, and the many interactions over many years involving Rasoul Ramezanian and Rahim Ramezanian, thanks to Mohammad Ardeshir's encouragement and facilitation. We thank a reviewer of the submission for useful comments.

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Chapter 6
Fuzzy Generalised Quantifiers for Natural Language in Categorical Compositional Distributional Semantics

Mătej Dostál, Mehrnoosh Sadrzadeh, and Gijs Wijnholds

Abstract Recent work on compositional distributional models shows that bialgebras over finite dimensional vector spaces can be applied to treat generalised quantifiers for natural language. That technique requires one to construct the vector space over powersets, and therefore is computationally costly. In this paper, we overcome this problem by considering fuzzy versions of quantifiers along the lines of Zadeh, within the category of many valued relations. We show that this category is a concrete instantiation of the compositional distributional model. We show that the semantics obtained in this model is equivalent to the semantics of the fuzzy quantifiers of Zadeh. As a result, we are now able to treat fuzzy quantification without requiring a powerset construction.

Keywords Category theory · Many valued relations · Fuzzy sets · Generalised quantifiers · Distributional semantics · Vector space models · Natural language data

6.1 Introduction

Distributional semantics is inspired by the idea of Firth that words can be represented by the company they keep (Firth 1957). This idea was formalised by computational linguistics and by information retrieval researchers; they represented words by the contexts in which they often occurred. Rubenstein and Goodenough introduced the concept of a co-occurrence matrix: a matrix whose columns are context words, whose

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M. Mojtahedi et al. (eds.), *Mathematics, Logic, and their Philosophies*,
Logic, Epistemology, and the Unity of Science 49,
https://doi.org/10.1007/978-3-030-53654-1_6

rows are target words, and whose entries are [a function of] the number of times the target and context words occurred together in a window of a fixed size (Rubenstein and Goodenough 1965). Later Salton, Wong and Yang employed similar ideas to index words in a document (Salton et al. 1975). In either setting, a word is represented by a vector: the row associated to it in a co-occurrence matrix, which is a vector in the vector space spanned by the context words. This representation has been applied to natural language tasks such as word similarity, disambiguation, and entailment (Rubenstein and Goodenough 1965; Curran 2004; Turney 2006; Schuetze 1998; Weeds et al. 2004; Geffet and Dagan 2005; Kotlerman et al. 2010) and information retrieval tasks, such as clustering, indexing, and search (Salton et al. 1975; Landauer and Dumais 1997; Lin 1998).

A challenge to distributional semantics was that its underlying hypothesis did make sense for words, but no longer for complex language units such as sentences. In an attempt to extend distributional semantics from words to sentences, Clark and Pulman put forward the idea of tracing the parse tree of a sentence, forming the tensor product of words and their grammatical roles, and representing the sentence by the resulting vector (Clark and Pulman 2007). Due to the high dimensionality of the resulting space (in which the vector of the sentence lives), this idea itself did not lead to tangible applications and experimental results in language tasks. It was however followed up by a series of related work, referred to by *compositional distributional semantics*, based on the principle of compositionality of Frege, that the meaning of a sentence is a function of the meanings of its parts. The approaches within this field, mapped the grammatical structure of sentences to linear maps that acted on the representations of the words therein, for example see the work of Baroni and Zamparelli (2010), Maillard et al. (2014), Krishnamurthy and Mitchell (2013), Lewis and Steedman (2013), Coecke et al. (2010). These models provided concrete vector constructions and were tested on natural language tasks such as sentence similarity, disambiguation, and entailment (Baroni et al. 2014; Grefenstette et al. 2013; Grefenstette and Sadrzadeh 2015; Kartsaklis and Sadrzadeh 2013, 2016; Bankova et al. 2016). They were however mostly focused on elementary fragments of language and left the treatment of logical operations such as conjunction, disjunction, and quantification to further work. Recently, quantification was tackled in Hedges and Sadrzadeh (2019), Sadrzadeh (2016) based on a model that sits within the setting of Coecke et al. (2010). This setting is based on theory of compact closed categories (Kelly and Laplaza 1980). It was shown that the bialgebras over these categories (McCurdy 2012; Bonchi et al. 2014) can be used to model the generalised quantifiers of Barwise and Cooper (1981). An instantiation of the abstract setting to category of sets and relations provided an equivalent semantics to the set theoretic semantics of generalised quantifiers.

This paper stems from a theoretical question and a practical concern. On the theoretical side we have a model for forming vectors for quantified sentences in distributional semantics (Hedges and Sadrzadeh 2019). On the practical side, the vector space instantiation of this model relies on vector spaces being spanned by a power set object; this leads to an exponential increase in the size of the vector space and thus implementing it becomes computationally costly. On the other hand, there is the work of Zadeh in fuzzy quantifiers for natural language (Zadeh 1983), which similar to previous work provides a quantitative interpretation for the generalised quantifiers of Barwise and Cooper. Fuzzy sets have been applied to a variety of different domains, including to computational linguistics and information retrieval, for example see Novák (1992), Cock et al. (2000), Zadeh (1996), Bezdek and Harris (1978). Given the mathematical equivalence between fuzzy sets and vectors, the question arises whether there is a connection between the two settings of vector representations of quantified sentences and their fuzzy set counterparts. In this paper we answer the question in positive and thus provide pathways to address the practical concern. As a result, we can now work with the fuzzy set counterparts of compositional distributional vectors and avoid the pitfall of having to compute within an exponentially sized vector space.

The outline of the paper is as follows: we recall basic definitions of compact closed categories and bialgebras over them and review how **Rel** and **V-Rel** are examples thereof. We then go through the fact that the category **V-Rel** of sets and many valued relations models fuzzy sets and a logic over them. We define in **V-Rel** the many valued versions of the abstract quantifier interpretations of the setting of previous wok, where we worked with non-fuzzy sets and quantifiers (Hedges and Sadrzadeh 2019). We show how Zadeh's fuzzy quantifiers can be recast categorically in this setting and prove that Zadeh's fuzzy semantics of quantified sentences is equivalent to their corresponding bialgebraic treatment. Whereas Zadeh's developments are not, at least explicitly, based on the grammatical structure of sentences, this result indicates that they do inherently follow the same composition principles as the ones employed in compositional distributional semantics. We conclude our theoretical contributions by remarking on how the degrees of truth obtained in the fuzzy interpretations relate to the absolute truth values of previous work. Overall, we have taken a step forward towards implementing and experimenting with quantifiers in distributional semantics, we, however, leave experimenting with this model to another paper.

This paper builds on the developments of a previous technical report (Dostal and Sadrzadeh 2016).

6.2 Dedication

In 2001, the second author of this article defended her masters thesis under the supervision of Mohammed Ardeshir in Sharif University of Technology, Tehran, Iran. The content of the thesis was based on Brouwer's interpretation of Intuitionistic Logic and the motivation behind it came from a mathematical logic course taught by Ardeshir on Gödel's incompleteness results, his work on intuitionistic logic, and his translation of intiotionistic logic to modal logic $S4$. The material taught in that course encouraged the second author to study the relationship between modal and constructive logics, work on a notion of constructive knowledge in epistemic logic

(Marion and Sadrzadeh 2004; Sadrzadeh 2003), and on dynamics of belief update in logics based on monoids and lattices, including Heyting Algebras (Sadrzaadeh 2006; Dyckhoff et al. 2013).

The connection between this article and Areshir's work is through the relationship between intuitionistic logic and many valued logics. Gödel's observation that intuitionistic loigic cannot be characterised by finite truth tables (Gödel 1932), led to the axiomatisaiton suggested by Dummett in what is nowadays known as the Gödel-Dummett logic (1959). Many valued or fuzzy logics formalise the theory of fuzzy sets, put forwards by Zadeh (1965). Fuzzy sets assign a degree of membership to the elements of a set. This degree is usually a real number in the unit interval [0, 1]. The degree of membership is used to define a truth-value between 0 and 1 for the formulae of a many valued logic. In these logics, 0 is still false and 1 is still true, a number between the two is a degree of truth.

Algebraically speaking, a many valued logic is put together by a bounded lattice and a monoid. The Gödel-Dummett logic has alternatively been seen as a many valued logic where the monoid multiplication is idempotent and thus it coincides with the lattice operation of least upper bound.

6.3 Generalised Quantifiers in Natural Language

We briefly review the theory of generalised quantifiers in natural language as presented in Barwise and Cooper (1981). Consider the fragment of English generated by the following context free grammar:

S	→	NP VP
VP	→	V NP
NP	→	Det N
NP	→	John, Mary, something, …
N	→	cat, dog, man, …
VP	→	sneeze, sleep, …
V	→	love, kiss, …
Det	→	some, all, no, most, almost all, several, …

A model for the language generated by this grammar is a pair $(U, [\![\ ]\!])$, where U is a universal reference set and $[\![\ ]\!]$ is an inductively defined interpretation function presented below.

1. $[\![\ ]\!]$ on terminals:

   a. The interpretation of a determiner d generated by 'Det → d' is a map with the following type:

   $$[\![d]\!] \colon \mathcal{P}(U) \to \mathcal{P}\mathcal{P}(U)$$

   It assigns to each $A \subseteq U$, a family of subsets of U. The images of these interpretations are referred to as *generalised quantifiers*. For logical quantifiers, they are defined as follows:

$$\llbracket \text{some} \rrbracket(A) = \{X \subseteq U \mid X \cap A \neq \emptyset\}$$
$$\llbracket \text{every} \rrbracket(A) = \{X \subseteq U \mid A \subseteq X\}$$
$$\llbracket \text{no} \rrbracket(A) = \{X \subseteq U \mid A \cap X = \emptyset\}$$
$$\llbracket n \rrbracket(A) = \{X \subseteq U \mid \mid X \cap A \mid = n\}$$

A similar method is used to define non-logical quantifiers, for example "most A" is defined to be the set of subsets of U that has 'most' elements of A, "few A" is the set of subsets of U that contain 'few' elements of A, and similarly for 'several' and 'many'.
Generalising the two cases above, provides us with the following definition for any generalised quantifier d:

$$\llbracket d \rrbracket(A) = \{X \subseteq U \mid \quad X \text{ has } d \text{ elements of } A\}$$

b. The interpretation of a terminal $y \in \{np, n, vp\}$ generated by either of the rules 'NP → np, N → n, VP → vp' is $\llbracket y \rrbracket \subseteq U$. That is, noun phrases, nouns and verb phrases are interpreted as subsets of the reference set.
c. The interpretation of a terminal y generated by the rule V → y is $\llbracket y \rrbracket \subseteq U \times U$. That is, verbs are interpreted as binary relations over the reference set.

2. $\llbracket\ \rrbracket$ on non-terminals:

a. The interpretation of expressions generated by the rule 'NP → Det N' is as follows:

$$\llbracket \text{Det N} \rrbracket = \llbracket d \rrbracket(\llbracket n \rrbracket) \text{ where } X \in \llbracket d \rrbracket(\llbracket n \rrbracket) \textbf{ iff}$$
$$X \cap \llbracket n \rrbracket \in \llbracket d \rrbracket(\llbracket n \rrbracket) \text{ for Det} \to d \text{ and N} \to n$$

b. The interpretations of expressions generated by other rules are as usual:

$$\llbracket \text{V NP} \rrbracket = \llbracket v \rrbracket(\llbracket np \rrbracket)$$
$$\llbracket \text{NP VP} \rrbracket = \llbracket vp \rrbracket(\llbracket np \rrbracket)$$

Here, for $R \subseteq U \times U$ and $A \subseteq U$, by $R(A)$ we mean the forward image of R on A, that is $R(A) = \{y \mid (x, y) \in R, \text{ for } x \in A\}$. To keep the notation unified, for R a unary relation $R \subseteq U$, we use the same notation and define $R(A) = \{y \mid y \in R, \text{ for } x \in A\}$, i.e. $R \cap A$.

The expressions generated by the rule 'NP → Det N' satisfy a property referred to by *living on* or *conservativity*, defined below.

Definition 1 For a terminal d generated by the rule 'Det → d', we say that $\llbracket d \rrbracket(A)$ lives on A whenever $X \in \llbracket d \rrbracket(A)$ iff $X \cap A \in \llbracket d \rrbracket(A)$, for $A, X \subseteq U$. Whenever this is the case, the quantifier $\llbracket d \rrbracket$ is called a **conservative** quantifier.

Barwise and Cooper argue that conservativity is a property of natural language quantifiers. This is certainly the case for the quantifiers generated from the grammar given above. Thus for the rest of the paper, we are assuming that our quantifiers are conservative. The 'meaning' of a sentence in this setting is its truth value. This is defined for any general sentence as follows:

Definition 2 A sentence is true iff $[\![\text{NP VP}]\!] \neq \emptyset$ and false otherwise.

For the special cases of quantified subject and object phrases of interest to this paper, a truth value is defined as follows:

Definition 3 1. A sentence of the form 'Det N VP' is *true* iff $[\![\text{Det N VP}]\!] = [\![vp]\!] \cap [\![n]\!] \in [\![\text{Det N}]\!]$ and *false* otherwise.
2. A sentence of the form 'NP V Det N' is *true* iff $[\![\text{NP V Det N}]\!] = [\![n]\!] \cap [\![v]\!]([\![np]\!]) \in [\![\text{Det N}]\!]$ and *false* otherwise.

As examples, first consdier sentence 'some men sneeze' with a quantifier at the subject phrase. This sentence is true iff $[\![\text{sneeze}]\!] \cap [\![\text{men}]\!] \in [\![\text{some men}]\!]$, that is, whenever the set of things that sneeze and are men is a non-empty set. Part of this meaning is obtained by following the inductive definition of $[\![\ ]\!]$ and part of it by applying Definition 3. The inductive case 2.b tells us that the semantics of this sentence is $[\![\text{NP VP}]\!] = [\![\text{sneeze}]\!]([\![\text{some men}]\!])$, where $[\![\text{NP}]\!]$ is obtained by case 2.a and by unfolding it to $[\![\text{Det N}]\!]$. These unfoldings, when used in Definition 3, provides us with the suggested meaning above, that is *true* iff $[\![vp]\!] \cap [\![n]\!] \in [\![\text{Det N}]\!]$ and false otherwise. Similarly, as an example of a sentence with a quantified phrase at its object position, consider 'John liked some trees'. This is true iff $[\![\text{trees}]\!] \cap [\![\text{like}]\!]([\![\text{John}]\!]) \in [\![\text{some trees}]\!]$, that is, whenever, the set of things that are liked by John and are trees is a non-empty set. Similarly, the sentence 'John liked five trees' is true iff the set of things that are liked by John and are trees has five elements in it.

6.4 Category Theoretic Definitions

This subsection briefly reviews compact closed categories and bialgebras. For a formal presentation, see Kelly and Laplaza (1980), Kock (1972), McCurdy (2012). A compact closed category **C** has objects A, B, morphisms $f: A \to B$, and a monoidal tensor $A \otimes B$ that has a unit I; that is, we have $A \otimes I \cong I \otimes A \cong A$. Furthermore, for each object A there are two objects A^r and A^l and the following morphisms:

$$A \otimes A^r \xrightarrow{\epsilon^r_A} I \xrightarrow{\eta^r_A} A^r \otimes A$$

$$A^l \otimes A \xrightarrow{\epsilon^l_A} I \xrightarrow{\eta^l_A} A \otimes A^l$$

These morphisms satisfy the following equalities, where 1_A is the identity morphism on object A:

$$(1_A \otimes \epsilon^l_A) \circ (\eta^l_A \otimes 1_A) = 1_A$$
$$(\epsilon^r_A \otimes 1_A) \circ (1_A \otimes \eta^r_A) = 1_A$$
$$(\epsilon^l_A \otimes 1_A) \circ (1_{A^l} \otimes \eta^l_A) = 1_{A^l}$$
$$(1_{A^r} \otimes \epsilon^r_A) \circ (\eta^r_A \otimes 1_{A^r}) = 1_{A^r}$$

These express the fact that A^l and A^r are the left and right adjoints, respectively, of A. A self adjoint compact closed category is one in which for every object A we have $A^l \cong A \cong A^r$.

Given two compact closed categories $\mathcal{C}$ and $\mathcal{D}$, a strongly monoidal functor $F: \mathcal{C} \to \mathcal{D}$ is defined as follows:

$$F(A \otimes B) = F(A) \otimes F(B) \qquad F(I) = I$$

One can show that this functor preserves the compact closed structure, that is we have:

$$F(A^l) = F(A)^l \qquad F(A^r) = F(A)^r$$

A bialgebra in a symmetric monoidal category $(\mathcal{C}, \otimes, I, \sigma)$ is a tuple $(X, \delta, \iota, \mu, \zeta)$ where, for X an object of $\mathcal{C}$, the triple (X, δ, ι) is an internal comonoid; i.e. the following are coassociative and counital morphisms of $\mathcal{C}$:

$$\delta\colon X \to X \otimes X \qquad\qquad \iota\colon X \to I$$

Moreover (X, μ, ζ) is an internal monoid; i.e. the following are associative and unital morphisms:

$$\mu\colon X \otimes X \to X \qquad\qquad \zeta\colon I \to X$$

Morphisms δ and μ satisfy four equations (McCurdy 2012):

$$\begin{aligned}
\iota \circ \mu &= \iota \otimes \iota \\
\delta \circ \zeta &= \zeta \otimes \zeta \\
\delta \circ \mu &= (\mu \otimes \mu) \circ (\mathrm{id}_X \otimes \sigma_{X,X} \otimes \mathrm{id}_X) \circ (\delta \otimes \delta) \\
\iota \circ \zeta &= \mathrm{id}_I
\end{aligned}$$

Informally, the co-multiplication δ copies the information contained in one object into two objects, and the multiplication μ merges the information of two objects into one.

Example Sets and Relations. An example of a compact closed category is **Rel**, the category of sets and relations. Here, $\otimes$ is cartesian product with the singleton set as its unit $I = \{\star\}$, and $A^l = A = A^r$. Hence **Rel** is self adjoint. Given a set S with elements $s_i, s_j \in S$, the epsilon and eta maps are given as follows:

$$\epsilon^l = \epsilon^r : S \times S \nrightarrow \{\star\} \quad \text{given by}$$
$$(s_i, s_j)\epsilon\star \iff s_i = s_j$$
$$\eta^l = \eta^r : \{\star\} \nrightarrow S \times S \quad \text{given by}$$
$$\star\eta(s_i, s_j) \iff s_i = s_j$$

These relations hold iff the first and second elements of the pair they are acting on are the same, i.e. whenever $s_i = s_j$, we related $\star$ to the pair (s_i, s_j) via η maps and relate the pair to the $\star$ via an epsilon map. They are designed to sift out pairs that are reflexive.

For an object in **Rel** of the form $W = \mathcal{P}(U)$, Hedges and Sadrzadeh (2019) gave W a bialgebra structure by taking

$$\delta : S \nrightarrow S \times S \quad \text{given by}$$
$$A\delta(B, C) \iff A = B = C$$
$$\iota : S \nrightarrow \{\star\} \quad \text{given by}$$
$$A\iota\star \iff \text{(always true)}$$
$$\mu : S \times S \nrightarrow S \quad \text{given by}$$
$$(A, B)\mu C \iff A \cap B = C$$
$$\zeta : \{\star\} \nrightarrow S \quad \text{given by}$$
$$\star\zeta A \iff A = U$$

It was shown in Hedges and Sadrzadeh (2019) that the four axioms of a bialgebra hold for the above definitions. In order to obtain an intuition, the δ map relates a subset A of the universe U to a pair of subsets (B, C) iff these three subsets are the same, i.e. iff $A = B = C$. This relation only holds when the set in its first input is the same as the pairs of sets in its second input, where as ι is meant to be the relation that always holds. The relation μ is more sophisticated, it is meant to enable the formalism to perform the set intersection operation, so given a pair of subsets of universe A and B, and another subset thereof C, the relation μ holds iff C is the intersection of A and B. The unit of this map, i.e. ζ only holds when its input subset A is actually the whole universe, the unit of intersection $A \cap U = A$.

In the next section we show how the category of sets and many valued relations is also an example of a self adjoint compact closed category with bialgebras over it. In fact both **Rel** and category of sets and many valued relations are also dagger compact closed and have other desirable properties, e.g. being partial order enriched, for elaborations on these properties see Marsden and Genovese (2017).

6.5 Category of Sets and Many Valued Relations

Definition 4 (*Commutative quantale*) A commutative quantale $\mathbf{V}$ is a complete lattice $(V, \bigwedge, \bigvee)$ with the structure of a commutative monoid $(V, \bullet, \mathsf{e})$ such that the tensor is monotone and distributes over arbitrary joins.

More in detail, $\mathbf{V}$ being a complete lattice means that V is partially ordered by $\leq$ and every subset $V' \subseteq V$ has an infimum (or meet) and a supremum (or join), denoted by $\bigwedge V'$ and $\bigvee V'$ respectively. From this it follows that V contains the greatest element $\top$ and the lowest element $\bot$. The fact that $(V, \bullet, \mathsf{e})$ is a commutative monoid means that $\bullet$ is commutative, associative, and e is the identity element:

$$v \bullet \mathsf{e} = v = \mathsf{e} \bullet v$$

The monotonicity of the tensor requires that $v \bullet w \leq v' \bullet w$ holds for $v \leq v'$, and distributivity of tensor over arbitrary joins means that the following equality is satisfied:

$$\left(\bigvee_i x_i\right) \bullet y = \bigvee_i (x_i \bullet y)$$

Definition 5 (*Complete Heyting algebra*) A complete Heyting algebra $\mathbf{V}$ is a commutative quantale where $\bullet = \wedge$ and $\mathsf{e} = \top$. In other words, it is a complete lattice $(V, \bigwedge, \bigvee)$ where the meet operation distributes over arbitrary joins:

$$\left(\bigvee_i x_i\right) \wedge y = \bigvee_i (x_i \wedge y)$$

Definition 6 (*Gödel chain*) We say that a complete Heyting algebra $\mathbf{V}$ is a *Gödel chain* if the ordering relation $\leq$ of the underlying lattice of $\mathbf{V}$ is a linear order, that is, for two elements $v \neq v'$ it either holds that $v \leq v'$ or $v' \leq v$.

Example Instances of commutative quantales:

1. The real interval [0, 1] with the usual lattice structure (given by computing suprema and infima), the tensor being the meet and the unit being 1, is a complete Heyting algebra, moreover a Gödel chain.
2. The real interval [0, 1] with the usual structure, the unit 1 and the tensor being defined as
$$a \bullet b = \max(0, a + b - 1)$$
is a commutative quantale.

3. The real interval [0, 1] with the usual structure, the unit 1 and the tensor being defined as
$$a \bullet b = a \cdot b$$
(multiplication) is a commutative quantale.
4. As a very special case, the 2-element Boolean algebra is a commutative quantale.

Definition 7 (*Many-valued relation*) For a given quantale **V**, a *many-valued relation* $R : A \nrightarrow B$ is a function $R : A \times B \to V$. We view this function as a **V**-valued matrix. We compose two relations $R : A \nrightarrow B$ and $S : B \nrightarrow C$ to get a relation $S \circ R : A \nrightarrow C$ such that
$$(S \circ R)(a, c) = \bigvee_{b \in B} (R(a, b) \bullet S(b, c))$$
holds in **V**.

Definition 8 (*The category of* **V**-*relations*) The collection of all sets and of **V**-relations between sets is a category. There is an identity **V**-relation id_A for every set A:
$$id_A(a, a') = \begin{cases} \mathsf{e} & \text{if } a = a' \\ \bot & \text{otherwise} \end{cases}$$

An easy computation yields that **V**-relation composition is associative. We denote the category of all sets and **V**-relations as **V-Rel**.

Remark 1 The associativity of **V**-relation composition follows from complete distributivity of **V**. For **V**-relations over finite sets, only finite distributivity of tensor over joins would be needed.

Example Some examples of **V-Rel** for various choices of **V**:

1. When **V** is the 2-element Boolean algebra, **V-Rel** is the category **Rel** of sets and (ordinary) relations.
2. When **V** is the real interval [0, 1] with Gödel operations min and max, the category **V-Rel** has sets as objects, and the composition of morphisms (**V**-relations) acts as follows. Given two **V**-relations $R : A \nrightarrow B$ and $S : B \nrightarrow C$ (so two functions $R : A \times B \to [0, 1]$ and $S : B \times C \to [0, 1]$), the composite $S \circ R : A \nrightarrow C$ is given by
$$(S \circ R)(a, c) = \max_{b \in B} \min(R(a, b), S(b, c)).$$
Given yet another **V**-relation $T : C \nrightarrow D$, the composite $T \circ S \circ R$ is then computed as follows:
$$(T \circ S \circ R)(a, d) = \max_{b \in B, c \in C} \min(R(a, b), S(b, c), T(c, d)).$$

Remark 2 Observe that there is an inclusion functor

$$\widetilde{(-)} : \mathbf{Rel} \to \mathbf{V\text{-}Rel}$$

for any **V** with more than one element. Indeed, let the functor act as an identity on objects, and assign to a relation $R : A \nrightarrow B$ the **V**-valued relation $\widetilde{R} : A \nrightarrow B$ defined as follows:

$$\widetilde{R}(a,b) = \begin{cases} \mathsf{e} & \text{if } R(a,b)\text{holds,} \\ \bot & \text{otherwise.} \end{cases}$$

An easy computation yields that $\widetilde{id_A} = id_A$ and that $\widetilde{S \circ R} = \widetilde{S} \circ \widetilde{R}$.

Lemma 1 *The category **V-Rel** is a self adjoint compact closed category with the tensor being the cartesian product* $\times$ *and the unit* I *being the singleton set* $\{\star\}$.

Proof Let us define the epsilon maps $\epsilon_S : S \times S \nrightarrow I$ for each S as follows

$$\epsilon_S((a,b),\star) = \begin{cases} \mathsf{e} & \text{if } a = b \\ \bot & \text{otherwise} \end{cases}$$

and define the eta maps $\eta_S : I \nrightarrow S \times S$ similarly:

$$\eta_S(\star,(a,b)) = \begin{cases} \mathsf{e} & \text{if} a = b \\ \bot & \text{otherwise} \end{cases}$$

Since with these definitions the epsilon and eta maps are the images of the epsilon and eta maps from **Rel** under the inclusion functor $\widetilde{(-)} : \mathbf{Rel} \to \mathbf{V\text{-}Rel}$, the axioms of a compact closed category hold in **V-Rel**. It remains to show that ϵ and η are natural; but this is straightforward.

Remark 3 Let us fix a set U. Very similarly to the case of **Rel**, we can define a bialgebra over the set $S = P(U)$ in **V-Rel** by the following data. The relation $\delta : S \nrightarrow S \times S$ is defined as

$$\delta(A,(B,C)) = \begin{cases} \mathsf{e} & \text{if} A = B = C \\ \bot & \text{otherwise.} \end{cases}$$

The relation $\mu : S \times S \nrightarrow S$ is defined as

$$\mu((A,B),C) = \begin{cases} \mathsf{e} & \text{if} A \cap B = C \\ \bot & \text{otherwise.} \end{cases}$$

The relation $\iota : S \nrightarrow I$ is defined as

$$\iota(A, \star) = \mathsf{e} \text{for every} A.$$

The relation $\zeta : I \nrightarrow S$ is defined as

$$\zeta(\star, A) = \begin{cases} \mathsf{e} & \text{if} A = U \\ \bot \text{otherwise.} \end{cases}$$

In fact, we obtain the structure of a bialgebra over $P(U)$ in **V-Rel** by taking the bialgebra structure over $P(U)$ in **Rel** and applying the inclusion functor $\widehat{(-)}$.

6.6 Fuzzy Sets and Fuzzy Quantifiers

In this section we review definitions of fuzzy sets and quantifiers, as done by Zadeh (1983). A fuzzy set is a set whose elements have a corresponding weight associated to them. For a set A, the weight μ_i of element u_i is interpreted as the degree of membership of u_i in A. The fuzzy set A is represented symbolically by the following sum:

$$A = \mu_1 u_1 + \mu_2 u_2 + \cdots + \mu_n u_n$$

standing for the following set of pairs of weights and elements:

$$\{(\mu_1, u_1), (\mu_2, u_2), \ldots, (\mu_n, u_n)\}$$

The sum above denotes a union operation on sets containing single $\mu_i u_i$ elements, where $\mu_i u_i$ stands for the pair (μ_i, u_i). Non-fuzzy, aka *crisp*, sets are special instances of fuzzy ones, where for every u_i of the set, we have $\mu_i = 1$, in other words:

$$A = u_1 + u_2 + \cdots + u_n$$

The *absolute* cardinality of a fuzzy set is defined via the notion of *sigma-count*, defined below:

$$\Sigma Count(A) = \Sigma_{i=1}^{n} \mu_i$$

This is the arithmetic sum of the degrees of membership in A; it is, if needed, rounded to the nearest integer. Terms whose degrees of membership fall below a certain threshold, may be omitted from the sum. This is to avoid a situation where a large number of terms with low degrees become equivalent to a small number of terms with high degrees. Following Zadeh, we denote the *absolute* cardinality of a non-fuzzy set by $Count(A)$. Observe that, under the interpretation of a non-fuzzy set as a special instance of a fuzzy set, we have that $Count(A) = \Sigma Count(A)$.

The *relative* cardinality of a fuzzy set is a possibility distribution over the cardinality of that set, denoted as follows

$$\Pi_{\Sigma Count(A)}$$

The quantified sentences Zadeh considers are built from two basic forms: "There are Q A's" and "Q A's are B's". Each of these propositions induces a possibility distribution. Zadeh provides the following insights for the analysis of these quantified propositions. "There are Q A's" implies that the probability of event A is a fuzzy probability equal to Q. "Q A's are B's" implies that the conditional probability of event B given event A is a fuzzy probability which is equal to Q. Most statements involving fuzzy probabilities may be replaced by semantically equivalent propositions involving fuzzy quantifiers. The connection between this two, Zadeh reports, plays an important role in expert systems and fuzzy temporal logic and has been developed in previous work of Zadeh (Barr and Feigenbaum 1982).

According to Zadeh, fuzzy quantifiers should be treated as fuzzy numbers. A fuzzy number provides a fuzzy characteristic of the absolute or relative cardinality of one or more fuzzy or non-fuzzy sets. As an example, consider the fuzzy quantifier "most" in the proposition "Most big men are kind". This proposition is interpreted as a fuzzily defined proportion of the fuzzy set "kind men" in the fuzzy set "big men". If our sentence was "Vickie has several credit cards", then "several" would be a fuzzy characterisation of the cardinality of the non-fuzzy set "Vickie's credit cards". The notion of the cardinality of a fuzzy set helps us compute the proposition "Vickie has several credit cards". Here, "most" is a fuzzy characterisation of the relative cardinality of the fuzzy set "kind men" in the fuzzy set "big men". It might not always be clear how a constituent fuzzy number relates to a fuzzy quantifier, but we will not go in details of these here, for examples see Zadeh (1983).

The fuzzy semantics of a proposition p is interpreted as "the degree of truth of p", or the possibility of p. In order to compute this, we translate p into a *possibility assignment equation*, which is denoted as follows

$$\Pi_{(X_1,\ldots,X_n)} = F$$

where F is a fuzzy subset of the universe of discourse U and $\Pi_{(X_1,\ldots,X_n)}$ is the joint possibility distribution over (explicit or implicit) variables $X_1, \ldots, X_n$ of p. For instance, the proposition "Vickie is tall" is translated as follows:

$$\Pi_{Height(Vickie)} = TALL$$

Here, $TALL$ is a fuzzy subset of the real line, $Height(Vickie)$ is a variable implicit in "Vickie is tall", and $\Pi_{Height(Vickie)}$ is the possibility distribution of this variable. The above possibility assignment equation implies that

$$Poss\{Height(Vickie) = u\} = \mu_{TALL}(u)$$

where $Poss\{X = u\}$ the possibility that X is u, for u a specified value. The above thus states that "the possibility that height of Vickie is u is equal to $\mu_{TALL}(u)$, that is, is the grade of membership of u in the fuzzy set $TALL$. Quantified sentences are translated in a similar way. For instance, "Vickie has several credit cards", is translated to the following:

$$\Pi_{Count(Credit-Cards(Vickie))} = SEVERAL$$

Suppose that 4 is compatible with the meaning of "several" with degree 0.8, then the above implies that, for instance, the possibility that Vickie has 4 credit cards is

$$Poss\{Count(Credit\text{-}Cards(Vickie)) = 4\} = 0.8$$

In order to analyse sentences of the general forms "There are Q A's" and 'Q A's are B's", Zadeh assumes that they are semantically equivalent to the following:

$$\text{There are } Q\ A'\text{s} \rightsquigarrow \Sigma Count(A) \text{ is } Q$$
$$Q\ A'\text{s are } B'\text{s} \rightsquigarrow Poportion(B|A) \text{ is } Q$$

Here, $Poportion(B|A)$ is the proportion of elements of B that are in A, aka the relative cardinality of B in A, formally defined as follows:

$$\Pi_{Poportion(B|A)} := \frac{\Sigma Count(A \cap B)}{\Sigma Count(A)}$$

Both $Proportion(B|A)$ and $\Sigma Count(A)$ may be fuzzy or non-fuzzy counts. Zadeh then formalises the above counts as possibility assignment equations as follows

$$\Sigma Count(A) \text{ is } Q \rightsquigarrow \Pi_{\Sigma Count(A)} = Q$$
$$Proportion(B|A) \text{ is } Q \rightsquigarrow \Pi_{Proportion(B|A)} = Q$$

In the spirit of truth-conditional semantics, the weight of each of the elements of the set can be interpreted as the degree of truth of the proposition denoted by the element. This weight is $Q(\Sigma Count(A))$ for sentences of the form "There are Q A's" and $Q(Proportion(B|A))$ for sentences of the form "Q A's are B's".

Writing $\mu_A(u)$ for the degree of membership of u in the fuzzy set A, we define the intersection of two fuzzy sets A and B as

$$A \cap B = \Sigma_i\ \min(\mu_A(u_i), \mu_B(u_i))\ u_i$$

where i is understood to range over all the elements in A and B (when an element is in A but not in B it will still be represented in A with a degree of 0). A similar version without the Σ is used to define it for the non-fuzzy case.

Example Let's say we have a universe

$$U = \{u_1, u_2, u_3, u_4, u_5\}$$

and fuzzy sets for "kind people" and "big men" as follows:

$$KP = 0.5u_1 + 0.8u_2 + 0.2u_3 + 0.6u_4$$
$$BM = 0.8u_1 + 0.3u_2 + 0.1u_3 + 0.9u_4 + 1u_5$$

The quantified sentence "Most big men are kind", is translated to the following possibility assignment equation:

$$\Pi_{Proportion(KP|BM)} = MOST$$

The intersection of KP and BM is computed as follows:

$$KP \cap BM = 0.5u_1 + 0.3u_2 + 0.1u_3 + 0.6u_4$$

The proportion of big men that are kind is computed as follows:

$$Proportion(KP|BM) = \frac{\Sigma Count(BM \cap KP)}{\Sigma Count(BM)} = \frac{0.5 + 0.3 + 0.1 + 0.6}{0.8 + 0.3 + 0.1 + 0.9 + 1} = \frac{1.5}{3.1}$$

Suppose that proportions between 0.6 and 0.7 are compatible with the meaning of $MOST$ with degree 0.75. Then, since $\frac{1.5}{3.1} = 0.48$, the degree of truth of our sentence is below 0.75. For the crisp quantifier ALL, the sentence "All big men are kind" is, since only the proportion 1 is compatible with the meaning of ALL with degree 1, which is not the case here.

Possibility distributions are encodable into vectors and indeed the possibility distributions of fuzzy quantifiers are learnt by Zadeh via a test-score procedure where as vectors by sampling from a database of related data.

6.7 Fuzzy Quantified Sentences in V-Rel

A non-fuzzy generalised quantifier d is interpreted as a relation $[\![d]\!]$ over the power set of the universe of discourse $P(U)$, where it relates a subset $A \subseteq U$ to subsets $u_i \subseteq U$, based on the cardinalities of A and u_i, as defined in Sect. 6.3. The fuzzy version of this quantifier is interpreted as a many valued relation over $P(U)$, where, in fuzzy set notation, it relates A to subsets $u_i \subseteq U$ and assigns to each such subset a degree of membership μ_i. The result is a fuzzy set whose weights come from a possibility distribution over the relative cardinalities of A and u_i's. In Zadeh's notation:

$$[\![d]\!](Proportion(u_i|A)) = \mu_i \tag{6.1}$$

We translate the above in the language of **V-Rel**, referring to the categorical version of the fuzzy generalised quantifier by $\overline{[\![d]\!]}$, which is a map with the type $P(U) \nrightarrow P(U)$. This is the map which was denoted by Q in the fuzzy generalised quantifier setting. In order to be coherent with the categorical semantics of sets and relations, we use the notation $\overline{[\![d]\!]}$ for it. Recall that in the categorical generalised quantifier theory, a quantifier was represented by $[\![d]\!]$. In what follows, we first define $\overline{[\![d]\!]}$ as a generalised fuzzy quantifier in the categorical setting of many valued relations, this is in Definition 9. Then, in Definition 11, extend it to quantified sentences of the fragment of language generated by the preliminary grammar of Sect. 6.3.

Definition 9 For $\mathbf{V} = [0, 1]$ and given a fuzzy generalised quantifier for which we have assumed $\Pi_{Proportion(B|A)} = \overline{[\![d]\!]}$, we define its **V-Rel** encoding to be the many valued relation $\overline{[\![d]\!]} : P(U) \nrightarrow P(U)$, with values coming from the possibility distribution of $\overline{[\![d]\!]}$, defined as follows:

$$\overline{[\![d]\!]}(A, B) = \mu_i \quad \text{for} \quad \mu_i = [\![d]\!](Proportion(B|A))$$

In order to obtain a many valued relation in **V-Rel**, we need a numerical value assigned to subsets A and B of universe. This number is nothing but the weight of $[\![d]\!](Proportion(B|A))$, denoted by μ_i in Eq. 6.1. If for any reason this number is unattainable, e.g. when B and A are not related to each other at all, we assign the $\perp$ to it.

Remark 4 Conservativity of a quantifier d in **V-Rel** is defined as follows:

$$\overline{[\![d]\!]}(A, B) = \overline{[\![d]\!]}(A, A \cap B)$$

and is implied by its conservativity in **Rel**. This is because

$$\begin{aligned} Proportion(A \cap B|A) &= \frac{\Sigma Count(A \cap B \cap A)}{\Sigma Count(A)} \\ &= \frac{\Sigma Count(A \cap B)}{\Sigma Count(B)} = Proportion(B|A) \end{aligned}$$

The absolute quantifiers such as “every” and “some” can still be interpreted in this setting, by defining them as follows:

$$\overline{[\![\text{every}]\!]}(A, B) = \begin{cases} \mathsf{e} & \text{if} A \subseteq B, \\ \perp & \text{otherwise.} \end{cases}$$

$$\overline{[\![\text{some}]\!]}(A, B) = \begin{cases} \mathsf{e} & \text{if } A \cap B \neq \emptyset, \\ \perp & \text{otherwise.} \end{cases}$$

The fuzzy version of a model generated by the grammar of Sect. 6.3, becomes as follows:

Definition 10 A fuzzy model $(U, [\![\ ]\!])_f$ is one where for $A \subseteq U$ we have:

$$[\![A]\!] := \mu_1 u_1 + \mu_2 u_2 + \cdots + \mu_n u_n \quad \text{for } u_i \in U$$

The fuzzy semantics of expressions of the grammar are many valued versions of those of $(U, [\![\ ]\!])$. For reasons of space, we only give them in **V-Rel** notation.

The following definition explains the many valued semantics of a sentence in our example grammar is computed.

Definition 11 A **V-Rel** fuzzy model is the tuple (**V-Rel**, $P(U)$, I, $\overline{[\![\ ]\!]}$) for which we have the following interpretation:

1. A terminal x of either category N,NP, or VP is interpreted as a many valued relation whose value is the degree to which a subset A of the universe is $[\![x]\!]$. This is the relative sigma count of the subset A in $[\![x]\!]$, that is:

$$\star\overline{[\![x]\!]}A := Proportion(A|[\![x]\!])$$

2. A terminal x of category V is interpreted as a many valued relation whose value is the degree to which its image on a subset A of universe is a subset B of the universe, that is the relative sigma count of B in $[\![x]\!](A)$:

$$\star\overline{[\![x]\!]}(A, B) = Proportion(B|[\![x]\!](A))$$

where $[\![x]\!](A)$ is the application of $[\![x]\!]$ to A, resulting in a set $\Sigma_{i=1}^{n}\mu_i b$ where the subscripts of the μ's vary over elements of fuzzy sets A and $[\![v]\!]$, so we have

$$\max_{a_i} \min(\mu_A(a_i), \mu_{[\![v]\!]}(a_i, b_i))$$

Here, μ_A and $\mu_{[\![v]\!]}$ are degrees of memberships of elements of fuzzy sets A and $[\![v]\!]$, respectively.

Given this definition, we compute the many valued semantics of quantified sentences and show that they are equivalent to the fuzzy quantifier definitions of Zadeh. Note that for U the universe of reference, the relative cardinality of A in U is the same as the cardinality of A. Thus "There are Q A's" has the same fuzzy meaning as "Q U's are A's".

Proposition 1 *The many valued semantics of a sentence with a quantified subject "*d np vp*" is the same as its fuzzy quantifier semantics in **V-Rel***

Proof The many valued semantics of "*d np vp*" is computed in four steps, according to the four composed morphisms of its **V-Rel** semantics:

$$\epsilon \circ (\overline{[\![d]\!]} \otimes \mu) \circ (\delta \otimes id) \circ (\overline{[\![np]\!]} \otimes \overline{[\![vp]\!]})$$

In the first step, we compute the following map

$$\overline{[\![np]\!]} \otimes \overline{[\![vp]\!]} \colon \{\star\} \otimes \{\star\} \nrightarrow \mathcal{P}(U) \otimes \mathcal{P}(U)$$

For $A, B \subseteq U$, the value returned by this map is

$$(\star, \star)(\overline{[\![np]\!]} \otimes \overline{[\![vp]\!]})(A, B) = \min\left(\star\overline{[\![np]\!]}A, \star\overline{[\![vp]\!]}B\right)$$

In the second step, we compute the following map

$$(\delta \otimes id) \circ (\overline{[\![np]\!]} \otimes \overline{[\![vp]\!]}) \colon \{\star\} \otimes \{\star\} \nrightarrow \mathcal{P}(U) \otimes \mathcal{P}(U) \otimes \mathcal{P}(U)$$

For $C, D, E \subseteq U$, it returns the following value

$$(\star, \star)(\delta \otimes id) \circ (\overline{[\![np]\!]} \otimes \overline{[\![vp]\!]})((C, D), E)$$

which is equal to

$$\max_{(A,B)} \min\left((\star, \star)(\overline{[\![np]\!]} \otimes \overline{[\![vp]\!]})(A, B),\ \ (A, B)(\delta \otimes id)((C, D), E)\ \right)$$

The maximum value of the above term is realised for (A, B)'s for which we have $A = C = D$ and $B = E$, in which case this value becomes equal to the following

$$\min\left(\star\overline{[\![np]\!]}A, \star\overline{[\![vp]\!]}B\right)$$

This is since the δ and id maps return e in their best case and e is the unit of the min operation.

In the third step, we compute the following map:

$$(\overline{[\![d]\!]} \otimes \mu) \circ (\delta \otimes id) \circ (\overline{[\![np]\!]} \otimes \overline{[\![vp]\!]}) \colon \{\star\} \otimes \{\star\} \nrightarrow \mathcal{P}(U) \otimes \mathcal{P}(U)$$

The value generated by this map is

$$(\star, \star)(\overline{[\![d]\!]} \otimes \mu) \circ (\delta \otimes id) \circ (\overline{[\![np]\!]} \otimes \overline{[\![vp]\!]})(F, G)$$

equal to

$$\max_{((C,D),E)} \min\left((\star, \star)(\delta \otimes id) \circ (\overline{[\![np]\!]} \otimes \overline{[\![vp]\!]})((C, D), E), (C, (D, E))(\overline{[\![d]\!]} \otimes \mu)(F, G)\right)$$

The maximum of the above is realised for the C, D, E that make $G = D \cap E$ true, in which case this value becomes equal to

$$\max_{(A,B)} \min\left(\star\overline{[\![np]\!]}A, \star\overline{[\![vp]\!]}B, A\overline{[\![d]\!]}F\right)$$

In the fourth step we compute the full map

$$\epsilon \circ (\overline{[\![d]\!]} \otimes \mu) \circ (\delta \otimes id) \circ (\overline{[\![np]\!]} \otimes \overline{[\![vp]\!]}) \colon \{\star\} \otimes \{\star\} \nrightarrow \{\star\}$$

the value generated by which is

$$(\star, \star)\, \epsilon \circ (\overline{[\![d]\!]} \otimes \mu) \circ (\delta \otimes id) \circ (\overline{[\![np]\!]} \otimes \overline{[\![vp]\!]})\star$$

which is equal to

$$\max_{(F,G)} \min \Big((\star, \star)(\overline{[\![d]\!]} \otimes \mu) \circ (\delta \otimes id) \circ (\overline{[\![np]\!]} \otimes \overline{[\![vp]\!]})(F, G), (F, G)\epsilon \star \Big)$$

The maximum of this term is realised when we have $F = G$, in which case it becomes equal to

$$\max_{(A,B)} \min \left(\star\overline{[\![np]\!]}A, \star\overline{[\![vp]\!]}B, A\overline{[\![d]\!]}A \cap B\right)$$

By applying Definition 11, the above unfolds as follows

$$\max_{(A,B)} \min \Big(Proportion([\![np]\!], A), Proportion([\![vp]\!], B), [\![d]\!][Proportion(A \cap B|A)]\Big)$$

where we are assuming $\Pi_{Proportion(A \cap B|A)} = d$. Since our quantifiers are conservative, we apply the simplifications computed in Remark 4, then given that the first two terms of the above max min are maximised when $A = [\![np]\!]$, $B = [\![vp]\!]$, the above simplifies to the following

$$\max_{(A,B)} \min \Big(\frac{\Sigma Count([\![np]\!] \cap [\![np]\!])}{\Sigma Count([\![np]\!])}, \frac{\Sigma Count([\![vp]\!] \cap [\![vp]\!])}{\Sigma Count([\![vp]\!])}, [\![d]\!]\Big[\frac{\Sigma Count(A \cap B)}{\Sigma Count(A)} \Big]\Big)$$

which simplifies to

$$[\![d]\!]\Big[\frac{\Sigma Count([\![np]\!] \cap [\![vp]\!])}{\Sigma Count([\![np]\!])} \Big]$$

Again, here we are assuming that $\Pi_{Proportion([\![vp]\!]|[\![np]\!])} = d$. Observe now that this equals the following

$$[\![d]\!][Proportion([\![vp]\!]|[\![np]\!])] \quad \text{for } \Pi_{Proportion([\![vp]\!]|[\![np]\!])} = d$$

which is the same as Zadeh's fuzzy quantifier semantics of "d np's are vp's".

Example Given Definition 10, the statement "several cats sleep" will be interpreted as

$$\max_{(A,B)} \min \Big(\star \overline{[\![cats]\!]}A, \star\overline{[\![sleep]\!]}B, A\overline{[\![several]\!]}A \cap B\Big)$$

This will be maximised for $A = [\![cats]\!]$, $B = [\![sleep]\!]$ and when assuming that $\Pi_{Proportion(A \cap B|A)}$ = several, in which case the value of the statement will become

$$[\![several]\!]\Big[\frac{\Sigma Count([\![cats]\!] \cap [\![sleep]\!])}{\Sigma Count([\![cats]\!])}\Big]$$

To compute this concretely, suppose that the fuzzy sets $[\![cats]\!]$ and $[\![sleep]\!]$ are defined as follows:

$$[\![cats]\!] = 0.2c_1 + 0.3c_2 + 0.8c_3$$
$$[\![sleep]\!] = 0.5c_1 + 0.4c_2 + 0.4c_3$$

Then the value for "several cats sleep" will be

$$[\![several]\!]\Big[\frac{\Sigma Count(0.2c_1 + 0.3c_2 + 0.4c_3)}{0.2c_1 + 0.3c_2 + 0.8c_3}\Big]$$
$$= [\![several]\!]\Big[\frac{0.9}{1.3}\Big]$$

Suppose that the possibility distribution [[several]] will map low values to low values and very high values to low values, but intermediate values would be mapped to a high number as they still represent "several". Thus the proportion $\frac{9}{13}$, which is a high number, will evaluate to a high number. Thus the many valued relation of this statement will be high (a number close to 1). For examples of possibility distributions of some other fuzzy quantifiers, see Zadeh (1983).

Proposition 2 *The many valued semantics of a sentence with quantified object "*np v d np'*" is the same as its fuzzy quantifier semantics in **V-Rel**.*

Proof After several steps of computation similar to those done in the proof of Proposition 1, we obtain the following value for the semantics of "*np v d np'*"

$$\max_{(A,B,C)} \min\Big(\star\,\overline{[\![np]\!]}A, \star\overline{[\![v]\!]}(A, B), \star\overline{[\![np']\!]}C, C\overline{[\![d]\!]}B \cap C\Big)$$

By applying Definition 11 and maximising the proportions, the above unfolds to

$$\max_{(A,B,C)} \min\left([\![d]\!]\Big[\frac{\Sigma Count([\![v]\!][[\![np]\!]] \cap [\![np']\!])}{\Sigma Count([\![np']\!])}\Big]\right)$$

for $\Pi_{Proportion(C|D)} = d$. Then the number computed above is the same as the one obtained from $[\![d]\!]\left[Proportion([\![v]\!][[\![np]\!]]|[\![np']\!])\right]$, which is the same as the fuzzy quantifier semantics of "d np"s are v-np's".

An example of this case is "Mice eat several plants" which has the same semantics as "Several plants are eaten by mice". Suppose we have fuzzy sets

$$[\![mice]\!] = 0.7c_1 + 0.6c_2 + 0.2c_3$$
$$[\![eat]\!] = 0.5(c_1, c_1) + 0.8(c_1, c_3) + 0.2(c_2, c_1)$$
$$\& + 0.3(c_2, c_3) + 0.9(c_3, c_3)$$
$$[\![plants]\!] = 0.2c_1 + 0.3c_2 + 0.6c_3$$

Then the semantics we get is

$$[\![several]\!]\left[\frac{\Sigma Count([\![eat]\!]([\![mice]\!]) \cap [\![plants]\!])}{\Sigma Count([\![plants]\!])}\right]$$

The application of the verb to its subject gives

$$[\![eat]\!]([\![mice]\!]) = 0.5c_1 + 0.7c_3$$

As a result, the whole expression now evaluates to

$$[\![several]\!]\left[\frac{\Sigma Count(0.2c_1 + 0.6c_3)}{\Sigma Count(0.2c_1 + 0.3c_2 + 0.6c_3)}\right] = [\![several]\!]\left[\frac{0.8}{1.1}\right]$$

This will yield another relatively high value for the many valued semantics of this sentence, as $Proportion([\![eat]\!]([\![mice]\!])|[\![plants]\!])$ certainly indicates a case of "several" mice eating plants.

Corollary 1 *The many valued semantics of a sentence with a quantified subject and a quantified object* "d np v d' np'" *is the same as its fuzzy quantifier semantics in* ***V-Rel***

Proof The proof is obtained by applying Propositions 1 and 2. After several steps of computation, we obtain that the many valued semantics of "*d np v d' np'*" is

$$\max_{(A,B),(C,D)} \min\left(\star\,\overline{[\![np]\!]}A,\ A\overline{[\![d]\!]}A \cap B, \star\overline{[\![v]\!]}(B, D), \star\overline{[\![np']\!]}C,\ C\overline{[\![d']\!]}C \cap D\right)$$

The above unfolds and simplifies as before. When the maximum of the min set is realised, we obtain equivalence with the following

$$[\![v]\!][[\![d]\!][[\![np]\!]], [\![d']\!][[\![np']\!]]]$$

for $\Pi_{Proportion([\![\tilde{v}]\!]|[\![d']\!][[\![np']\!]])} = d'$ and $\Pi_{Proportion([\![v]\!][[\![np]\!]]|[\![np]\!])} = d$ and where $[\![\tilde{v}]\!]$ is the image of v on "*d np*", given by

$$[\![\tilde{v}]\!] = [\![v]\!][[\![d]\!]([\![np]\!])]$$

An example of this case is "Several mice eat most plants" which has the same semantics as "most plants are eaten by several mice". Given that the fuzzy sets representing mice and plants are as before and taking the same fuzzy relation for ⟦eat⟧, we compute the meaning of this sentence. Suppose further that ⟦most⟧ is a possibility distribution that assigns the value 0 to numbers below 0.5, and gradually increasing the value for numbers from 0.5 to 1.

First, we compute the application of the quantifiers to their respective noun phrases:

$$\begin{aligned}&[\![several]\!][[\![mice]\!]] =\\ &\quad \arg\max_B \left([\![several]\!]\left[\frac{\Sigma Count([\![mice]\!] \cap B)}{\Sigma Count([\![mice]\!])}\right]\right)\end{aligned}$$

If we assume that "several" has the highest value for 0.4, then it would for instance assign to the set $0.4[\![mice]\!]$ the value $\Sigma_i 0.4\mu_i u_i$ for $\mu_i u_i$ in $[\![mice]\!]$. The second application gives

$$\begin{aligned}&[\![most]\!][[\![plants]\!]] =\\ &\quad \arg\max_A \left([\![most]\!]\left[\frac{A \cap [\![plants]\!]}{\Sigma Count([\![plants]\!])}\right]\right)\end{aligned}$$

This will set $A = [\![plants]\!]$, given that 1 has the highest probability of being "most".

The value of the whole sentence will be the verb applied to the quantified subject and object, hence we obtain

$$\begin{aligned}&[\![eat]\!]\Big[[\![several]\!][[\![mice]\!]], [\![most]\!][[\![plants]\!]]\Big]\\ =&[\![eat]\!]\Big[0.4[\![mice]\!], [\![plants]\!]\Big]\\ =&\max_{a,b} \min(\mu_{0.4[\![mice]\!]}(a), \mu_{[\![eat]\!]}(a,b), \mu_{[\![plants]\!]}(b))\\ =&\max\Big(\min(0.28, 0.5, 0.2), \min(0.28, 0.8, 0.6),\\ &\min(0.24, 0.2, 0.2), \min(0.24, 0.3, 0.6),\\ &\min(0.08, 0.9, 0.6)\Big)\\ =&\max(0.2, 0.28, 0.2, 0.24, 0.08)\\ =&0.28\end{aligned}$$

This means that the extent to which several mice eat most plants is 28%.

We conclude this section by defining the notion of a degree of truth for sentences in **V-Rel** and noting that in the absolute case, it is the same as the truth value of the sentence in **Rel**.

Definition 12 A quantified sentence s has a degree of truth r iff $\overline{[\![s]\!]} = r$ in $(\textbf{V-Rel}, \mathcal{P}(U), \{\star\}, \overline{[\![\ ]\!]})$.

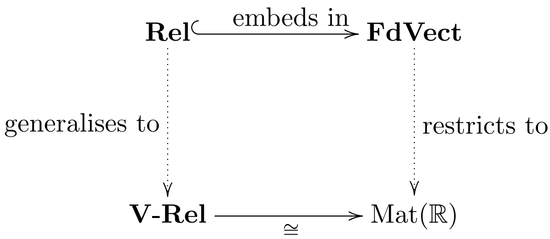
Remark 5 Suppose $\overline{[\![s]\!]} = 1$ in $(\textbf{V-Rel}, \mathcal{P}(U), \{\star\}, \overline{[\![\ ]\!]})$ and consider a sentence of the form "d np v"; as proved in Proposition 1 and by the above definition, this is the case iff we have $[\![d]\!][Proportion([\![vp]\!]/[\![np]\!])] = 1$ for $\Pi_{Proportion([\![vp]\!]/[\![np]\!])} = d$. This means that the proportion of elements of $[\![vp]\!]$ that are in $[\![np]\!]$ is d. Recall that our quantifiers are conservative, thus the proportion of elements of $[\![vp]\!] \cap [\![np]\!]$ that are in $[\![np]\!]$ is also d, which means $[\![vp]\!] \cap [\![np]\!] \in [\![d]\!][[\![np]\!]]$, and according to Definition 2 this makes the sentence "d np vp" have truth value *true*. In Hedges and Sadrzadeh (2019), authors showed that this is equivalent to $\overline{[\![s]\!]} = t$ in $(\textbf{Rel}, \mathcal{P}(U), \{\star\}, \overline{[\![\ ]\!]})$. The other direction holds in a similar fashion: suppose $\overline{[\![s]\!]} = t$ in $(\textbf{Rel}, \mathcal{P}(U), \{\star\}, \overline{[\![\ ]\!]})$, this is iff (as shown in Hedges and Sadrzadeh (2019)), the interpretation of "d np vp" is true in **Rel**, which is iff $[\![vp]\!] \cap [\![np]\!] \in [\![d]\!][[\![np]\!]]$, which by generalised quantifier theory means that $[\![vp]\!] \cap [\![np]\!]$ has d elements of $[\![np]\!]$, which will then make $[\![d]\!][Proportion([\![vp]\!]/[\![np]\!])]$ to be 1. The case for sentences of the form "np v d np" and "d np v d' np" are similar.

6.8 Conclusions and Future Work

In recent work Hedges and Sadrzadeh (2019) showed how one can reason about generalised quantifiers using bialgebras over the category of sets and relations over a fixed powerset object (powerset of a universe of discourse). They developed an abstract categorical semantics and instantiated it to category of sets and relations. Via the *Set-to-Vector Space* and *Relation-to-Linear Map* embedding, they transferred this semantics from sets and relations to vectors and linear maps. Their resulting vectorial semantics, however, is hard to reason with and costly to implement, a fault mainly due to the fact that in order to keep the maps linear, they had to work with vector spaces over powerset objects.

The reason for transferring the formal semantics of natural language from sets and relations to vector spaces and linear maps in compositional distributional semantics is to allow for quantitative reasoning in terms of the statistical data provided in distributional semantics. Another way to work with quantities and distributions of data is to move to a fuzzy setting, as done in Zadeh (1983). But then the question arises whether these two semantics are the same. This is the question to which this paper answers in positive. Concretely, in this paper we have shown that the categorical version of fuzzy sets, that is category **V-Rel** of sets and many valued relations, is compact closed and defined over it the required bialgebras. We developed, within this category, a many valued version of the abstract compact closed categorical semantics of Hedges and Sadrzadeh (2019) with Zadeh's fuzzy quantifiers and showed that the two semantics amount to the same degrees of truth for quantified sentences. As a result, in order to do quantification in compositional distributional semantics, one is not restricted to working with vector spaces over powerset objects and furthermore, fuzzy quantification is now also added to the existing setting.

A practical question that arises is what the empirical statistical consequences of embedding **FdVect** in **V-Rel** are. In order to answer this question, we have to work alongside intuitions such as "a distributional vector for a target word w yields a fuzzy set whose degrees of membership are the degrees of co-occurrences of w with a set of context words c, or the degrees of similarity of w to c, or the degrees of contextual relevance of w to c", or other similar readings. Formally, one has to work alongside the following diagram and with category of matrices over reals; these are equivalent to **V-Rel** and a special case of category of **FdVect**.



Building on the above ideas and implementing our model on real data and experimenting with it constitutes future work.

Another future direction is to build on the above intuitions and use the logic of fuzzy sets to develop a logic for distributional data. The quantified vectorial setting of Hedges and Sadrzadeh (2019) did not allow for a natural notion of logic: the main vector space, which was spanned by a power set object, did not have a natural interpretation of union and intersection of basis vectors in terms of basic set theoretic operations. A fuzzy setting, however, gives rise to a fuzzy logic and provides semantics for coordination in natural language, via operations such fuzzy conjunction and disjunction, e.g. see Novák (1992). Exploring the theoretical corollaries of this fact and experimenting with them constitutes future work.

Acknowledgements Support by EPSRC for Career Acceleration Fellowship EP/J002607/2, and by AFOSR for International Scientific Collaboration Grant FA9550-14-1- 0079 is gratefully acknowledged by M. S. Support by a QMUL Ph.D. Scholarship is acknowledged by G. W. The authors are grateful for the fruitful comments of an anonymous reviewer.

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Chapter 7
Implication via Spacetime

Amirhossein Akbar Tabatabai

Abstract In this paper we intend to study implications in their most general form, generalizing different classes of implications including the Heyting implication, sub-structural implications and weak strict implications. Following the topological interpretation of the intuitionistic logic, we will introduce non-commutative spacetimes to provide a more dynamic and subjective interpretation of an intuitionistic proposition. These combinations of space and time are natural sources for well-behaved implications and we will show that their spatio-temporal implications represent any other reasonable abstract implication. Then to provide a faithful well-behaved syntax for abstract implications, we will develop a logical system for the non-commutative spacetimes for which we will present both topological and Kripke semantics. These logics unify sub-structural and sub-intuitionist logics by embracing them as their special fragments.

Keywords Weak implications · Sub-intuitionistic logics · Sub-structural logics · Temporal logics · Topological semantics

7.1 Introduction

> *I remember that back in 1980, as an undergraduate, I was disappointed in logic, and was thinking of shifting to topology. Then Van Dalen came along and gave a course at the University of Amsterdam on sheaves and their relation to logic (the first such course in Holland), and subsequently organised a stimulating seminar on the subject. A course of lectures on Kripke-Joyal semantics by Michael Fourman formed part of this seminar. I was immediately fascinated by the subject, and still am.* (Moerdijk 1995)

Replacing topology with algebraic geometry and categorical logic with Brouwer's liberating revolution, I can hardly imagine a more vivid explanation of Mohammad Ardeshir's eye-opening influence on my life, both academic and personal, than what

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M. Mojtahedi et al. (eds.), *Mathematics, Logic, and their Philosophies*,
Logic, Epistemology, and the Unity of Science 49,
https://doi.org/10.1007/978-3-030-53654-1_7

Ieke Moerdijk is drawing in Dirk van Dalen festschrift. Through Ardehsir's fascinating explanation of the intutionistic philosophy and its huge impact on the everyday practice of mathematics, I found the realm of constructive mathematics and its *implications* haunting and hence decided to leave not only my possible future in algebraic geometry but the whole discipline of everyday mathematics, altogether. However, a true revolution knows no border and Brouwer's was no exception. Starting from the second half of the last century, the anti-realistic interpretation of mathematics has emerged unexpectedly and as a technical inevitable necessity in the mainstream mathematics, first in algebraic geometry through Alexander Grothendieck's inexhaustible quest for the generalized space and then in higher geometry, homotopy theory and the so-called homotopical mathematics. Following this historical thread, my fascination for intuitionism and more specifically the intuitionistic implication, is now slowly bringing me back to algebraic geometry again, where intuitionism might play its most deserved technical role. In this introduction I intend to explain how such a seemingly unrelated notion of space can be useful to understand intuitionism and hence intuitionistic implications. Far better, I will explain how intuitionism and geometry, interpreted in its most general sense, are nothing but the two sides of the same coin.

To establish this connection, we have to first understand the spatial interpretation of the notion of construction. For that purpose, let us start with the easier notion of constructibility rather than the explicit constructions, themselves. This means that we are interested in propositions and the provability relation between them rather than the actual proofs. Let us start with the creative subject's mind that may have many possible *states*. These states may encode many different data including the knowledge that she possesses in that mental state. It generally consists of all the constructions to which she has some reasonable access. For an intuitionist, a proposition is simply an entity that in every state of the creative subject's mind, it possesses a truth value and if the proposition happens to be true at some point, it must be possible to verify this truth in a finite number of steps. The truth value checks whether the proposition is derivable from the knowledge in a given state or not. Interpreting the knowledge as the story that has been told, a true proposition is exactly what the story can imply. Note that the finite verifiability condition is different from the decidability of a proposition in a mental state. For instance, let the knowledge content of a mental state be the axioms of Peano arithmetic. Then if something is not derivable from this theory, there is no a priory way to verify that.

The key point in the connection between intuitionism and topology is the set of these finitely verifiable propositions. This set has exactly the structure of the open subsets of a topological space and conversely, for any topological space, the set of its open subsets can be interpreted as the set of finitely verifiable propositions in a given theory.[1] To explain how to interpret the set of all finitely verifiable propositions as

[1] Technically, this holds for a pointfree version of topological spaces that are called locales. However, for the sake of simplicity, in this introduction we limit ourselves only to topological spaces.

the open subsets of a topological space, let us explain the three main structures that this set possess. Let S be the set of all possible mental states. Then a proposition can be identified by a subset of S, consisting of all the mental states for which the proposition holds. First, note that these subsets are ordered by the partial order $A \vdash B$ that encodes the situation that the truth of A in any state implies the truth of B in the same state. The second structure is the finite meets of the poset, called conjunctions. The reason is that if both A and B are finitely verifiable propositions, then so is $A \wedge B$. Because, if $A \wedge B$ holds in a state, there are finite verifications for both of them and the combination of these verifications is also finite. Note that the same claim is not necessarily true for infinite conjunctions, because, if the infinite conjunction is true, we need possibly infinite number of verifications that may exceed any possible finite memory. The last and the third structure is the arbitrary joins called disjunctions. For some set I, if A_i is finitely verifiable for any $i \in I$, then so is $\bigvee_{i \in I} A_i$. Because, if $\bigvee_{i \in I} A_i$ holds in a state, then one of them must hold and since it has a finite verification, the verification also works for the whole disjunction. Note that the semi-decidability condition and the existential nature of validity allows arbitrary disjunctions while it prohibits infinite conjunctions.[2] These ingredients are nothing but the conditions on a topology of a topological space. Therefore, the set of all finitely verifiable propositions is actually the set of opens of the space of the mental states. Therefore, it should not be surprising that intuitionistic propositional logic is sound and complete with respect to its topological interpretation that reads a proposition as an open subset of a given topological space; see (McKinsey and Tarski 1944). In this sense, intuitionism may be interpreted as the logic of space as opposed to the classical logic that corresponds to the logic of sets or discrete spaces. Compare the set of all opens of a space to the opens of a discrete space, namely the Boolean algebra of all subsets.

Now let us leave the truncated constructibility to address the actual explicit constructions. In this move, for any state we need a **Set**-like world to encode the constructions of the propositions and not just their truth values. In this setting, the three structures that we have explained transform to the following higher order notions: First, a poset transforms into a category whose objects and morphisms are propositions and the constructions between them. Secondly, for conjunctions we need the categorical version of finite meets, i.e., finite limits. And finally, for disjunctions we have to bring categorical joins, i.e., small colimits. Together with some technical

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[2] The reader may argue that using infinite sets and sequences may be somewhat problematic in the intuitionistic tradition. That is a very reasonable objection but at the same time it is also worth noting that the real meaning of the set I and the sequence of propositions $\{A_i\}_{i \in I}$ is somehow open to meta-mathematical interpretations and therefore they can be chosen completely constructively. For instance, the set I can be just the set of natural numbers and the sequence $\{A_i\}_{i \in I}$ can be a computable sequence of finite subsets. More mathematically, it means that everything in the argument is internalized in an elementary topos that formalizes what the intuitionist means by a set. For instance, the effective topos for the Russian school may be a reasonable choice for the universe. Having all said, the main point here is that while the conjunctions must be finite, the disjunctions can be arbitrary and this arbitrariness is something up to interpretation.

conditions, this *new space* is nothing but a Grothendieck topos. In this sense, the generalized notion of space is canonically conceivable from the pure intuitionistic conception of a proposition—a truly borderless revolution, indeed! Moreover, it implies that we should not be surprised that Grothendieck topoi or their elementary version can serve as the models for intuitionistic set theories or type theories, since the latter is simply the syntactic axiomatization of the constructions that the former formalizes model-theoretically. Unfortunately, this paper does not have enough space to explain all the details of this interpretation. However, we strongly encourage the reader to pursue this logical/philosophical path to geometry and read any geometrical construction by keeping an eye on the foregoing interpretation. This briefly explained connection between constructivism and the different incarnations of the notion of space is a very well-established tradition and here we only had time to see the tip of the iceberg. To see how this connection may lead to some useful interpretations in topos theory, higher geometry and even computer science, see (Joyal and Tierney 1984), (Anel and Joyal 2019), (Abramsky 1987, Abramsky 1991), (Abramsky and Vickers 1993) and (Vickers 1989).

Now, considering propositions as the open subsets of a (new) space, we are ready to address the complex, ubiquitous and hard to comprehend notion of implication. First note that any sophisticated anti-realistic philosophy needs an act of internalization; the way by which the creative subject internalizes her own notion of construction to be able to bring them to her consideration as the object of the study and not just its instrument. This internalization is actually what the implication is developed for. It transforms the provability order between propositions, $A \vdash B$, a meta-mathematical property, into the validity of another proposition, i.e., $A \to B$. In the case we also care about the explicit constructions, the implication or in this case the function space, implements the same idea to transform the set of constructions from A to B to the constructions of $A \to B$.

What is an internalizer? For the sake of simplicity, let us limit ourselves only to the constructibility case. Therefore, we have the provability order which we intend to internalize. There are many different structures that we can expect an implication to internalize. For instance, the order is reflexive, i.e., $A \vdash A$ for any proposition A and it is transitive, i.e., "$A \vdash B$ *and* $B \vdash C$ *implies* $A \vdash C$" for any propositions A, B, and C. The internalizations for these basic properties are $\vdash A \to A$ and

$$(A \to B) \wedge (B \to C) \vdash (A \to C),$$

for any propositions A, B, and C. The order has also all finite conjunctions meaning that for any two propositions B and C, there exists a proposition $B \wedge C$ such that for any A we have "$A \vdash B \wedge C$ *iff* "$A \vdash B$ *and* $A \vdash C$"" whose internalization is:

$$A \to (B \wedge C) = (A \to B) \wedge (A \to C),$$

and for all finite disjunctions it means the existence of $A \vee B$ such that for any C we have "$A \vee B \vdash C$ *iff* "$A \vdash C$ *and* $B \vdash C$"" whose internalization is:

$$(A \vee B) \to C = (A \to C) \wedge (B \to C)$$

As we can observe by the foregoing instances, there can be many structures or properties that we may want to internalize and depending on that, there can be many different possible implications. The usual Heyting implications in posets, exponential objects in categories, the many-valued, the relevant and the linear implications and the monoidal internal hom structures in monoidal categories are only some of these many implications. See (Mac Lane 1998), (Borceux 1994a), (Restall 2002). There are also some non-substructural internalizations. One of the early examples that also motivated the present work was introduced first in (Visser 1981b) and (Visser 1981a) and re-emerged in a more philosophically motivated form in (Ruitenburg 1991) to address the impredicativity problem of the implication. This implication is morally the Heyting implication without its modus ponens rule; see (Ardeshir 1995), (Ardeshir and Ruitenburg 2001), (Ardeshir and Ruitenburg 1998), (Celani and Jansana 2001). The emergence of these weak implications then set the scene for a plethora of other and sometimes even weaker implications emerging philosophically (Ruitenburg 1992); algebraically (Restall 1994); (Celani and Jansana 2005); (Alizadeh and Ardeshir 2006a); (Alizadeh 2009); (Alizadeh and Ardeshir 2006b); (Alizadeh and Ardeshir 2012); (Alizadeh and Ardeshir 2004); (Ardeshir and Ruitenburg 2018); proof theoretically (Corsi 1987); (Došen 1993); (Suzuki 1999); (Sasaki 1999); via provability interpretations (Visser 2002); (Iemhoff 2003); (Iemhoff et al. 2005) and relational semantics (Ardeshir and Hesaam 2008); (Litak and Visser 2018), almost everywhere in the logical realm. Apart from the philosophically oriented reasons, the weak implications raise also some independent mathematical interests. In their propositional form, they appear in different logical disciplines including provability logic (Visser 1981b) and preservability logic (Visser 2002); (Iemhoff 2003); (Iemhoff et al. 2005), (Litak and Visser 2018). In their higher categorical form, they capture some type constructors called arrows by the functional programming community. Arrows were first introduced in (Hughes 2000) to encode some natural types of function-like entities that are not really functions. For instance, the type of all partial functions from A to B, for the given types A and B is such an arrow type. Categorically speaking, they generalize monads, used elegantly to formalize the computational effects in (Moggi 1991). For the categorical formalizations of arrows see (Jacobs et al. 2009) and for more information on their role in programming and type theory see (Paterson 2003) and (Lindley et al. 2011).

Coming back to the spatial interpretation, we are facing a question: If the notion of space is powerful enough to formalize constructions, why not using them to also understand implications and exponentials? For this purpose, we have to bring in another important intuitionistic notion, different from the usual constructions. This notion is time. Assume that the mental states encode not only the current knowledge of the mind, but also the relevant temporal data including the actual moment that the mental state occupies in the time line. To encode this temporal structure, we add a temporal modality, ∇, to construct a proposition ∇A from a proposition A, meaning "*A holds at some point in the past*". First note that ∇A is a proposition itself. Since, if ∇A holds in a mental state, there is some point in the past in which A holds. But A is a proposition and hence has a finite verification at that point. Therefore, it is easy to bring that verification to the current mental state and save it as some temporal information of the past. Secondly, ∇ is clearly monotone and union preserving. The reason for the latter is the existential nature of ∇. More precisely, if $\nabla(\bigvee_{i\in I} A_i)$ holds at some state, then there exists some point in the past in which $\bigvee_{i\in I} A_i$ holds. Hence, one of A_i's must hold in that point which implies ∇A_i holds at the current state. The converse is similar and easy. This completes the data we need for the temporal modality.

Back to the implications, using ∇ as the temporal modality, it is possible to design an implication that brings the temporal structure to the scene. Define the implication by

$$A \to_\nabla B = \bigcup\{C \mid \nabla C \wedge A \vdash B\}. \quad (*)$$

By this definition and the fact that ∇ preserves all disjunctions, it is not hard to prove

$$\nabla C \wedge A \vdash B \quad \text{iff} \quad C \vdash A \to_\nabla B, \qquad (**)$$

which can be read as a pair of the introduction-elimination rules that defines the implication. Note that the definition $(*)$ has been dictated by the equivalence $(**)$ in a unique way. The introduction-elimination rules state that $A \to_\nabla B$ is a consequence of C if the fact that C constructed before plus the truth of A at this moment implies the truth of B. Note that the only role that ∇ plays is delaying the implication. Philosophically speaking, it is the machinery to ensure a delay between constructing an implication and using it. For instance, based on the introduction-elimination rules, we know that $\nabla(A \to_\nabla B) \wedge A \vdash B$ while there is no reason to have $(A \to_\nabla B) \wedge A \vdash B$. The former means that $A \to_\nabla B$ holds (constructed) before and hence, at this moment we can argue that in the presence of A, we can use the implication to show B. While in the latter case, $A \to_\nabla B$ is just constructed and it can not be applicable at the moment. Now, identifying the set of propositions by the opens of a topological space, we have a mathematical formalization of the foregoing discussion. It is enough to have a topological space and a monotone and union preserving map $\nabla : \mathcal{O}(X) \to \mathcal{O}(X)$ encoding the temporal modality. Calling such a data a spacetime, we can ensure that all spacetimes have their canonical implications, as defined above. Admittedly, these implications define a special class of all possible implications. However, we will show that any reasonable implication is actually representable by these temporal implications. The advantage of a temporal implication is the full introduction-elimination rules that it possesses. These rules make a natural machinery for internalization and leads to a very well-behaved implication as opposed to the arbitrary selection of structures that an implication may randomly internalize. In sum, our motto is that the study of the notion of time can almost be the study of the notion of implication. In this paper and in its sequel, we intend to follow this motto to investigate the general notion of implication via its incarnations in the above-mentioned spacetimes. Here, we will focus on the algebraic side of the story and leave the full general categorical setting and its categorical spacetimes as the more structured Grothendieck topoi to the forthcoming work.

The structure of the present paper is as follows. In Sect. 7.2, we will present a rather intense section on preliminaries to make the paper self-contained and hence accessible for a wider range of audience. In Sect. 7.3 quantales will be presented as the natural generalization of the notion of space. We will also discuss how to capture a more subjective formalization of finitely verifiable propositions in which even observing the truth of a proposition changes the mental state. In Sect. 7.4, we will define an abstract implication as an order internalizing operation. Then in Sect. 7.5, we will develop a generalized version of spacetimes via quantales as developed in Sect. 7.3. Section 7.6 is devoted to the representation theorems to show that a considerable class of abstract implications are essentially the implications of the generalized spacetimes. In Sect. 7.7, we will continue by developing a series of sub-structural logics for spacetimes and we will study their topological semantics. Their Kripke semantics will be introduced in Sect. 7.8. And finally, in Sect. 7.9, we will show how to embed the sub-intuitionistic logics, the logics of weak implications into these more well-behaved logics of spacetime.

7.2 Preliminaries

In this section we will review some basic facts and some useful constructions, including the notions of poset, adjunction, the monoidal posets, quantales and some completion techniques. These are very well-known facts and constructions. However, for the sake of completeness and being accessible to a wider range of audience, we prefer to briefly explain some necessary parts here. For more information, see (Johnstone 1982), (Vickers 1989) and (Borceux 1994b) on locales and completions and (Rosenthal 1990) on quantales.

Definition 1 By a monoid $\mathcal{M} = (M, \otimes, e)$, we mean a set M equipped with a binary multiplication function $\otimes : M \times M \to M$ and an element $e \in M$ such that the multiplication is associative, i.e., for all $m, n, k \in M$ we have $(m \otimes n) \otimes k = m \otimes (n \otimes k)$ and e is the identity element, i.e., for all $m \in M$ we have $e \otimes m = m = m \otimes e$. If $\mathcal{M} = (M, \otimes_M, e_M)$ and $\mathcal{N} = (N, \otimes_N, e_N)$ are two monoids, by a homomorphism $f : \mathcal{M} \to \mathcal{N}$ we mean a structure preserving function $f : M \to N$, i.e., $f(e_M) = e_N$ and for any $m, n \in M$, $f(m \otimes_M n) = f(m) \otimes_N f(n)$.

Definition 2 By a poset we mean a pair $\mathcal{A} = (A, \leq)$, where A is a set and $\leq$ is a reflexive, anti-symmetric and transitive binary relation over A. By $\mathcal{A}^{op}$ we mean the opposite poset of $\mathcal{A}$, consisting of A with the opposite order. When there is no risk of confusion, we denote $\mathcal{A}^{op}$ simply by A^{op}. By a downset of $\mathcal{A}$, we mean a subset of A that is $\leq$-downward closed, i.e., a subset S such that if $a \leq b$ and $b \in S$, then $a \in S$. By an upset we mean a $\leq$-upward closed subset, i.e., a subset S such that if $a \leq b$ and $a \in S$, then $b \in S$.

By the join (the meet) of a subset $S \subseteq A$, we mean the greatest lower bound (the least upper bound) of S in A, if it exists. We denote it by $\bigvee S$ ($\bigwedge S$). If S has at most two elements $a, b \in A$, we use the notation $a \vee b$ for the join ($a \wedge b$ for the meet) and we denote the join of the empty set by 0 (the meet of the empty set by 1). A poset is called join semi-lattice or finitely cocomplete (meet-semilattice or finitely complete) if the join (meet) of all finite subsets of A exist. It is called cocomplete (complete) if the join (meet) of all subsets of A exist. And finally by a map between two posets $\mathcal{A} = (A, \leq_A)$ and $\mathcal{B} = (B, \leq_B)$, denoted by $f : \mathcal{A} \to \mathcal{B}$, we simply mean an order preserving function $f : A \to B$ meaning $f(a) \leq_B f(b)$ for any $a \leq_A b$. An order-preserving map is called an embedding if for any $a, b \in A$, the inequality $f(a) \leq_B f(b)$ implies $a \leq_A b$.

Remark 1 Note that any cocomplete poset is also complete and vice versa. It is easy to see that if $(A, \leq)$ is cocomplete and $S \subseteq A$ then $\bigvee\{x \in A | \forall s \in S \ (x \leq s)\}$ exists and serves as the meet $\bigwedge S$. The converse is similar.

Definition 3 Let $\mathcal{A} = (A, \leq_A)$ and $\mathcal{B} = (B, \leq_B)$ be two posets and $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{A}$ be two maps. The map f is called a left adjoint for g (or equivalently g is a right adjoint for f), if for all $a \in A$ and $b \in B$,

$$f(a) \leq_B b \quad \text{iff} \quad a \leq_A g(b)$$

In such situation the pair (f, g) is called an adjunction and it is denoted by $f \dashv g : \mathcal{B} \to \mathcal{A}$ or simply $f \dashv g$.

Remark 2 Note that given $f \dashv g : \mathcal{B} \to \mathcal{A}$, we have $fg(b) \leq_B b$, for all $b \in B$ because $g(b) \leq_B g(b)$. Similarly, $a \leq_A gf(a)$, for all $a \in A$. Moreover, in any adjunction situation, we have $fgf = f$. The reason is that since for any a, $a \leq_A gf(a)$, by applying f on both sides we have $f(a) \leq_B fgf(a)$. On the other hand, $fg(b) \leq_B b$, for all $b \in B$. Hence, for $b = f(a)$ we have $fg(f(a)) \leq_B f(a)$. Therefore, $fgf(a) = f(a)$. Similarly, $gfg = g$.

Theorem 1 (Adjoint Functor Theorem for Posets) *Let $\mathcal{A} = (A, \leq_A)$ be a complete poset and $\mathcal{B} = (B, \leq_B)$ be a poset. Then an order preserving map $f : \mathcal{A} \to \mathcal{B}$ has a right (left) adjoint iff it preserves all joins (meets).*

Proof See (Borceux 1994a). □

Definition 4 A monoidal poset is a structure $\mathcal{A} = (A, \leq, \otimes, e)$ where $(A, \leq)$ is a poset and $(A, \otimes, e)$ is a monoid whose multiplication is compatible with the order, i.e., $\otimes$ is order-preserving in each of its arguments. A monoidal poset is called distributive if its poset is a join-semilattice and its multiplication distributes over all finite joins in each of its arguments.

Definition 5 Let $\mathcal{A} = (A, \leq_A, \otimes_A, e_A)$ and $\mathcal{B} = (B, \leq_B, \otimes_B, e_B)$ be two monoidal posets. By a lax monoidal map $f : \mathcal{A} \to \mathcal{B}$ we mean an order preserving function $f : A \to B$ such that $f(e_A) \geq e_B$ and for any $a, b \in A$ we have $f(a \otimes_A b) \geq f(a) \otimes_B f(b)$. A map is called oplax monoidal if it is order preserving and the last two inequalities are in the reverse order, i.e., $f(e_A) \leq e_B$ and for any $a, b \in A$ we have $f(a \otimes_A b) \leq f(a) \otimes_B f(b)$. A map is called strict monoidal if it is both lax monoidal and oplax monoidal. It is called strict monoidal embedding if it is strict monoidal and if $f(a) \leq f(b)$ implies $a \leq b$, for any $a, b \in A$.

Theorem 2 *Let $\mathcal{A} = (A, \leq_A, \otimes_A, e_A)$ and $\mathcal{B} = (B, \leq_B, \otimes_B, e_B)$ be two monoidal posets, $f : \mathcal{A} \to \mathcal{B}$ be an oplax monoidal (lax monoidal) map and $g : \mathcal{B} \to \mathcal{A}$ be its right (left) adjoint. Then g is lax monoidal (oplax monoidal).*

Proof We prove the case when f is oplax and $f \dashv g$. The other case is similar. Since f is oplax we have $f(e_A) \leq_B e_B$ from which and by using the adjunction we have $e_A \leq_A g(e_B)$. For the other condition, note that by the Remark 2, the adjunction implies $f(g(a)) \leq_B a$ and $f(g(b)) \leq_B b$. By the fact that f is oplax, we have

$$f(g(a) \otimes g(b)) \leq_B f(g(a)) \otimes f(g(b)) \leq_B a \otimes b$$

and by the adjunction again, we have $g(a) \otimes g(a) \leq_B g(a \otimes b)$, which completes the proof. □

Definition 6 A monoidal poset $\mathcal{X}$ is called a quantale if its order is cocomplete and its multiplication distributes over all joins on both sides. A quantale is a locale if its monoidal structure is the meet structure of the poset. In other words, a locale is a cocomplete poset whose meet distributes over all of its joins.

Remark 3 Note that quantales are also complete. This provides the enough structure to interpret conjunctions in a quantale, as we will see later.

Here are some prototypical examples of locales and quantales that help to develop the intuition:

Example 1 Let $\mathcal{S} = (S, \leq)$ be a cocomplete poset and define X as the set of all join preserving functions $f : \mathcal{S} \to \mathcal{S}$ with the pointwise order $\leq_X$. Then $\mathcal{X} = (X, \leq_X, \circ, id)$ is a quantale where $\circ$ is the usual composition and $id : \mathcal{S} \to \mathcal{S}$ is the identity map.

Example 2 Let X be a set and $\mathcal{R}$ be a set of binary relations over X that includes the equality and is closed under composition and arbitrary union. Then $\mathcal{X} = (\mathcal{R}, \subseteq, \circ, =)$ is a quantale where $\circ$ is the relation composition.

Example 3 Let X be a topological space. Then $\mathcal{X} = (\mathcal{O}(X), \subseteq, \cap, X)$ is a locale where $\mathcal{O}(X)$ is the set of all open subsets of X.

Example 4 Let $\mathcal{M} = (M, \otimes, e)$ be a monoid. Consider $I(\mathcal{M})$ as the set of all ideals of $\mathcal{M}$, i.e., the subsets of M closed under arbitrary left and right multiplication. Then $(I(\mathcal{M}), \subseteq, \cdot, M)$ is a quantale where

$$I \cdot J = \{i \otimes j | i \in I, j \in J\}$$

The reason is that the union of any set of ideals is an ideal again and the multiplication clearly distributes over the union.

Remark 4 Note that if $\mathscr{X}$ is a quantale, then for any fixed $a \in \mathscr{X}$, the functions $l_a, r_a : \mathscr{X} \to \mathscr{X}$ mapping x into $a \otimes x$ and $x \otimes a$, respectively, preserve all joins and since the poset is cocomplete, by the adjoint functor theorem, Theorem 1, they both have right adjoints. Because of some technical reasons, we are only interested in l_a. Therefore, it will be useful to have a name and a notation for l_a's right adjoint. We denote it by $a \Rightarrow (-)$ and we call the binary operator $\Rightarrow$, *the canonical implication* of the qunatale $\mathscr{X}$. Spelling out the adjunction conditions, it means that for any $a, b, c \in \mathscr{X}$, we have $a \otimes b \leq c$ iff $b \leq a \Rightarrow c$. Note that if $\mathscr{X}$ is a locale, its canonical implication is just the usual Heyting implication of $\mathscr{X}$.

Definition 7 Let $\mathscr{X}, \mathscr{Y}$ be two quantales. Then by a lax/oplax/strict geometric morphism $f : \mathscr{X} \to \mathscr{Y}$, we mean a lax/oplax/strict monoidal join preserving map $f : \mathscr{X} \to \mathscr{Y}$.

Example 5 Let X and Y be two topological spaces, $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ be the poset of all open subsets of X and Y, respectively and $f : X \to Y$ be a continuous function. Then $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ is a strict geometric morphism.

It is worth mentioning that over locales, any join preserving map $f : \mathscr{X} \to \mathscr{X}$ is an oplax geometric morphism because f is order preserving which implies $f(a \wedge b) \leq f(a) \wedge f(b)$.

Example 6 Let X and Y be two sets and $f : X \to Y$ be a function. Then f induces a lax geometric morphism $f^* : P(Y \times Y) \to P(X \times X)$ by $f^*(R) = F^{-1}(R)$, where $F : X \times X \to Y \times Y$ and $F(x, x') = (f(x), f(x'))$. The map f^* is clearly union preserving. Moreover, for any two relations $R, S \subseteq Y \times Y$, we have $F^{-1}(R) \circ F^{-1}(S) \subseteq F^{-1}(R \circ S)$, because, if $(x, x') \in F^{-1}(R) \circ F^{-1}(S)$ then there is $z \in X$ such that $(x, z) \in F^{-1}(S)$ and $(z, x') \in F^{-1}(R)$. Therefore, $(f(x), f(z)) \in S$ and $(f(z), f(x')) \in R$ which implies $(f(x), f(x')) \in R \circ S$ from which $(x, x') \in F^{-1}(R \circ S)$.

The function f also induces an oplax geometric morphism. Define $f_* : P(X \times X) \to P(Y \times Y)$ by $f_*(R) = F[R]$ as the F-image of R. This is also union preserving. Moreover, we have $f_*(R \circ S) \subseteq f_*(R) \circ f_*(S)$, because, if $(y, y') \in F[R \circ S]$ then there is $x, x', z \in X$ such that $y = f(x)$, $y' = f(x')$, $(x, z) \in S$ and $(z, x') \in R$. Therefore, $(f(x), f(z)) \in F[S]$ and $(f(z), f(x')) \in F[R]$. Hence, $(y, y') = (f(x), f(x')) \in f_*(R) \circ f_*(S)$.

Example 7 Let $\mathcal{M} = (M, \otimes_M, e_M)$ and $\mathcal{N} = (N, \otimes_N, e_N)$ be two monoids and $f : \mathcal{M} \to \mathcal{N}$ be a homomorphism. Consider $I(\mathcal{M})$ and $I(\mathcal{N})$, defined in Example 4. Then f induces a lax geometric morphism $f^* : I(\mathcal{N}) \to I(\mathcal{M})$ by $f^*(I) = f^{-1}(I)$. It is clearly union preserving. Moreover, we have $f^*(I)f^*(J) \subseteq f^*(IJ)$ because if $x \in f^*(I)f^*(J)$ then there are $y \in f^*(I)$ and $z \in f^*(J)$ such that $x = y \otimes_M z$. Since f is a homomorphism we have $f(x) = f(y) \otimes_N f(z) \in IJ$. Therefore, $x \in f^*(IJ)$. The homomorphism f also induces an oplax geometric morphism defined by $f_* : I(\mathcal{M}) \to I(\mathcal{N})$ by $f_*(I) = Nf[I]N$, where $f[I]$ is the image of I and $Nf[I]N$ is the generated ideal of the image of I. This map clearly preserves union. Moreover, $f_*(IJ) \subseteq f_*(I)f_*(J)$, because if $x \in f_*(IJ)$, then there are $m, n \in N$, $i \in I$ and $j \in J$ such that $x = m \otimes_N f(i \otimes_M j) \otimes_N n$. Since f is a homomorphism we have $x = m \otimes_N f(i) \otimes_N f(j) \otimes_N n \in f_*(I)f_*(J)$.

In the rest of this section, we will recall some of the main completion techniques for the monoidal posets. We will address the details of constructions as we need them later in some other constructions of the paper.

Theorem 3 (Downset and Ideal Completions) *Let $\mathcal{A} = (A, \leq, \otimes, e)$ be a monoidal poset. Then there exists a quantale $D(\mathcal{A})$, called the downset completion of $\mathcal{A}$ and a strict monoidal embedding $i : \mathcal{A} \to D(\mathcal{A})$. If $\mathcal{A}$ has all finite joins and distributive, then there exists another quantale $I(\mathcal{A})$, called the ideal completion of $\mathcal{A}$ and a finite join-preserving strict monoidal embedding $i : \mathcal{A} \to I(\mathcal{A})$. If $\mathcal{A}$ has all finite meets, then in both cases i preserves all finite meets.*

Proof First let us explain the downset completion that works for monoidal posets that do not necessarily have the join structure. Later we will also address the joins and the distributive case. Define $\mathcal{X} = D(\mathcal{A})$ as the set of all downsets of A with the inclusion as its order. Since downsets are closed under arbitrary union and intersection, they are the joins and the meets of the poset, respectively. Define the map $i : A \to \mathcal{X}$ by $i(a) = \{x \in A | x \leq a\}$ and the monoidal structure of $\mathcal{X}$ by $e_{\mathcal{X}} = i(e)$ and

$$I \otimes_{\mathcal{X}} J = \{x \in A |\ \exists i \in I \exists j \in J\ (x \leq i \otimes j)\},$$

for any downsets I and J. Note that $I \otimes_{\mathcal{X}} J$ is also a downset. Moreover, it is not hard to prove that this multiplication is associative with the identity element $e_{\mathcal{X}}$ and it distributes over all unions. Therefore, $(\mathcal{X}, \otimes_{\mathcal{X}}, e_{\mathcal{X}})$ is actually a quantale. Moreover, i is a strict monoidal map because by definition, $e_{\mathcal{X}} = i(e)$ and

$$i(a) \otimes_{\mathcal{X}} i(b) = \{x \in A | \exists i \leq a \exists j \in b\ (x \leq i \otimes j)\} = \{x \in A | x \leq a \otimes b\}.$$

Finally, note that i is clearly an embedding, because,

$$i(a) \subseteq i(b) \quad \text{iff} \quad \{x \in A | x \leq a\} \subseteq \{x \in A | x \leq b\} \quad \text{iff} \quad a \leq b,$$

and if $\mathcal{A}$ has all finite meets, i preserves them because, $i(1) = \{x \in A | x \leq 1\} = A$ and

$$x \in i(a) \cap i(b) \quad \text{iff} \quad (x \in a \text{ and } x \in b) \quad \text{iff} \quad x \leq a \wedge b \quad \text{iff} \quad x \in i(a \wedge b),$$

which implies $i(a) \cap i(b) = i(a \wedge b)$.

Now, let us move to the distributive case, where $\mathcal{A} = (A, \leq, \otimes, e)$ has all finite joins. Then the foregoing function i does not necessarily preserve the join structure of $\mathcal{A}$. To handle this issue, we have to change $\mathscr{X}$ a little bit: Define $\mathscr{Y} = I(\mathcal{A})$ as the poset of all ideals of A, i.e., all downsets $I \subseteq A$ such that $0 \in I$ and $a \vee b \in I$, for any $a, b \in I$. We want to show that $\mathscr{Y}$ with the join

$$\bigvee_{i \in N} I_i = \{x \in A | \exists x_1, \ldots x_n \in \bigcup_{i \in N} I_i \ (x \leq \bigvee_{j=1}^{n} x_j)\}$$

and the same monoidal structure as of $\mathscr{X}$'s is a quantale and the previous function i is again an embedding that also preserves all finite joins. First, it is not hard to prove that $\bigvee$ maps ideals to ideals and is actually the join of the family $\{I_i\}_{i \in N}$ in the inclusion order over ideals. Secondly, note that the original $i : A \to \mathscr{X}$ actually lands into the set of ideals $\mathscr{Y}$, because, $\{x \in A | x \leq a\}$ is closed under all finite joins. Note also that i preserves all finite joins because,

$$i(a) \vee i(b) = \{x \in A | \exists i \leq a \exists j \leq b \ (x \leq i \vee j)\} = \{x \in A | x \leq a \vee b\}.$$

Since the intersection of ideals is also an ideal, the meet structure for ideals is also the intersection. Hence, the same argument for meet preservation by i works here, as well. Thirdly, note that the defined $\otimes$ on $\mathscr{X}$ maps ideal to ideals, meaning that if I and J are ideals then so is $I \otimes J$. To prove this claim, first note that $0 \otimes 0 \leq 0 \otimes e = 0$ from which $0 \otimes 0 = 0$ and hence $0 \in I \otimes J$. Secondly, assume that $x, y \in I \otimes J$. We want to show that $x \vee y \in I \otimes J$. By definition, there exist $i, i' \in I$ and $j, j' \in J$ such that $x \leq i \otimes j$ and $y \leq i' \otimes j'$. By monotonicity of $\otimes$ we have $x \leq (i \vee i') \otimes (j \vee j')$ and $y \leq (i \vee i') \otimes (j \vee j')$ and hence $x \vee y \leq [(i \vee i') \otimes (j \vee j')]$. Since both I and J are closed under finite joins, $i \vee i' \in I$ and $j \vee j' \in J$ and hence, $x \vee y \in I \otimes J$.

Finally, we show that the multiplication distributes over joins, i.e.,

$$\bigvee_{n \in N} (I_n \otimes J) = (\bigvee_{n \in N} I_n) \otimes J \quad \text{and} \quad I \otimes (\bigvee_{n \in N} J_n) = \bigvee_{n \in N} (I \otimes J_n).$$

We will prove the left equality. The right one is similar. There are two directions to prove. $\bigvee_{n \in N} (I_n \otimes J) \subseteq (\bigvee_{n \in N} I_n) \otimes J$ is clear by monotonicity. For the other direction, assume $x \in (\bigvee_{n \in N} I_n) \otimes J$. By definition, there exist $y \in \bigvee_{n \in N} I_n$ and $j \in J$ such that $x \leq y \otimes j$. Again by definition, there exist $i_1, i_2, \ldots, i_k \in \bigcup_{n \in N} I_n$ such that $y \leq i_1 \vee \ldots \vee i_k$. By distributivity, we have

$$x \leq (i_1 \otimes j) \vee (i_2 \otimes j) \vee \ldots \vee (i_k \otimes j).$$

But since each i_r is in at least one I_{m_r}, we have

$$i_r \otimes j \in (I_{m_r} \otimes J) \subseteq \bigvee_{n \in N} (I_n \otimes J).$$

Since $\bigvee_{n \in N}(I_n \otimes J)$ is closed under finite joins, we have $x \in \bigvee_{n \in N}(I_n \otimes J)$. □

Remark 5 Note that in the both downset and ideal completions, if the monoidal structure of $\mathcal{A}$ is just the meet structure, i.e., $\otimes = \wedge$ and $e = 1$, then $\otimes_{\mathcal{X}}$ is the intersection because

$$I \otimes_{\mathcal{X}} J = \{x \in A \mid \exists i \in I \exists j \in J \ (x \leq i \wedge j)\} = I \cap J,$$

which is the meet of $\mathcal{X}$ and also $e_{\mathcal{X}}$ is $i(1)$ which is the top element $1_{\mathcal{X}} = i(1) = A$.

Theorem 4 (Lifting Monoidal Maps) *Let $\mathcal{A} = (A, \leq_A, \otimes_A, e_A)$ and $\mathcal{B} = (B, \leq_B, \otimes_B, e_B)$ be two monoidal posets and $f : \mathcal{A} \to \mathcal{B}$ be a lax (oplax) monoidal map. Then there exists a lax (oplax) geometric map $f_! : D(\mathcal{A}) \to D(\mathcal{B})$ such that $f_! i_A = i_B f$, where i_A and i_B are the canonical embeddings of the downset completions of $\mathcal{A}$ and $\mathcal{B}$, respectively. Moreover, if both $\mathcal{A}$ and $\mathcal{B}$ have all finite joins and are distributive, and if $f : \mathcal{A} \to \mathcal{B}$ is finite join preserving, then the same holds for some map $f_! : I(\mathcal{A}) \to I(\mathcal{B})$.*

Proof First let us prove the downset case. We will address the ideal case later. Define

$$f_!(I) = \{x \in A \mid \exists i \in I \ (x \leq_B f(i))\}.$$

This set is clearly a downset, hence $f_!$ is well-defined. Moreover, note that

$$f_!(i_A(a)) = \{x \in A \mid \exists i \leq_A a \ (x \leq_B f(i))\} = \{x \in A \mid (x \leq_B f(a))\} = i_B(f(a)).$$

The map $f_!$ obviously preserves all unions. We have to prove that if f is lax (oplax), then so is $f_!$. Assume f is lax monoidal. The other case is similar. We have to prove that $i_B(e_B) \subseteq f_!(i_A(e_A))$ and $f_!(I) \otimes f_!(J) \subseteq f_!(I \otimes J)$, for any downsets I and J of $\mathcal{A}$. For the first, assume $x \in i_B(e_B)$, then $x \leq_B e_B \leq_B f(e_A)$. Hence, $x \in f_!(i_A(e_A))$. For the second, if $x \in f_!(I) \otimes f_!(J)$, then there are $y \in f_!(I)$ and $z \in f_!(J)$ such that $x \leq y \otimes z$. Since $y \in f_!(I)$ and $z \in f_!(J)$ there are $i \in I$ and $j \in J$ such that $y \leq f(i)$ and $z \leq f(j)$. Hence, $x \leq f(i) \otimes f(j) \leq f(i \otimes j)$, which implies $x \in f_!(I \otimes J)$.

For the ideal completion case, we define the same $f_!$. However, we have to check whether it is ideal and join preserving. It is an ideal because, $0 \leq f(0)$ and since $0 \in I$ we have $0 \in f_!(I)$. Moreover, if $x, y \in f_!(I)$ then there are $i, j \in I$ such that $x \leq f(i)$ and $y \leq f(j)$. Since f is monotone, we have $x \vee y \leq f(i \vee j)$. Since I is an ideal we have $i \vee j \in I$ and hence $x \vee y \in f_!(I)$. Furthermore, we have to check that $f_!$ is join preserving. For that matter, we have to show $f_!(\bigvee_{n \in N} I_n) = \bigvee_{n \in N} f_!(I_n)$. From right to left is easy by monotonicity of $f_!$. For the left to right, assume $x \in f_!(\bigvee_{n \in N} I_n)$. Hence, there are $i_1, \ldots, i_k \in \bigcup_{n \in N} I_n$ such that $x \leq f(i_1 \vee \ldots \vee i_k)$. Since f is join preserving we have $x \leq f(i_1) \vee \ldots \vee f(i_k)$ which implies that $x \in \bigvee_{n \in N} f_!(I_n)$. □

Upset and Filter Completions. Using two ideal completions in an appropriate way leads to a very useful construction that we call the upset construction. The details follow. Let $\mathcal{A} = (A, \leq, \otimes, e)$ be a monoidal poset and denote the downset qunatale of $\mathcal{A}$ by $D(\mathcal{A})$ and the opposite of $\mathcal{A}$, the same structure with the reverse order, by $\mathcal{A}^{op}$. Then by the downset completion for $\mathcal{A}^{op}$, there exists a strict monoidal embedding $i : \mathcal{A}^{op} \to D(\mathcal{A}^{op})$ or equivalently $i : \mathcal{A} \to D(\mathcal{A}^{op})^{op}$. It is useful to observe that $D(\mathcal{A}^{op})$ is nothing but the poset of all upsets of $\mathcal{A}$ with the multiplication:

$$P \otimes Q = \{x \in A | \exists y \in P \exists z \in Q \ (x \geq y \otimes z)\}.$$

Denote this poset by $U(\mathcal{A})$. Now we use the same operation again to embed $D(\mathcal{A}^{op})^{op}$ into $D(D(\mathcal{A}^{op})^{op})$. Combining these two embeddings, we reach a strict monoidal embedding of $\mathcal{A}$ into $D(D(\mathcal{A}^{op})^{op})$ which we call the upset completion of $\mathcal{A}$. Spelling out the construction of the upset completion, the set consists of all the upsets of the upsets of $\mathcal{A}$ with the inclusion as its order and the following multiplication for any upsets of upsets X and Y:

$$X \otimes Y = \{P \in U(\mathcal{A}) | \exists Q \in X \exists R \in Y \ (P \supseteq Q \otimes R)\}.$$

Moreover, the embedding is simply expressible by $i(a) = \{P \in U(\mathcal{A}) | a \in P\}$.

In the case that the monoidal poset is a meet semi-lattice $\mathcal{A} = (A, \leq, \wedge, 1)$, there is another construction that is called the canonical construction $C(\mathcal{A})$ and an embedding $i : \mathcal{A} \to C(\mathcal{A})$ that respects all finite meets. A non-empty upset of A is called a filter if it is closed under all finite meets. Denote the class of all filters of $\mathcal{A}$ by $F(\mathcal{A})$ and then define $C(\mathcal{A})$ as the poset of all upsets of filters and use the same i as defined before. The embedding $i : \mathcal{A} \to C(\mathcal{A})$ preserves all finite meets. First note that all filters include 1, thus

$$i(1) = \{P \in F(\mathcal{A}) | 1 \in P\} = F(\mathcal{A}).$$

Secondly, note that the filters are closed under meets. Hence,

$$i(a \wedge b) = \{P \in F(\mathcal{A}) | a \wedge b \in P\} = \{P \in F(\mathcal{A}) | a \in P \text{ and } b \in P\} = i(a) \cap i(b).$$

In case $\mathcal{A}$ has all finite joins and it is distributive, it is also possible to change the canonical construction so that i also preserves the finite joins. The construction is as follows: A filter is called prime if it is proper and for any $a, b \in A$, the assumption $a \vee b \in P$ implies either $a \in P$ or $b \in P$. Denote the set of all prime filters by $P(\mathcal{A})$. If we change $C(\mathcal{A})$ to the poset of all upsets of $P(\mathcal{A})$ with the same i, then i preserves both finite joins and finite meets. The reasoning for the meet is the same as before. For the joins, since prime filters are proper, we have $0 \notin P$, which implies

$i(0) = \{P \in P(\mathcal{A}) | 0 \in P\} = \emptyset$ and

$$i(a \vee b) = \{P \in P(\mathcal{A}) | a \vee b \in P\} = \{P \in P(\mathcal{A}) | a \in P \text{ or } b \in P\} = i(a) \cup i(b).$$

7.3 Intuitionism via Quantales

In the Introduction, we have seen that any finitely verifiable proposition can be interpreted as an open subset of a topological space. In this interpretation, the corresponding open subset captures the set of all the mental states in which the proposition actually holds. More operationally, a finitely verifiable proposition A is just an *observation* that reads a mental state and finds the truth value of A in that state, in the same way that a physical quantity like the speed or the temperature can be seen as an observation that reads a physical state to find the value of the quantity.

Reading propositions as observations suggests that we silently believe in some sort of an independent objective mind. Let us assume that the creative subject observes her mental state to check the validity of a proposition. It seems that this introspection only observes a mental state and extracts some needed information from it but it does not affect the mental state at all. The situation is similar to the classical assumption that the physical observations do not affect the physical phenomenon that they are observing. It measures a quantity ideally without distorting the picture or interfering with any other observation. This may be the case when we interpret the knowledge content of a mental state as a set of propositions and the validity of a proposition as its provability. Then it is just a real factual situation and it is not important what, when and in what order we are observing the validity of the propositions. However, it is totally possible to imagine a more subjective, more dynamic and more interactive formalization of knowledge. One possible scenario to show how natural such a situation could be is the following: Interpret the knowledge content of a mental state as a set of propositions as before but change the validity from provability to immediate provability. It means that a valid proposition is either in the set or provable in one step via some given proving methods from the set. Observing a proposition in this scenario clearly affects the mental state. If a proposition holds in a mental state, it is provable in at most one step. Then since the creative subject thinks about the proposition and finds out the proof, it is totally reasonable to assume that she then modifies her knowledge to add this new proposition to the set she had before. The observation process is also interactive. In each step, there could be many one-step provable propositions and hence it could be important to choose which way she wants to proceed. This choice may change her path forever. It is also non-commutative because A may be immediately provable and its presence makes B also immediately provable while the proposition B is not immediately provable without using A. Hence, proving A after B may not be even possible.

This is only one possible scenario. Now let us find a more formal way to express not only this scenario but its essential dynamic, interactive and non-commutative nature. We will begin by a toy example to be prepared to find the algebraic abstract formalization later. Let S be the set of all the mental states and identify a proposition not by a subset of S but by a binary relation $A \subseteq S \times S$ that includes (s, t) if the proposition A holds in the state s, its truth is verifiable in a finite number of steps and this verification changes the mental state s to t. Using this example, we can also identify the previous static interpretation of knowledge as the non-state-changing relations, i.e., the relations like A with the property that if $(s, t) \in A$ then $s = t$. These A's are simply identifiable by the subset $\{s \in S | (s, s) \in A\}$ of the mental states where they are valid. This is simply our previous proposition-as-subset formalization.

To formalize the calculus of this new interpretation of finitely verifiable propositions, we try to provide an algebraic axiomatization reflecting the main intuitive properties of this toy example. Here again we have three main structures. The first obvious structure is the order between the propositions encoding how a proposition implies another one. This order in our toy example is the inclusion order between the binary relations. Secondly, propositions has a natural notion of composition. Philosophically speaking, for any two propositions A and B, we can imagine $A \otimes B$ as the composition of observations, first applying B and then A. $A \otimes B$ changes the state s to t if there exists a state r such that B holds in s and maps s to r where A holds and A changes this r to t. In our toy example composition is simply the composition of relations. Note that this composition is clearly associative and has an identity element. The identity element is simply the do-nothing observation. In our toy example it is the equality relation over S. Moreover, note that in the static interpretation of propositions when A and B are encoded by subsets $\{s \in S | (s, s) \in A\}$ and $\{s \in S | (s, s) \in B\}$, their composition $A \otimes B$ will be $\{s \in S | (s, s) \in A\} \cap \{s \in S | (s, s) \in B\}$ which is nothing but the intersection. This shows how this dynamic approach really generalizes the static topological interpretation of the Introduction.

Finally, let us address the finiteness condition. Note that the poset of propositions is cocomplete as we explained in the Introduction, simply because for any set I, if all A_i's are all finitely verifiable, then their disjunction $\bigvee_{i \in I} A_i$ is also finitely verifiable. The main point is that for verifying a disjunction it is enough to verify one of them. How does a disjunction act on the states? It just combines the actions of all A_i's, since observing the validity of $\bigvee_{i \in I} A_i$ is just observing one of A_i's and hence it changes a states s to one of the states that one of A_i's may dictate. The disjunction in our toy example is just the union of relations. Moreover, note that the composition distributes over all joins because doing the observation B after "at least one of A_i's" is nothing different than doing one of "A_i's before B". The same also goes for the other argument of the multiplication. Therefore, to make a calculus for finitely verifiable propositions in its dynamic interactive sense, we need a cocomplete monoidal poset whose multiplication distributes over all joins on both sides. This is nothing but a quantale. Note that if we collapse the monoidal structure to the meet structure as in the non-state-changing-observation interpretation dictates, then the quantale turns into a locale, the point-free version of a topological space. Interpreting locales as the calculus of non-state-changing observations were developed in (Abramsky 1987, Abramsky 1991), and (Vickers 1989). This generalization to quantales has its roots even in (Mulvey 1986) where quantales first appeared to provide an algebraic formalization for non-commutative C^*-algebras. However, in its explicit form, the state-changing interpretation is developed in (Abramsky and Vickers 1993) and has been important in the connection between the quantales and their categorical monoidal versions on the one hand and the formalization of processes and observations in computer science and quantum physics on the other.

7.4 Abstract Implications

Philosophically speaking, an implication is a conditional proposition internalizing the provability order of the poset of all propositions. Traditionally, the internalization has been implemented via Heyting implications or in a more general setting of monoidal posets via residuations for right and left multiplications. We argue that this tradition is far more restricting than what a basic internalization task demands. As we have seen already in Introduction, internalizations can take place in many different levels to internalize many different structures. For instance, if we have a meet-semilattice, the implication may internalize the basic structures of reflexivity and transitivity via the axioms $a \to a = 1$ and

$$(a \to b) \wedge (b \to c) \leq (a \to c),$$

or it can go one step further to also internalize the finite meet structure via

$$a \to (b \wedge c) = (a \to b) \wedge (a \to c),$$

or in the case that the meet-semilattice has all finite joins, the join structure via

$$(a \vee b) \to c = (a \to c) \wedge (b \to c).$$

We propose that the minimum reasonable conditions for any internalization is the inernalization of reflexivity of the order and its transitivity. However, it does not need to be over meet-semilattices. We can use a more general setting where we only have a monoidal poset:

Definition 8 Let $\mathcal{A} = (A, \leq, \otimes, e)$ be a monoidal poset. By an implication on $\mathcal{A}$, denoted by the symbol $\to$, we mean a function from $A^{op} \times A$ to A such that it is order preserving in its both arguments and:

(i) $e \leq a \to a$,
(ii) $(a \to b) \otimes (b \to c) \leq (a \to c)$,

The structure $\mathcal{A} = (A, \leq, \otimes, e, \to)$ is called a strong algebra if $\to$ is an implication. And if $\mathcal{A} = (A, \leq_A, \otimes_A, e_A, \to_A)$ and $\mathcal{B} = (B, \leq_B, \otimes_B, e_B, \to_B)$ are two strong algebras, by a strong algebra morphism we mean a strict monoidal map $f : A \to B$ that also preserves $\to$, i.e., $f(a \to_A b) = f(a) \to_B f(b)$, for any $a, b \in A$.

Remark 6 Based on the order preservability of the implications in their second arguments, it is possible to strengthen the axiom (i) by the following more general axiom: (i'): If $a \leq b$ then $e \leq a \to b$.

Remark 7 Different versions of strong algebras are defined in the literature under many different names. Usually, the definitions use lattices and the meet structure as the monoidal structure, i.e., $\otimes = \wedge$ and $e = 1$. They also start with relatively more internalization axioms, including the internalization of finite meets and finite joins, as mentioned above. These algebras are the natural algebraic models for sub-intuitionistic logics. See for instance (Restall 1994); (Celani and Jansana 2005); (Ardeshir and Ruitenburg 2018); (Alizadeh and Ardeshir 2006a); (Alizadeh 2009); (Alizadeh and Ardeshir 2006b) and (Alizadeh and Ardeshir 2012) for the algebraic notions and (Ardeshir 1995); (Ardeshir and Ruitenburg 1998); (Alizadeh and Ardeshir 2004) and (Restall 1994) for their role in sub-intuitionistic logics.

Example 8 By a left residuated algebra we mean a monoidal poset $\mathcal{A} = (A, \leq, \otimes, e)$ with a binary operation $\Rightarrow$ such that $x \otimes y \leq z$ is equivalent to $y \leq x \Rightarrow z$, for all $x, y, z \in A$. As a special case, a finitely complete and finitely cocomplete left residuated algebra with the meet structure as its monoidal structure is called a Heyting algebra. Spelling out, a Heyting algebra is a finitely complete and finitely cocomplete poset $\mathcal{H} = (H, \leq, \wedge, \vee, 1, 0)$ with a binary operation $\Rightarrow$ such that $x \wedge y \leq z$ is equivalent to $y \leq x \Rightarrow z$, for all $x, y, z \in H$. It is clear that $\Rightarrow$ in any left residuated algebra is an implication. Note that if $\mathscr{X}$ is a quantale, then $(\mathscr{X}, \Rightarrow_{\mathscr{X}})$ is a left residuated algebra where $\Rightarrow_{\mathscr{X}}$ is the canonical implication of $\mathscr{X}$. Therefore, $\Rightarrow_{\mathscr{X}}$ is also an implication.

7.4.1 Constructing New Implications from the Old

There are some simple methods to make new implications from the old. Two of these methods play an important role in our future investigations. Here we will explain them. See also (Ardeshir and Ruitenburg 2018).

The First Method. For the first method, let $\mathcal{A} = (A, \leq, \otimes, e, \to)$ be a strong algebra and $F : A \to A$ be a monotone function (not necessarily lax or oplax). Then $\mathcal{A} = (A, \leq, \otimes, e, \to^F)$ where $a \to^F b = F(a) \to F(b)$ is a strong algebra. Since $\to$ is an implication, we have $e \leq F(a) \to F(a)$. The other axiom is trivial, because

$$(F(a) \to F(b)) \otimes (F(b) \to F(c)) \leq (F(a) \to F(c))$$

The Second Method. Let $\mathcal{A} = (A, \leq, \otimes, e, \to)$ be a strong algebra and let $G : A \to A$ be a lax monoidal map. Then the structure $\mathcal{A} = (A, \leq, \otimes, e, \to_G)$ where $a \to_G b = G(a \to b)$ is also a strong algebra. The reason is the following. Since $\to$ is an implication, then $e \leq a \to a$. Since G is monotone $G(e) \leq G(a \to a)$. Since G is lax we have $e \leq G(e)$ which implies $e \leq G(a \to a)$. For the second axiom, since G is lax and $\to$ is an implication, we have

$$G(a \to b) \otimes G(b \to c) \leq G((a \to b) \otimes (b \to c)) \leq G(a \to c)$$

Later in Theorem 8, we will prove a representation theorem to show that any implication is essentially the result of applying these two methods on the canonical implication of a quantale.

Example 9 Let $\mathcal{H} = (H, \leq, \wedge, \vee, 1, 0, \Rightarrow)$ be a Heyting algebra. Then for some $a \in H$, consider $M_a(x) = a \wedge x$ and $J_a : H \to H$ as $J_a(x) = a \vee x$. Then since M_a and J_a are monotone, the following operations are implications: $[x \to^{M_a} y = (x \wedge a \Rightarrow y \wedge a)]$ and $[x \to^{J_a} y = (x \vee a \Rightarrow y \vee a)]$.

Example 10 Let X be a topological space, $f : X \to X$ be a continuous function and $\mathcal{O}(X)$ be the locale of all open subsets of X. Since $f^{-1} : \mathcal{O}(X) \to \mathcal{O}(X)$ preserves all unions, by the adjoint functor theorem, Theorem 1, it has a right adjoint. Call it $g : \mathcal{O}(X) \to \mathcal{O}(X)$. Since g is a right adjoint, it preserves all meets. Therefore, it is lax monoidal. Therefore, the operation $U \to V = g(U \Rightarrow V)$, where $\Rightarrow$ is the Heyting implication on $\mathcal{O}(X)$ is an implication by the second construction.

Definition 9 Let $\mathcal{A} = (A, \leq, \otimes, e, \to)$ be a strong algebra. It internalizes its monoidal structure if for all $a, b, c \in A$:

$$a \to b \leq c \otimes a \to c \otimes b$$

$\mathcal{A}$ is called closed if it has the left residuation, i.e., the operation $\Rightarrow$ such that $a \otimes b \leq c$ iff $b \leq a \Rightarrow c$, for any $a, b, c \in A$. A strong algebra internalizes the closed monoidal structure if it is closed, it internalizes the monoidal structure and for all $a, b, c \in A$:

$$a \otimes b \to c \leq b \to (a \Rightarrow c)$$

Remark 8 For strong algebras for which the monoidal structure is the meet structure, internalizing the monoidal structure simply means $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$, for all $a, b, c \in A$. First note that we always have $a \to (b \wedge c) \leq (a \to b) \wedge (a \to c)$ because, $\to$ is order preserving in its second argument. Now, assume that $\mathcal{A}$ internalizes its monoidal structure, then we have

$$(a \to b) \leq (a \wedge a \to a \wedge b) \quad \text{and} \quad (a \to c) \leq (b \wedge a \to b \wedge c)$$

implying

$$(a \to b) \wedge (a \to c) \leq (a \wedge a \to a \wedge b) \wedge (b \wedge a \to b \wedge c) \leq (a \to b \wedge c)$$

Therefore, $(a \to b) \wedge (a \to c) \leq a \to (b \wedge c)$ and hence

$$a \to (b \wedge c) = (a \to b) \wedge (a \to c)$$

Conversely, since $c \wedge a \leq c$ we have $c \wedge a \to c = 1$. Moreover, $c \wedge a \leq a$ implies $(a \to b) \leq (c \wedge a) \to b$. Hence,

$$(a \to b) \leq [(c \wedge a) \to c] \wedge [(c \wedge a) \to b] = (c \wedge a \to c \wedge b)$$

Example 11 Let X be a set and $f : X \to X$ be a function. Consider $P(X)$, the poset of all subsets of X and $F : P(X) \to P(X)$ defined by $F(A) = f[A]$, where $f[A]$ is the image of A. Since F is monotone, $A \to^F B = F(A) \Rightarrow F(B)$ is an implication, where $\Rightarrow$ is the usual Boolean implication on $P(X)$. In a special case, if we choose X and f such that f is surjective and for some subsets of X such as A, B we have $f[A \cap B] \neq f[A] \cap f[B]$, then $\to^F$ does not internalize the monoidal structure (the meet) because,

$$[1 \to^F (A \cap B)] = [F(1) \Rightarrow F(A \cap B)] = F(A \cap B)$$

$$[(1 \to^F A) \cap (1 \to^F B)] = [(F(1) \Rightarrow F(A)) \cap (F(1) \Rightarrow F(B))] = [F(A) \cap F(B)]$$

are not equal. There are many such arrangements. For instance, take $X = \mathbb{N}$, $f(n) = \lfloor \frac{n}{2} \rfloor$ and $A = 2\mathbb{N}$ and $B = 2\mathbb{N} + 1$ as the set of even and odd natural numbers, respectively. Then $A \cap B = \emptyset$ and hence $f[A \cap B] = \emptyset$, while $0 \in f[A] \cap f[B]$. This example provides an implication that does not internalize the monoidal structure.

7.5 Non-commutative Spacetimes

As we have discussed in Sect. 7.3, quantales provide a natural formalization for a more subjective notion of intuitionistic proposition. However, to address the full intuitionistic picture, along the constructibility formalized by the order, we also need to formalize the independent notion of time. How can we formalize such a temporal structure? The answer is the modality ∇ that we introduced in the Introduction. Recall that ∇a must be read as the proposition "a hold at some point in the past".

Definition 10 A pair $\mathcal{S} = (\mathscr{X}, \nabla)$ is called a non-commutative spacetime if $\mathscr{X}$ is a quantale and $\nabla : \mathscr{X} \to \mathscr{X}$ is an oplax geometric map, i.e., a monotone and join preserving map such that $\nabla e \leq e$ and $\nabla(a \otimes b) \leq \nabla a \otimes \nabla b$, for all $a, b \in \mathscr{X}$. A non-commutative spacetime is called a spacetime if its monoidal structure is a meet structure. Spelling out, $\mathcal{S} = (\mathscr{X}, \nabla)$ is a spacetime if $\mathscr{X}$ is a locale and $\nabla : \mathscr{X} \to \mathscr{X}$ is just a join preserving map. Note that the oplax condition is a consequence of monotonicity of ∇ and the fact that 1 is the greatest element.

Remark 9 Our notion of spacetime is similar to dynamic topological spaces studied in (Kremer and Mints 2007). However, in spacetimes, we are interested in the combination of both adjoints rather than the $\Box$ as the right adjoint of ∇, alone. Moreover, we depart from topological spaces and the inverse image of continuous functions to quantales and oplax join preserving maps. The latter is extremely more general than the former.

Example 12 Assume that X is a topological space and $f : X \to X$ is a continuous function. Then $\mathcal{S} = (\mathcal{O}(X), f^{-1})$ is a spacetime where $\mathcal{O}(X)$ is the locale of the open subsets of X.

Example 13 By a *Kripke frame*, we mean a tuple $\mathcal{K} = (W, \leq, R)$ where $(W, \leq)$ is a poset and $R \subseteq W \times W$ is a relation compatible with the order $\leq$, meaning that for all $u, v, u', v' \in W$, if $(u, v) \in R$, $u' \leq u$ and $v \leq v'$ then $(u', v') \in R$. For any Kripke frame $\mathcal{K}$, define $\nabla_{\mathcal{K}} : U(W, \leq) \to U(W, \leq)$ as $\nabla_{\mathcal{K}}(U) = \{v \in W | \exists u \in U \;\; R(u, v)\}$ where $U(W, \leq)$ is the poset of all upsets of $(W, \leq)$. The map $\nabla_{\mathcal{K}}$ is trivially monotone and join preserving. For the latter, note that $w \in \bigcup_{i \in I} \nabla U_i$ iff $\exists i \in I \;\; (w \in \nabla U_i)$ iff

$$\exists i \in I \exists u \in W \;\; ((u, w) \in R \;\wedge (u \in U_i))$$

$$\text{iff} \;\; \exists u \in W \;\; (u \in \bigcup_{i \in I} U_i \wedge (u, w) \in R) \;\; \text{iff} \;\; w \in \nabla(\bigcup_{i \in I} U_i).$$

Therefore, $\mathcal{S}_{\mathcal{K}} = (U(W, \leq), \nabla_{\mathcal{K}})$ is a spacetime. Note that if we take equality $=_W$ for $\leq$, it transform any usual Kripke frame (W, R) with arbitrary R to a spacetime. Philosophically speaking, in an arbitrary Kripke frame, W can be interpreted as the set of the creative subject's mental states, $\leq$ as an encoding of the order on the knowledge content of states and R as an encoding of the order of time on the states. Note that by this interpretation, the compatibility condition between $\leq$ and R is nothing but the compatibility between knowledge and time.

Example 14 Let X be a set, $f : X \to X$ be a function and $P(X \times X)$ be the quantale of all binary relations over X. Consider $f_* : P(X \times X) \to P(X \times X)$ defined as $f_*(R) = \{(f(x), f(y)) | x, y \in X \text{ and } (x, y) \in R\}$. By Example 6, the map f_* is an oplax geometric morphism and hence $(P(X, X), f_*)$ is a non-commutative spacetime.

Example 15 Let $\mathcal{M} = (M, \otimes, e)$ be a monoid, $I(\mathcal{M})$ be the quantale of its ideals and $f : M \to M$ be an endomorphism. Consider $f_* : I(\mathcal{M}) \to I(\mathcal{M})$ defined as $f_*(I) = Mf[I]M = \{m \otimes f(i) \otimes n | i \in I, m, n \in M\}$. By Example 7, the map f_* is an oplax geometric morphism and hence $(I(\mathcal{M}), f_*)$ is a non-commutative spacetime.

Any non-commutative spacetime has its own canonical implication. It is constructible via the second method we have explained in Sect. 7.4.1. This implication is nothing but the usual implication, delayed by the passage of time. The main point of these canonical implications is the full adjunctions that they present. This means that the structure is complete enough to fully capture the behaviour of the implication. Throughout the rest of this paper, we will see how this completeness makes the non-commutative spacetimes and their implications extremely well-behaved.

Theorem 5 *Let $\mathcal{S} = (\mathscr{X}, \nabla)$ be a non-commutative spacetime. Then there exists an implication $\to_{\mathcal{S}} : \mathscr{X}^{op} \times \mathscr{X} \to \mathscr{X}$ such that*

$$a \otimes \nabla b \leq c \;\; \textit{iff} \;\; b \leq a \to_{\mathcal{S}} c$$

Proof Since $\mathscr{X}$ is a quantale and $\nabla : \mathscr{X} \to \mathscr{X}$ is a join preserving monotone map, by the adjoint functor theorem, Theorem 1, it has a right adjoint $\Box : \mathscr{X} \to \mathscr{X}$. Now, define $a \to_{\mathcal{S}} b = \Box(a \Rightarrow b)$ where $\Rightarrow$ is the canonical implication of $\mathscr{X}$. This map has the desired property since

$$a \otimes \nabla b \leq c \;\; \text{iff} \;\; \nabla b \leq a \Rightarrow c \;\; \text{iff} \;\; b \leq \Box(a \Rightarrow c)$$

Moreover, note that $\Box$ is the right adjoint of ∇. Therefore, since ∇ is oplax, by Theorem 2, $\Box$ must be lax monoidal and hence by the second construction method for implications, the operation $\to_{\mathcal{S}}$ must be an implication. □

It is worth defining an elementary version of the previous adjunction situation. This is similar to how Heyting algebras provide an elementary version of locales:

Definition 11 Let $(A, \leq, \otimes, e)$ be a monoidal poset and $\nabla : A \to A$ and $\to : A^{op} \times A \to A$ be two order preserving functions. Then the structure $\mathcal{A} = (A, \leq, \otimes, e, \nabla, \to)$ is called a temporal algebra if for any $a, b, c \in A$, we have $a \otimes \nabla b \leq c$ iff $b \leq a \to c$. A temporal algebra is called distributive if $(A, \leq, \otimes, e)$ is a distributive monoidal poset. A temporal algebra is called a left residuated algebra if ∇ is the identity map. A strong algebra $(A, \leq, \otimes, e, \to)$ is called a reduct of a temporal algebra if there exists $\nabla : A \to A$ such that $(A, \leq, \otimes, e, \nabla, \to)$ is a temporal algebra. And finally, if $\mathcal{A} = (A, \leq_A, \otimes_A, e_A, \nabla_A, \to_A)$ and $\mathcal{B} = (B, \leq_B, \otimes_B, e_B, \nabla_B, \to_B)$ are two temporal algebras, by a temporal algebra morphism we mean a strict monoidal map $f : A \to B$ that also preserves ∇ and $\to$, i.e., $f \nabla_A = \nabla_B f$ and $f((-) \to_A (-)) = f(-) \to_B f(-)$.

Interpreting a temporal algebra $\mathcal{A} = (A, \leq, \otimes, e, \nabla, \to)$ as the world of propositions and ∇a as "a happened at some point in the past", $a \to b$ must be interpreted as "a implies b at any point in the future." Therefore, it is reasonable to assume that the combination $\nabla(\to)$ forgets the temporal delay and provides a usual left residuation. This is almost true. It is almost, because ∇ is the approximate inverse of $\to$, namely its adjoint rather than its real inverse and hence $\nabla(\to)$ can not be the real identity but its best approximation. To make the adjunction pair a real inverse pair, it is enough to move from A to $\nabla[A] = \{\nabla a | a \in A\}$, as we will see in the next theorem. (See also Remark 2.) In this sense we can claim that a temporal algebra (with meet structure for the monoidal part) is a refined version of the usual left residual algebra (Heyting algebra).

Theorem 6 *Let $\mathcal{A} = (A, \leq, \otimes, e, \nabla, \to)$ be a temporal algebra and ∇ preserves all finite multiplications. Then, the structure $\nabla\mathcal{A} = (\nabla[A], \leq, \otimes, e, \nabla, \to')$ is a left residuated algebra where $\nabla[A] = \{\nabla x | x \in A\}$ and $a \to' b = \nabla(a \to b)$, for any $a, b \in \nabla[A]$. Moreover, if $\mathcal{A}$ is finitely complete (finitely cocomplete), so is $\nabla\mathcal{A}$. The same is also true for completeness. Finally, if the monoidal structure of $\mathcal{A}$ is the meet structure, $\nabla\mathcal{A}$ is a Heyting algebra.*

Proof Since ∇ preserves the monoidal structure, the set $\nabla[A]$ is closed under all finite multiplications. Therefore, the only thing to prove is the adjunction $a \otimes (-) \dashv (a \to' (-))$, for any $a \in \nabla[A]$. It means $a \otimes b \leq c$ iff $b \leq \nabla(a \to c)$, for all $a, b, c \in \nabla[A]$. From left to right, since $b \in \nabla[A]$, there exists $b' \in A$ such that $b = \nabla b'$. Since $a \otimes \nabla b' \leq c$ we have $b' \leq a \to c$ which implies $b = \nabla b' \leq \nabla(a \to c)$. From right to left, if $b \leq \nabla(a \to c)$ then $a \otimes b \leq a \otimes \nabla(a \to c) \leq c$.

Note that if A has also all (finite) joins or all (finite) meets, so does $\nabla[A]$. For joins, since ∇ has a right adjoint and preserves all joins, $\nabla[A]$ is closed under all (finite) joins. Therefore, $\nabla[A]$ has also all (finite) joins. For meet, the situation is a bit more complex. We will address the binary meet. The rest is similar. For any $a, b \in \nabla[A]$, we claim that $\nabla\Box(a \wedge b) \in \nabla[A]$ is the meet of a and b in $\nabla\mathcal{A}$. Because, $\nabla\Box(a \wedge b) \leq (a \wedge b) \leq a$ and similarly for b we also have $\nabla\Box(a \wedge b) \leq b$. If for some $c \in \nabla[A]$ we have $c \leq a$ and $c \leq b$, then $c \leq a \wedge b$ and hence $\nabla\Box c \leq \nabla\Box(a \wedge b)$. Since $c \in \nabla[A]$, there exists $c' \in A$ such that $c = \nabla c'$. Hence, $\nabla\Box c = \nabla\Box\nabla c' = \nabla c' = c$. Thus, $c \leq \nabla\Box(a \wedge b)$. $\square$

Definition 12 Let $\mathcal{S} = (\mathscr{X}, \nabla_{\mathcal{S}})$ and $\mathcal{T} = (\mathscr{Y}, \nabla_{\mathcal{T}})$ be two non-commutative spacetimes. By a geometric map $f : \mathcal{S} \to \mathcal{T}$, we mean a strict geometric morphism $f : \mathscr{X} \to \mathscr{Y}$ such that $f\nabla_{\mathcal{S}} = \nabla_{\mathcal{T}} f$. A geometric map is called logical if it also preserves the implication, i.e., $f[(-) \to_{\mathcal{S}} (-)] = f(-) \to_{\mathcal{T}} f(-)$.

Example 16 Let $\mathcal{K} = (W, =_W, R)$ and $\mathcal{L} = (V, =_V, S)$ be two Kripke frames. A map $p : W \to V$ is called a p-morphism if $(u, v) \in R$ implies $(p(u), p(v)) \in S$, for any $u, v \in W$ and for any $w \in W$ and $s, t \in V$ if $p(w) = s$ and $(s, t) \in S$, then there exists $u \in W$ such that $p(u) = t$ and $(w, u) \in R$. A map $p : W \to V$ is a p-morphism iff $p^{-1} : \mathcal{S}_{\mathcal{L}^{op}} \to \mathcal{S}_{\mathcal{K}^{op}}$ is a geometric morphism, where $\mathcal{K}^{op} = (W, R^{op})$ and $(v, u) \in R^{op}$ iff $(u, v) \in R$ and similarly for $\mathcal{L}$. We only prove the left to right direction. The other direction is similar. First note that p^{-1} preserves all unions and all finite intersections. Therefore, the only thing we have to prove is the preservability of ∇, i.e., $p^{-1}\nabla_{\mathcal{L}^{op}} = \nabla_{\mathcal{K}^{op}} p^{-1}$. Let U be a subset of V. Then if $u \in \nabla_{\mathcal{K}^{op}} p^{-1}(U)$, then there exists $w \in W$ such that $(w, u) \in R^{op}$ or equivalently $(u, w) \in R$ and $p(w) \in U$. Since p is a p-morphism we have $(p(u), p(w)) \in S$ which means $(p(w), p(u)) \in S^{op}$. Hence, $p(u) \in \nabla_{\mathcal{L}^{op}}(U)$ which implies $u \in p^{-1}\nabla_{\mathcal{L}^{op}}(U)$. Conversely, if $u \in$

$p^{-1}\nabla_{\mathcal{L}^{op}}(U)$, we have $p(u) \in \nabla_{\mathcal{L}^{op}}(U)$ from which, there exists $v \in U$ such that $(v, p(u)) \in S^{op}$ or equivalently $(p(u), v) \in S$. Since p is a p-morphism, there exists $w \in W$ such that $p(w) = v$ and $(u, w) \in R$. Hence, $(w, u) \in R^{op}$ from which, $u \in \nabla_{\mathcal{K}^{op}}(p^{-1}(U))$.

Theorem 7 *Let $\mathcal{S}$ and $\mathcal{T}$ be two non-commutative spacetimes and $f : \mathcal{S} \to \mathcal{T}$ be a geometric morphism with a left adjoint $f_!$. Then f is logical iff $f_!(fb \otimes \nabla_{\mathcal{T}} a) = b \otimes \nabla_{\mathcal{S}} f_! a$.*

Proof Using the adjunctions $x \otimes \nabla_{\mathcal{T}}(-) \dashv x \to_{\mathcal{T}} (-)$, $y \otimes \nabla_{\mathcal{S}}(-) \dashv y \to_{\mathcal{S}} (-)$ and $f_! \dashv f$ we have

$$f_!(fb \otimes \nabla_{\mathcal{T}} a) \leq c \;\text{ iff }\; fb \otimes \nabla_{\mathcal{T}} a \leq fc \;\text{ iff }\; a \leq fb \to_{\mathcal{T}} fc$$

and

$$b \otimes \nabla_{\mathcal{S}} f_! a \leq c \;\text{ iff }\; f_! a \leq b \to_{\mathcal{S}} c \;\text{ iff }\; a \leq f(b \to_{\mathcal{S}} c)$$

These equivalences imply exactly what we wanted. Because, if $f_!(fb \otimes \nabla_{\mathcal{T}} a) = b \otimes \nabla_{\mathcal{S}} f_! a$, then the left hand sides of the above lines are equivalent which implies the equivalence of the right hand sides from which $fb \to_{\mathcal{T}} fc = f(b \to_{\mathcal{S}} c)$. The converse is similar. □

Sometimes, it would be reasonable to investigate the pure spatial behaviour of a non-commutative space, meaning the properties that hold for *all* possible time structures or more formally *all* possible ∇'s over a fixed space. The following corollary provides a method to transfer these properties along certain geometric morphisms. We will use this corollary when we have a suitable syntax for non-commutative spacetimes to formally address what we mean by a "property".

Corollary 1 *Let $\mathcal{S} = (\mathscr{X}, \nabla_{\mathcal{S}})$ be a non-commutative spacetime, $\mathscr{Y}$ be a quantale and $f : \mathscr{X} \to \mathscr{Y}$ be a strict geometric embedding with a left adjoint $f_!$. Then there exists ∇ on $\mathscr{Y}$ such that $\mathcal{T} = (\mathscr{Y}, \nabla)$ is a non-commutative spacetime and $f : \mathcal{S} \to \mathcal{T}$ is a logical morphism.*

Proof Define $\nabla = f \nabla_{\mathcal{S}} f_!$. Since $f_!$ is a left adjoint and both f and $\nabla_{\mathcal{S}}$ preserves all joins, the operator ∇ is also join preserving. Moreover, since f is strict monoidal, its left adjoint, $f_!$ is oplax, by Theorem 2. Therefore, ∇ as a composition of three oplax monoidal maps is also oplax. To prove the geometricity of $f : (\mathscr{X}, \nabla_{\mathcal{S}}) \to (\mathscr{Y}, \nabla)$, since $f_! \dashv f$, by Remark 2, $f f_! f = f$. Since f is an embedding we have $f_! f = id$. Therefore, $\nabla f = f \nabla_{\mathcal{S}} f_! f = f \nabla_{\mathcal{S}}$. Hence $f : (\mathscr{X}, \nabla_{\mathcal{S}}) \to (\mathscr{Y}, \nabla)$ is geometric. To prove it is logical, by Theorem 7 we have to show that $f_!(\nabla a \otimes fb) = f_!(f \nabla_{\mathcal{S}} a \otimes fb)$. Since f is strict monoidal, the right hand side is equivalent to $f_! f(\nabla_{\mathcal{S}} a \otimes b)$. Since $f_! f = id$, the latter is equivalent to $\nabla_{\mathcal{S}} f_! a \otimes b$. Hence, the geometric map $f : (\mathscr{X}, \nabla_{\mathcal{S}}) \to (\mathscr{Y}, \nabla)$ is logical. □

Corollary 1 is useful in case the quantales are the open posets of topological spaces and the space for $\mathscr{X}$ is an Alexandroff space. Recall that a topological space is Alexandroff if any arbitrary intersection of its open subsets is also open.

Corollary 2 *Let X be a topological space, Y be an Alexandroff space, $f : X \to Y$ be a continuous surjection and $\mathcal{S} = (\mathcal{O}(Y), \nabla_Y)$ be a spacetime. Then there exists $\nabla_X : \mathcal{O}(X) \to \mathcal{O}(X)$ such that $\mathcal{T} = (\mathcal{O}(X), \nabla_X)$ is a spacetime and $f^{-1} : \mathcal{S} \to \mathcal{T}$ is a logical morphism.*

Proof Since the space Y is Alexandroff, $\mathcal{O}(Y)$ is closed under all intersections. Therefore, since f^{-1} preserves arbitrary intersections it also preserves arbitrary meets. Hence, by the adjoint functor theorem, Theorem 1, it has a left adjoint $f_!$. Moreover, note that $f : X \to Y$ is surjective which means that $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ is an embedding. Hence, it is enough to use Corollary 1. □

7.6 Representation Theorems

In this section we will present some quantale-based representations for different classes of strong algebras. The main motive is embedding an abstract strong algebra in a quantale in a way that the implication presents a possible well-behaved left adjunction. We call this process *resolving the implication*. In a technical sense, these left adjoints make the implications easier to handle as it is usual all over mathematics. However, resolutions have a very philosophical role, as well. We know that adjunctions are the algebraic term for the usual proof theoretical situation in which we have a pair of introduction and elimination rules for a logical connective that we try to capture. For instance, think about the intuitionistic implication and its natural deduction rules. Following Gentzen, a connective is fully captured if it enjoys a pair of introduction and elimination rules. In this sense, resolving an implication is an attempt to fully identify an abstract implication as a logical connective.

Having all said, resolving all the implications is unfortunately impossible. We will explain the reason later in this section. We will also see some necessary and partially sufficient conditions to make resolutions possible. But first let us begin by a general yet weak resolution-type result. We will prove that any strong algebra is embeddable in a quantale equipped with an implication. The implication is not necessarily a non-commutative spacetime implication but it is a substitution of it. We can think of the implication as the result of the application of the two construction methods that we explained before, applied on the canonical implication of the quantale.

Let $\mathcal{A} = (A, \leq, \otimes, e, \to)$ be a strong algebra. A priory, there is no reason to assume that the structure $\mathcal{A}$ has the power (enough elements or structure) to resolve the implication and find an adjunction-type situation. However, if we extend the domain to also include the *relative elements*, meaning the monotone functions $A^{op} \to A$, then we can provide the following characterization for the implication:

$$c \leq a \to b \quad \text{iff} \quad (x \to a) \otimes c \leq (x \to b)$$

where x is a variable and the right-hand side consists of the functions for which the order and the monoidal structure are both defined pointwise. The reason is simple. From left to right, note that

$$c \leq a \to b \quad \text{implies} \quad (x \to a) \otimes c \leq (x \to a) \otimes (a \to b) \leq (x \to b)$$

and from right to left, it is enough to put $x = a$ to have

$$c = e \otimes c \leq (a \to a) \otimes c \leq a \to b$$

Note that while this adjunction-type characterization handles all the elements of A, it can not handle the functions that it adds. To solve this problem we simply need infinitely many of such variables:

Theorem 8 *For any strong algebra $\mathcal{A}$ there exists a non-commutative spacetime $\mathcal{S} = (\mathscr{X}, \nabla)$, a monotone map $F : \mathscr{X} \to \mathscr{X}$ and a strict monoidal embedding $i : \mathcal{A} \to \mathscr{X}$ such that $i(a \to_{\mathcal{A}} b) = F(i(a)) \to_{\mathcal{S}} F(i(b))$.*

Proof Define E as the set of all monotone functions $f : (\Pi_{n\in\mathbb{N}} A^{op}) \to A$ with finite support, i.e., all order preserving maps that depend only on some finitely many of their arguments. Define $\leq_E$ as the pointwise order on E and use $\otimes_E$ and e_E to represent the pointwise monoidal structure of E. Then the structure $\mathcal{E} = (E, \leq_E, \otimes_E, e_E)$ is clearly a monoidal poset. Define $j : \mathcal{A} \to \mathcal{E}$ by mapping any element $a \in A$ to the constant function with the value a. Since the structure of $\mathcal{E}$ is defined pointwise, it is clear that j is a strict monoidal embedding.

Define the shift map $r : \Pi_{n\in\mathbb{N}} A^{op} \to \Pi_{n\in\mathbb{N}} A^{op}$ by $r(\langle a_n \rangle_{n=0}^{\infty}) = \langle a_{n+1} \rangle_{n=0}^{\infty}$. Then define $s : E \to E$ as the coordinate shift map induced by r, i.e., $s(f) = f \circ r$. Spelling out, s sends the function $f(\langle x_n \rangle_{n=0}^{\infty})$ to $f(\langle x_{n+1} \rangle_{n=0}^{\infty})$. This map is clearly strictly monoidal. Moreover, define $l : E \to E$ mapping $f(\langle x_n \rangle_{n=0}^{\infty}) \mapsto (x_0 \to f(\langle x_{n+1} \rangle_{n=0}^{\infty}))$. Now use the downset completion on $\mathcal{E} = (E, \leq_E, \otimes_E, e_E)$ to construct our $\mathscr{X}$. Let $k : \mathcal{E} \to \mathscr{X}$ be the canonical strict monoidal embedding from the downset completion. Define

$$\nabla I = s_! = \{f \in E | \exists g \in I \ (f \leq_E s(g))\}$$

and

$$F(I) = \{f \in E | \exists g \in I \ [f \leq_E l(s(g))]\}$$

The map F is clearly monotone and mapping downsets to downsets. By Theorem 4, ∇ preserves the joins and it is oplax because s is oplax. Therefore, $\mathcal{S} = (\mathscr{X}, \nabla)$ is a non-commutative spacetime. We claim that $i = kj : \mathcal{A} \to \mathscr{X}$ is the strict monoidal embedding that we are looking for. The only thing to prove is that $i(a \to_{\mathcal{A}} b) = F(i(a)) \to_{\mathcal{S}} F(i(b))$. To prove that, we show

$$J \subseteq i(a \to b) \ \text{ iff } \ F(i(a)) \otimes \nabla J \subseteq F(i(b)), \qquad (*)$$

for any downset J of E. First to simplify the proof, note that for any $c \in A$

$$f \in F(i(c)) \quad \text{iff} \quad f \leq (x_0 \to c)$$

The reason is that $f \in F(i(c))$ iff there exists a function $f' \in i(c)$ such that $f \leq (x_0 \to s(f'))$. This is equivalent to $f \leq (x_0 \to c)$.

Now to prove $(*)$, for left to right, if $J \subseteq i(a \to b)$ and $f \in F(i(a)) \otimes \nabla J$ by the definition of the multiplication on downsets, there exist $g \in F(i(a))$ and $h \in \nabla J$ such that $f \leq g \otimes_E h$. By the above point, since $g \in F(i(a))$ we have $g \leq (x_0 \to a)$. By definition of ∇ there exists $h' \in J$ such that $h \leq s(h')$. Since $h' \in J \subseteq i(a \to b)$ we have $h' \leq a \to b$ and hence $h \leq s(a \to b) = a \to b$. Therefore, $g \otimes_E h \leq (x_0 \to a) \otimes_E (a \to b) \leq x_0 \to b$. Therefore, by the above-mentioned point we have $g \otimes_E h \in F(i(b))$ and since $f \leq g \otimes_E h$ and $F(i(b))$ is a downset, we have $f \in F(i(b))$. For the converse, assume $F(i(a)) \otimes \nabla J \subseteq F(i(b))$ and we want to show that $J \subseteq i(a \to b)$. Assume $f \in J$. Then by the definition of ∇, we have $s(f) \in \nabla J$. Moreover, by the above mentioned point we have $(x_0 \to a) \in F(i(a))$. Hence, $(x_0 \to a) \otimes_E s(f) \in F(i(a)) \otimes_{\mathscr{X}} \nabla J$. Therefore, $(x_0 \to a) \otimes_E s(f) \in F(i(b))$. Hence, $(x_0 \to a) \otimes_E s(f) \leq (x_0 \to b)$. Since the order of E is pointwise, put $x_0 = a$ and keep the other variables intact. Since $s(f)$ does not depend on x_0, it does not change after the substitution. Hence, $(a \to a) \otimes_E s(f) \leq (a \to b)$. Since $e \leq a \to a$, we have $s(f) \leq a \to b = s(a \to b)$. Since s is an embedding, $f \leq a \to b$ and hence $f \in i(a \to b)$. □

Although, the previous theorem provides a weak resolution for any abstract implication, it can only resolve it up to a factor F which breaks the full adjunction situation. This F is inevitable, simply because it is impossible to embed any implication into a non-commutative spacetime. The reason is that for any non-commutative spacetime $\mathcal{S} = (\mathscr{X}, \nabla)$, its implication, $\to_{\mathcal{S}}$, internalizes the closed monoidal structure of $\mathcal{S}$, i.e., for all $a, b, c \in \mathscr{X}$ we have

$$a \to_{\mathcal{S}} b \leq c \otimes a \to_{\mathcal{S}} c \otimes b$$

because by the associativity and the adjunction

$$c \otimes a \otimes \nabla(a \to_{\mathcal{S}} b) \leq c \otimes b$$

Therefore, if we seek an embedding into a non-commutative spacetime we have to restrict our domain to the implications that internalize their monoidal structure. Unfortunately, we do not know if this necessary condition is also sufficient. However, if the multiplication has left residuation and the implication internalizes the closed monoidal structure, we will have the following representation. Here our main ingredient is the ternary frames introduced first in (Routley and Meyer 1973) as the Kripke models for the relevant logics.

Theorem 9 *For any strong algebra $\mathcal{A}$ whose multiplication has left residual and $\mathcal{A}$ internalizes its closed monoidal structure, there exists a non-commutative spacetime $\mathcal{S} = (\mathscr{X}, \nabla)$ and a strong algebra embedding $i : \mathcal{A} \to \mathcal{S}$.*

Proof Recall that $U(\mathcal{A})$ is the poset of all upsets of $\mathcal{A}$ with inclusion. Define $\mathcal{R}$ as a ternary relation over $U(\mathcal{A})$ as: $(P, Q, R) \in \mathcal{R}$ iff for all $a, b \in A$ if $a \to b \in P$ and $a \in Q$ then $b \in R$. Note that the relation $\mathcal{R}$ is order-reversing in its first two arguments while it is order preserving in its third argument. Consider $\mathscr{X} = U(U(\mathcal{A}))$ and $i : \mathcal{A} \to \mathscr{X}$ by defining $i(a) = \{P \in U(\mathcal{A}) |\ a \in P\}$. As we have observed in Preliminaries, this i is clearly a strict monoidal embedding. Our strategy is first defining an implication on $\mathscr{X}$ and showing how i maps the implication of $\mathcal{A}$ to this implication and then finding an oplax ∇ such that $X \otimes \nabla(-) \dashv (X \to (-))$ for any $X \in \mathscr{X}$.

For any upsets of $U(\mathcal{A})$ such as X and Y define $X \to Y$ as:

$$\{P \in U(\mathcal{A}) |\ \forall Q, R \in U(\mathcal{A}),\ \text{if } (P, Q, R) \in \mathcal{R} \text{ and } Q \in X \text{ then } R \in Y\}$$

Since $\mathcal{R}$ is order-reversing in its first argument, $X \to Y$ is an upset. To prove that i maps the implication of $\mathcal{A}$ into this implication, i.e., $i(a \to b) = i(a) \to i(b)$, we need to address the following two directions:

For $i(a \to b) \subseteq i(a) \to i(b)$, if $P \in i(a \to b)$ then $a \to b \in P$. To show that $P \in i(a) \to i(b)$, assume for some $Q, R \in U(\mathcal{A})$ we have $(P, Q, R) \in \mathcal{R}$ and $Q \in i(a)$. Then by the definition of i we have $a \in Q$ and since $a \to b \in P$, by the definition of $\mathcal{R}$ we have $b \in R$ implying $R \in i(b)$. Conversely, for $i(a) \to i(b) \subseteq i(a \to b)$, if $P \in i(a) \to i(b)$ define $Q = \{x \in A | x \geq a\}$ and $R = \{y \in A | a \to y \in P\}$. We have $(P, Q, R) \in \mathcal{R}$ because if $x \to y \in P$ and $x \in Q$ then $x \geq a$ and hence $x \to y \leq a \to y$ which implies $a \to y \in P$. Therefore, by definition $y \in R$. Finally, Since $a \in Q$ we have $Q \in i(a)$ and since $(P, Q, R) \in \mathcal{R}$ we have $R \in i(b)$ which implies $b \in R$. By definition of R it means that $a \to b \in R$.

To complete the proof, we have to introduce an oplax ∇ and show that for any upsets of $U(\mathcal{A})$ such as X, Y, Z we have $X \subseteq Y \to Z$ iff $Y \otimes \nabla X \subseteq Z$. Define ∇ as:

$$\nabla X = \{R \in U(\mathcal{A}) |\ \exists P, Q \in U(\mathcal{A})\ [(P, Q, R) \in \mathcal{R}, (P \in X) \text{ and } (e \in Q)]\}$$

Since $\mathcal{R}$ is order-preserving in its third argument, ∇ is an upset. To prove the adjunction condition and the fact that it is oplax, we need a claim first:

(i) For any upsets $P, Q, R, S \in U(\mathcal{A})$, if $(P, Q, R) \in \mathcal{R}$ then $(P, S \otimes Q, S \otimes R) \in \mathcal{R}$.
(ii) For any upsets $P, Q, R \in U(\mathcal{A})$, if $(P, Q, R) \in \mathcal{R}$ then $(P, E, Q \Rightarrow R) \in \mathcal{R}$ where $E = \{x \in A | x \geq e\}$ and $\Rightarrow$ is the canonical implication of the quantale $U(\mathcal{A})$.
(iii) For any upsets $P_1, P_2, Q, R \in U(\mathcal{A})$, if $(P_1 \otimes P_2, Q, R) \in \mathcal{R}$ and $e \in Q$, then there are upsets $Q_1, Q_2, R_1, R_2 \in U(\mathcal{A})$ such that $e \in Q_1$, $e \in Q_2$, $(P_1, Q_1, R_1) \in \mathcal{R}$, $(P_2, Q_2, R_2) \in \mathcal{R}$ and $R_1 \otimes R_2 \subseteq R$.

Proof of the Claim. For (i), if $x \to y \in P$ and $x \in S \otimes Q$ then there are $z \in S, w \in Q$ such that $x \geq z \otimes w$. Since $x \geq z \otimes w$ and $x \to y \in P$ we have $z \otimes w \to y \in P$. Since, $\mathcal{A}$ internalizes its closed monoidal structure we have

$$z \otimes w \to x \leq w \to (z \Rightarrow_A x)$$

where $\Rightarrow_A$ is the left residual of multiplication in $\mathcal{A}$. Since P is an upset, $w \to (z \Rightarrow_A x) \in P$. Since $(P, Q, R) \in \mathcal{R}$ and $w \in Q$ we have $z \Rightarrow_A x \in R$. Since $z \in S$ and $z \otimes (z \Rightarrow_A x) \leq x$ we have $x \in S \otimes R$.

For (ii), assume $x \to y \in P$ and $x \geq e$ then we have to show that $y \in Q \Rightarrow R$. Equivalently, it means $Y \subseteq Q \Rightarrow R$ where $Y = \{x \in A | x \geq y\}$. The latter is equivalent to $Q \otimes Y \subseteq R$ because $\Rightarrow$ is the left residual in $U(\mathcal{A})$. Assume $z \in Q \otimes Y$. Therefore, there exist $w \in Q$ and $u \geq y$ such that $z \geq w \otimes u$ implying $z \geq w \otimes y$. Since $\mathcal{A}$ internalizes its monoidal structure, we have

$$x \to y \leq w \otimes x \to w \otimes y$$

Hence, $w \otimes x \to w \otimes y \in P$. Since $e \leq x$ we have $w = w \otimes e \leq w \otimes x$. By $w \in Q$ we have $w \otimes x \in Q$. Since $(P, Q, R) \in \mathcal{R}$ we have $w \otimes y \in R$. Since $z \geq w \otimes y$ we conclude $z \in R$ that completes the proof.
To prove (iii), if $(P_1 \otimes P_2, Q, R) \in \mathcal{R}$ and $e \in Q$, then define $Q_1 = Q_2 = \{x \in A | x \geq e\}$ and $R_i = \{x \in A | e \to x \in P_i\}$ for $i \in \{1, 2\}$. By definition it is clear that $(P_1, Q_1, R_1) \in \mathcal{R}$ and $(P_2, Q_2, R_2) \in \mathcal{R}$, because if $u \to v \in P_i$ and $u \geq e$ then $e \to v \in P_i$ which by definition means $v \in R_i$. Finally, to prove $R_1 \otimes R_2 \subseteq R$, assume $z \in R_1 \otimes R_2$. Therefore, there are $x \in R_1$ and $y \in R_2$ such that $z \geq x \otimes y$. Since $x \in R_1$ and $y \in R_2$ we have $e \to x \in P_1$ and $e \to y \in P_2$. Therefore, $(e \to x) \otimes (e \to y) \in P_1 \otimes P_2$. Since $\mathcal{A}$ internalizes its monoidal structure we have

$$e \to y \leq (x \otimes e \to x \otimes y) = (x \to x \otimes y)$$

Therefore,

$$(e \to x) \otimes (e \to y) \leq (e \to x) \otimes (x \to x \otimes y) \leq e \to x \otimes y$$

Hence, $e \to x \otimes y \in P_1 \otimes P_2$. Since $e \in Q$ and $(P, Q, R) \in \mathcal{R}$ we have $x \otimes y \in R$ and since $z \geq x \otimes y$ we have $z \in R$.

Now let us come back to prove that ∇ is a join preserving oplax map. We have to show that $\nabla i(e) \subseteq i(e)$ and for any upsets of $U(\mathcal{A})$ such as X, Y we have $\nabla(X \otimes Y) \subseteq \nabla X \otimes \nabla Y$. For the first one, if $R \in \nabla i(e)$, by definition there exist upsets P and Q such that $(P, Q, R) \in \mathcal{R}$, $e \in Q$ and $P \in i(e)$. Therefore, $e \in P$. Since $e \leq e$, we have $e \leq e \to e$. Since P is an upset we have $e \to e \in P$. Then since $e \in Q$ and $(P, Q, R) \in \mathcal{R}$ we have $e \in R$ which means that $R \in i(e)$. For $\nabla(X \otimes Y) \subseteq \nabla X \otimes \nabla Y$, assume $R \in \nabla(X \otimes Y)$ then again by definition there exist upsets $P \in X \otimes Y$ and Q such that $e \in Q$ and $(P, Q, R) \in \mathcal{R}$. Since $P \in X \otimes Y$, there are $P_1 \in X$ and $P_2 \in Y$ such that $P_1 \otimes P_2 \subseteq P$. Since $\mathcal{R}$ is order reversing in its first argument and $(P, Q, R) \in \mathcal{R}$ we have $(P_1 \otimes P_2, Q, R) \in \mathcal{R}$. By the part (iii) of the claim, there are upsets Q_1, Q_2, R_1, R_2 such that $e \in Q_1$, $e \in Q_2$, $(P_1, Q_1, R_1) \in \mathcal{R}$ and $(P_2, Q_2, R_2) \in \mathcal{R}$ and $R_1 \otimes R_2 \subseteq R$. Hence, by definition $R_1 \in \nabla X$ and $R_2 \in \nabla Y$ and since $R_1 \otimes R_2 \subseteq R$ we have $R \in \nabla X \otimes \nabla Y$. Therefore, $\nabla(X \otimes Y) \subseteq \nabla X \otimes \nabla Y$.

For the adjunction conditions, i.e., $X \subseteq Y \to Z$ iff $Y \otimes \nabla X \subseteq Z$, we need to address the following two directions. For left to right, if $X \subseteq Y \to Z$ and $P \in Y \otimes \nabla X$ we have to show that $P \in Z$. Since $P \in Y \otimes \nabla X$, by definition there exist Q, R such that $Q \otimes R \subseteq P$ and $Q \in Y$ and $R \in \nabla X$. Again by definition since $R \in \nabla X$ there exist P', Q' such that $(P', Q', R) \in \mathcal{R}$, $P' \in X$ and $e \in Q'$. Since $e \in Q'$ for any $q \in Q$ we have $q = q \otimes e \in Q \otimes Q'$. Therefore, $Q \subseteq Q \otimes Q'$. Since $Q \in Y$ we have $Q \otimes Q' \in Y$. Since $(P', Q', R) \in \mathcal{R}$ by the part (i) of the Claim, we have $(P', Q \otimes Q', Q \otimes R) \in \mathcal{R}$ and since $P' \in X \subseteq Y \to Z$ and $Q \otimes Q' \in Y$, we have $Q \otimes R \in Z$. Finally since Z is an upset and $Q \otimes R \subseteq P$ we have $P \in Z$.

For right to left, if $Y \otimes \nabla X \subseteq Z$ and $P \in X$ we want to show that $P \in Y \to Z$. Pick Q and R such that $(P, Q, R) \in \mathcal{R}$ and $Q \in Y$. We have to show that $R \in Z$. By the part (ii) of the Claim, since $(P, Q, R) \in \mathcal{R}$ we have $(P, E, Q \Rightarrow R) \in \mathcal{R}$ where $e \in E$. Hence, by definition of ∇, we have $Q \Rightarrow R \in \nabla X$ and hence $Q \otimes (Q \Rightarrow R) \in Y \otimes \nabla X$. Since $Y \otimes \nabla X \subseteq Z$ we have $Q \otimes (Q \Rightarrow R) \in Z$. Finally, since $Q \otimes (Q \Rightarrow R) \subseteq R$ and Z is an upset we have $R \in Z$. □

Fortunately, if the monoidal structure is just the meet structure, it is possible to show that the internalization of the monoidal structure is sufficient for resolution. Moreover, it is possible to show that the quantale is actually a locale or even better an Alexandroff space:

Theorem 10 *For any (distributive) strong algebra $\mathcal{A} = (A, \leq, \wedge, 1, \to)$ that internalizes its monoidal structure [not necessarily its closed structure if it has any] (and its join structure), there exists a Kripke frame $\mathcal{K}$ and a (join preserving) strong algebra embedding $i : \mathcal{A} \to \mathcal{S}_{\mathcal{K}}$. Moreover, if $\mathcal{A}$ is a reduct of a (distributive) temporal algebra, i also preserves ∇.*

Proof See Theorem 17. □

And finally, in case that we already have a nice left adjoint for the implication, it is possible to make the algebra cocomplete, preserving the temporal structure. This will be useful in topological completeness theorem, Theorem 14.

Theorem 11 *Let $\mathcal{A} = (A, \leq, \otimes, e, \nabla, \to)$ be a (distributive) temporal algebra. Then there exists a non-commutative spacetime $\mathcal{S} = (\mathcal{X}, \nabla)$ and a (join preserving) temporal algebra embedding $i : \mathcal{A} \to \mathcal{S}$. Moreover, if $\mathcal{A}$ has all finite meets, then i also preserves them.*

Proof See Theorem 13. □

7.7 Logics of Spacetime

In the previous section we presented some methods to represent some classes of implications via a diamond-type modality ∇, encoding the abstract notion of time. In this section we bring the adjunction into the syntax of logic to provide a more expressible language to address non-standard weak implications. Later, we will see how this new language provides a conservative extension for some weak implication logics including Visser-Ruitenburg's basic logic, introduced in (Visser 1981b) and (Ruitenburg 1991). However, the fully captured implications of these new logics make the non-standard implications more suitable for foundational studies. We will present an embedding of a fragment of full Lambek calculus, (Galatos et al. 2007), i.e., $\{\top, \bot, \wedge, \vee, \otimes, 1, \backslash\}$ into our logic and full intuitionistic logic into our logic equipped with the structural rules. Therefore, the logics of spacetime can be interpreted as a unification of sub-structural and sub-intuitionist logics.

Let $\mathcal{L}_\nabla$ be the usual language of propositional logic equipped with a new unary modal operator ∇. To introduce some formal systems in this language, consider the following set of sequent-style rules in which the left side of a sequent is a sequence of formulas and if $\Gamma = \langle A_i \rangle_{i=0}^n$ by $\nabla\Gamma$ we mean $\langle \nabla A_i \rangle_{i=0}^n$:

Axioms:

$$\frac{}{A \Rightarrow A} \quad \frac{}{\Rightarrow 1} \quad \frac{}{\nabla 1 \Rightarrow 1} \quad \frac{}{\Gamma \Rightarrow \top} \quad \frac{}{\Gamma, \bot, \Sigma \Rightarrow A}$$

Cut:

$$\frac{\Gamma \Rightarrow A \qquad \Pi, A, \Sigma \Rightarrow B}{\Pi, \Gamma, \Sigma \Rightarrow B}\, cut$$

Conjunction Rules:

$$\frac{\Gamma, A, \Sigma \Rightarrow C}{\Gamma, A \wedge B, \Sigma \Rightarrow C}\, L\wedge \quad \frac{\Gamma, B, \Sigma \Rightarrow C}{\Gamma, A \wedge B, \Sigma \Rightarrow C}\, L\wedge \quad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\, R\wedge$$

Disjunction Rules:

$$\frac{\Gamma, A, \Sigma \Rightarrow C \qquad \Gamma, B, \Sigma \Rightarrow C}{\Gamma, A \vee B, \Sigma \Rightarrow C}\, L\vee \quad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\, R\vee \quad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}\, R\vee$$

Rule for 1:

$$\frac{\Gamma, \Sigma \Rightarrow A}{\Gamma, 1, \Sigma \Rightarrow A}\, L1$$

Multiplication Rules:

$$\frac{\Gamma, A, B, \Sigma \Rightarrow C}{\Gamma, A \otimes B, \Sigma \Rightarrow C}\, L\otimes \quad \frac{\Gamma \Rightarrow A \qquad \Sigma \Rightarrow B}{\Gamma, \Sigma \Rightarrow A \otimes B}\, R\otimes$$

Modal Rules:

$$\frac{A \Rightarrow B}{\nabla A \Rightarrow \nabla B}\ \nabla \qquad \frac{\nabla A, \nabla B \Rightarrow C}{\nabla(A \otimes B) \Rightarrow C}\ Oplax$$

Implication Rules:

$$\frac{\Gamma \Rightarrow A \qquad \Pi, B, \Sigma \Rightarrow C}{\Pi, \Gamma, \nabla(A \to B), \Sigma \Rightarrow C}\ L\to \qquad \frac{A, \nabla\Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}\ R\to$$

Now define the logic of spacetime, **STL**, as the logic of the proof system consisting of all the axioms, cut and propositional rules. The provability of a sequent $\Gamma \Rightarrow A$ in **STL** is denoted by $\mathbf{STL} \vdash \Gamma \Rightarrow A$ or $\Gamma \vdash_{\mathbf{STL}} A$.

By the basic rule schemes $\{N, H, P, F, wF\}$, we mean one of the following schemes:

Rule Schemes:

$$\frac{\Gamma \Rightarrow A}{\nabla\Gamma \Rightarrow \nabla A}\ N \quad \frac{\Gamma \Rightarrow \nabla A}{\Gamma \Rightarrow A}\ P \quad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \nabla A}\ F \quad \frac{\nabla A \Rightarrow \bot}{A \Rightarrow \bot}\ wF$$

$$\frac{\Gamma, \{A_i \to B_i\}_{i \in I} \Rightarrow C}{\nabla\Gamma, \{\nabla A_i \to \nabla B_i\}_{i \in I} \Rightarrow \nabla C}\ H$$

Also consider the structural rules:

Structural Rules:

$$\frac{\Gamma, \Sigma \Rightarrow B}{\Gamma, A, \Sigma \Rightarrow B}\ Lw \quad \frac{\Gamma, A, A, \Sigma \Rightarrow B}{\Gamma, A, \Sigma \Rightarrow B}\ Lc \quad \frac{\Gamma, A, B, \Sigma \Rightarrow C}{\Gamma, B, A, \Sigma \Rightarrow C}\ Le$$

For any $\mathcal{R} \subseteq \{N, H, P, F, wF\}$, by the logic $\mathbf{STL}(\mathcal{R})$ we mean the logic of all rules of **STL** plus the rules of $\mathcal{R}$. By $i\mathbf{STL}(\mathcal{R})$ we mean $\mathbf{STL}(\mathcal{R})$ with all structural rules. And finally we denote $\mathbf{STL}(\{P, F\})$ by $\mathbf{FL}_l$ and $i\mathbf{STL}(\{P, F\})$ by **IPC**.

Remark 10 Note that in the presence of all the structural rules, the connective $\otimes$ collapses to $\wedge$ and the constant 1 is reduced to $\top$. Therefore, it is possible to axiomatize the structural logics of spacetime by eliminating the connective $\otimes$ and 1 from the language and the axiom $\Rightarrow 1$ and the rules $L\otimes$, $R\otimes$, $L1$ and $Oplax$ from the system.

Remark 11 Note that in the presence of both (F) and (P), the connective ∇ trivializes to identity. Therefore, in such logics and more specifically in $\mathbf{FL}_l$ and **IPC**, it is possible to formalize the logics without the axiom $\nabla 1 \Rightarrow 1$ and the rules ∇ and $Oplax$, by eliminating ∇ in the implication rules. In such a situation, the implication rules become the usual left implication rules in **FL**. This explains our terminology. In fact, our logic is exactly the fragment of **FL** excluding the right implication and 0 from both the language and the rules. For **IPC**, it is easy to see that the system becomes the original system **LJ** for intuitionistic propositional logic if we forget the collapsed $\otimes$. See Remark 10.

Remark 12 Note that the following sequents are provable in the system. First $\nabla(A \otimes B) \Rightarrow \nabla A \otimes \nabla B$ stating the oplax condition for ∇:

$$\dfrac{\dfrac{\nabla A \Rightarrow \nabla A \qquad \nabla B \Rightarrow \nabla B}{\nabla A, \nabla B \Rightarrow \nabla A \otimes \nabla B}\,{\scriptstyle \otimes R}}{\nabla(A \otimes B) \Rightarrow \nabla A \otimes \nabla B}\,{\scriptstyle Oplax}$$

Secondly, **STL** proves the distributivity of multiplication over disjunction, on both sides, i.e. $(A \otimes B) \vee (A \otimes C) \Rightarrow A \otimes (B \vee C)$ and $A \otimes (B \vee C) \Rightarrow (A \otimes B) \vee (A \otimes C)$. The first is a simple consequence of monotonicity of $\otimes$. For the second:

$$\dfrac{\dfrac{\dfrac{A, B \Rightarrow A \otimes B}{A, B \Rightarrow A \otimes B \vee A \otimes C} \qquad \dfrac{A, C \Rightarrow A \otimes C}{A, C \Rightarrow A \otimes B \vee A \otimes C}}{A, (B \vee C) \Rightarrow A \otimes B \vee A \otimes C}}{A \otimes (B \vee C) \Rightarrow A \otimes B \vee A \otimes C}\,{\scriptstyle L\otimes}$$

Thirdly, the system proves the sequent $A \otimes \nabla(A \to B) \Rightarrow B$:

$$\dfrac{\dfrac{A \Rightarrow A \qquad B \Rightarrow B}{A, \nabla(A \to B) \Rightarrow B}\,{\scriptstyle L\to}}{A \otimes \nabla(A \to B) \Rightarrow B}\,{\scriptstyle L\otimes}$$

Therefore, the sequents $A, \nabla B \Rightarrow C$ and $B \Rightarrow A \to C$ are equivalent. From left to right is just one application of the rule $R \to$. From right to left, by the rule ∇, we have $\nabla B \Rightarrow \nabla(A \to C)$. Using cut with $A, \nabla(A \to C) \Rightarrow C$ we reach what we wanted. Note that this adjunction situation simply implies that ∇ preserves all disjunctions, i.e., $\nabla\bot \Rightarrow \bot$, $\nabla(A \vee B) \Rightarrow \nabla A \vee \nabla B$ and $\nabla A \vee \nabla B \Rightarrow \nabla(A \vee B)$. Fourthly, the system proves the sequent $A \to B \Rightarrow C \otimes A \to C \otimes B$:

$$\dfrac{\dfrac{\dfrac{C \Rightarrow C \qquad \dfrac{A \Rightarrow A \qquad B \Rightarrow B}{A, \nabla(A \to B) \Rightarrow B}\,{\scriptstyle L\to}}{C, A, \nabla(A \to B) \Rightarrow C \otimes B}\,{\scriptstyle R\otimes}}{C \otimes A, \nabla(A \to B) \Rightarrow C \otimes B}\,{\scriptstyle L\otimes}}{A \to B \Rightarrow C \otimes A \to C \otimes B}\,{\scriptstyle R\to}$$

Remark 13 Note that the defined extensions of the system **STL** can be also axiomatized with some axioms instead of rules. For (N) the axioms are $\Rightarrow \nabla 1$ and $\nabla A \otimes \nabla B \Rightarrow \nabla(A \otimes B)$. These are provable by the rule (N) because:

$$\dfrac{\Rightarrow 1}{\Rightarrow \nabla 1}\,{\scriptstyle (N)} \qquad \dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad B \Rightarrow B}{A, B \Rightarrow A \otimes B}\,{\scriptstyle R\otimes}}{\nabla A, \nabla B \Rightarrow \nabla(A \otimes B)}\,{\scriptstyle (N)}}{\nabla A \otimes \nabla B \Rightarrow \nabla(A \otimes B)}\,{\scriptstyle L\otimes}$$

The converse is also true. For the empty Γ, if $\Rightarrow A$, then by $(L1)$, we have $1 \Rightarrow A$. By ∇ we have $\nabla 1 \Rightarrow \nabla A$. Hence, by $\Rightarrow \nabla 1$ we have $\Rightarrow \nabla A$. For Γ with at least one element, by induction, it is possible to use the axiom to prove that $\bigotimes(\nabla\Gamma) \Rightarrow \nabla(\bigotimes \Gamma)$, where by $\bigotimes \Pi$ we mean $\bigotimes_{i=0}^{n} A_i$ when $\Pi = \langle A_i \rangle_{i=0}^{n}$. Hence,

$$\dfrac{\bigotimes(\nabla\Gamma) \Rightarrow \nabla(\bigotimes\Gamma) \qquad \dfrac{\dfrac{\Gamma \Rightarrow A}{\bigotimes\Gamma \Rightarrow A}\,{\scriptstyle L\otimes}}{\nabla(\bigotimes\Gamma) \Rightarrow \nabla A}\,{\scriptstyle \nabla}}{\dfrac{\bigotimes(\nabla\Gamma) \Rightarrow \nabla A}{\overline{\nabla\Gamma \Rightarrow \nabla A}}}\,{\scriptstyle cut}$$

where the double line means the existence of an easy omitted proof tree there. Therefore, since $\nabla 1 \Rightarrow 1$ and $\nabla(A \otimes B) \Rightarrow \nabla A \otimes \nabla B$ are already provable in **STL** without (N), the rule (N) just states the strictness of ∇, i.e., for any sequence Γ, the sequents $\bigotimes(\nabla\Gamma)$ and $\nabla(\bigotimes\Gamma)$ are equivalent. This justifies the name of the rule, (N), that stands for normality, reflecting the normality condition of the usual conjunction-preserving modalities. For (H), note that this rule implies the rule (N) for $I = \emptyset$. It also implies that $\nabla A \to \nabla B \Rightarrow \nabla(A \to B)$ because:

$$\dfrac{A \to B \Rightarrow A \to B}{\nabla A \to \nabla B \Rightarrow \nabla(A \to B)}\,{\scriptstyle H}$$

Therefore, H implies $(\Rightarrow \nabla 1)$, $(\nabla A \otimes \nabla B \Rightarrow \nabla(A \otimes B))$ and $(\nabla A \to \nabla B \Rightarrow \nabla(A \to B))$. These are enough to prove (H) because the first part implies the rule (N) and then

$$\dfrac{\{\nabla A_i \to \nabla B_i\}_{i \in I} \Rightarrow \bigotimes_{i \in I} \nabla(A_i \to B_i) \qquad \dfrac{\dfrac{\Gamma, \{A_i \to B_i\}_{i \in I} \Rightarrow C}{\nabla\Gamma, \{\nabla(A_i \to B_i)\}_{i \in I} \Rightarrow \nabla C}\,{\scriptstyle (N)}}{\nabla\Gamma, \bigotimes_{i \in I} \nabla(A_i \to B_i) \Rightarrow \nabla C}\,{\scriptstyle L\otimes}}{\nabla\Gamma, \{\nabla A_i \to \nabla B_i\}_{i \in I} \Rightarrow \nabla C}\,{\scriptstyle \nabla}$$

Moreover, in the presence of (H) or even (N) we also have:

$$\dfrac{\dfrac{A, \nabla(A \to B) \Rightarrow B}{\nabla A, \nabla\nabla(A \to B) \Rightarrow \nabla B}\,{\scriptstyle (N)}}{\nabla(A \to B) \Rightarrow \nabla A \to \nabla B}\,{\scriptstyle R\to}$$

Therefore, the rule (H) is equivalent to the strictness of ∇ and the equivalence between $\nabla(A \to B)$ and $\nabla A \to \nabla B$. We will see that these conditions when applied on a locale of the open subsets of a topological space is equivalent to the condition that ∇ be the inverse image of a homeomorphism. This justifies the name of the rule, (H). For (P) and (F), they are equivalent to $\nabla A \Rightarrow A$ and $A \Rightarrow \nabla A$, respectively. (P) stands for past and (F) for future, reflecting the temporal nature of the modality ∇. We will see the details in Sect. 7.8. Finally, (wF) is equivalent to $1 \to \bot \Rightarrow \bot$. It is provable via (wF) because

$$\dfrac{\dfrac{\Rightarrow 1 \qquad \overline{\overline{1, \nabla(1 \to \bot) \Rightarrow \bot}}}{\nabla(1 \to \bot) \Rightarrow \bot}\,{\scriptstyle cut}}{1 \to \bot \Rightarrow \bot}\,{\scriptstyle (wF)}$$

Conversely, if we have the axiom $1 \to \bot \Rightarrow \bot$, then

$$\dfrac{\dfrac{\dfrac{\nabla A \Rightarrow \bot}{1, \nabla A \Rightarrow \bot}\,{\scriptstyle L1}}{A \Rightarrow 1 \to \bot}\,{\scriptstyle R\to} \qquad 1 \to \bot \Rightarrow \bot}{A \Rightarrow \bot}\,{\scriptstyle cut}$$

In this rule, (wF) stands for "weak future", since the rule (F) clearly implies (wF). The reason is that (F) implies $A \Rightarrow \nabla A$. Hence, using cut $\nabla A \Rightarrow \bot$ implies $A \Rightarrow \bot$.

Definition 13 (Topological Semantics) Let $\mathcal{S} = (\mathscr{X}, \nabla_{\mathcal{S}})$ be a non-commutative spacetime and $V : \mathcal{L}_\nabla \to \mathscr{X}$ an assignment. A tuple $(\mathcal{S}, V)$ is called a topological model for the language $\mathcal{L}_\nabla$ if:

- $V(1) = e$, $V(\bot) = 0$ and $V(\top) = 1$,
- $V(A \wedge B) = V(A) \wedge V(B)$,
- $V(A \vee B) = V(A) \vee V(B)$,
- $V(A \otimes B) = V(A) \otimes V(B)$,
- $V(\nabla A) = \nabla_{\mathcal{S}} V(A)$,
- $V(A \to B) = V(A) \to_{\mathcal{S}} V(B)$.

We say $(\mathcal{S}, V) \vDash \Gamma \Rightarrow A$ when $\bigotimes_{\gamma \in \Gamma} V(\gamma) \leq V(A)$ and $\mathcal{S} \vDash \Gamma \Rightarrow A$ when for all V, $(\mathcal{S}, V) \vDash \Gamma \Rightarrow A$. For a class C of non-commutative spacetimes, we write $C \vDash \Gamma \Rightarrow A$ if for any $\mathcal{S} \in C$ we have $\mathcal{S} \vDash \Gamma \Rightarrow A$. Moreover, if for some fixed $\mathscr{X}$ and for all $(\mathscr{X}, \nabla)$ in some class C we have $(\mathscr{X}, \nabla) \vDash \Gamma \Rightarrow A$, we write $\mathscr{X} \vDash_C \Gamma \Rightarrow A$. If $\mathscr{X}$ is $\mathcal{O}(X)$ for some topological space, we simplify it more to $X \vDash_C \Gamma \Rightarrow A$. Furthermore, we omit the symbol $\Rightarrow$ whenever Γ is empty.

Definition 14 Let $\mathcal{A} = (A, \leq, \otimes, e, \to, \nabla)$ be a temporal algebra. Then for any rule scheme $R \in \{N, H, P, F, wF\}$, we say $\mathcal{A}$ satisfies R if:

(N) ∇ preserves all finite multiplications,
(H) ∇ preserves all the structure including the implication,
(P) For any $a \in A$ we have $\nabla a \leq a$,
(F) For any $a \in A$ we have $a \leq \nabla a$,
(wF) $\mathcal{A}$ has zero and for any $a \in A$, if $\nabla a = 0$ then $a = 0$.

Definition 15 For any set of rule schemes $\mathcal{R} \subseteq \{N, H, P, F, wF\}$, by the class $\mathbf{ST}(\mathcal{R})$ we mean the class of all non-commutative sapcetimes $(\mathscr{X}, \nabla)$ that satisfies all the rule schemes in $\mathcal{R}$. The class $i\mathbf{ST}(\mathcal{R})$ is defined similarly for spacetimes.

Remark 14 Note that the condition (H) implies that ∇ is an isomorphism with the inverse $\Box = e \to (-)$. The proof is the following. Since $\nabla e = e$ we have

$$e \to \nabla a = \nabla e \to \nabla a = \nabla(e \to a)$$

but since $\nabla \dashv e \to (-)$, we have $\nabla(e \to a) \leq a \leq e \to \nabla a$. Hence, $\nabla(e \to a) = a = e \to \nabla a$. This means that ∇ and $\Box$ are inverses of each other over A.

Remark 15 Note that for non-commutative spacetimes, the conditions (N) and (H) are equivalent to "∇ is a strict geometric morphism" and "∇ is a strict geometric isomorphism", respectively. The reason for the first one is that ∇ has a right adjoint and hence preserves all joins. Hence, the only geometricity condition is the preservation of multiplications. For the second, we have to show that if ∇ is a strict geometric isomorphism, then it also preserves the implication. Let $\mathcal{S} = (\mathscr{X}, \nabla)$ be a non-commutative spacetime where ∇ is a strict geometric isomorphism. Then, to reduce the risk of confusion, let us denote ∇ by f. We know that f has an inverse. Call it g. Since they are inverses, we have $g \dashv f$. Then since f preserves ∇, it can be seen as a geometric map between non-commutative spaces, i.e., $f : \mathcal{S} \to \mathcal{S}$. Finally, by Theorem 7, to prove it is logical meaning that it respects the implication, it is enough to check that $g(fb \otimes \nabla a) = b \otimes \nabla ga$. Since $f = \nabla$ is strict and $gf = id = fg$ we have $g(fb \otimes fa) = gf(b \otimes a) = b \otimes fga$. Therefore, $f = \nabla$ preserves the implication.

Theorem 12 (Soundness) *For any set of rule schemes* $\mathcal{R} \subseteq \{N, H, P, F, wF\}$*, if* $\mathbf{STL}(\mathcal{R}) \vdash \Gamma \Rightarrow A$ *then* $\mathbf{ST}(\mathcal{R}) \vDash \Gamma \Rightarrow A$*. Specially, if* $\Gamma \vdash_{i\mathbf{STL}(\mathcal{R})} A$ *then* $i\mathbf{ST}(\mathcal{R}) \vDash \Gamma \Rightarrow A$.

Proof Since the logics are just the syntactical elementary representations of the structure of the non-commutative spacetimes, the soundness theorem is clear and we will leave the details to the reader. There are only four points to make. First about the rule $Oplax$ and the axiom $\nabla 1 \Rightarrow 1$. They are clearly valid whenever the interpretation of ∇ is oplax. Hence, they are valid in our topological interpretation. Secondly, consider the rule $R \to$. If $\Gamma \Rightarrow A \to B$ is proved by $A, \nabla\Gamma \Rightarrow B$, then by induction hypothesis, for any non-commutative spacetime $\mathcal{S} = (\mathscr{X}, \nabla_{\mathcal{S}})$ and any $V : \mathcal{L}_{\nabla} \to \mathscr{X}$ we have: $V(A) \otimes \bigotimes_{\gamma \in \Gamma} \nabla_{\mathcal{S}} V(\gamma) \leq V(B)$. Since $\nabla_{\mathcal{S}}$ is oplax, we have $V(A) \otimes \nabla_{\mathcal{S}}(\bigotimes_{\gamma \in \Gamma} V(\gamma)) \leq V(B)$. By adjunction, we have $\bigotimes_{\gamma \in \Gamma} V(\gamma) \leq V(A) \to_{\mathcal{S}} V(B)$. Therefore, the rule $R \to$ is also valid. Thirdly, note that all the rule schemes are equivalent to some axioms and those axioms are exactly the corresponding conditions on the non-commutative spacetimes. Hence, their validity is evident. Finally, note that for the spacetimes $\otimes = \wedge$ and $e = 1$. Therefore, it is clear that all the structural rules are valid. □

To prove the completeness theorem, we need the Lindenbaum construction together with a completion technique. For the former, set $L = \mathbf{STL}(\mathcal{R})$. Define $\mathcal{B}(L)$ to be the set of all formulas of the language $\mathcal{L}_{\nabla}$ with the equivalence relation $\equiv$ as $A \equiv B$ iff $L \vdash A \Rightarrow B$ and $L \vdash B \Rightarrow A$. It is clear that $(\mathcal{B}(L)/\equiv, \vdash)$ is a monoidal poset with all finite meets and all finite joins. Moreover it is also a distributive temporal algebra with its canonical ∇ and $\to$ such that $[A] \otimes \nabla(-)$ is a left adjoint to $[A] \to (-)$. See Remark 12. For the completion technique we have the following representation theorem, presented in Sect. 7.6. Here we present it in a slightly stronger form to also address the rule schemes.

Theorem 13 *Let* $\mathcal{A} = (A, \leq_A, \otimes_A, e_A, \nabla_A, \to_A)$ *be a (distributive) temporal algebra. Then there exists a non-commutative spacetime* $\mathcal{S} = (\mathscr{X}, \nabla)$ *and a (join preserving) temporal algebra embedding* $i : \mathcal{A} \to \mathcal{S}$*. Moreover, if the algebra has all finite meets, i preserves them and if* $\mathcal{A}$ *satisfies a rule scheme* $\mathcal{R} \subseteq \{N, H, P, F\}$*, then so does* $\mathcal{S}$*. The same is also true for* (wF) *if* $\mathcal{A}$ *is distributive.*

Proof First, let us address the case in which the temporal algebra does not necessarily have all the joins. Let $\mathscr{X} = D(\mathcal{A})$ be the downset completion of $\mathcal{A}$ and define

$$\nabla I = (\nabla_A)_! = \{x \in A | \exists i \in I \ (x \leq \nabla_A i)\}$$

First, observe that ∇ maps downsets to downsets. Secondly, note that by Theorem 4, ∇ is join preserving and since ∇_A is oplax, $(\nabla_A)_!$ is also oplax. Therefore, ∇ has a right adjoint by adjoint functor theorem, Theorem 1. Now let us provide the explicit adjoint. Define

$$I \to J = \{x \in A | \ \forall i \in I \ (i \otimes \nabla_A x \in J)\}$$

Again observe that $\to$ maps downsets to downsets. Then note that for any $I \in \mathscr{X}$, the map $I \otimes \nabla(-) \dashv (I \to (-))$ because for any $I, J, K \in \mathscr{X}$ we have

$$I \otimes \nabla J \subseteq K \ \text{ iff } \ I \subseteq J \to K$$

For the left to right, note that if $i \in I$, then for any $j \in J$, we have $i \otimes \nabla j \in I \otimes \nabla J \subseteq K$ and hence $i \otimes \nabla j \in K$. Hence, $I \subseteq J \to K$. Conversely, if $I \subseteq J \to K$ and $x \in I \otimes \nabla J$, then there exist $i \in I$ and $j \in J$ such that $x \leq i \otimes \nabla j$. Since $i \in I \subseteq J \to K$, by the definition of the implication we have $i \otimes \nabla j \in K$. Hence, $x \in K$.

Finally, define $i(a) = \{x \in A | x \leq a\}$. Then by Theorem 3, the map i is a monoidal poset's embedding that preserves finite meets (if they exist). Moreover, by Theorem 4, i also preserves ∇ i.e., $i \nabla_A a = \nabla i(a)$, for any $a \in A$. For implication:

$$i(a \to b) = \{x \in A | x \leq a \to b\} = \{x \in A | a \otimes \nabla x \leq b\} =$$

$$\{x \in A | \forall y \leq a \ (y \otimes \nabla x \leq b)\} = i(a) \to i(b)$$

Now, let us move to the distributive case. In this case, we have to move from the downset completion to the ideal completion with the same monoidal structure. By Theorem 4, since ∇ is join preserving so does $(\nabla_A)_!$. Moreover, the same i as before is a join preserving monoidal embedding that respects ∇ and finite meets (if they exist). The only thing we have to check is the stability of the ideals under the implication. This implies that the previous proofs for adjunction $I \otimes \nabla(-) \dashv (I \to (-))$, for any ideal I and preservability of implication under i work again. First note that $0 \in I \to J$ because for any $i \in I$, we have $i \otimes \nabla 0 = i \otimes 0 = 0 \in J$. The last equality is the consequence of distributivity of $\mathcal{A}$. And secondly, note that if $x, y \in I \to J$, then for all $i \in I$, we have $i \otimes \nabla x \in J$ and $i \otimes \nabla y \in J$. Since J is an ideal, ∇ preserves joins and $\mathcal{A}$ is distributive, we have

$$[i \otimes \nabla x] \vee [i \otimes \nabla y] = [i \otimes \nabla(x \vee y)] \in J$$

which proves that $I \to J$ is an ideal.

Finally, for the rule schemes, we have to show that the previous downset or ideal construction respects the rule schemes. For all schemes, except (wF), it is enough to prove the scheme for all downsets. The scheme for the ideals is just its special case.

For (N), note that ∇_A is lax and hence by Theorem 4, $(\nabla_A)_!$ is also lax. Being lax is nothing but satisfying (N).

For (H), note that if $\mathcal{A}$ satisfies (H), by Remark 14, ∇ and $\Box$ are inverses of each other over $\mathcal{A}$. This fact lifts also to $\mathcal{S}$. It is enough to prove that for any ideal I, we have $\nabla\Box I = I = \Box\nabla I$. We prove $I \subseteq \nabla\Box I$. The rest is similar. Assume $i \in I$, then $i = \nabla\Box i$. For the sake of readability, let $j = \Box i$. Then $\nabla j = i$. We have $j \in \Box I$ because $e \otimes \nabla j = i \in I$. Therefore, $i = \nabla j \in \nabla\Box I$. Finally, since ∇ has an inverse and is join preserving and strict, it will be a strict geometric isomorphism. The claim follows from Remark 15.

For (P), we have $\nabla I \subseteq I$ because if $x \in \nabla I$, then there exists $i \in I$ such that $x \leq \nabla i$. Since $\nabla i \leq i$, we have $x \leq i \in I$ which implies $x \in I$. For (F), we have $I \subseteq \nabla I$, because for any $i \in I$ we have $i \leq \nabla i$ which implies $i \in \nabla I$. Finally, for (wF), if $\nabla I = \{0\}$ and $i \in I$, we have $\nabla i \in \nabla I = \{0\}$ which implies $\nabla i = 0$. Since $\mathcal{A}$ satisfies (wF), we have $i = 0$ that proves $I = \{0\}$. $\square$

Theorem 14 (Completeness) *For any rule scheme $\mathcal{R} \subseteq \{N, H, P, F, wF\}$, there exists a non-commutative spacetime $\mathcal{S} \in \mathbf{ST}(\mathcal{R})$ such that if $\mathcal{S} \vDash \Gamma \Rightarrow A$ then $\Gamma \vdash_{\mathbf{STL}(\mathcal{R})} A$. The same is also true for $i\mathbf{ST}(\mathcal{R})$ and $i\mathbf{STL}(\mathcal{R})$.*

Proof Since the Lindenbaum algebra for $\mathbf{STL}(\mathcal{R})$ is clearly a finitely complete distributive temporal algebra, by Theorem 13, there exists a non-commutative spacetime $\mathcal{S} = (\mathcal{X}, \nabla)$ and a finite meet and finite join preserving temporal embedding $i : \mathcal{B}(L) \to \mathcal{S}$. Define $V(p) = i([p])$. It is easy to check that for all formula $C \in \mathcal{L}_\nabla$, we have $V(C) = i([C])$. Since $(\mathcal{S}, V) \vDash \Gamma \Rightarrow A$ we have $\bigotimes_{\gamma\in\Gamma} V(\gamma) \leq V(A)$. Hence, $\bigotimes_{\gamma\in\Gamma} i([\gamma]) \leq i([A])$. Since i preserves the monoidal structure and is an embedding, $\bigotimes_{\gamma\in\Gamma}[\gamma] \leq [A]$ or equivalently $\Gamma \vdash_{\mathbf{STL}\mathcal{R})} A$. For the structural version, note that by Remark 5, the ideal construction in Theorem 13, applied on a monoidal poset with meet structure as its monoidal structure, produces a locale for $\mathcal{X}$. $\square$

One of the main advantages of the spacetime logics over the usual sub-intuitionistic logics is their complete pairs of introduction-elimination rules. This well-behaved nature may find some evidence by the following translation that interprets the seemingly more powerful logics into the weaker ones. The translation is the syntactical version of Theorem 6. It helps to import what we have in sub-structural and intuitionistic tradition to the spacetime logics. It also shows that **STL** and i**STL** are in some sense more powerful than the usual $\mathbf{FL}_l$ and **IPC**, respectively. In this sense the former refine the timeless spatial structure of the latter by bringing the more temporal and hence more expressive power.

Definition 16 Define the translation $(-)^\nabla : \mathcal{L} \to \mathcal{L}_\nabla$ as the following, where $\mathcal{L} = \{\wedge, \vee, \top, \bot, 1, \otimes, \to\}$:

- $p^\nabla = \nabla\Box p$, $\bot^\nabla = \bot$, $\top^\nabla = \nabla\Box\top$ and $1^\nabla = 1$.
- $(A \wedge B)^\nabla = \nabla\Box(A^\nabla \wedge B^\nabla)$.
- $(A \vee B)^\nabla = A^\nabla \vee B^\nabla$.
- $(A \otimes B)^\nabla = A^\nabla \otimes B^\nabla$.
- $(A \to B)^\nabla = \nabla(A^\nabla \to B^\nabla)$.

Theorem 15 *For any* $\Gamma \cup \{A\} \subseteq \mathcal{L}$,

(i) $\Gamma \vdash_{\mathbf{FL}_l} A$ *iff* $\Gamma^\nabla \vdash_{\mathbf{STL}(N)} A^\nabla$.
(ii) $\Gamma \vdash_{\mathbf{IPC}} A$ *iff* $\Gamma^\nabla \vdash_{i\mathbf{STL}(N)} A^\nabla$ *(Originally proved in (Akbar Tabatabai et al. 2017)).*

Proof We will prove (i), the proof for (ii) is the same. For that matter, we will first prove a claim that for any formula $A \in \mathcal{L}$, there exists a formula $A' \in \mathcal{L}_\nabla$ such that $A^\nabla \vdash_{\mathbf{STL}(N)} \nabla A'$ and $\nabla A' \vdash_{\mathbf{STL}(N)} A^\nabla$. The proof for the claim is by induction on the structure of A. For atomic cases, considering the fact that $\nabla\bot$ is equivalent to $\bot$, there is nothing to prove. The claim for conjunction and implication is clear by definition of the translation. Finally, for $\otimes$ and $\vee$, note that the translation $(-)^\nabla$ commutes with these connectives. Therefore, if there exist A' and B' for A^∇ and B^∇, respectively, for $A \otimes B$ it is enough to consider $A' \otimes B'$. The reason is that ∇ commutes with $\otimes$ because of (N). For $\vee$ the same strategy works. Therefore, the existence of A' is proved. This property implies the following useful fact: For any $B \in \mathcal{L}_\nabla$, if $\Gamma^\nabla \vdash_{\mathbf{STL}(N)} B$, then $\Gamma^\nabla \vdash_{\mathbf{STL}(N)} \nabla\Box B$. The reason is the following. Since the formula $\bigotimes \Gamma^\nabla$ is equivalent to $(\bigotimes \Gamma)^\nabla$ and the latter is also equivalent to ∇C for some $C \in \mathcal{L}_\nabla$, it is enough to prove the claim for ∇C. Now, since $\nabla C \vdash_{\mathbf{STL}(N)} B$, by $(L1)$ we have $1, \nabla C \vdash_{\mathbf{STL}(N)} B$. By implication introduction we have $C \vdash_{\mathbf{STL}(N)} \Box B$ and hence by the rule (∇), we have $\nabla C \vdash_{\mathbf{STL}(N)} \nabla\Box B$.
Coming back to the proof of the theorem, for the soundness part it is enough to use an induction on the $\mathbf{FL}_l$-proof length of $\Gamma \Rightarrow A$. For axioms, all cases are clear, except $\Gamma \Rightarrow \top$. For this case we have to prove $\Gamma^\nabla \vdash \nabla\Box\top$ which is clear from what we observed above.

For the conjunction rule $(R\wedge)$, assume $\Gamma \Rightarrow A \wedge B$ is proved via $\Gamma \Rightarrow A$ and $\Gamma \Rightarrow B$. Then by IH, we have $\Gamma^\nabla \Rightarrow A^\nabla$ and $\Gamma^\nabla \Rightarrow B^\nabla$. Then $\Gamma^\nabla \Rightarrow A^\nabla \wedge B^\nabla$. Therefore, by what we have above $\Gamma^\nabla \Rightarrow \nabla\Box(A^\nabla \wedge B^\nabla)$. For the conjunction rule $(L\wedge)$, assume $\Gamma, A \wedge B, \Sigma \Rightarrow C$ is proved via $\Gamma, A, \Sigma \Rightarrow C$. Then by IH, $\Gamma^\nabla, A^\nabla, \Sigma^\nabla \Rightarrow C^\nabla$. Then $\Gamma^\nabla, A^\nabla \wedge B^\nabla, \Sigma^\nabla \Rightarrow C^\nabla$. Since $\nabla\Box(A^\nabla \wedge B^\nabla) \Rightarrow A^\nabla \wedge B^\nabla$, we have $\Gamma^\nabla, (A \wedge B)^\nabla, \Sigma^\nabla \Rightarrow C^\nabla$.

For implication rule $(R \to)$, assume $\Gamma \Rightarrow A \to B$ is proved via $A, \Gamma \Rightarrow B$. Then by IH, we have $A^\nabla, \Gamma^\nabla \Rightarrow B^\nabla$. Since Γ^∇ is equivalent to $(\bigotimes \Gamma)^\nabla$, it is also equivalent to ∇C for some C. We have $A^\nabla, \nabla C \Rightarrow B^\nabla$. Hence, $C \Rightarrow (A^\nabla \to B^\nabla)$. Hence, by (∇) we have $\nabla C \Rightarrow \nabla(A^\nabla \to B^\nabla)$. Since $\bigotimes \Gamma^\nabla$ is equivalent to ∇C, we have $\Gamma^\nabla \Rightarrow \nabla(A^\nabla \to B^\nabla)$. For implication rule $(L \to)$, assume $\Pi, \Gamma, (A \to B), \Sigma \Rightarrow C$ is proved via $\Gamma \Rightarrow A$ and $\Pi, B, \Sigma \Rightarrow C$. Then by IH, $\Gamma^\nabla \Rightarrow A^\nabla$ and $\Pi^\nabla, B^\nabla, \Sigma^\nabla \Rightarrow C^\nabla$. Hence, $\Pi^\nabla, \Gamma^\nabla, \nabla(A^\nabla \to B^\nabla), \Sigma^\nabla \Rightarrow C^\nabla$.

For completeness, note that if $\Gamma^\nabla \Rightarrow A^\nabla$ is provable in $\mathbf{STL}(N)$, then it is also provable in the greater logic $\mathbf{FL}_l = \mathbf{STL}(N, P, F)$. Since for any $B \in \mathcal{L}$, the formulas B^∇ and B are equivalent in $\mathbf{STL}(N, P, F)$, the sequent $\Gamma \Rightarrow A$ is also provable in $\mathbf{FL}_l$. □

7.8 Kripke Models

In this section we will focus on the structural logics of spacetime and their Kripke semantics. This semantics is essentially the usual Kripke semantics for the intuitionistic modal and implication logics (Simpson 1994; Fischer-Servi 1977 and Litak and Visser 2018). However, to also address ∇, we will add a natural forcing condition using the same accessibility relation that the model uses for $\Box$. In this sense, the structural logics of spacetime are actually the result of a faithful extension of the language and logics to have a better reflection of the Kripke models into the pure syntax.

Definition 17 By a Kripke model for the language $\mathcal{L}_\nabla$, we mean a tuple $\mathcal{K} = (W, \leq, R, V)$ where $(W, \leq)$ is a poset, $R \subseteq W \times W$ is a relation over W (not necessarily transitive or reflexive) compatible with $\leq$, i.e., for all $u, u', v, v' \in W$ if $(u, v) \in R$ and $u' \leq u$ and $v \leq v'$ then $(u', v') \in R$ and $V : At(\mathcal{L}_\nabla) \to U((W, \leq))$ where $At(\mathcal{L}_\nabla)$ is the set of atomic formulas of $\mathcal{L}_\nabla$ and $U((W, \leq))$ is the set of all upsets of $(W, \leq)$. Define the forcing relation as usual using the relation R and for the ∇ let $u \Vdash \nabla A$ if there exists $v \in W$ such that $(v, u) \in R$ and $v \Vdash A$. A Kripke model is called normal if there exists an order preserving function $\pi : W \to W$ such that $(u, v) \in R$ iff $u \leq \pi(v)$. It is clear that if this π exists, it would be unique. Finally, a sequent $\Gamma \Rightarrow A$ is valid in a Kripke model if for all $w \in W, \forall B \in \Gamma\ (w \Vdash B)$ implies $w \Vdash A$.

Lemma 1 (Monotonicity Lemma) *For any formula $A \in \mathcal{L}_\nabla$, any Kripke model $\mathcal{K} = (W, \leq, R, V)$ and any $u, v \in W$, if $u \leq v$ and $u \Vdash A$ then $v \Vdash A$.*

Proof The proof is a routine induction on the structure of A. The only case to mention is when $A = \nabla B$. Then if $u \Vdash \nabla B$, there exist $u' \in W$ such that $(u', u) \in R$ and $u' \Vdash B$. Since $u \leq v$ and R is compatible with $\leq$, we have $(u', v) \in R$. Therefore, $v \Vdash \nabla A$. $\Box$

Remark 16 Note that in a normal Kripke model $w \Vdash \nabla A$ iff $\pi(w) \Vdash A$. One direction is clear, for the other, if there exists $u \in W$ such that $(u, w) \in R$ and $u \Vdash A$, then since $u \leq \pi(w)$, by the monotonicity lemma we have $\pi(w) \Vdash A$. This means that the normal Kripke models are the models in which we have a canonical way to witness the existential quantifier in the forcing condition of ∇.

Definition 18 For any rule scheme in the set $\{N, H, P, F, wF\}$, we define a corresponding condition on a Kripke model as:

- (N) The model is normal.
- (H) The model is normal and its π is a poset isomorphism.
- (P) $R \subseteq \leq$. For a normal model, it is equivalent to $\forall w \in W\ (\pi(w) \leq w)$.
- (F) R is reflexive, i.e., for all $w \in W$ we have $(w, w) \in R$. For a normal model, is equivalent to $\forall w \in W\ (w \leq \pi(w))$.
- (wF) R is serial, i.e., for all $u \in W$ there exists $v \in W$ such that $(u, v) \in R$. For a normal model, it is equivalent to $\forall u \in W \exists v \in W\ (u \leq \pi(v))$.

Moreover, if $\mathcal{R} \subseteq \{N, H, P, F, wF\}$, by a $\mathbf{K}(\mathcal{R})$-Kripke model we mean a model satisfying the conditions corresponding to all the schemes in $\mathcal{R}$.

Theorem 16 (Soundness) *For any rule scheme $\mathcal{R} \subseteq \{N, H, P, F, wF\}$, the logic $i\mathbf{STL}(\mathcal{R})$ is sound for all $\mathbf{K}(\mathcal{R})$-Kripke models.*

Proof Our strategy is reducing the soundness for Kripke models to soundness for topological models. It is also possible to prove it directly. However, we follow this strategy to also show how Kripke models must be considered as the special case of the topological models. For that purpose, we show how to assign a topological model to a Kripke model with the same valid sequents. Moreover, we will show that this construction respects the schema conditions. Let $\mathcal{K} = (W, \leq, R, V)$ be a Kripke model. Define the spacetime $\mathcal{S}_{\mathcal{K}} = (U(W, \leq), \nabla_{\mathcal{K}})$ as in Example 13 by

$$\nabla_{\mathcal{K}} P = \{w \in W | \exists u \in P \text{ such that } (u, w) \in R\}$$

For any formula $B \in \mathcal{L}_{\nabla}$ define $[B]$ as the set $\{w \in W | w \Vdash B\}$. By the monotonicity lemma, $[B]$ is an upset of W. If we define the topological valuation $\bar{V}(p) = V(p)$, it is easy to see that $\bar{V}(B) = [B]$ for any formula $B \in \mathcal{L}_{\nabla}$. Hence, for any sequent $\Gamma \Rightarrow A$, it is valid in $(U(W, \leq), \nabla_{\mathcal{K}}, \bar{V})$ iff $\bigwedge_{\gamma \in \Gamma} \bar{V}(\gamma) \subseteq \bar{V}(A)$ iff $\bigcap_{\gamma \in \Gamma} [\gamma] \subseteq [A]$ which is nothing but the validity of $\Gamma \Rightarrow A$ in $\mathcal{K}$.

It is remaining to prove the preservation of the schema conditions. First note that for (N), the existence of π means that $\nabla_{\mathcal{K}} = \pi^{-1}$. Therefore, ∇ preserves all intersections and hence is a strict geometric map. For (H), since π is an order isomorphism, it has an inverse ρ. Then $\pi^{-1}, \rho^{-1} : U(W, \leq) \to U(W, \leq)$ are each other's inverses. Hence, $\nabla_{\mathcal{K}} = \pi^{-1} : U(W, \leq) \to U(W, \leq)$ is a strict geometric isomorphism. For (P), we have $\nabla_{\mathcal{K}} P \subseteq P$. The reason is that if $w \in \nabla_{\mathcal{K}} P$, there exist $u \in W$ such that $(u, w) \in R$ and $u \in P$. Since $R \subseteq \leq$, we have $u \leq w$. Since P is an upset we have $w \in P$. For (F), we have $P \subseteq \nabla_{\mathcal{K}} P$ because if $w \in P$ then since $(w, w) \in R$ we have $w \in \nabla_{\mathcal{K}} P$. And finally, for (wF), if $\nabla_{\mathcal{K}} P = \emptyset$, then $P = \emptyset$ because if $w \in P$ then since R is serial, there exists $u \in W$ such that $(w, u) \in R$ which means that $u \in \nabla_{\mathcal{K}} P = \emptyset$. This is a contradiction and hence $P = \emptyset$. □

Definition 19 Let $\mathcal{A} = (A, \leq, \wedge, 1, \to)$ be a strong algebra where $(A, \leq)$ is finitely cocomplete. Then $\mathcal{A}$ is called join internalizing if $(a \vee b \to c) = (a \to c) \wedge (b \to c)$, for every $a, b, c \in A$.

For completeness, we need the following representation theorem, presented before as Theorem 10. The proof is essentially the canonical extension construction in (Celani and Jansana 2001) expanded to also cover both weaker and stronger cases. In fact, in (Akbar Tabatabai et al. 2017), we modified this construction to also address the operator ∇. Since (Akbar Tabatabai et al. 2017) is not accessible yet, we restate the full details and we add the proofs for some other cases that are absent in (Akbar Tabatabai et al. 2017).

Theorem 17 *For any strong algebra $\mathcal{A} = (A, \leq, \wedge, 1, \to)$ that internalizes its monoidal structure [not necessarily its closed monoidal structure if it has any],* *there exists a Kripke frame $\mathcal{K}$ and a strong algebra embedding $i : \mathcal{A} \to \mathcal{S}_{\mathcal{K}}$. Moreover, if $\mathcal{A}$ is distributive and its implication internalizes the joins, the map i can be chosen join preserving, as well. Finally, if $\mathcal{A}$ is a reduct of a temporal algebra, i also preserves ∇ and for any rule scheme $\mathcal{R} \subseteq \{N, H, P, F, wF\}$, if $\mathcal{A}$ satisfies $\mathcal{R}$, then so does $\mathcal{K}$.*

Proof We split the proof to four cases depending on the presence of joins and ∇. For all cases, we need the following constructions. Recall that $F(\mathcal{A})$ is the poset of all filters of $\mathcal{A}$ and define $\mathcal{R}$ as a binary relation over $F(\mathcal{A})$ as: $(P, Q) \in \mathcal{R}$ iff for all $a, b \in A$ if $a \to b \in P$ and $a \in Q$ then $b \in Q$.

Case I. In this case both joins and ∇ are not necessarily present. Define $W = F(\mathcal{A})$ and its order as the equality on W. Then it is clear that $\mathcal{K}_1 = (W, =, \mathcal{R})$ is a Kripke frame in the sense of Example 13. Consider $i : \mathcal{A} \to U(W, =_W)$ defined by $i(a) = \{P \in F(\mathcal{A}) | \; a \in P\}$. As we observed in the Preliminaries, i is clearly a meet-semilattice embedding. Note that for any X and Y as the upsets of $(W, =_W)$, the implication in $\mathcal{S}_{\mathcal{K}_1}$ is defined by:

$$X \to Y = \{P \in F(\mathcal{A}) | \forall Q \in F(\mathcal{A}) \text{ if } (P, Q) \in \mathcal{R} \text{ and } Q \in X \text{ then } Q \in Y\}$$

To prove that i preserves the implication, i.e., $i(a \to b) = i(a) \to i(b)$, we have to check the following two directions:

To prove $i(a \to b) \subseteq i(a) \to i(b)$, if $P \in i(a \to b)$ then $a \to b \in P$. Then assume $(P, Q) \in \mathcal{R}$ and $Q \in i(a)$. Hence, $a \in Q$ and since $a \to b \in P$, by the definition of $\mathcal{R}$, we have $b \in Q$, meaning $Q \in i(b)$. Therefore, $P \in i(a) \to i(b)$. Conversely, if $P \in i(a) \to i(b)$, then consider $Q = \{x \in A | a \to x \in P\}$. Since $\mathcal{A}$ internalizes its meet structure, by Remark 8, we have

$$(a \to x) \wedge (a \to y) = (a \to x \wedge y)$$

which means that Q is a filter. Moreover, $a \in Q$ because $a \to a = 1 \in P$. Note that $(P, Q) \in \mathcal{R}$ because if $x \to y \in P$ and $x \in Q$, then $a \to x \in P$ and since

$$(a \to x) \wedge (x \to y) \leq (a \to y)$$

and P is a filter, we have $a \to y \in P$ which means $y \in Q$. Therefore, $(P, Q) \in \mathcal{R}$. Now, since $a \in Q$ we have $Q \in i(a)$. Since $P \in i(a) \to i(b)$ and $(P, Q) \in \mathcal{R}$ we have $Q \in i(b)$, meaning $b \in Q$ which by the definition of Q means $a \to b \in P$.

Case II. In this case, again joins are not necessarily present. However, the algebra $\mathcal{A}$ is a reduct of a temporal algebra. Therefore, there exists $\nabla : A \to A$ such that for any $a \in A$, $a \wedge \nabla(-) \dashv (a \to -)$. First note that the relation $\mathcal{R}$ on $F(\mathcal{A})$ is also definable by ∇ as $(P, Q) \in \mathcal{R}$ iff $\nabla[P] = \{\nabla x | x \in P\} \subseteq Q$. The reason is the following: If $(P, Q) \in \mathcal{R}$ and $x \in P$, since $x \leq 1 \to \nabla x$ and P is a filter, $1 \to \nabla x \in P$. Therefore, by $1 \in Q$ and $(P, Q) \in \mathcal{R}$ we have $\nabla x \in Q$. Hence, $\nabla[P] \subseteq$

Q. Conversely, if $\nabla[P] \subseteq Q$, given $a \to b \in P$ and $a \in Q$ we have $\nabla(a \to b) \in \nabla[P] \subseteq Q$ and since $a \wedge \nabla(a \to b) \leq b$ we have $b \in Q$. Therefore, $(P, Q) \in \mathcal{R}$.

Defining $\mathcal{R}$ in terms of ∇ has the advantage to make $\mathcal{R}$ monotone also in its second argument, i.e, if $(P, Q) \in \mathcal{R}$ and $Q \subseteq Q'$, then $(P, Q') \in \mathcal{R}$. For this part, pick $W = F(\mathcal{A})$ as before and change the order on W to $\subseteq$. Since $\mathcal{R}$ is compatible with $\subseteq$, the tuple $\mathcal{K}_2 = (W, \subseteq, \mathcal{R})$ is a Kripke frame. Moreover, note that $i(a)$ for any $a \in A$ is an upset with respect to $\subseteq$. For the preservation of the implication, since it does not depend on the order on W, the argument for the previous case also works here. Therefore, we only have to show that i preserves ∇, i.e., $i(\nabla a) = \nabla i(a)$. If $P \in i(\nabla a)$, then $\nabla a \in P$. Pick $Q = \{x \in A | x \geq a\}$. This is clearly a filter and $(Q, P) \in \mathcal{R}$ because $\nabla[Q] \subseteq \nabla\{x \in A | x \geq a\} \subseteq P$ because $\nabla a \in P$. Therefore, there exists Q that includes a and $(Q, P) \in \mathcal{R}$. Therefore, $P \in \nabla i(a)$. Conversely, if $P \in \nabla i(a)$, then there exists Q such that $a \in Q$ and $(Q, P) \in \mathcal{R}$. Therefore, $\nabla a \in \nabla[Q] \subseteq P$ and hence $\nabla a \in P$. Therefore, $P \in i(\nabla a)$.

Case III. Now, we move to the case where $\mathcal{A}$ is distributive and the implication internalizes the finite joins while ∇ is not necessarily present. Here, we want to construct a Kripke frame and a join preserving map i. For that matter, as we observe in Preliminaries, it is sufficient to change W from the set of filters of $\mathcal{A}$ to the set of all prime filters of $\mathcal{A}$, denoted by $P(\mathcal{A})$. The same i works as an embedding and it preserves both finite meets and finite joins. Define $\mathcal{R}$ as before and $\mathcal{K}_3 = (P(\mathcal{A}), =_W, \mathcal{R})$. The only thing to check is whether i preserves both implication and ∇, again.

For the implication, by the definition of $\mathcal{R}$ and as we had in Case I, $i(a \to b) \subseteq i(a) \to i(b)$ is clear. For the converse, assume $a \to b \notin P$ but $P \in i(a) \to i(b)$. Define $Q = \{x \in A | a \to x \in P\}$. The problem is that this Q is not necessarily prime. The strategy is extending it to a suitable prime filter. Since $a \to b \notin P$ then $b \notin Q$. Define

$$\Sigma = \{S \in F(\mathcal{A}) \mid (P, S) \in \mathcal{R}, a \in S \text{ and } b \notin S\}.$$

The set Σ is non-empty because $Q \in \Sigma$, as we have checked in Case I. Moreover, in Σ any chain has an upper bound because if for all $i \in I$ we have $(P, S_i) \in \mathcal{R}$ then $(P, \bigcup_{i \in I} S_i)$. The reason is the following: If $x \to y \in P$ and $x \in \bigcup_{i \in I} S_i$ then for some $i \in I$ we have $x \in S_i$. Since $(P, S_i) \in \mathcal{R}$, we have $y \in S_i$ from which $y \in \bigcup_{i \in I} S_i$. Therefore, by Zorn's lemma, Σ has a maximal element M. We will prove that M is prime. First note that $0 \notin M$ because if so, $M = A$ which contradicts with $b \notin M$. Now for the sake of contradiction, let us assume that $x \vee y \in M$ and $x, y \notin M$. Then we claim that either for all $m \in M$ we have $(m \wedge x \to b) \notin P$ or for all $m \in M$ we have $(m \wedge y \to b) \notin P$. The reason is that if for some $m, n \in M$ both $(m \wedge x \to b) \in P$ and $(n \wedge y \to b) \in P$ happen, we would have $(m \wedge n \wedge x \to b) \in P$ and $(m \wedge n \wedge y \to b) \in P$. Then by distributivity and the fact that the implication internalizes the finite joins, we reach

$$[m \wedge n \wedge (x \vee y) \to b] = [(m \wedge n \wedge x \to b) \wedge (m \wedge n \wedge y \to b)] \in P$$

and since $[m \wedge n \wedge (x \vee y)] \in M$ and $(P, M) \in \mathcal{R}$ we have $b \in M$ which is a contradiction. Hence, w.l.o.g. we can assume that for all $m \in M$ we have $(m \wedge x \to b) \notin P$. Then define

$$N = \{z \in A |\ \exists m \in M\ (m \wedge x \to z \in P)\}$$

First, note that $M \subseteq N$, because for any $m \in M$ we have $m \wedge x \to m = 1 \in P$. Similarly, we have $x \in N$. Therefore, N is a proper extension of M because $x \notin M$. Secondly, note that N is a filter because $1 = [(1 \wedge 1) \to 1] \in P$ which implies $1 \in N$ and if $z, w \in N$ then there are $m, n \in M$ such that $(m \wedge x \to z) \in P$ and $(n \wedge x \to w) \in P$. Therefore, $(m \wedge n \wedge x \to z) \in P$ and $(m \wedge n \wedge x \to w) \in P$. Since P is a filter and $\mathcal{A}$ internalizes its monoidal structure, by Remark 8, we have

$$(m \wedge n \wedge x) \to (z \wedge w) \in P$$

Since M is a filter we have $m \wedge n \in M$ which implies $z \wedge w \in N$. Thirdly, note that we have $(P, N) \in \mathcal{R}$ because if $z \to w \in P$ and $z \in N$ there exists $m \in M$ such that $m \wedge x \to z \in P$ which implies $m \wedge x \to w \in P$ meaning that $w \in N$. And finally, note that $b \notin N$, because for all $m \in M$ we have $m \wedge x \to b \notin P$. Hence, $N \in \Sigma$ while it is a proper extension of M. This contradicts with the maximality of M which implies that M is prime. Finally, since $a \in M$ and $b \notin M$, we have $M \in i(a)$ and $M \notin i(b)$. Since $(P, M) \in \mathcal{R}$, this contradicts with $P \in i(a) \to i(b)$.

Case IV. In this case, the algebra $\mathcal{A}$ is assumed to be a reduct of a distributive temporal algebra and we have to show that i also preserves the ∇ operator, i.e., $i(\nabla a) = \nabla i(a)$. Define $\mathcal{R}$ as before and $\mathcal{K}_4 = (P(\mathcal{A}), \subseteq, \mathcal{R})$. As we have seen in Case II, $\mathcal{R}$ is compatible with $\subseteq$ and hence $\mathcal{K}_4$ is a Kripke frame. Again, since the implication does not depend on the order, the proof of preservability of implication in the Case III works here, as well. The only thing to check is whether i preserves ∇. As we have observed in Case II, $\nabla i(a) \subseteq i(\nabla a)$ is an easy consequence of the definition of $\mathcal{R}$. To show $i(\nabla a) \subseteq \nabla i(a)$, if $Q \in i(\nabla a)$ then $\nabla a \in Q$. Define

$$\Sigma = \{S \in F(\mathcal{A}) |\ (S, Q) \in \mathcal{R}, a \in S \text{ and } 0 \notin S\}$$

It is clear that $P = \{x \in A | x \geq a\} \in \Sigma$, because $a \in P$, since $\nabla a \in Q$, we have

$$\nabla[P] = \{\nabla x | x \geq a\} \subseteq Q$$

and $0 \notin P$ because if $0 \in P$ then $0 \geq a$ which implies $a = 0$ and hence $\nabla a = 0 \in Q$ which is impossible since Q is proper. Since $P \in \Sigma$, the set Σ is non-empty. Any chain in Σ has an upper bound because if for all $i \in I$ we have $(S_i, Q) \in \mathcal{R}$ then $\nabla[S_i] \subseteq Q$ from which $\nabla[\bigcup_{i \in I} S_i] = \bigcup_{i \in I} \nabla[S_i] \subseteq Q$ and hence $(\bigcup_{i \in I} S_i, Q) \in \mathcal{R}$. By Zorn's lemma, Σ has a maximal element. Call it M. We will prove that M is prime which completes the proof. First note that $M \in \Sigma$ which implies that $0 \notin M$. Hence, M is proper. Now assume $x \vee y \in M$ and $x \notin M$, $y \notin M$. The filters M_x and M_y generated by $M \cup \{x\}$ and $M \cup \{y\}$ are proper extensions of M. Therefore, they are not in Σ which means that either one of them includes zero or we have both $\nabla M_x \nsubseteq Q$ and $\nabla M_y \nsubseteq Q$. The first is impossible because if $0 \in M_x$, then there is $m \in M$ such that $m \wedge x \leq 0$. Since $\mathcal{A}$ is distributive, we have

$$m \wedge (x \vee y) = (m \wedge x) \vee (m \wedge y) = m \wedge y$$

Since $x \vee y \in M$ and M is a filter, we have $m \wedge (x \vee y) \in M$ which implies $m \wedge y \in M$. Therefore, since $m \wedge y \leq y$ we have $y \in M$ which is a contradiction. A similar argument also works for the case $0 \in M_y$. Hence, we are in the case that $\nabla M_x \nsubseteq Q$ and $\nabla M_y \nsubseteq Q$.
Therefore, there are $z, w \in A$ such that $\nabla z, \nabla w \notin Q$ and $z \in M_x$ and $w \in M_y$. Hence, there are $m, n \in M$ such that $m \wedge x \leq z$ and $n \wedge y \leq w$. Therefore, $\nabla(m \wedge x) \notin Q$ and $\nabla(n \wedge y) \notin Q$. Since M is a filter, $m \wedge n \in M$ and since $x \vee y \in M$, we have

$$m \wedge n \wedge (x \vee y) = (m \wedge n \wedge x) \vee (m \wedge n \wedge y) \in M$$

which by $(M, Q) \in \mathcal{R}$ implies $\nabla[(m \wedge n \wedge x) \vee (m \wedge n \wedge y)] \in Q$ and hence

$$\nabla(m \wedge n \wedge x) \vee \nabla(m \wedge n \wedge y) \in Q$$

and since Q is prime, either $\nabla(m \wedge n \wedge x) \in Q$ or $\nabla(m \wedge n \wedge y) \in Q$. If $\nabla(m \wedge n \wedge x) \in Q$ then since $\nabla(m \wedge n \wedge x) \leq \nabla(m \wedge x)$ we have $\nabla(m \wedge x) \in Q$ which is a contradiction. A similar argument also works for the other case. Hence, M is prime. Finally, since $a \in M$ and $(M, Q) \in \mathcal{R}$ we have $Q \in \nabla i(a)$ which completes the proof.

Finally, we have to address the preservability of the validity of the rule schemes. For (N), if $\mathcal{A}$ satisfies the scheme (N), ∇ commutes with all finite meets. We want to find an order preserving function $\pi(P)$ such that $(P, Q) \in \mathcal{R}$ iff $P \subseteq \pi(Q)$. Define $\pi(P) = \nabla^{-1}[P]$. It is clearly order preserving. Note that $(P, Q) \in \mathcal{R}$ is equivalent to $\nabla[P] \subseteq Q$ which is equivalent to $P \subseteq \pi(Q)$. The only thing to show is that $\pi(P)$ is actually a filter if we are in Case II and it is a prime filter if we are in Case IV. First, $\nabla^{-1}[P]$ is clearly an upset. Since $1 = \nabla 1$ and P is a filter, we have $1 \in \nabla^{-1}[P]$. Moreover, if $x, y \in \nabla^{-1}[P]$ then $\nabla x, \nabla y \in P$. Since P is a filter and $\nabla x \wedge \nabla y = \nabla(x \wedge y)$, we have $\nabla(x \wedge y) \in P$ and hence $x \wedge y \in \nabla^{-1}[P]$. Moreover, if $\mathcal{A}$ has all finite joins, then $\nabla^{-1}[P]$ is prime because if $x \vee y \in \nabla^{-1}[P]$, then $\nabla(x \vee y) \in P$. Since ∇ has a right adjoint, it commutes with all joins and hence $\nabla x \vee \nabla y \in P$. Since P is prime, either $\nabla x \in P$ or $\nabla y \in P$. Therefore, either $x \in \nabla^{-1}[P]$ or $y \in \nabla^{-1}[P]$. Moreover, $0 \notin \nabla^{-1}[P]$ because otherwise, $\nabla 0 = 0 \in P$ which is impossible, since P is prime.

For (H), note that ∇ and $\Box$ as two operators over the Lindenbaum algebra are inverse of each other. Hence, ∇^{-1} as an operation over all filters or prime filters is an isomorphism. For (P), given $(P, Q) \in \mathcal{R}$, we have $P \subseteq Q$. Because, given $a \in P$ and the fact that (P, Q), we have $\nabla[P] \subseteq Q$ which implies $\nabla a \in \nabla[P] \subseteq Q$. Hence, $\nabla a \in Q$. Finally, we have $a \in Q$ Since $\nabla a \leq a$. For (F), we have to show

$(P, P) \in \mathcal{R}$. The reason is that we have $\nabla[P] \subseteq P$, because for any $a \in P$, we have $a \leq \nabla a$ which implies $\nabla a \in P$.

$(*)$ [We denote this part by $(*)$ for the future reference.] Finally, for (wF), first note that this rule scheme is also expressible by implication via $1 \to 0 = 0$. The reason is that if we have (wF), then by adjunction $\nabla(1 \to 0) \leq 0$ from which $\nabla(1 \to 0) = 0$ and by (wF) we have $1 \to 0 = 0$. Conversely, if $1 \to 0 = 0$ and $\nabla a = 0$, then $a \leq 1 \to \nabla a = 1 \to 0 = 0$ from which $a = 0$. Now let us prove that even in the Case III where ∇ is not present, and we have a distributive join internalizing strong algebra $\mathcal{A} = (A, \leq, \wedge, 1, \to)$ if $1 \to 0 = 0$, then the defined $\mathcal{R}$ is serial. This generality will be useful later in the last section. For the proof, let P be a prime filter. We have to find a prime filter M such that $(P, M) \in \mathcal{R}$. Define $Q = \{x \in A | 1 \to x \in P\}$. Similar to what we had in the four cases above, Q is a filter and $(P, Q) \in \mathcal{R}$. Note that $0 \notin Q$, because otherwise, $1 \to 0 = 0 \in P$ which is impossible. Define

$$\Sigma = \{S \in F(\mathcal{A}) \mid (P, S) \in \mathcal{R} \text{ and } 0 \notin S\}.$$

The set Σ is non-empty because $Q \in \Sigma$. Moreover, in Σ any chain has an upper bound. The proof is similar to the Case III. Hence, by Zorn's lemma, Σ has a maximal element. Similar to the proof of the Case III, this M is prime which completes the proof. □

Theorem 18 (Completeness) *For any rule scheme $\mathcal{R} \subseteq \{N, H, P, F, wF\}$, the logic $i\mathbf{STL}(\mathcal{R})$ is complete with respect to the class of all $\mathbf{K}(\mathcal{R})$-Kripke models.*

Proof Since the Lindenbaum algebra of the logic $i\mathbf{STL}(\mathcal{R})$ is clearly a distributive join internalizing temporal algebra, then if we apply Theorem 17 on it, it produces a Kripke frame $\mathcal{K} = (W, \leq, R)$ and an embedding i. Then define $V : At(\mathcal{L}_\nabla) \to U(W, \leq)$ by $V(p) = i([p])$ where $[p]$ is the equivalence class of p in the Lindenbaum algebra. It is routine to check that $\{w \in W |\ w \Vdash B\} = i([B])$ for any formula $B \in \mathcal{L}_\nabla$. Therefore, if $\Gamma \Rightarrow A$ is valid in all $\mathbf{K}(\mathcal{R})$-Kripke models including $(W, \leq, R, V)$, we will have $i([\Gamma]) \subseteq i([A])$. Since i is an embedding, it implies $[\Gamma] \leq [A]$ which simply means that $i\mathbf{STL}(\mathcal{R}) \vdash \Gamma \Rightarrow A$. □

Lemma 2 *In Corollary 1, if $\mathcal{S}$ satisfies any rule scheme in $\{F, wF\}$, then so does $\mathcal{T}$.*

Proof Note that we defined $\nabla = f \nabla_\mathcal{S} f_!$. If $\mathcal{S} \in \mathbf{ST}(wF)$, then $\mathcal{T} \in \mathbf{ST}(wF)$ because for any $a \in \mathcal{Y}$, if $\nabla a = 0$, then $f \nabla_\mathcal{S} f_! a = 0$. Since f is an embedding, $\nabla_\mathcal{S} f_! a = 0$. Since $\mathcal{S} \in \mathbf{ST}(wF)$, we have $f_! a = 0$. Then $f_! a \leq 0$ implies $a \leq f(0)$. But $f(0) = 0$ because f is join preserving. Hence, $a = 0$. For (F), if $\mathcal{S} \in \mathbf{ST}(F)$, then we have $\nabla_\mathcal{S} f_! a \geq f_! a$ from which $\nabla a = f \nabla_\mathcal{S} f_! a \geq f f_! a \geq a$. The last inequality is from the adjunction $f_! \dashv f$. Hence, $\mathcal{T} \in \mathbf{ST}(F)$. □

Theorem 19 *Let X be a topological space, Y be an Alexandroff space and $f : X \to Y$ be a continuous surjection. Then for any ∇_Y over $\mathcal{O}(Y)$ and any valuation $V : At(\mathcal{L}_\nabla) \to \mathcal{O}(Y)$, there exist ∇_X over $\mathcal{O}(X)$ and a valuation $U : At(\mathcal{L}_\nabla) \to \mathcal{O}(X)$ such that for any sequent $\Gamma \Rightarrow A$, we have $(\mathcal{O}(X), \nabla_X, U) \vDash \Gamma \Rightarrow A$ iff*

$(\mathcal{O}(Y), \nabla_Y, V) \vDash \Gamma \Rightarrow A$. *Moreover, for any class* $C \in \{i\mathbf{ST}(F), i\mathbf{ST}(wF)\}$, *if* $(\mathcal{O}(Y), \nabla_Y) \in C$ *then* $(\mathcal{O}(X), \nabla_X) \in C$. *Hence, if* $X \vDash_C \Gamma \Rightarrow A$ *then* $Y \vDash_C \Gamma \Rightarrow A$.

Proof Let $\nabla_Y : \mathcal{O}(Y) \to \mathcal{O}(Y)$ be a join preserving map and $V : At(\mathcal{L}_\nabla) \to \mathcal{O}(Y)$. By Corollary 2, since f is a continuous surjection and Y is Alexandroff, there exists a join preserving map $\nabla_X : \mathcal{O}(X) \to \mathcal{O}(X)$ such that $f^{-1} : (\mathcal{O}(Y), \nabla_Y) \to (\mathcal{O}(X), \nabla_X)$ becomes a logical morphism. Therefore, f^{-1} commutes with all connectives of the language $\mathcal{L}_\nabla$. Define $U(p) = f^{-1}(V(p))$. For any formula $B \in \mathcal{L}_\nabla$, it is evident that $U(B) = f^{-1}(V(B))$. Now note that $(\mathcal{O}(X), \nabla_X, U) \vDash \Gamma \Rightarrow A$ iff $U(\Gamma) \subseteq U(A)$ iff $f^{-1}(V(\Gamma)) \subseteq f^{-1}(V(A))$. Since f is surjective, f^{-1} is an embedding. Thus, the last is equivalent to $V(\Gamma) \subseteq V(A)$ iff $(\mathcal{O}(Y), \nabla_Y, V) \vDash \Gamma \Rightarrow A$. Finally, note that if for any class C from the classes $i\mathbf{ST}(F)$ and $i\mathbf{ST}(wF)$, if $(\mathcal{O}(Y), \nabla_Y) \in C$ then $(\mathcal{O}(X), \nabla_X) \in C$, from Lemma 2. □

The following theorem uses the Kripke completeness to show that for the topological completeness theorem and for logics $i\mathbf{ST}$, $i\mathbf{ST}(F)$ and $i\mathbf{ST}(wF)$, even one fixed and large enough discrete space is sufficient. This means that despite the intuitionistic logic, **IPC**, these logics can not understand the difference between discrete sets (complete for classical logic) and topological spaces (complete for intuitionistic logic).

Theorem 20 (Topological Completeness Theorem, Strong version) *Let X be a set with cardinality at least $2^{\aleph_0}$. Consider X as a discrete space. Then:*

(i) If $X \vDash_{i\mathbf{ST}} A$ *then* $i\mathbf{STL} \vdash A$.
(ii) If $X \vDash_{i\mathbf{ST}(F)} A$ *then* $i\mathbf{STL}(F) \vdash A$.
(iii) If $X \vDash_{i\mathbf{ST}(wF)} A$ *then* $i\mathbf{STL}(wF) \vdash A$.

Proof For (i), let $\mathcal{K} = (W, \leq, R, V)$ be the Kripke model in the proof of Kripke completeness theorem. Note that $U(W, \leq)$ is Alexandroff. The cardinality of this space is at most $2^{\aleph_0}$, since the Lindenbaum algebra is countable. Hence, there exists a surjective function $f : X \to Y$. Since X is discrete, f is also continuous. Therefore, the claim follows from the last part of Theorem 19. The proofs for the other parts are similar. □

Remark 17 Note that the Theorem 20 is not true without the size condition. Interestingly, it is not true for a singleton set $X = \{0\}$. The reason is that in this space we always have $p \vee \neg p$. There are only two possibilities for $\nabla : \{0, 1\} \to \{0, 1\}$. Since $\nabla 0 = 0$, we have either $\nabla 1 = 0$ or $\nabla 1 = 1$. In the second case, ∇ collapses to identity and hence $p \vee \neg p$ is valid because validity is just the boolean validity. In the first case, since $\nabla 1 = 0$, we have $(\nabla 1 \cap V(p)) = 0 \leq 0$ which implies $1 \leq (V(p) \to 0)$. Hence, $(V(p) \to 0) = 1$ from which $[(V(p) \to 0) \cup V(p)] = 1$. However, $p \vee \neg p$ is not provable in neither of the logics $i\mathbf{ST}$, $i\mathbf{ST}(F)$ and $i\mathbf{ST}(wF)$, because all of them are sub-logics of **IPC**.

7.9 Sub-intuitionistic Logics

Sub-intuitionistic logics are the propositional logics of the weak implications. They are usually defined by weakening certain axioms and rules for the intuitionistic implication including the modus ponens rule and the implication introduction rule in the natural deduction system. As we have mentioned before, the logics of spacetime are also designed for the same purpose. In this section we will show how the structural logics of spacetime provide a well-behaved conservative extension for sub-intuitionistic logics. Moreover, we will also use spacetimes to provide a topological semantics for these logics.

First let us review some important sub-intuitionistic logics, introduced in (Visser 1981b), (Visser 1981a), (Restall 1994), (Celani and Jansana 2001), (Ardeshir and Hesaam 2008), (Ruitenburg 1991), (Corsi 1987), (Okada 1987), and (Došen 1993) and investigated extensively in (Ardeshir and Ruitenburg 1998), (Ardeshir 1995), (Alizadeh and Ardeshir 2006a), (Alizadeh 2009), (Alizadeh and Ardeshir 2006b), (Alizadeh and Ardeshir 2012), (Alizadeh and Ardeshir 2004), (Celani and Jansana 2005), (Sasaki 1999), and (Suzuki 1999). To complete the list we also define one new logic, **EKPC** and we will explain its behaviour later. Consider the following rules of the usual natural deduction system on sequents in the form $\Gamma \vdash A$, where $\Gamma \cup \{A\}$ is a finite set of formulas in the usual propositional language, i.e., $\{\top, \bot, \wedge, \vee, \rightarrow\}$:

Propositional Rules:

$$\frac{}{\Gamma \vdash \top}\ \top \qquad\qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash A}\ \bot$$

$$\frac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma \vdash C}\ \vee E \quad (i=0,1)\ \frac{\Gamma \vdash A_i}{\Gamma \vdash A_0 \vee A_1}\ \vee I$$

$$(i=0,1)\ \frac{\Gamma \vdash A_0 \wedge A_1}{\Gamma \vdash A_i}\ \wedge E \qquad\qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge I$$

$$\frac{A \vdash B}{\Gamma \vdash A \rightarrow B} \rightarrow I$$

Formalized Rules:

$$\frac{\Gamma \vdash A \rightarrow B \qquad \Gamma \vdash A \rightarrow C}{\Gamma \vdash A \rightarrow B \wedge C}\ (\wedge I)_f \quad \frac{\Gamma \vdash A \rightarrow C \qquad \Gamma \vdash B \rightarrow C}{\Gamma \vdash A \vee B \rightarrow C}\ (\vee E)_f$$

$$\frac{\Gamma \vdash A \rightarrow B \qquad \Gamma \vdash B \rightarrow C}{\Gamma \vdash A \rightarrow C}\ tr_f$$

Additional Rules:

$$\frac{\Gamma \vdash \top \rightarrow \bot}{\Gamma \vdash \bot}\ E \quad \frac{\Gamma \vdash A \qquad \Gamma \vdash A \rightarrow B}{\Gamma \vdash B}\ MP \quad \frac{\Gamma \vdash A}{\Gamma \vdash \top \rightarrow A}\ Cur$$

The logic **KPC** is defined as the logic of the system of all the propositional and formalized rules. **BPC** is defined as **KPC** + Cur; **EKPC** as **KPC** plus the rule E; **EBPC** as **BPC** plus the rule E; **KTPC** as **KPC** plus the rule MP and finally **IPC** is defined as **BPC** plus the rule MP.

Remark 18 First note that in the algebraic terminology, the rules state that the connective $\to$ is an implication that internalizes both the monoidal structure, i.e., the meet and the finite joins. Secondly, note that in defining the consequence relation $\vdash$ for sub-intuitionistic logics, we mostly follow (Celani and Jansana 2001), where **KPC** and **KTPC** are called wK_σ and $wK_\sigma(MP)$. Here, we follow the modal naming tradition to call them **KPC** and **KTPC** since, they are sound and complete with respect to the class of all and reflexive Kripke models, respectively. The final point to make is on the axiomatization of **BPC**. This logic can be also defined as **KPC** plus the relaxed version of $\to I$ as defined in (Ardeshir 1995):

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$$

To prove the equivalence, it is clear that the rule Cur is provable by this more strong version of $\to I$. Moreover, it is easy to show that the new system with this rule admits the weakening rule. Hence, the original $\to I$ is provable. For the converse, first we will show that using the rule Cur, $C \vdash D \to C$ is provable, for all the formulas C and D. First use Cur on C to prove $C \vdash \top \to C$ and since $D \vdash \top$, we have $C \vdash D \to \top$. By formalized tr, we have $C \vdash D \to C$. Coming back to the proof of the converse part, assume $\Gamma, A \vdash B$. It is easy to see that $\bigwedge \Gamma \wedge A \vdash B$ and then $\vdash \bigwedge \Gamma \wedge A \to B$, by the original version of $\to I$. By the foregoing point and the formalized $\wedge I$, we can prove $\bigwedge \Gamma \vdash A \to \bigwedge \Gamma \wedge A$, which implies $\bigwedge \Gamma \vdash A \to B$, by tr_f. Therefore, $\Gamma \vdash A \to B$.

Definition 20 By a propositional Kripke model for the usual propositional language $\mathcal{L}_p$, we mean a tuple $\mathcal{K} = (W, R, V)$, where W is a set, $R \subseteq W \times W$ is a binary relation over W (not necessarily transitive or reflexive) and $V : At(\mathcal{L}_p) \to P(W)$, where $At(\mathcal{L}_p)$ is the set of atomic formulas of $\mathcal{L}_p$ and $P(W)$ is the powerset of W. A propositional Kripke model is called persistent if $V(p)$ is R-upward closed, i.e., if $u \in V(p)$ and $(u, v) \in R$ then $v \in V(p)$. The model is called serial if R is serial, i.e., for all $u \in W$ there exists $v \in W$ such that $(u, v) \in R$. It is called reflexive if R is reflexive, i.e., $(w, w) \in R$, for all $w \in W$. It is called transitive if R is transitive, i.e., for all $u, v, w \in W$ if $(u, v) \in R$ and $(v, w) \in R$ then $(u, w) \in R$. It is called a rooted tree if it has an element r such that for any $w \neq r$ we have $(r, w) \in R$, it is transitive and for any $u, v, w \in W$, if $(u, w), (v, w) \in R$ and $u \neq v$ then exactly one of the cases $(u, v) \in R$ or $(v, u) \in R$ happens. The forcing relation for a propositional Kripke model is defined as usual using the relation R for implication, i.e., $u \Vdash A \to B$ if for any $v \in W$ that $(u, v) \in R$, if $v \Vdash A$ then $v \Vdash B$. A sequent $\Gamma \Rightarrow A$ is valid in a propositional Kripke model if for all $w \in W$, $\forall B \in \Gamma\ (w \Vdash B)$ implies $w \Vdash A$.

Theorem 21 (Soundness-Completeness for Sub-intuitionistic Logics)

(i) **KPC** *is sound and complete with respect to the class of all propositional Kripke models (Celani and Jansana 2001).*

(ii) **EKPC** *is sound and complete with respect to the class of all serial propositional Kripke models.*

(iii) **KTPC** *is sound and complete with respect to the class of all reflexive propositional Kripke models (Celani and Jansana 2001).*

(iv) **BPC** *is sound and complete with respect to the class of all transitive persistent propositional rooted Kripke trees. If $\Gamma = \emptyset$, the finite rooted transitive trees are sufficient (Ardeshir and Ruitenburg 1998).*

(v) **EBPC** *is sound and complete with respect to the class of all transitive serial persistent propositional rooted Kripke trees. If $\Gamma = \emptyset$, the finite rooted transitive serial trees are sufficient (Ardeshir and Hesaam 2008).*

(vi) **IPC** *is sound and complete with respect to the class of all transitive reflexive persistent propositional rooted Kripke trees. If $\Gamma = \emptyset$, the finite rooted transitive reflexive persistent trees are sufficient.*

Proof We have to prove the case of **EKPC**. For soundness, note that the rule E is valid in all serial Kripke models. Let (W, R, V) be such a model. If

$$\frac{\Gamma \vdash \top \to \bot}{\Gamma \vdash \bot}\ E$$

and for some $u \in W$, $u \Vdash \Gamma$, then by the validity of the premise, $u \Vdash \top \to \bot$. Since R is serial, there exists $v \in W$ such that $(u, v) \in R$. Hence, $v \Vdash \bot$, which is impossible. Hence, $u \nVdash \Gamma$ from which $u \Vdash \Gamma \Rightarrow \bot$. For completeness, use the Lindenbaum algebra for **EKPC**. This algebra is clearly a distributive join internalizing strong algebra that satisfies $1 \to 0 = 0$. Therefore, by part $(*)$ in the proof of Theorem 17, it is possible to embed the algebra into its canonical Kripke model with a serial relation R. Note that the Kripke frame from the proof of Theorem 17 is in the form $(W, =_W, R)$. Therefore, since the validity for ∇-free sequents in any model of the form $(W, =_W, R, V)$ is equivalent to its validity in the propositional Kripke model (W, R, V), the completeness follows. $\square$

Note that the language $\mathcal{L}_p$ is a fragment of the full language $\mathcal{L}_\nabla$. Therefore, it is meaningful to use spacetimes and Kripke models (not propositional Kripke models we have just defined) as models for sub-intuitionistic logics.

Theorem 22 (Embedding Theorem) *Assume $\Gamma \cup \{A\} \subseteq \mathcal{L}_p$, where $\mathcal{L}_p$ is the usual language of propositional logic. Then:*

(i) $\Gamma \vdash_{\mathbf{KPC}} A$ *iff* $\Gamma \vdash_{i\mathbf{STL}} A$ *iff* $i\mathbf{ST} \vDash \Gamma \Rightarrow A$ *iff* $\mathbf{K} \vDash \Gamma \Rightarrow A$.

(ii) $\Gamma \vdash_{\mathbf{EKPC}} A$ *iff* $\Gamma \vdash_{i\mathbf{STL}(wF)} A$ *iff* $i\mathbf{ST}(wF) \vDash \Gamma \Rightarrow A$ *iff* $\mathbf{K}(wF) \vDash \Gamma \Rightarrow A$.

(iii) $\Gamma \vdash_{\mathbf{KTPC}} A$ *iff* $\Gamma \vdash_{i\mathbf{STL}(F)} A$ *iff* $i\mathbf{ST}(F) \vDash \Gamma \Rightarrow A$ *iff* $\mathbf{K}(F) \vDash \Gamma \Rightarrow A$.

(iv) $\Gamma \vdash_{\mathbf{BPC}} A$ *iff* $\Gamma \vdash_{i\mathbf{STL}(P)} A$ *iff* $i\mathbf{ST}(P) \vDash \Gamma \Rightarrow A$ *iff* $\mathbf{K}(P) \vDash \Gamma \Rightarrow A$.

(v) $\Gamma \vdash_{\mathbf{EBPC}} A$ *iff* $\Gamma \vdash_{i\mathbf{STL}(P,wF)} A$ *iff* $i\mathbf{ST}(P, wF) \vDash \Gamma \Rightarrow A$ *iff* $\mathbf{K}(P, wF) \vDash \Gamma \Rightarrow A$.

(vi) $\Gamma \vdash_{\mathbf{IPC}} A$ *iff* $\Gamma \vdash_{i\mathbf{STL}(P,F)} A$ *iff* $i\mathbf{ST}(P, F) \vDash \Gamma \Rightarrow A$ *iff* $\mathbf{K}(P, F) \vDash \Gamma \Rightarrow A$.

Proof Let us start with the embedding of the sub-intuitionist logics into the logics of spacetime. This part is just the syntactical version of the algebraic fact that the connective $\to$ in a temporal algebra is really an implication which internalizes both the monoidal structure and the finite joins. However, to show the proof theoretical flavour of the system, let us present the proof trees for all sub-intuitionistic rules. This hopefully shows the more natural adjoint-based approach to implication compared to the sub-intuitionistic proposal.

To prove the embedding, we use induction on the length of the sub-intuitionistic proof. Note that all the axioms and the propositional rules except $\to I$ are available in the basic system i**STL**. Therefore, it remains to prove the formalized rules and the rule $\to I$. This is what we will do in the following proof trees. Note that by a double line rule, we mean the existence of an easy omitted proof tree between the upper part and the lower part of the double line and by the label S together with a double line, we mean that the omitted tree is a simple combination of the structural rules. For the formalized $\wedge I$, we have:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\overline{\nabla(A \to B), A \Rightarrow B}}}{\nabla(A \to B), \nabla(A \to C), A \Rightarrow C}\,S \qquad \dfrac{\overline{\overline{\nabla(A \to C), A \Rightarrow C}}}{\nabla(A \to B), \nabla(A \to C), A \Rightarrow C}\,S}{\nabla(A \to B), \nabla(A \to C), A \Rightarrow B \wedge C}\,R\wedge}{\nabla[(A \to B) \wedge (A \to C)], A \Rightarrow B \wedge C}}{(A \to B) \wedge (A \to C) \Rightarrow A \to (B \wedge C)}\,R\to}{(A \to B), (A \to C) \Rightarrow A \to (B \wedge C)}
$$

and for the formalized $\vee I$, we have:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\overline{\nabla(A \to C), A \Rightarrow C}}}{\nabla(A \to C), \nabla(B \to C), A \Rightarrow C}\,S \qquad S\,\dfrac{\overline{\overline{\nabla(B \to C), B \Rightarrow C}}}{\nabla(A \to C), \nabla(B \to C), B \Rightarrow C}}{\nabla(A \to C), \nabla(B \to C), A \vee B \Rightarrow C}\,L\vee}{\nabla[(A \to C) \wedge (B \to C)], A \vee B \Rightarrow C}}{(A \to C) \wedge (B \to C) \Rightarrow A \vee B \to C}\,R\to}{(A \to C), (B \to C) \Rightarrow A \vee B \to C}
$$

for the formalized tr, we have:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overline{\overline{\nabla(A \to B), A \Rightarrow B}} \qquad \overline{\overline{\nabla(B \to C), B \Rightarrow C}}}{\nabla(A \to B), \nabla(B \to C), A \Rightarrow C}\,cut}{\nabla[(A \to B) \wedge (B \to C)], A \Rightarrow C}}{(A \to B) \wedge (B \to C) \Rightarrow A \to C}\,\to I}{(A \to B), (B \to C) \Rightarrow A \to C}
$$

And finally for $\to I$ we have:

$$
\dfrac{\Rightarrow \top \qquad \dfrac{\dfrac{A \Rightarrow B}{\nabla\top, A \Rightarrow B}\,LW}{\top \Rightarrow A \to B}\,R\to}{\Rightarrow A \to B}\,cut
$$

Now we have to show that the additional rules are provable by their corresponding additional rules in the logics of spacetime. For Cur, we will use its characterization based on $\to I$ as mentioned in the Remark 18.

$$\dfrac{\dfrac{\nabla(\bigwedge\Gamma) \Rightarrow \nabla(\bigwedge\Gamma)}{\nabla(\bigwedge\Gamma) \Rightarrow \bigwedge\Gamma}\,P \qquad \dfrac{\Gamma, A \Rightarrow B}{\bigwedge\Gamma, A \Rightarrow B}}{\dfrac{\dfrac{\nabla(\bigwedge\Gamma), A \Rightarrow B}{\bigwedge\Gamma \Rightarrow A \to B}\,R\to}{\Gamma \Rightarrow A \to B}}\,cut$$

For MP and E we have:

$$\dfrac{\dfrac{A \to B \Rightarrow A \to B}{A \to B \Rightarrow \nabla(A \to B)}\,F \qquad \dfrac{}{A, \nabla(A \to B) \Rightarrow B}}{A, A \to B \Rightarrow B}\,L\to \qquad \dfrac{\dfrac{\dfrac{}{\top, \nabla(\top \to \bot) \Rightarrow \bot}}{\nabla(\top \to \bot) \Rightarrow \bot}}{\top \to \bot \Rightarrow \bot}\,wF$$

This completes the embedding part of the theorem. To complete the equivalences, it is enough to close the circle by coming back from the validity in the Kripke models to provability in the sub-intuitionistic logics. For **KPC**, by Theorem 21, it is sufficient to prove $\Gamma \Rightarrow A$ is valid in all propositional Kripke models. Let (W, R, V) be a propositional Kripke model. Consider the tuple $(W, =, R, V)$, where the order is just equality. This tuple is a Kripke model, since R is compatible with the equality and V maps atomic formulas to $=$-upward closed subsets of W that are just all subsets. Since $\Gamma \Rightarrow A$ is valid in all Kripke models, it is valid in $(W, =, R, V)$. However, the forcing in this model and the original propositional model is the same for ∇-free formulas. Therefore, $\Gamma \Rightarrow A$ is also valid in (W, R, V). For (ii) and (iii) the argument is similar. For (iv), again by Theorem 21, it is sufficient to prove the validity of $\Gamma \Rightarrow A$ in all transitive persistent Kripke trees. Let (W, R, V) be such a tree. Define $\leq_R$ as the reflexive extension of R, i.e., $R \cup \{(w, w) \in W^2 | w \in W\}$. Since the model is a tree, $\leq_R$ is a partial order. Since, R is transitive, R is also compatible with $\leq_R$ and hence $(W, \leq_R, R, V)$ is a Kripke frame. Moreover, note that $R \subseteq \leq_R$ and if a set is R-upward closed, it is also $\leq_R$ upward closed. Therefore, $(W, \leq_R, R, V)$ is a $\mathbf{K}(P)$-Kripke model and hence $\Gamma \Rightarrow A$ is valid in $(W, \leq_R, R, V)$. Again since the validity of $\Gamma \Rightarrow A$ in (W, R, V) is the same as validity in $(W, \leq_R, R, V)$ for ∇-free formulas, the theorem follows. The remained cases are similar to (iv). □

In the presence of the rule Cur, it is also possible to strenghten the topological completeness to capture the logics via one arbitrary infinite fixed Hausdorff space. For that matter, we need the following topological lemma:

Lemma 3 *Let X be an infinite Hausdorff space. Then every finite rooted tree is a surjective continuous image of X.*

Proof Let us first prove the following claims:

Claim I. For any natural numbers N and K, there exists a natural number $M = M_{N,K}$ such that for any Hausdorff space X with cardinality greater than or equal to

M, there are K many open mutually disjoint subspaces of X each of which has at least N elements.

Proof of the Claim I. We prove the claim by induction on N. For $N = 1$, pick $M_{1,K} = K$ and prove the claim by induction on K. For $K = 1$, it is enough to pick the whole space as the open subset. To prove the claim for $K + 1$, by IH, since $M_{1,K+1} = K + 1 \geq K$, it is possible to find at least K non-empty mutually disjoint open subsets $\{U_i\}_{i=1}^{K}$. Pick $\{x_i\}_{i=1}^{K}$ as some elements such that $x_i \in U_i$. It is possible because they are not empty. Since the space has at least $K + 1$ elements, there should be some point $x \notin \{x_i\}_{i=1}^{K}$. Now, use the Hausdorff condition to find a sequence $\{V_i\}_{i=1}^{K+1}$ of non-empty mutually disjoint open subsets. The argument is as follows. For any $1 \leq i \leq K$, there exist disjoint open subsets A_i and B_i such that $x \in A_i$ and $x_i \in B_i$. For any $1 \leq i \leq K$, take $V_i = U_i \cap B_i$ and also define $V_{K+1} = \bigcap_{i=1}^{K} A_i$. They are clearly open non-empty subsets that are mutually disjoint.

Now, if we have the claim for N, we want to prove it for $N + 1$. By IH we know that there exists M' that works for N and $K' = 2K$. We claim that $M = M'$ works for $N + 1$ and K. If X has at least M' elements, then there are at lest $2K$ mutually disjoint opens such that each of them has at least N elements. If we arrange these $2K$, to K pairs and compute their unions, then we have K opens, each of which contains at least $2N$ elements, which is greater than or equal to $N + 1$.

Claim II. For any natural number n, there exists a natural number m such that for any Hausdorff space with at least m elements and any finite rooted tree with at most n elements, there exists a continuous surjection from the space to the tree.

Proof of the Claim II. We will prove the claim by induction on n. For $n = 1$ pick $m = 1$ and use the constant function. For $n + 1$, by IH, we know that for n there exists an m'. Pick m as the number in the claim 1, for $N = m'$ and $K = n$. Therefore, the space X has at least n opens each of which contains at least m' elements. Call them $\{U_i\}_{i=1}^{n}$. Since the tree has $n + 1$ elements, there are at most n branches for the root such that each of them has at most n nodes. Call these branches $\{T_j\}_{j=1}^{r}$ for some $r \leq n$. By IH, we can find a surjective continuous function $f_i : U_i \to T_i$ for any $1 \leq i \leq r$. Now define $f : X \to T$ as the extension of the union of f_i's such that it sends any $x \notin \bigcup_{i=1}^{r} U_i$ to the root of the tree. The function is clearly surjective. For continuity, note that any open subset of the tree is an upward-closed subset which means that it is either equal to T or is a union of the upward-closed subsets of the T_i's. For the first case, $f^{-1}(T) = X$ which is open. For the second case, it is implied from the continuity of f_i and the condition that U_i is open.

To prove the theorem, let T be a rooted tree with n elements. Then by Claim II, there exists a bound m such that for any Hausdorff space X with at least m elements, there exists a continuous surjection from X to the tree. The theorem follows from the fact that X is infinite and hence has at least m elements. □

Definition 21 Let $\mathcal{R} \subseteq \{P, F, wF\}$ and X be a topological space. By $X \vDash^g_{\mathcal{R}} A$, we mean that for any spacetime $(\mathcal{O}(X), \nabla)$ and any $V : At(\mathcal{L}_p) \to \mathcal{O}(X)$, if $(\mathcal{O}(X), \nabla, V) \vDash i\mathbf{STL}(\mathcal{R})$ then $(\mathcal{O}(X), \nabla, V) \vDash A$.

Theorem 23 (Topological Completeness Theorem, Strong version) *Let X be an infinite Hausdorff space. Then:*

(i) *If $X \vDash^g_P A$ then* $\mathbf{BPC} \vdash A$.
(ii) *If $X \vDash^g_{P,wF} A$ then* $\mathbf{EBPC} \vdash A$.
(iii) *If $X \vDash^g_{P,F} A$ then* $\mathbf{IPC} \vdash A$.

Proof The proof is a truth transformation sequence starting from a propositional Kripke tree, going to an appropriate Kripke model and then to a suitable spacetime to finally land in a spacetime over X, using Theorem 19. More precisely, for (i), let (W, R, V) be a finite transitive rooted tree. To prove $\mathbf{BPC} \vdash A$, by Theorem 21, it is enough to show that $(W, R, V) \Vdash A$. As we have seen in the proof of Theorem 22, it is possible to define the Kripke model $\mathcal{K} = (W, \leq_R, R, V)$ such that $\mathcal{K} \vDash i\mathbf{STL}(P)$ and the validity of ∇-free formulas in (W, R, V) and $\mathcal{K}$ are equivalent. Therefore, it is enough to prove $\mathcal{K} \vDash A$. By Example 13, it is possible to turn the Kripke model $\mathcal{K}$ to the spacetime $\mathcal{S}_{\mathcal{K}}$ equipped with a valuation $\bar{V}$, again with the same validity for every sequents. Hence, we will show that $(\mathcal{S}_{\mathcal{K}}, \bar{V}) \vDash A$. By Lemma 3, there exists a surjective continuous function $f : X \to W$ where W is considered with the upset topology by the order $\leq_R$. By Theorem 19 and the fact that the order topology is Alexandroff, there are $\nabla : \mathcal{O}(X) \to \mathcal{O}(X)$ and $U : At(\mathcal{L}_\nabla) \to \mathcal{O}(X)$ such that the validity of any sequent in $(\mathcal{S}_{\mathcal{K}}, V)$ and $(\mathcal{O}(X), \nabla, U)$ are the same. Hence, it is enough to prove $(\mathcal{O}(X), \nabla, U) \vDash A$. Since, $(\mathcal{O}(X), \nabla, U)$, the topological model $(\mathcal{S}_{\mathcal{K}}, \bar{V})$ and the Kripke model $\mathcal{K}$ have the same validity and $\mathcal{K} \vDash i\mathbf{STL}(P)$, we have $(\mathcal{O}(X), \nabla, U) \vDash i\mathbf{STL}(P)$. Finally, since, $X \vDash^g_P A$, we have $(\mathcal{O}(X), \nabla, U) \vDash A$. The proofs for (ii) and (iii) are exactly the same. □

Acknowledgements We are really grateful to Majid Alizadeh, Mohammad Ardeshir, Raheleh Jalali and Masoud Memarzadeh for their thoughtful remarks and the invaluable discussions that we have had. We are also thankful to the anonymous reviewer for his/her detailed and helpful suggestions and comments. We also gratefully acknowledge the support provided by the Netherlands Organisation for Scientific Research under the grant 639.073.807.

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Chapter 8
Bounded Distributive Lattices with Two Subordinations

Sergio Celani and Ramon Jansana

Abstract In this paper we consider the notion of subordination on distributive lattices, equivalent to that of quasi-modal operator for distributive lattices introduced by Castro and Celani in 2004. We provide topological dualities for categories of distributive lattices with a subordination and then for some categories of distributive lattices with two subordinations, structures that we name bi-subordination lattices. We investigate three classes of bi-subordination lattices. In particular that of positive bi-subordination lattices.

Keywords Subordination relations on distributive lattices · Contact relations · Distributive lattices · Distributive lattices with operators · Quasi-modal operators

8.1 Introduction

Subordination algebras and contact algebras originate in the duality for compact Hausdorff spaces developed by de Vries (1962) where the algebraic duals of the spaces are complete Boolean algebras with a proximity relation. The relations on arbitrary Boolean algebras that satisfy the conditions in the definition of de Vries proximity relation are known as compingent relations. Deleting some of the conditions we have the subordination relations of Bezhanishvili et al. (2016). These relations also originate in the Region-based theory of space, where precontact rela-

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M. Mojtahedi et al. (eds.), *Mathematics, Logic, and their Philosophies*,
Logic, Epistemology, and the Unity of Science 49,
https://doi.org/10.1007/978-3-030-53654-1_8

tions on a Boolean algebra, as are called in Dimov and Vakarelov (2006c), were introduced by Düntsch and Vakarelov (2003, 2007) under the name of proximity relations as a weakening of the relations of the contact algebras studied in Dimov and Vakarelov (2006a, b). The precontact relations and the subordination relations of Bezhanishvili et al. (2016) are dual notions (*a* is related to *b* in one relation if and only if *a* is not related to the complement of *b* in the other). An equivalent concept to those of subordination relation and precontact relation is that of quasi-modal operator introduced by Celani in Celani (2001), where a topological duality for Boolean algebras with a quasi-modal operator is given.

The definition of subordination relation on a Boolean algebra does not mention the complement operation and therefore it can be considered for bounded distributive lattices as well as its equivalent concept of quasi-modal operator. This is done in Castro and Celani (2004) where the concept of quasi-modal operator for bounded distributive lattices is studied and a topological Priestley duality is given for bounded distributive lattices with two quasi-modal operators. From the results proved in Castro and Celani (2004) one easily obtains a duality for bounded distributive lattices with a subordination.

In this paper we study three kinds of distributive lattices with two subordination relations that we call bi-subordination lattices: the bi-subordination lattices where one subordination is included in the other, the bi-subordination lattices where one subordination is the converse of the other, and the positive bi-subordination lattices where the relation between one subordination and the other is similar to that between the box operation and the diamond operator in positive modal algebras. We present topological dualities for these classes of bi-subordination lattices. In order to be able to do it we introduce in detail topological dualities for several categories of bounded distributive lattices with a subordination given by different choices of morphisms between them. The dual objects are Priestley spaces endowed with a binary relation. Some of the results we report can be found in Castro and Celani (2004) but for completeness we decided to present them with full proofs, besides phrasing them in terms of subordination relations instead of quasi-modal operators.

After the preliminaries section we present in Sect. 8.3 the concepts of subordination, Δ-quasi-modal operator and ∇-quasi-modal operator for bounded distributive lattices as well as the concept of bi-subordination lattice. We also introduce some tools necessary for the dualities we present in Sect. 8.5. In Sect. 8.4 we discuss some examples of bi-subordination lattices and define the concept of positive bi-subordination lattice. In Sect. 8.5 we present the dualities for different categories of subordination lattices. The objects of the dual categories are Priestley spaces with two binary relations, one for each subordination. We extend the dualities to bi-subordination lattices in the natural way. Finally, in Sect. 8.6 we first discus the dualities for positive bi-subordination lattices that naturally result when we dualize each subordination by a relation. Then we present a different duality where the objects are Priestley spaces with a single binary relation in a similar way as one can obtain a Priestley duality for positive modal algebras by considering only one relation on the Priestley space dual to the distributive lattice reduct instead of considering one relation for the box operation and another one for the diamond operator.

8.2 Preliminaries

In this preliminaries section we introduce the most basic concepts and notation we need related to posets, lattices, and binary relations. The other concepts assumed to be known in the paper, like Priestley space, will be introduced when needed.

Let $\langle X, \leq \rangle$ be a partially ordered set (or poset). A set $U \subseteq X$ is an *upset* of X if for every $x, y \in X$, if $x \in U$ and $x \leq y$, then $y \in U$. The dual notion is that of downset, that is, a set $V \subseteq X$ is a *downset* of X if for every $x, y \in X$ such that $x \in V$ and $y \leq x$, we have $y \in V$.

We assume knowledge of bounded distributive lattices (Balbes and Dwinger 1974; Davey and Priestley 2002; Grätzer 2009). Let L be a bounded distributive lattice. Recall that a *filter* of L is a nonempty subset of L that is an upset w.r.t. the order of the lattice and is closed under the operation of meet. Dually, an *ideal* of L is a nonempty subset of L that is a downset w.r.t. the order of the lattice and is closed under the operation of join. A filter F of L is said to be *prime* if for every $a, b \in L$ such that $a \vee b \in F$ it holds that $a \in F$ or $b \in F$. If L is a Booelan lattice (i.e. a lattice where every element has a complement) the prime filters are known as *ultrafilters*. The filter generated by a set $H \subseteq L$ will be denoted by $[H)$ or by $\mathrm{Fg}(H)$ and the ideal generated by H by $(H]$ or $\mathrm{Ig}(H)$. Given $a \in L$, we write $[a)$ or $\mathrm{Fg}(a)$ for the filter generated by $\{a\}$ and $(a]$ or $\mathrm{Ig}(a)$ for the ideal generated by $\{a\}$. The set, and the lattice, of ideals of L will be denoted by $\mathrm{Id}(L)$ and that of its filters by $\mathrm{Fi}(L)$.

For every set X, we use $\mathcal{P}(X)$ to denote the powerset of X as well as the powerset lattice and the powerset Booelan algebra of (the subsets of) X.

If X is an arbitrary set and R a binary relation on X, then for every $x \in X$ we let

$$R(x) := \{y \in X : \langle x, y \rangle \in R\} \quad \text{and} \quad R^{-1}(x) := \{y \in X : \langle y, x \rangle \in R\}$$

and for every set $Y \subseteq X$ we let

$$\begin{aligned} R[Y] &:= \{y \in X : (\exists x \in Y)\, xRy\}, \\ R^{-1}[Y] &:= \{x \in X : (\exists y \in Y)\, xRy\}, \\ \Box_R(Y) &:= \{x \in X : R(x) \subseteq Y\}. \end{aligned}$$

Note that

$$R[Y] = \bigcup \{R(y) : y \in Y\} \quad \text{and} \quad R^{-1}[Y] = \{x \in X : R(x) \cap Y \neq \emptyset\}.$$

We also refer to $R^{-1}[Y]$ by $\Diamond_R(Y)$. Note that then $\Box_R(Y) = [\Diamond_R(Y^c)]^c$ and $\Diamond_R(Y) = [\Box_R(Y^c)]^c$. Moreover, we denote by R^{-1} the converse of the relation R, i.e., $R^{-1} = \{\langle x, y \rangle : yRx\}$.

8.3 Subordination Relations and Quasi-modal Operators on Distributive Lattices

The notion of subordination on a Boolean algebra defined in Bezhanishvili et al. (2016) is equivalent to the notion of precontact or proximity relation on a Boolean algebra given in Dimov and Vakarelov (2006c) and Düntsch and Vakarelov (2007). It can be exported to bounded distributive lattices since it does not involve the operation of complement.

Definition 1 A *subordination* on a bounded distributive lattice L is a binary relation $\prec$ on L satisfying the following conditions for every $a, b, c, d \in L$:

(S1) $0 \prec 0$ and $1 \prec 1$;
(S2) $a \prec b, c$ implies $a \prec b \wedge c$;
(S3) $a, b \prec c$ implies $a \vee b \prec c$;
(S4) $a \leq b \prec c \leq d$ implies $a \prec d$.

A *subordination lattice* is a pair $\langle L, \prec\rangle$ where L is a bounded distributive lattice and $\prec$ a subordination on L. A *bi-subordination lattice* is a triple $\langle L, \prec, \lhd\rangle$ where L is a bounded distributive lattice and $\prec, \lhd$ are subordinations on L.

We will denote by SLat the class of subordination lattices and by BSLat the class of bi-subordination lattices.

In the case of Boolean algebras, the subordination relations are equivalent to the quasi-modal operators of Celani (2001). Similarly, on bounded distributive lattices they are equivalent to the quasi-modal operators on bounded distributive lattices introduced in Castro and Celani (2004).

Definition 2 (Castro and Celani 2004) A Δ-*quasi-modal operator* on a bounded distributive lattice L is a map $\Delta : L \to \mathrm{Id}(L)$ satisfying the conditions:

(QM1) $\Delta(a \wedge b) = \Delta(a) \cap \Delta(b)$, for every $a, b \in L$,
(QM2) $\Delta(1) = L$,

that is, it is a meet-homomorphism (preserving also the top element) from the lattice L to the lattice of its ideals.

Dually, a ∇-*quasi-modal operator* on a bounded distributive lattice L is a map $\nabla : L \to \mathrm{Fi}(L)$ satisfying the conditions:

(QM3) $\nabla(a \vee b) = \nabla(a) \cap \nabla(b)$, for every $a, b \in L$,
(QM4) $\nabla(0) = L$,

that is, it is a join-homomorphism (preserving also the bottom element) from L to the dual of the lattice of the filters of L.

Remark 1 A dual modal operator $\Box$ on a bounded distributive lattice L is a unary operation on L that is a meet-homomorphism from L to L preserving the top element. The map that sends every element of L to the principal ideal it generates is an embedding from L to the lattice of the ideals of L. Thus we can look at a dual modal operator $\Box$ on a bounded distributive lattice L as a meet-homomorphism from L to the lattice $\mathrm{Id}(L)$ of the ideals of L that preserves also the top element and has the property that the elements of its range are principal ideals. In this way, the concept of Δ-quasi-modal operator on a bounded distributive lattice is a natural generalization of the notion of dual modal operator. Dually, an operator $\Diamond$ on a bounded distributive lattice L is a unary operation on L that is a join-homomorphism from L to L that preserves the bottom element and since L is dually embeddable into the lattice of the filters of L by the map that sends every element of L to the principal filter it generates, an operator $\Diamond$ on a bounded distributive lattice L can be seen as a join-homomorphism from L to the dual lattice of the lattice $\mathrm{Fi}(L)$ of the filters of L that in addition preserves the bottom element. Therefore, the concept of ∇-quasi-modal operator on a bounded distributive lattice is a natural generalization of the notion of modal operator.

Quasi-modal operators and subordination relations are strictly connected in the way we proceed to describe. Recall the well-known fact that any map $f : L \to \mathcal{P}(L)$ determines two relations $R_f, R_f^+ \subseteq L \times L$, one the converse of the other, defined by the conditions

$$aR_f b \quad \text{iff} \quad a \in f(b) \qquad \text{and} \qquad aR_f^+ b \quad \text{iff} \quad b \in f(a).$$

Conversely, every relation $R \subseteq L \times L$ determines two maps $f_R, f_R^+ : L \to \mathcal{P}(L)$ defined by the conditions

$$f_R(a) := R^{-1}(a) = \{b \in L : bRa\} \qquad \text{and} \qquad f_R^+(a) := R(a) = \{b \in L : aRb\}.$$

It is immediate to see that if $f : L \to \mathcal{P}(L)$, then $f_{R_f} = f$ and $f^+_{R_f^+} = f$ and that if $R \subseteq L \times L$, then $R_{f_R} = R$ and $R^+_{f_R^+} = R$.

We apply these facts to Δ-quasi-modal operators, ∇-quasi-modal operators and subordinations on L.

Let $f : L \to \mathcal{P}(L)$ be a map. It is easy to see that f is a Δ-quasi-modal operator if and only if its associated relation R_f is a subordination on L, and that f is a ∇-quasi-modal operator if and only if R_f^+ is a subordination on L.

If $\Delta : L \to \mathcal{P}(L)$ is a Δ-quasi-modal operator, then we denote the relation R_Δ by $\prec_\Delta$. Thus for every $a, b \in L$

$$a \prec_\Delta b \quad \text{iff} \quad a \in \Delta(b).$$

Analogously, if $\nabla : L \to \mathcal{P}(L)$ is a ∇-quasi-modal operator, then we denote the relation R_∇^+ by $\prec_\nabla$ and we have for every $a, b \in L$

$$a \prec_\nabla b \quad \text{iff} \quad b \in \nabla(a).$$

Consider now a binary relation R on L. It is easy to see that the function $f_R : L \to \mathcal{P}(L)$ is a Δ-quasi-modal operator on L if and only if R is a subordination and that this holds if and only if $f_R^+ : L \to \mathcal{P}(L)$ is a ∇-quasi-modal operator on L.

If $\prec$ is a subordination on L, then we denote the map $f_\prec$ by $\Delta_\prec$ and the map $f_\prec^+$ by $\nabla_\prec$. Hence, for every $a \in L$

$$\Delta_\prec(a) := \{b \in B : b \prec a\} \qquad \text{and} \qquad \nabla_\prec(a) := \{b \in B : a \prec b\}\,.$$

Since Δ-quasi-modal operators correspond to subordinations and these to ∇-quasi-modal operators, the procedures just described above allow us to associate with every Δ-quasi-modal operator a ∇-quasi-modal operator and conversely, in the following way.

Let L be a bounded distributive lattice and Δ a Δ-quasi-modal operator on L. The ∇-quasi-modal operator $\nabla_{\prec_\Delta}$ of the subordination $\prec_\Delta$ is then given for each $a \in L$ by

$$\nabla_{\prec_\Delta}(a) := \{b \in L : a \in \Delta(b)\}\,.$$

In a similar way, given a ∇-quasi-modal operator ∇, the Δ-quasi-modal operator of the subordination $\prec_\nabla$ is given for each $a \in L$ by

$$\Delta_{\prec_\nabla}(a) := \{b \in L : a \in \nabla(b)\}\,.$$

It immediately follows that $\Delta_{\nabla_{\prec_\Delta}} = \Delta$ and $\nabla_{\Delta_{\prec_\nabla}} = \nabla$.

Note that due to the equivalence between subordinations and Δ-(∇-)modal operators, Remark 1 shows that subordinations can be taken as generalizations of modal operators.

Remark 2 If L is a bounded distributive lattice, $\Box$ a dual modal operator on L and $\Diamond$ a modal operator on L, then it is easy to see that the binary relations $\prec_\Box$ and $\prec_\Diamond$ defined on L by setting for every $a, b \in L$

$$a \prec_\Box b \iff a \leq \Box b$$

and

$$a \prec_\Diamond b \iff \Diamond a \leq b$$

are subordinations on L.

The Δ-quasi-modal operator $\Delta_{\prec_\Box}$ associated with $\prec_\Box$ satisfies that $\Delta_{\prec_\Box}(a) = (\Box a]$ for all $a \in L$. The ∇-quasi-modal operator $\nabla_{\prec_\Box}$ of $\prec_\Box$ is then given by the condition $b \in \nabla_{\prec_\Box}(a)$ if and only if $a \leq \Box b$. Therefore, $\nabla_{\prec_\Box}(a) = \Box^{-1}[[a)]$ for every $a \in B$.

Similarly, the ∇-quasi-modal operator associated with $\prec_\Diamond$ satisfies for every $a \in L$ that $\nabla_{\prec_\Diamond}(a) = [\Diamond a)$. The Δ-quasi-modal operator of $\prec_\Diamond$ is then given for every $a, b \in L$ by the condition $b \in \Delta_{\prec_\Diamond}(a)$ if and only if iff $\Diamond b \leq a$. Thus, for every $a \in L$ we have $\Delta_{\prec_\Diamond}(a) = \Diamond^{-1}[(a]]$.

Being the notions of subordination relation and Δ-quasi-modal operator equivalent, as well as equivalent to that of ∇-quasi-modal operator, we can take any of them as a primitive notion. We decided to take the notion of subordination as primitive in this paper; nevertheless we will make use of the associated quasi-modal operators on some proofs and statements.

In Castro and Celani (2004) the authors introduce and study quasi-modal lattices which consist of a bounded distributive lattice together with both a Δ-quasi-modal operator and a ∇-quasi-modal operator. Thus they consider in disguise bounded distributive lattices with two subordinations, i.e., bi-subordination lattices.

We proceed to introduce in the remaining part of this section some tools that are essential to the presentation of the results in the paper.

8.3.1 Two Maps on the Power Set of a Subordination Lattice Determined by the Subordination Relation

Given a bounded distributive lattice with a subordination we define two maps on the poset of all subsets of the lattice determined by the subordination and present the properties we need. One is a modal operator and the other its dual. Using them we will define two relations on the set of prime filters of a bounded distributive lattice with a subordination.

Let L be a bounded distributive lattice and $\prec$ a subordination on L. The maps $\Delta_{\prec}^{-1} : \mathcal{P}(L) \to \mathcal{P}(L)$ and $\nabla_{\prec}^{-1} : \mathcal{P}(L) \to \mathcal{P}(L)$ are defined by setting for every $C \subseteq L$:

1. $\Delta_{\prec}^{-1}(C) := \{a \in L : \Delta_{\prec}(a) \cap C \neq \emptyset\}$,
2. $\nabla_{\prec}^{-1}(C) := \{a \in L : \nabla_{\prec}(a) \subseteq C\}$.

These two maps are obviously monotone (w.r.t. inclusion), $\nabla_{\prec}^{-1}$ distributes over intersections, $\Delta_{\prec}^{-1}$ over unions, $\Delta_{\prec}^{-1}(\emptyset) = \emptyset$, and $\nabla_{\prec}^{-1}(L) = L$. It is easy to see that for every $C \subseteq L$,

$$\Delta_{\prec}^{-1}(C) = (\nabla_{\prec}^{-1}(C^c))^c \quad \text{and} \quad \nabla_{\prec}^{-1}(C) = (\Delta_{\prec}^{-1}\left(C^c\right))^c.$$

Hence, $\Delta_{\prec}^{-1}$ is a modal operator on the powerset lattice $\mathcal{P}(L)$ and $\nabla_{\prec}^{-1}$ is its dual.

The first two items of the next lemma are proved in Castro and Celani (2004).

Lemma 1 *For every filter F, every ideal I, and every prime filter P of L:*

1. *$\Delta_{\prec}^{-1}(F)$ is a filter,*
2. *$\nabla_{\prec}^{-1}(I)$ is an ideal,*
3. *$\left(\nabla_{\prec}^{-1}(P)\right)^c$ is an ideal.*

Proof We abbreviate all along the proof $\Delta_{\prec}$ by Δ and $\nabla_{\prec}$ by ∇.

(1) Suppose that $a, b \in \Delta^{-1}(F)$. Then $\Delta(a) \cap F \neq \emptyset$ and $\Delta(b) \cap F \neq \emptyset$. Let $c \in \Delta(a) \cap F$ and $d \in \Delta(b) \cap F$. Then $c \wedge d \in F$ and $c \wedge d \in \Delta(a) \cap \Delta(b)$, because

these sets are ideals. Hence $c \wedge d \in \Delta(a \wedge b)$. Therefore, $\Delta(a \wedge b) \cap F \neq \emptyset$ and so $a \wedge b \in \Delta^{-1}(F)$. Suppose now that $a \in \Delta^{-1}(F)$ and $a \leq b$. Then $\Delta(a) \cap F \neq \emptyset$. Since $\Delta(a) \subseteq \Delta(b)$, $\Delta(b) \cap F \neq \emptyset$. Hence $b \in \Delta^{-1}(F)$.

(2) Suppose that $a, b \in \nabla^{-1}(I)$. Then $\nabla(a) \subseteq I$ and $\nabla(b) \subseteq I$. Therefore $\nabla(a \vee b) = \nabla(a) \cap \nabla(b) \subseteq I$. Hence $a \vee b \in \nabla^{-1}(I)$. Suppose that $a \in \nabla^{-1}(I)$ and $b \leq a$. Then $\nabla(a) \subseteq I$ and $\nabla(a) \subseteq \nabla(b)$. Therefore, $\nabla(b) \subseteq I$. Thus, $b \in \nabla^{-1}(I)$.

(3) Let P be a prime filter of L. First note that since $\nabla(0) = L$ and $P \neq L$, we have $\nabla(0) \nsubseteq P$. Therefore $0 \notin \nabla^{-1}(P)$. Suppose now that $a, b \notin \nabla^{-1}(P)$. Hence there are $c \in \nabla(a)$ and $d \in \nabla(b)$ such that $c, d \notin P$. Since P is a prime filter it follows that $c \vee d \notin P$. But since $\nabla(a), \nabla(b)$ are filters $c \vee d \in \nabla(a) \cap \nabla(b) = \nabla(a \vee b)$. Hence $\nabla(a \vee b) \nsubseteq P$ and therefore $a \vee b \notin \nabla^{-1}(P)$. Finally, suppose that $a \notin \nabla^{-1}(P)$ and $b \leq a$. Then $\nabla(a) \nsubseteq P$. But since $\nabla(a) \subseteq \nabla(b)$, $\nabla(b) \nsubseteq P$ which implies that $b \notin \nabla^{-1}(P)$. □

8.3.2 The Two Relations on the Set of Prime Filters of a Lattice Determined by a Subordination

Let L be a bounded distributive lattice and $\prec$ a subordination on L. We define the binary relation $R_{\prec}^{\Delta}$ on the set $X(L)$ of the prime filters of L by setting for every $P, Q \in X(L)$

$$(P, Q) \in R_{\prec}^{\Delta} \iff \Delta_{\prec}^{-1}(P) \subseteq Q.$$

In a similar way, we define the binary relation $R_{\prec}^{\nabla}$ on $X(L)$ by setting for every $P, Q \in X(L)$

$$(P, Q) \in R_{\prec}^{\nabla} \iff Q \subseteq \nabla_{\prec}^{-1}(P).$$

Proposition 1 *Let $\prec$ be a subordination on a bounded distributive lattice L. Then $R_{\prec}^{\Delta}$ is the converse of the relation $R_{\prec}^{\nabla}$.*

Proof Suppose that $P R_{\prec}^{\Delta} Q$, i.e., that $\{a \in L : \Delta_{\prec}(a) \cap P \neq \emptyset\} \subseteq Q$. To prove that $Q R_{\prec}^{\nabla} P$ we have to show that $P \subseteq \{a \in L : \nabla_{\prec}(a) \subseteq Q\}$. Suppose that $a \in P$ and $\nabla_{\prec}(a) \nsubseteq Q$. Let $b \in \nabla_{\prec}(a)$ be such that $b \notin Q$. Thus $b \notin \{a \in L : \Delta_{\prec}(a) \cap P \neq \emptyset\}$, that is, $\Delta_{\prec}(b) \cap P = \emptyset$. Note that since $b \in \nabla_{\prec}(a)$, $a \in \Delta_{\prec}(b)$. Therefore $\Delta_{\prec}(b) \cap P \neq \emptyset$. Hence $b \in Q$, a contradiction.

Conversely, suppose that $Q R_{\nabla_{\prec}} P$, so that $P \subseteq \{a \in L : \nabla_{\prec}(a) \subseteq Q\}$. To prove that $P R_{\prec}^{\Delta} Q$, suppose that $\Delta_{\prec}(a) \cap P \neq \emptyset$ and let $b \in \Delta_{\prec}(a) \cap P$. Then $a \in \nabla_{\prec}(b)$ and $\nabla_{\prec}(b) \subseteq Q$. Therefore, $a \in Q$. □

Lemma 2 *Let L be a bounded distributive lattice and $\prec$ a subordination on L. The relations $R_{\prec}^{\Delta}$ and $R_{\prec}^{\nabla}$ satisfy the following conditions:*

1. $R_{\prec}^{\Delta} = (\subseteq \circ R_{\prec}^{\Delta})$,
2. $R_{\prec}^{\Delta} = (R_{\prec}^{\Delta} \circ \subseteq)$,
3. $R_{\prec}^{\nabla} = (\subseteq^{-1} \circ R_{\prec}^{\nabla})$,
4. $R_{\prec}^{\nabla} = (R_{\prec}^{\nabla} \circ \subseteq^{-1})$.

Proof (1) The inclusion $R^{\Delta}_{\prec} \subseteq (\subseteq \circ R^{\Delta}_{\prec})$ is obvious. To prove the other inclusion assume that $P \subseteq Q'$ and $Q' R^{\Delta}_{\prec} Q$. Then $\{a \in L : \Delta_{\prec}(a) \cap Q' \neq \emptyset\} \subseteq Q$. Since $P \subseteq Q'$, we have $\{a \in L : \Delta_{\prec}(a) \cap P \neq \emptyset\} \subseteq Q$, and we are done. (2) follows easily form the definitions involved. (3) and (4) follow from (2) and (1) respectively using that $R^{\Delta}_{\prec}$ is the converse of $R^{\nabla}_{\prec}$. □

Lemma 2 is basically Lemma 5 in Castro and Celani (2004) and the next lemma is Lemma 6 in Castro and Celani (2004).

Lemma 3 *Let L be a bounded distributive lattice and $\prec$ a subordination on L. Let $a \in L$ and $P \in X(L)$. Then*

1. $a \in \Delta^{-1}_{\prec}(P)$ iff $(\forall Q \in X(L))$(if $\Delta^{-1}_{\prec}(P) \subseteq Q$, then $a \in Q$),
2. $a \in \nabla^{-1}_{\prec}(P)$ iff $(\exists Q \in X(L))(Q \subseteq \nabla^{-1}_{\prec}(P)$ and $a \in Q)$.

Proof (1) follows from the fact that $\Delta^{-1}_{\prec}(P)$ is a filter. (2) follows from the fact that $\nabla^{-1}_{\prec}(P)^c$ is an ideal. The direction from right to left is obvious. Assume that $a \in \nabla^{-1}_{\prec}(P)$. Hence $a \notin \nabla^{-1}_{\prec}(P)^c$. Thus since this last set is an ideal, there exist $Q \in X(L)$ such that $a \in Q$ and $\nabla^{-1}_{\prec}(P)^c \cap Q = \emptyset$. Hence $Q \subseteq \nabla^{-1}_{\prec}(P)$. □

8.4 Some Kinds of Bi-Subordination Lattices

We are interested in some kinds of bi-subordination lattices $L = \langle L, \prec, \lhd \rangle$. In one kind $\prec \subseteq \lhd$, in another $\lhd = \prec^{-1}$. Finally, we are interested in positive bi-subordination lattices where the link between the subordinations $\prec$ and $\lhd$ is similar to the link between the $\Box$ and $\Diamond$ in positive modal algebras.

Definition 3 A bi-subordination lattice $L = \langle L, \prec, \lhd \rangle$ is a *positive bi-subordination lattice* if the following conditions hold for all $a, b, c \in L$:

(P1) $c \prec a \vee b \implies (\forall d \in L)(a \lhd d \Rightarrow (\exists e \in L)(e \prec b \;\&\; c \leq e \vee d))$
(P2) $a \wedge b \lhd c \implies (\forall d \in L)(d \prec a \Rightarrow (\exists e \in L)(b \lhd e \;\&\; e \wedge d \leq c))$.

The conditions (P1) and (P2) can be stated in an equivalent form using the operators $\Delta_{\prec}$ and $\nabla_{\lhd}$. To do it we need to introduce the following operations between filters and ideals and between ideals and filters of a bounded distributive lattice.

Let L be a bounded distributive lattice, $F \in \mathrm{Fi}(L)$ and $I \in \mathrm{Id}(L)$. We define the following ideal and filter, respectively

$$F \odot I := \bigcap \{(I \cup \{f\}] : f \in F\}$$

and

$$I \oplus F := \bigcap \{[F \cup \{i\}) : i \in I\}.$$

In terms of the operators $\Delta_{\prec}$ and $\nabla_{\lhd}$ the conditions (P1) and (P2) respectively say that for all $a, b \in L$,

1. $\Delta_{\prec}(a \vee b) \subseteq \nabla_{\lhd}(a) \odot \Delta_{\prec}(b)$,
2. $\nabla_{\lhd}(a \wedge b) \subseteq \Delta_{\prec}(a) \oplus \nabla_{\lhd}(b)$.

We proceed to provide examples of the three kinds of bi-subordination lattices we are interested in.

Example 1 Let $\langle X, \tau \rangle$ be a topological space. The relations $\prec$ and $\lhd$ defined on $\mathcal{P}(X)$ by

$$U \prec V \;\Leftrightarrow\; U \subseteq \operatorname{int}(V)$$

and

$$U \lhd V \;\Leftrightarrow\; \operatorname{cl}(U) \subseteq V$$

are easily seen to be subordinations. Thus $\langle \mathcal{P}(X), \prec, \lhd \rangle$ is bi-subordination lattice. We note that the quasi-modal operators $\Delta_{\prec}$ and $\nabla_{\lhd}$ satisfy that

$$\Delta_{\prec}(U) = (\operatorname{int}(U)]$$

and

$$\nabla_{\lhd}(U) = [\operatorname{cl}(U))$$

for each $U \in \mathcal{P}(X)$.

If we restrict $\prec$ and $\lhd$ respectively to the distributive lattices of the open sets of X and of the closed sets of X we obtain bounded distributive lattices with two subordinations, which are one included in the other. Indeed, if U, V are closed then

$$U \prec V \;\Leftrightarrow\; U \subseteq \operatorname{int}(V) \;\Leftrightarrow\; \operatorname{cl}(U) \subseteq \operatorname{int}(V) \;\Rightarrow\; U \lhd V.$$

Also, if U, V are open, then

$$U \lhd V \;\Leftrightarrow\; \operatorname{cl}(U) \subseteq V \;\Leftrightarrow\; \operatorname{cl}(U) \subseteq \operatorname{int}(V) \;\Rightarrow\; U \subseteq \operatorname{int}(V) \;\Leftrightarrow\; U \prec V.$$

Example 2 Recall that a distributive double p-algebra $\langle L, \vee, \wedge, ^*, ^+, 0, 1 \rangle$, see Katriňák (1973), is a double *Stone algebra* if $a^* \vee a^{**} = 1$ and $a^+ \wedge a^{++} = 0$. In a double Stone algebra L the following properties are valid:

1. $a^* \leq a^+$.
2. $a^{+*} = a^{++} \leq a \leq a^{**} = a^{*+}$.
3. $(a \wedge b)^* = a^* \vee b^*$ and $(a \vee b)^+ = a^+ \wedge b^+$.

Double Stone algebras are considered by Katriňák in Katriňák (1974) and in several papers by the same author. For information on Stone algebras see Grätzer (2009) and for double Stone algebras see also Balbes and Dwinger (1974).

If $\langle L, \vee, \wedge, ^*, ^+, 0, 1\rangle$ is distributive double p-algebra it is easily seen that the relation $\prec$ on L defined by

$$a \prec b \;\Leftrightarrow\; a^* \vee b = 1,$$

is a subordination and that the relation $\lhd$ defined by

$$a \lhd b \;\Leftrightarrow\; b^+ \wedge a = 0$$

is also a subordination.

On a double Stone algebra both subordination relations are equal. In fact, a distributive double p-algebra L is a double Stone algebra if and only if $\prec = \lhd$.

Proposition 2 *Let $\langle L, \vee, \wedge, ^*, ^+, 0, 1\rangle$ be a distributive double p-algebra. Then L is a double Stone algebra if and only if for every $a, b \in L$,*

$$a^* \vee b = 1 \;\Leftrightarrow\; b^+ \wedge a = 0$$

Proof Suppose that L is a double Stone algebra. Then for every $a, b \in L$ we have:

$$\begin{aligned} a^* \vee b = 1 \;&\Rightarrow\; (a^* \vee b)^+ = 1^+ \\ &\Leftrightarrow\; a^{*+} \wedge b^+ = 0 \\ &\Rightarrow\; a \wedge b^+ = 0 \end{aligned}$$

and

$$\begin{aligned} a \wedge b^+ = 0 \;&\Rightarrow\; (a \wedge b^+)^* = 0^* \\ &\Leftrightarrow\; (a^* \vee b^{+*}) = 1 \\ &\Rightarrow\; a^* \vee b = 1. \end{aligned}$$

Now assume that for every $a, b \in L$, $a^* \vee b = 1$ if and only if $b^+ \wedge a = 0$. Let $a \in L$. Since $a^{*+} \wedge a^* = 0$, we obtain that $a^{**} \vee a^* = 1$. And since $a^{+*} \vee a^+ = 1$ we obtain $a^{++} \wedge a^+ = 0$. Hence, L is a double Stone algebra. □

The quasi-modal operators $\Delta_\prec$ and $\nabla_\prec$ associated with $\prec$ have the following description:

$$\Delta_\prec(a) = \{x \in L : x^* \vee a = 1\}$$

$$\nabla_\prec(a) = \{x \in L : x^+ \wedge a = 0\}.$$

Proposition 3 *Let L be a double Stone algebra. Then the bi-subordination lattice $\langle L, \prec, \prec\rangle$ is a positive bi-subordination lattice.*

Proof We proceed to prove that it satisfies the conditions (P1) and (P2) in Definition 3. We will work with the equivalent conditions stated in terms of the delta and nabla operators.

To prove that the condition (P1) holds, suppose that $a, b, c \in L$ are such that $c \in \Delta_{\prec}(a \vee b)$ but $c \notin \nabla_{\prec}(a) \odot \Delta_{\prec}(b)$. Then there exists $d \in \nabla_{\prec}(a)$ such that $c \notin (\Delta_{\prec}(b) \cup \{d\}]$. So, there exists $P \in X(L)$ such that $c \in P$, $\Delta_{\prec}(b) \cap P = \emptyset$ and $d \notin P$. Since $d \vee d^{+} = 1 \in P$, we get $d^{+} \in P$, and as $d \in \nabla_{\prec}(a)$, we have $d^{+} \wedge a = 0$. So

$$0 = 0^{**} = (d^{+} \wedge a)^{**} = d^{+**} \wedge a^{**} = d^{+++} \wedge a^{**} = d^{+++} \wedge a^{**} = d^{+} \wedge a^{**}.$$

Then, since $d^{+} \in P$, $a^{**} \notin P$. So $a^{*} \in P$, because L is a Stone algebra, and since $c \in P$, we get $c^{**} \in P$. Thus, $a^{*} \wedge c^{**} \in P$. Moreover,

$$1 = c^{*} \vee a \vee b \leq b \vee a^{**} \vee c^{*} = b \vee (a^{*} \wedge c^{**})^{*},$$

so that $b \vee (a^{*} \wedge c^{**})^{*} = 1$ and therefore $a^{*} \wedge c^{**} \in \Delta_{\prec}(b)$. Now since $\Delta_{\prec}(b) \cap P = \emptyset$, it follows that $a^{*} \wedge c^{**} \notin P$, which is a contradiction.

Now, to prove that the condition (P2) holds, suppose that there are elements $a, b, c \in L$ such that $c \in \nabla_{\prec}(a \wedge b)$, but $c \notin \Delta_{\prec}(a) \oplus \nabla_{\prec}(b)$. Therefore $c^{+} \wedge (a \wedge b) = 0$ and there exists $P \in X(L)$ and $d \in \Delta_{\prec}(a)$ such that $c \notin P$, $\nabla(b) \subseteq P$, and $d \in P$. So, $1 = d^{*} \vee a$, and therefore $1 = 1^{++} = (d^{*} \vee a)^{++} = d^{*++} \vee a^{++} = d^{*} \vee a^{++} \in P$. Since $d \in P$, $d^{+} \notin P$. Therefore, $a^{++} \in P$. We note that $c^{+} \in P$, because $c \notin P$. So, $a \vee c^{+} \in P$ and therefore, $a^{+} \vee c^{++} \notin P$. As $c \in \nabla_{\prec}(a \wedge b)$,

$$\begin{aligned} 0 &= a \wedge b \wedge c^{+} = a^{++} \wedge c^{+} \wedge b \\ &= (a^{+} \vee c^{++})^{+} \wedge b. \end{aligned}$$

Then $a^{+} \vee c^{++} \in \nabla_{\prec}(b) \subseteq P$, which is a contradiction.

Thus we have that for every double Stone algebra L the bi-subordination lattice $\langle L, \prec, \prec\rangle$ is a positive bi-subordination lattice. □

Example 3 This example is given in Bezhanishvili (2013) for bounded sublattices of Boolean algebras. It can be extended to bounded sublattices of bounded distributive lattices. Let L be a bounded distributive lattice and let S be a bounded sublattice of L. We consider the relations $\prec_S$ and $\lhd_S$ defined by

$$a \prec_S b \iff (\exists c \in S)\, a \leq c \leq b$$

and

$$a \lhd_S b \iff (\exists c \in S)\, b \leq c \leq a.$$

These two relations are easily seen to be subordination relations and each one is the converse relation of the other.

The operators associated with the relations $\prec_S$ and $\lhd_S$ are given by

$$\Delta_{\prec_S}(a) = \{b \in L : S \cap [b) \cap (a] \neq \emptyset\}$$

and

$$\nabla_{\lhd_S}(a) = \{b \in L : S \cap [a) \cap (b] \neq \emptyset\}\,.$$

An element a of a bounded lattice L is said to be *complemented* if there is $b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$. The set of all complemented elements of L is called the *center* of L. The center of L contains 0 and 1. Moreover, if L is distributive, the complements when they exist are unique. This implies that the center of a bounded distributive lattice L is a bounded sublattice of L and a Boolean lattice.

Proposition 4 *Let L be a bounded distributive lattice. If S is a bounded sublattice of the center of L, then $\langle L, \prec_S, \lhd_S \rangle$ is a positive bi-subordination lattice.*

Proof We note that S is a bounded sublattice of the center of L if and only if it is a Boolean lattice. Thus, $P \cap S$ is an ultrafilter of S for each prime filter P of L.

We note that $\Delta^{-1}_{\prec_S}(P) \subseteq Q$ if and only if $P \cap S \subseteq Q$, for all $P, Q \in X(L)$. Indeed: If $a \in P \cap S$, then $a \in \Delta_{\prec_S}(a)$, and so $\Delta_{\prec_S}(a) \cap P \neq \emptyset$ having then that $a \in Q$. Conversely, if $\Delta_{\prec_S}(a) \cap P \neq \emptyset$, there exists $b \in \Delta_{\prec_S}(a) \cap P$ and there exits $s \in S$ such that $b \leq s \leq a$. Then $s \in P \cap S \subseteq Q$, and thus $a \in Q$.

Suppose that there are elements $a, b, c \in L$ such that $c \in \Delta_{\prec_S}(a \vee b)$, but $c \notin \nabla_{\lhd_S}(a) \odot \Delta_{\prec_S}(b)$. So, there exists $d \in \nabla_{\lhd_S}(a)$ such that $c \notin \left(\Delta_{\prec_S} b \cup \{d\}\right]$. Then there exists $P \in X(L)$ such that $c \in P$, $\Delta_{\prec_S} b \cap P = \emptyset$ and $d \notin P$. Therefore, there exists $Q \in X(L)$ such that $\Delta^{-1}_{\prec_S}(P) \subseteq Q$ and $b \notin Q$., i.e., $P \cap S \subseteq Q$. As S is a Boolean lattice, $P \cap S = Q \cap S$. Since $c \in \Delta_{\prec_S}(a \vee b) \cap P$ and $b \notin Q$, we have $a \in Q$. And since $d \in \nabla_{\lhd_S}(a)$, there exists $e \in S$ such that $a \leq e \leq y$. So, $s' \in Q \cap S = P \cap S$, and thus $d \in P$, which is impossible. Therefore $\Delta_{\prec_S}(a \vee b) \subseteq \nabla_{\lhd_S}(a) \odot \Delta_{\prec_S} b$, for all $a, b \in L$. The proof of the inclusion $\nabla_{\lhd_S}(a \wedge b) \subseteq \Delta_{\prec_S} a \oplus \nabla(b)$ is similar. $\square$

8.5 Duality for Subordination Lattices and Bi-Subordination Lattices

We recall first the Priestley topological duality between bounded distributive lattices and Priestley spaces (see for example Davey and Priestley 2002) and then we expand it to subordination lattices and Priestley subordination spaces. The duality for subordiantion lattices we present can be extracted from that in Castro and Celani (2004) which is for distributive lattices with a Δ and a ∇ quasi-modal operator and Priestley spaces with two binary relations. A duality for bi-subordination lattices, which is basically the duality obtained in Castro and Celani (2004), easily follows from the duality we describe for subordination lattices. For completeness we opted to give the details.

A *totally order-disconnected* topological space is a triple $X = \langle X, \leq, \tau_X \rangle$ where $\langle X, \leq \rangle$ is a poset, $\langle X, \tau_X \rangle$ is a topological space, and given $x, y \in X$ such that $x \nleq y$ there exists a clopen upset U of X such that $x \in U$ and $y \notin U$. A *Priestley space* is a compact and totally order-disconnected topological space.

If X is a Priestley space, the set of all clopen upsets of X is denoted by $D(X)$. It is well-known that $D(X) = \langle D(X), \cup, \cap, \emptyset, X\rangle$ is a bounded distributive lattice, which is a sublattice of the complete lattice $\mathcal{P}_u(X)$ of all the upsets of X. The lattice $D(X)$ is the dual of the Priestley space X.

If $L = \langle L, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice, we denote the set of all prime filters of L by $X(L)$ and recall that we denote the families of all ideals and all filters of L by $\mathrm{Id}(L)$ and $\mathrm{Fi}(L)$, respectively. Given a bounded distributive lattice L, the representation map is the function $\sigma_L : L \to \mathcal{P}_u(X(L))$ given for every $a \in L$ by

$$\sigma_L(a) := \{P \in X(L) : a \in L\}.$$

It is a one-to-one lattice homomorphism, i.e. $L \cong \sigma_L[L]$. Moreover, the topological space $\left\langle X(L), \subseteq, \tau_{X(L)}\right\rangle$ where the topology $\tau_{X(L)}$ has the set

$$\sigma[L] \cup \{X(L) \smallsetminus \sigma(a) : \sigma(a) \in \sigma[L]\}$$

as a subbase, is a Priestley space such that the domain of $D(X(L))$ is $\sigma_L[L]$ and therefore the map σ_L establishes an isomorphism between L and the lattice $D(X(L))$. The Priestley space $X(L) := \left\langle X(L), \subseteq, \tau_{X(L)}\right\rangle$ is the dual of L.

Let X be a Priestley space. The map $\varepsilon_X : X \to X(D(X))$ defined for every $x \in X$ by

$$\varepsilon_X(x) := \{U \in D(X) : x \in U\}$$

is a homeomorphism between the Priestley space X and the Priestley space $X(D(X))$ of the bounded distributive lattice of the clopen upsets of X and it is also an isomorphism between the posets $\langle X, \leq\rangle$ and $\langle X(D(X)), \subseteq\rangle$.

A *homomorphism* from a bounded distributive lattice L_1 to a bounded distributive lattice L_2 is a map that preserves the infimums of finite sets and the supremums of finite sets; thus it preserves the bounds. A *Priestley morphism* from a Priestley space $\langle X_1, \leq_1, \tau_1\rangle$ to a Priestley space $\langle X_2, \leq_2, \tau_2\rangle$ is a continuous map from $\langle X_1, \tau_1\rangle$ to $\langle X_2, \tau_2\rangle$ that is order preserving w.r.t. the orders $\leq_1$ and $\leq_2$.

Let L_1, L_2 be bounded distributive lattices and $h : L_1 \to L_2$ a homomorphism. The map $X(h) : X(L_2) \to X(L_1)$ defined for every $P \in X(L_2)$ by

$$X(h)(P) := h^{-1}[P]$$

is a continuos and order preserving function, thus a morphism from $X(L_2)$ to $X(L_1)$.

If X_1 and X_2 are Priestley spaces and $f : X_1 \to X_2$ is a Priestley morphism, then the map $D(f) : D(X_2) \to D(X_1)$ defined for every $U \in D(X_2)$ by

$$D(f)(U) := f^{-1}[U]$$

is a homomorphism from $D(X_2)$ to $D(X_1)$.

Let PriSp be the category with objects the Priestley spaces and arrows the Priestley morphisms and let DLat be the cateogry of the bounded distributive lattices with the homomorphisms as its arrowss. The Priestley duality says that the maps D from PriSp to DLat and X from DLat to PriSp given by the definitions above are functors that establish a dual equivalence between the two categories whit natural transformations given by the maps σ_L and ε_X.

We proceed to expand the duality between DLat and PriSp to a duality between subordination lattices and Priestley spaces augmented with a binary relation.

We recall some facts we need on Priestley duality. Let L be a bounded distributive lattice. We denote by $\mathcal{O}_u(X(L))$ the set of all open upsets of $X(L)$, which is a lattice when ordered by inclusion, and by $\mathcal{C}_d(X(L))$ the set of all closed downsets sets of $X(L)$, which is also a lattice when ordered by inclusion. The map $\varphi : \mathrm{Id}(L) \to \mathcal{O}_u(X(L))$ given by

$$\varphi(I) := \{P \in X(L) : P \cap I \neq \emptyset\},$$

for every $I \in \mathrm{Id}(L)$ is a lattice isomorphism. Similarly, the function $\psi : \mathrm{Fi}(L) \to \mathcal{C}_d(X(L))$ given by

$$\psi(F) := \{P \in X(L) : F \subseteq P\},$$

for every $F \in \mathrm{Fi}(L)$ is a dual lattice isomorphism. These functions can be expressed in terms of σ_L as follows:

$$\varphi(I) = \bigcup \{\sigma_L(a) : a \in I\}$$

for each $I \in \mathrm{Id}(L)$ and

$$\psi(F) = \bigcap \{\sigma_L(a) : a \in F\}$$

for each $F \in \mathrm{Fi}(L)$.

If $L = \langle L, \prec \rangle$ is a subordination lattice, the relations $R_\prec^\Delta$ and $R_\prec^\nabla$ on $X(L)$ will be used to obtain the dual structures of L. We proceed to see the relevant topological properties that they have on the Priestley space $X(L)$. To this end we first note that Lemma 3 can be stated using $R_\prec^\Delta$ and $R_\prec^\nabla$ as follows:

Lemma 4 *Let L be a bounded distributive lattice and $\prec$ a subordination on L. Let $a \in L$ and $P \in X(L)$. For every $P \in X(L)$ and $a \in L$*

1. $R_\prec^\Delta(P) \subseteq \sigma_L(a)$ *iff* $a \in \Delta_\prec^{-1}(P)$,
2. $P \in (R_\prec^\nabla)^{-1}[\sigma_L(a)]$ *iff* $a \in \nabla_\prec^{-1}(P)$.

The lemma implies the next corollary, which is Lemma 7 in Castro and Celani (2004).

Corollary 1 *Let L be a bounded distributive lattice and $\prec$ a subordination on L. Then*

1. $\Box_{R_\prec^\Delta}(\sigma_L(a)) = \varphi(\Delta_\prec(a))$,
2. $\Diamond_{R_\prec^\nabla}(\sigma_L(a)) = \psi(\nabla_\prec(a))$.

Proof (1) Let $P \in X(L)$. Then $P \in \Box_{R^{\Delta}_{\prec}}(\sigma_L(a))$ if and only if $R^{\Delta}_{\prec}(P) \subseteq \sigma_L(a)$. By Lemma 4 the last condition holds if and only if $a \in \Delta^{-1}_{\prec}(P)$, which means that $\Delta_{\prec}(a) \cap P \neq \emptyset$. This can be restated saying that $P \in \varphi(\Delta_{\prec}(a))$, because $\Delta_{\prec}(a)$ is an ideal.

(2) Let $P \in X(L)$. Then $P \in \Diamond_{R^{\nabla}_{\prec}}(\sigma_L(a))$ if and only if $a \in \nabla^{-1}_{\prec}(P)$. This holds if and only if $\nabla_{\prec}(a) \subseteq P$, which is equivalent to say that $P \in \psi(\nabla_{\prec}(a))$, because $\nabla_{\prec}(a)$ is a filter. □

Let X be a Priestley space. A binary relation R on X is *point-closed* if $R(x)$ is a closed set for every $x \in X$. We say that R is *up point-closed* if it is point-closed and for every $x \in X$ the set $R(x)$ is an upset of X. Similarly, we say that R is *down point-closed* if it is point-closed and $R(x)$ is a downset of X, for every $x \in X$.

Proposition 5 *If L is a bounded distributive lattice and $\prec$ a subordination on L, then*

1. *$R^{\Delta}_{\prec}$ is an up point-closed relation on $X(L)$,*
2. *$\Box_{R^{\Delta}_{\prec}}(U)$ is an open upset for each $U \in D(X(L))$,*
3. *$R^{\nabla}_{\prec}$ is a down point-closed relation on $X(L)$,*
4. *$\Diamond_{R^{\nabla}_{\prec}}(U)$ is a closed upset for each $U \in D(X(L))$.*

Proof (1) Let $P \in X(L)$. Then $R^{\Delta}_{\prec}(P) = \{Q \in X(L) : \Delta^{-1}_{\prec}(P) \subseteq Q\}$. Since $\Delta^{-1}_{\prec}(P)$ is a filter, $R^{\Delta}_{\prec}(P) = \psi(\Delta^{-1}_{\prec}(P))$. Hence, $R^{\Delta}_{\prec}(P)$ is a closed upset of $X(L)$.

(2) If $U \in D(X(L))$, then $U = \sigma_L(a)$ for some $a \in L$. Since $\Delta_{\prec}(a)$ is an ideal and $\Box_{R^{\Delta}_{\prec}}(U) = \varphi(\Delta_{\prec}(a))$ we obtain that $\Box_{R^{\Delta}_{\prec}}(U)$ is an open upset.

(3) Let $P \in X(L)$. Then $R^{\nabla}_{\prec}(P) = \{Q \in X(L) : Q \subseteq \nabla^{-1}_{\prec}(P)\} = \{Q \in X(L) : Q \cap \nabla^{-1}(P)^c = \emptyset\}$. Therefore $R_{\nabla}(P)^c = \{Q \in X(L) : Q \cap \nabla^{-1}(P)^c \neq \emptyset\}$. Since $\nabla^{-1}(P)^c$ is an ideal, $R_{\nabla}(P)^c = \varphi(\nabla^{-1}(P)^c)$ and hence it is an open upset. Therefore, $R_{\nabla}(P)^c$ is a closed downset.

(4) If $U \in D(X(L))$, then $U = \sigma_L(a)$ for some $a \in L$. Hence, since then $\Diamond_{R^{\nabla}_{\prec}}(U) = \psi(\nabla_{\prec}(a))$ and moreover $\nabla_{\prec}(a)$ is an ideal, $\Diamond_{R^{\nabla}_{\prec}}(U)$ is a closed upset. □

Definition 4 Let X be a Priestley space. We say that a binary relation R on X is the Δ*-dual of a subordination* if R is up point-closed and $\Box_R(U)$ is an open upset for every $U \in D(X)$. Similarly, we say that a binary relation R on X is the ∇*-dual of a subordination* if R is down point-closed and $\Diamond_R(U)$ is a closed upset for every $U \in D(X)$.

We have two choices to obtain the dual objects of subordination lattices. One is to consider Priestley spaces X endowed with a binary relation R which is the Δ-dual of a subordination and the other is to take Priestley spaces X endowed with a binary relation which is the ∇-dual of a subordination. In this way we will end up with two equivalent categories for every choice of morphisms between subordination lattices we take. We will see that the functor that transforms an object of one category into an object of the other simply changes the relation to its converse.

For every Priestley space X and binary relation R on X note that $R(x)$ is an upset for every $x \in X$ if and only if $(R \circ \leq) = R$, and that $R(x)$ is an downset for every

$x \in X$ if and only if $(R \circ \leq^{-1}) = R$. In general, $R \subseteq (R \circ \leq)$ and $R \subseteq (R \circ \leq^{-1})$ because $\leq$ is reflexive.

Lemma 5 *Let X be a Priestley space and R a binary relation on X.*

1. *If R is the Δ-dual of a subordination, then*
   a. $(\leq \circ R) = (R \circ \leq) = R$,
   b. *if $x \leq y$, then $R(y) \subseteq R(x)$, for all $x, y \in X$.*
2. *If R is the ∇-dual of a subordination, then*
   a. $(\leq^{-1} \circ R) = (R \circ \leq^{-1}) = R$,
   b. *if $x \leq y$, then $R(x) \subseteq R(y)$, for all $x, y \in X$.*

Proof (1). We first prove (a). Assume that $x, y, z \in X$ are such that $x \leq y$ and $(y, z) \in R$. This implies that $R(x) \neq \emptyset$. Otherwise, since $R(x) \subseteq \emptyset$ and $\emptyset \in D(X)$ we have $x \in \Box_R(\emptyset)$. Therefore, $y \in \Box_R(\emptyset)$ so that $R(y) \subseteq \emptyset$ and this is not possible since $z \in R(y)$. Suppose in search of a contradiction that $w \not\leq z$ for all $w \in R(x)$. Then for each $w \in R(x)$ there exists $U_w \in D(X)$ such that $w \in U_w$ and $z \notin U_w$. So, $R(x) \subseteq \bigcup\{U_w : w \in R(x)\}$. As $R(x)$ is closed, and hence compact, there exists a finite family $\{U_1, \ldots, U_n\}$ such that $R(x) \subseteq U_1 \cup \cdots \cup U_n = U$. Thus $x \in \Box_R(U)$ and $U \in D(X)$. As $\Box_R(U)$ is an upset, $y \in \Box_R(U)$. This yields $R(y) \subseteq U$, and since $(y, z) \in R$, $z \in U$, which is impossible. Thus there exists $w \in X$ such that $(x, w) \in R$ and $w \leq z$. We conclude that $(\leq \circ R) \subseteq (R \circ \leq)$. The inclusion $(R \circ \leq) \subseteq R$ follows from the assumption that $R(x)$ is an upset for every $x \in X$. And $R \subseteq (\leq \circ R)$ follows from the fact that $\leq$ is reflexive.

(b) follows from (a). Let $x \leq y$ and $z \in R(y)$, so that $(x, z) \in \leq \circ R$. Hence, by (a), $(x, z) \in R$, i.e., $z \in R(x)$.

(2). To prove (a) let $x, y \in X$ be such that $(x, y) \in \leq^{-1} \circ R$. Then there exists $z \in X$ such that $z \leq x$ and $(z, y) \in R$. It follows that $R(x) \neq \emptyset$. Otherwise, $R(x) \cap X = \emptyset$ and therefore $x \notin \Diamond_R(X)$. Thus, since this set is an upset, $z \notin \Diamond_R(X)$ so that $R(z) \cap X = \emptyset$ which is not possible because $y \in R(z)$. Suppose now that $w \not\leq y$ for all $w \in R(x)$. Then for each $w \in R(x)$, there exists $U_w \in D(X)$ such that $w \notin U_i$ and $y \in U_w$. So, $R(x) \subseteq \bigcup \left\{U_w^c : w \in R(x)\right\}$. As $R(x)$ is closed, and hence compact, there exists a finite family $\{U_1, \ldots, U_n\}$ such that $R(x) \subseteq U_1^c \cup \cdots \cup U_n^c = U^c$. Hence, $x \notin \Diamond_R(U)$. Since $U = U_1 \cap \cdots \cap U_n \in D(X)$, $\Diamond_R(U)$ is an upset by assumption; thus $z \notin \Diamond_R(U)$. i.e., $R(z) \cap U = \emptyset$. But $y \in R(z) \cap U$, which is a contradiction. Thus there exists $w \in X$ such that $(x, w) \in R$ and $y \leq w$. We conclude that $(\leq^{-1} \circ R) \subseteq (R \circ \leq^{-1})$. The inclusion $R \circ \leq^{-1} \subseteq R$ follows from the fact that $R(x)$ is a downset for every $x \in X$. Item (b) follows from (a). $\Box$

In the next proof we use Esakia's lemma (2019) that says that if X is a Priestley space and R is a point-closed relation on X, then for every down-directed family C of closed sets of X,

$$R^{-1}[\bigcap C] = \bigcap\{R^{-1}[U] : U \in C\}.$$

Lemma 6 *Let X be a Priestley space and R a binary relation on X.*

1. *The following two conditions are equivalent:*

   a. *R is the Δ-dual of a subordination.*
   b. *For every closed set Y in X, $R[Y]$ is a closed upset, and for every closed downset Z of X, $R^{-1}[Z]$ is a closed downset of X.*

2. *The following two conditions are equivalent:*

   a. *R is the ∇-dual of a subordination,*
   b. *For every closed subset Y of X, $R[Y]$ is a closed downset, and for every closed upset Z of X, $R^{-1}[Z]$ is a closed upset of X.*

Proof (1). Assume (a). Let Y be a closed subset of X. If $Y = \emptyset$, then since $R[Y] = \emptyset$ we are done. If $Y \neq \emptyset$, suppose that $x \notin R[Y]$. Then $x \notin R(y)$ for all $y \in Y$. Therefore, as R is up point-closed, for each $y \in Y$ there exists $U_y \in D(X)$ such that $R(y) \subseteq U_y$ and $x \notin U_y$. We fix such an U_y for each $y \in Y$. So, $y \in \Box_R(U_y)$, for every $y \in Y$. Therefore, $Y \subseteq \bigcup \left\{\Box_R(U_y) : y \in Y\right\}$, and as Y is closed and X is compact, Y is compact. Using that from the assumption, for every $y \in Y$ the set $\Box_R(U_y)$ is open and the fact that Y is compat, there exist $y_1, \ldots, y_n \in Y$ such that

$$Y \subseteq \Box_R(U_{y_1}) \cup \cdots \cup \Box_R(U_{y_n}) \subseteq \Box_R(U_{y_1} \cup \cdots \cup U_{y_n}).$$

We choose $y_1, \ldots, y_n \in Y$ with that property and we let $U_x = U_{y_1} \cup \cdots \cup U_{y_n}$. Then $Y \subseteq \Box_R(U_x)$ and therefore $R[Y] \subseteq U_x$. Moreover, $x \notin U_x$. It easily follows that $R[Y] = \bigcap\{U_x : x \notin R[Y]\}$. Thus, $R[Y]$ is a closed upset of X.

Let now Z be a closed downset of X. Then

$$Z = \bigcap \left\{U^c : Z \subseteq U^c \text{ and } U \in D(X)\right\}.$$

Note that the family $\{U^c : Z \subseteq U^c \text{ and } U \in D(X)\}$ is a downdirected family of closed sets. Thus by Esakia's lemma we have

$$R^{-1}[Z] = \bigcap \left\{R^{-1}[U^c] : Z \subseteq U^c \text{ and } U \in D(X)\right\}.$$

But by assumption $\Box_R(U) = (R^{-1}[U^c])^c$ is an open upset for every $U \in D(X)$. Thus $R^{-1}[U^c]$ is a closed downset for every $U \in D(X)$. This implies that $R^{-1}[Z]$ is a closed downset, as desired.

Assume now (b). As X is Hausdorff, for every $x \in X$, $\{x\}$ is closed. Thus, by (b), $R(x) = R[\{x\}]$ is a closed upset. For each $U \in D(X)$ we have $\Box_R(U) = R^{-1}[U^c]^c$. Thus, $\Box_R(U)$ is an open upset, because U^c is a closed downset.

(2). Assume (a) and let Y be a closed subset of X. If $Y = \emptyset$, then since $R[Y] = \emptyset$ we are done. Assume that $Y \neq \emptyset$ and that $x \notin R[Y] = \bigcup\{R(y) : y \in Y\}$. Then $x \notin R(y)$ for all $y \in Y$. As R is point-closed, we have that for every $y \in Y$ there exists $U_y \in D(X)$ such that $R(y) \cap U_y = \emptyset$ and $x \in U_y$. We fix one such U_y for every $y \in Y$. So, $y \in \Diamond_R(U_y)^c$, for every $y \in Y$, i.e., $Y \subseteq \bigcup \left\{\Diamond_R(U_y)^c : y \in Y\right\}$.

As Y is closed and X is compact, Y is compact. Thus there exists $y_1, \ldots, y_n \in Y$ such that

$$Y \subseteq \Diamond_R(U_{y_1})^c \cup \cdots \cup \Diamond_R(U_{y_n})^c \subseteq \Diamond_R(U_{y_1} \cap \cdots \cap U_{y_n})^c.$$

We choose $y_1, \ldots, y_n \in Y$ with that property and we let $U_x = U_{y_1} \cap \cdots \cap U_{y_n}$. Then, $Y \subseteq \Diamond_R(U_x)^c$ and so $Y \cap \Diamond_R(U_x) = \emptyset$. Therefore, $R[Y] \cap U_x = \emptyset$, i.e., $R[Y] \subseteq U_x^c$. Moreover, $x \in U_x$. It easily follows that $R[Y] = \bigcap \left\{U_x^c : x \notin R[Y]\right\}$. Thus, $R[Y]$ is a closed downset.

Let now Z be a closed upset of X. Then $Z = \bigcap \{U : Z \subseteq U \in D(X)\}$. Therefore,

$$R^{-1}[Z] = R^{-1}\left[\bigcap \{U \in D(X) : Z \subseteq U\}\right].$$

Note that the set $\{U : Z \subseteq U \in D(X)\}$ is a filter of $D(X)$, thus a downdirected family of closed sets. By Esakia's lemma we have

$$R^{-1}\left[\bigcap \{U \in D(X) : Z \subseteq U\}\right] = \bigcap \left\{R^{-1}[U] : Z \subseteq U \in D(X)\right\}.$$

As $R^{-1}[U] = \Diamond_R(U)$ is a closed upset for every closed upset U, we obtain that $R^{-1}[Z]$ is a closed upset.

Now we assume (b). As X is Hausdorff, $\{x\}$ is closed. Thus $R(x) = R[\{x\}]$ is a closed downset. For each $U \in D(X)$ we get that $R^{-1}[U]$ is a closed upset, because U is a closed upset. □

Lemma 7 *Let X be a Priestley space and R a binary relation on X. The following statements are equivalent:*

1. *R^{-1} is the Δ-dual of a subordination,*
2. *R is the ∇-dual of a subordination.*

Proof Assume (2). Note that for every $x \in X$, the set $(x] = \{y \in X : y \leq x\}$ is a closed downset. Using (1) in Lemma 5, we have $R^{-1}(x) = R^{-1}[\{x\}] = R^{-1}[(x]]$. Then using (1) in Lemma 6 we obtain that $R^{-1}(x)$ is closed and a downset. Now, given $U \in D(X)$, note that $\Diamond_{R^{-1}}(U) = R[U]$. Using again (1) in Lemma 5 we obtain that $\Diamond_{R^{-1}}(U)$ is a closed upset.

The proof of the implication from (1) to (2) is similar, using now (2) in Lemmas 5 and 6. □

Proposition 6 *Let X be a Priestley space and R a binary relation on X. If R is the Δ-dual of a subordination or the ∇-dual of a subordination, then R is a closed relation (i.e., a closed set of the product space).*

Proof Suppose that R is up point-closed and $\Box_R(U)$ is an open upset for each $U \in D(X)$. Suppose that $\langle x, y\rangle \notin R$. Using Lemma 5 it is easy to see that $R(x) = R[[x)]$. Moreover, the set $[x)$ is closed, so applying Lemma 6 it follows that $R(x)$

is a closed upset. Hence there is $U \in D(X)$ such that $R(x) \subseteq U$ and $y \notin U$. It follows that $x \in \Box_R(U)$. Consider now the complement U^c of U, which is a clopen downset and $y \in U^c$. By Lemma 6, $R^{-1}[U^c]$ is a closed downset and therefore $(R^{-1}[U^c])^c$ an open upset. Note that $z \in (R^{-1}[U^c])^c$ if and only if $R(z) \subseteq U$. Let $O := (R^{-1}[U^c])^c$. Then $\langle x, y\rangle \in O \times U^c$ and $O \times U^c$ is an open set in the product topology. We show that $R \cap (O \times U^c) = \emptyset$. If $\langle u, v\rangle \in R \cap (O \times U^c)$, then since $u \in O$ we have $v \in R(u) \subseteq U$ and $v \notin U$, a contradiction.

Now suppose that R is down point-closed and $\Diamond_R(U)$ is a closed downset for each $U \in D(X)$. Then by Lemma 7, R^{-1} is up point-closed and $\Box_{R^{-1}}(U)$ is an open upset for each $U \in D(X)$. Therefore by the first part of the proof, R^{-1} is a closed relation. It is easy to see that the converse of a closed relation is a closed relation. Thus R is a closed relation. □

Remark 3 A closed relation on a Priestley space need not be up point-closed nor down point-closed.

We proceed to see how a binary relation on a set determines two subordinations on its powerset lattice. Thus, a binary relation R on a Priestley space X determines two subordinations on the lattice of the clopen upsets of X by restricting to this lattice the subordinations determined by R on the powerset of X.

Let X be a set and R a binary relation on X. The map $\Box_R : \mathcal{P}(X) \to \mathcal{P}(X)$ is a dual modal operator and the map $\Diamond_R : \mathcal{P}(X) \to \mathcal{P}(X)$ a modal operator. That is, we have for all $U, V \subseteq X$ that $\Box_R(U \cap V) = \Box_R(U) \cap \Box_R(V)$, $\Box_R(X) = X$, $\Diamond_R(U \cup V) = \Diamond_R(U) \cap \Diamond_R(V)$, and $\Diamond_R(\emptyset) = \emptyset$. Considering Remark 2, the relations $\prec_R$ and $\prec_R^*$ defined on $\mathcal{P}(X)$ by

$$U \prec_R V \iff U \subseteq \Box_R(V)$$

and

$$U \prec_R^* V \iff \Diamond_R(U) \subseteq V.$$

are subordination relations. Note that since $\Diamond_R(U) = \Box_R(U^c)^c$ for every $U \subseteq X$, we have

$$U \prec_R V \iff V^c \prec_R^* U^c.$$

Remark 4 It is well known, and easy to check, that the maps $\Diamond_{R^{-1}}$ and $\Box_R$ form an adjoint pair, that is, for every $U, V \subseteq X$, $U \subseteq \Box_R(V)$ if and only if $\Diamond_{R^{-1}}(U) \subseteq V$. Therefore, $U \prec_R V$ if and only if $U \prec_{R^{-1}}^* V$. Hence, $\prec_R = \prec_{R^{-1}}^*$ and, similarly, $\prec_R^* = \prec_{R^{-1}}$.

If X is a Priestley space and R a binary relation on X, the subordinations $\prec_R$ and $\prec_R^*$ on $\mathcal{P}(X)$ restrict to subordinations on the sublattice $D(X)$ of $\mathcal{P}(X)$. Therefore, give a Priestley space X and a binary relation R on X we have two natural ways to obtain a subordination on $D(X)$.

In the next two propositions we proceed to find a necessary and sufficient condition that R has to satisfy in order that the homeomorphism $\varepsilon : X \to X(D(X))$ is an isomorphism between $\langle X, R\rangle$ and $\langle X(D(X)), R^{\Delta}_{\prec_R}\rangle$ and a necessary and sufficient condition to be an isomorphism between $\langle X, R\rangle$ and $\langle X(D(X)), R^{\nabla}_{\prec^*_R}\rangle$.

Proposition 7 *Let X be a Priestley space and R a binary relation on X. The following statements are equivalent:*

1. *R is the Δ-dual of a subordination.*
2. *for every $x, y \in X$, xRy if and only if $\varepsilon_X(x)R^{\Delta}_{\prec_R}\varepsilon_X(y)$.*

Proof We omit the subscript X in ε_X all along the proof.

(1) $\Rightarrow$ (2). Assume that xRy. We have to prove that $\Delta^{-1}_{\prec_R}(\varepsilon(x)) \subseteq \varepsilon(y)$. Suppose that $U \in \Delta^{-1}_{\prec_R}(\varepsilon(x))$. Hence, $\Delta_{\prec_R}(U) \cap \varepsilon(x) \neq \emptyset$. Thus, there is $V \in D(X)$ such that $V \subseteq \Box_R(U)$ and $x \in V$. Therefore, $R(x) \subseteq U$. This implies that $y \in U$ and so $U \in \varepsilon(y)$. Conversely, suppose that $\varepsilon(x)R^{\Delta}_{\prec_R}\varepsilon(y)$ and $x \not R y$. Then, since $R(x)$ is a closed upset, there is $V \in D(X)$ such that $y \notin V$ and $R(x) \subseteq V$. Hence, $V \notin \varepsilon(y)$ and $x \in \Box_R(V)$. Thus, $V \notin \Delta^{-1}_{\prec_R}(\varepsilon(x))$, which means that $\Delta_{\prec_R}(V) \cap \varepsilon(x) = \emptyset$. But $\Box_R(V)$ is an open upset. So $\Box_R(V) = \bigcup\{U \in D(X) : U \subseteq \Box_R(V)\}$. Hence, there is $U \in D(X)$ such that $x \in U$ and $U \subseteq \Box_R(V)$. Therefore, $V \in \Delta^{-1}_{\prec_R}(\varepsilon(x))$. Since $V \notin \varepsilon(y)$, it follows that $\varepsilon(x) \not R^{\Delta}_{\prec_R}\varepsilon(y)$. (2) $\Rightarrow$ (1). Let $x, y \in X$ be such that $y \in \mathrm{Cl}(R(x))$ and $y \notin R(x)$. Then, by (2), $\varepsilon(y) \notin R^{\Delta}_{\prec_R}(\varepsilon(x))$. Therefore, there exist $U \in D(X)$ such that $\Delta_{\prec_R}(U) \cap \varepsilon(x) \neq \emptyset$ and $y \notin U$. Let $V \in \Delta_{\prec_R}(U) \cap \varepsilon(x)$. Then $V \in D(X)$, $V \subseteq \Box_R(U)$, and $x \in V$. Therefore $R(x) \subseteq U$. Hence, $y \notin \mathrm{Cl}(R(x))$, a contradiction. Thus R is point-closed. Now to prove that it is an upset, suppose that $y \in R(x)$ and $y \leq z$. Since ε is an order isomorphism, $\varepsilon(y) \subseteq \varepsilon(z)$. Moreover, since xRy, by the assumption (2) we have $\Delta^{-1}_{\prec}(\varepsilon(x)) \subseteq \varepsilon(y)$. Thus, $\Delta^{-1}_{\prec}(\varepsilon(x)) \subseteq \varepsilon(z)$. This, again by the assumption (2), implies that xRz.

Now let $U \in D(X)$. Using (2), the definitions involved, and the fact that ε is a bijection, it is easy to see that $\Box_{R^{\Delta}_{\prec}}(\varepsilon[U]) = \varepsilon[\Box_R(U)]$. Hence, considering that $\Box_{R^{\Delta}_{\prec_R}}(\varepsilon[U])$ is an open upset of $X(D(X))$, because $\prec_R$ is a subordination on $D(X)$, and the fact that ε is an order isomorphism and a homeomorphism we obtain that $\Box_R(U)$ is an open upset. □

The next proposition is proved in Castro and Celani (2004).

Proposition 8 *Let X be a Priestley space and R a binary relation on X. The following are equivalent:*

1. *R is the ∇-dual of a subordination,*
2. *for every $x, y \in X$, xRy if and only if $\varepsilon_X(x)R^{\nabla}_{\prec^*_R}\varepsilon_X(y)$.*

Proof As in the last proof, we omit the subscript X in ε_X.

(1) $\Rightarrow$ (2). Suppose that xRy. We have to prove that $\varepsilon(y) \subseteq \nabla^{-1}_{\prec^*_R}(\varepsilon(x))$. Suppose that $U \in \varepsilon(y)$. To prove that $U \in \nabla^{-1}_{\prec^*_R}(\varepsilon(x))$ we have to show that $\nabla_{\prec_R}(U) \subseteq \varepsilon(x)$. To this end suppose that $U \prec^*_R V$ which means that $\Diamond_R(U) \subseteq V$. Since $y \in U$ and xRy, $x \in \Diamond_R(U)$. Therefore, $V \in \varepsilon(x)$ and we are done. Conversely, suppose that $x \not R y$. Then since $R(x)$ is a closed downset there is $U \in D(X)$ such that $y \in U$ and $R(x) \cap U = \emptyset$. Then $x \notin \Diamond_R(U)$ and $U \in \varepsilon(y)$. Since $\Diamond_R(U)$ is a closed upset,

there exists $V \in D(X)$ such that $\Diamond_R(U) \subseteq V$ and $x \notin V$. Then $V \in \nabla_{\prec_R^*}(U)$ and therefore $\nabla_{\prec_R^*}(U) \nsubseteq \varepsilon(x)$, which implies that $U \notin \nabla_{\prec_R^*}^{-1}(\varepsilon(x))$. Hence we obtain that $\varepsilon(x) \not R^{\nabla}_{\prec_R^*} \varepsilon(y)$.

(2) $\Rightarrow$ (1). Let $x, y \in X$ be such that $y \in \mathrm{Cl}(R(x))$ and $y \notin R(x)$. Then, by (2), $\varepsilon(y) \notin R^{\nabla}_{\prec_R^*}(\varepsilon(x))$. Therefore, there exist $U \in D(X)$ such that $y \in U$ and $\nabla_{\prec_R^*}(U) \nsubseteq \varepsilon(x)$. Let then $V \in \nabla_{\prec_R^*}(U)$, namely that $\Diamond_R(U) \subseteq V$, and such that $x \notin V$. Hence, $x \notin \Diamond_R(U)$ and so $R(x) \cap U = \emptyset$. Since $R(x)$ is a closed downset and $y \in U$, $y \notin \mathrm{Cl}(R(x))$, a contradiction. Thus $R(x)$ is closed. Now to prove that $R(x)$ is a downset, suppose that $y \in R(x)$ and $z \leq y$. Then $\varepsilon(z) \subseteq \varepsilon(y)$. By the assumption (2), since xRy we have $\varepsilon(x) R^{\nabla}_{\prec_R^*} \varepsilon(y)$, namely $\varepsilon(y) \subseteq \nabla_{\prec_R^*}^{-1}(\varepsilon(x))$. It follows that $\varepsilon(z) R^{\nabla}_{\prec_R^*} \varepsilon(y)$ and therefore that $z \in R(x)$.

Now let $U \in D(X)$. Using (2), the definitions involved, and the fact that ε is a bijection, it is easy to see that $\Diamond_{R^{\nabla}_{\prec_R^*}}(\varepsilon[U]) = \varepsilon[\Diamond_R(U)]$. Hence, considering that $\Diamond_{R^{\nabla}_{\prec_R^*}}(\varepsilon[U])$ is a closed upset of $X(D(X))$, because $\prec_R^*$ is a subordination on $D(X)$, and the fact that ε is an order isomorphism and a homeomorphism we obtain that $\Diamond_R(U)$ is a closed upset. $\square$

Proposition 9 *Let L be a bounded distributive lattice and $\prec$ a subordination on L. The representation isomorphism $\sigma_L : L \to D(X(L))$ satisfies for every $a, b \in L$ the following two conditions:*

$$b \prec a \iff \sigma_L(b) \prec_{R^{\Delta}_{\prec}} \sigma_L(a) \quad \textit{and} \quad b \prec a \iff \sigma_L(b) \prec^*_{R^{\nabla}_{\prec}} \sigma_L(a).$$

In terms of the associated Δ-quasi-modal operators the conditions say that for every $a \in L$, $\Delta_{\prec_{R^{\Delta}_{\prec}}}(\sigma_L(a)) = \sigma_L[\Delta_{\prec}(a)]$ and $\nabla_{\prec^*_{R^{\nabla}_{\prec}}}(\sigma_L(b)) = \sigma_L[\nabla_{\prec}(b)]$.

Proof First note that by Corollary 1, $\Box_{R^{\Delta}_{\prec}}(\sigma_L(a)) = \varphi(\Delta_{\prec}(a))$ and $\Diamond_{R^{\nabla}_{\prec}}(\sigma_L(a)) = \psi(\nabla_{\prec}(a))$. Now for every $b \in L$,

$$\begin{aligned} \sigma_L(b) \prec_{R^{\Delta}_{\prec}} \sigma_L(a) &\Leftrightarrow \sigma_L(b) \subseteq \Box_{R^{\Delta}_{\prec}}(\sigma_L(a)) \\ &\Leftrightarrow \sigma_L(b) \subseteq \varphi(\Delta_{\prec}(a)) \\ &\Leftrightarrow b \in \Delta_{\prec}(a) \\ &\Leftrightarrow b \prec a. \end{aligned}$$

The equivalence before the last one holds because if $b \in \Delta_{\prec}(a)$, then by the definition of φ, $\sigma_L(b) \subseteq \varphi(\Delta_{\prec}(a))$, and if $b \notin \Delta_{\prec}(a)$, then there exists $P \in X(L)$ such that $b \in P$ and $\Delta_{\prec}(a) \cap P = \emptyset$, which implies that $P \notin \varphi(\Delta_{\prec}(a))$ and hence we have that $\sigma_L(b) \nsubseteq \varphi(\Delta_{\prec}(a))$. This proves the first condition.

To prove the second condition we have for every $b \in L$,

$$
\begin{aligned}
\sigma_L(b) \prec^*_{R^{\nabla}_{\prec}} \sigma_L(a) &\Leftrightarrow \Diamond_{R^{\nabla}_{\prec}}(\sigma_L(b)) \subseteq \sigma_L(a) \\
&\Leftrightarrow \psi(\nabla_{\prec}(b)) \subseteq \sigma_L(a) \\
&\Leftrightarrow a \in \nabla_{\prec}(b) \\
&\Leftrightarrow b \prec a.
\end{aligned}
$$

The equivalence before the last one holds because if $a \in \nabla_{\prec}(b)$, then by the definition of ψ, $\psi(\nabla_{\prec}(b)) \subseteq \sigma_L(a)$, and if $a \notin \nabla_{\prec}(b)$, then, since $\nabla_{\prec}(b)$ is a filter, there exists $P \in X(L)$ such that $a \notin P$ and $\nabla_{\prec}(b) \subseteq P$, which implies that $P \in \psi(\nabla_{\prec}(b))$ and hence we have $\psi(\nabla_{\prec}(b)) \nsubseteq \sigma_L(a)$. □

Definition 5 We say that a pair $\langle X, R\rangle$ is a *Priestley* Δ*-subordination space* (a Priestley Δ-space, for short) if X is a Priestley space and R is the Δ-dual of a subordination. Similarly, we say a pair $\langle X, R\rangle$ is a *Priestley* ∇*-subordination space* (a Priestley ∇-space, for short) if X is a Priestley space and R is the ∇-dual of a subordination.

Proposition 5 establishes that if $L = \langle L, \prec\rangle$ is a subordination lattice, then $\langle X(L), R^{\Delta}_{\prec}\rangle$ is a Priestley Δ-subordination space and $\langle X(L), R^{\nabla}_{\prec}\rangle$ is a Priestley ∇-subordination space. Moreover, Proposition 9 shows that the map σ_L is an isomorphism between the subordination lattices $\langle L, \prec\rangle$ and $\langle D(X(L)), \prec_{R^{\Delta}_{\prec}}\rangle$ and between $\langle L, \prec\rangle$ and $\langle D(X(L)), \prec^*_{R^{\nabla}_{\prec}}\rangle$. Conversely, Proposition 7 shows that if $\langle X, R\rangle$ is a Priestley Δ-subordination space, then $\langle D(X), \prec_R\rangle$ is a subordination lattice such that the map ε_X an isomorphism between $\langle X, R\rangle$ and $\langle X(D(X)), R^{\Delta}_{\prec_R}\rangle$ and Proposition 8 shows that if $\langle X, R\rangle$ is a Priestley ∇-subordination space, then $\langle D(X), \prec^*_R\rangle$ is a subordination lattice such that the map ε_X an isomorphism between $\langle X, R\rangle$ and $\langle X(D(X)), R^{\nabla}_{\prec^*_R}\rangle$.

To complete the duality we have to introduce the morphisms. We will consider three kinds of morphisms on subordination lattices and four kinds of morphisms on Priestley spaces with a binary relation.

Definition 6 Let L_1 and L_2 be subordination lattices. A *subordination homomorphism* from L_1 to L_2 is a homomorphism $h : L_1 \to L_2$ such that for every $a, b \in L_1$, if $a \prec_1 b$, then $h(a) \prec_2 h(b)$. A subordination homomorphism h from L_1 to L_2 is *strong* if for every $a \in L_1$ and $c \in L_2$, if $c \prec_2 h(a)$, then there exists $b \in L_1$ such that $b \prec_1 a$ and $c \leq h(b)$. We say that it is *dually strong* if for every $a \in L_1$ and $c \in L_2$, if $h(a) \prec_2 c$, then there exists $b \in L_1$ such that $a \prec_1 b$ and $h(b) \leq c$.

Remark 5 It is easy to see that if $\prec$ is a subordination on a lattice L, then $\prec^{-1}$ is a subordination on the dual lattice L^{∂} of L. The condition that defines dually strong subordination homomorphism in the definition above is then the same as that for strong subordination homomorphism but between L_1 and $\langle L_2^{\partial}, \prec_2^{-1}\rangle$.

Definition 7 Let X_1 and X_2 be Priestley spaces with binary relations R_1 and R_2 respectively. A Priestley morphism $f : X_1 \to X_2$ is *stable* if for every $x, y \in X_1$ such that $x R_1 y$ we have $f(x) R_2 f(y)$, and it is *strongly stable* if in addition for every $x \in X_1$ and $y \in X_2$, if $f(x)R_2y$, then there exists $z \in X_1$ such that xR_1z and $f(z) \leq_2 y$. Moreover, we say that f is *reversely strongly stable* if it is stable and for every $x \in X_1$ and $y \in X_2$, if $yR_2f(x)$, then there exists $z \in X_1$ such that zR_1x and $f(z) \leq_2 y$. Also we say that f is *dually strongly stable* if it is stable and for every $x \in X_1$ and $y \in X_2$, if $yR_2f(x)$, then there exists $z \in X_1$ such that zR_1x and $y \leq_2 f(z)$.

The strongly stable Priestley morphisms are the morphisms considered in the duality for quasi-modal distributive lattices given in Castro and Celani (2004).

Proposition 10 *Let L_1, L_2 be subordination lattices. For every map $h : L_1 \to L_2$ the following conditions are equivalent:*

1. *h is a (strong) subordination homomorphism from L_1 to L_2,*
2. *the dual map $X(h)$ is a (strongly) stable Priestley morphism from the Priestley Δ-space $\langle X(L_2), R^{\Delta}_{\prec_2}\rangle$ to the Priestley Δ-space $\langle X(L_1), R^{\Delta}_{\prec_1}\rangle$,*
3. *the dual map $X(h)$ is a (reversely strongly) stable Priestley morphism from the Priestley ∇-space $\langle X(L_2), R^{\nabla}_{\prec_2}\rangle$ to the Priestley ∇-space $\langle X(L_1), R^{\nabla}_{\prec_1}\rangle$.*

Proof (1) $\Rightarrow$ (2). Assume that h is a subordination homomorphism from L_1 to L_2. Suppose that $\langle P, Q\rangle \in R^{\Delta}_{\prec_2}$. Then $\Delta^{-1}_{\prec_2}(P) \subseteq Q$. We prove that $\Delta^{-1}_{\prec_1}(h^{-1}[P]) \subseteq h^{-1}[Q]$. Suppose that $a \in \Delta^{-1}_{\prec_1}(h^{-1}[P])$. Then $\Delta_{\prec_1}(a) \cap h^{-1}[P] \neq \emptyset$. Let then $b \in \Delta_{\prec_1}(a) \cap h^{-1}[P]$. Thus $b \prec_1 a$ and $h(b) \in P$. Hence $h(b) \prec_2 h(a)$ and we obtain that $h(b) \in \Delta_{\prec_1}(h(a)) \cap P$. Thus, $h(a) \in \Delta^{-1}_{\prec_2}(P)$. It follows that $b \in h^{-1}[Q]$. If h is in addition strong, then suppose that $P \in X(L_2)$ and $Q \in X(L_1)$ are such that $h^{-1}[P]R^{\Delta}_{\prec_1}Q$, i.e., $\Delta^{-1}_{\prec_1}(h^{-1}[P]) \subseteq Q$. We prove that $\Delta^{-1}_{\prec_2}(P) \cap (h[L_1 \setminus Q]] = \emptyset$. On the contrary, let $a \in L_2$ and $b \in L_1 \setminus Q$ be such that $a \in \Delta^{-1}_{\prec_2}(P)$ and $a \leq_2 h(b)$. Then $\Delta_{\prec_2}(a) \cap P \neq \emptyset$. So, let $c \prec_2 a$ be such that $c \in P$. It follows that $c \prec_2 h(b)$. Since h is strong, there exists $d \in L_1$ such that $d \prec_1 b$ and $c \leq_2 h(d)$. Thus $h(d) \in P$ and $d \in h^{-1}[P]$, therefore $\Delta_{\prec_1}(b) \cap h^{-1}[P] \neq \emptyset$. Hence $b \in \Delta^{-1}_{\prec_1}(h^{-1}[P])$ and so $b \in Q$, a contradiction. By the Prime filter theorem there is $P' \in X(L_2)$ such that $\Delta^{-1}_{\prec_2}(P) \subseteq P'$ and $h^{-1}[P'] \subseteq Q$. Hence, there is $P' \in X(L_2)$ such that $PR^{\Delta}_{\prec_2}P'$ and $h^{-1}[P'] \subseteq Q$. This shows that $X(h)$ is strong.

(2) $\Rightarrow$ (1). Suppose now that $X(h)$ is a stable Priestley morphism from the space $\langle X(L_2), R^{\Delta}_{\prec_2}\rangle$ to $\langle X(L_1), R^{\Delta}_{\prec_1}\rangle$ and suppose that $a, b \in L_1$ are such that $a \prec_1 b$. If $h(a) \not\prec_2 h(b)$, then, since $h(a) \notin \Delta_{\prec_2}(h(b))$ and $\Delta_{\prec_2}(h(b))$ is an ideal, there is $P \in X(L_2)$ such that $h(a) \in P$ and $P \cap \Delta_{\prec_2}(h(b)) = \emptyset$. Hence, $h(b) \notin \Delta^{-1}_{\prec_2}(P)$. Since $\Delta^{-1}_{\prec_2}(P)$ is a filter, there is $Q \in X(L_2)$ such that $\Delta^{-1}_{\prec_2}(P) \subseteq Q$ and $h(b) \notin Q$. Thus, $PR^{\Delta}_{\prec_2}Q$. The stability of $X(h)$ implies that $h^{-1}[P]R^{\Delta}_{\prec_1}h^{-1}[Q]$. Since $b \notin h^{-1}[Q]$, we have $b \notin \Delta^{-1}_{\prec_1}(h^{-1}[P])$. And since $a \in h^{-1}[P]$ it follows that $a \notin \Delta_{\prec_1}(b)$, which is not possible because by assumption $a \prec_1 b$. If $X(h)$ is in addition strongly stable, to prove that h is strong assume that $a \in L_1$ and $c \in L_2$ are such that $c \prec_2 h(a)$. Consider the ideal I of L_2 generated by $h[\Delta_{\prec_1}(a)]$. Assume that $c \notin I$. Then let $P \in X(L_2)$ be such that $c \in P$ and $I \cap P = \emptyset$. Thus $\Delta_{\prec_1}(a) \cap h^{-1}[P] = \emptyset$. It follows that $a \notin \Delta^{-1}_{\prec_1}(h^{-1}[P])$. Thus there exists $Q \in X(L_1)$ such that $\Delta^{-1}_{\prec_1}(h^{-1}[P]) \subseteq Q$, so that $h^{-1}[P]R^{\Delta}_{\prec_1}Q$ and $a \notin Q$. Since $X(h)$ is strongly stable, there exists $P' \in X(L_2)$

such that $PR^{\Delta}_{\prec_2}P'$ and $h^{-1}[P'] \subseteq Q$. Since $c \in P$ and $c \prec_2 h(a)$, $h(a) \in \Delta^{-1}_{\prec_2}(P)$; therefore, $h(a) \in P'$ and $a \in h^{-1}[P']$. Hence $a \in Q$, a contradiction. We conclude that $c \in I$. Therefore there exists $b \in \Delta_{\prec_1}(a)$ such that $c \leq_2 h(b)$ and we are done.

The equivalence between (3) and (2) can be proved using Proposition 1. □

Proposition 11 *Let $\langle X_1, R_1\rangle$ and $\langle X_2, R_2\rangle$ be two Priestley Δ-subordination spaces and $f : X_1 \to X_2$ a map. Then f is a (strongly) stable Priestley morphism if and only if the map $D(f) : D(X_2) \to D(X_1)$ is a (strong) subordination homomorphism from $\langle D(X_2), \prec_{R_2}\rangle$ to $\langle D(X_1), \prec_{R_1}\rangle$.*

Proof From Priestley duality we have that f is a Priestley morphism if and only if $D(f) : D(X_2) \to D(X_1)$ is a homomorphism. Moreover, for every $x \in X_1$ it holds that $X(D(f))(\varepsilon_1(x)) = \varepsilon_2(f(x))$. Thus, using Propositions 7 and 10 we have that f is (strongly) stable if and only is $D(f)$ a (strong) subordination homomorphism. □

In a similar way we have:

Proposition 12 *Let $\langle X_1, R_1\rangle$ and $\langle X_2, R_2\rangle$ be two Priestley ∇-subordination spaces and $f : X_1 \to X_2$ a map. Then f is a (reversely strongly) stable Priestley morphism if and only if the map $D(f) : D(X_2) \to D(X_1)$ is a (strong) subordination homomorphism from $\langle D(X_2), \prec^*_{R_2}\rangle$ to $\langle D(X_1), \prec^*_{R_1}\rangle$.*

Proof From Priestley duality we have that f is a Priestley morphism if and only if $D(f) : D(X_2) \to D(X_1)$ is a homomorphism. And moreover for every $x \in X_1$ it holds that $X(D(f))(\varepsilon_1(x)) = \varepsilon_2(f(x))$. Thus, using Propositions 8 and 10 we have that f is (reversely strongly) stable if and only if $D(f)$ is a (strong) subordination homomorphism. □

Lemma 8 *Let $\langle X_1, R_1\rangle$ and $\langle X_2, R_2\rangle$ be Priestley Δ-subordination spaces. A Priestley morphism $f : X_1 \to X_2$ is dually strongly stable if and only if for every $U \in D(X_2)$,*

$$\Diamond_{R_1^{-1}}(f^{-1}[U]) = f^{-1}[\Diamond_{R_2^{-1}}(U)].$$

Proof Assume that f is a Priestley morphism that is dually strongly stable from $\langle X_1, R_1\rangle$ to $\langle X_2, R_2\rangle$. Let $U \in D(X_2)$. If $x \in \Diamond_{R_1^{-1}}(f^{-1}[U])$, then there exists $y \in f^{-1}[U]$ such that $xR_1^{-1}y$. Therefore yR_1x and, since f is stable, $f(y)R_2f(x)$. Since $f(y) \in U$ it follows that $f(x) \in \Diamond_{R_2^{-1}}(U)$ and hence $x \in f^{-1}[\Diamond_{R_2^{-1}}(U)]$. Conversely, if $x \in f^{-1}[\Diamond_{R_2^{-1}}(U)]$, we have $f(x)R_2^{-1}y$ for some $y \in U$, so that $yR_2f(x)$. Applying that f is dually strongly stable, there is $z \in X_1$ such that zR_1x and $y \leq_2 f(z)$. Since U is an upset, $z \in f^{-1}[U]$. Hence, since $xR_1^{-1}z$ we obtain that $x \in \Diamond_{R_1^{-1}}(f^{-1}[U])$.

Suppose now that for every $U \in D(X_2)$, $\Diamond_{R_1^{-1}}(f^{-1}[U]) = f^{-1}[\Diamond_{R_2^{-1}}(U)]$. Assume that $x, y \in X_1$ are such that xR_1y and $f(x) \not R_2 f(y)$, so that $f(x) \notin R_2^{-1}(f(y))$. Since $R_2^{-1}(f(y))$ is a closed set and a downset, there exists $U \in D(X_2)$ such that $R_2^{-1}(f(y)) \subseteq U^c$ and $f(x) \in U$, and therefore $x \in f^{-1}[U]$. Hence, $y \in$

$\Diamond_{R_1^{-1}}(f^{-1}[U])$. The assumption implies that $f(y) \in \Diamond_{R_2^{-1}}(U)$, a contradiction with $R_2^{-1}(f(y)) \subseteq U^c$. This shows that f is stable. To prove that f is dually strong assume that $x \in X_1$ and $y \in X_2$ are such that $yR_2 f(x)$. Let $U \in D(X_2)$ be such that $y \in U$. Then $x \in f^{-1}[\Diamond_{R_2^{-1}}(U)]$ and therefore $x \in \Diamond_{R_1^{-1}}(f^{-1}[U])$. This implies that $R_1^{-1}(x)$ is nonempty. Suppose that $R_1^{-1}(x) \cap f^{-1}[\uparrow y] = \emptyset$. For every $z \in R_1^{-1}(x)$, since $y \not\leq_2 f(z)$, let $U_z \in D(X_2)$ be such that $y \in U_z$ and $f(z) \notin U_z$ so that $z \in f^{-1}[U_z^c]$. Then $R_1^{-1}(x) \subseteq \bigcup\{f^{-1}[U_z^c] : z \in R_1^{-1}(x)\}$. Since $R_1^{-1}(x) = R_1^{-1}[(x]]$, because $R_1 \circ \leq_1 = R_1$ and $(x]$ is a closed downset, we have that $R_1^{-1}(x)$ is a closed downset. Therefore $R_1^{-1}(x)$ is compact. Thus there are $z_1, \ldots, z_n \in R_1^{-1}(x)$ with $R_1^{-1}(x) \subseteq f^{-1}[U_{z_1}^c] \cup \cdots \cup f^{-1}[U_{z_n}^c]$. Let $U = U_{z_1} \cap \cdots \cap U_{z_n}$. Then $U \in D(X_2)$, $f^{-1}[U_{z_1}^c] \cup \cdots \cup f^{-1}[U_{z_n}^c] = f^{-1}[U^c]$, and $y \in U$. Therefore, $R_1^{-1}(x) \cap f^{-1}[U] = \emptyset$. If follows that $x \notin \Diamond_{R_1^{-1}}(f^{-1}[U])$ but since $yR_2 f(x)$ and $y \in U$, $x \in f^{-1}[\Diamond_{R_2^{-1}}(U)]$, a contradiction. □

Proposition 13 *Let L_1, L_2 be subordination lattices and $h : L_1 \to L_2$ a map. If h is a dually strong subordination homomorphism from L_1 to L_2, then the dual map $X(h)$ is a dually strongly stable Priestley morphism from the Priestley Δ-space $\langle X(L_2), R^{\Delta}_{\prec_2}\rangle$ to the Priestley Δ-space $\langle X(L_1), R^{\Delta}_{\prec_1}\rangle$.*

Proof Suppose that $h : L_1 \to L_2$ is a dually strong subordination homomorphism. Since it is a subordination homomorphism we know that $X(h)$ is a stable Priestley morphism from $\langle X(L_2), R^{\Delta}_{\prec_2}\rangle$ to $\langle X(L_1), R^{\Delta}_{\prec_1}\rangle$. To prove that $X(h)$ is dually strongly stable assume that $QR^{\Delta}_{\prec_1} h^{-1}[P]$. Then $\Delta^{-1}_{\prec_1}(Q) \subseteq h^{-1}[P]$. Let I be the ideal generated by $\bigcup_{a \in L_2 \setminus P} \Delta_{\prec_2}(a)$ and let F be the filter generated by $h[Q]$. We claim that $F \cap I = \emptyset$. Assume the contrary and let $d \in F \cap I$. Then let $b \in Q$ such that $h(b) \leq_2 d$ and let $a \in L_2 \setminus P$ and $c \in \Delta_{\prec_2}(a)$ such that $d \leq_2 c$. It follows that $h(b) \prec_2 a$. Then by (1) there exists $e \in L_1$ such that $b \prec_1 e$ and $h(e) \leq_2 a$. Since $b \in Q$ we have $e \in \Delta^{-1}_{\prec_1}(Q)$ and therefore $e \in h^{-1}[P]$. Thus, $h(e) \in P$ and hence $a \in P$, a contradiction. We conclude that $F \cap I = \emptyset$. Let then $P' \in X(L_2)$ be such that $F \subseteq P'$ and $I \cap P' = \emptyset$. It follows that $\Delta^{-1}_{\prec_2}(P') \subseteq P$ and $Q \subseteq h^{-1}[P']$. Thus $P' R^{\Delta}_{\prec_2} P$ and $Q \subseteq h^{-1}[P']$. □

Proposition 14 *Let $\langle X_1, R_1\rangle, \langle X_2, R_2\rangle$ be Priestley Δ-subordination spaces and $f : X_1 \to X_2$ a Priestley morphism that is dually strongly stable from $\langle X_1, R_1\rangle$ to $\langle X_2, R_2\rangle$ if and only if $D(f) : D(X_2) \to D(X_1)$ is a dually strong subordination homomorphism from $\langle D(X_2), \prec_{R_2}\rangle$ to $\langle D(X_1), \prec_{R_1}\rangle$.*

Proof Assume that f is a Priestley morphism that is dually strongly stable from $\langle X_1, R_1\rangle$ to $\langle X_2, R_2\rangle$. We know from Lemma 8 that for every $U \in D(X_2)$,

$$\Diamond_{R_1^{-1}}(f^{-1}[U]) = f^{-1}[\Diamond_{R_2^{-1}}(U)].$$

We show that $D(f)$ is a subordination homomorphism from the subordination lattice $\langle D(X_2), \prec_{R_2}\rangle$ to $\langle D(X_1), \prec_{R_1}\rangle$. Suppose that $U, V \in D(X_2)$ are such that $U \prec_{R_2} V$. By Remark 4 we have $\Diamond_{R_2^{-1}}(U) \subseteq V$. Then $f^{-1}[\Diamond_{R_2^{-1}}(U)] \subseteq f^{-1}[V]$. Therefore,

$\Diamond_{R_1^{-1}}(f^{-1}[U]) \subseteq f^{-1}[V]$ and using Remark 4 again we have $f^{-1}[U] \prec_{R_1} f^{-1}[V]$. Now we prove that $D(f)$ is dually strong. Suppose that $U \in D(X_2)$ and $V \in D(X_1)$ are such that $f^{-1}[U] \prec_{R_1} V$. Thus, using Remark 4, $\Diamond_{R_1^{-1}}(f^{-1}[U]) \subseteq V$ and therefore $f^{-1}[\Diamond_{R_2^{-1}}(U)] \subseteq V$. The set $\Diamond_{R_2^{-1}}(U)$ is a closed upset. Therefore there is a family $\{U_j : j \in J\} \subseteq D(X_2)$ such that $\Diamond_{R_2^{-1}}(U) = \bigcap_{j\in J} U_j$ and hence $f^{-1}[\bigcap_{j\in J} U_j] \subseteq V$. Thus $\bigcap_{j\in J} f^{-1}[U_j] \subseteq V$. Since the sets $f^{-1}[U_j]$ are closed and V is open, by compactness of the space follows that there exists a finite $J' \subseteq J$ such that $\bigcap_{j\in J'} f^{-1}[U_j] \subseteq V$. Let $U' = \bigcap_{j\in J'} U_j$. Then $f^{-1}[U'] \subseteq V$ and since $\Diamond_{R_2^{-1}}(U) \subseteq U'$ we obtain, using Remark 4, that $U \prec_{R_2} U'$. Hence, we conclude that $D(f) : D(X_2) \to D(X_1)$ is a dually strong subordination homomorphism from $\langle D(X_2), \prec_{R_2}\rangle$ to $\langle D(X_1), \prec_{R_1}\rangle$.

Conversely, assume that $D(f) : D(X_2) \to D(X_1)$ is a dually strong subordination homomorphism from $\langle D(X_2), \prec_{R_2}\rangle$ to $\langle D(X_1), \prec_{R_1}\rangle$. Then Proposition 13 implies that $X(D(f))$ is a dually strongly stable Priestley morphism from the Priestley Δ-s-space $\langle X(D(L_1)), R^{\Delta}_{\prec_{R_1}}\rangle$ to the Priestley Δ-s-space $\langle X(D(L_2)), R^{\Delta}_{\prec_{R_2}}\rangle$. Using Proposition 7 and Priestley duality it easily follows that f is a dually strongly stable Priestley morphism from $\langle X_1, R_1\rangle$ to $\langle X_2, R_2\rangle$. □

Proposition 15 *Let L_1, L_2 be subordination lattices and $h : L_1 \to L_2$ a map. Then h is a dually strong subordination homomorphism from L_1 to L_2 if and only if the dual map $X(h)$ is a dually strongly stable Priestley morphism from the Priestley Δ-space $\langle X(L_2), R^{\Delta}_{\prec_2}\rangle$ to the Priestley Δ-space $\langle X(L_1), R^{\Delta}_{\prec_1}\rangle$.*

Proof The implication from left to right is Proposition 13. To prove the other implication, if $X(h)$ is a dually strongly stable Priestley morphism from the Priestley Δ-space $\langle X(L_2), R^{\Delta}_{\prec_2}\rangle$ to the Priestley Δ-space $\langle X(L_1), R^{\Delta}_{\prec_1}\rangle$, then by Proposition 14, $D(X(h))$ is a dually strong subordination homomorphism from the subordination lattice $\langle D(X(L_2)), \prec_{R^{\Delta}_{\prec_2}}\rangle$ to $\langle D(X(L_1)), \prec_{R^{\Delta}_{\prec_1}}\rangle\rangle$. By Proposition 9 the map σ_{L_i} is an isomorphism between $\langle L_i, \prec_i\rangle$ and $\langle D(X(L_i)), \prec_{R^{\Delta}_{\prec_i}}\rangle$ for $i = 1, 2$. Using Priestley duality, it follows that h is a dually strong subordination homomorphism from L_1 to L_2. □

We consider the following categories:

- ΔPriSp: the category of Priestley Δ-subordination spaces with the stable Priestley morphisms as its arrows.
- ΔPriSp^s: the category of Priestley Δ-subordination spaces with the strongly stable Priestley morphisms as its arrows.
- ΔPriSp^{ds}: the category of Priestley Δ-subordination spaces with the dually strongly stable Priestley morphisms as its arrows.
- ∇PriSp: the category of Priestley ∇-subordination spaces with the stable Priestley morphisms as its arrows.
- ∇PriSp^s: the category of Priestley ∇-subordination spaces with the reversely strongly stable Priestley morphisms as its arrows.
- SLat: the category of the subordination lattices with the subordination homomorphisms as its arrows.

- SLat^s: the category of the subordination lattices with the strong subordination homomorphisms as its arrows.
- SLat^{ds}: the category of the subordination lattices with the dually strong subordination homomorphisms as its arrows.

Remark 6 The categories $\Delta\mathsf{PriSp}$ and $\nabla\mathsf{PriSp}$ are equivalent as well as the categories $\Delta\mathsf{PriSp}^s$ and $\nabla\mathsf{PriSp}^s$. The functors that witness the equivalence are defined as follows. The functor from $\Delta\mathsf{PriSp}$ to $\nabla\mathsf{PriSp}$ maps a Priestley Δ-subordination space $\langle X, R\rangle$ to the Priestley ∇-subordination space $\langle X, R^{-1}\rangle$, and the functor from $\nabla\mathsf{PriSp}$ to $\Delta\mathsf{PriSp}$ does the same. For morphisms the functors leave the functions as they are. The same happens with $\Delta\mathsf{PriSp}^s$ and $\nabla\mathsf{PriSp}^s$.

The results above show that the functor D from the category PriSp to the category DLat can be expanded to a functor from $\Delta\mathsf{PriSp}$ to SLat, to a functor from $\Delta\mathsf{PriSp}^s$ to SLat^s and to a functor from $\Delta\mathsf{PriSp}^{ds}$ to SLat^{ds} by mapping any Δ-Priestley space $\langle X, R\rangle$ to its subordination lattice $\langle D(X), \prec_R\rangle$ and every morphism in the corresponding category of spaces to its Priestley dual. Also the results above show that the functor X from DLat to PriSp can be expanded to a functor from SLat to $\Delta\mathsf{PriSp}$, to a functor from SLat^s to $\Delta\mathsf{PriSp}^s$, and to a functor from SLat^{ds} to $\Delta\mathsf{PriSp}^{ds}$ by sending a subordination lattice $\langle L, \prec\rangle$ to the Priestley Δ-subordination space $\langle X(L), R^{\Delta}_{\prec}\rangle$ and every morphism in the corresponding category of subordination lattices to its Priestley dual. Doing it, we have that the two categories in the pairs ($\Delta\mathsf{PriSp}$, SLat), ($\Delta\mathsf{PriSp}^s$, SLat^s), and ($\Delta\mathsf{PriSp}^{ds}$, SLat^{ds}) are dually equivalent.

In a similar way, the functor D from PriSp to DLat can be expanded to a functor from $\nabla\mathsf{PriSp}$ to SLat and to a functor from $\Delta\mathsf{PriSp}^s$ to SLat^s by mapping any ∇-Priestley space $\langle X, R\rangle$ to its subordination lattice $\langle D(X), \prec^*_R\rangle$. The functor X from DLat to PriSp can also be expanded to a functor from SLat to $\nabla\mathsf{PriSp}$ and to a functor from SLat^s to $\nabla\mathsf{PriSp}^s$ that sends a subordination lattice $\langle L, \prec\rangle$ to the Priestley ∇-subordination space $\langle X(L), R^{\nabla}_{\prec}\rangle$. The morphisms are mapped in each case to their Priestley duals.

From the dualities discussed above for categories of subordination lattices we can obtain dualities for categories of bi-subordination lattices in the natural way. Let us introduce the dual objects of bi-subordination lattices.

Definition 8 We say that a triple $\langle X, R, S\rangle$ is a *Priestley bi-subordination space* if X is a Priestley space and R and S are binary relations on X each one of which is the Δ-dual of a subordination.

Note that $\langle X, R, S\rangle$ is a Priestley bi-subordination space if and only if $\langle X, R, S^{-1}\rangle$ is a quasi-modal space in the terminology of Castro and Celani (2004).

By combining the properties of subordination homomorphism, strong subordination homomorphism and dually strong subordination homomorphism we can consider several categories of bi-subordination lattices by taking as morphisms maps that are of one of these kinds for the first subordination and of another for the second. Similarly, we can consider several categories of Priestley bi-subordination spaces. Once we fix a choice of morphisms for a category of bi-subordination lattices we can consider the category of Priestley bi-subordination spaces with the corresponding choice of morphisms and in this way we obtain two categories that are dually equivalent.

For example, the category of bi-subordination lattices with morphisms the lattice homomorphisms that are a subordination homomorphism w.r.t. the first subordination and a strong subordination homomorphism w.r.t. the second subordination is dually equivalent to the category of Priestley bi-subordination spaces with the Priestley morphisms that are a stable morphism w.r.t. the first relation and a strongly stable morphism w.r.t. the second relation.

Now we turn to discuss the duals of the bi-subordination lattices with the properties we considered in the examples given in Sect. 8.3.

First we consider the bi-subordination lattices where the first subordination is included in the second. They include the bi-subordinations lattices in Example 1.

Proposition 16 *Let $\langle L, \prec, \lhd \rangle$ be a bi-subordination lattice. Then*

$$\prec \subseteq \lhd \iff R_\lhd \subseteq R_\prec,$$

(where $R_\lhd = R_{\Delta_\lhd}$ and $R_\prec = R_{\Delta_\prec}$).

Proof Assume that $\leq \subseteq \lhd$. Suppose that $P, Q \in X(L)$ are such that $P R_\lhd Q$. To prove that $P R_\prec Q$, suppose that $a \in \Delta_\prec^{-1}(P)$. Then $\Delta_\prec(a) \cap P \neq \emptyset$. So, let $b \in \Delta_\prec(a) \cap P$. Then $b \prec a$ and therefore $b \lhd a$. Therefore, $b \in \Delta_\lhd(a) \cap P$ and so $a \in \Delta_\prec^{-1}(P)$. Since $P R_\lhd Q$ it follows that $a \in Q$. We conclude that $\Delta_\prec^{-1}(P) \subseteq Q$, which by definition implies that $P R_\lhd Q$.

Assume now that $R_\lhd \subseteq R_\prec$, that $a \prec b$, and that it is not the case that $a \lhd b$. Then $a \notin \Delta_\lhd(b)$. Therefore there exists $P \in X(L)$ such that $a \in P$ and $P \cap \Delta_\lhd(b) = \emptyset$. Then $b \notin \Delta_\lhd^{-1}(P)$. Let then $Q \in X(L)$ such that $\Delta_\lhd^{-1}(P) \subseteq Q$ and $b \notin Q$. So we have $P R_\lhd Q$ and hence, by the assumption, $P R_\prec Q$, namely $\Delta_\prec^{-1}(P) \subseteq Q$. Since $a \in \Delta_\prec(b) \cap P$, $b \in \Delta_\prec^{-1}(P)$. Therefore $b \in Q$, a contradiction. We conclude that $a \lhd b$. Hence, $\prec \subseteq \lhd$. □

The proposition allows us to consider categories of Priestley bi-subordination spaces where the first relation is included in the second and obtain dualities for the categories of bi-subordination lattices with the first subordination included in the second.

Now we can consider the bi-subordination lattices where the second subordination is the converse of the first. They include the bi-subordination lattices in Example 3.

Let $L = \langle L, \prec, \lhd \rangle$ be a bi-subordination lattice such that $\lhd$ is the converse relation of $\prec$. Then, since for every $a, b \in L$

$$\sigma(a) \prec_{R_\prec} \sigma(b) \Leftrightarrow \sigma(a) \subseteq \Box_{R_\prec}(\sigma(b))$$

and

$$\sigma(a) \prec_{R_\lhd} \sigma(b) \Leftrightarrow \sigma(a) \subseteq \Box_{R_\lhd}(\sigma(b)),$$

we have

$$\sigma(a) \subseteq \Box_{R_\prec}(\sigma(b)) \Leftrightarrow \sigma(b) \subseteq \Box_{R_\lhd}(\sigma(a)).$$

This suggest considering the Priestley bi-subordination spaces $\langle X, R_1, R_2\rangle$ where R_1 and R_2 satisfy for all clopen upsets U, V the following condition:

$$U \subseteq \Box_{R_1}(V) \Leftrightarrow V \subseteq \Box_{R_2}(U).$$

Using this observation we can obtain dualities for the categories of bi-subordination lattices with each subordination being the converse of the other by takin as duals of these bi-subordination lattices the Priestley bi-subordination spaces that satisfy the above condition.

In the next section we discuss the dualities for positive bi-subordination lattices.

8.6 Positive Bi-Subordination Lattices

In this section we present first the dualities for positive subordination lattices that follow from the general facts described in the previous section. Then we present a different duality where positive subordination lattices are represented by a Priestley space endowed with a single binary relation.

The conditions that define positive bi-subordination lattices in Definition 3 can be characterized by properties of the relations associated with the subordinations as shown in the next two propositions.

Proposition 17 *Let $\langle L, \prec, \lhd\rangle$ be a bi-subordination lattice. The following conditions are equivalent:*

1. $\Delta_\prec(a \vee b) \subseteq \nabla_\lhd(a) \odot \Delta_\prec(b)$, *for all* $a, b \in L$,
2. $R_\prec^\Delta = (R_L \circ \subseteq)$,

where $R_L = R_\prec^\Delta \cap R_\lhd^\nabla$.

Proof (1) $\Rightarrow$ (2). To prove the inclusion $(R_L \circ \subseteq) \subseteq R_\prec^\Delta$, note that $(R_L \circ \subseteq) \subseteq R_\prec^\Delta \circ \subseteq$ and that by Lemma 2, $R_\prec^\Delta \circ \subseteq = R_\prec^\Delta$. To prove the other inclusion, suppose that $P, Q \in X(L)$ are such that $\Delta_\prec^{-1}(P) \subseteq Q$. We recall, by Lemma 1, that the set $\Delta_\prec^{-1}(P)$ is a filter of L. Consider the ideal $\left(Q^c \cup \nabla_\lhd^{-1}(P)^c\right]$. We prove that

$$\Delta_\prec^{-1}(P) \cap \left(Q^c \cup \nabla_\lhd^{-1}(P)^c\right] = \emptyset. \tag{8.1}$$

We assume the contrary. Then let $c \in \Delta_\prec^{-1}(P)$, $b \notin Q$ and $a \notin \nabla_\lhd^{-1}(P)$ such that $c \leq a \vee b$. We note that as $b \notin Q$, we have $b \notin \Delta_\prec^{-1}(P)$ and hence

$$\Delta_\prec(b) \cap P = \emptyset. \tag{8.2}$$

Also, as $a \notin \nabla_\lhd^{-1}(P)$, there exists $d \in L$ such that

$$d \in \nabla_{\lhd}(a) \text{ and } d \notin P. \tag{8.3}$$

Since $c \leq a \vee b$, and $\Delta_{\prec}$ is monotonic, $\Delta_{\prec}(c) \subseteq \Delta_{\prec}(a \vee b)$, and thus $\Delta_{\prec}(a \vee b) \cap P \neq \emptyset$, i.e., there exists $e \in \Delta_{\prec}(a \vee b)$ such that $e \in P$. Since by the hypothesis, $\Delta_{\prec}(a \vee b) \subseteq \nabla_{\lhd}(a) \odot \Delta_{\prec}(b)$ we have $e \in \nabla_{\lhd}(a) \odot \Delta_{\prec}(b)$ and as $d \in \nabla_{\lhd}(a)$, there exists $w \in \Delta_{\prec}(b)$ such that $e \leq d \vee w$. Since $e \in P$ and $d \notin P$, it follows that $w \in P$; hence $\Delta_{\prec}(b) \cap P \neq \emptyset$, in contradiction with (8.2). Thus, we obtain (8.1). Then there exists $D \in X(L)$ such that

$$\Delta_{\prec}^{-1}(P) \subseteq D \subseteq \nabla_{\lhd}^{-1}(P) \text{ and } D \subseteq Q.$$

This implies that $(P, Q) \in (R_L \circ \subseteq)$.

(2) $\Rightarrow$ (1) Assume that there exists $c \in \Delta_{\prec}(a \vee b)$ such that $c \notin \nabla_{\lhd}(a) \odot \Delta_{\prec}(b)$. Then there exists $d \in \nabla_{\lhd}(a)$ such that $c \notin (\Delta_{\prec}(b) \cup \{d\}]$. Therefore there exists $P \in X(L)$ satisfying that $c \in P$, $\Delta_{\prec}(b) \cap P = \emptyset$, and $d \notin P$. So, there exists $Q \in X(L)$ such that $\Delta_{\prec}^{-1}(P) \subseteq Q$ and $b \notin Q$. By hypothesis, there exists $D \in X(L)$ such that $\Delta_{\prec}^{-1}(P) \subseteq D \subseteq \nabla_{\lhd}^{-1}(P)$ and $D \subseteq Q$. As $c \in \Delta_{\prec}(a \vee b) \cap P$ and $\Delta_{\prec}^{-1}(P) \subseteq D$, we get that $a \vee b \in D \subseteq Q$, but since $b \notin Q$, it follows that $b \notin D$. Therefore, $a \in D$ and hence $a \in \nabla_{\lhd}^{-1}(P)$, which means that $\nabla_{\lhd}(a) \subseteq P$. Since $d \in \nabla_{\lhd}(a)$, it follows that $d \in P$, which is a contradiction. □

Proposition 18 *Let $\langle L, \prec, \lhd \rangle$ be a bi-subordination lattice. The following conditions are equivalent:*

1. $\nabla_{\lhd}(a \wedge b) \subseteq \Delta_{\prec}(a) \oplus \nabla_{\lhd}(b)$, *for all* $a, b \in L$,
2. $R_{\lhd}^{\nabla} = (R_L \circ \subseteq^{-1})$,

where $R_L = R_{\prec}^{\Delta} \cap R_{\lhd}^{\nabla}$.

Proof (1) $\Rightarrow$ (2) The inclusion $(R_L \circ \subseteq^{-1}) \subseteq R_{\lhd}^{\nabla}$ follows from Lemma 2. To prove the other inclusion, assume that $P, Q \in X(L)$ are such that $Q \subseteq \nabla_{\lhd}^{-1}(P)$. By Lemma 1, the set $\nabla_{\lhd}^{-1}(P)^c$ is an ideal of L. Consider the filter $\left[\Delta^{-1}(P) \cup Q\right)$. We prove that.

$$\left[\Delta_{\prec}^{-1}(P) \cup Q\right) \cap \nabla_{\lhd}^{-1}(P)^c = \emptyset. \tag{8.4}$$

Suppose the contrary. Then let $a \in \Delta_{\prec}^{-1}(P)$, $b \in Q$ and $c \notin \nabla_{\lhd}^{-1}(P)$ such that $a \wedge b \leq c$. Since $\nabla_{\lhd}$ is antimonotonic, $\nabla_{\lhd}(c) \subseteq \nabla_{\lhd}(a \wedge b)$, and since $c \notin \nabla_{\lhd}^{-1}(P)$, $\nabla_{\lhd}(c) \nsubseteq P$; hence $\nabla_{\lhd}(a \wedge b) \nsubseteq P$. Thus, let $d \in \nabla_{\lhd}(a \wedge b)$ be such that $d \notin P$. By hypothesis, $\nabla_{\lhd}(a \wedge b) \subseteq \Delta_{\prec}(a) \oplus \nabla_{\lhd}(b)$, so $d \in \Delta_{\prec}(a) \oplus \nabla_{\lhd}(b)$. Since $a \in \Delta_{\prec}^{-1}(P)$, there exists $e \in \Delta_{\prec}(a) \cap P$. Then there exists $w \in \nabla_{\lhd}(b)$ such that $e \wedge w \leq d$. Since $b \in Q \subseteq \nabla_{\lhd}^{-1}(P)$, we have $\nabla_{\lhd}(b) \subseteq P$. Thus $w \in P$, and hence, since then $e \wedge w \in P$, we have $d \in P$, which is a contradiction. Therefore, (8.4) is valid. Then there exists $D \in X(L)$ such that

$$\Delta_{\prec}^{-1}(P) \subseteq D \subseteq \nabla_{\lhd}^{-1}(P) \text{ and } Q \subseteq D,$$

i.e., $(P, Q) \in (R_L \circ \subseteq^{-1})$.

(2) $\Rightarrow$ (1) Assume (2) and suppose that $a, b, c \in L$ are such that $c \in \nabla_{\lhd}(a \wedge b)$. Suppose that $c \notin \Delta_{\prec}(a) \oplus \nabla_{\lhd}(b)$. Then there exists $d \in \Delta_{\prec}(a)$ such that $c \notin [\nabla_{\lhd}(b) \cup \{d\})$. Then there exists $P \in X(L)$ such that $\nabla_{\lhd}(b) \subseteq P$, $d \in P$ and $c \notin P$. By Lemma 3 there exists $Q \in X(L)$ such that $Q \subseteq \nabla_{\lhd}^{-1}(P)$ and $b \in Q$. By hypothesis, there exists $D \in X(L)$ such that $\Delta_{\prec}^{-1}(P) \subseteq D \subseteq \nabla_{\lhd}^{-1}(P)$ and $Q \subseteq D$. As $d \in \Delta_{\prec}(a) \cap P$, we have $a \in D$, and as $b \in Q$, we get that $a \wedge b \in D$. So, $\nabla_{\lhd}(a \wedge b) \subseteq P$, but this implies that $c \in P$, which is impossible. Therefore, $\nabla_{\lhd}(a \wedge b) \subseteq \Delta_{\prec}(a) \oplus \nabla_{\lhd}(b)$. □

Corollary 2 *Let $\langle L, \prec, \lhd\rangle$ be a bi-subordination lattice. Then L is a positive bi-subordination lattice if and only if the following two conditions hold*

1. $R_{\prec}^{\Delta} = (R_L \circ \subseteq)$,
2. $R_{\lhd}^{\nabla} = (R_L \circ \subseteq^{-1})$,

where $R_L = R_{\prec}^{\Delta} \cap R_{\lhd}^{\nabla}$. Equivalently, if and only if

1. $R_{\prec}^{\Delta} = (R_L \circ \subseteq)$,
2. $(R_{\lhd}^{\Delta})^{-1} = (R_L \circ \subseteq^{-1})$,

where $R_L = R_{\prec}^{\Delta} \cap (R_{\lhd}^{\Delta})^{-1}$.

The corollary motivates the next definition.

Definition 9 A Priestley bi-subordination space $\langle X, R_1, R_2\rangle$ is *positive* if R_1 and R_2 satisfy the following conditions:

1. $R_1 = (R_1 \cap R_2^{-1}) \circ \leq$,
2. $R_2^{-1} = (R_1 \cap R_2^{-1}) \circ \leq^{-1}$.

From the results obtained up to now we easily can prove the next two propositions.

Proposition 19 *Let L be a bi-subordination lattice. Then L is a positive subordination lattice if and only if the Priestley bi-subordination space $\langle X(L), R_{\prec}^{\Delta}, R_{\lhd}^{\Delta}\rangle$ is positive.*

Proof It follows from Corollary 2. □

Proposition 20 *A Priestley bi-subordination space $\langle X, R_1, R_2\rangle$ is positive if and only if $\langle D(X), \prec_{R_1}, \prec_{R_2}\rangle$ is a positive subordination lattice.*

Proof It follows from Proposition 7, Corollary 2, and Proposition 9. □

As for other classes of bi-subordination lattices, once we fix as objects the positive subordination lattices we obtain different categories by taking as arrows maps that are of one kind of morphism (subordination homomorphism, strong subordination homomorphism and dually strong subordination homomorphism) for the first subordination and of another for the second. Then, moving to the corresponding categories of positive Priestley bi-subordination spaces, the results in Sect. 8.4 provide us with the corresponding duality results.

Now we turn to find a category of Priestley spaces with a single binary relation dually equivalent to the category of the positive subordination lattices with arrows the maps that are a subordination homomorphism for both subordinations and also a category of Priestley spaces with a single binary relation dually equivalent to the category of the positive subordination lattices with arrows the maps that are a strong subordination homomorphism for the first subordination and a dually strong subordination homomorphism for the second subordination.

Proposition 21 *If $L = \langle L, \prec, \lhd \rangle$ is a positive bi-subordination lattice, $a \in L$, and $P \in X(L)$, then using the relation $R_L = R^{\Delta}_{\prec} \cap R^{\nabla}_{\lhd} = R^{\Delta}_{\prec} \cap (R^{\Delta}_{\prec})^{-1}$ we have:*

1. *$\Delta_{\prec}(a) \cap P = \emptyset$ iff there exists $Q \in X(L)$ such that $(P, Q) \in R_L$ and $a \notin Q$.*
2. *$\nabla_{\lhd}(a) \subseteq P$ iff there exists $Q \in X(L)$ such that $(P, Q) \in R_L$ and $a \in Q$.*

Proof (1) Assume that $\Delta_{\prec}(a) \cap P = \emptyset$. By Lemma 3 there exists $Q \in X(L)$ such that $P R^{\Delta}_{\prec} Q$ and $a \notin Q$. By Corollary 2, there exists $Q' \in X(L)$ such that $P R_L Q'$ and $Q' \subseteq Q$. The converse follows from the fact that $R_L \subseteq R^{\Delta}_{\prec}$.

The proof of (2) is similar. □

Lemma 9 *If $\langle X, R_1, R_2 \rangle$ is a positive Priestley bi-subordination space and $R = R_1 \cap R_2^{-1}$, then*

1. *$R(x)$ is a closed subset of X, for each $x \in X$,*
2. *$R = (R \circ \leq) \cap (R \circ \leq^{-1})$,*
3. *$\Box_R(U) = \Box_{R_1}(U)$, for each $U \in D(X)$,*
4. *$\Diamond_R(U) = \Diamond_{R_2^{-1}}(U)$, for each $U \in D(X)$.*

Proof The proof of (1) is immediate because both $R_1(x)$ and $R_2^{-1}(x)$ are closed sets. For (2) we note that as $R_1 = (R_1 \cap R_2^{-1}) \circ \leq = R \circ \leq$ and $R_2^{-1} = (R_1 \cap R_2^{-1}) \circ \leq^{-1} = R \circ \leq^{-1}$, then $R = R_1 \cap R_2^{-1} = (R \circ \leq) \cap (R \circ \leq^{-1})$.

For the proof of (3) assume that $x \in \Box_R(U)$. If $x \notin \Box_{R_1}(U)$, then there exists $y \in X$ such that $(x, y) \in R_1$ but $y \notin U$. As $R_1 = R \circ \leq$, there exist $z \in X$ such that $(x, z) \in R$ and $z \leq y$. So, since $x \in \Box_R(U)$, we have $z \in U$, and since U is an upset, $y \in U$, which is a contradiction. Suppose now that $x \in \Box_{R_1}(U)$. Note that $R \subseteq R \circ \leq = R_1$. Therefore $R(x) \subseteq R_1(x)$. It follows that $x \in \Box_R(U)$.

To prove (4) assume that $x \in \Diamond_R(U)$. Then $R(x) \cap U \neq \emptyset$. Therefore, $R_2^{-1}(x) \cap U \neq \emptyset$ and so $x \in \Diamond_{R_2^{-1}}(U)$. Conversely, if $x \in \Diamond_{R_2^{-1}}(U)$, let $y \in R_2^{-1}(x) \cap U$. Then, since $R_2^{-1} = (R_1 \cap R_2^{-1}) \circ \leq^{-1}$, there is $u \in (R_1 \cap R_2^{-1})(x)$ such that $y \leq u$. Since U is an upset, it follows that $u \in R(x) \cap U$; therefore $x \in \Diamond_R(U)$. □

Proposition 22 *Let $\langle X, R \rangle$ be a relational structure such that X is a Priestley space and R is a binary relation on X satisfying the following conditions:*

1. *For every $x \in X$, $R(x)$ is a closed set,*
2. *$R = (R \circ \leq) \cap (R \circ \leq^{-1})$,*
3. *$\Box_R(U)$ is an open upset for every $U \in D(X)$, and $\Diamond_R(U)$ is closed upset for every $U \in D(X)$.*

Then the structure $\langle X, R_1, (R_2)^{-1}\rangle$ *is a positive bi-subordination space, where* R_1 *and* R_2 *are defined as* $R_1 = R \circ \leq$ *and* $R_2 = R \circ \leq^{-1}$*, respectively.*

Proof We prove that $R_1(x)$ is a closed set and an upset. If it is empty it is clear. If it is nonempty, suppose that $y \notin R_1(x)$. Then, since $R_1 = R \circ \leq$, for each $z \in R(x)$ we have $z \nleq y$. Also $R(x)$ is nonempty. So, for each $z \in R(x)$ there exists $U_z \in D(X)$ such that $z \in U_z$, and $y \notin U_z$. Then, $R(x) \subseteq \bigcup\{U_z : z \in R(x)\}$ and since $R(x)$ is closed, there exists a finite subfamily $\{U_{z_1}, \ldots, U_{z_n}\}$ of $\{U_z : z \in R(x)\}$ such that $R(x) \subseteq U_{z_1} \cup \cdots \cup U_{z_n}$. Let $U = U_{z_1} \cup \cdots \cup U_{z_n}$. It is clear that $y \notin U$ and that U is an upset. Therefore, $R_1(x) \subseteq U$. Thus we have proved that for each $y \notin R_1(x)$ there exists $U \in D(X)$ such that $R_1(x) \subseteq U$ and $y \notin U$. It follwos that $R_1(x)$ is closed and the definition of R_1 implies that it is an upset. Similarly, we can prove that $R_2(x)$ is a closed downset.

Now we prove that for every $U \in D(X)$, $\Box_R(U) = \Box_{R_1}(U)$ and $\Diamond_R(U) = \Diamond_{R_2}(U)$. Since every $U \in D(X)$ is an upset, it easily follows that for every $U \in D(X)$ and $x \in X$, $R(x) \subseteq U$ if and only if $(R \circ \leq)(x) \subseteq U$. Hence, $\Box_R(U) = \Box_{R_1}(U)$ for every $U \in D(X)$. Moreover, for every $U \in D(X)$ it also holds that $\Diamond_R(U) = \Diamond_{R_2}(U)$. Indeed, since $R_2 = R \circ \leq^{-1}$, for every $x \in X$ and $U \in D(X)$, since U is an upset it follows that $R(x) \cap U \neq \emptyset$ if and only if $R_2(x) \cap U \neq \emptyset$. Therefore $\Diamond_R(U) = \Diamond_{R_2}(U)$.

Using the fact we have just proved, (3) of the assumption implies that R_1 is the Δ-dual of a subordination and R_2 is the ∇-dual of a subordination. Therefore, R_2^{-1} is the Δ-dual of a subordination. Then (2) of the assumption implies that $R = R_1 \cap R_2$. Therefore $R_1 = R \circ \leq = (R_1 \cap R_2) \circ \leq$ and $R_2 = R \circ \leq^{-1} = (R_1 \cap R_2) \circ \leq^{-1}$. It follows form the definition of positive bi-subordination space that $\langle X, R_1, (R_2)^{-1}\rangle$ is a positive bi-subordination space. □

Definition 10 A *positive Priestley space* is a pair $\langle X, R\rangle$ where X is a Priestley space and R is a relation on X that satisfies the conditions of Proposition 22.

By the above results, if $\langle X, R\rangle$ is a positive Priestley space, then the structure $\langle X, R_1, (R_2)^{-1}\rangle$ defined as in Proposition 22 is a positive Priestley bi-subordination space such that the pair $\langle X, R_1 \cap R_2\rangle$ satisfies the conditions in Proposition 22. Conversely, if $\langle X, R_1, R_2\rangle$ is a positive Priestley bi-subordination space, then the structure $\langle X, R_1 \cap R_2^{-1}\rangle$ satisfies the conditions in Proposition 22, and therefore it is a positive Priestley space such that the triple $\langle X, (R_1 \cap R_2^{-1}) \circ \leq, (R_1 \cap R_2^{-1}) \circ \leq^{-1}\rangle$ is a positive Priestley bi-subordination space where $R_1 = (R_1 \cap R_2^{-1}) \circ \leq$ and $R_2 = (R_1 \cap R_2^{-1}) \circ \leq^{-1}$. Thus, we have that there exists a bijective correspondence between positive Priestley bi-subordination spaces and positive Priestley spaces.

Definition 11 Let $\langle X, R\rangle$ and $\langle Y, S\rangle$ be positive Priestley spaces. A Priestley morphism $f : X \to Y$ from $\langle X, R\rangle$ to $\langle Y, S\rangle$ is *stable* if for every $x, y \in X$ such that xRy it holds that $f(x)Sf(y)$ and it is *doubly strongly stable* if it is stable and for all $x \in X$ and all $y \in Y$, if $f(x)Sy$ then there exist $z_1, z_2 \in X$ such that xRz_1, xRz_2 and $f(z_1) \leq y \leq f(z_2)$.

Proposition 23 *Let $\langle X, R\rangle$ and $\langle Y, S\rangle$ be positive Priestley spaces and consider the relations $R_1 = R \circ \leq$, $R_2 = R \circ \leq^{-1}$, $S_1 = S \circ \leq$, and $S_2 = S \circ \leq^{-1}$. Then $f : X \to Y$ is a (doubly strongly) stable Priestley morphism from $\langle X, R\rangle$ to $\langle Y, S\rangle$ if and only if f is a (strongly) stable morphism from $\langle X, R_1\rangle$ to $\langle Y, S_1\rangle$ and a (dually strongly) stable morphism from $\langle X, R_2^{-1}\rangle$ to $\langle Y, S_2^{-1}\rangle$.*

Proof Let $f : X \to Y$ be a Priestley morphism. Suppose that f is stable from $\langle X, R\rangle$ to $\langle Y, S\rangle$. To prove that f is stable from $\langle X, R_1\rangle$ to $\langle Y, S_1\rangle$ suppose that $x, y \in X$ are such that $x R_1 y$. Then let $z \in X$ be such that $x R z$ and $z \leq y$. Hence, by the assumption of stability, $f(x) S f(z)$ and since f is a Priestley morphism $f(z) \leq f(y)$. Therefore $f(x) S_1 f(y)$. A similar proof shows that f is stable from $\langle X, R_2^{-1}\rangle$ to $\langle Y, S_2^{-1}\rangle$. Assume now that f is doubly strongly stable. We proceed to prove that f is strongly stable from $\langle X, R_1\rangle$ to $\langle Y, S_1\rangle$. Assume that $x \in X$ and $y \in Y$ are such that $f(x) S_1 y$. Then there is $u \in Y$ such that $f(x) S u$ and $u \leq y$. Thus, since f is double strongly stable there are $z_1, z_2 \in X$ such that $x R z_1$, $x R z_2$ and $f(z_1) \leq u \leq f(z_2)$. Then, as $u \leq y$, $x R z_1$ and $f(z_1) \leq y$. This shows that f is strongly stable from $\langle X, R_1\rangle$ to $\langle Y, S_1\rangle$. We now show that f is dually strongly stable from $\langle X, R_2^{-1}\rangle$ to $\langle Y, S_2^{-1}\rangle$. Suppose now that $x \in X$ and $y \in Y$ are such that $f(x) S_2 y$. Then there is $u \in Y$ such that $f(x) S u$ and $y \leq u$. Since f is double strongly stable there are $z_1, z_2 \in X$ such that $x R z_1, x R z_2$ and $f(z_1) \leq u \leq f(z_2)$. Then, as $y \leq u$, $x R z_2$ and $y \leq f(z_2)$. Thus we obtain that f is dually strongly stable from $\langle X, R_2^{-1}\rangle$ to $\langle Y, S_2^{-1}\rangle$.

Conversely, assume that f is a stable morphism from $\langle X, R_1\rangle$ to $\langle Y, S_1\rangle$ and a stable morphism from $\langle X, R_2\rangle$ to $\langle Y, S_2\rangle$. To prove that f is stable from $\langle X, R\rangle$ to $\langle Y, S\rangle$, suppose that $x, y \in X$ are such that $x R y$. Note that since $\langle X, R\rangle$ and $\langle Y, S\rangle$ are positive Priestley spaces $R = R_1 \cap R_2$ and $S = S_1 \cap S_2$. Hence, $x R_1 y$ and $x R_2 y$. Therefore the assumption implies that $f(x) S_1 f(y)$ and $f(x) S_2 f(y)$. Thus we have $f(x) S f(y)$. Suppose now that f is a strongly stable morphism from $\langle X, R_1\rangle$ to $\langle Y, S_1\rangle$ and a dually strongly stable morphism from $\langle X, R_2^{-1}\rangle$ to $\langle Y, S_2^{-1}\rangle$. Assume that $x \in X$ and $y \in Y$ are such that $f(x) S y$. Then $f(x) S_1 y$ and $f(x) S_2 y$. Thus, let $z_1, z_2 \in X$ such that $x R_1 z_1$ and $f(z_1) \leq y$ and $x R_2 z_2$ and $y \leq f(z_2)$. Let then $u_1, u_2 \in X$ such that $x R u_1$, $u_1 \leq z_1$, $x R u_2$, and $z_2 \leq u_2$. Then $x R u_1$, $x R u_2$ $f(u_1) \leq f(z_1) \leq y$, and $y \leq f(z_2) \leq f(u_2)$. We obtain the desired conclusion. □

Let PBiSLat be the category with objects the positive subordination lattices and arrows the maps between them that are a subordination homomorphism w.r.t. the two subordinations. Let $\mathsf{PBiSLat}^{\mathsf{s}}$ be the category with objects the positive subordination lattices and arrows the maps between them that are a strong subordination homomorphism w.r.t. the first subordination and a dual strong subordination homomorphism w.r.t. the second. Similarly, let PPriSp be the category with objects the positive Priestley spaces and arrows the stable Priestley morphisms and let $\mathsf{PPriSp}^{\mathsf{s}}$ be the category with objects the positive Priestley spaces and arrows the doubly strongly stable Priestley morphisms. From the results above the next theorem follows.

Proposition 24 *The categories* PBiSLat *and* PPriSp *are dually equivalent as well as the categories* $\mathsf{PBiSLat}^{\mathsf{s}}$ *and* $\mathsf{PPriSp}^{\mathsf{s}}$.

Acknowledgements This project has received funding from the European Union Horizon 2020 research and innovation program under the Marie Sklodowska-Curie Grant Agreement No 689176. The second author was also partially supported by the research Grants 2014 SGR 788 and 2016 SGR 95 from the Generalitat de Catalunya and by the research project MTM2016-74892-P from the government of Spain, which include FEDER funds from the European Union.

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Chapter 9
Hard Provability Logics

Mojtaba Mojtahedi

Abstract Let $\mathsf{PL}(T, T')$ and $\mathsf{PL}_{\Sigma_1}(T, T')$ respectively indicate the provability logic and Σ_1-provability logic of T relative in T'. In this paper we characterise the following relative provability logics: $\mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathbb{N})$, $\mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{PA})$, $\mathsf{PL}_{\Sigma_1}(\mathsf{HA}^*, \mathbb{N})$, $\mathsf{PL}_{\Sigma_1}(\mathsf{HA}^*, \mathsf{PA})$, $\mathsf{PL}(\mathsf{PA}, \mathsf{HA})$, $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{HA})$, $\mathsf{PL}(\mathsf{PA}^*, \mathsf{HA})$, $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{HA})$, $\mathsf{PL}(\mathsf{PA}^*, \mathsf{PA})$, $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{PA})$, $\mathsf{PL}(\mathsf{PA}^*, \mathbb{N})$, $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathbb{N})$ (see Table 9.3). It turns out that all of these provability logics are decidable. The notion of *reduction* for provability logics, first informally considered in (Ardeshir and Mojtahedi 2015). In this paper, we formalize a generalization of this notion (Definition 9.4.1) and provide several reductions of provability logics (see Diagram 9.5). The interesting fact is that $\mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathbb{N})$ is the hardest provability logic: the arithmetical completenesses of all provability logics listed above, as well as well-known provability logics like $\mathsf{PL}(\mathsf{PA}, \mathsf{PA})$, $\mathsf{PL}(\mathsf{PA}, \mathbb{N})$, $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{PA})$, $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathbb{N})$ and $\mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{HA})$, are all propositionally reducible to the arithmetical completeness of $\mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathbb{N})$.

Keywords Provability logic · Relative provability logic · Standard model · Heyting arithmetic HA · Peano arithmetic PA · Intuitionistic logic · Rreduction

9.1 Dedication

My works in general and the present paper in particular have been highly inspired by Mohammad Ardeshir's outstanding contributions to mathematical logic. The ideas that I developed in this paper originate from a joint paper (Ardeshir and Mojtahedi 2015) which was initially motivated by him. My first recollection of Dr. Ardeshir as we call him (out of awe and admiration, even when refer to him in his absence), goes

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M. Mojtahedi et al. (eds.), *Mathematics, Logic, and their Philosophies*, Logic, Epistemology, and the Unity of Science 49,
https://doi.org/10.1007/978-3-030-53654-1_9

back to 2003 when, in the second semester of my undergraduate studies, I attended his course on the foundations of mathematics. I was impressed by his knowledge and by the style of his teaching which encouraged me to attend most of his other courses during my undergraduate and graduate studies. I still vividly remember how deeply I was fascinated by his graduate course on Gödel's incompleteness theorems in Fall 2006. It was this course which made me determine to do my Ph.D. in mathematical logic and under the supervision of Dr. Ardeshir. His influence on me is not restricted to my academic work, as he has been a source of inspiration on many aspects of my life; and that is why dedicating this paper to him is the least thing I can do to thank him.

9.2 Introduction

There are two excellent surveys on provability logic: (Beklemishev and Visser 2006; Artemov and Beklemishev 2004). To be self-contained, I bring some selected subjects from them here, and then review some related recent results on this subject.

The provability interpretation for the modal operator $\Box$, first considered by Kurt Gödel (Gödel 1933), intending to provide a semantics for Heyting's formalization of the intuitionistic logic, IPC. On the other hand, and again by celebrated Gödel's incompleteness results (Gödel 1931), for a recursively enumerable theory T and a sentence in the language of T, one may formalize "A is provable in T" via a simple (Σ_1) formula $\mathsf{Prov}_\mathsf{T}(\ulcorner A \urcorner)$ in the first-order language of arithmetic, in which $\ulcorner A \urcorner$ is the Gödel number of A. Let $\mathsf{PL}(T, T')$ and $\mathsf{PL}_{\Sigma_1}(T, T')$ respectively indicate the provability logic and Σ_1-provability logic of T relative in T' (Definition 9.3.2). Here is a list of results on provability logics with arithmetical flavour:

1. $\neg\Box\bot \notin \mathsf{PL}(\mathsf{PA}, \mathsf{PA})$ (Gödel 1931).
2. $\Box(\Box A \to A) \to \Box A \in \mathsf{PL}(\mathsf{PA}, \mathsf{PA})$ (Löb 1955).
3. $A \in \mathsf{PL}(\mathsf{HA}, \mathsf{HA})$ for a nonmodal proposition A, iff A is valid in the intuitionistic logic IPC (de Jongh 1970; de Jongh et al. 2011).
4. $\mathsf{GL} = \mathsf{PL}(\mathsf{PA}, \mathsf{PA})$ and $\mathsf{GL\underline{S}} = \mathsf{PL}(\mathsf{PA}, \mathbb{N}) = \mathsf{PL}(\mathsf{PA}, \mathsf{ZF})$ (Solovay 1976), in which GL is Gödel-Löb logic, as defined in Definition 9.3.6.
5. $\Box(A \vee B) \to (\Box A \vee \Box B) \notin \mathsf{PL}(\mathsf{HA}, \mathsf{HA})$ (Myhill 1973; Friedman 1975).
6. $\Box(A \vee B) \to \Box(\boxdot A \vee \boxdot B) \in \mathsf{PL}(\mathsf{HA}, \mathsf{HA})$, in which $\boxdot A$ is a shorthand for $A \wedge \Box A$ (Leivant 1975).
7. $\mathsf{iGLCT} = \mathsf{PL}(\mathsf{PA}^*, \mathsf{PA}^*)$ (Visser 1981, 1982), in which iGLCT is as defined in Definition 9.3.6.
8. $\Box\neg\neg\Box A \to \Box\Box A \in \mathsf{PL}(\mathsf{HA}, \mathsf{HA})$ and $\Box(\neg\neg\Box A \to \Box A) \to \Box(\Box A \vee \neg\Box A) \in \mathsf{PL}(\mathsf{HA}, \mathsf{HA})$ (Visser 1981, 1982).
9. Rosalie Iemhoff 2001 introduced a uniform axiomatization of all known axiom schemas of $\mathsf{PL}(\mathsf{HA}, \mathsf{HA})$ in an extended language with a bimodal operator $\rhd$. In her Ph.D. dissertation (Iemhoff 2001), Iemhoff raised a conjecture that implies

directly that her axiom system, iPH, restricted to the normal modal language, is equal to $\mathsf{PL}(\mathsf{HA}, \mathsf{HA})$ (Iemhoff 2001).
10. $\mathsf{PL}_{\{\top,\bot\}}(\mathsf{HA}, \mathsf{HA})$ is decidable (Visser 2002). In other words, Visser introduced a decision algorithm for $A \in \mathsf{PL}(\mathsf{HA}, \mathsf{HA})$, for all A not containing any atomic variable.
11. $\mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{HA}) = \mathsf{iH}_\sigma$ (Definition 9.3.30) is decidable (Ardeshir and Mojtahedi 2018; Visser and Zoethout 2019).
12. $\mathsf{PL}_{\Sigma_1}(\mathsf{HA}^*, \mathsf{HA}^*) = \mathsf{iH}_\sigma^*$ (Definition 9.3.30) is decidable, Ardeshir and Mojtahedi (2019).

As it is known in the literature (Troelstra and van Dalen 1988), Heyting Arithmetic HA, enjoys disjunction property: if $\mathsf{HA} \vdash A \vee B$, then either $\mathsf{HA} \vdash A$ or $\mathsf{HA} \vdash B$. Regrettably, HA is not able to prove this (Friedman 1975; Myhill 1973). Hence such properties, are not reflected in the provability logic of HA, as a valid principle $\Box(A \vee B) \to (\Box A \vee \Box B)$. A natural question arises here: *is there any other valid rule?*

One way to systematically answer this question, is to characterise the truth provability logic of HA. In the case of classical arithmetic PA, Robert Solovay in his original ingenious paper (Solovay 1976), characterised the truth provability logic of PA. He showed that the only extra valid axiom is the soundness principle $\Box A \to A$, which is known to be true and unprovable in PA. In this paper we show that, in the Σ_1-provablity logic of HA, the same thing happens: The truth Σ_1-provablity logic of HA is decidable and only has the extra axiom schema $\Box A \to A$. The disjunction property, which we mentioned before, will be deduced from Leivant's principle $\Box(A \vee B) \to \Box(\Box A \vee \Box B)$ and the soundness principle.

The author of this paper in his joint paper with (Ardeshir and Mojtahedi 2015), showed that the arithmetical completeness of the modal logic GL is reducible to the arithmetical completeness of $\mathsf{GL} + p \to \Box p$ for Σ_1 interpretations. The reduction involves only a propositional argument. In this paper, I show that all relative provability logics, discussed in this paper, are reducible to the truth Σ_1-provability logic of HA (see Diagram 9.5). So, in a sense, $\mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathbb{N})$ is the hardest among them.

With the handful propositional reductions, we will characterise several relative provability logics for HA, PA, HA^* and PA^*, the self-completions of HA and PA (Visser 1982).

9.3 Definitions and Preliminaries

The propositional non-modal language $\mathcal{L}_0$ contains atomic variables, $\vee$, $\wedge$, $\to$, $\bot$ and the propositional modal language $\mathcal{L}_\Box$ has an additional operator $\Box$. In this paper, the atomic propositions (in the modal or non-modal language) include atomic variables and $\bot$. For an arbitrary proposition A, $\mathsf{Sub}(A)$ is defined to be the set of all subformulae of A, including A itself. We take $\mathsf{Sub}(X) := \bigcup_{A \in X} \mathsf{Sub}(A)$ for a set of propositions X. We use $\boxdot A$ as a shorthand for $A \wedge \Box A$ and $\neg A$ for $A \to \bot$. The logic IPC is the intuitionistic propositional non-modal logic over the usual propositional non-modal language. The theory $\mathsf{IPC}_\Box$ is the same theory IPC in the extended language of the propositional modal language, i.e. its language is the propositional modal language and its axioms and rules are same as IPC. Because we have no axioms for $\Box$ in $\mathsf{IPC}_\Box$, it is obvious that $\Box A$ for each A, behaves exactly like an atomic variable inside $\mathsf{IPC}_\Box$. First-order intuitionistic logic is denoted by IQC and the logic CQC is its classical closure, i.e. IQC plus the principle of excluded middle. For a set of sentences and rules $\Gamma \cup \{A\}$ in the propositional non-modal, propositional modal or first-order language, $\Gamma \vdash A$ means that A is derivable from Γ in the system IPC, $\mathsf{IPC}_\Box$, IQC, respectively. For an arithmetical formula, $\ulcorner A \urcorner$ represents the Gödel number of A. For an arbitrary arithmetical theory T with a $\Delta_0(\exp)$-set of axioms, as far as we work in strong enough theories which is the case in this paper, we have the $\Delta_0(\exp)$-predicate $\mathsf{Proof_T}(x, \ulcorner A \urcorner)$, that is a formalization of "x is the code of a proof for A in T". Note that by (inspection of the proof of) Craig's theorem, every recursively enumerable theory has a $\Delta_0(\mathsf{exp})$-axiomatization. We also have the provability predicate $\mathsf{Prov_T}(\ulcorner A \urcorner) := \exists x\ \mathsf{Proof_T}(x, \ulcorner A \urcorner)$. The set of natural numbers is denoted by $\omega := \{0, 1, 2, \ldots\}$.

Definition 9.3.1 Suppose T is a $\Delta_0(\mathsf{exp})$-axiomatized theory and σ is a substitution i.e. a function from atomic variables to arithmetical sentences. We define the interpretation σ_T which extends the substitution σ to all modal propositions A, inductively:

- $\sigma_T(A) := \sigma(A)$ for atomic A,
- σ_T distributes over $\wedge, \vee, \to$,
- $\sigma_T(\Box A) := \mathsf{Prov_T}(\ulcorner \sigma_T(A) \urcorner)$.

We call σ a Γ-substitution (in some theory T), if for every atomic A, $\sigma(A) \in \Gamma$ ($T \vdash \sigma(A) \leftrightarrow A'$ for some $A' \in \Gamma$). We also say that σ_T is a Γ-interpretation if σ is a Γ-substitution.

Definition 9.3.2 The *relative provability logic* of T in some sufficiently strong theory U restricted to a set of first-order sentences Γ, is defined to be a modal propositional theory $\mathsf{PL}_\Gamma(T, U)$ such that $\mathsf{PL}_\Gamma(T, U) \vdash A$ iff for all arithmetical substitutions σ in Γ, we have $U \vdash \sigma_T(A)$. We make this convention: $\mathsf{PL}_\Gamma(T, \mathbb{N})$ indicates $\mathsf{PL}_\Gamma(T, \mathsf{Theory}(\mathbb{N}))$, in which $\mathsf{Theory}(\mathbb{N})$ is the set of all true sentences in the standard model of arithmetic.

Define NOI (No Outside Implication) as the set of modal propositions A, such that any occurrence of $\to$ is in the scope of some $\Box$. To be able to state an extension of Leivant's Principle, we need a translation on the modal language which we call *Leivant's translation*.

Definition 9.3.3 Define the Leivant's translation $(.)^l$, inductively on modal propositions:

- $A^l := A$ for atomic or boxed A,
- $(A \wedge B)^l := A^l \wedge B^l$,
- $(A \vee B)^l := \boxdot A^l \vee \boxdot B^l$,

- $(A \to B)^l$ is defined by cases: If $A \in \mathsf{NOI}$, we define $(A \to B)^l := A \to B^l$, otherwise we define $(A \to B)^l := A \to B$.

Definition 9.3.4 Let us inductively define the box translation $(.)^{\Box}$ and some variants of it:

- $A^{\Box\uparrow} := A^{\Box} := \boxdot A$ and $A^{\Box\downarrow} := A$ for atomic A or $A := \top, \bot$,
- $(\Box A)^{\Box\uparrow} := \Box A$ and $(\Box A)^{\Box} := (\Box A)^{\Box\downarrow} := \Box A^{\Box}$,
- $(.)^{\Box\uparrow}$, $(.)^{\Box}$ and $(.)^{\Box\downarrow}$ commute with $\wedge$ and $\vee$,
- $(B \to C)^{\Box\uparrow} := \boxdot(B^{\Box\uparrow} \to C^{\Box\uparrow})$, $(B \to C)^{\Box} := \boxdot(B^{\Box} \to C^{\Box})$ and $(B \to C)^{\Box\downarrow} := B^{\Box\downarrow} \to C^{\Box\downarrow}$.

Remark 9.3.5 For every A we have $A^{\Box} = (A^{\Box\downarrow})^{\Box\uparrow}$. Also $\mathsf{iK4} \vdash A^{\Box} \leftrightarrow (A^{\Box\uparrow})^{\Box\downarrow}$.

Proof Both statements are proved easily by induction on the complexity of A, and we leave them to the reader. □

Definition 9.3.6 Define the following list of axiom schemas:

- $\underline{\mathsf{i}}$: A, for every theorem A of $\mathsf{IPC}_{\Box}$,
- $\underline{\mathsf{K}}$: $\Box(A \to B) \to (\Box A \to \Box B)$,
- $\underline{\mathsf{4}}$: $\Box A \to \Box\Box A$,
- The Löb's axiom, $\underline{\mathsf{Löb}}$ or $\underline{\mathsf{L}}$: $\Box(\Box A \to A) \to \Box A$,
- The Completeness Principle, $\underline{\mathsf{CP}}$ or $\underline{\mathsf{C}}$: $A \to \Box A$.
- Restriction of Completeness Principle to atomic variables, $\underline{\mathsf{CP_a}}$ or $\underline{\mathsf{C_a}}$: $p \to \Box p$, for atomic p.
- The reflection principle, $\underline{\mathsf{S}}$: $\Box A \to A$.
- The complete reflection principle, $\underline{\mathsf{S}^*}$: $\Box A \to A^{\Box}$.
- The Principle of Excluded Middle, $\underline{\mathsf{PEM}}$ or $\underline{\mathsf{P}}$: $A \vee \neg A$.
- Leivant's Principle, $\underline{\mathsf{Le}}$: $\Box(B \vee C) \to \Box(\Box B \vee C)$ (Leivant 1975).
- Extended Leivant's Principle, $\underline{\mathsf{Le}^+}$: $\Box A \to \Box A^l$ (Ardeshir and Mojtahedi 2018).
- Trace Principle, $\underline{\mathsf{TP}}$: $\Box(A \to B) \to (A \vee (A \to B))$ (Visser 1982).
- For an axiom schema $\underline{\mathsf{A}}$, the axiom schema $\overline{\mathsf{A}}$ indicates the box of every axiom instance of $\underline{\mathsf{A}}$. Also A indicates $\underline{\mathsf{A}} \wedge \overline{\mathsf{A}}$.

All modal systems which will be defined here, only have one inference rule: modus ponens $\frac{B \quad B\to A}{A}$. Also, celebrated modal logics, like $\mathsf{K4}$, which have the necessitation rule of inference, $\frac{A}{\Box A}$, by abuse of notation, are considered here with the same name and with the same set of theorems, however without the necessitation rule. The reason for this alternate definition of systems, is quite technical. Of course one may define them with the necessitation rule, but at the cost of loosing the uniformity of definitions. So in the rest of this paper, all modal systems, are considered with the modus ponens rule of inference. Note that in the presence of the axiom schema 4, one may finitely axiomatize logics such as $\mathsf{iK4}$ and extensions, without necessitation rule.

Consider a list $\mathsf{A}_1, \ldots, \mathsf{A}_n$ of axiom schemas and also L is a modal logic. The notation $\mathsf{A}_1\mathsf{A}_2\ldots\mathsf{A}_n$ ($L\mathsf{A}_1\mathsf{A}_2\ldots\mathsf{A}_n$) will be used in this paper for a modal system containing all axiom instances of all axiom schemas A_i (and all axioms of L), and is closed under modus ponens. This general notation makes things uniform and easy to remember for later usage. However, we make the following exceptions:

- $\mathsf{iGL} := \mathsf{iKL}$,
- $\mathsf{GL} := \mathsf{iKLP}$.

We also gathered the list of axioms and theories in Tables 9.1 and 9.3.

Lemma 9.3.7 *For every modal proposition A, we have* $\mathsf{iK4} \vdash A^{\Box} \leftrightarrow \boxdot A^{\Box}$.

Proof Use induction on the complexity of A. $\Box$

Lemma 9.3.8 *For every modal proposition A, we have* $\mathsf{iK4} + \boxdot\mathsf{CP} \vdash A \leftrightarrow A^{\Box}$ *and* $\mathsf{iK4} + \Box\mathsf{CP} \vdash A \leftrightarrow A^{\Box\downarrow}$.

Proof Note that the first assertion implies the second one. To prove the equivalence of A and $A^{\Box}$ in $\mathsf{iK4} + \boxdot\mathsf{CP}$, one must use induction on the complexity of A. All cases are simple and left to the reader. $\Box$

Lemma 9.3.9 *For every modal proposition A, we have* $\mathsf{iGL} \vdash A$ *implies* $\mathsf{iGL} \vdash A^{\Box\uparrow} \wedge A^{\Box\downarrow} \wedge A^{\Box}$. *The same holds for* $\mathsf{iGLC_a}$.

Proof Use induction on the complexity of proof $\mathsf{iGL} \vdash A$. $\Box$

Lemma 9.3.10 *Let A be some proposition and $E \in \mathsf{sub}(A^{\Box})$. Then* $\mathsf{iK4} + \mathsf{CP_a} \vdash E^{\Box\uparrow} \to \boxdot E$.

Proof Use induction on the complexity of E. All cases are trivial except for $E = F^{\Box} \to G^{\Box}$. In this case we have $E^{\Box\uparrow} = \boxdot((F^{\Box})^{\Box\uparrow} \to (G^{\Box})^{\Box\uparrow})$. One may observe that $(A^{\Box})^{\Box\uparrow} \leftrightarrow A^{\Box}$ is valid in $\mathsf{iK4}$ and hence we have $\mathsf{iK4} \vdash E^{\Box\uparrow} \leftrightarrow \boxdot E$. $\Box$

9.3.1 Preliminaries from Arithmetic

The first-order language of arithmetic contains three functions (successor, addition and multiplication), one predicate symbol and a constant: $(\mathsf{S}, +, \cdot, \leq, 0)$. First-order intuitionistic arithmetic (HA) is the theory over IQC with the axioms:

Q1 $\mathsf{S}x \neq 0$,
Q2 $\mathsf{S}x = \mathsf{S}(y) \to x = y$,
Q3 $x + 0 = x$,
Q4 $x + \mathsf{S}y = \mathsf{S}(x + y)$,
Q5 $x.0 = 0$,
Q6 $x.\mathsf{S}y = (x.y) + x$,
Q7 $x \leq y \leftrightarrow \exists z\, z + x = y$,
Ind: For each formula $A(x)$:

$$\mathsf{Ind}(A, x) := \mathcal{UC}[(A(0) \wedge \forall x(A(x) \to A(\mathsf{S}x))) \to \forall x A(x)]$$

In which $\mathcal{UC}(B)$ is the universal closure of B.

Peano Arithmetic PA, has the same axioms of HA over CQC.

Notation 9.3.11 From now on, when we are working in the first-order language of arithmetic, for a first-order sentence A, the notations $\Box A$ and $\Box^+ A$ are shorthand for $\mathsf{Prov}_{\mathsf{HA}}(\ulcorner A \urcorner)$ and $\mathsf{Prov}_{\mathsf{PA}}(\ulcorner A \urcorner)$, respectively. Let $i\Sigma_1$ be the theory HA, where the induction principle is restricted to Σ_1-formulae. We also define HA_x to be the theory with axioms of HA, in which the induction principle is restricted to formulae satisfying at least one of the following conditions:

- Σ_1-formulas,
- formulae with Gödel number less than x.

We define PA_x similarly. Also define $\Box_x A$ and $\Box^+_x A$ to be provability predicates in HA_x and PA_x, respectively.

Lemma 9.3.12 *For every formula A, we have* $\mathsf{PA} \vdash \forall x\, \Box^+(\Box^+_x A \to A)$ *and* $\mathsf{HA} \vdash \forall x\, \Box(\Box_x A \to A)$.

Proof The case of PA is well known (Hájek and Pudlák 1993). For the case HA, see (Smoryński 1973) or (Visser 2002, Theorem 8.1). □

Lemma 9.3.13 HA *proves all true Σ_1 sentences. Moreover this argument is formalizable and provable in* HA, *i.e. for every Σ_1-formula $A(x_1, \ldots, x_k)$ we have* $\mathsf{HA} \vdash A(x_1, \ldots, x_k) \to \Box A(\dot{x}_1, \ldots, \dot{x}_k)$.

Proof It is a well-known fact that any true (in the standard model $\mathbb{N}$) Σ_1-sentence is provable in HA (Visser 2002). Moreover this argument is constructive and formalizable in HA. □

Lemma 9.3.14 *For any $\Delta_0(\exp)$-formula $A(\bar{x})$, we have* $\mathsf{HA} \vdash \forall \bar{x}(A(\bar{x}) \vee \neg A(\bar{x}))$.

Proof This is well-known in the literature (Troelstra and van Dalen 1988). □

Lemma 9.3.15 *Let A, B be Σ_1-formulae such that* $\mathsf{PA} \vdash A \to B$. *Then* $\mathsf{HA} \vdash A \to B$.

Proof Observe that every implication of Σ_1-sentences in HA is equivalent to a Π_2 sentence and use the Π_2-conservativity of PA over HA (Troelstra and van Dalen 1988)(3.3.4). □

Definition 9.3.16 For a first-order theory T and first-order arithmetical formula A, the Beeson-Visser translation A^T is defined as follows:

- $A^T := A$ for atomic A,
- $(.)^T$ commutes with $\wedge$, $\vee$ and $\exists$,
- $(A \to B)^T := (A^T \to B^T) \wedge \mathsf{Prov}_{\mathsf{T}}(\ulcorner A^T \to B^T \urcorner)$

- $(\forall x\, A)^T := \forall x\, A^T \wedge \mathsf{Prov}_{\mathsf{T}}(\ulcorner \forall x\, A^T \urcorner)$.

HA^* and PA^* were first introduced in (Visser 1982). These theories are defined as

$$\mathsf{HA}^* := \{A \mid \mathsf{HA} \vdash A^{\mathsf{HA}}\} \quad \text{and} \quad \mathsf{PA}^* := \{A \mid \mathsf{PA} \vdash A^{\mathsf{PA}}\}.$$

Visser in (Visser 1982) showed that the (Σ_1-)provability logic of PA^* is iGLCT, i.e. $\mathsf{iGLCT} \vdash A$ iff for all arithmetical substitution σ, $\mathsf{PA}^* \vdash \sigma_{\mathsf{PA}^*}(A)$. That means that

$$\mathsf{PL}(\mathsf{PA}^*) = \mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*) = \mathsf{iGLCT}.$$

Lemma 9.3.17 *For any arithmetical Σ_1-formula A*

1. $\mathsf{HA} \vdash A \leftrightarrow A^{\mathsf{HA}}$,
2. $\mathsf{HA} \vdash A \leftrightarrow A^{\mathsf{PA}}$.

Proof See (Visser 1982, 4.6.iii). □

Lemma 9.3.18 *For every arithmetical sentence A we have*

- $\mathsf{HA} \vdash \mathsf{Prov}_{\mathsf{HA}}(\ulcorner A \urcorner) \to \mathsf{Prov}_{\mathsf{HA}^*}(\ulcorner A \urcorner)$,
- $\mathsf{HA}^* \vdash A \to \mathsf{Prov}_{\mathsf{HA}^*}(\ulcorner A \urcorner)$,
- $\mathsf{PA}^* \vdash A \to \mathsf{Prov}_{\mathsf{PA}^*}(\ulcorner A \urcorner)$.

Proof For the first item, consider some A such that $\mathsf{HA} \vdash A$. By induction on the proof of A in HA, one may prove that $\mathsf{HA} \vdash A^{\mathsf{HA}}$. Moreover this argument is formalizable and provable in HA. We refer the reader to (Visser 1982) for details.
For the proof of second and third items, one may use induction on the complexity of A, and we leave the routine induction to the reader. □

Lemma 9.3.19 *For any Σ_1-substitution σ and each propositional modal sentence A, we have* $\mathsf{HA} \vdash (\sigma_{\mathsf{HA}^*}(A))^{\mathsf{HA}} \leftrightarrow \sigma_{\mathsf{PA}^*}(A^{\Box\uparrow})$ *and* $\mathsf{PA} \vdash (\sigma_{\mathsf{PA}^*}(A))^{\mathsf{PA}} \leftrightarrow \sigma_{\mathsf{PA}^*}(A^{\Box\uparrow})$.

Proof Use induction on the complexity of A. All cases are easily derived by Lemma 9.3.17. □

Lemma 9.3.20 *For any Σ_1-substitution σ and each propositional modal sentence A, we have* $\mathsf{HA} \vdash \sigma_{\mathsf{HA}}(A^{\Box}) \leftrightarrow (\sigma_{\mathsf{HA}^*}(A))^{\mathsf{HA}}$ *and* $\mathsf{HA} \vdash \sigma_{\mathsf{PA}}(A^{\Box}) \leftrightarrow (\sigma_{\mathsf{PA}^*}(A))^{\mathsf{PA}}$.

Proof Use induction on the complexity of A. All cases are easily derived by Lemma 9.3.17. □

Lemma 9.3.21 *For any Σ_1-substitution σ and each propositional modal sentence A, we have* $\mathsf{HA} \vdash \sigma_{\mathsf{HA}}(A^{\Box\downarrow}) \leftrightarrow \sigma_{\mathsf{HA}^*}(A)$ *and* $\mathsf{HA} \vdash \sigma_{\mathsf{PA}}(A^{\Box\downarrow}) \leftrightarrow \sigma_{\mathsf{PA}^*}(A)$.

Proof We use induction on the complexity of A. All cases are easy, except for boxed case, which holds by Lemma 9.3.20. □

Lemma 9.3.22 *For any Σ_1-substitution σ and each propositional modal sentence A, we have* $\mathsf{HA} \vdash \sigma_{\mathsf{PA}}(A^{\Box\downarrow}) \leftrightarrow \sigma_{\mathsf{PA}^*}(A)$.

Proof We use induction on the complexity of A. All cases are easy, except for boxed case, which holds by Lemma 9.3.20. □

9.3.1.1 Kripke Models of HA

A first-order Kripke model for the language of arithmetic is a triple $\mathcal{K} = (K, \preccurlyeq, \mathfrak{M})$ such that:

- The frame of $\mathcal{K}$, i.e. $(K, \preccurlyeq)$, is a non-empty partially ordered set,
- $\mathfrak{M}$ is a function from K to the first-order classical structures for the language of the arithmetic, i.e. $\mathfrak{M}(\alpha)$ is a first-order classical structure, for each $\alpha \in K$,
- For any $\alpha \preccurlyeq \beta \in K$, $\mathfrak{M}(\alpha)$ is a weak substructure of $\mathfrak{M}(\beta)$.

For any $\alpha \in K$ and first-order formula $A \in \mathcal{L}_\alpha$ (the language of arithmetic augmented with constant symbols $\bar{a}$ for each $a \in |\mathfrak{M}(\alpha)|$), we define $\mathcal{K}, \alpha \Vdash A$ (or simply $\alpha \Vdash A$, if no confusion is likely) inductively as follows:

- For atomic A, $\mathcal{K}, \alpha \Vdash A$ iff $\mathfrak{M}(\alpha) \models A$. Note that in the structure $\mathfrak{M}(\alpha)$, $\bar{a}$ is interpreted as a,
- $\mathcal{K}, \alpha \Vdash A \vee B$ iff $\mathcal{K}, \alpha \Vdash A$ or $\mathcal{K}, \alpha \Vdash B$,
- $\mathcal{K}, \alpha \Vdash A \wedge B$ iff $\mathcal{K}, \alpha \Vdash A$ and $\mathcal{K}, \alpha \Vdash B$,
- $\mathcal{K}, \alpha \Vdash A \to B$ iff for all $\beta \succcurlyeq \alpha$, $\mathcal{K}, \beta \Vdash A$ implies $\mathcal{K}, \beta \Vdash B$,
- $\mathcal{K}, \alpha \Vdash \exists x A$ iff $\mathcal{K}, \alpha \Vdash A[x : \bar{a}]$, for some $a \in |\mathfrak{M}(\alpha)|$,
- $\mathcal{K}, \alpha \Vdash \forall x A$ iff for all $\beta \succcurlyeq \alpha$ and $b \in |\mathfrak{M}(\beta)|$, we have $\mathcal{K}, \beta \Vdash A[x : \bar{b}]$.

It is well-known in the literature (Troelstra and van Dalen 1988) that HA is complete for first-order Kripke models.

Lemma 9.3.23 *Let $\mathcal{K} = (K, \preccurlyeq, \mathfrak{M})$ be a Kripke model of* HA *and A be an arbitrary Σ_1-formula. Then for each $\alpha \in K$, we have $\alpha \Vdash A$ iff $\mathfrak{M}(\alpha) \models A$.*

Proof Use induction on the complexity of A to show that for each $\alpha \in K$, we have $\alpha \Vdash A$ iff $\mathfrak{M}(\alpha) \models A$. In the inductive step for $\to$ and $\forall$, use Lemma 9.3.14. $\square$

9.3.1.2 Interpretability

Let T and S be two first-order theories. Informally speaking, we say that T interprets S ($T \rhd S$) if there exists a translation from the language of S to the language of T such that T proves the translation of all of the theorems of S. For a formal definition see (Visser 1998). It is well-known that for recursive theories T and S containing PA, the assertion $T \rhd S$ is formalizable in first-order language of arithmetic. For two arithmetical sentences A and B, we use the notation $A \rhd B$ to mean that $\mathsf{PA} + A$ interprets $\mathsf{PA} + B$. The following theorem due to Orey, first appeared in (Feferman 1960).

Theorem 9.3.24 *For recursive theories T and S containing* PA*, we have:*

$$\mathsf{PA} \vdash (T \rhd S) \leftrightarrow \forall x\, \Box_T \mathsf{Con}(S^x),$$

in which S^x is the restriction of the theory S to axioms with Gödel number $\leq x$ and $\mathsf{Con}(U) := \neg \Box_U \bot$.

Proof See (Feferman 1960), p. 80 or (Berarducci 1990). □

Convention. From Theorem 9.3.24, one can easily observe that $\mathsf{PA} \vdash (A \rhd B) \leftrightarrow \forall x\, \Box^{+}(A \to \neg \Box^{+}_{x} \neg B)$. So from now on, $A \rhd B$ means its Π_2-equivalent $\forall x\, \Box^{+}(A \to \neg \Box^{+}_{x} \neg B)$, even when we are working in weaker theories like HA, for which the above theorem (Theorem 9.3.24) doesn't hold. We remind the reader that $\Box^{+}$ stands for provability in PA.

9.3.1.3 Smoryński's Method for Constructing Kripke Models of HA

With the general method of constructing Kripke models for HA, invented by Smoryński (Smoryński 1973), interpretability of theories containing PA plays an important role in constructing Kripke models of HA.

Definition 9.3.25 A triple $\mathcal{I} := (K, \preccurlyeq, T)$ is called an I-frame iff it has the following properties:

- $(K, \preccurlyeq)$ is a finite tree,
- T is a function from K to arithmetical r.e. consistent theories containing PA,
- if $\beta \preccurlyeq \gamma$, then T_β interprets T_γ ($T_\beta \rhd T_\gamma$).

Theorem 9.3.26 *For every I-frame $\mathcal{I} := (K, \preccurlyeq, T)$ there exists a first-order Kripke model $\mathcal{K} = (K, c, \mathfrak{M})$ such that $\mathcal{K} \Vdash \mathsf{HA}$ and moreover $\mathfrak{M}(\alpha) \models T_\alpha$, for any $\alpha \in K$. Note that both of the I-frame and Kripke model are sharing the same frame $(K, \preccurlyeq)$.*

Proof See (Smoryński 1973, pp. 372–377). For more detailed proof of a generalization of this theorem, see (Ardeshir and Mojtahedi 2014, Theorem 4.8) □

9.3.2 The NNIL Formulae and Related Topics

The class of *No Nested Implications to the Left*, NNIL formulae in a propositional language was introduced in (Visser et al. 1995), and more explored in (Visser 2002). The crucial result of (Visser 2002) is providing an algorithm that as input, gives a non-modal proposition A and returns its best NNIL approximation A^* from below, i.e., $\mathsf{IPC} \vdash A^* \to A$ and for all NNIL formulae B such that $\mathsf{IPC} \vdash B \to A$, we have $\mathsf{IPC} \vdash B \to A^*$. Also for all Σ_1-substitutions σ, we have $\mathsf{HA} \vdash \sigma_{\mathsf{HA}}(\Box A \leftrightarrow \Box A^*)$ (Visser 2002).

The precise definition of the class NNIL of modal propositions is $\mathsf{NNIL} := \{A \mid \rho A \leq 1\}$, in which the complexity measure ρ, is defined inductively as follows:

- $\rho(\Box A) = \rho(p) = \rho(\bot) = \rho(\top) = 0$, for an arbitrary atomic variables p and modal proposition A,
- $\rho(A \wedge B) = \rho(A \vee B) = \max(\rho A, \rho B)$,
- $\rho(A \to B) = \max(\rho A + 1, \rho B)$,

Definition 9.3.27 For any two modal propositions A and B, we define $[A]B$ by induction on the complexity of B:

- $[A]B = B$, for atomic or boxed B,
- $[A](B_1 \circ B_2) = [A](B_1) \circ [A](B_2)$ for $\circ \in \{\vee, \wedge\}$,
- $[A](B_1 \to B_2) = A' \to (B_1 \to B_2)$, in which $A' = A[B_1 \to B_2 \mid B_2]$, i.e., replace each *outer occurrence* of $B_1 \to B_2$ (by outer occurrence we mean that it is not in the scope of any $\Box$) in A by B_2,

For a set X of modal propositions, we also define $[A]X := \bigvee_{B \in X} [A]B$.

9.3.2.1 The NNIL-Algorithm

For each modal proposition A, the proposition A^* is defined inductively as follows (Visser 2002):

1. A is atomic or boxed, take $A^* := A$.
2. $A = B \wedge C$, take $A^* := B^* \wedge C^*$.
3. $A = B \vee C$, take $A^* := B^* \vee C^*$.
4. $A = B \to C$, we have several sub-cases. In the following, an occurrence of E in D is called an *outer occurrence*, if E is neither in the scope of an implication nor in the scope of a boxed formula.

   (a) C contains an outer occurrence of a conjunction. In this case, there is some formula $J(q)$ such that

   - q is a propositional variable not occurring in A.
   - q is outer in J and occurs exactly once.
   - $C = J[q|(D \wedge E)]$.

   Now set $C_1 := J[q|D]$, $C_2 := J[q|E]$ and $A_1 := B \to C_1$, $A_2 := B \to C_2$ and finally, define $A^* := A_1^* \wedge A_2^*$.

   (b) B contains an outer occurrence of a disjunction. In this case, there is some formula $J(q)$ such that

   - q is a propositional variable not occurring in A.
   - q is outer in J and occurs exactly once.
   - $B = J[q|(D \vee E)]$.

   Now set $B_1 := J[q|D]$, $B_2 := J[q|E]$ and $A_1 := B_1 \to C$, $A_2 := B_2 \to C$ and finally, define $A^* := A_1^* \wedge A_2^*$.

   (c) $B = \bigwedge X$ and $C = \bigvee Y$ and X, Y are sets of implications, atomics or boxed formulas. We have several sub-cases:

      (i) X contains an atomic variable or a boxed formula E. We set $D := \bigwedge(X \setminus \{E\})$ and take $A^* := E^* \to (D \to C)^*$.

      (ii) X contains $\top$. Define $D := \bigwedge(X \setminus \{\top\})$ and take $A^* := (D \to C)^*$.

      (iii) X contains $\bot$. Take $A^* := \top$.

(iv) X contains only implications. For any $D = E \to F \in X$, define

$$B \downarrow D := \bigwedge((X \setminus \{D\}) \cup \{F\}).$$

Let $Z := \{E \mid E \to F \in X\} \cup \{C\}$ and define:

$$A^* := \bigwedge\{((B \downarrow D) \to C)^* | D \in X\} \wedge \bigvee\{([B]E)^* \mid E \in Z\}$$

Lemma 9.3.28 *If* $\mathsf{IPC}_\Box \vdash A \to B$ *then* $\mathsf{IPC}_\Box \vdash A^* \to B^*$.

Proof See (Ardeshir and Mojtahedi 2018, Theorem. 4.5). □

9.3.2.2 The TNNIL-Algorithm

Definition 9.3.29 TNNIL (Thoroughly NNIL) is the smallest class of propositions such that

- TNNIL contains all atomic propositions,
- if $A, B \in$ TNNIL, then $A \vee B$, $A \wedge B$, $\Box A \in$ TNNIL,
- if all $\to$ occurring in A are contained in the scope of a $\Box$ (or equivalently $A \in$ NOI) and $A, B \in$ TNNIL, then $A \to B \in$ TNNIL.

Alternatively, one may define TNNIL and NOI, as follows: (A defines TNNIL and B defines NOI)

- $A ::= At \mid \top \mid \bot \mid (A \wedge A) \mid (A \vee A) \mid (B \to A)$.
- $B ::= At \mid \top \mid \bot \mid (B \wedge B) \mid (B \vee B)$.

In which At varies in the set of all atomic variables.

Let $\mathsf{TNNIL}^\Box$ indicate the set of all the propositions like $A(\Box B_1, \ldots, \Box B_n)$, such that $A(p_1, \ldots, p_n)$ is an arbitrary non-modal proposition and $B_1, \ldots, B_n \in$ TNNIL.

Here we define A^+ to be the TNNIL-formula approximating A. Informally speaking, to find A^+, we first compute A^* and then replace all outer boxed formula $\Box B$ in A by $\Box B^+$. More precisely, we define A^+ by induction on the maximum number of nesting $\Box$'s. Suppose that $A'(p_1, \ldots, p_n)$ and $\Box B_1, \ldots, \Box B_n$ are such that $A = A'[p_1|\Box B_1, \ldots, p_n|\Box B_n]$, where A' is a non-modal proposition and $p_1, \ldots, p_n$ are fresh atomic variables (not occurred in A). It is clear that each B_i has less number of nesting $\Box$'s and then we can define $A^+ := (A')^*[p_1|\Box B_1^+, \ldots, p_n|\Box B_n^+]$.
For a modal proposition A, let $B(p_1, \ldots, p_n)$ is the unique (modulo permutation of p_i) non-modal proposition such that $A := B(\Box C_1, \ldots, \Box C_n)$. Then define $A^- := B(\Box C_1^+, \ldots, \Box C_n^+)$. Next we may define the theory iH_σ as follows:

Definition 9.3.30 We define Visser's axiom schema

$$\underline{\mathsf{V}} := A \leftrightarrow A^{-}$$

Then define the following modal systems:

- $\mathsf{iH}_\sigma := \mathsf{iGLLe}^{+}\mathsf{V}$,
- $\mathsf{iH}_\sigma^{**} := \{A : \mathsf{iH}_\sigma \vdash A^{\Box}\}$,
- $\mathsf{iH}_\sigma^{*} := \{A : \mathsf{iH}_\sigma \vdash A^{\Box\downarrow}\}$.

Remark 9.3.31 The definitions of iH_σ in (Ardeshir and Mojtahedi 2018, Sect. 4.3) and iH_σ^{**} in (Ardeshir and Mojtahedi 2019, Definition 3.16) (which were called iH_σ^{*} there) are presented in some other equivalent way. For the sake of simplicity of definitions, we preferred Definition 9.3.30 here. To see an axiomatization for iH_σ^{**}, we refer the reader to (Ardeshir and Mojtahedi 2019).

Lemma 9.3.32 $\mathsf{IPC}_\Box \vdash (A^{+} \wedge (A \to B)^{+}) \to B^{+}$.

Proof By definition of $(.)^{+}$, for every C we have $C^{+} = (C^{-})^{*}$. Since $\mathsf{IPC}_\Box \vdash (A^{-} \wedge (A \to B)^{-}) \to B^{-}$, by Lemma 9.3.28 we have $\mathsf{IPC}_\Box \vdash (A^{-} \wedge (A \to B)^{-})^{*} \to (B^{-})^{*}$. Then we have $\mathsf{IPC}_\Box \vdash (A^{+} \wedge (A \to B)^{+}) \to B^{+}$ by the argument at the beginning of proof. $\Box$

Lemma 9.3.33 *Let A be a modal proposition. Then* $\mathsf{iK4} \vdash A^{\Box\uparrow} \leftrightarrow \boxdot A^{\Box\uparrow}$.

Proof Use induction on the complexity of A. $\Box$

Lemma 9.3.34 *For arbitrary $A \in \mathsf{TNNIL}^{\Box}$ we have* $\mathsf{iK4} + \mathsf{CP_a} \vdash \boxdot A^{l} \leftrightarrow \boxdot A^{\Box\uparrow}$.

Proof We use induction on the complexity of A:

- A is atomic: then $A^{l} = A$ and $A^{\Box\uparrow} = \boxdot A$. Hence by $\mathsf{CP_a}$ we have the desired equivalency.
- A is boxed: $(\Box A)^{l} = \Box A = (\Box A)^{\Box\uparrow}$.
- $A = B \wedge C$: then $(B \wedge C)^{l} = B^{l} \wedge C^{l}$ and $(B \wedge C)^{\Box\uparrow} = B^{\Box\uparrow} \wedge C^{\Box\uparrow}$. Hence by induction hypothesis we have the desired result.
- $A = B \vee C$: then $(B \vee C)^{l} = \boxdot B^{l} \vee \boxdot C^{l}$ and $(B \vee C)^{\Box\uparrow} = B^{\Box\uparrow} \vee C^{\Box\uparrow}$. Using Lemma 9.3.33 we have $\mathsf{iK4} \vdash (B \vee C)^{\Box\uparrow} \leftrightarrow (\boxdot B^{\Box\uparrow} \vee \boxdot C^{\Box\uparrow})$ and hence induction hypothesis implies the desired result.
- $A = B \to C$ and $B \in \mathsf{NOI}$: then $(B \to C)^{l} = B \to C^{l}$ and $(B \to C)^{\Box\uparrow} = \boxdot(B^{\Box\uparrow} \to C^{\Box\uparrow})$. Observe that

  1. $\mathsf{iK4} \vdash \boxdot\boxdot E \leftrightarrow \boxdot E$ for any A,
  2. $\mathsf{iK4} + \mathsf{CP_a} \vdash B^{\Box\uparrow} \leftrightarrow B$,
  3. $\mathsf{iK4} \vdash \boxdot(B \to E) \leftrightarrow \boxdot(B \to \boxdot E)$ for any $B \in \mathsf{NOI}$ and arbitrary E.

We have the following equivalences in $\mathsf{iK4} + \mathsf{CP_a}$:

$$
\begin{array}{rcll}
\boxdot A^{\Box\uparrow} & \leftrightarrow & \boxdot(B^{\Box\uparrow} \to C^{\Box\uparrow}) & \text{by first observation} \\
\boxdot(B^{\Box\uparrow} \to C^{\Box\uparrow}) & \leftrightarrow & \boxdot(B \to C^{\Box\uparrow}) & \text{by second observation} \\
\boxdot(B \to C^{\Box\uparrow}) & \leftrightarrow & \boxdot(B \to \boxdot C^{\Box\uparrow}) & \text{by third observation} \\
\boxdot(B \to \boxdot C^{\Box\uparrow}) & \leftrightarrow & \boxdot(B \to \boxdot C^{l}) & \text{by induction hypothesis} \\
\boxdot(B \to \boxdot C^{l}) & \leftrightarrow & \boxdot(B \to C^{l}) & \text{by third observation}
\end{array}
$$

□

Lemma 9.3.35 *For* $A \in \mathsf{TNNIL}^{\Box}$ *we have* $\mathsf{iK4} + \mathsf{Le}^{+} + \mathsf{CP_a} \vdash A \leftrightarrow A^{\Box\downarrow}$.

Proof Use induction on the complexity of A. The only nontrivial case is when $A = \Box B$. We have the following equivalences in $\mathsf{iK4} + \mathsf{Le}^{+} + \mathsf{CP_a}$:

$$
\begin{array}{rcll}
(\Box B)^{\Box\downarrow} & \leftrightarrow & \Box(B^{\Box}) & \text{by definition} \\
\Box(B^{\Box}) & \leftrightarrow & \Box\left((B^{\Box\downarrow})^{\Box\uparrow}\right) & \text{by Remark 3.5} \\
\Box\left((B^{\Box\downarrow})^{\Box\uparrow}\right) & \leftrightarrow & \Box(B^{\Box\uparrow}) & \text{by induction hypothesis} \\
\Box(B^{\Box\uparrow}) & \leftrightarrow & \Box B^{l} & \text{by Lemma 3.34} \\
\Box B^{l} & \leftrightarrow & \Box B & \text{by the axiom schema} \mathsf{Le}^{+}
\end{array}
$$

□

Theorem 9.3.36 *For any* TNNIL*-proposition* A*,* $\mathsf{iGLC} \vdash A$ *implies* $\mathsf{iGLLe}^{+} \vdash A$.

Proof See (Ardeshir and Mojtahedi 2018) Theorem 4.24. □

Theorem 9.3.37 *For any* $\mathsf{TNNIL}^{\Box}$*-proposition* A*,* $\mathsf{iGL\overline{C}\underline{P}C_a} \vdash A$ *implies* $\mathsf{iGLLe}^{+}\underline{\mathsf{P}} \vdash A$. *Also* $\mathsf{iGL\overline{C}\underline{SP}C_a} \vdash A$ *implies* $\mathsf{iGLLe}^{+}\underline{\mathsf{SP}} \vdash A$.

Proof Both statements proved by induction on proofs. The only non-trivial case is when A is an axiom instance of the form $\Box A$ such that $\mathsf{iGLC} \vdash A$. In this case, Theorem 9.3.36 implies $\mathsf{iGLLe}^{+} \vdash A$. Hence by necessitation which is available in iGLLe^{+} we have $\mathsf{iGLLe}^{+} \vdash \Box A$. Hence $\mathsf{iGLLe}^{+}\underline{\mathsf{P}} \vdash A$ and $\mathsf{iGLLe}^{+}\underline{\mathsf{SP}} \vdash A$. □

9.3.3 Intuitionistic Modal Kripke Semantics

Let us first review results and notations from (Iemhoff 2001) which will be used here. Assume two binary relations R and S on a set. Define $\alpha(R;S)\gamma$ iff there exists some β such that $\alpha R \beta$ and $\beta S \gamma$. We use the binary relation symbol $\preccurlyeq$ always as a reflexive relation and $\prec$ for the irreflexive part of $\preccurlyeq$, i.e. $u \prec v$ holds iff $u \preccurlyeq v$ and

$u \neq v$. Moreover we use the mirror image of a relational symbol for its inverse, e.g. $\succ$ for $\prec^{-1}$ and so on.

A Kripke model $\mathcal{K}$, for intuitionistic modal logic, is a quadruple $(K, \preccurlyeq, \sqsubset, V)$, such that K is a set (we call its elements as nodes), $(K, \prec)$ is a partial ordering, $\sqsubset$ is a binary relation on K such that $(\preccurlyeq ; \ \sqsubset) \subseteq \sqsubset$, and V is a binary relation between nodes and atomic variables such that $\alpha V p$ and $\alpha \preccurlyeq \beta$ implies $\beta V p$. Then we can extend V to the modal language with $\sqsubset$ corresponding to $\Box$ and $\preccurlyeq$ for intuitionistic $\rightarrow$. More precisely, we define $\Vdash$ inductively as an extension of V as follows:

- $\mathcal{K}, \alpha \Vdash p$ iff $\alpha V p$, for atomic variable p,
- $\mathcal{K}, \alpha \Vdash A \vee B$ iff $\mathcal{K}, \alpha \Vdash A$ or $\mathcal{K}, \alpha \Vdash B$,
- $\mathcal{K}, \alpha \Vdash A \wedge B$ iff $\mathcal{K}, \alpha \Vdash A$ and $\mathcal{K}, \alpha \Vdash B$,
- $\mathcal{K}, \alpha \nVdash \bot$ and $\mathcal{K}, \alpha \Vdash \top$,
- $\mathcal{K}, \alpha \Vdash A \rightarrow B$ iff for all $\beta \succcurlyeq \alpha$, $\mathcal{K}, \beta \Vdash A$ implies $\mathcal{K}, \beta \Vdash B$,
- $\mathcal{K}, \alpha \Vdash \Box A$ iff for all β with $\alpha \sqsubset \beta$, we have $\mathcal{K}, \beta \Vdash A$.

Also we define the local truth in this way:

- $\mathcal{K}, \alpha \models p$ iff $\alpha V p$, for atomic variable p,
- $\mathcal{K}, \alpha \models A \vee B$ iff $\mathcal{K}, \alpha \models A$ or $\mathcal{K}, \alpha \models B$,
- $\mathcal{K}, \alpha \models A \wedge B$ iff $\mathcal{K}, \alpha \models A$ and $\mathcal{K}, \alpha \models B$,
- $\mathcal{K}, \alpha \not\models \bot$ and $\mathcal{K}, \alpha \models \top$,
- $\mathcal{K}, \alpha \models A \rightarrow B$ iff either $\mathcal{K}, \alpha \not\models A$ or $\mathcal{K}, \alpha \models B$,
- $\mathcal{K}, \alpha \models \Box A$ iff for all β with $\alpha \sqsubset \beta$, we have $\mathcal{K}, \beta \Vdash A$.

The classical truth $\mathcal{K}, \alpha \models_c A$ is defined similar to $\mathcal{K}, \alpha \models A$, except for the boxed case:

- $\mathcal{K}, \alpha \models_c \Box A$ iff for all $\beta \sqsupset \alpha$ we have $\mathcal{K}, \beta \models_c A$.

For a boolean interpretation I, we also define the local I-truth $\mathcal{K}, \alpha, I \models A$ and the classical I-truth $\mathcal{K}, \alpha, I \models_c A$, similar to $\mathcal{K}, \alpha \models A$, and $\mathcal{K}, \alpha \models A$, except for atomic variables p which we define:

- $\mathcal{K}, \alpha, I \models p$ iff $I \models p$ iff $\mathcal{K}, \alpha, I \models_c p$.

Remark 9.3.38 Note that when we consider the classical truth for a Kripke model $\mathcal{K} = (K, \sqsubset, \preccurlyeq, V)$, we are ignoring the $\preccurlyeq$ from $\mathcal{K}$ and it would collapse to the well known Kripke semantics for the classical modal logic $\mathcal{K}_c := (K, \sqsubseteq, V)$. The same argument holds for the classical I-truth, except for the valuation V, which should be modified according to I. More precisely, $\mathcal{K}, \alpha, I \models_c A$ iff $\mathcal{K}_c^I, \alpha \models A$, in which $\mathcal{K}_c^I := (K, \sqsubset, V_\alpha^I)$ is a classical Kripke semantics for classical modal logic with

$$\beta\, V_\alpha^I\, p \quad \Leftrightarrow \quad (\beta \neq \alpha \wedge \beta\, V\, p) \vee (\beta = \alpha \wedge I \models p)$$

In the rest of paper, we may simply write $\alpha \Vdash A$ for $\mathcal{K}, \alpha \Vdash A$, if no confusion is likely. By an induction on the complexity of A, one can observe that $\alpha \Vdash A$ implies $\beta \Vdash A$ for all A and $\alpha \preccurlyeq \beta$. We define the following notions.

- If $\alpha \preccurlyeq \beta$, β is *above* α and α is *beneath* β. If $\alpha \sqsubset \beta$, β is a *successor* of α. We say that β is an immediate successor of α, if $\alpha \sqsubset \beta$ and there is no γ such that $\alpha \sqsubset \gamma \sqsubset \beta$.
- We say that α is $\sqsubset$-branching, if the set of immediate successors of α is not singleton.
- A Kripke model is finite if its set of nodes is finite.
- $(\alpha\sqsubset)$ indicates the set of successors of α, and $(\alpha\prec)$ and $(\alpha\preccurlyeq)$ are defined similarly.
- α is classical, if $(\alpha\prec) = \emptyset$.
- α is quasi-classical, if $(\alpha\prec) = (\alpha\sqsubset)$.
- α is complete if $(\alpha\sqsubset) \subseteq (\alpha\prec)$. Also we say that α is atom-complete if $\alpha \Vdash p$ and $\alpha \sqsubset \beta$ implies $\beta \Vdash p$, for every atomic variable p.
- Let φ indicate some property for nodes in $\mathcal{K}$ and $X \subseteq K$. We say that $\mathcal{K}$ is X-φ, if every $\alpha \in X$ has the property φ. If $X = \{\alpha\}$, we may use α-φ instead. We say that $\mathcal{K}$ has the property φ, or simply "is φ", if it is K-φ. For example if we set $\mathsf{Suc} := \bigcup_{\alpha \in K}(\alpha\sqsubset)$, Suc-classical means that every $\sqsubset$-accessible node is classical.
- $\mathcal{K}$ is called *neat* iff $\alpha \sqsubset \gamma$ and $\alpha \preccurlyeq \beta \preccurlyeq \gamma$ implies $\alpha \sqsubset \beta$ or $\beta \sqsubset \gamma$.
- $\mathcal{K}$ is called *brilliant* iff $(\sqsubset\, ;\, \preccurlyeq) \subseteq \sqsubset$ (Iemhoff 2001). Note that $\alpha \sqsubset ; \preccurlyeq \beta$ iff there is some δ such that $\alpha \sqsubset \delta \preccurlyeq \beta$.
- We say that $\mathcal{K}$ has tree frame, if $(K, \prec \cup \sqsubset)$ is tree. A tree is a partial order $(X, <)$ such that for every $x \in X$, the set $\{y \in X : y \leq x\}$ is finite linearly ordered.
- $\mathcal{K}$ is called *semi-perfect* iff it is (1) with finite tree frame, (2) brilliant, (3) neat and (4) $\sqsubset$ is irreflexive and transitive. We say that $\mathcal{K}$ is perfect if it is semi-perfect and complete. Note that every quasi-classical Kripke model with finite tree frame is perfect.
- We say that a Kripke model $\mathcal{K}$ is A-sound at α (α is A-sound), if for every boxed subformula $\Box B$ of A we have $\mathcal{K}, \alpha \models \Box B \to B$.
- Suppose X is a set of propositions that is closed under sub-formulae (we call such X *adequate*). An X-saturated set of propositions Γ with respect to some logic L, is a consistent subset of X such that
  - For each $A \in X$, $\Gamma \vdash_L A$ implies $A \in \Gamma$.
  - For each $A \vee B \in X$, $\Gamma \vdash_L A \vee B$ implies $A \in \Gamma$ or $B \in \Gamma$.

Lemma 9.3.39 *Let* $\nvdash_L A$ *and let* X *be an adequate set. Then there is an* X*-saturated set* Γ *such that* $\Gamma \nvdash A$.

Proof See (Iemhoff 2001). □

Theorem 9.3.40 iGLC *is sound and complete for perfect Kripke models. Also* iGLCT *is sound and complete for perfect quasi-classical Kripke models.*

Proof See (Ardeshir and Mojtahedi 2018, Theorem 4.26) for iGLC and (Visser 1982, Lemma 6.14) for iGLCT. □

Since iGLC and iGLCT have finite model property, as it is expected, we can easily deduce the decidability of iGLC and iGLCT:

Corollary 9.3.41 iGLC *and* iGLCT *are decidable.*

Proof For iGLC see (Ardeshir and Mojtahedi 2018, Corollary 4.27). iGLCT is similar and left to the reader. □

Lemma 9.3.42 *Let A be a modal proposition and $\mathcal{K} = (K, \preccurlyeq, \sqsubset, V)$ be a semi-perfect Kripke model. Then for every quasi-classical node $\alpha \in K$ we have*

$$\mathcal{K}, \alpha \Vdash A^{\Box} \iff \mathcal{K}, \alpha \models A^{\Box}$$

Proof We use induction on the complexity of A. The only non-trivial case is when $A = B \to C$. Let $\mathcal{K}, \alpha \nVdash \boxdot(B^{\Box} \to C^{\Box})$. If $\mathcal{K}, \alpha \nVdash \Box(B^{\Box} \to C^{\Box})$ then evidently $\mathcal{K}, \alpha \not\models \Box(B^{\Box} \to C^{\Box})$ and we are done. If $\mathcal{K}, \alpha \nVdash B^{\Box} \to C^{\Box}$, then there exists some $\beta \succcurlyeq \alpha$ such that $\mathcal{K}, \beta \Vdash B^{\Box}$ and $\mathcal{K}, \beta \nVdash C^{\Box}$. Since α is quasi classical, hence $\beta \sqsupset \alpha$ or $\beta = \alpha$. If $\beta \sqsupset \alpha$, we have $\mathcal{K}, \alpha \not\models \Box(B^{\Box} \to C^{\Box})$ and we are done. Otherwise, $\mathcal{K}, \alpha \Vdash B^{\Box}$ and $\mathcal{K}, \alpha \nVdash C^{\Box}$ and hence by induction hypothesis we have $\mathcal{K}, \alpha \models B^{\Box}$ and $\mathcal{K}, \alpha \not\models C^{\Box}$ and we are done. For the other way around, let $\mathcal{K}, \alpha \not\models \boxdot(B^{\Box} \to C^{\Box})$. If $\mathcal{K}, \alpha \not\models \Box(B^{\Box} \to C^{\Box})$, evidently we have $\mathcal{K}, \alpha \nVdash \Box(B^{\Box} \to C^{\Box})$ and we are done. Otherwise, let $\mathcal{K}, \alpha \not\models B^{\Box} \to C^{\Box}$. Then $\mathcal{K}, \alpha \models B^{\Box}$ and $\mathcal{K}, \alpha \not\models C^{\Box}$. Induction hypothesis implies $\mathcal{K}, \alpha \Vdash B^{\Box}$ and $\mathcal{K}, \alpha \nVdash C^{\Box}$ and hence $\mathcal{K}, \alpha \nVdash \boxdot(B^{\Box} \to C^{\Box})$. □

Corollary 9.3.43 *Let A be a modal proposition and $\mathcal{K}$ is a semi-perfect quasi-classical Kripke model. Then for every node α we have*

$$\mathcal{K}, \alpha \Vdash A^{\Box} \iff \mathcal{K}, \alpha \models A^{\Box} \iff \mathcal{K}, \alpha \models_c A^{\Box}$$

$$\mathcal{K}, \alpha \models A^{\Box\downarrow} \iff \mathcal{K}, \alpha \models_c A^{\Box\downarrow} \quad \textit{and} \quad \mathcal{K}, \alpha, I \models A^{\Box\downarrow} \iff \mathcal{K}, \alpha, I \models_c A^{\Box\downarrow}$$

Proof By Lemma 9.3.42, for every node α we have $\mathcal{K}, \alpha \Vdash A^{\Box}$ iff $\mathcal{K}, \alpha \models A^{\Box}$. One can easily observe by induction on the height of the node $\alpha \in K$ that $\mathcal{K}, \alpha \Vdash A^{\Box}$ iff $\mathcal{K}, \alpha \models_c A^{\Box}$. □

Corollary 9.3.44 *Let A be a modal proposition and $\mathcal{K}$ is a semi-perfect quasi-classical Kripke model. Then for every node α we have*

$$\mathcal{K}, \alpha \Vdash A \iff \mathcal{K}, \alpha \models A^{\Box\uparrow}$$

Proof Observe that $\mathcal{K}, \alpha \models B^{\Box\downarrow} \leftrightarrow B$, $\mathcal{K}, \alpha \Vdash B^{\Box} \leftrightarrow B$ and $B^{\Box} = (B^{\Box\uparrow})^{\Box\downarrow}$. Hence by Corollary 9.3.43 we have $\mathcal{K}, \alpha \Vdash A$ iff $\mathcal{K}, \alpha \Vdash A^{\Box}$ iff $\mathcal{K}, \alpha \models (A^{\Box\uparrow})^{\Box\downarrow}$ iff $\mathcal{K}, \alpha \models A^{\Box\uparrow}$. □

9.3.3.1 The Smoryński Operation

In this subsection, we define the Smoryński operation on Kripke models (Smoryński 1985). Given a Kripke model $\mathcal{K} = (K, \preccurlyeq, \sqsubset, V)$ and some fixed node $\alpha \in K$, define $\mathcal{K}' := (K', \preccurlyeq', \sqsubset', V')$ as the Kripke model constituted by adding one fresh node α' to $\mathcal{K}$. All nodes of $\mathcal{K}'$ other than α', forces the same atomic variables and have the same accessibility relationships as they did in $\mathcal{K}$. Also α' imitates all relationships of α. More precisely $\mathcal{K}'$ is constituted as follows:

- $K' := K \cup \{\alpha'\}$, in which $\alpha' \notin K$,
- $\beta \preccurlyeq' \gamma$ iff $\beta \preccurlyeq \gamma$ for every $\beta, \gamma \in K$,
- $\beta \sqsubset' \gamma$ iff $\beta \sqsubset \gamma$ for every $\beta, \gamma \in K$,
- $\beta\ V'\ p$ iff $\beta\ V\ p$ for every $\beta \in K$,
- $\alpha'\ V'\ p$ iff $\alpha\ V\ p$,
- $\alpha' \preccurlyeq' \beta$ iff ($\alpha \preccurlyeq \beta$ or $\beta = \alpha'$). Also $\beta \preccurlyeq' \alpha'$ iff $\beta = \alpha$,
- $\alpha' \sqsubset' \beta$ iff $\alpha \sqsubseteq \beta$. Also $\beta \not\sqsubset' \alpha'$ for every $\beta \in K'$.

Then we define $\mathcal{K}^{(n)}$ and α_n inductively:

- $\mathcal{K}^{(0)} := \mathcal{K}$ and $\alpha_0 := \alpha$,
- $\mathcal{K}^{(n+1)} := \left(\mathcal{K}^{(n)}\right)'$ and α_{n+1} is defined as the fresh node which is added to $\mathcal{K}^{(n)}$ in the definition of $\left(\mathcal{K}^{(n)}\right)'$.

Lemma 9.3.45 *Let $\mathcal{K}$ be a Kripke model which is $A^{\Box\downarrow}$-sound at the quasi-classical node α. Then for every subformula B of $A^{\Box\downarrow}$ and arbitrary boolean interpretation I we have*

1. *$\mathcal{K}, \alpha \models B$ iff $\mathcal{K}', \alpha' \models B$.*
2. *$\mathcal{K}, \alpha, I \models B$ iff $\mathcal{K}', \alpha', I \models B$.*
3. *α' is quasi-classical and $\mathcal{K}'$ is $A^{\Box\downarrow}$-sound at α'.*
4. *If $\mathcal{K}$ is semi-perfect, perfect or quasi-classical, then $\mathcal{K}'$ is so.*

Proof 1. Use induction on the complexity of B. All cases are trivial, except for the case $B = \Box C^{\Box}$. If $\alpha' \models \Box C^{\Box}$, evidently $\alpha \models \Box C^{\Box}$ as well. If $\alpha \models \Box C^{\Box}$, then by $A^{\Box\downarrow}$-soundness, $\alpha \models C^{\Box}$, and by Lemma 9.3.42, $\alpha \Vdash C^{\Box}$. Hence $\alpha' \models \Box C^{\Box}$.
2. Similar to first item and left to the reader.
3. The fact that α' is quasi-classical can easily be observed by the definition of $\mathcal{K}'$ and left to the reader. The $A^{\Box\downarrow}$-soundness, is derived from first item.
4. Easy and left to the reader.

□

9.4 Reduction of Arithmetical Completenesses

Let us define $[\![A; T, U; \Gamma]\!]$ as the set of all Γ-substitutions σ such that $U \nvdash \sigma_{T}(A)$. Hence $\mathsf{PL}_{\Gamma}(T, U) = \{A : [\![A; T, U; \Gamma]\!] = \emptyset\}$. For an arithmetical substitution σ, let

$[\![\sigma]\!]$ indicate the propositional closure of σ, i.e. the smallest set X of arithmetical substitutions with the following conditions:

- $\sigma \in X$,
- if $\alpha \in X$, τ is some $\mathcal{L}_\Box$-substitution and T is some recursively axiomatizable arithmetical theory, then $\alpha_T \circ \tau \in X$.

Note that the substitution $\alpha_T \circ \tau$ is defined on atomic variable p in this way: $\alpha_T \circ \tau(p) := \alpha_T(\tau(p))$.

Let V_0 be a modal theory. We define the arithmetical Γ-completeness of V_0 with respect to T relative in U as follows:

$$\mathcal{AC}_\Gamma(V_0; T, U) \quad \equiv \quad A \in \mathsf{PL}_\Gamma(T, U) \text{implies} V_0 \vdash A, \text{for every} A \in \mathcal{L}_\Box$$

Similarly we define the arithmetical Γ-soundness $\mathcal{AS}_\Gamma(V_0; T, U)$ as follows:

$$\mathcal{AS}_\Gamma(V_0; T, U) \quad \equiv \quad V_0 \vdash A \text{ implies} A \in \mathsf{PL}_\Gamma(T, U), \text{for every} A \in \mathcal{L}_\Box$$

When Γ is the set of all arithmetical sentences, we may omit the subscript Γ in the notations $\mathsf{PL}_\Gamma(T, U)$, $\mathcal{AC}_\Gamma(V_0; T, U)$ and $\mathcal{AS}_\Gamma(V_0; T, U)$.
Note that $\mathsf{PL}_\Gamma(T, U) = V_0$ iff $\mathcal{AC}_\Gamma(V_0; T, U)$ and $\mathcal{AS}_\Gamma(V_0; T, U)$.

In the following definition, we formalize reduction of the arithmetical completeness of V_0 to V_0'.

Definition 9.4.1 Let T and T' be consistent recursively axiomatizable and U and U' be strong enough arithmetical theories. Also let Γ and Γ' be sets of arithmetical sentences and V_0, V_0' be modal theories. We say that $f, \bar{f}$ propositionally reduces $\mathcal{AC}_{\Gamma'}(V_0'; T', U')$ to $\mathcal{AC}_\Gamma(V_0; T, U)$, with the notation $\mathcal{AC}_\Gamma(V_0; T, U) \leq^{\text{Prop}}_{f,\bar{f}} \mathcal{AC}_{\Gamma'}(V_0'; T', U')$, if:

R0. $f : \mathcal{L}_\Box \longrightarrow \mathcal{L}_\Box$ and $\bar{f} = \{\bar{f}_A\}_A$ is a family of functions,
R1. $V_0 \vdash f(A)$ implies $V_0' \vdash A$,
R2. for every $A \in \mathcal{L}_\Box$, $\bar{f}_A$ is a function on arithmetical substitutions and

$$\bar{f}_A : [\![f(A); T, U; \Gamma]\!] \longrightarrow [\![A; T', U'; \Gamma']\!] \quad \text{and for every} \sigma : \quad \bar{f}_A(\sigma) \in [\![\sigma]\!].$$

We say that $\mathcal{AC}_{\Gamma'}(V_0'; T', U')$ is propositionally reducible to $\mathcal{AC}_\Gamma(V_0; T, U)$, with the notation

$$\mathcal{AC}_\Gamma(V_0; T, U) \leq^{\text{Prop}} \mathcal{AC}_{\Gamma'}(V_0'; T', U'),$$

if there exists some $f, \bar{f}$ such that $\mathcal{AC}_\Gamma(V_0; T, U) \leq^{\text{Prop}}_{f,\bar{f}} \mathcal{AC}_{\Gamma'}(V_0'; T', U')$.

Following theorems are what one expect from the reduction:

Theorem 9.4.2 *The reduction of arithmetical completenesses is a transitive reflexive relation.*

Proof The reflexivity is trivial and left to the reader. For the transitivity, let

$$\mathcal{AC}_{\Gamma}(V_0; T, U) \leq^{\text{Prop}}_{f,\bar{f}} \mathcal{AC}_{\Gamma'}(V_0'; T', U') \leq^{\text{Prop}}_{g,\bar{g}} \mathcal{AC}_{\Gamma''}(V_0''; T'', U'')$$

and observe that

$$\mathcal{AC}_{\Gamma}(V_0; T, U) \leq^{\text{Prop}}_{h,\bar{h}} \mathcal{AC}_{\Gamma''}(V_0''; T'', U'')$$

in which $h := f \circ g$ and $\bar{h}_A := \bar{g}_A \circ \bar{f}_{g(A)}$. □

Theorem 9.4.3 *$\mathcal{AC}_{\Gamma}(V_0; T, U) \leq^{Prop}_{f,\bar{f}} \mathcal{AC}_{\Gamma'}(V_0'; T', U')$ and $\mathcal{AC}_{\Gamma}(V_0; T, U)$ implies $\mathcal{AC}_{\Gamma'}(V_0'; T', U')$.*

Proof Let $V_0' \nvdash A$. Then by R1 in Definition 9.4.1, $V_0 \nvdash f(A)$. Hence by $\mathcal{AC}_{\Gamma}(V_0; T, U)$, there exists some Γ-substitution σ such that $U \nvdash \sigma_T(f(A))$, or in other words $\sigma \in [\![f(A); T, U; \Gamma]\!]$. Hence by R2 we have $\bar{f}_A(\sigma) \in [\![A; T', U'; \Gamma']\!]$, which implies $A \notin \mathsf{PL}_{\Gamma'}(T', U')$. □

Remark 9.4.4 Note that the requirement $\bar{f}_A(\alpha) \in [\![\alpha]\!]$, was not used in the proof of arithmetical completeness of V_0 in Theorem 9.4.3. The only use of this condition is to restrict the way one may compute $\bar{f}_A(\alpha)$ from α: only propositional substitutions are allowed to be composed with α to produce $\bar{f}_A(\alpha)$. If we remove this restriction from the definition, we would have a trivial reduction: every arithmetical completeness would be reducible to everyone.

Corollary 9.4.5 *If $\mathcal{AC}_{\Gamma}(V_0; T, U) \leq^{Prop}_{f,\bar{f}} \mathcal{AC}_{\Gamma'}(V_0'; T', U')$ and $\mathcal{AC}_{\Gamma}(V_0; T, U)$, then we have*

$$V_0 \vdash f(A) \iff V_0' \vdash A$$

Proof The direction $\Longleftarrow$ holds by definition. For the other way around, use Theorem 9.4.3. □

Remark 9.4.6 Note that $V_0 \vdash A \iff V_0' \vdash f(A)$ is not enough for reduction of arithmetical completenesses. This is simply because f does not have anything to do with arithmetical substitutions. So one may not be able to translate an arithmetical refutation from $\mathsf{PL}_{\Gamma'}(T', U')$ to a refutation from $\mathsf{PL}_{\Gamma}(T, U)$, via propositional translations. If we remove R2 and replace R1 by $V_0 \vdash A \iff V_0' \vdash f(A)$ in Definition 9.4.1, $\mathcal{AC}_{\Gamma}(V_0; T, U)$ would be reducible to every arithmetical completeness via the following vicious reduction:

$$f(A) := \begin{cases} \top & : \text{ if } V_0 \vdash A \\ \bot & : \text{ otherwise} \end{cases}$$

Notation 9.4.7 In the rest of the paper, we are going to characterise several provability logics. Our main tool for proving their arithmetical completeness is the reduction of arithmetical completenesses and using Theorem 9.4.3. The notation $\mathsf{PL}_{\Gamma}(T, U) \leq^{\text{Prop}} \mathsf{PL}_{\Gamma'}(T', U')$ is a shorthand for

$$\mathcal{AC}_{\Gamma}(\mathsf{PL}_{\Gamma}(T, U); T, U) \leq^{\text{Prop}} \mathcal{AC}_{\Gamma}'(\mathsf{PL}_{\Gamma'}(T', U'); T', U')$$

Theorem 9.4.8 *Let* $\mathsf{PL}_{\Gamma}(T, U) \leq_{f,\bar{f}}^{Prop} \mathsf{PL}_{\Gamma'}(T', U')$ *for some computable function* f*. Then the decidability of* $\mathsf{PL}_{\Gamma}(T, U)$ *implies the decidability of* $\mathsf{PL}_{\Gamma'}(T', U')$*.*

Proof Direct consequence of Corollary 9.4.5 and computability of f. □

9.4.1 Two Special Cases

In later applications, always we consider two simple cases of reduction $f, \bar{f}$ (Definition 9.4.1) to provide new arithmetical completenesses:

- Substitution: in this case, we let $f(A)$ as some $\mathcal{L}_{\Box}$-substitution, possibly depending on A. Also $\bar{f}_A(\sigma) := \sigma_{T'} \circ \tau$. This kind of reductions, in this paper are only used to reduce provability logics to their Σ_1-provability logics, which are labelled with τ in Diagram 9.5.
- Identity: in this case we consider $\bar{f}_A$ as the identity function and $f(A)$ is some propositional translation like $(.)^{\Box}$, $(.)^{\Box\downarrow}$, $(.)^{\neg\uparrow}$, $(.)^{\dagger}$ and $\Box(.)$. See Diagram 9.5.

9.5 Relative Σ_1-provability Logics for HA

In this section, we will characterise $\mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathbb{N})$, i.e. the truth Σ_1-provability logic of HA, and $\mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{PA})$, i.e. the Σ_1-provability logic of HA, relative to PA. We also show that $\mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathbb{N})$ is hardest among the Σ_1-provability logics of HA relative in HA, PA, $\mathbb{N}$ (see Diagram 9.1).

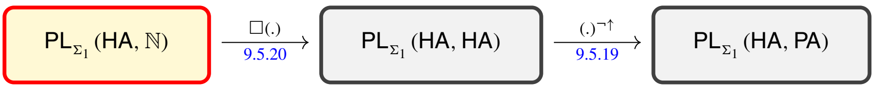

Diagram 9.1 Reductions for relative provability logics of HA

9.5.1 Kripke Semantics

Lemma 9.5.1 *For every* A *we have*

$$\mathsf{iGLC} \vdash A \iff \mathsf{iGL} \vdash \left[\boxdot \bigwedge_{E \in \mathsf{sub}(A)} (E \to \Box E)\right] \to A$$

Proof For the simplicity of notations, in this proof, let

$$\varphi := \boxdot \bigwedge_{E \in \mathsf{sub}(A)} (E \to \Box E)$$

and $\vdash$ indicate derivablity in $\mathsf{iGL} + \varphi$.

One side is trivial. For the other way around, assume that $\mathsf{iGL} \nvdash \varphi \to A$. We will construct some perfect Kripke model $\mathcal{K} = (K, \sqsubset, \preccurlyeq, V)$ such that $\mathcal{K}, \alpha \nVdash A$, which by soundness of iGLC for finite brilliant models with $\sqsubset \subseteq \prec$, we have the desired result. The proof is almost identical to the proof of Theorem 9.3.40 in (Ardeshir and Mojtahedi 2018, Theorem 4.26), but to be self-contained, we repeat it here.

Let $\mathsf{Sub}(A)$ be the set of sub-formulae of A. Then define

$$X := \{B, \Box B \mid B \in \mathsf{Sub}(A)\}$$

It is obvious that X is a finite adequate set. We define $\mathcal{K} = (K, \preccurlyeq, \sqsubset, V)$ as follows. Take K as the set of all X-saturated sets with respect to $\mathsf{iGL} + \varphi$, and $\preccurlyeq$ is the subset relation over K. Define $\alpha \sqsubset \beta$ iff for all $\Box B \in X$, $\Box B \in \alpha$ implies $B \in \beta$, and also there exists some $\Box C \in \beta \setminus \alpha$. Finally define $\alpha V p$ iff $p \in \alpha$, for atomic p.

It only remains to show that $\mathcal{K}$ is a finite brilliant Kripke model with $\sqsubset \subseteq \prec$ which refutes A. To this end, we first show by induction on $B \in X$ that $B \in \alpha$ iff $\alpha \Vdash B$, for each $\alpha \in K$. The only non-trivial case is $B = \Box C$. Let $\Box C \notin \alpha$. We must show $\alpha \nVdash \Box C$. The other direction is easier to prove and we leave it to reader. Let $\beta_0 := \{D \in X \mid \alpha \vdash \Box D\}$. If $\beta_0, \Box C \vdash C$, since by definition of β_0, we have $\alpha \vdash \Box \beta_0$ and hence by Löb's axiom, $\alpha \vdash \Box C$, which is in contradiction with $\Box C \notin \alpha$. Hence $\beta_0, \Box C \nvdash C$ and so there exists some X-saturated set β such that $\beta \nvdash C, \beta \supseteq \beta_0 \cup \{\Box C\}$. Hence $\beta \in K$ and $\alpha \sqsubset \beta$. Then by the induction hypothesis, $\beta \nVdash C$ and hence $\alpha \nVdash \Box C$.

Since $\mathsf{iGL} + \varphi \nvdash A$, by Lemma 9.3.39, there exists some X-saturated set $\alpha \in K$ such that $\alpha \nvdash A$, and hence by the above argument we have $\alpha \nVdash A$.

$\mathcal{K}$ trivially satisfies all the properties of finite brilliant Kripke model with $\sqsubset \subseteq \prec$. As a sample, we show that why $\sqsubset \subseteq \prec$ holds. Assume $\alpha \sqsubset \beta$ and let $B \in \alpha$. If $B = \Box C$ for some C, then by definition, $C \in \beta$ and since $C \to \Box C$ is a conjunct in φ, we have $\beta \vdash \Box C$ and we are done. So assume B is not a boxed formula. Then by definition of X, we have $\Box B \in X$ and since $B \to \Box B$ is a conjunct in φ, we have $\alpha \vdash \Box B$ and hence by definition of $\sqsubset$, it is the case that $B \in \beta$. This shows $\alpha \subseteq \beta$ and hence $\alpha \preccurlyeq \beta$. But α is not equal to β, because $\alpha \sqsubset \beta$ implies existence of some $\Box C \in \beta \setminus \alpha$. Hence $\alpha \prec \beta$, as desired. $\Box$

Lemma 9.5.2 *For any proposition A, if* $\mathsf{iGLC} \vdash A^{\Box}$ *then* $\mathsf{iGL} + \mathsf{CP_a} + \Box\mathsf{CP} \vdash A^{\Box}$.

Proof Let $\mathsf{iGLC} \vdash A$. Hence by Lemma 9.5.1 for some finite set X of subformulas of $A^{\Box}$ we have

$$\mathsf{iGL} \vdash \boxdot\left(\bigwedge_{E\in X} E \to \Box E\right) \to A^{\Box}$$

Lemma 9.3.9 implies

$$\mathsf{iGL} \vdash \Box\left(\bigwedge_{E\in X} E \to \Box E\right) \wedge \left(\bigwedge_{E\in X} \boxdot(E^{\Box\uparrow} \to \Box E)\right) \to A^{\Box}$$

By Lemma 9.3.10 we have $\mathsf{iGL} + \mathsf{CP_a} + \Box\mathsf{CP} \vdash A^{\Box}$. □

Theorem 9.5.3 $\mathsf{iGL\overline{C}\underline{P}C_a}$ *is sound and complete for local truth at quasi-classical nodes in perfect Kripke models. More precisely, we have* $\mathsf{iGL\overline{C}\underline{P}C_a} \vdash A$ *iff* $\mathcal{K}, \alpha \models A$ *for every perfect Kripke model* $\mathcal{K}$ *and the quasi-classical node* α.

Proof The soundness part easily derived by the soundness of iGLC and left to the reader.

Since local truth at α is not affected by changing the set of $\preccurlyeq$-accessible nodes from α, it is enough to prove the completeness part only for the perfect Kripke models. Let $\mathsf{iGL\overline{C}\underline{P}C_a} \nvdash A$. Let A' be a boolean equivalent of A which is a conjunction of implications $E \to F$ in which E is a conjunction of a set of atomics or boxed propositions and F is a disjunction of atomics or boxed proposition. Evidently such A' exists for every A. Hence $\mathsf{iGL\overline{C}\underline{P}C_a} \nvdash A'$. Then there must be some conjunct $E \to F$ of A' such that $\mathsf{iGL\overline{C}\underline{P}C_a} \nvdash (E \to F)^{\Box}$, E is a conjunction of atomic and boxed propositions and F is a disjunction of atomic and boxed propositions. Hence $\mathsf{iGL} + \mathsf{CP_a} + \Box\mathsf{CP} \nvdash (E \to F)^{\Box}$ and by Lemma 9.5.2 we have $\mathsf{iGLC} \nvdash (E \to F)^{\Box}$. By Theorem 9.3.40, there exists some perfect Kripke model $\mathcal{K} = (K, \preccurlyeq, \sqsubset, V)$ such that $\mathcal{K}, \alpha \nVdash (E \to F)^{\Box}$ for some $\alpha \in K$. Since iGLC is sound for $\mathcal{K}$, we have $\mathcal{K}, \alpha \nVdash E \to F$. Hence there exists some $\beta \succcurlyeq \alpha$ such that $\mathcal{K}, \beta \Vdash E$ and $\mathcal{K}, \beta \nVdash F$. Then by definition of local truth we have $\mathcal{K}, \beta \models E$ and $\mathcal{K}, \beta \not\models F$. Then $\mathcal{K}, \beta \not\models E \to F$. Hence $\mathcal{K}, \beta \not\models A$, as desired. □

Corollary 9.5.4 $\mathsf{iGL\overline{C}\underline{P}C_a}$ *is decidable.*

Proof Direct consequence of the proof of Theorem 9.5.3 and decidability of iGLC (Corollary 9.3.41). □

Theorem 9.5.5 $\mathsf{iGL\overline{C}\underline{SP}C_a} \vdash A^{\Box\Downarrow}$ *iff* $\mathcal{K}, \alpha \models A^{\Box\Downarrow}$ *for every perfect Kripke models* $\mathcal{K}$ *and quasi-classical* $A^{\Box\Downarrow}$*-sound nodes* α.

Proof Both directions are non-trivial and proved contra-positively. For the soundness part, assume that $\mathcal{K}, \alpha \not\models A^{\Box\Downarrow}$ for some perfect Kripke model $\mathcal{K} := (K, \preccurlyeq, \sqsubset, V)$ which is $A^{\Box\Downarrow}$-sound at the quasi-classical node $\alpha \in K$. Since derivability is finite, it is enough to show that for every finite set Γ of modal propositions we have

$$\mathsf{iGL\overline{C}\underline{P}C_a} \nvdash \bigwedge_{B\in\Gamma}(\Box B \to B) \to A^{\Box\downarrow}.$$

By Theorem 9.5.3 and Lemma 9.3.45, it is enough to find some number i such that

$$\mathcal{K}^{(i)}, \alpha_i \nvDash \bigwedge_{B\in\Gamma}(\Box B \to B) \to A^{\Box\downarrow}.$$

Let us define n_i and m_i as the number of propositions in the sets $N_i := \{B \in \Gamma : \mathcal{K}^{(i)}, \alpha_i \vDash B \wedge \Box B\}$ and $M_i := \{B \in \Gamma : \mathcal{K}^{(i)}, \alpha_i \vDash \Box B \wedge \neg B\}$, respectively.
We use induction as follows. As induction hypothesis, assume that for any number i with $n_i < k$ there is some $0 \le j \le 1 + n_i$ such that

$$\mathcal{K}^{(i+j)}, \alpha_{i+j} \nvDash \bigwedge_{B\in\Gamma}(\Box B \to B) \to A^{\Box\downarrow} \tag{9.1}$$

Let $n_i = k$. If $m_i = 0$, we may let $j = 0$ and by Lemma 9.3.45 we have Eq. (9.1) as desired. So let $B \in \Gamma$ such that $\mathcal{K}^{(i)}, \alpha_i \vDash \Box B \wedge \neg B$. We have two sub-cases:

- $m_{i+1} = 0$: observe in this case that Eq. (9.1) holds for $j = 1$.
- $m_{i+1} > 0$: in this case we have $n_{i+1} < k$ and hence by application of the induction hypothesis with $i := i + 1$, we get some $0 \le j' \le 1 + n_{i+1}$ such that

$$\mathcal{K}^{(i+1+j')}, \alpha_{i+1+j'} \nvDash \bigwedge_{B\in\Gamma}(\Box B \to B) \to A^{\Box\downarrow}$$

Hence if we let $j := j' + 1$ we have $0 \le j \le 1 + n_i$ and eq. (9.1), as desired.

For the completeness part, assume that $\mathsf{iGL\overline{C}\underline{SP}C_a} \nvdash A^{\Box\downarrow}$. Hence

$$\mathsf{iGL\overline{C}\underline{P}C_a} \nvdash \left(\bigwedge_{\Box B\in \mathsf{Sub}(A)}(\Box B \to B)\right) \to A^{\Box\downarrow}$$

Hence Theorem 9.5.3 implies the desired result. □

Corollary 9.5.6 $\mathsf{iGL\overline{C}\underline{SP}C_a}$ *is decidable.*

Proof First observe that by Theorem 9.5.5, 9.5.3, we have $\mathsf{iGL\overline{C}\underline{SP}C_a} \vdash A^{\Box\downarrow}$ iff

$$\mathsf{iGL\overline{C}\underline{P}C_a} \vdash \bigwedge_{\Box B\in \mathsf{Sub}(A^{\Box\downarrow})}(\Box B \to B) \to A^{\Box\downarrow}.$$

Hence the decidability of $\mathsf{iGL\overline{C}\underline{P}C_a}$ (Corollary 9.5.4) implies the decidability of $\mathsf{iGL\overline{C}\underline{SP}C_a}$. □

9.5.2 Arithmetical Interpretations

The following theorem is the main result in (Ardeshir and Mojtahedi 2018):

Theorem 9.5.7 iH_σ *is the* Σ_1*-provability logic of* HA*, i.e.* $\mathsf{iH}_\sigma \vdash A$ *iff for all* Σ_1*-substitution* σ *we have* $\mathsf{HA} \vdash \sigma_{\mathsf{HA}}(A)$*. Moreover* iH_σ *is decidable.*

Here we present some essential facts and definitions from (Ardeshir and Mojtahedi 2018). Let us fix some perfect Kripke model $\mathcal{K}_0 = (K_0, \sqsubset_0, \preccurlyeq_0, V_0)$ with quasi-classical root α_0 and its extension $\mathcal{K} := \mathcal{K}'_0 = (K, \preccurlyeq, \sqsubset, V)$ by the Smoryński operation with the new quasi-classical root α_1 (which was called α_0 in (Ardeshir and Mojtahedi 2018)) and define a recursive function F, called Solovay function, as we did in (Ardeshir and Mojtahedi 2018). We have the following definitions and facts from (Ardeshir and Mojtahedi 2018): (later we refer to them simply as e.g. "item 1"):

1. The function F is provably total in HA and hence we may use the function symbol F inside HA and stronger theories.
2. The Σ_1-substitution σ is defined in this way:
$$\sigma(p) := \bigvee_{\mathcal{K},\alpha \Vdash p} \exists x\, (F(x) = \alpha)$$
3. Define $L = \alpha$ as $\exists x\, \forall y \geq x\; F(y) = \alpha$.
4. $\mathsf{PA} \vdash \exists x\, (F(x) = \alpha) \to \bigvee_{\beta \succcurlyeq \alpha} L = \beta$ (Ardeshir and Mojtahedi 2018, Lemma 5.2)
5. For a modal proposition A when we use A in a context which it is expected to be some first-order formula, like $\mathsf{HA} \vdash A$, we should replace A with the first-order sentence $\sigma_{\mathsf{HA}}(A)$.
6. For every $A \in \mathsf{sub}(\Gamma) \cap \mathsf{TNNIL}$ and $\alpha \in K_0$ such that $\mathcal{K}_0, \alpha \Vdash A$, we have $\mathsf{HA} \vdash \exists x\, F(x) = \alpha \to A$ (Ardeshir and Mojtahedi 2018, Lemma 5.18 & 5.19)
7. For each $B \in \mathsf{Sub}(\Gamma) \cap \mathsf{TNNIL}$ and $\alpha \in K$ such that $\alpha \nVdash \Box B$,
$$\mathsf{HA} \vdash L{=}\alpha \to \neg\Box B.$$
8. $\mathbb{N} \models L = \alpha_1$ and $\mathsf{PA} + L = \alpha$ is consistent for every $\alpha \in K$ (Ardeshir and Mojtahedi 2018, Corollaries 5.20 & 5.24 and Lemma 5.23).

Lemma 9.5.8 *For every* $A \in \mathsf{NOI} \cap \mathsf{sub}(\Gamma)$ *such that* $\mathcal{K}, \alpha_1 \nVdash A$*, we have*
$$\mathsf{HA} \vdash A \leftrightarrow \bigvee_{\alpha \in K \text{ and } \mathcal{K},\alpha \Vdash A} \exists x\, F(x) = \alpha$$

Proof First observe that by Π_2-conservativity of PA over HA (Lemma 9.3.15), it is enough to prove this lemma in PA instead of HA. Then by "item 4", it is enough to show that

$$\mathsf{PA} \vdash A \leftrightarrow \bigvee_{\alpha \in K \text{ and } \mathcal{K},\alpha \Vdash A} L = \alpha$$

We use induction on the complexity of A. Since $A \in \mathsf{NOI}$ we do not consider the $\rightarrow$ case in the induction steps:

- A is atomic: by definition of the arithmetical substitution, we have

$$\sigma(A) = \bigvee_{\alpha \in K \text{ and } \mathcal{K},\alpha \Vdash A} \exists x\, F(x) = \alpha.$$

- $A = B \circ C$ and $\circ \in \{\vee, \wedge\}$: easy and left to the reader.
- $A = \Box B$: first note that by "item 6", $\mathsf{PA} \vdash \exists x\, F(x) = \alpha \rightarrow A$ for every $\alpha \Vdash A$ (here actually we need $\alpha_1 \nVdash A$). Hence

$$\mathsf{PA} \vdash \bigvee_{\alpha \in K \text{ and } \mathcal{K},\alpha \Vdash A} \exists x\, F(x) = \alpha \rightarrow A$$

  For the other direction, it is enough (by "item 4") to show that for every $\beta \in K$ such that $\mathcal{K}, \beta \nVdash A$ we have $\mathsf{PA} \vdash A \rightarrow L \neq \beta$ or equivalently $\mathsf{PA} \vdash L = \beta \rightarrow \neg A$, which holds by "item 7". □

□

Lemma 9.5.9 *For every $A \in \mathsf{sub}(\Gamma)$ and $\alpha \in K_0$, we have*

$$\begin{cases} \mathcal{K}, \alpha \models A & \Longrightarrow \quad \mathsf{HA} \vdash L = \alpha \rightarrow A \\ \mathcal{K}, \alpha \not\models A & \Longrightarrow \quad \mathsf{HA} \vdash L = \alpha \rightarrow \neg A \end{cases}$$

Proof We use induction on the complexity of A. All cases are simple and we only treat the case $A = \Box B$ here. If $\mathcal{K}, \alpha \models \Box B$, by definition, $\mathcal{K}, \alpha \Vdash \Box B$ and hence by "item 6" we have the desired result. If also $\mathcal{K}, \alpha \not\models \Box B$, by definition, $\mathcal{K}, \alpha \nVdash \Box B$ and hence by "item 7" we have the desired result. □

Lemma 9.5.10 *Let $\mathcal{K}$ be $A^{\Box\downarrow}$-sound at α_0 for some $A^{\Box\downarrow} \in \mathsf{sub}(\Gamma)$. Then for every $B \in \mathsf{sub}(A^{\Box\downarrow})$ we have*

$$\begin{cases} \mathcal{K}, \alpha_1 \models B & \Longrightarrow \quad \mathsf{HA} \vdash L = \alpha_1 \rightarrow B \\ \mathcal{K}, \alpha_1 \not\models B & \Longrightarrow \quad \mathsf{HA} \vdash L = \alpha_1 \rightarrow \neg B \\ \mathcal{K}, \alpha_1 \Vdash B & \Longrightarrow \quad \mathsf{HA} \vdash B \end{cases}$$

Proof We prove this by induction on the complexity of $B \in \mathsf{sub}(A^{\Box\downarrow})$.

- B is atomic, conjunction or disjunction: easy and left to the reader.
- $B = E \rightarrow F$: it is easy to show the first two derivations and we leave them to the reader. For the third one, assume that $\mathcal{K}, \alpha_1 \Vdash E \rightarrow F$. If $\mathcal{K}, \alpha_1 \Vdash F$ we have the

desired result by induction hypothesis. So let $\mathcal{K}, \alpha_1 \nVdash F$ and Hence $\mathcal{K}, \alpha_1 \nVdash E$. Hence by Lemma 9.5.8, we have $\mathsf{HA} \vdash E \to \bigvee_{\alpha \Vdash E} \exists x\, F(x) = \alpha$. On the other hand by "item 6" we have $\mathsf{HA} \vdash \bigvee_{\alpha \Vdash E} \exists x\, F(x) = \alpha \to F$. Hence we have $\mathsf{HA} \to E \to F$.

- $B = \Box C^{\Box}$: Let $\mathcal{K}, \alpha_1 \models \Box C^{\Box}$. Then by Lemma 9.3.45 we have $\mathcal{K}, \alpha_1 \models C^{\Box}$ and hence by Lemma 9.3.42 $\mathcal{K}, \alpha_1 \Vdash C^{\Box}$. Then by induction hypothesis $\mathsf{HA} \vdash C^{\Box}$ and hence $\mathsf{HA} \vdash L = \alpha_1 \to \Box C^{\Box}$.
  For the second derivation, Let $\mathcal{K}, \alpha_1 \not\models \Box C^{\Box}$. Then by "item 7" we have the desired result.
  For the third derivation, let $\mathcal{K}, \alpha_1 \Vdash \Box C^{\Box}$. Then by Lemma 9.3.42 we have $\mathcal{K}, \alpha_1 \models \Box C^{\Box}$ and hence Lemma 9.3.45 implies $\mathcal{K}, \alpha_1 \models C^{\Box}$ and then again by Lemma 9.3.42 $\mathcal{K}, \alpha_1 \Vdash C^{\Box}$. Then by induction hypothesis $\mathsf{HA} \vdash C^{\Box}$ and hence $\mathsf{HA} \vdash \Box C^{\Box}$.

$\Box$

9.5.3 Arithmetical Completeness

Definition 9.5.11 Define the following modal systems:

- $\mathsf{iH}_\sigma\underline{\mathsf{P}} := \mathsf{iH}_\sigma$ plus $\underline{\mathsf{P}}$,
- $\mathsf{iH}_\sigma\underline{\mathsf{SP}} := \mathsf{iH}_\sigma\underline{\mathsf{P}}$ plus $\underline{\mathsf{S}}$,
- $\mathsf{iH}_\sigma\underline{\mathsf{P}}^* := \{A \in \mathcal{L}_\Box : \mathsf{iH}_\sigma\underline{\mathsf{P}} \vdash A^{\Box\downarrow}\}$,
- $\mathsf{iH}_\sigma\underline{\mathsf{SP}}^* := \{A \in \mathcal{L}_\Box : \mathsf{iH}_\sigma\underline{\mathsf{SP}} \vdash A^{\Box\downarrow}\}$.

Obviously $\mathsf{iH}_\sigma\underline{\mathsf{SP}}^*$ and $\mathsf{iH}_\sigma\underline{\mathsf{P}}^*$ are closed under modus ponens.

Theorem 9.5.12 $\mathsf{iH}_\sigma\underline{\mathsf{P}} = \mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{PA})$, *i.e.* $\mathsf{iH}_\sigma\underline{\mathsf{P}}$ *is the relative* Σ_1*-provability logic of* HA *in* PA.

Proof The soundness easily deduced by use of the soundness of the iH_σ for arithmetical interpretations in HA (see Theorem 6.3 in (Ardeshir and Mojtahedi 2018)).

For the other way around, let $\mathsf{iH}_\sigma\underline{\mathsf{P}} \nvdash A$. Then $\mathsf{iH}_\sigma\underline{\mathsf{P}} \nvdash A^-$ in which $A^- \in \mathsf{TNNIL}^{\Box}$ and $\mathsf{iH}_\sigma \vdash A \leftrightarrow A^-$. Then $\mathsf{iGLLe^+}\underline{\mathsf{P}} \nvdash A^-$ and hence by Theorem 9.3.37 we have $\mathsf{iGL\overline{C}\underline{P}C_a} \nvdash A^-$. By Theorem 9.5.3, there is some perfect Kripke model $\mathcal{K}_0$ with the quasi-classical root α_0 such that $\mathcal{K}_0, \alpha_0 \not\models A^-$. Let σ be the Σ_1-substitution as provided in Sect. 9.5.2 for the Kripke model $\mathcal{K}_0$ and its Smoryński extension $\mathcal{K}$ with $\Gamma := \{A^-\}$. Then by Lemma 9.5.9 we have $\mathsf{HA} \vdash L = \alpha_0 \to \sigma_{\mathsf{HA}}(\neg A^-)$. Since $\mathsf{iH}_\sigma \vdash A \leftrightarrow A^-$, by soundness part of Theorem 9.5.7 we have $\mathsf{HA} \vdash L = \alpha_0 \to \sigma_{\mathsf{HA}}(\neg A)$. Hence by "item 8" we may deduce $\mathsf{PA} \nvdash \sigma_{\mathsf{HA}}(A)$, as desired. $\Box$

Theorem 9.5.13 $\mathsf{iH}_\sigma\underline{\mathsf{SP}} = \mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathbb{N})$, *i.e.* $\mathsf{iH}_\sigma\underline{\mathsf{SP}}$ *is the truth* Σ_1*-provability logic of* HA.

Proof The soundness easily deduced by use of the soundness of the iH_σ for arithmetical interpretations in HA (see Theorem 6.3 in (Ardeshir and Mojtahedi 2018)).

For the other way around, let $\mathsf{iH}_\sigma\underline{\mathsf{SP}} \nvdash A$. Then by Lemma 9.3.35 we have $\mathsf{iH}_\sigma\underline{\mathsf{SP}} \nvdash (A^-)^{\Box\downarrow}$ in which $(A^-)^{\Box\downarrow} \in \mathsf{TNNIL}^\Box$ and $\mathsf{iH}_\sigma \vdash A \leftrightarrow (A^-)^{\Box\downarrow}$. Then $\mathsf{iGLLe}^+\underline{\mathsf{SP}} \nvdash (A^-)^{\Box\downarrow}$ and hence by Theorem 9.3.37 we have $\mathsf{iGL}\overline{\mathsf{C}}\underline{\mathsf{SP}}\mathsf{C_a} \nvdash (A^-)^{\Box\downarrow}$.

By Theorem 9.5.5, there is some perfect Kripke model $\mathcal{K}_0$ with the quasi-classical $(A^-)^{\Box\downarrow}$-sound root α_0 such that $\mathcal{K}_0, \alpha_0 \not\models (A^-)^{\Box\downarrow}$. Let σ be the Σ_1-substitution as provided in Sect. 9.5.2 for the Kripke model $\mathcal{K}_0$ and its Smoryński extension $\mathcal{K}$ with $\Gamma := \{(A^-)^{\Box\downarrow}\}$. Then by Lemma 9.5.10 we have $\mathsf{HA} \vdash L = \alpha_1 \to \sigma_{\mathsf{HA}}(\neg(A^-)^{\Box\downarrow})$. Since $\mathsf{iH}_\sigma \vdash A \leftrightarrow (A^-)^{\Box\downarrow}$, by soundness part of the Theorem 9.5.7 we have $\mathsf{HA} \vdash L = \alpha_1 \to \sigma_{\mathsf{HA}}(\neg A)$. Hence by "item 8" we may deduce $\mathbb{N} \not\models \sigma_{\mathsf{HA}}(A)$, as desired. □

9.5.4 Reductions

In this subsection we will show that

$$\mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathbb{N}) \leq^{\text{Prop}} \mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{HA}) \leq^{\text{Prop}} \mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{PA})$$

First some definition:

Definition 9.5.14 For $A \in \mathcal{L}_\Box$ we define $A^{\neg\uparrow}$ and $A^\neg$ as follows:

- $(A \circ B)^\neg := \neg\neg(A^\neg \circ B^\neg)$ and $(A \circ B)^{\neg\uparrow} := \neg\neg(A^{\neg\uparrow} \circ B^{\neg\uparrow})$ for $\circ \in \{\vee, \wedge, \to\}$,
- $(\Box A)^\neg := \neg\neg\Box A^\neg$ and $(\Box A)^{\neg\uparrow} := \neg\neg\Box A$,
- $p^\neg := p^{\neg\uparrow} := \neg\neg p$ for atomic p.

For an arithmetical formula A we have these additional clauses for the definition of $A^\neg$:

- $(\forall x\, A)^\neg := \neg\neg\forall x\, A^\neg$,
- $(\exists x\, A)^\neg := \neg\neg\exists x\, A^\neg$.

Lemma 9.5.15 *For every formula A, we have $\mathsf{PA} \vdash A$ iff $\mathsf{HA} \vdash A^\neg$.*

Proof The direction from right to left is trivial. For the other way around, one may use induction on the proof $\mathsf{PA} \vdash A$. For details see (Troelstra and van Dalen 1988). □

Lemma 9.5.16 *For every Σ_1-formula A, we have $\mathsf{HA} \vdash A^\neg \leftrightarrow \neg\neg A$.*

Proof Easy by use of the decidability of Δ_0-formulas in HA (Lemma 9.3.14). □

Lemma 9.5.17 *For every $A \in \mathcal{L}_\Box$, we have $\mathsf{iH}_\sigma\underline{\mathsf{P}} \vdash A$ iff $\mathsf{iH}_\sigma \vdash A^{\neg\uparrow}$.*

Proof The direction from right to left holds by the classically valid $A \leftrightarrow A^{\neg\uparrow}$. For the other way around, one must use induction on the length of the proof $\mathsf{iH}_\sigma\underline{\mathsf{P}} \vdash A$. All cases are easy and left to the reader. □

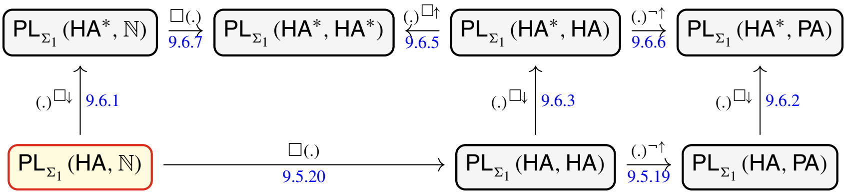

Diagram 9.2 Reductions for relative provability logics of HA^*

Lemma 9.5.18 *For every $A \in \mathcal{L}_\Box$, recursively axiomatizable theory T and any Σ_1-substitution σ, we have* $\mathsf{HA} \vdash (\sigma_T(A))^\neg \leftrightarrow \sigma_T(A^{\neg\uparrow})$.

Proof We use induction on the complexity of A. All cases are simple. For atomic and boxed cases, use Lemma 9.5.16. $\Box$

Theorem 9.5.19 $\mathsf{iH}_\sigma = \mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{HA}) \leq^{Prop} \mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{PA}) = \mathsf{iH}_\sigma\underline{\mathsf{P}}$.

Proof By Theorems 9.5.12 and 9.5.7 we have $\mathsf{iH}_\sigma\underline{\mathsf{P}} = \mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{PA})$ and $\mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{HA}) = \mathsf{iH}_\sigma$. We must show $\mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma; \mathsf{HA}, \mathsf{HA}) \leq^{\text{Prop}}_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma\underline{\mathsf{P}}; \mathsf{HA}, \mathsf{PA})$. Given $A \in \mathcal{L}_\Box$, define $f(A) := A^{\neg\uparrow}$ and observe by Lemma 9.5.17 we have R1 (see Definition 9.4.1). Also define $\bar{f}_A$ as the identity function. Then by Lemmas 9.5.18, 9.5.15 the condition R2 holds. $\Box$

Theorem 9.5.20 $\mathsf{iH}_\sigma\underline{\mathsf{SP}} = \mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathbb{N}) \leq^{Prop} \mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{HA}) = \mathsf{iH}_\sigma$.

Proof By Theorems 9.5.7, 9.5.13 we have $\mathsf{iH}_\sigma = \mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{HA})$ and $\mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathbb{N}) = \mathsf{iH}_\sigma\underline{\mathsf{SP}}$. We must show $\mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma\underline{\mathsf{SP}}; \mathsf{HA}, \mathbb{N}) \leq^{\text{Prop}}_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma; \mathsf{HA}, \mathsf{HA})$. Given $A \in \mathcal{L}_\Box$, define $f(A) = \Box A$ and $\bar{f}_A$ as the identity function.

R1. Let $\mathsf{iH}_\sigma\underline{\mathsf{SP}} \vdash \Box A$. By soundness of $\mathsf{iH}_\sigma\underline{\mathsf{SP}} = \mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathbb{N})$, for every Σ_1-substitution σ we have $\mathbb{N} \models \sigma_{\mathsf{HA}}(\Box A)$ and hence $\mathsf{HA} \vdash \sigma_{\mathsf{HA}}(A)$. Then by arithmetical completeness of $\mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{HA})$, we have $\mathsf{iH}_\sigma \vdash A$.
One also may prove this item with a direct propositional argument. For simplicity reasons, we chose the indirect way.

R2. Let $\mathbb{N} \not\models \sigma_{\mathsf{HA}}(\Box A)$. Then $\mathsf{HA} \nvdash \sigma_{\mathsf{HA}}(A)$, as desired. $\Box$

9.6 Relative Σ_1-provability Logics for HA^*

The σ_1-provability logic of HA^*, $\mathsf{PL}_{\Sigma_1}(\mathsf{HA}^*, \mathsf{HA}^*)$, is already characterised (Ardeshir and Mojtahedi 2019). In this section, we characterise the Σ_1-provability logic of HA^*, relative in PA and $\mathbb{N}$. We also show that reductions in (Diagram 9.2) holds.

Each arrow in the above diagram, indicates a reduction of the completeness of the left hand side to the right one. Note that the diagram of the first row is already known by Theorems 9.5.19 and 9.5.20.

Theorem 9.6.1 $\mathsf{iH}_\sigma\underline{\mathsf{SP}} = \mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathbb{N}) \leq^{Prop} \mathsf{PL}_{\Sigma_1}(\mathsf{HA}^*, \mathbb{N}) = \mathsf{iH}_\sigma\underline{\mathsf{SP}}^*$. (See Definition 9.5.11)

Proof By Theorem 9.5.13 we have $\mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathbb{N}) = \mathsf{iH}_\sigma\underline{\mathsf{SP}}$. It is enough to prove the arithmetical soundness $\mathcal{AS}_{\Sigma_1}(\mathsf{iH}_\sigma\underline{\mathsf{SP}}^*; \mathsf{HA}^*, \mathbb{N})$ and the reduction $\mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma\underline{\mathsf{SP}}; \mathsf{HA}, \mathbb{N}) \leq^{\text{Prop}} \mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma\underline{\mathsf{SP}}^*; \mathsf{HA}^*, \mathbb{N})$.

$\mathcal{AS}_{\Sigma_1}(\mathsf{iH}_\sigma\underline{\mathsf{SP}}^*; \mathsf{HA}^*, \mathbb{N})$: Let $\mathsf{iH}_\sigma\underline{\mathsf{SP}}^* \vdash A$ and σ is a Σ_1-substitution. Then $\mathsf{iH}_\sigma\underline{\mathsf{SP}} \vdash A^{\Box\downarrow}$, and then by arithmetical soundness of $\mathsf{iH}_\sigma\underline{\mathsf{SP}}$ Theorem 9.5.13, we have $\mathbb{N} \models \sigma_{\mathsf{HA}}(A^{\Box\downarrow})$. Hence Lemma 9.3.21 implies $\mathbb{N} \models \sigma_{\mathsf{HA}^*}(A)$, as desired.

For the proof of $\mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma\underline{\mathsf{SP}}; \mathsf{HA}, \mathbb{N}) \leq^{\text{Prop}}_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma\underline{\mathsf{SP}}^*; \mathsf{HA}^*, \mathbb{N})$, define $f(A) := A^{\Box\downarrow}$ and $\bar{f}_A$ as the identity function.

R1. Let $\mathsf{iH}_\sigma\underline{\mathsf{SP}} \vdash A^{\Box\downarrow}$. Then by definition we have $\mathsf{iH}_\sigma\underline{\mathsf{SP}}^* \vdash A$.
R2. Let $\mathbb{N} \not\models \sigma_{\mathsf{HA}^*}(A^{\Box\downarrow})$. Hence by Lemma 9.3.21 $\mathbb{N} \not\models \sigma_{\mathsf{HA}}(A)$, as desired. □

Theorem 9.6.2 $\mathsf{iH}_\sigma\underline{\mathsf{P}} = \mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{PA}) \leq^{Prop} \mathsf{PL}_{\Sigma_1}(\mathsf{HA}^*, \mathsf{PA}) = \mathsf{iH}_\sigma\underline{\mathsf{P}}^*$. (See Definition 9.5.11).

Proof Similar to the proof of Theorem 9.6.1 and left to the reader. □

Theorem 9.6.3 $\mathsf{iH}_\sigma = \mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{HA}) \leq^{Prop} \mathsf{PL}_{\Sigma_1}(\mathsf{HA}^*, \mathsf{HA}) = \mathsf{iH}_\sigma^*$. (See Definition 9.3.30).

Proof Similar to the proof of theorem 9.6.1 and left to the reader. □

Lemma 9.6.4 *For every* $A \in \mathcal{L}_\Box$ *we have* $\mathsf{iH}_\sigma^{**} \vdash A$ *iff* $\mathsf{iH}_\sigma^* \vdash A^{\Box\uparrow}$. (See Definition 9.3.30).

Proof We have the following equivalents: $\mathsf{iH}_\sigma^{**} \vdash A$ iff $\mathsf{iH}_\sigma \vdash A^{\Box}$ iff (by Remark 9.3.5) $\mathsf{iH}_\sigma \vdash (A^{\Box\uparrow})^{\Box\downarrow}$ iff $\mathsf{iH}_\sigma^* \vdash A^{\Box\uparrow}$. □

Theorem 9.6.5 $\mathsf{iH}_\sigma^* = \mathsf{PL}_{\Sigma_1}(\mathsf{HA}^*, \mathsf{HA}) \leq^{Prop} \mathsf{PL}_{\Sigma_1}(\mathsf{HA}^*, \mathsf{HA}^*) = \mathsf{iH}_\sigma^{**}$.

Proof By Theorem 9.6.3 we have $\mathsf{PL}_{\Sigma_1}(\mathsf{HA}^*, \mathsf{HA}) = \mathsf{iH}_\sigma^*$. It is enough to prove the arithmetical soundness $\mathcal{AS}_{\Sigma_1}(\mathsf{iH}_\sigma^{**}; \mathsf{HA}^*, \mathsf{HA}^*)$ and the reduction $\mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma^*; \mathsf{HA}^*, \mathsf{HA}) \leq^{\text{Prop}} \mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma^{**}; \mathsf{HA}^*, \mathsf{HA}^*)$.
$\mathcal{AS}_{\Sigma_1}(\mathsf{iH}_\sigma^{**}; \mathsf{HA}^*, \mathsf{HA}^*)$: Let $\mathsf{iH}_\sigma^{**} \vdash A$ and σ is a Σ_1-substitution. Then $\mathsf{iH}_\sigma \vdash A^{\Box}$, and then by arithmetical soundness of iH_σ Theorem 9.5.7, we have $\mathsf{HA} \vdash \sigma_{\mathsf{HA}}(A^{\Box})$. Hence Lemma 9.3.20 implies $\mathsf{HA} \vdash \sigma_{\mathsf{HA}^*}(A)^{\mathsf{HA}}$, which implies $\mathsf{HA}^* \vdash \sigma_{\mathsf{HA}^*}(A)$.
For the proof of $\mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma\underline{\mathsf{SP}}; \mathsf{HA}, \mathbb{N}) \leq^{\text{Prop}}_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma\underline{\mathsf{SP}}^*; \mathsf{HA}^*, \mathbb{N})$, define $f(A) := A^{\Box\uparrow}$ and $\bar{f}_A$ as the identity function.

R1. Let $\mathsf{iH}_\sigma^* \vdash A^{\Box\uparrow}$. Then by Lemma 9.6.4 we have $\mathsf{iH}_\sigma^{**} \vdash A$, as desired.

R2. Let $\mathsf{HA} \nvdash \sigma_{\mathsf{HA}*}(A^{\Box\uparrow})$. Hence by Lemma 9.3.19 we have $\mathsf{HA} \nvdash (\sigma_{\mathsf{HA}*}(A))^{\mathsf{HA}}$, which implies $\mathsf{HA}^* \nvdash \sigma_{\mathsf{HA}*}(A)$, as desired. □

Theorem 9.6.6 $\mathsf{iH}_\sigma^* = \mathsf{PL}_{\Sigma_1}(\mathsf{HA}^*, \mathsf{HA}) \leq^{Prop} \mathsf{PL}_{\Sigma_1}(\mathsf{HA}^*, \mathsf{PA}) = \mathsf{iH}_\sigma\underline{\mathsf{P}}^*$.

Proof $\mathsf{iH}_\sigma\underline{\mathsf{P}}^* = \mathsf{PL}_{\Sigma_1}(\mathsf{HA}^*, \mathsf{PA})$ and $\mathsf{iH}_\sigma^* = \mathsf{PL}_{\Sigma_1}(\mathsf{HA}^*, \mathsf{HA})$, by Theorems 9.6.2 and 9.6.3 holds. Given A, define $f(A) := A^{\neg\uparrow}$ and $\bar{f}_A$ as the identity function.

R1. By definition of $\mathsf{iH}_\sigma\underline{\mathsf{P}}^*$, we have $\mathsf{iH}_\sigma\underline{\mathsf{P}}^* \vdash A$ iff $\mathsf{iH}_\sigma\underline{\mathsf{P}} \vdash A^{\Box\downarrow}$. The latter, by Lemma 9.5.17 is equivalent to $\mathsf{iH}_\sigma \vdash (A^{\Box\downarrow})^{\neg\uparrow}$. Since $(A^{\Box\downarrow})^{\neg\uparrow} = (A^{\neg\uparrow})^{\Box\downarrow}$, the latter is equivalent to $\mathsf{iH}_\sigma^* \vdash A^{\neg\uparrow}$.
R2. By Lemmas 9.5.18, 9.5.15. □

Theorem 9.6.7 $\mathsf{iH}_\sigma\underline{\mathsf{SP}}^* = \mathsf{PL}_{\Sigma_1}(\mathsf{HA}^*, \mathbb{N}) \leq^{Prop} \mathsf{PL}_{\Sigma_1}(\mathsf{HA}^*, \mathsf{HA}^*) = \mathsf{iH}_\sigma^{**}$.

Proof By Theorems 9.6.1 and 9.6.5 we have $\mathsf{PL}_{\Sigma_1}(\mathsf{HA}^*, \mathbb{N}) = \mathsf{iH}_\sigma\underline{\mathsf{SP}}^*$ and $\mathsf{iH}_\sigma^{**} = \mathsf{PL}_{\Sigma_1}(\mathsf{HA}^*, \mathsf{HA}^*)$. We must show $\mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma\underline{\mathsf{SP}}^*; \mathsf{HA}^*, \mathbb{N}) \leq^{\text{Prop}}_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma^{**}; \mathsf{HA}^*, \mathsf{HA}^*)$. Given $A \in \mathcal{L}_\Box$, define $f(A) = \Box A$ and $\bar{f}_A$ as the identity function.

R1. Let $\mathsf{iH}_\sigma\underline{\mathsf{SP}}^* \vdash \Box A$. By soundness of $\mathsf{iH}_\sigma\underline{\mathsf{SP}}^* = \mathsf{PL}_{\Sigma_1}(\mathsf{HA}^*, \mathbb{N})$, for every Σ_1-substitution σ we have $\mathbb{N} \models \sigma_{\mathsf{HA}*}(\Box A)$ and hence $\mathsf{HA}^* \vdash \sigma_{\mathsf{HA}*}(A)$. Then by arithmetical completeness of $\mathsf{PL}_{\Sigma_1}(\mathsf{HA}^*, \mathsf{HA}^*)$, we have $\mathsf{iH}_\sigma^{**} \vdash A$.
One also may prove this item with a direct propositional argument. For simplicity reasons, we chose the indirect way.
R2. If $\mathbb{N} \not\models \sigma_{\mathsf{HA}*}(\Box A)$ evidently we have $\mathsf{HA}^* \nvdash \sigma_{\mathsf{HA}*}(A)$. □

9.7 Relative Provability Logics for PA

In this section, we characterise $\mathsf{PL}(\mathsf{PA}, \mathsf{HA})$ and $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{HA})$, the provability logic and Σ_1-provability logic of PA relative in HA. We show that $\mathsf{PL}(\mathsf{PA}, \mathsf{HA}) = \mathsf{iGL}\overline{\mathsf{P}}$ and $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{HA}) = \mathsf{iGL}\overline{\mathsf{P}}\mathsf{C}_\mathsf{a}$. Also we show that all of the six (Σ_1-) provability logics of PA relative in PA, HA, $\mathbb{N}$ are reducible to $\mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathbb{N})$ (see Diagram 9.3):

Let us first review some wellspsknown results:

Theorem 9.7.1 *We have the following provability logics:*

- GL *is the provability logic of* PA, *i.e.* $\mathsf{PL}(\mathsf{PA}, \mathsf{PA}) = \mathsf{GL}$ *(Solovay 1976).*
- $\mathsf{GL}\underline{\mathsf{S}}$ *is the truth provability logic of* PA, *i.e.* $\mathsf{PL}(\mathsf{PA}, \mathbb{N}) = \mathsf{GL}\underline{\mathsf{S}}$ *(Solovay 1976).*
- GLC_a *is the* Σ_1*-provability logic of* PA, *i.e.* $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{PA}) = \mathsf{GLC}_\mathsf{a}$ *(Visser 1982).*
- $\mathsf{GL}\underline{\mathsf{S}}\mathsf{C}_\mathsf{a}$ *is the truth* Σ_1*-provability logic of* PA, *i.e.* $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathbb{N}) = \mathsf{GL}\underline{\mathsf{S}}\mathsf{C}_\mathsf{a}$ *(Visser 1982).*

Definition 9.7.2 A propositional modal substitution τ is called $(.)^{\Box\downarrow}$-substitution, if for every atomic variable p, there is some B such that $\mathsf{iK4} + \mathsf{CP}_\mathsf{a} \vdash \tau(p) \leftrightarrow B^{\Box\downarrow}$ and $\mathsf{iK4} \vdash \boxdot B^{\Box\downarrow} \leftrightarrow B^{\Box}$.

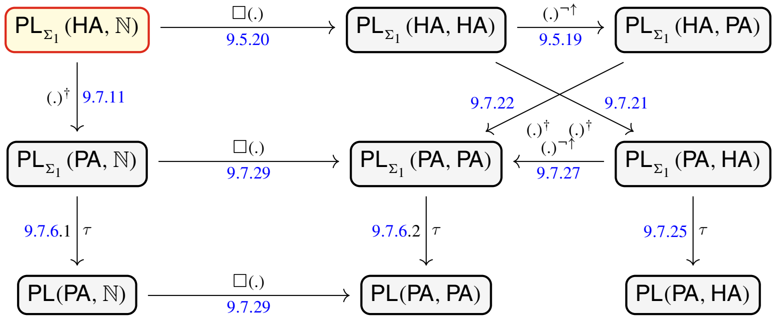

Diagram 9.3 Reductions for relative provability logics of PA

Lemma 9.7.3 *For every* $(.)^{\Box\downarrow}$*-substitution* τ *and every modal proposition* A*, we have* $\mathsf{iK4V} \vdash \tau(A^{\Box}) \leftrightarrow \tau(A)^{\Box}$ *and* $\mathsf{iK4V} \vdash \tau(A^{\Box\downarrow}) \leftrightarrow \tau(A)^{\Box\downarrow}$.

Proof First by induction on the complexity of B we show $\mathsf{iK4V} \vdash \tau(B^{\Box}) \leftrightarrow \tau(B)^{\Box}$. All cases are easy, except for atomic B, which holds by existence of some C such that $\mathsf{iK4V} \vdash \tau(B) \leftrightarrow C^{\Box\downarrow}$ and $\mathsf{iK4} \vdash \boxdot C^{\Box\downarrow} \leftrightarrow C^{\Box}$.

Then we use induction on the complexity of A to deduce the second assertion of this lemma. The only non-trivial cases are atomic and boxed cases:

- A is atomic. Since $\mathsf{iK4} \vdash B^{\Box\downarrow} \leftrightarrow (B^{\Box\downarrow})^{\Box\downarrow}$ for every B, and $\mathsf{iK4V} \vdash \tau(A) \leftrightarrow B^{\Box\downarrow}$, we have the desired result.
- $A = \Box B$. Easily deduced by $\mathsf{iK4V} \vdash \tau(B^{\Box}) \leftrightarrow \tau(B)^{\Box}$.

□

The following remark, will be helpful for later reductions of provability logics in Sect. 9.8.

Remark 9.7.4 For every modal proposition A, $\mathsf{GL} \vdash A$ ($\mathsf{GL\underline{S}} \vdash A$) iff for every $(.)^{\Box\downarrow}$-substitution τ we have $\mathsf{GLC_a} \vdash \tau(A)$ ($\mathsf{GL\underline{S}C_a} \vdash \tau(A)$).

Proof See (Ardeshir and Mojtahedi 2015, Lemmas. 3.1 and 3.3). □

Lemma 9.7.5 *For every* $A \in \mathcal{L}_{\Box}$*,*

- $\mathsf{GL\underline{S}} \vdash \Box A$ *iff* $\mathsf{GL} \vdash A$*,*
- $\mathsf{GL\underline{S}C_a} \vdash \Box A$ *iff* $\mathsf{GLC_a} \vdash A$.

Proof The proof of second item is similar to the first one. Here we only treat the first item. Obviously, $\mathsf{GL} \vdash A$ implies $\mathsf{GL\underline{S}} \vdash \Box A$. For a direct proof of the other way around, one may use of Smoryński's operation. However, now that we enjoy the arithmetical soundness of $\mathsf{PL}(\mathsf{PA}, \mathbb{N}) = \mathsf{GL\underline{S}}$, from $\mathsf{GL\underline{S}} \vdash \Box A$ for every σ we have $\mathbb{N} \models \sigma_{\mathsf{PA}}(\Box A)$ and hence $\mathsf{PA} \vdash \sigma_{\mathsf{PA}}(A)$. From the arithmetical completeness of $\mathsf{GL} = \mathsf{PL}(\mathsf{PA}, \mathsf{PA})$, we get $\mathsf{GL} \vdash A$. □

In the following theorem, we will show that $\mathsf{GL\underline{S}C_a}$ is the hardest provability logic among GL, $\mathsf{GLC_a}$, $\mathsf{GL\underline{S}}$ and $\mathsf{GL\underline{S}C_a}$.

Theorem 9.7.6 *We have the following reductions:*

1. $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathbb{N}) \leq^{Prop} \mathsf{PL}(\mathsf{PA}, \mathbb{N})$,
2. $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{PA}) \leq^{Prop} \mathsf{PL}(\mathsf{PA}, \mathsf{PA})$.

Proof We prove each item separately:

1. We must show that $\mathcal{AC}_{\Sigma_1}(\mathsf{GL\underline{S}C_a}; \mathsf{PA}, \mathbb{N}) \leq^{\text{Prop}}_{f,\bar{f}} \mathcal{AC}(\mathsf{GL\underline{S}}; \mathsf{PA}, \mathbb{N})$. Consider some $A \in \mathcal{L}_\Box$. If $\mathsf{GL\underline{S}} \nvdash A$, by Remark 9.7.4, there exists some $\mathcal{L}_\Box$-substitution τ such that $\mathsf{GL\underline{S}C_a} \nvdash \tau(A)$. Let

$$f(A) := \begin{cases} \tau(A) & : \mathsf{GL\underline{S}} \nvdash A \\ A & : \text{otherwise} \end{cases}$$

   Hence R1 (Definition 9.4.1) holds. Also let $\bar{f}_A(\sigma) := \sigma_{\mathsf{PA}} \circ \tau$, which belongs to $[\![\sigma]\!]$. Then obviously R2 holds.
2. Similar to first item and left to the reader. ☐

9.7.1 Reducing PL_{Σ_1} (PA,$\mathbb{N}$) *to* PL_{Σ_1} (HA,$\mathbb{N}$)

In this subsection, we illustrate how to reduce the arithmetical completeness of $\mathsf{GL\underline{S}C_a}$ to that of $\mathsf{iH_\sigma\underline{SP}}$. First some definitions and lemmas:

Definition 9.7.7 For a modal proposition A let $A^{\rightarrow}$ indicate a classically equivalent formula of the form

$$A^{\rightarrow} := \bigwedge_i (B_i \to C_i) \quad \text{in which} \quad B_i = \bigwedge_j E_{i,j} \text{ and } C_i = \bigvee_j F_{i,j}$$

such that $E_{i,j}$, $F_{i,j}$ are atomic or boxed formulas occurring in A. Then define $A^{\ddagger}$ as follows: first compute $A^{\rightarrow}$ and then replace every outer occurrences of boxed subformulas $\Box B$ in $A^{\rightarrow}$ with $\Box(B^{\ddagger})$. Note that since $\Box B$ also occurs in A, then B has lower complexity than A and hence this is a valid inductive definition. Also define $A^{\dagger}$ as follows:

- $(.)^{\dagger}$ commutes with $\vee, \wedge, \to$,
- $p^{\dagger} = p$ for atomic p,
- $(\Box A)^{\dagger} = \Box A^{\ddagger}$

Lemma 9.7.8 *For every modal proposition A and arithmetical substitution α, we have*

$$\mathsf{HA} \vdash \alpha_{\mathsf{HA}}(A^{\dagger}) \leftrightarrow \alpha_{\mathsf{PA}}(A)$$

Proof Easy and left to the reader. □

Lemma 9.7.9 *For every $A \in \mathcal{L}_\Box$, if* $\mathsf{iH}_\sigma\underline{\mathsf{SP}} \vdash A^\dagger$ *then* $\mathsf{GL\underline{S}C_a} \vdash A$.

Proof Let $\mathsf{GL\underline{S}C_a} \nvdash A$. Since in classical logic we have $A \leftrightarrow A^\dagger$, then $\mathsf{GL\underline{S}C_a} \nvdash A^\dagger$. Hence by $\mathcal{AC}_{\Sigma_1}(\mathsf{GL\underline{S}C_a}; \mathsf{PA}, \mathbb{N})$ from 9.7.1, we have some Σ_1-substitution σ such that $\mathbb{N} \not\models \sigma_{\mathsf{PA}}(A^\dagger)$. Then Lemma 9.7.8 implies $\mathbb{N} \not\models \sigma_{\mathsf{HA}}(A^\dagger)$, and hence by arithmetical soundness of $\mathsf{iH}_\sigma\underline{\mathsf{SP}}$ (Theorem 9.5.13) we have $\mathsf{iH}_\sigma\underline{\mathsf{SP}} \nvdash A^\dagger$, as desired. □

Lemma 9.7.10 *For every $A \in \mathcal{L}_\Box$, if* $\mathsf{iH}_\sigma\underline{\mathsf{P}} \vdash A^\dagger$ *then* $\mathsf{GLC_a} \vdash A$.

Proof Let $\mathsf{GLC_a} \nvdash A$. Since in classical logic we have $A \leftrightarrow A^\dagger$, then $\mathsf{GLC_a} \nvdash A^\dagger$. Hence by $\mathcal{AC}_{\Sigma_1}(\mathsf{GLC_a}; \mathsf{PA}, \mathsf{PA})$ from 9.7.1, we have some Σ_1-substitution σ such that $\mathsf{PA} \nvdash \sigma_{\mathsf{PA}}(A^\dagger)$. Then Lemma 9.7.8 implies $\mathsf{PA} \nvdash \sigma_{\mathsf{HA}}(A^\dagger)$, and hence by arithmetical soundness of $\mathsf{iH}_\sigma\underline{\mathsf{P}}$ (Theorem 9.5.12) we have $\mathsf{iH}_\sigma\underline{\mathsf{P}} \nvdash A^\dagger$, as desired. □

Theorem 9.7.11 $\mathsf{iH}_\sigma\underline{\mathsf{SP}} = \mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathbb{N}) \leq^{Prop} \mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathbb{N}) = \mathsf{GL\underline{S}C_a}$.

Proof By Theorems 9.5.13, 9.7.1 we have $\mathsf{iH}_\sigma\underline{\mathsf{SP}} = \mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathbb{N})$ and $\mathsf{GL\underline{S}C_a} = \mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathbb{N})$. For the reduction, let $f(A) := A^\dagger$ and $\bar{f}_A$ as the identity function.

R1. If $\mathsf{iH}_\sigma\underline{\mathsf{SP}} \vdash A^\dagger$, by Lemma 9.7.9 we have $\mathsf{GL\underline{S}C_a} \vdash A$.
R2. Holds by Lemma 9.7.8. □

9.7.2 Kripke Semantics

Let $\mathsf{Suc}_\mathcal{K}$ or simply Suc, when no confusion is likely, indicate the set of all $\sqsubset$-accessible nodes in the Kripke model $\mathcal{K}$.

Theorem 9.7.12 $\mathsf{iGL\overline{P}}$ *is sound and complete for semi-perfect* Suc*-classical* $\sqsubset$*-branching Kripke models.*

Proof The soundness is easy and left to the reader. For the completeness, we first show the completeness for finite brilliant irreflexive transitive Suc-classical Kripke models. Let $\mathsf{iGL\overline{P}} \nvdash A$. Let

$$X := \{B, \neg B, B \vee \neg B : B \in \mathsf{Sub}(A)\} \cup \{\bot\}$$

and define the Kripke model $\mathcal{K} = (K, \preccurlyeq, \sqsubset, V)$ as follows:

- K is the family of all X-saturated sets with respect to $\mathsf{iGL\overline{P}}$.
- $\alpha \preccurlyeq \beta$ iff $\alpha \subseteq \beta$.
- $\alpha \sqsubset \beta$ iff β is a maximally consistent set and $\{B, \Box B : \Box B \in \alpha\} \subseteq \beta$ and there is some $\Box B \in \beta \setminus \alpha$.

It is straightforward to show that $\mathcal{K}$ is actually a finite brilliant irreflexive Suc-classical Kripke model, and we leave all of them to the reader.

It is enough to show that $\mathcal{K}, \alpha \Vdash B$ iff $B \in \alpha$ for every $\alpha \in K$ and $B \in X$. Then we may use Lemma 9.3.39 and find some α such that $\mathcal{K}, \alpha \nVdash A$. We use induction on the complexity of $B \in X$. All inductive steps are trivial, except for $B = \Box C$. If $\Box C \in \alpha$ and $\alpha \sqsubset \beta$, then by definition, $C \in \beta$ and hence by induction hypothesis $\beta \Vdash C$. This implies $\alpha \Vdash \Box C$. For the other way around, let $\Box C \notin \alpha$. Consider the set $\Delta := \{E, \Box E : \Box E \in \alpha\}$. If $\mathsf{GL} \vdash \bigwedge \Delta \to (\Box C \to C)$, then $\mathsf{iGL} + \Box\mathsf{PEM} \vdash \Box(\bigwedge \Delta) \to \Box C$. Since $\mathsf{iGL\overline{P}} \vdash \alpha \to \Box \bigwedge \Delta$ and $\mathsf{iGL\overline{P}} \vdash \Box\mathsf{PEM}$, we have $\mathsf{iGL\overline{P}} + \alpha \vdash \Box C$ and hence $\Box C \in \alpha$, a contradiction. Hence we have $\mathsf{GL} \nvdash (\bigwedge \Delta \wedge \Box C) \to C$. Then by Lemma 9.3.39 there is some X-saturated set $\beta \supseteq \Delta \cup \{\Box C\} \cup \{E \vee \neg E : E \in \mathsf{Sub}(A)\}$ such that $C \notin \beta$. Hence $\beta \sqsupset \alpha$ and $\beta \nVdash C$. Then $\alpha \nVdash \Box C$, as desired.

Next we use the construction method (Iemhoff 2001), to fulfil the other conditions: $\sqsubset$-branching, neat and tree. Let $\mathcal{K}_t := (K_t, \preccurlyeq_t, \sqsubset_t, V_t)$ as follows:

- K_t is the set of all finite sequences of pairs $r := \langle(\alpha_0, a_0), \ldots (\alpha_n, a_n)\rangle$ such that for any $i \leq n$: (1) $\alpha_i \in K$, (2) $a_i \in \{0, 1\}$, (3) for $i < n$ either we have $\alpha_i \prec \alpha_{i+1}$ or $\alpha_i \sqsubset \alpha_{i+1}$. Let $f_1(r)$ and $f_2(r)$ indicate the left and right elements in the final element of the sequence r. In other words, we let $(f_1(r), f_2(r))$ be the final element of the sequence r.
- $r \preccurlyeq_t s$ iff r is an initial segment of s and $f_1(r) \preccurlyeq f_1(s)$.
- $r \sqsubset_t s$ iff r is an initial segment of $s = \langle(\alpha_0, a_0), \ldots, (\alpha_n, a_n)\rangle$, e.g. $r = \langle(\alpha_0, a_0), \ldots, (\alpha_k, a_k)\rangle$ for some $k < n$ and $\alpha_i \sqsubset \alpha_{i+1}$ for some $k \leq i < n$.
- $r\, V_t\, p$ iff $f_1(r)\, V\, p$.

It is straightforward to show that $\mathcal{K}_t$ is semi-perfect $\sqsubset$-branching Suc-classical Kripke model and for every $r \in K_t$ and formula B we have

$$\mathcal{K}_t, r \Vdash B \quad \Longleftrightarrow \quad \mathcal{K}, f(r) \Vdash B.$$

□

Theorem 9.7.13 $\mathsf{iGL\overline{P}C_a}$ *is sound and complete for semi-perfect* Suc*-classical atom-complete Kripke models.*

Proof The proof is almost identical to the one for Theorem 9.7.12. We only explain the differences here. Define

$$X := \{B, \neg B, B \vee \neg B : B \in \mathsf{Sub}(A)\} \cup \{\bot\} \cup \{\Box p : p \in \mathsf{Sub}(A) \text{ and } p \text{ is atomic}\}$$

and K, the set of the nodes of Kripke model, is defined as the set of all X-saturated sets with respect to $\mathsf{iGL\overline{P}C_a}$. We show that every $\alpha \in K$ is atom-complete. Let p be an atomic variable such that $\alpha \Vdash p$. Hence $p \in \alpha$ which implies $p \in \mathsf{Sub}(A)$, and since $\mathsf{iGL\overline{P}C_a} \vdash p \to \Box p$ and α is closed under deduction, we have $\Box p \in \alpha$. Then $\alpha \Vdash \Box p$ and hence for every $\beta \sqsupset \alpha$ we have $\beta \Vdash p$, as desired. □

9.7.3 Arithmetical Completeness

Theorem 9.7.14 $\mathsf{iGL\overline{P}C_a}$ *is the relative* Σ_1*-provability logic of* PA *in* HA*, i.e.* $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{HA}) = \mathsf{iGL\overline{P}C_a}$.

Proof The soundness is straightforward and left to the reader. For the completeness part, let $\mathsf{iGL\overline{P}C_a} \nvdash A$. Then by Theorem 9.7.13, there is some semi-perfect atom-complete Suc-classical Kripke model $\mathcal{K} = (K, \preccurlyeq, \sqsubset, V)$ such that $\mathcal{K}, \alpha_0 \nVdash A$ for some $\alpha_0 \in K$. Without loss of generality, we may assume that $K = (\alpha_0 \preccurlyeq) \cup (\alpha_0 \sqsubset)$. Let $\mathcal{K}' = (K', \preccurlyeq', \sqsubset', V')$ indicate the Smorýnski's extension of $\mathcal{K}$ at α_0 with the fresh node α_1. For the simplicity of notations, we may use $\preccurlyeq$ and $\sqsubset$ instead of $\preccurlyeq'$ and $\sqsubset'$. Define the recursive function F as follows. Since K' is a finite set, we might assign a unique number $\bar{\alpha}$ to each node α and speak about K' and its relationships $\preccurlyeq$ and $\sqsubset$ inside the language of arithmetic. For simplicity of notations, we may simply use $\alpha \preccurlyeq \beta$ and $\alpha \sqsubset \beta$ corresponding to its equivalent arithmetical formula.

Define $F(0) := \alpha_1$ and

$$F(n+1) := \begin{cases} \beta & : F(n) \sqsubset \beta \quad \text{and} \quad r(\beta, n+1) < n+1 \quad \text{and} \quad (n)_0 = \beta \\ \beta & : F(n) \prec \beta \quad \text{and} \quad F(n) \not\sqsubset \beta \quad \text{and} \quad F(r(\beta, n+1)) = \alpha_1 \\ & \quad \text{and} \quad r(\beta, n+1) < r(F(n), n+1) \quad \text{and} \quad (n)_0 = \beta \\ F(n) & : \text{otherwise} \end{cases}$$

in which $L = \beta$ is shorthand for $\exists x \forall y \geq x (F(y) = F(x))$, $(n)_0$ is the exponent of 2 in n and

$$r(\alpha, n) := \min\left(\{x \in \mathbb{N} : \exists t \leq n\, \mathsf{Proof}_{\mathsf{PA}_x}(t, \ulcorner L \neq \alpha \urcorner)\} \cup \{n\}\right)$$

Note that $r(\alpha, n) < n$ implies $\Box^+(L \neq \alpha)$. F is a provably total recursive function in HA, i.e. $F(x) = y$ could be expressed as a Σ_1-formula in the language of arithmetic and all of its expected properties are provable in HA. Hence we may use the function symbol F in the language of arithmetic.

Define the arithmetical substitution $\sigma(p)$ in this way:

$$\sigma(p) := \bigvee_{\mathcal{K}, \alpha \Vdash p} \exists x\, F(x) = \alpha$$

Consider the triple $\mathcal{I} := (K^*, \preccurlyeq^*, T)$ as follows:

- $K^* := \{\alpha \in K : \nexists \beta \in K(\beta \sqsubset \alpha)\}$.
- $\alpha \preccurlyeq^* \beta$ iff $\alpha \preccurlyeq \beta$ for every $\alpha, \beta \in K^*$. Again, by abuse of notations, we use $\preccurlyeq$ instead of $\preccurlyeq^*$.
- $T(\alpha) := \mathsf{PA} + (L = \alpha)$.

By Theorem 9.3.26 and Lemma 9.7.16, we have some first-order Kripke model $\mathcal{K}^* = (K^*, \preccurlyeq, \mathfrak{M})$ such that $\mathcal{K}^* \Vdash \mathsf{HA}$ and $\mathcal{K}^*, \alpha \models T(\alpha)$. By Lemma 9.3.23

$$\mathcal{K}^*, \alpha \Vdash \exists x\, F(x) = \beta \implies \beta \preccurlyeq \alpha \tag{9.1}$$

Hence by Lemma 9.3.23, for every $\alpha \in K^*$

$$\mathcal{K}^*, \alpha \Vdash \sigma_{\mathsf{PA}}(p) \iff \mathcal{K}, \alpha \Vdash p \tag{9.2}$$

For every classical node $\alpha \in K^*$, since the Kripke model above α is just a classical Kripke model, one may repeat the Solovay's argument and show that for every modal proposition B we have

$$\begin{cases} \mathcal{K}, \alpha \Vdash B \implies \mathsf{PA} \vdash L = \alpha \to \sigma_{\mathsf{PA}}(B) \\ \mathcal{K}, \alpha \nVdash B \implies \mathsf{PA} \vdash L = \alpha \to \neg\sigma_{\mathsf{PA}}(B) \end{cases} \tag{9.3}$$

We may use Lemmas 9.7.18, 9.7.19 and Eq. 9.2 to conclude

$$\mathcal{K}^*, \alpha \Vdash \sigma_{\mathsf{PA}}(B) \iff \mathcal{K}, \alpha \Vdash B$$

for every modal proposition B and $\alpha \in K^*$. Since $\mathcal{K}, \alpha_0 \nVdash A$, we have $\mathcal{K}^*, \alpha \nVdash \sigma_{\mathsf{PA}}(A)$, and hence $\mathsf{HA} \nvdash \sigma_{\mathsf{PA}}(A)$, as desired. □

Lemma 9.7.15 *For arbitrary $\alpha, \beta \in K'$ we have*

1. $\mathsf{PA} \vdash \exists x F(x) = \alpha \to \bigvee_{\alpha(\preccurlyeq \cup \sqsubseteq)\beta} L = \beta$,
2. $\mathsf{PA} \vdash L = \alpha \to \neg\Box^+(L \neq \beta)$*, for every* $\alpha \sqsubset \beta$,
3. $\mathsf{PA} \vdash (L = \alpha) \rhd (L = \beta)$*, for every* $\alpha \prec \beta$,
4. $\mathbb{N} \models L = \alpha_1$,
5. $\mathsf{PA} \vdash L = \alpha \to \Box^+(L \neq \alpha \wedge \exists x F(x) = \alpha)$*, for every* $\alpha \neq \alpha_1$.

Proof All proofs are straightforward and left to the reader. □

Lemma 9.7.16 *$\mathcal{I}$, as defined in the proof of Theorem 9.7.14, is an I-frame* (see Definition 9.3.25).

Proof Use Theorem 9.3.24 and the items 2,3 and 4, of Lemma 9.7.15. □

Lemma 9.7.17 *For every $\alpha \in K$ we have* $\mathsf{PA} \vdash L = \alpha \to \Box^+(\bigvee_{\alpha \sqsubset \beta} L = \beta)$.

Proof It is enough to show that $\mathsf{PA} \vdash L = \alpha \to \Box^+(L \neq \beta)$ for every $\beta \succcurlyeq \alpha$ such that $\beta \not\sqsupseteq \alpha$, holds. Consider some $\beta \succcurlyeq \alpha$ with $\beta \not\sqsupseteq \beta$. If $\beta = \alpha$, by item 5 in Lemma 9.7.15 we have the desired result. So we may let $\beta \neq \alpha$. We reason inside PA. Let $L = \alpha$. Hence for some x we have $F(x) = \alpha$. Then we reason inside $\Box^+$. By Σ_1-completeness of PA (seeLemma 9.3.13), we have $F(x) = \alpha$. Assume that $L = \beta$. Let x_0 be the first number such that $F(x_0) = \beta$. Hence for some r such that $\Box_r^+(L \neq \beta)$ holds, we have $F(r) = \alpha_1$. Then $r \leq x$ and hence by Lemma 9.3.12 we may deduce $L \neq \beta$, in contradiction with $L = \beta$. □

Lemma 9.7.18 *For every α in K and proposition B we have*

$$\mathcal{K}, \alpha \Vdash \Box B \quad \Longrightarrow \quad \mathsf{PA} \vdash L = \alpha \rightarrow \sigma_{\mathsf{PA}}(\Box B) \tag{9.4}$$

Proof Let $\mathcal{K}, \alpha \Vdash \Box B$. Hence for every $\beta \sqsupset \alpha$ we have $\mathcal{K}, \beta \Vdash B$. Since every $\beta \sqsupset \alpha$ is classical, by Eq. 9.3 we have $\mathsf{PA} \vdash \bigvee_{\alpha \sqsubset \beta} L = \beta \rightarrow \sigma_{\mathsf{PA}}(B)$. Hence $\mathsf{PA} \vdash \Box^+(\bigvee_{\alpha \sqsubset \beta} L = \beta) \rightarrow \sigma_{\mathsf{PA}}(\Box B)$. Lemma 9.7.17 implies $\mathsf{PA} \vdash L = \alpha \rightarrow \sigma_{\mathsf{PA}}(\Box B)$. $\Box$

Lemma 9.7.19 *For every α in K and proposition B we have*

$$\mathcal{K}, \alpha \nVdash \Box B \quad \Longrightarrow \quad \mathsf{PA} \vdash L = \alpha \rightarrow \neg\sigma_{\mathsf{PA}}(\Box B) \tag{9.5}$$

Proof Let $\mathcal{K}, \alpha \nVdash \Box B$. Hence for every $\beta \sqsupset \alpha$ we have $\mathcal{K}, \beta \nVdash B$. Since every $\beta \sqsupset \alpha$ is classical, by Eq. 9.3 we have $\mathsf{PA} \vdash L = \beta \rightarrow \neg\sigma_{\mathsf{PA}}(B)$. Hence $\mathsf{PA} \vdash \sigma_{\mathsf{PA}}(B) \rightarrow L \neq \beta$ and then $\mathsf{PA} \vdash \Box^+\sigma_{\mathsf{PA}}(B) \rightarrow \Box^+ L \neq \beta$ and $\mathsf{PA} \vdash \neg\Box^+ L \neq \beta \rightarrow \neg\Box^+\sigma_{\mathsf{PA}}(B)$. Hence item 2 of Lemma 9.7.15 implies $\mathsf{PA} \vdash L = \alpha \rightarrow \neg\Box^+\sigma_{\mathsf{PA}}(B)$. $\Box$

9.7.4 Reductions

Lemma 9.7.20 *For every $A \in \mathcal{L}_\Box$, if $\mathsf{iH}_\sigma \vdash A^\dagger$ then $\mathsf{iGL\overline{P}C_a} \vdash A$.*

Proof Let $\mathsf{iGL\overline{P}C_a} \nvdash A$. Since in $\mathsf{iK4} + \Box\mathsf{PEM}$ we have $A \leftrightarrow A^\dagger$, then $\mathsf{iGL\overline{P}C_a} \nvdash A^\dagger$. Hence by $\mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{P}C_a}; \mathsf{PA}, \mathsf{PA})$ from Theorem 9.7.14, we have some Σ_1-substitution σ such that $\mathsf{HA} \nvdash \sigma_{\mathsf{PA}}(A^\dagger)$. Then Lemma 9.7.8 implies $\mathsf{HA} \nvdash \sigma_{\mathsf{HA}}(A^\dagger)$, and hence by arithmetical soundness of iH_σ (Theorem 9.5.7) we have $\mathsf{iH}_\sigma \nvdash A^\dagger$, as desired. $\Box$

Theorem 9.7.21 $\mathsf{iH}_\sigma = \mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{HA}) \leq^{Prop} \mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{HA}) = \mathsf{iGL\overline{P}C_a}$.

Proof The soundness of $\mathsf{iGL\overline{P}C_a}$ is straightforward and left to the reader. Also by Theorem 9.5.7, we have $\mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{HA}) = \mathsf{iH}_\sigma$. So, it is enough to show $\mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma; \mathsf{HA}, \mathsf{HA}) \leq^{\mathrm{Prop}}_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{P}C_a}; \mathsf{PA}, \mathsf{HA})$.
Define $f(A) := A^\dagger$ and $\bar{f}_A$ as the identity function.

R1. Use Lemma 9.7.10.
R2. Use Lemma 9.7.8. $\Box$

Theorem 9.7.22 $\mathsf{iH}_\sigma\underline{\mathsf{P}} = \mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{PA}) \leq^{Prop} \mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{PA}) = \mathsf{GLC_a}$.

Proof We already have $\mathsf{GLC_a} = \mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{PA})$ and $\mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{PA}) = \mathsf{iH}_\sigma\underline{\mathsf{P}}$ by Theorem 9.5.12,9.7.1. So, it is enough to show $\mathcal{AC}_{\Sigma_1}(\mathsf{iH}_\sigma\underline{\mathsf{P}}; \mathsf{HA}, \mathsf{PA}) \leq^{\mathrm{Prop}}_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{GLC_a}; \mathsf{PA}, \mathsf{PA})$.
Define $f(A) := A^\dagger$ and $\bar{f}_A$ as the identity function.

R1. Let $\mathsf{iH}_\sigma\underline{\mathsf{P}} \vdash A^\dagger$. By Lemma 9.7.10 we have $\mathsf{GLC_a} \vdash A$.
R2. Use Lemma 9.7.8. □

The arithmetical completeness of $\mathsf{iGL\overline{P}}$ will be reduced to the one for $\mathsf{iGL\overline{P}C_a}$ via the following lemma. This argument is similar to the one explained in (Ardeshir and Mojtahedi 2015). One may use a direct proof for the arithmetical completeness of $\mathsf{iGL\overline{P}}$, similar to what we do for $\mathsf{iGL\overline{P}C_a}$. However this is not enough for our later use in Sect. 9.8 of the arithmetical completeness of $\mathsf{iGL\overline{P}}$.

Lemma 9.7.23 *For every modal proposition A, $\mathsf{iGL\overline{P}} \vdash A$ iff for every propositional modal $(.)^{\Box\downarrow}$-substitution τ (Definition 9.7.2) we have $\mathsf{iGL\overline{P}C_a} \vdash \tau(A)$.*

Proof One direction holds since $\mathsf{iGL\overline{P}}$ is closed under substitutions and is included in $\mathsf{iGL\overline{P}C_a}$. For the other way around, let $\mathsf{iGL\overline{P}} \nvdash A$. By Theorem 9.7.12, there is some Suc-classical, semi-perfect $\sqsubset$-branching Kripke model $\mathcal{K} = (K, \preccurlyeq, \sqsubset, V)$ such that $\mathcal{K} \nVdash A$. For every $\alpha \in K$, let $p_{_\alpha}$ be a fresh atomic variable such that for every $\alpha \neq \beta$ we have $p_{_\alpha} \neq p_{_\beta}$. For every $\alpha \in K$, define A_α via induction on the $\prec$-height of α (the maximum number n such that a sequence $\alpha = \alpha_0 \prec \ldots \prec \alpha_n$ exists). So as induction hypothesis, let A_β for every $\beta \succ \alpha$ is defined.

$$A_\alpha^+ := \bigvee_{\alpha \prec \beta} A_\beta \quad , \quad A_\alpha := p_{_\alpha} \wedge \bigwedge_{\alpha \sqsubseteq \beta} \Box\neg\boxdot p_{_\beta} \to A_\alpha^+$$

Let $\bar{\mathcal{K}} = (K, \preccurlyeq, \sqsubset, \bar{V})$, in which $\alpha \bar{V} p$ iff $p = p_{_\beta}$ for some $\beta(\preccurlyeq \cup \sqsubset)\alpha$. Define

$$\tau(p) := \bigvee_{\mathcal{K},\alpha \Vdash p} A_\alpha$$

Then by induction on the complexity of the modal proposition B, we show

$$\mathcal{K}, \alpha \Vdash B \iff \bar{\mathcal{K}}, \alpha \Vdash \tau(B)$$

- B is atomic variable: For every $\alpha \in K$ such that $\mathcal{K}, \alpha \Vdash B$, by Lemma 9.7.24 we have $\bar{\mathcal{K}}, \alpha \Vdash A_\alpha$ and hence $\bar{\mathcal{K}}, \alpha \Vdash \tau(p)$. Also if $\bar{\mathcal{K}}, \alpha \Vdash \tau(B)$, then for some $\beta \in K$ we have $\mathcal{K}, \beta \Vdash B$ and $\bar{K}, \alpha \Vdash A_\beta$. Hence by Lemma 9.7.24 we have $\beta \preccurlyeq \alpha$, which implies $\mathcal{K}, \alpha \Vdash B$, as desired.
- All the other cases are trivial and left to the reader.

Then we have $\bar{\mathcal{K}} \nVdash \tau(A)$. Obviously the Kripke model $\bar{\mathcal{K}}$ inherits all properties from $\mathcal{K}$ and moreover it is atom-complete. Hence by soundness part of the Theorem 9.7.13, $\mathsf{iGL\overline{P}C_a} \nvdash \tau(A)$, as desired. □

Lemma 9.7.24 *Let $\bar{\mathcal{K}}$ and A_α, as defined in the proof of Lemma 9.7.23. For every $\alpha, \beta \in K$ we have $\bar{\mathcal{K}}, \alpha \Vdash A_\beta$ iff $\alpha \succcurlyeq \beta$.*

Proof We use induction on the $\prec$-height of β. As induction hypothesis, let for every $\beta \succ \beta_0$ and $\alpha \in K$ we have $\bar{\mathcal{K}}, \alpha \Vdash A_\beta$ iff $\beta \succcurlyeq \alpha$. Note that by induction hypothesis we have $\bar{\mathcal{K}}, \beta \Vdash A_{\beta_0}^+$ iff $\beta \succ \beta_0$.

- ($\alpha \succcurlyeq \beta_0$ implies $\bar{\mathcal{K}}, \alpha \Vdash A_{\beta_0}$): It is enough to show that $\bar{\mathcal{K}}, \beta_0 \Vdash A_{\beta_0}$. Then for evey $\alpha \succcurlyeq \beta_0$ we have $\bar{\mathcal{K}}, \alpha \Vdash A_{\beta_0}$, as desired. By definition of $\bar{\mathcal{K}}$, we have $\bar{\mathcal{K}}, \beta_0 \Vdash p_{\beta_0}$. Consider some $\gamma \sqsupseteq \beta_0$. Again by definition of $\bar{\mathcal{K}}$, we have $\bar{\mathcal{K}}, \beta_0 \nVdash \Box\neg p_\gamma$ and for every $\delta \succ \beta_0$ we have $\bar{\mathcal{K}}, \delta \Vdash A^+_{\beta_0}$. Hence $\bar{\mathcal{K}}, \beta_0 \Vdash \Box\neg p_\gamma \to A^+_{\beta_0}$. This argument shows that $\bar{\mathcal{K}}, \beta_0 \Vdash A_{\beta_0}$, as desired.
- ($\bar{\mathcal{K}}, \alpha \Vdash A_{\beta_0}$ implies $\alpha \succcurlyeq \beta_0$): Let $\bar{\mathcal{K}}, \alpha \Vdash A_{\beta_0}$. Since $\bar{\mathcal{K}}, \alpha \Vdash p_{\beta_0}$, we have $\beta_0(\preccurlyeq \cup \sqsubset)\alpha$. If $\beta_0 \preccurlyeq \alpha$, we are done. So let $\beta_0 \not\preccurlyeq \alpha$ and $\beta_0 \sqsubset \alpha$. Hence for arbitrary $\gamma \sqsupseteq \beta_0$ we have $\bar{\mathcal{K}}, \alpha \Vdash \neg\Box\neg p_\gamma$. This by Suc-classicality, implies that there is some $\delta \sqsupset \alpha$ such that $\bar{\mathcal{K}}, \delta \Vdash p_\gamma$. Then we have $\gamma(\preccurlyeq \cup \sqsubset)\delta$. By Suc-classicality, we have $\gamma \sqsubseteq \delta$. Since $\bar{\mathcal{K}}$ is with tree frame, we have either $\alpha \sqsubseteq \gamma$ or $\gamma \sqsubseteq \alpha$. On the other hand, since $\bar{\mathcal{K}}$ is $\sqsubset$-branching, there must be some $\gamma \sqsupset \beta_0$ which is $\sqsubset$-incomparable with α, a contradiction with our previous argument. □

Theorem 9.7.25 $\mathsf{iGL\overline{P}C_a} = \mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{HA}) \leq^{Prop} \mathsf{PL}(\mathsf{PA}, \mathsf{HA}) = \mathsf{iGL\overline{P}}$.

Proof The arithmetical soundness of $\mathsf{iGL\overline{P}}$ is straightforward and left to the reader. Also by Eq. 9.7.21 we have $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{HA}) = \mathsf{iGL\overline{P}C_a}$. It remains to show

$$\mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{P}}; \mathsf{PA}, \mathsf{HA}) \leq^{\rm Prop}_{f,\bar{f}} \mathcal{AC}(\mathsf{iGL\overline{P}}; \mathsf{PA}, \mathsf{HA})$$

Let $A \in \mathcal{L}_\Box$ such that $\mathsf{iGL\overline{P}} \nvdash A$. Then by Lemma 9.7.23 there is some substitution τ such that $\mathsf{iGL\overline{P}C_a} \nvdash \tau(A)$. Define the function f as follows:

$$f(A) := \begin{cases} \tau(A) & : \mathsf{iGL\overline{P}} \nvdash A \\ \text{whatever you like} & : \text{otherwise} \end{cases}$$

Also let $\bar{f}_A(\sigma) := \sigma_{\mathsf{PA}} \circ \tau$. Then one may easily observe that R0, R1 and R3 holds for this f, $\bar{f}$. □

Lemma 9.7.26 *For* $A \in \mathcal{L}_\Box$, *if* $\mathsf{iGL\overline{P}C_a} \vdash A^{\neg\uparrow}$ *then* $\mathsf{GLC_a} \vdash A$.

Proof Let $\mathsf{iGL\overline{P}C_a} \vdash A^{\neg\uparrow}$. Then $\mathsf{GLC_a} \vdash A^{\neg\uparrow}$ and since $A^{\neg\uparrow}$ is classically equivalent to A we have $\mathsf{GLC_a} \vdash A$. □

Theorem 9.7.27 $\mathsf{iGL\overline{P}C_a} = \mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{HA}) \leq^{Prop} \mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{PA}) = \mathsf{GLC_a}$.

Proof By Theorems 9.7.1 and 9.7.21 we have $\mathsf{GLC_a} = \mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{PA})$ and $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{HA}) = \mathsf{iGL\overline{P}C_a}$. We must show $\mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{P}C_a}; \mathsf{PA}, \mathsf{HA}) \leq^{\rm Prop}_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{GLC_a}; \mathsf{PA}, \mathsf{PA})$. Given $A \in \mathcal{L}_\Box$, define $f(A) := A^{\neg\uparrow}$ and $\bar{f}_A$ as the identity function.

R1. If $\mathsf{iGL\overline{P}C_a} \vdash A^{\neg\uparrow}$, then by Lemma 9.7.26 we have $\mathsf{GLC_a} \vdash A$.
R2. Holds by Lemmas 9.5.18, 9.5.15. □

Theorem 9.7.28 $\mathsf{GL\underline{S}C_a} = \mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathbb{N}) \leq^{Prop} \mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{HA}) = \mathsf{iGL\overline{P}C_a}$.

Proof By Theorems 9.7.1 and 9.7.25 we have $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathbb{N}) = \mathsf{GL\underline{S}C_a}$ and $\mathsf{iGL\overline{P}C_a} = \mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{HA})$. We must show $\mathcal{AC}_{\Sigma_1}(\mathsf{GL\underline{S}C_a}; \mathsf{PA}, \mathbb{N}) \leq^{\text{Prop}}_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{P}}; \mathsf{PA}, \mathsf{HA})$. Given $A \in \mathcal{L}_\Box$, define $f(A) = \Box A$ and $\bar{f}_A$ as the identity function.

R1. Let $\mathsf{GL\underline{S}C_a} \vdash \Box A$. By soundness of $\mathsf{iH_\sigma \underline{SP}} = \mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathbb{N})$, for every Σ_1-substitution σ we have $\mathbb{N} \models \sigma_{\mathsf{HA}}(\Box A)$ and hence $\mathsf{HA} \vdash \sigma_{\mathsf{HA}}(A)$. Then by arithmetical completeness of $\mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{HA})$, we have $\mathsf{iH_\sigma} \vdash A$.
One also may prove this item with a direct propositional argument. For simplicity reasons, we chose the indirect way.

R2. Let $\mathbb{N} \not\models \sigma_{\mathsf{HA}}(\Box A)$. Then $\mathsf{HA} \nvdash \sigma_{\mathsf{HA}}(A)$, as desired. □

Theorem 9.7.29 $\mathsf{GL\underline{S}C_a} = \mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathbb{N}) \leq^{Prop} \mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{PA}) = \mathsf{GLC_a}$.

Proof By Theorem 9.7.1 we have $\mathsf{GLC_a} = \mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{PA})$ and $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathbb{N}) = \mathsf{GL\underline{S}C_a}$. We must show $\mathcal{AC}_{\Sigma_1}(\mathsf{GL\underline{S}C_a}; \mathsf{PA}, \mathbb{N}) \leq^{\text{Prop}}_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{GLC_a}; \mathsf{PA}, \mathsf{PA})$. Given $A \in \mathcal{L}_\Box$, define $f(A) = \Box A$ and $\bar{f}_A$ as the identity function.

R1. Let $\mathsf{GL\underline{S}C_a} \vdash \Box A$. By soundness of $\mathsf{GL\underline{S}C_a} = \mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathbb{N})$, for every Σ_1-substitution σ we have $\mathbb{N} \models \sigma_{\mathsf{PA}}(\Box A)$ and hence $\mathsf{PA} \vdash \sigma_{\mathsf{PA}}(A)$. Then by arithmetical completeness of $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{PA})$, we have $\mathsf{GLC_a} \vdash A$.
One also may prove this item with a direct propositional argument, using Kripke semantics. For simplicity reasons, we chose the indirect way.

R2. Let $\mathbb{N} \not\models \sigma_{\mathsf{PA}}(\Box A)$. Then $\mathsf{PA} \nvdash \sigma_{\mathsf{PA}}(A)$, as desired. □

Theorem 9.7.30 $\mathsf{GL\underline{S}} = \mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathbb{N}) \leq^{Prop} \mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{PA}) = \mathsf{GL}$.

Proof Similar to the proof of Theorem 9.7.29 and left to the reader. □

9.8 Relative Provability Logics for PA*

In this section, we characterise several relative provability logics for PA^* via reductions. All reductions are shown at once in the Diagram 9.4. The head of arrow reduces to its tail, via some simple reduction (Sect. 9.4.1). The translation f in the reduction, is shown over the arrow lines and the number which appears under arrow, is the corresponding theorem.

9.8.1 Kripke Semantics

In the following lemma, we will show that the axioms CP and TP are local over iGL, i.e. whenever we can deduce some proposition A from $\mathsf{CP} + \mathsf{TP}$ in iGL, then we may deduce it by those instances of CP and TP which use the subformulas of A:

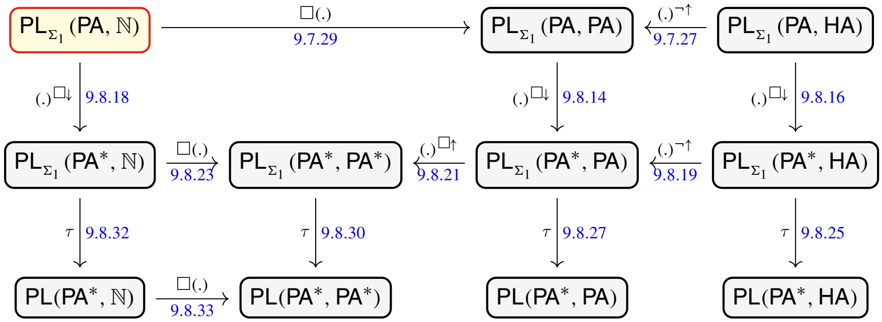

Diagram 9.4 Reductions for relative provability logics of PA^*

Lemma 9.8.1 *For every A, if* $\mathsf{iGLCT} \vdash A$ *then*

$$\mathsf{iGL} \vdash \boxdot \left[\bigwedge_{E \to F \in \mathsf{sub}(A)} \Box(E \to F) \to (E \vee (E \to F)) \wedge \bigwedge_{E \in \mathsf{sub}(A)} (E \to \Box E) \right] \to A$$

Proof For the simplicity of notations, in this proof, let

$$\varphi := \boxdot \bigwedge_{E \to F \in \mathsf{sub}(A)} \Box(E \to F) \to (E \vee (E \to F)) \wedge \boxdot \bigwedge_{E \in \mathsf{sub}(A)} (E \to \Box E)$$

and $\vdash$ indicate derivablity in $\mathsf{iGL} + \varphi$.
One side is trivial. For the other way around, assume that $\mathsf{iGL} \nvdash \varphi \to A$. We will construct some finite Kripke model $\mathcal{K} = (K, \sqsubset, \preccurlyeq, V)$ with $\sqsubset = \prec$ such that $\mathcal{K}, \alpha \nVdash A$, which by soundness of iGLCT for finite Kripke models with $\prec = \sqsubset$, we have the desired result. The proof is almost identical to the proof of Theorem 9.3.40 in (Ardeshir and Mojtahedi 2018, Theorem 4.26). To be self-contained, we elaborate it here.

Let $\mathsf{Sub}(A)$ be the set of sub-formulae of A. Then define

$$X := \{B, \Box B \mid B \in \mathsf{Sub}(A)\}$$

It is obvious that X is a finite adequate set. We define $\mathcal{K} = (K, \preccurlyeq, \sqsubset, V)$ as follows. Define

- K as the set of all X-saturated sets with respect to $\mathsf{iGL} + \varphi$,
- $\alpha \sqsubset \beta$ iff $\{D : \Box D \in \alpha\} \subseteq \beta$ and $\alpha \subsetneqq \beta$,
- $\alpha \preccurlyeq \beta$ iff $\alpha \sqsubset \beta$ or $\alpha = \beta$,
- $\alpha V p$ iff $p \in \alpha$, for atomic p.

$\mathcal{K}$ trivially satisfies all the properties of finite Kripke model with $\sqsubset = \prec$. So we must only show that $\mathcal{K} \nVdash A$. To this end, we first show by induction on $B \in X$ that $B \in \alpha$ iff $\alpha \Vdash B$, for each $\alpha \in K$. The only non-trivial cases are $B = \Box C$ and $B = E \to F$.

- $B = \Box C$: Let $\Box C \notin \alpha$. We must show $\alpha \nVdash \Box C$. The other direction is easier to prove and we leave it to reader. Let $\beta_0 := \{D \in X \mid \alpha \vdash \Box D\}$. If $\beta_0, \Box C \vdash C$, since by definition of β_0, we have $\alpha \vdash \Box\beta_0$ and hence by Löb's axiom, $\alpha \vdash \Box C$, which is in contradiction with $\Box C \notin \alpha$. Hence $\beta_0, \Box C \nvdash C$ and so there exists some X-saturated set β such that $\beta \nvdash C$, $\beta \supseteq \beta_0 \cup \{\Box C\}$. Hence $\beta \in K$ and $\alpha \sqsubset \beta$. Then by the induction hypothesis, $\beta \nVdash C$ and hence $\alpha \nVdash \Box C$.
- Let $E \to F \notin \alpha$. Then $F \notin \alpha$. If $E \in \alpha$, by induction hypothesis we have $\alpha \Vdash E$ and $\alpha \nVdash F$, and hence $\alpha \nVdash E \to F$, as desired. So we may let $E \notin \alpha$. Define $\beta_0 := \{D : \Box D \in \alpha\}$. If $\alpha \vdash \bigwedge \beta_0 \to (E \to F)$, then $\alpha \vdash \Box(E \to F)$ and hence by TP, either we have $\alpha \vdash E$ or $\alpha \vdash E \to F$, a contradiction. So we may let $\alpha \nvdash \bigwedge \beta_0 \to (E \to F)$, and use Lemma 9.3.39 to find $\beta \supseteq \beta_0 \cup \alpha \cup \{E\}$ as some X-saturated node in $\mathcal{K}$. Hence $\alpha \sqsubset \beta$ which implies $\alpha \prec \beta$ and by induction hypothesis $\beta \Vdash E$ and $\beta \nVdash F$, which implies $\alpha \nVdash E \to F$, as desired.

Since $\mathsf{iGL} + \varphi \nvdash A$, by Lemma 9.3.39, there exists some X-saturated set $\alpha \in K$ such that $\alpha \nvdash A$, and hence by the above argument we have $\alpha \nVdash A$. $\Box$

Lemma 9.8.2 *For arbitrary proposition A*

$$\mathsf{iGLCT} \vdash A^\Box \textit{ implies } \mathsf{iGL} + \Box\mathsf{CP} + \mathsf{TP} \vdash A^\Box.$$

Proof Let $\mathsf{iGLCT} \vdash A^\Box$. Hence by Lemma 9.8.1 the following is derivable in iGL

$$\left[\boxdot \overbrace{\bigwedge_{B \in \mathsf{sub}(A^\Box)} B \to \Box B}^{G} \wedge \boxdot \overbrace{\bigwedge_{E \to F \in \mathsf{sub}(A^\Box)} \Box(E \to F) \to (E \vee (E \to F))}^{H} \right] \to A^\Box$$

Hence by Lemma 9.3.9 $\mathsf{iGL} \vdash G^{\Box\uparrow} \wedge H^{\Box\uparrow} \to A^\Box$. By Lemma 9.3.10, $\mathsf{iK4} \vdash G^{\Box\uparrow}$. Also $(\Box G)^{\Box\uparrow} = \Box G$ which is an instance of $\Box\mathsf{CP}$. Let us consider some arbitrary conjunct $\Box(E \to F) \to (E \vee (E \to F))$ in H. Since $E \to F$ is a subformula of $A^\Box$, we have $E = E_0^\Box$ and $F = F_0^\Box$. Hence inside $\mathsf{iK4}$, the $H^{\Box\uparrow}$ is equivalent to some instance of TP. Hence $\mathsf{iGL} + \Box\mathsf{CP} + \mathsf{TP} \vdash A^\Box$. $\Box$

Theorem 9.8.3 $\mathsf{iGL\overline{CT}}$ *is sound and complete for semi-perfect* Suc*-quasi-classical Kripke models.*

Proof The soundness is easy and left to the reader. For the completeness, we first show the completeness for finite brilliant irreflexive transitive Suc-quasi-classical Kripke models. Let $\mathsf{iGL\overline{CT}} \nvdash A$. Let

$$X := \{B, \Box B : B \in \mathsf{Sub}(A)\}$$

and define the Kripke model $\mathcal{K} = (K, \preccurlyeq, \sqsubset, V)$ as follows:

- K is the family of all X-saturated sets with respect to $\mathsf{iGL\overline{CT}}$.
- $\alpha \sqsubset \beta$ iff $\alpha \neq \beta$ and $\{B, \Box B : \Box B \in \alpha\} \subseteq \beta$ and β is X-saturated with respect to iGLCT.
- $\alpha \prec \beta$ iff $\alpha \subsetneqq \beta$ and either $\alpha \sqsubset \beta$ or $\gamma \not\sqsubset \alpha$, for every $\gamma \in K$.
- $\alpha V p$ iff $p \in \alpha$.

It is straightforward to show that $\mathcal{K}$ is a finite brilliant irreflexive transitive Suc-quasi-classical Kripke model. We leave them to the reader. We only show that $\mathcal{K}, \alpha \Vdash B$ iff $B \in \alpha$ for every $\alpha \in K$ and $B \in X$. Then by Lemma 9.3.39 one may find some $\alpha \in K$ such that $\mathcal{K}, \alpha \nVdash A$, as desired.

Use induction on the complexity of $B \in X$. All inductive steps are trivial, except for:

- $B = \Box C$: If $\Box C \in \alpha$ and $\alpha \sqsubset \beta$, then by definition, $C \in \beta$ and hence by induction hypothesis $\beta \Vdash C$. This implies $\alpha \Vdash \Box C$. For the other way around, let $\Box C \notin \alpha$. Consider the set $\Delta := \{E, \Box E : \Box E \in \alpha\}$. If $\mathsf{iGLCT} \vdash \bigwedge \Delta \to (\Box C \to C)$, then $\mathsf{iGL\overline{CT}} \vdash \Box(\bigwedge \Delta) \to \Box C$. Since $\mathsf{iK4} + \alpha \vdash \Box(\bigwedge \Delta)$, we may deduce $\mathsf{iGL\overline{CT}} + \alpha \vdash \Box C$, a contradiction. Hence $\mathsf{iGLCT} \nvdash (\bigwedge \Delta \wedge \Box C) \to C$. By Lemma 9.3.39, there exists some X-saturated set $\beta \supseteq \Delta \cup \{\Box C\}$ with respect to iGLCT such that $C \notin \beta$. Hence $\beta \in K$ and $\alpha \sqsubset \beta$ and $C \notin \beta$. Induction hypothesis implies that $\beta \nVdash C$ and hence $\alpha \nVdash \Box C$.
- $B = C \to D$: If $C \to D \in \alpha$ and $\alpha \preccurlyeq \beta$ and $\beta \Vdash C$, by induction hypothesis $C \in \beta$ and hence $D \in \beta$. Again by induction hypothesis we have $\beta \Vdash D$. This shows that $\alpha \Vdash C \to D$. For the other way around, let $C \to D \notin \alpha$. We have two cases:
  - There is some $\gamma \sqsubset \alpha$: Hence α is X-sturated w.r.t iGLCT. Let $\Delta := \{E : \Box E \in \alpha\}$. We have tow subcases:
    - If $\mathsf{iGLCT} + \Delta + \alpha \vdash C \to D$, then $\mathsf{iGLCT} + \Box\alpha + \Box\Delta \vdash \Box(C \to D)$. By the completeness principle, we have $\mathsf{iGLCT} + \alpha \vdash \Box(C \to D)$. By TP we have $\mathsf{iGLCT} + \alpha \vdash C \vee (C \to D)$. Since α is X-saturated with respect to iGLCT, we have either $C \in \alpha$ or $C \to D \in \alpha$. The latter is impossible, hence $C \in \alpha$. Again by X-saturatedness of α, we can deduce $D \notin \alpha$. Hence by induction hypothesis we have $\alpha \Vdash C$ and $\alpha \nVdash D$, which implies $\alpha \nVdash C \to D$, as desired.
    - If $\mathsf{iGLCT} + \Delta + \alpha \nvdash C \to D$, then by Lemma 9.3.39 there exists some X-saturated $\beta \supseteq \alpha \cup \Delta \cup \{C\}$ w.r.t iGLCT (and a fortiori $\mathsf{iGL\overline{CT}}$) such that $D \notin \beta$. Induction hypothesis implies $\beta \Vdash C$ and $\beta \nVdash D$. One may observe that $\alpha \prec \beta$ or $\alpha = \beta$, and hence $\alpha \nVdash C \to D$, as desired.
  - There is no $\gamma \sqsubset \alpha$: since $\mathsf{iGL\overline{CT}} + \alpha \nvdash C \to D$, by Lemma 9.3.39, there exists some X-saturate set $\beta \supseteq \alpha \cup \{C\}$ with respect to $\mathsf{iGL\overline{CT}}$ such that $D \notin \beta$. Hence by induction hypothesis $\beta \Vdash C$ and $\beta \nVdash D$. One may observe that $\beta \succcurlyeq \alpha$ and hence $\alpha \nVdash C \to D$.

Next we use the construction method (Iemhoff 2001) to fulfil the other conditions: being neat and tree. Let $\mathcal{K}_t := (K_t, \preccurlyeq_t, \sqsubset_t, V_t)$ as follows:

- K_t is the set of all finite sequences $r := \langle \alpha_0, \dots \alpha_n \rangle$ such that for any $i < n$ either we have $\alpha_i \prec \alpha_{i+1}$ or $\alpha_i \sqsubset \alpha_{i+1}$. Let $f(r)$ indicate the final element of the sequence r.
- $r \preccurlyeq_t s$ iff r is an initial segment of s and $f(r) \preccurlyeq f(s)$.
- $r \sqsubset_t s$ iff r is an initial segment of $s = \langle \alpha_0, \dots \alpha_n \rangle$, e.g. $r = \langle \alpha_0, \dots \alpha_k \rangle$ for some $k < n$ and $\alpha_i \sqsubset \alpha_{i+1}$ for some $k \leq i < n$.
- $r \; V_t \; p$ iff $f(r) \; V \; p$.

It is straightforward to show that $\mathcal{K}_t$ is semi-perfect Suc-quasi-classical Kripke model and for every $r \in K_t$ and formula B we have

$$\mathcal{K}_t, r \Vdash B \qquad \Longleftrightarrow \qquad \mathcal{K}, f(r) \Vdash B.$$

$\square$

Theorem 9.8.4 $\mathsf{iGL\overline{C}TC_a}$ *is sound and complete for semi-perfect* Suc*-quasi-classical atom-complete Kripke models.*

Proof Similar to the proof of Theorem 9.8.3 and left to the reader. $\square$

Theorem 9.8.5 *For every proposition A, we have* $\mathsf{iGL\overline{C}T\underline{P}} \vdash A$ *iff for every quasi-classical perfect Kripke model $\mathcal{K}$ and every boolean interpretation I and arbitrary node α in $\mathcal{K}$ we have $\mathcal{K}, \alpha, I \models A$.*

Proof The soundness is easy and left to the reader. For the completeness part, let $\mathsf{iGL\overline{C}T\underline{P}} \nvdash A$. Let A' be a boolean equivalent of A which is a conjunction of implications $E \to F$ in which E is a conjunction of a set of atomics or boxed propositions and F is a disjunction of atomics or boxed proposition. Evidently such A' exists for every A. Hence $\mathsf{iGL\overline{C}T\underline{P}} \nvdash A'$. Then there must be some conjunct $E \to F$ of A' such that $\mathsf{iGL\overline{C}T\underline{P}} \nvdash E \to F$, E is a conjunction of atomic and boxed propositions and F is a disjunction of atomic and boxed propositions. Let X_E be the set of atomic conjuncts in E and X_F the set of atomic disjuncts in F. Note here that X_E and X_F are disjoint sets. Define $\bar{E}$ and $\bar{F}$ as the replacement of X_E and X_F by $\top$ and $\bot$ in E and F, respectively. Hence $\bar{E} \to \bar{F}$, does not have any outer atomics and then $(\bar{E} \to \bar{F})^\Box$ is equivalent in $\mathsf{iGL} + \Box\mathsf{CP}$ with $\bar{E} \to \bar{F}$. Then $\mathsf{iGL} + \mathsf{TP} + \Box\mathsf{CP} \nvdash (E \to F)^\Box$ and by Lemma 9.8.2 we have $\mathsf{iGLCT} \nvdash \bar{E} \to \bar{F}$. Then by Theorem 9.3.40, there is some perfect, quasi-classical Kripke model $\mathcal{K}$ such that $\mathcal{K}, \alpha \nVdash \bar{E} \to \bar{F}$. Hence there is some $\beta \succcurlyeq \alpha$ such that $\mathcal{K}, \beta \Vdash \bar{E}$ and $\mathcal{K}, \beta \nVdash \bar{F}$. Let the boolean interpretation I defined such that:

$$I(p) := \begin{cases} \text{true} & : p \in X_E \\ \text{false} & : p \in X_F \\ \text{no matter, true or false} & : \text{otherwise} \end{cases}$$

One may observe that $\mathcal{K}, \beta, I \not\models E \to F$ and hence $\mathcal{K}, \beta, I \not\models A$. $\square$

Theorem 9.8.6 *For every proposition A, we have* $\mathsf{iGL\overline{C}T\underline{P}C_a} \vdash A$ *iff for every quasi-classical perfect Kripke model* $\mathcal{K}$ *and arbitrary node* α *in* $\mathcal{K}$ *we have* $\mathcal{K}, \alpha \models A$.

Proof The proof is very similar to the one for Theorem 9.8.5, except for the argument for X_E and X_F and $\bar{E}$ and $\bar{F}$ and the boolean interpretation I, which are unnecessary here with the presence of the $\mathsf{CP_a}$. For readability reasons, we bring the adapted proof here.

The soundness is straightforward and left to the reader. For the completeness, let $\mathsf{iGL\overline{C}T\underline{P}C_a} \nvdash A$. Let A' be a boolean equivalent of A which is a conjunction of implications $E \to F$ in which E is a conjunction of a set of atomics or boxed propositions and F is a disjunction of atomics or boxed proposition. Evidently such A' exists for every A. Hence $\mathsf{iGL\overline{C}T\underline{P}C_a} \nvdash A'$. Then there must be some conjunct $E \to F$ of A' such that $\mathsf{iGL\overline{C}T\underline{P}} \nvdash E \to F$, E is a conjunction of atomic and boxed propositions and F is a disjunction of atomic and boxed propositions. Hence $E^{\Box} \to F^{\Box}$ is equivalent in $\mathsf{iK4} + \mathsf{CP_a} + \Box\mathsf{CP}$ with $E \to F$. Then $\mathsf{iGL} + \mathsf{TP} + \Box\mathsf{CP} + \mathsf{CP_a} \nvdash (E \to F)^{\Box}$ and by Lemma 9.8.2 we have $\mathsf{iGLCT} \nvdash E \to F$. Then by Theorem 9.3.40, there is some perfect, quasi-classical Kripke model $\mathcal{K}$ such that $\mathcal{K}, \alpha \nVdash E \to F$. Hence there is some $\beta \succcurlyeq \alpha$ such that $\mathcal{K}, \beta \Vdash E$ and $\mathcal{K}, \beta \nVdash F$. Then $\mathcal{K}, \beta \not\models E \to F$ and hence $\mathcal{K}, \beta \not\models A$. $\Box$

Theorem 9.8.7 $\mathsf{iGL\overline{C}T\underline{S^*P}} \vdash A^{\Box\downarrow}$ *iff for every quasi-classical perfect Kripke model* $\mathcal{K}$ *and every boolean interpretation* I *and arbitrary* $A^{\Box\downarrow}$*-sound node* α *in* $\mathcal{K}$ *we have* $\mathcal{K}, \alpha, I \models A^{\Box\downarrow}$.

Proof Both directions are proved contra-positively. For the soundness part, assume that $\mathcal{K}, \alpha, I \not\models A^{\Box\downarrow}$ for some boolean interpretation I and quasi-classical perfect Kripke model $\mathcal{K} := (K, \preccurlyeq, \sqsubset, V)$ which is $A^{\Box\downarrow}$-sound at $\alpha \in K$. Since derivability is finite, it is enough to show that for every finite set Γ of modal propositions we have

$$\mathsf{iGL\overline{C}T\underline{P}} \nvdash \bigwedge_{B\in\Gamma} (\Box B^{\Box} \to B^{\Box}) \to A^{\Box\downarrow}.$$

By Theorem 9.8.5 and Lemma 9.3.45, it is enough to find some number i such that

$$\mathcal{K}^{(i)}, \alpha_i, I \not\models \bigwedge_{B\in\Gamma} (\Box B^{\Box} \to B^{\Box}) \to A^{\Box\downarrow}.$$

Let us define n_i and m_i as the number of propositions in the sets $N_i := \{B \in \Gamma : \mathcal{K}^{(i)}, \alpha_i, I \models B^{\Box} \wedge \Box B^{\Box}\}$ and $M_i := \{B \in \Gamma : \mathcal{K}^{(i)}, \alpha_i, I \models \Box B^{\Box} \wedge \neg B^{\Box}\}$, respectively. We use induction on k and prove the following statement:

$$\varphi(k) := \text{for every} i, \text{ if } n_i < k \text{then there is some} 0 \le j \le 1 + n_i \text{such that}$$
$$\mathcal{K}^{(i+j)}, \alpha_{i+j}, I \models \bigwedge_{B\in\Gamma} (\Box B^{\Box} \to B^{\Box}) \tag{9.1}$$

Then by $\varphi(n_0+1)$, one may find some number j such that $\mathcal{K}^j, \alpha_j, I \models \bigwedge_{B\in\Gamma}(\Box B^{\Box} \to B^{\Box})$, and by Lemma 9.3.45 we also have $\mathcal{K}^j, \alpha_j, I \not\models A^{\Box\downarrow}$, as desired.

$\varphi(0)$ trivially holds. As induction hypothesis, let $\varphi(k)$ holds and show that $\varphi(k+1)$ holds as follows. Let some number i such that $n_i < k+1$. If $n_i < k$, by induction hypothesis we have the desired conclusion. So let $n_i = k$. If $m_i = 0$, we may let $j = 0$ and we have Eq. 9.1. So let $B \in \Gamma$ such that $\mathcal{K}^{(i)}, \alpha_i, I \models \Box B^{\Box} \wedge \neg B^{\Box}$. We have two sub-cases:

- $m_{i+1} = 0$: observe in this case that Eq. 9.1 holds for $j = 1$.
- $m_{i+1} > 0$: in this case we have $n_{i+1} < k$ and hence by application of the induction hypothesis with $i := i+1$, we get some $0 \leq j' \leq 1 + n_{i+1}$ such that $\mathcal{K}^{(i+1+j')}, \alpha_{i+1+j'} \models \bigwedge_{B\in\Gamma}(\Box B^{\Box} \to B^{\Box})$. Hence if we let $j := j'+1$ we have $0 \leq j \leq 1 + n_i$ and Eq. 9.1, as desired.

For the completeness part, assume that $\mathsf{iGL\overline{C}T\underline{S^*P}} \nvdash A^{\Box\downarrow}$. Hence

$$\mathsf{iGL\overline{C}T\underline{P}} \nvdash \left(\bigwedge_{\Box B^{\Box} \in \mathsf{Sub}(A^{\Box\downarrow})} (\Box B^{\Box} \to B^{\Box}) \right) \to A^{\Box\downarrow}$$

Hence Theorem 9.8.5 implies the desired result. □

Theorem 9.8.8 $\mathsf{iGL\overline{C}T\underline{S^*P}C_a} \vdash A^{\Box\downarrow}$ *iff for every quasi-classical perfect Kripke model $\mathcal{K}$ and arbitrary A-sound node α in $\mathcal{K}$ we have $\mathcal{K}, \alpha \models A$.*

Proof The proof is similar to the one for Theorem 9.8.7. One must use Theorem 9.8.6 instead of Theorem 9.8.5 in the proof. □

9.8.2 Reductions

Lemma 9.8.9 $\mathsf{iGLCT} \vdash A$ *implies* $\mathsf{GL} \vdash A^{\Box}$.

Proof Use induction on the proof $\mathsf{iGLCT} \vdash A$. □

Lemma 9.8.10 $\mathsf{GL} \vdash A$ *implies* $\mathsf{iGL\overline{P}} \vdash \Box A$.

Proof Let $\mathsf{GL} \vdash A$. Hence $\mathsf{iGL} \vdash \boxdot \mathsf{PEM} \to A$. Since necessitation is admissible to iGL, we have $\mathsf{iGL} \vdash \Box\mathsf{PEM} \to \Box A$ which implies $\mathsf{iGL\overline{P}} \vdash \Box A$. □

Definition 9.8.11 For a Kripke model $\mathcal{K} = (K, \preccurlyeq, \sqsubset, V)$, let $\widetilde{\mathcal{K}}$, indicate the Kripke model derived from $\mathcal{K}$ by making every $\sqsubset$-accessible node as a classical node. More precisely, we define $\widetilde{\mathcal{K}} := (K, \widetilde{\preccurlyeq}, \sqsubset, V))$ in this way:

$$\alpha \widetilde{\preccurlyeq} \beta \text{ iff “}\alpha \text{ is not } \sqsubset\text{-accessible } (\alpha \notin \mathsf{Suc}) \text{ and } \alpha \preccurlyeq \beta\text{” or}$$
$$\text{“}\alpha \text{ is } \sqsubset\text{-accessible } (\alpha \notin \mathsf{Suc}) \text{ and } \alpha = \beta\text{”}$$

Lemma 9.8.12 *For every* Suc*-quasi-classical semi-perfect Kripke model* $\mathcal{K} = (K, \preccurlyeq, \sqsubset, V)$ *and* $\alpha \notin \mathsf{Suc}$ *and arbitrary proposition* A *we have*

$$\mathcal{K}, \alpha \Vdash A^{\Box\downarrow} \iff \widetilde{\mathcal{K}}, \alpha \Vdash A^{\Box\downarrow}.$$

Proof First observe that for every $\alpha \in \mathsf{Suc}$ and every proposition B we have

$$\widetilde{\mathcal{K}}, \alpha \Vdash B \iff \widetilde{\mathcal{K}}, \alpha \models_c B \iff \mathcal{K}, \alpha \models_c B$$

Then we may use Corollary 9.3.43 and for $\alpha \in \mathsf{Suc}$ deduce

$$\mathcal{K}, \alpha \Vdash B^{\Box} \iff \widetilde{\mathcal{K}}, \alpha \Vdash B^{\Box}. \tag{9.2}$$

We use induction on the complexity of A and prove the assertion of the lemma. All cases are obvious except for the cases $A = \Box B$ in which we have $A^{\Box\downarrow} = \Box B^{\Box}$. We have

$$\begin{aligned} \mathcal{K}, \alpha \nVdash \Box B^{\Box} &\iff \text{there exists some } \beta \sqsupset \alpha \text{ such that } \mathcal{K}, \beta \nVdash B^{\Box} \\ &\iff \text{there exists some } \beta \sqsupset \alpha \text{ such that } \widetilde{\mathcal{K}}, \beta \nVdash B^{\Box} \\ &\iff \widetilde{\mathcal{K}}, \alpha \nVdash \Box B^{\Box} \end{aligned}$$

in which in the second line we use Eq. 9.2. □

Lemma 9.8.13 *For every* $A \in \mathcal{L}_{\Box}$ *we have* $\mathsf{iGL\overline{C}T\underline{P}C_a} \vdash A$ *iff* $\mathsf{GLC_a} \vdash A^{\Box\downarrow}$.

Proof We use induction on the proof $\mathsf{iGL\overline{C}T\underline{P}C_a} \vdash A$ and show $\mathsf{GLC_a} \vdash A^{\Box\downarrow}$. All cases are similar to the one for $\mathsf{iGL\overline{C}T\underline{P}}$, except for

- $A = p \to \Box p$: then $\mathsf{iK4} \vdash A^{\Box\downarrow} \leftrightarrow A$ and hence $\mathsf{GLC_a} \vdash A^{\Box\downarrow}$.

For the other way around, let $\mathsf{iGL\overline{C}T\underline{P}C_a} \nvdash A$. Then by $\Box\mathsf{CP}$ we have $A^{\Box\downarrow} \leftrightarrow A$, and then we may deduce $\mathsf{iGL\overline{C}T\underline{P}C_a} \nvdash A^{\Box\downarrow}$. By Theorem 9.8.6, there exists some quasi-classical perfect Kripke model $\mathcal{K}$ such that $\mathcal{K}, \alpha \nvDash A^{\Box\downarrow}$. Corollary 9.3.43 implies $\mathcal{K}, \alpha \nvDash_c A^{\Box\downarrow}$, which by soundness of $\mathsf{GLC_a}$ for classical Kripke models with the property of truth-ascending (i.e. if p is true at some node, then it is true also at all accessible nodes), implies $\mathsf{GLC_a} \nvdash A^{\Box\downarrow}$. □

Theorem 9.8.14 $\mathsf{GLC_a} = \mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{PA}) \leq^{Prop} \mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{PA}) = \mathsf{iGL\overline{C}T\underline{P}C_a}$.

Proof The arithmetical soundness of $\mathsf{iGL\overline{C}T\underline{P}C_a}$ is straightforward and left to the reader. Also $\mathsf{GLC_a} = \mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{PA})$ holds by Theorem 9.7.1. It is enough here to show that

$$\mathcal{AC}_{\Sigma_1}(\mathsf{GLC_a}; \mathsf{PA}, \mathsf{PA}) \leq^{\mathrm{Prop}}_{f, \bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{C}TC_a}; \mathsf{PA}^*, \mathsf{PA}).$$

Given $A \in \mathcal{L}_{\Box}$, let $f(A) := (A)^{\Box\downarrow}$ and $\bar{f}_A$ as the identity function.

R1. Lemma 9.8.13.
R2. If $\mathsf{PA} \nvdash \sigma_{\mathsf{PA}}(A^{\Box\downarrow})$, for a Σ_1-substitution σ, then by Lemma 9.3.22 we have $\mathsf{PA} \nvdash \sigma_{\mathsf{PA}*}(A)$. □

Lemma 9.8.15 *For every $A \in \mathcal{L}_\Box$ we have* $\mathsf{iGL\overline{CT}C_a} \vdash A$ *iff* $\mathsf{iGL\overline{P}C_a} \vdash A^{\Box\downarrow}$.

Proof We use induction on the proof $\mathsf{iGL\overline{CT}C_a} \vdash A$ and show $\mathsf{iGL\overline{P}C_a} \vdash A^{\Box\downarrow}$. All cases are identical to the corresponding on in the previous proof, except for when $A = p \to \Box p$, which trivially we have $\mathsf{iGL\overline{P}C_a} \vdash A^{\Box\downarrow}$.

For the other way around, let $\mathsf{iGL\overline{CT}C_a} \nvdash A$. Then by Lemma 9.3.8 we have $A^{\Box\downarrow} \leftrightarrow A$, and hence $\mathsf{iGL\overline{CT}C_a} \nvdash A^{\Box\downarrow}$. By Theorem 9.8.4, there exists some Suc-quasi-classical semi-perfect atom-complete Kripke model $\mathcal{K}$ such that $\mathcal{K}, \alpha \nVdash A^{\Box\downarrow}$, for some node α. We may assume $\alpha \notin \mathsf{Suc}$, otherwise eliminate all nodes not in $(\alpha \preccurlyeq) \cup (\alpha \sqsubset)$ and consider this new Kripke model instead of $\mathcal{K}$. Obviously the new Kripke model still refutes $A^{\Box\downarrow}$ at α and is Suc-quasi-classical semi-perfect and atom-complete. Hence Lemma 9.8.12 implies that $\widetilde{\mathcal{K}}, \alpha \nVdash A^{\Box\downarrow}$, in which $\widetilde{\mathcal{K}}$ indicates the Kripke model derived from $\mathcal{K}$ by making every $\sqsubset$-accessible node as a classical node. Precise definition of $\widetilde{\mathcal{K}}$ came before Lemma 9.8.12. It is obvious that $\widetilde{\mathcal{K}}$ is a Suc-classical semi-perfect atom-complete Kripke model. Hence Theorem 9.7.13 implies $\mathsf{iGL\overline{P}C_a} \nvdash A^{\Box\downarrow}$, as desired. □

Theorem 9.8.16 $\mathsf{iGL\overline{P}C_a} = \mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{HA}) \leq^{Prop} \mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{HA}) = \mathsf{iGL\overline{CT}C_a}$.

Proof The arithmetical soundness of $\mathsf{iGL\overline{CT}C_a}$ is straightforward and left to the reader. Also $\mathsf{iGL\overline{P}C_a} = \mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{HA})$ holds by Theorem 9.7.14. It is enough here to show that

$$\mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{P}C_a}; \mathsf{PA}, \mathsf{HA}) \leq^{\mathrm{Prop}}_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{CT}C_a}; \mathsf{PA}^*, \mathsf{HA}).$$

Given $A \in \mathcal{L}_\Box$, let $f(A) := (A)^{\Box\downarrow}$ and $\bar{f}_A$ as the identity function.

R1. Lemma 9.8.15.
R2. If $\mathsf{HA} \nvdash \sigma_{\mathsf{PA}}(A^{\Box\downarrow})$, for a Σ_1-substitution σ, then by Lemma 9.3.22 we have $\mathsf{HA} \nvdash \sigma_{\mathsf{PA}*}(A)$. □

Lemma 9.8.17 *For every $A \in \mathcal{L}_\Box$ we have* $\mathsf{iGL\overline{C}T\underline{S^*P}C_a} \vdash A$ *iff* $\mathsf{GL\underline{S}C_a} \vdash A^{\Box\downarrow}$.

Proof We use induction on the proof $\mathsf{iGL\overline{C}T\underline{S^*P}C_a} \vdash A$ and show $\mathsf{GL\underline{S}C_a} \vdash A^{\Box\downarrow}$. All cases are similar to the one for $\mathsf{iGL\overline{C}T\underline{S^*P}}$, except for

- $A = p \to \Box p$: then $\mathsf{iK4} \vdash A^{\Box\downarrow} \leftrightarrow A$ and hence $\mathsf{GLC_a} \vdash A^{\Box\downarrow}$.

For the other way around, let $\mathsf{iGL\overline{C}T\underline{S^*P}C_a} \nvdash A$. Then by $\Box\mathsf{CP}$ we have $A^{\Box\downarrow} \leftrightarrow A$, and then we may deduce $\mathsf{iGL\overline{C}T\underline{S^*P}C_a} \nvdash A^{\Box\downarrow}$. By Theorem 9.8.8, there exists some quasi-classical perfect Kripke model $\mathcal{K}$ such that $\mathcal{K}, \alpha \not\models A^{\Box\downarrow}$ and $\mathcal{K}$ is $A^{\Box\downarrow}$-sound at α. Corollary 9.3.43 implies $\mathcal{K}, \alpha \not\models_c A^{\Box\downarrow}$, which by soundness of $\mathsf{GL\underline{S}C_a}$ for classical Kripke models with the property of truth-ascending (i.e. if p is true at some node, then it is true also at all accessible nodes), implies $\mathsf{GL\underline{S}C_a} \nvdash A^{\Box\downarrow}$. □

Theorem 9.8.18 $\mathsf{GL\underline{S}C_a} = \mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathbb{N}) \leq^{Prop} \mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathbb{N}) = \mathsf{iGL\overline{C}T\underline{S^*P}C_a}$.

Proof The arithmetical soundness of $\mathsf{iGL\overline{C}T\underline{S^*P}C_a}$ is straightforward and left to the reader. Also $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathbb{N}) = \mathsf{GL\underline{S}C_a}$ holds by Theorem 9.7.1. It is enough here to show that

$$\mathcal{AC}_{\Sigma_1}(\mathsf{GL\underline{S}C_a}; \mathsf{PA}, \mathbb{N}) \leq^{\text{Prop}}_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{C}T\underline{S^*P}C_a}; \mathsf{PA}^*, \mathbb{N}).$$

Given $A \in \mathcal{L}_\Box$, let $f(A) := (A)^{\Box\downarrow}$ and $\bar{f}_A$ as the identity function.

R1. Lemma 9.8.17.
R2. If $\mathbb{N} \not\models \sigma_{\mathsf{PA}}(A^{\Box\downarrow})$, for a Σ_1-substitution σ, then by Lemma 9.3.22 we have $\mathbb{N} \not\models \sigma_{\mathsf{PA}^*}(A)$. □

Theorem 9.8.19 $\mathsf{iGL\overline{CT}C_a} = \mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{HA}) \leq^{Prop} \mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{PA}) = \mathsf{iGL\overline{C}T\underline{P}C_a}$.

Proof We already have $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{PA}) = \mathsf{iGL\overline{C}T\underline{P}C_a}$ and $\mathsf{iGL\overline{C}T\underline{P}C_a} = \mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{PA})$ by Theorems 9.8.16 and 9.8.14. It is enough here to show that

$$\mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{CT}C_a}; \mathsf{PA}^*, \mathsf{HA}) \leq^{\text{Prop}}_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{C}T\underline{P}C_a}; \mathsf{PA}^*, \mathsf{PA}).$$

Given $A \in \mathcal{L}_\Box$, let $f(A) := (A)^{\neg\uparrow}$ and $\bar{f}_A$ as the identity function.

R1. If $\mathsf{iGL\overline{CT}C_a} \vdash A^{\neg\uparrow}$ then $\mathsf{iGL\overline{C}T\underline{P}C_a} \vdash A^{\neg\uparrow}$, and since we have PEM in $\mathsf{iGL\overline{C}T\underline{P}C_a}$, we may conclude $\mathsf{iGL\overline{C}T\underline{P}C_a} \vdash A$.
R2. If $\mathsf{HA} \nvdash \sigma_{\mathsf{PA}^*}(A^{\neg\uparrow})$, for a Σ_1-substitution σ, then by Lemma 9.5.18 we have $\mathsf{HA} \nvdash (\sigma_{\mathsf{PA}^*}(A))^{\neg}$. Hence by Lemma 9.5.15 we have $\mathsf{PA} \nvdash \sigma_{\mathsf{PA}^*}(A)$. □

Lemma 9.8.20 *For every* $A \in \mathcal{L}_\Box$, *if* $\mathsf{iGL\overline{C}T\underline{P}C_a} \vdash A^{\Box\uparrow}$, *then* $\mathsf{iGLCT} \vdash A$.

Proof Let $\mathsf{iGLCT} \nvdash A$. Hence by Theorem 9.3.40, there is some perfect quasi-classical Kripke model $\mathcal{K}$ such that $\mathcal{K}, \alpha \not\Vdash A$. Then Corollary 9.3.44 implies $\mathcal{K}, \alpha \not\models A^{\Box\uparrow}$, and hence by soundness of $\mathsf{iGL\overline{C}T\underline{P}C_a}$ (Theorem 9.8.6) implies $\mathsf{iGL\overline{C}T\underline{P}C_a} \nvdash A^{\Box\uparrow}$.
□

Theorem 9.8.21 $\mathsf{iGL\overline{C}T\underline{P}C_a} = \mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{PA}) \leq^{Prop} \mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{PA}^*) = \mathsf{iGLCT}$.

Proof The soundness of iGLCT is straightforward and left to the reader. By Theorem 9.8.14 we have $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{PA}) = \mathsf{iGL\overline{C}T\underline{P}C_a}$. We must show

$$\mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{C}T\underline{P}C_a}; \mathsf{PA}^*, \mathsf{PA}) \leq^{\text{Prop}}_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{iGLCT}; \mathsf{PA}^*, \mathsf{PA}^*).$$

Given $A \in \mathcal{L}_\Box$, define $f(A) = A^{\Box\uparrow}$ and $\bar{f}_A$ as the identity function.

R1. Lemma 9.8.20.
R2. Let $\mathsf{PA} \nvdash \sigma_{\mathsf{PA}^*}(A^{\Box\uparrow})$. Then by Lemma 9.3.19, $\mathsf{PA} \nvdash \sigma_{\mathsf{PA}^*}(A)^{\mathsf{PA}}$, and hence by definition of PA^*, we have $\mathsf{PA}^* \nvdash \sigma_{\mathsf{PA}^*}(A)$. □

Corollary 9.8.22 *For every $A \in \mathcal{L}_\Box$, we have* $\mathsf{iGLCT} \vdash A$ *iff* $\mathsf{iGL\overline{C}T\underline{P}C_a} \vdash A^{\Box\uparrow}$.

Proof Use Corollary 9.4.5, 9.8.21. □

Theorem 9.8.23 $\mathsf{iGL\overline{C}T\underline{S^*P}C_a} = \mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathbb{N}) \leq^{Prop} \mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{PA}^*) = \mathsf{iGLCT}$.

Proof By Theorems 9.8.18 and 9.8.21 we have $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathbb{N}) = \mathsf{iGL\overline{C}T\underline{S^*P}C_a}$ and $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{PA}^*) = \mathsf{iGLCT}$. We must show

$$\mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{C}T\underline{S^*P}C_a}; \mathsf{PA}^*, \mathbb{N}) \leq^{\mathrm{Prop}}_{f,\bar{f}} \mathcal{AC}_{\Sigma_1}(\mathsf{iGLCT}; \mathsf{PA}^*, \mathsf{PA}^*).$$

Given $A \in \mathcal{L}_\Box$, define $f(A) = \Box A$ and $\bar{f}_A$ as the identity function.

R1. Let $\mathsf{iGL\overline{C}T\underline{S^*P}C_a} \vdash \Box A$. By soundness of $\mathsf{iGL\overline{C}T\underline{S^*P}C_a} = \mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathbb{N})$, for every Σ_1-substitution σ we have $\mathbb{N} \models \sigma_{\mathsf{PA}^*}(\Box A)$ and hence $\mathsf{PA}^* \vdash \sigma_{\mathsf{PA}^*}(A)$. Then by arithmetical completeness of $\mathsf{iGLCT} = \mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{PA}^*)$, we have $\mathsf{iGLCT} \vdash A$.
One also may prove this item with a direct propositional argument, using Kripke semantics. For simplicity reasons, we chose the indirect way.

R2. Let $\mathbb{N} \not\models \sigma_{\mathsf{PA}^*}(\Box A)$. Then $\mathsf{PA}^* \nvdash \sigma_{\mathsf{PA}^*}(A)$, as desired. □

Lemma 9.8.24 *For every $A \in \mathcal{L}_\Box$, we have* $\mathsf{iGL\overline{CT}} \vdash A$ *iff* $\mathsf{iGL\overline{P}} \vdash A^{\Box\downarrow}$.

Proof We use induction on the proof $\mathsf{iGL\overline{CT}} \vdash A$ and show $\mathsf{iGL\overline{P}} \vdash A^{\Box\downarrow}$:

- $\mathsf{iGL} \vdash A$: by Lemma 9.3.9 we have $\mathsf{iGL} \vdash A^{\Box\downarrow}$.
- A is an axiom instance of $\Box\mathsf{CP}$ or $\Box\mathsf{TP}$: Then $A = \Box B$ and $\mathsf{iGLCT} \vdash B$ and by Lemma 9.8.9 we have $\mathsf{GL} \vdash B^{\Box}$. By Lemma 9.8.10 we have $\mathsf{iGL\overline{P}} \vdash \Box B^{\Box}$.
- $\mathsf{iGL\overline{CT}} \vdash B$ and $\mathsf{iGL\overline{CT}} \vdash B \to A$ with lower proof length: by induction hypothesis we have $\mathsf{iGL\overline{P}} \vdash B^{\Box\downarrow}$ and $\mathsf{iGL\overline{P}} \vdash B^{\Box\downarrow} \to A^{\Box\downarrow}$, which implies $\mathsf{iGL\overline{P}} \vdash A^{\Box\downarrow}$, as desired.

For the other way around, let $\mathsf{iGL\overline{CT}} \nvdash A$. Then by Lemma 9.3.8 we have $A^{\Box\downarrow} \leftrightarrow A$, and hence $\mathsf{iGL\overline{CT}} \nvdash A^{\Box\downarrow}$. By Theorem 9.8.3, there exists some Suc-quasi-classical semi-perfect Kripke model $\mathcal{K}$ such that $\mathcal{K}, \alpha \nVdash A^{\Box\downarrow}$, for some node α. We may let $\alpha \notin \mathsf{Suc}$, otherwise eliminate all nodes not in $(\alpha \preccurlyeq) \cup (\alpha \sqsubset)$ and consider this new Kripke model instead of $\mathcal{K}$. Obviously the new Kripke model still refutes $A^{\Box\downarrow}$ at α and is Suc-quasi-classical semi-perfect. Hence Lemma 9.8.12 implies that $\widetilde{\mathcal{K}}, \alpha \nVdash A^{\Box\downarrow}$, in which $\widetilde{\mathcal{K}}$ indicates the Kripke model derived from $\mathcal{K}$ by making every $\sqsubset$-accessible node as a classical node. Precise definition of $\widetilde{\mathcal{K}}$ came before Lemma 9.8.12. It is obvious that $\widetilde{\mathcal{K}}$ is a Suc-classical semi-perfect Kripke model. Hence Theorem 9.7.12 implies $\mathsf{iGL\overline{P}} \nvdash A^{\Box\downarrow}$, as desired. □

Theorem 9.8.25 $\mathsf{iGL\overline{CT}C_a} = \mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{HA}) \leq^{Prop} \mathsf{PL}(\mathsf{PA}^*, \mathsf{HA}) = \mathsf{iGL\overline{CT}}$.

Proof The arithmetical soundness of $\mathsf{iGL\overline{CT}C_a}$ is straightforward and left to the reader. By Theorem 9.8.16 we have $\mathsf{PL}(\mathsf{PA}^*, \mathsf{HA}) = \mathsf{iGL\overline{CT}C_a}$. We must show

$$\mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{CT}C_a};\mathsf{PA}^*,\mathsf{HA}) \leq^{\text{Prop}}_{f,\bar{f}} \mathcal{AC}(\mathsf{iGL\overline{CT}};\mathsf{PA}^*,\mathsf{HA}).$$

Given $A \in \mathcal{L}_\Box$, if $\mathsf{iGL\overline{CT}} \vdash A$, define $f(A) := \top$. If $\mathsf{iGL\overline{CT}} \nvdash A$, by Lemma 9.8.24 we have $\mathsf{iGL\overline{P}} \nvdash A^{\Box\downarrow}$, and hence by Lemma 9.7.23 there exists some propositional $(.)^{\Box\downarrow}$-substitution τ such that $\mathsf{iGL\overline{P}C_a} \nvdash \tau(A^{\Box\downarrow})$. Define $f(A) := \tau(A)$ and $\bar{f}_A(\sigma) := \sigma_{\mathsf{PA}^*} \circ \tau$.

R1. Let $\mathsf{iGL\overline{CT}} \nvdash A$. By Lemma 9.8.24 we have $\mathsf{iGL\overline{P}} \nvdash A^{\Box\downarrow}$ and then Lemma 9.7.23 implies $\mathsf{iGL\overline{P}C_a} \nvdash \tau(A^{\Box\downarrow})$, in which τ is as used for the definition of $f(A)$. Since τ is a $(.)^{\Box\downarrow}$-substitution, by Lemma 9.7.3 we have $\mathsf{iGL\overline{P}C_a} \nvdash (\tau(A))^{\Box\downarrow}$. Then Lemma 9.8.15 implies that $\mathsf{iGL\overline{CT}C_a} \nvdash \tau(A)$, or in other words $\mathsf{iGL\overline{CT}C_a} \nvdash f(A)$.

R2. Let $\mathsf{HA} \nvdash \sigma_{\mathsf{PA}^*}(f(A))$ for some Σ_1-substitution σ. By definition of $f(A)$, we must have $\mathsf{iGL\overline{CT}} \nvdash A$, otherwise $f(A) := \top$, which contradicts $\mathsf{HA} \nvdash \sigma_{\mathsf{PA}^*}(f(A))$. Hence $f(A) = \tau(A)$ for some propositional $(.)^{\Box\downarrow}$ -substitution τ. By Lemma 9.3.21 we have $\mathsf{HA} \nvdash \sigma_{\mathsf{PA}}(\tau(A)^{\Box\downarrow})$. Since $\mathsf{iK4} + \mathsf{CP_a}$ is included in $\mathsf{PL}_{\Sigma_1}(\mathsf{PA},\mathsf{HA}) = \mathsf{iGL\overline{P}C_a}$ (Theorem 9.7.21), we have $\mathsf{HA} \nvdash \sigma_{\mathsf{PA}}(\tau(A^{\Box\downarrow}))$. This implies that $\mathsf{HA} \nvdash [\bar{f}_A(\sigma)]_{\mathsf{PA}}(A^{\Box\downarrow})$ and again by Lemma 9.3.21 we have $\mathsf{HA} \nvdash [\bar{f}_A(\sigma)]_{\mathsf{PA}^*}(A)$.

□

Lemma 9.8.26 *For every $A \in \mathcal{L}_\Box$, we have* $\mathsf{iGL\overline{CT}\underline{P}} \vdash A$ *iff* $\mathsf{GL} \vdash A^{\Box\downarrow}$.

Proof We use induction on the proof $\mathsf{iGL\overline{CT}\underline{P}} \vdash A$ and show $\mathsf{GL} \vdash A^{\Box\downarrow}$:

- $A = \Box B$ and $\mathsf{iGLCT} \vdash B$: by Lemma 9.8.9 we have $\mathsf{GL} \vdash B^{\Box}$ and hence by necessitation $\mathsf{GL} \vdash \Box B^{\Box}$.
- $\mathsf{iGL} \vdash A$: by Lemma 9.3.9 we have $\mathsf{iGL} \vdash A^{\Box\downarrow}$.
- $A = B \vee \neg B$: Then $A^{\Box\downarrow} = B^{\Box\downarrow} \vee \neg B^{\Box\downarrow}$ which is valid in GL.
- $\mathsf{iGL\overline{CT}\underline{P}} \vdash B$ and $\mathsf{iGL\overline{CT}\underline{P}} \vdash B \to A$ with lower proof length than the one for A: by induction hypothesis we have $\mathsf{GL} \vdash B^{\Box\downarrow}$ and $\mathsf{GL} \vdash B^{\Box\downarrow} \to A^{\Box\downarrow}$, which implies $\mathsf{GL} \vdash A^{\Box\downarrow}$, as desired.

For the other way around, let $\mathsf{iGL\overline{CT}\underline{P}} \nvdash A$. Then by $\Box\mathsf{CP}$ we have $A^{\Box\downarrow} \leftrightarrow A$, and then we may deduce $\mathsf{iGL\overline{CT}\underline{P}} \nvdash A^{\Box\downarrow}$. By Theorem 9.8.5, there exists some quasi-classical perfect Kripke model $\mathcal{K}$ and some boolean interpretation I such that $\mathcal{K}, \alpha, I \not\models A^{\Box\downarrow}$. Corollary 9.3.43 implies $\mathcal{K}, \alpha, I \not\models_c A^{\Box\downarrow}$, which by soundness of GL for classical Kripke models, implies $\mathsf{GL} \nvdash A^{\Box\downarrow}$. □

Theorem 9.8.27 $\mathsf{iGL\overline{CT}\underline{P}C_a} = \mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*,\mathsf{PA}) \leq^{Prop} \mathsf{PL}(\mathsf{PA}^*,\mathsf{PA}) = \mathsf{iGL\overline{CT}\underline{P}}$.

Proof The arithmetical soundness of $\mathsf{iGL\overline{CT}\underline{P}}$ is straightforward and left to the reader. By Theorem 9.8.14 we have $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*,\mathsf{PA}) = \mathsf{iGL\overline{CT}\underline{P}C_a}$. We must show

$$\mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{CT}\underline{P}C_a};\mathsf{PA}^*,\mathsf{PA}) \leq^{\text{Prop}}_{f,\bar{f}} \mathcal{AC}(\mathsf{iGL\overline{CT}\underline{P}};\mathsf{PA}^*,\mathsf{PA}).$$

Given $A \in \mathcal{L}_\Box$, if $\mathsf{iGL\overline{C}T\underline{P}} \vdash A$, define $f(A) := \top$. If $\mathsf{iGL\overline{C}T\underline{P}} \nvdash A$, by Lemma 9.8.26 we have $\mathsf{GL} \nvdash A^{\Box\downarrow}$, and hence by Remark 9.7.4 there exists some propositional $(.)^{\Box\downarrow}$-substitution τ such that $\mathsf{GLC_a} \nvdash \tau(A^{\Box\downarrow})$. Define $f(A) := \tau(A)$ and $\bar{f}_A(\sigma) := \sigma_{\mathsf{PA}^*} \circ \tau$.

R1. Let $\mathsf{iGL\overline{C}T\underline{P}} \nvdash A$. By Lemma 9.8.26 we have $\mathsf{GL} \nvdash A^{\Box\downarrow}$ and then Remark 9.7.4 implies $\mathsf{GLC_a} \nvdash \tau(A^{\Box\downarrow})$, in which τ is as used for the definition of $f(A)$. Since τ is a $(.)^{\Box\downarrow}$-substitution, by Lemma 9.7.3 we have $\mathsf{GLC_a} \nvdash (\tau(A))^{\Box\downarrow}$. Then Lemma 9.8.13 implies that $\mathsf{iGL\overline{C}T\underline{P}C_a} \nvdash \tau(A)$, or in other words $\mathsf{iGL\overline{C}T\underline{P}C_a} \nvdash f(A)$.

R2. Let $\mathsf{PA} \nvdash \sigma_{\mathsf{PA}^*}(f(A))$ for some Σ_1-substitution σ. By definition of $f(A)$, we must have $\mathsf{iGL\overline{C}T\underline{P}} \nvdash A$, otherwise $f(A) := \top$, which contradicts $\mathsf{PA} \nvdash \sigma_{\mathsf{PA}^*}(f(A))$. Hence $f(A) = \tau(A)$ for some propositional $(.)^{\Box\downarrow}$-substitution τ. By Lemma 9.3.21 we have $\mathsf{PA} \nvdash \sigma_{\mathsf{PA}}(\tau(A)^{\Box\downarrow})$. Since $\mathsf{iK4} + \mathsf{CP_a}$ is included in $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{PA}) = \mathsf{GLC_a}$ (Theorem 9.7.1), by Lemma 9.7.3 we have $\mathsf{PA} \nvdash \sigma_{\mathsf{PA}}(\tau(A^{\Box\downarrow}))$. This implies that $\mathsf{PA} \nvdash [\bar{f}_A(\sigma)]_{\mathsf{PA}}(A^{\Box\downarrow})$ and again by Lemma 9.3.21 we have $\mathsf{PA} \nvdash [\bar{f}_A(\sigma)]_{\mathsf{PA}^*}(A)$. □

Lemma 9.8.28 *For every $A \in \mathcal{L}_\Box$, we have* $\mathsf{iGLCT} \vdash A$ *iff* $\mathsf{GL} \vdash A^\Box$.

Proof One may use induction on the proof $\mathsf{iGLCT} \vdash A$ to show that $\mathsf{GL} \vdash A^\Box$. For the other direction, we reason contrapositively. Let $\mathsf{iGLCT} \nvdash A$. Since in iGLC we have $A \leftrightarrow A^\Box$, we have $\mathsf{iGLCT} \nvdash A^\Box$. Hence by Theorem 9.3.40 there is some perfect quasi-classical model $\mathcal{K}$ such that $\mathcal{K}, \alpha \nVdash A^\Box$. Hence by Corollary 9.3.43 $\mathcal{K}, \alpha \not\models_c A^\Box$. Since $\models_c$ is just a classical semantics for the modal logic GL, by the soundness of GL for finite irreflexive Kripke models (Smoryński 1985, Sect. 2.2) we may deduce $\mathsf{GL} \nvdash A^\Box$, as desired. □

Lemma 9.8.29 *For every $A \in \mathcal{L}_\Box$, we have* $\mathsf{iGLCT} \vdash A$ *iff* $\mathsf{GLC_a} \vdash A^\Box$.

Proof There are at least two options for the proof. First is that one repeat a similar argument of the proof o for Lemma 9.8.28. Second proof follows: By Corollary 9.8.22, $\mathsf{iGLCT} \vdash A$ iff $\mathsf{iGL\overline{C}T\underline{P}C_a} \vdash A^{\Box\uparrow}$, and Lemma 9.8.13 implies $\mathsf{iGL\overline{C}T\underline{P}C_a} \vdash A^{\Box\uparrow}$ iff $\mathsf{GLC_a} \vdash (A^{\Box\uparrow})^{\Box\downarrow}$. Since $\mathsf{iK4} \vdash A^\Box \leftrightarrow (A^{\Box\uparrow})^{\Box\downarrow}$, we have the desired result. □

Theorem 9.8.30 $\mathsf{iGLCT} = \mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{PA}^*) \leq^{Prop} \mathsf{PL}(\mathsf{PA}^*, \mathsf{PA}^*) = \mathsf{iGLCT}$.

Proof The arithmetical soundness of iGLCT for general substitutions, i.e. $\mathcal{AS}(\mathsf{iGLCT}; \mathsf{PA}^*, \mathsf{PA}^*)$, is straightforward and left to the reader. By Theorem 9.8.21 we have $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathsf{PA}^*) = \mathsf{iGLCT}$. We must show

$$\mathcal{AC}_{\Sigma_1}(\mathsf{iGLCT}; \mathsf{PA}^*, \mathsf{PA}^*) \leq^{\text{Prop}}_{f,\bar{f}} \mathcal{AC}(\mathsf{iGLCT}; \mathsf{PA}^*, \mathsf{PA}^*).$$

Given $A \in \mathcal{L}_\Box$, if $\mathsf{iGLCT} \vdash A$, define $f(A) := \top$. If $\mathsf{iGLCT} \nvdash A$, by Lemma 9.8.28 we have $\mathsf{GL} \nvdash A^\Box$, and hence by Remark 9.7.4 there exists some propositional $(.)^{\Box\downarrow}$-substitution τ such that $\mathsf{GLC_a} \nvdash \tau(A^\Box)$. Define $f(A) := \tau(A)$ and $\bar{f}_A(\sigma) := \sigma_{\mathsf{PA}^*} \circ \tau$.

R1. Let $\mathsf{iGLCT} \nvdash A$. By Lemma 9.8.28 we have $\mathsf{GL} \nvdash A^{\Box}$ and then Remark 9.7.4 implies $\mathsf{GLC_a} \nvdash \tau(A^{\Box})$, in which τ is as used for the definition of $f(A)$. Since τ is a $(.)^{\Box\downarrow}$-substitution, by Lemma 9.7.3 we have $\mathsf{GLC_a} \nvdash \tau(A)^{\Box}$. Then Lemma 9.8.29 implies that $\mathsf{iGLCT} \nvdash \tau(A)$, or in other words $\mathsf{iGLCT} \nvdash f(A)$.

R2. Let $\mathsf{PA}^* \nvdash \sigma_{\mathsf{PA}^*}(f(A))$ for some Σ_1-substitution σ. By definition of $f(A)$, we must have $\mathsf{iGLCT} \nvdash A$, otherwise $f(A) := \top$, which contradicts $\mathsf{PA}^* \nvdash \sigma_{\mathsf{PA}^*}(f(A))$. Hence $f(A) = \tau(A)$ for some propositional $(.)^{\Box\downarrow}$-substitution τ. Then we have $\mathsf{PA} \nvdash \sigma_{\mathsf{PA}^*}(\tau(A))^{\mathsf{PA}}$ and by Lemma 9.3.20 we have $\mathsf{PA} \nvdash \sigma_{\mathsf{PA}}(\tau(A)^{\Box})$. Since $\mathsf{iK4} + \mathsf{CP_a}$ is included in $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathsf{PA}) = \mathsf{GLC_a}$ (Theorem 9.7.1), by Lemma 9.7.3 we have $\mathsf{PA} \nvdash \sigma_{\mathsf{PA}}(\tau(A^{\Box}))$. This implies that $\mathsf{PA} \nvdash [\bar{f}_A(\sigma)]_{\mathsf{PA}}(A^{\Box})$ and again by Lemma 9.3.20 we have $\mathsf{PA} \nvdash ([\bar{f}_A(\sigma)]_{\mathsf{PA}^*}(A))^{\mathsf{PA}}$. Hence $\mathsf{PA}^* \nvdash [\bar{f}_A(\sigma)]_{\mathsf{PA}^*}(A)$. ◻

Lemma 9.8.31 *For every $A \in \mathcal{L}_{\Box}$, we have* $\mathsf{iGL\overline{C}T\underline{S^*P}} \vdash A$ *iff* $\mathsf{GL\underline{S}} \vdash A^{\Box\downarrow}$.

Proof We use induction on the proof $\mathsf{iGL\overline{C}T\underline{S^*P}} \vdash A$ and show $\mathsf{GL\underline{S}} \vdash A^{\Box\downarrow}$. All cases are similar to the one for item 3 above, except for

- $A = \Box B \to B^{\Box}$: since $\mathsf{iK4} \vdash A^{\Box\downarrow} \leftrightarrow (\Box B^{\Box} \to B^{\Box})$, we may deduce $\mathsf{GL\underline{S}} \vdash A^{\Box\downarrow}$.

For the other way around, let $\mathsf{iGL\overline{C}T\underline{S^*P}} \nvdash A$. Then by $\Box\mathsf{CP}$ we have $A^{\Box\downarrow} \leftrightarrow A$, and then we may deduce $\mathsf{iGL\overline{C}T\underline{S^*P}} \nvdash A^{\Box\downarrow}$. By Theorem 9.8.7, there exists some quasi-classical perfect Kripke model $\mathcal{K}$ and some boolean interpretation I such that $\mathcal{K}, \alpha, I \not\models A^{\Box\downarrow}$ and $\mathcal{K}$ is $A^{\Box\downarrow}$-sound at α. Corollary 9.3.43 implies $\mathcal{K}, \alpha, I \not\models_c A^{\Box\downarrow}$, which by soundness of $\mathsf{GL\underline{S}}$ (restricted to sub-formulas of $A^{\Box\downarrow}$) for $A^{\Box\downarrow}$-sound classical Kripke models, implies $\mathsf{GL\underline{S}} \nvdash A^{\Box\downarrow}$. ◻

Theorem 9.8.32 $\mathsf{iGL\overline{C}T\underline{S^*PC}_a} = \mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathbb{N}) \leq^{Prop} \mathsf{PL}(\mathsf{PA}^*, \mathbb{N}) = \mathsf{iGL\overline{C}T\underline{S^*P}}$.

Proof The arithmetical soundness of $\mathsf{iGL\overline{C}T\underline{S^*P}}$ is straightforward and left to the reader. By Theorem 9.8.18 we have $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*, \mathbb{N}) = \mathsf{iGL\overline{C}T\underline{S^*PC}_a}$. We must show

$$\mathcal{AC}_{\Sigma_1}(\mathsf{iGL\overline{C}T\underline{S^*PC}_a}; \mathsf{PA}^*, \mathbb{N}) \leq^{\mathrm{Prop}}_{f,\bar{f}} \mathcal{AC}(\mathsf{iGL\overline{C}T\underline{S^*P}}; \mathsf{PA}^*, \mathbb{N}).$$

Given $A \in \mathcal{L}_{\Box}$, if $\mathsf{iGL\overline{C}T\underline{S^*P}} \vdash A$, define $f(A) := \top$. If $\mathsf{iGL\overline{C}T\underline{S^*P}} \nvdash A$, by Lemma 9.8.31 we have $\mathsf{GL\underline{S}} \nvdash A^{\Box\downarrow}$, and hence by Remark 9.7.4 there exists some propositional $(.)^{\Box\downarrow}$-substitution τ such that $\mathsf{GL\underline{SC}_a} \nvdash \tau(A^{\Box\downarrow})$. Define $f(A) := \tau(A)$ and $\bar{f}_A(\sigma) := \sigma_{\mathsf{PA}^*} \circ \tau$.

R1. Let $\mathsf{iGL\overline{C}T\underline{S^*P}} \nvdash A$. By Lemma 9.8.31 we have $\mathsf{GL\underline{S}} \nvdash A^{\Box\downarrow}$ and then Remark 9.7.4 implies $\mathsf{GL\underline{S}C_a} \nvdash \tau(A^{\Box\downarrow})$, in which τ is as used for the definition of $f(A)$. Since τ is a $(.)^{\Box\downarrow}$-substitution, by Lemma 9.7.3 we have $\mathsf{GL\underline{S}C_a} \nvdash \tau(A)^{\Box\downarrow}$. Then Lemma 9.8.17 implies that $\mathsf{iGL\overline{C}T\underline{S^*P}C_a} \nvdash \tau(A)$, or in other words $\mathsf{iGL\overline{C}T\underline{S^*P}C_a} \nvdash f(A)$.

R2. Let $\mathbb{N} \not\models \sigma_{\mathsf{PA}^*}(f(A))$ for some Σ_1-substitution σ. By definition of $f(A)$, we must have $\mathsf{iGL\overline{C}T\underline{S^*P}} \nvdash A$, otherwise $f(A) := \top$, which contradicts $\mathbb{N} \not\models \sigma_{\mathsf{PA}^*}(f(A))$. Hence $f(A) = \tau(A)$ for some propositional $(.)^{\Box\downarrow}$-substitution τ. We have $\mathbb{N} \not\models \sigma_{\mathsf{PA}^*}(\tau(A))$ and by Lemma 9.3.21we have $\mathbb{N} \not\models \sigma_{\mathsf{PA}}(\tau(A)^{\Box\downarrow})$. Since $\mathsf{iK4} + \mathsf{CP_a}$ is included in $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}, \mathbb{N}) = \mathsf{GL\underline{S}C_a}$ (Theorem 9.7.1), by Lemma 9.7.3 we have $\mathbb{N} \not\models \sigma_{\mathsf{PA}}(\tau(A^{\Box\downarrow}))$. This implies that $\mathbb{N} \not\models [\bar{f}_A(\sigma)]_{\mathsf{PA}}(A^{\Box\downarrow})$ and again by Lemma 9.3.21 we have $\mathbb{N} \not\models [\bar{f}_A(\sigma)]_{\mathsf{PA}^*}(A)$. $\Box$

Theorem 9.8.33 $\mathsf{iGL\overline{C}T\underline{S^*P}} = \mathsf{PL}(\mathsf{PA}^*, \mathbb{N}) \leq^{Prop} \mathsf{PL}(\mathsf{PA}^*, \mathsf{PA}^*) = \mathsf{iGLCT}$.

Proof By Theorems 9.8.32 and 9.8.30 we have $\mathsf{PL}(\mathsf{PA}^*, \mathsf{PA}^*) = \mathsf{iGLCT}$ and $\mathsf{PL}(\mathsf{PA}^*, \mathbb{N}) = \mathsf{iGL\overline{C}T\underline{S^*P}}$. We must show

$$\mathcal{AC}(\mathsf{iGL\overline{C}T\underline{S^*P}}; \mathsf{PA}^*, \mathbb{N}) \leq^{\text{Prop}}_{f,\bar{f}} \mathcal{AC}(\mathsf{iGLCT}; \mathsf{PA}^*, \mathsf{PA}^*).$$

Given $A \in \mathcal{L}_\Box$, define $f(A) = \Box A$ and $\bar{f}_A$ as the identity function.

R1. Let $\mathsf{iGL\overline{C}T\underline{S^*P}} \vdash \Box A$. By soundness of $\mathsf{iGL\overline{C}T\underline{S^*P}} = \mathsf{PL}(\mathsf{PA}^*, \mathbb{N})$, for every substitution σ we have $\mathbb{N} \models \sigma_{\mathsf{PA}^*}(\Box A)$ and hence $\mathsf{PA}^* \vdash \sigma_{\mathsf{PA}^*}(A)$. Then by arithmetical completeness of $\mathsf{iGLCT} = \mathsf{PL}(\mathsf{PA}^*, \mathsf{PA}^*)$, we have $\mathsf{iGLCT} \vdash A$.

R2. Let $\mathbb{N} \not\models \sigma_{\mathsf{PA}^*}(\Box A)$. Then $\mathsf{PA}^* \nvdash \sigma_{\mathsf{PA}^*}(A)$, as desired. $\Box$

Theorem 9.8.34 $\mathsf{iGL\overline{CT}} = \mathsf{PL}(\mathsf{PA}^*, \mathsf{HA}) \leq^{Prop} \mathsf{PL}(\mathsf{PA}^*, \mathsf{PA}) = \mathsf{iGL\overline{C}T\underline{P}}$.

Proof We already have $\mathsf{iGL\overline{CT}} = \mathsf{PL}(\mathsf{PA}^*, \mathsf{HA})$ and $\mathsf{PL}(\mathsf{PA}^*, \mathsf{PA}) = \mathsf{iGL\overline{C}T\underline{P}}$ by Theorems 9.8.27 and 9.8.25. It is enough here to show that $\mathcal{AC}(\mathsf{iGL\overline{C}T\underline{P}}; \mathsf{PA}^*, \mathsf{PA}) \leq_{f,\bar{f}} \mathcal{AC}(\mathsf{iGL\overline{CT}}; \mathsf{PA}^*, \mathsf{HA})$. Given $A \in \mathcal{L}_\Box$, let $f(A) := (A)^{\neg\uparrow}$ and $\bar{f}_A$ as the identity function.

R1. If $\mathsf{iGL\overline{CT}} \vdash A^{\neg\uparrow}$ then $\mathsf{iGL\overline{C}T\underline{P}} \vdash A^{\neg\uparrow}$, and since we have PEM in $\mathsf{iGL\overline{C}T\underline{P}C_a}$, we may conclude $\mathsf{iGL\overline{C}T\underline{P}C_a} \vdash A$.

R2. If $\mathsf{HA} \nvdash \sigma_{\mathsf{PA}^*}(A^{\neg\uparrow})$, for a substitution σ, then by Lemma 9.5.18 we have $\mathsf{HA} \nvdash (\sigma_{\mathsf{PA}^*}(A))^{\neg}$. Hence by Lemma 9.5.15 we have $\mathsf{PA} \nvdash \sigma_{\mathsf{PA}^*}(A)$. $\Box$

9.9 Conclusion

From Diagram 9.5, it turns out that the truth Σ_1-provability logic of HA, is the hardest provability logic among the provability logics in Table 9.3. Closer inspection in the reductions provided in previous sections, reveals that all propositional reductions, i.e. the functions f, are computable. Hence by decidability of $\mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathbb{N})$ (Corollary 9.5.6) and Theorem 9.4.8, we have the decidability of all provability logics in Table 9.3.

Corollary 9.9.1 *All provability logics in the Table 9.3, are decidable.*

So far, we have seen many reductions of provability logics. The reductions, helped out to prove new arithmetical completeness results, have a more general view of all provability logics and intuitively say which provability logic is *harder*. The reader may wonder what other reductions hold, beyond the transitive closure of the Diagram 9.5. However it seems more likely that no other reductions hold, at the moment we can not say anything more than that. This question calls for more work.

Conjecture 9.9.2 We conjecture that the following characterizations and reductions hold:

1. $\mathsf{iH}_\sigma = \mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{HA}) \leq^{\mathrm{Prop}} \mathsf{PL}(\mathsf{HA}, \mathsf{HA}) = \mathsf{iH}$. This will solve a traditional great open problem which calls for the characterization and decidability of the provability logic of HA with which all this research and also (Ardeshir and Mojtahedi 2018, 2015) started.
2. $\mathsf{iH}\underline{\mathsf{SP}} = \mathsf{PL}(\mathsf{HA}, \mathbb{N}) \leq^{\mathrm{Prop}} \mathsf{PL}(\mathsf{HA}, \mathsf{HA}) = \mathsf{iH}$.
3. $\mathsf{iH}_\sigma\underline{P} = \mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathsf{PA}) \leq^{\mathrm{Prop}} \mathsf{PL}(\mathsf{HA}, \mathsf{PA}) = \mathsf{iH}\underline{\mathsf{P}}$.
4. $\mathsf{iH}_\sigma\underline{\mathsf{SP}} = \mathsf{PL}_{\Sigma_1}(\mathsf{HA}, \mathbb{N}) \leq^{\mathrm{Prop}} \mathsf{PL}(\mathsf{HA}, \mathbb{N}) = \mathsf{iH}\underline{\mathsf{SP}}$.
5. $\mathsf{iH}^{**}_\sigma = \mathsf{PL}_{\Sigma_1}(\mathsf{HA}^*, \mathsf{HA}^*) \leq^{\mathrm{Prop}} \mathsf{PL}(\mathsf{HA}^*, \mathsf{HA}^*) = \mathsf{iH}^*$.

Moreover, all reductions are computable and hence all provability logics are conjectured to be decidable. In which

- iH is as defined in (Iemhoff 2001),
- iH^* as defined in (Ardeshir and Mojtahedi 2019),
- $\mathsf{iH}\underline{\mathsf{P}}$ is iH plus $\underline{\mathsf{P}}$,
- $\mathsf{iH}\underline{\mathsf{SP}}$ is iH plus $\underline{\mathsf{S}}$ and $\underline{\mathsf{P}}$,

Acknowledgements I would like to thank Mohammad Ardeshir for reading of the first draft of this paper and his very helpful comments, remarks and corrections. I am very grateful to the referee for her/his very helpful corrections, comments and suggestions.

Appendices

Table 9.1 List of axiom schemas

Name(s)	Axiom scheme	Name(s)	Axiom scheme
$\underline{\mathsf{K}}$	$\Box(A \to B) \to (\Box A \to \Box B)$	$\underline{\mathsf{4}}$	$\Box A \to \Box\Box A$
$\underline{\mathsf{L\ddot{o}b}}$, $\underline{\mathsf{L}}$	$\Box(\Box A \to A) \to \Box A$	$\underline{\mathsf{CP}}$, $\underline{\mathsf{C}}$	$A \to \Box A$
$\underline{\mathsf{S}}$	$\Box A \to A$	$\underline{\mathsf{CP_a}}$, $\underline{\mathsf{C_a}}$	$p \to \Box p$ for atomic variable p
$\underline{\mathsf{S}^*}$	$\Box A \to A^{\Box}$	$\underline{\mathsf{PEM}}$, $\underline{\mathsf{P}}$	$A \vee \neg A$
$\underline{\mathsf{Le}}$	$\Box(A \vee B) \to \Box(\boxdot A \vee \boxdot B)$	$\underline{\mathsf{Le}^+}$	$\Box A \to \Box A^l$
$\underline{\mathsf{TP}}$, $\underline{\mathsf{T}}$	$\Box(A \to B) \to (A \vee (A \to B))$	$\underline{\mathsf{i}}$	All theorems of $\mathsf{IPC}_{\Box}$
		$\underline{\mathsf{V}}$	$A \leftrightarrow A^-$

For an axiom scheme $\underline{\mathsf{A}}$, let $\overline{\mathsf{A}}$ indicate $\Box\underline{\mathsf{A}}$ and A indicates $\overline{\mathsf{A}} \wedge \underline{\mathsf{A}}$

Table 9.2 List of translations

Translation	Description	Reference
$\Box(.)$	Modal operator	
$(.)^{\Box\downarrow}$	Replace every inner (inside some $\Box$) subformula B with $B \wedge \Box B$	Definition 9.3.4
$(.)^{\Box\uparrow}$	Replace every outer (not inside any $\Box$) subformula B with $B \wedge \Box B$	Definition 9.3.4
$(.)^{\neg\uparrow}$	Insert $\neg\neg$ behind every outer (not inside any $\Box$) subformula	Definition 9.5.14
$(.)^{\dagger}$	Recursively replaces every subformula $\Box B$ with $\Box B^{\to}$, in which in which $B^{\to}$ is the classically equivalent proposition of the form $\bigwedge_i(\bigwedge \Gamma_i \to \bigvee \Delta_i)$, and Γ_i, Δ_i only contains atomics or boxed	Definition 9.7.7
τ	Not a fixed translation. It is a substitution and for every reduction which uses τ, it is defined in the proof of its reference theorem	

Table 9.3 List of all provability logics

Theory	Axioms	Provability Logic(s)	Reference
$\mathsf{iK4}$	$\mathsf{i,K,4}$		
iGL	$\mathsf{iK4,L}$		
GL	$\mathsf{iGL,P}$	$\mathsf{PL}(\mathsf{PA},\mathsf{PA})$	(Solovay 1976)
$\mathsf{GLC_a}$	$\mathsf{GL,CP_a}$	$\mathsf{PL}_{\Sigma_1}(\mathsf{PA},\mathsf{PA})$	(Visser 1982)
$\mathsf{GL\underline{S}}$	$\mathsf{GL,\underline{S}}$	$\mathsf{PL}(\mathsf{PA},\mathbb{N})$	(Solovay 1976)
$\mathsf{GL\underline{S}C_a}$	$\mathsf{GLC_a,\underline{S}}$	$\mathsf{PL}_{\Sigma_1}(\mathsf{PA},\mathbb{N})$	(Visser 1982)
iGLCT	$\mathsf{iGL,C,T}$	$\mathsf{PL}(\mathsf{PA}^*,\mathsf{PA}^*)$ $\mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*,\mathsf{PA}^*)$	(Visser 1982)
iH_σ	$\mathsf{iGL,V,Le}^+$	$\mathsf{PL}_{\Sigma_1}(\mathsf{HA},\mathsf{HA})$	(Ardeshir and Mojtahedi 2018) (Visser and Zoethout 2019)
iH_σ^{**}	$\{A : \mathsf{iH}_\sigma \vdash A^{\Box}\}$	$\mathsf{PL}_{\Sigma_1}(\mathsf{HA}^*,\mathsf{HA}^*)$	(Ardeshir and Mojtahedi 2019)
$\mathsf{iH}_\sigma\mathsf{\underline{P}}$	$\mathsf{iH}_\sigma,\mathsf{\underline{P}}$	$\mathsf{PL}_{\Sigma_1}(\mathsf{HA},\mathsf{PA})$	Theorem 9.5.12
$\mathsf{iH}_\sigma\mathsf{\underline{SP}}$	$\mathsf{iH}_\sigma,\mathsf{\underline{S}},\mathsf{\underline{P}}$	$\mathsf{PL}_{\Sigma_1}(\mathsf{HA},\mathbb{N})$	Theorem 9.5.13
$\mathsf{iH}_\sigma\mathsf{\underline{SP}}^*$	$\{A : \mathsf{iH}_\sigma\mathsf{\underline{SP}} \vdash A^{\Box\downarrow}\}$	$\mathsf{PL}_{\Sigma_1}(\mathsf{HA}^*,\mathbb{N})$	Theorem 9.6.1
$\mathsf{iH}_\sigma\mathsf{\underline{P}}^*$	$\{A : \mathsf{iH}_\sigma\mathsf{\underline{P}} \vdash A^{\Box\downarrow}\}$	$\mathsf{PL}_{\Sigma_1}(\mathsf{HA}^*,\mathsf{PA})$	Theorem 9.6.2
iH_σ^*	$\{A : \mathsf{iH}_\sigma \vdash A^{\Box\downarrow}\}$	$\mathsf{PL}_{\Sigma_1}(\mathsf{HA}^*,\mathsf{HA})$	Theorem 9.6.3
$\mathsf{iGL\overline{P}C_a}$	$\mathsf{iGL,\overline{P},C_a}$	$\mathsf{PL}_{\Sigma_1}(\mathsf{PA},\mathsf{HA})$	Theorem 9.7.21
$\mathsf{iGL\overline{P}}$	$\mathsf{iGL,\overline{P}}$	$\mathsf{PL}(\mathsf{PA},\mathsf{HA})$	Theorem 9.7.25
$\mathsf{iGL\overline{C}T\underline{P}C_a}$	$\mathsf{iGL,\overline{C},T,\underline{P},C_a}$	$\mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*,\mathsf{PA})$	Theorem 9.8.14
$\mathsf{iGL\overline{C}\overline{T}C_a}$	$\mathsf{iGL,\overline{C},\overline{T},C_a}$	$\mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*,\mathsf{HA})$	Theorem 9.8.16
$\mathsf{iGL\overline{C}T\underline{S^*}\underline{P}C_a}$	$\mathsf{iGL,\overline{C},T,\underline{S^*},\underline{P},C_a}$	$\mathsf{PL}_{\Sigma_1}(\mathsf{PA}^*,\mathbb{N})$	Theorem 9.8.18
$\mathsf{iGL\overline{C}T\underline{P}}$	$\mathsf{iGL,\overline{C},T,\underline{P}}$	$\mathsf{PL}(\mathsf{PA}^*,\mathsf{PA})$	Theorem 9.8.27
$\mathsf{iGL\overline{C}\overline{T}}$	$\mathsf{iGL,\overline{C},\overline{T}}$	$\mathsf{PL}(\mathsf{PA}^*,\mathsf{HA})$	Theorem 9.8.25
$\mathsf{iGL\overline{C}T\underline{S^*}\underline{P}}$	$\mathsf{iGL,\overline{C},T,\underline{S^*},\underline{P}}$	$\mathsf{PL}(\mathsf{PA}^*,\mathbb{N})$	Theorem 9.8.32

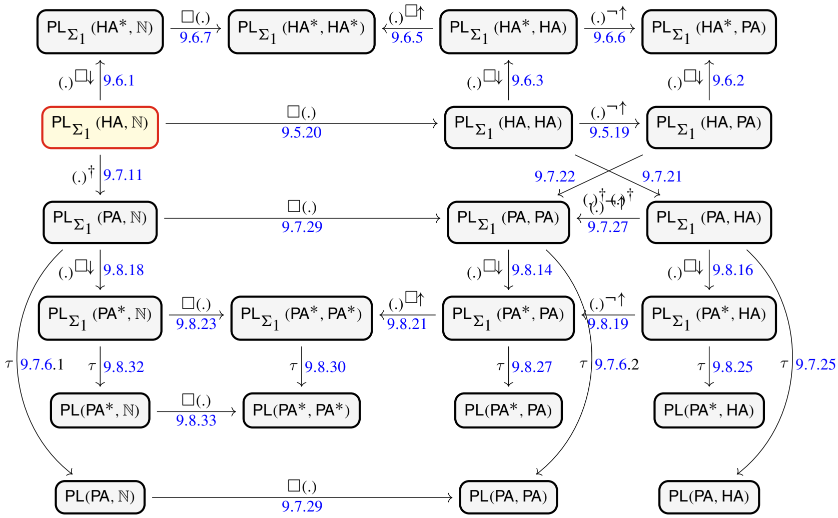

Diagram 9.5 Reductions of all provability logics. Arrows indicate a reduction of the completeness of the right hand side to the left one. The propositional reduction is shown over the arrow line and the theorem number proving this, is shown under arrow line. For definitions see Tables 9.1, 9.2, and 9.3

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Chapter 10
On PBZ*–Lattices

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Abstract We continue our investigation of paraorthomodular BZ*-lattices (PBZ*–lattices), started in Giuntini et al. (2016, 2017, 2018, 2020), Mureşan (2019). We shed further light on the structure of the subvariety lattice of the variety $\mathbb{PBZL}^*$ of PBZ*–lattices; in particular, we provide axiomatic bases for some of its members. Further, we show that some distributive subvarieties of $\mathbb{PBZL}^*$ are term-equivalent to well-known varieties of expanded Kleene lattices or of nonclassical modal algebras. By so doing, we somehow help the reader to locate PBZ*–lattices on the atlas of algebraic structures for nonclassical logics.

Keywords PBZ*-lattices · Brouwer-Zadeh lattices · Quantum logic · Orthomodular lattices · Nonclassical modal algebras

10.1 Introduction

One of the core topics within the impressive *corpus* of Mohammad Ardeshir's contributions to mathematical logic is the algebraic semantics of nonclassical logics. In particular, Ardeshir and his collaborators intensively investigated the relationships between Visser's basic propositional calculus (Visser 1981) and its algebraic counterpart, *basic algebras*, generalisations of Heyting algebras where only the left-to-right direction of the residuation equivalence $x \wedge y \leq z \iff x \leq y \to z$ is retained (Alizadeh and Ardeshir 2006; Ardeshir and Ruitenburg 2001, 1998). Also, in a basic algebra $\mathbf{A}$ there may be $a \in A$ such that $1 \to a \neq a$. Crucially, the introduction of these structures is not motivated by abstraction *per se*: Ardeshir argues that basic algebras can contribute to a deeper understanding of constructive mathematics, whence they can have a paramount *foundational* interest.

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M. Mojtahedi et al. (eds.), *Mathematics, Logic, and their Philosophies*,
Logic, Epistemology, and the Unity of Science 49,
https://doi.org/10.1007/978-3-030-53654-1_10

The approach that led to the introduction of paraorthomodular BZ*-lattices (*PBZ*–lattices*) (Giuntini et al. 2016, 2017, 2018, 2020; Mureşan 2019) is similar. The key motivation for this particular generalisation of orthomodular lattices, in fact, comes from the foundations of quantum mechanics. Consider the structure

$$\mathbf{E}\left(\mathbf{H}\right)=\left(\mathcal{E}\left(\mathbf{H}\right),\wedge_{s},\vee_{s},^{\prime},^{\sim},\mathbb{O},\mathbb{I}\right),$$

where:

- $\mathcal{E}\left(\mathbf{H}\right)$ is the set of all effects of a given complex separable Hilbert space $\mathbf{H}$, i.e., positive linear operators of $\mathbf{H}$ that are bounded by the identity operator $\mathbb{I}$;
- $\wedge_s$ and $\vee_s$ are the meet and the join, respectively, of the *spectral ordering* $\leq_s$ so defined for all $E, F \in \mathcal{E}\left(\mathbf{H}\right)$:

$$E \leq_s F \text{ if and only if } \forall\lambda \in \mathbb{R}: \ M^F(\lambda) \leq M^E(\lambda),$$

  where for any effect E, M^E is the unique spectral family (Kreyszig 1978, Chap. 7) such that $E = \int_{-\infty}^{\infty} \lambda \, dM^E(\lambda)$ (the integral is here meant in the sense of norm-converging Riemann-Stieltjes sums (Stroock 1998, Chap. 1));
- $\mathbb{O}$ and $\mathbb{I}$ are the null and identity operators, respectively;
- $E' = \mathbb{I} - E$ and $E^{\sim} = P_{\ker(E)}$ (the projection onto the kernel of E).

The operations in $\mathbf{E}\left(\mathbf{H}\right)$ are well-defined. The spectral ordering is indeed a lattice ordering (Olson 1971; de Groote 2005) that coincides with the usual ordering of effects induced via the trace functional when both orderings are restricted to the set of projection operators of the same Hilbert space.

A PBZ*–lattice can be viewed as an abstraction from this concrete physical model, much in the same way as an orthomodular lattice can be viewed as an abstraction from a certain structure of projection operators in a complex separable Hilbert space. The faithfulness of PBZ*–lattices to the physical model whence they stem is further underscored by the fact that they reproduce at an abstract level the “collapse” of several notions of *sharp physical property* that can be observed in $\mathbf{E}\left(\mathbf{H}\right)$.

Referring the reader to Giuntini et al. (2016) for a more detailed discussion of the previous issues, we now summarise the discourse of the present paper. In Sect. 10.2 we collect some preliminaries, with the twofold aim of fixing the notation to be used throughout the article and of making the article itself sufficiently self-contained—although we will occasionally need to refer the reader to results included in the previous papers on the subject. In Sect. 10.3 we zoom in on some subvarieties of the variety $\mathbb{PBZL}^*$ of PBZ*–lattices. First, we axiomatise the subvariety of $\mathbb{PBZL}^*$ generated by a particular algebra whose role in the context of $\mathbb{PBZL}^*$ is analogous to the role of the benzene ring in the context of ortholattices. Next, we prove that the subvariety of $\mathbb{PBZL}^*$ generated by the (unique PBZ*–lattice over the) 4-element Kleene chain is the unique antiorthomodular cover of the variety generated by the (unique PBZ*–lattice over the) 3-element Kleene chain. Finally, we put to good use the construction of subdirect products of varieties of PBZ*–lattices, employing them to characterise some joins of subvarieties of PBZ*–lattices. Section 10.4 is devoted to term-equivalence results that establish connections between *distributive* varieties of PBZ*–lattices and some known expansions of Kleene lattices, on the one hand, and nonclassical modal algebras—i.e., modal algebras whose nonmodal reducts are generic De Morgan algebras rather than Boolean algebras—on the other. We hope that these equivalences can help readers to make out the whereabouts of PBZ*–lattices in the vast landscape of algebraic structures for nonclassical logic, a territory whose exploration has been decisively aided by the research work of Mohammad Ardeshir.

10.2 Preliminaries

For further information on the notions recalled in this section, we refer the reader to Giuntini et al. (2016, 2017, 2018, 2020), Mureşan (2019).

We denote by $\mathbb{N}$ the set of the natural numbers and by $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. If $\mathbf{A}$ is an algebra, then A will denote its universe. We call *trivial algebras* the singleton algebras. For any $n \in \mathbb{N}^*$, $\mathbf{D}_n$ will denote the n–element chain, as well as any bounded lattice-ordered structure having this chain as a bounded lattice reduct. For any lattice $\mathbf{L}$, we denote by $\mathbf{L}^d$ the dual of $\mathbf{L}$. For any bounded lattices $\mathbf{L}$ and $\mathbf{M}$, we denote by $\mathbf{L} \oplus \mathbf{M}$ the ordinal sum of $\mathbf{L}$ with $\mathbf{M}$, obtained by glueing together the top element of $\mathbf{L}$ and the bottom element of $\mathbf{M}$, thus stacking $\mathbf{M}$ on top of $\mathbf{L}$, and by $L \oplus M$ the universe of the bounded lattice $\mathbf{L} \oplus \mathbf{M}$; clearly, the ordinal sum of bounded lattices is associative.

Let $\mathbb{V}$ be a variety of algebras of similarity type τ and $\mathbb{C}$ a class of algebras with τ–reducts. We denote by $\mathrm{I}_{\mathbb{V}}(\mathbb{C})$, $\mathrm{H}_{\mathbb{V}}(\mathbb{C})$, $\mathrm{S}_{\mathbb{V}}(\mathbb{C})$ and $\mathrm{P}_{\mathbb{V}}(\mathbb{C})$ the classes of the isomorphic images, homomorphic images, subalgebras and direct products of τ–reducts of members of $\mathbb{C}$, respectively, and by $V_{\mathbb{V}}(\mathbb{C}) = \mathrm{H}_{\mathbb{V}}\mathrm{S}_{\mathbb{V}}\mathrm{P}_{\mathbb{V}}(\mathbb{C})$ the subvariety of $\mathbb{V}$ generated by the τ–reducts of the members of $\mathbb{C}$. For any class operator O and any $\mathbf{A} \in \mathbb{C}$, the notation $\mathrm{O}_{\mathbb{V}}(\{\mathbf{A}\})$ will be streamlined to $\mathrm{O}_{\mathbb{V}}(\mathbf{A})$. If $\mathbf{A}$ is an algebra having a τ–reduct, $n \in \mathbb{N}$ and $\kappa_1, \ldots, \kappa_n$ are constants over τ, then we denote by $\mathrm{Con}_{\mathbb{V}}(\mathbf{A})$ the complete lattice of the congruences of the τ–reduct of $\mathbf{A}$, as well as the set reduct of this congruence lattice, and by $\mathrm{Con}_{\mathbb{V}_{\kappa_1,\ldots,\kappa_n}}(\mathbf{A})$ the complete sublattice of $\mathrm{Con}_{\mathbb{V}}(\mathbf{A})$ consisting of the congruences with singleton classes of $\kappa_1^{\mathbf{A}}, \ldots, \kappa_n^{\mathbf{A}}$, as well as its set reduct. If $\mathbb{V}$ is the variety of lattices or that of bounded lattices, then the subscript $_{\mathbb{V}}$ will be eliminated from the previous notations. If $\mathbb{C} \subseteq \mathbb{V}$, then we denote by $Si(\mathbb{C})$ the class of the members of $\mathbb{C}$ which are subdirectly irreducible in $\mathbb{V}$. The lattice of subvarieties of $\mathbb{V}$ and its set reduct will be denoted by $\mathrm{Subvar}(\mathbb{V})$.

An *involution lattice* (in brief, *I–lattice*) is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot')$ of type $(2, 2, 1)$ such that $(L, \wedge, \vee)$ is a lattice and $\cdot' : L \to L$ is an order–reversing operation that satisfies $a'' = a$ for all $a \in L$. This makes $\cdot'$ a dual lattice automorphism of $\mathbf{L}$, called *involution*.

A *bounded involution lattice* (in brief, *BI–lattice*) is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \wedge, \vee, 0, 1)$ is a bounded lattice and

$(L, \wedge, \vee, \cdot')$ is an involution lattice. A distributive bounded involution lattice is called a *De Morgan algebra*.

For any BI–lattice $\mathbf{L}$, we denote by $S(\mathbf{L})$ the set of the *sharp elements* of $\mathbf{L}$, that is: $S(\mathbf{L}) = \{x \in L : x \vee x' = 1\}$. A BI–lattice $\mathbf{L}$ is called an *ortholattice* if and only if all its elements are sharp, and it is called an *orthomodular lattice* if and only if, for all $a, b \in L$, $a \leq b$ implies $b = (b \wedge a') \vee a$.

A *pseudo–Kleene algebra* is a BI–lattice $\mathbf{L}$ that satisfies $a \wedge a' \leq b \vee b'$ for all $a, b \in L$. The involution of a pseudo–Kleene algebra is called *Kleene complementation*. Distributive pseudo–Kleene algebras are called *Kleene algebras* or *Kleene lattices*.

Clearly, for any bounded lattice $\mathbf{L}$ and any BI–lattice $\mathbf{K}$, if $\mathbf{K}_l$ is the bounded lattice reduct of $\mathbf{K}$, then the bounded lattice $\mathbf{L} \oplus \mathbf{K}_l \oplus \mathbf{L}^d$ becomes a BI–lattice with the involution that restricts to the involution of $\mathbf{K}$ on K, to a dual lattice isomorphism from $\mathbf{L}$ to $\mathbf{L}^d$ on L and to the inverse of this lattice isomorphism on L^d. This BI–lattice, which we denote by $\mathbf{L} \oplus \mathbf{K} \oplus \mathbf{L}^d$, is a pseudo–Kleene algebra if and only if $\mathbf{K}$ is a pseudo–Kleene algebra.

We denote by $\mathbb{BA}$, $\mathbb{OML}$, $\mathbb{OL}$, $\mathbb{KA}$, $\mathbb{PKA}$, $\mathbb{BI}$ and $\mathbb{I}$ the varieties of Boolean algebras, orthomodular lattices, ortholattices, Kleene algebras, pseudo–Kleene algebras, BI–lattices and I–lattices, respectively. Note that $\mathbb{BA} \subsetneq \mathbb{OML} \subsetneq \mathbb{OL} \subsetneq \mathbb{PKA} \subsetneq \mathbb{BI}$ and $\mathbb{BA} \subsetneq \mathbb{KA} \subsetneq \mathbb{PKA}$.

An algebra $\mathbf{A}$ having a BI–lattice reduct is said to be *paraorthomodular* if and only if, for all $a, b \in A$, if $a \leq b$ and $a' \wedge b = 0$, then $a = b$. Note that orthomodular lattices are paraorthomodular and that paraorthomodular ortholattices are orthomodular lattices.

A *Brouwer–Zadeh lattice* (in brief, *BZ–lattice*) is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot', \cdot^{\sim}, 0, 1)$ of type $(2, 2, 1, 1, 0, 0)$ such that $(L, \wedge, \vee, \cdot', 0, 1)$ is a pseudo–Kleene algebra and $\cdot^{\sim} : L \to L$ is an order–reversing operation, called *Brouwer complementation*, that satisfies: $a \wedge a^{\sim} = 0$ and $a \leq a^{\sim\sim} = a^{\sim\prime}$ for all $a \in L$. In any BZ–lattice $\mathbf{L}$, we denote by $\Box a = a'^{\sim}$ and by $\Diamond a = a^{\sim\sim}$ for all $a \in L$. Note that, in any BZ–lattice $\mathbf{L}$, we have, for all $a, b \in L$: $a^{\sim\sim\sim} = a^{\sim} \leq a'$, $(a \vee b)^{\sim} = a^{\sim} \wedge b^{\sim}$ and $(a \wedge b)^{\sim} \geq a^{\sim} \vee b^{\sim}$. The class of BZ-lattices is a variety, hereafter denoted by $\mathbb{BZL}$.

We consider the following identities over $\mathbb{BZL}$, out of which SDM (the *Strong De Morgan* identity) clearly implies $(*)$, as well as SK, while J0 implies J2:

$(*)$ $(x \wedge x')^{\sim} \approx x^{\sim} \vee x'^{\sim}$
SDM $(x \wedge y)^{\sim} \approx x^{\sim} \vee y^{\sim}$
SK $x \wedge \Diamond y \leq \Box x \vee y$
DIST $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$
J0 $(x \wedge y^{\sim}) \vee (x \wedge \Diamond y) \approx x$
J2 $(x \wedge (y \wedge y')^{\sim}) \vee (x \wedge \Diamond(y \wedge y')) \approx x$

A *PBZ*–lattice* is a paraorthomodular BZ–lattice that satisfies identity $(*)$. In any PBZ*–lattice $\mathbf{L}$,

$$S(\mathbf{L}) = \{a^{\sim} : a \in L\} = \{a \in L : a^{\sim\sim} = a\} = \{a \in L : a' = a^{\sim}\}$$

and $S(\mathbf{L})$ is the universe of the largest orthomodular subalgebra of $\mathbf{L}$, that we denote by $\mathbf{S}(\mathbf{L})$.

We denote by $\mathbb{PBZL}^*$ the variety of PBZ*–lattices; note that paraorthomodularity becomes an equational condition under the $\mathbb{BZL}$ axioms and condition $(*)$. We also denote by $\mathbb{DIST} = \{\mathbf{L} \in \mathbb{PBZL}^* : \mathbf{L} \vDash \text{DIST}\}$. By the above, $\mathbb{OML}$ can be identified with the subvariety $\{\mathbf{L} \in \mathbb{PBZL}^* : \mathbf{L} \vDash x' \approx x^{\sim}\}$ of $\mathbb{PBZL}^*$, by endowing each orthomodular lattice, in particular every Boolean algebra, with a Brouwer complement equalling its Kleene complement. With the same extended signature, $\mathbb{OL}$ becomes the subvariety $\{\mathbf{L} \in \mathbb{BZL} : \mathbf{L} \vDash x' \approx x^{\sim}\}$ of $\mathbb{BZL}$.

A PBZ*–lattice $\mathbf{A}$ with no nontrivial sharp elements, that is with $S(\mathbf{A}) = \{0, 1\}$, is called an *antiortholattice*. A PBZ*–lattice $\mathbf{A}$ is an antiortholattice if and only if it is endowed with the following Brouwer complement, called the *trivial Brouwer complement*: $0^{\sim} = 1$ and $a^{\sim} = 0$ for all $a \in A \setminus \{0\}$. Every paraorthomodular pseudo–Kleene algebra with no nontrivial sharp elements becomes an antiortholattice when endowed with the trivial Brouwer complement. In particular, any BZ–lattice with the 0 meet–irreducible, and thus any BZ–chain, is an antiortholattice. Moreover, BZ–lattices with the 0 meet–irreducible are exactly the antiortholattices that satisfy SDM. Also, if $\mathbf{L}$ is a nontrivial bounded lattice and $\mathbf{K}$ is a pseudo–Kleene algebra, then the pseudo–Kleene algebra $\mathbf{L} \oplus \mathbf{K} \oplus \mathbf{L}^d$, endowed with the trivial Brouwer complement, becomes an antiortholattice, that we will also denote by $\mathbf{L} \oplus \mathbf{K} \oplus \mathbf{L}^d$.

Antiortholattices form a proper universal class, denoted by $\mathbb{AOL}$. Clearly, $\mathbb{AOL} \cup \mathbb{OML} \subsetneq \mathbb{PBZL}^* \subsetneq \mathbb{BZL} \supsetneq \mathbb{OL}$. Note, also, that $\mathbb{OML} \cap V_{\mathbb{BZL}}(\mathbb{AOL}) = \mathbb{OML} \cap \mathbb{DIST} = \mathbb{BA}$, hence $\mathbb{DIST} \subsetneq V_{\mathbb{BZL}}(\mathbb{AOL})$. We denote by $\mathbb{SDM} = \{\mathbf{L} \in \mathbb{PBZL}^* : \mathbf{L} \vDash \text{SDM}\}$ and by $\mathbb{SAOL} = \mathbb{SDM} \cap V_{\mathbb{BZL}}(\mathbb{AOL})$.

If $\mathbf{L}$ is a nontrivial bounded lattice and $\mathbb{C}$ is a class of bounded lattices, BI–lattices or pseudo–Kleene algebras, then we denote by $\mathbf{L} \oplus \mathbb{C} \oplus \mathbf{L}^d$ the following class of bounded lattices, BI–lattices or antiortholattices:

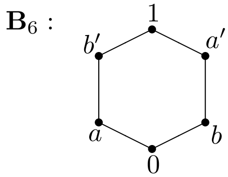
$$\mathbf{L} \oplus \mathbb{C} \oplus \mathbf{L}^d = \{\mathbf{L} \oplus \mathbf{A} \oplus \mathbf{L}^d : \mathbf{A} \in \mathbb{C}\}.$$

10.3 A Study of Some Subvarieties

Throughout this section, the results cited from Mureşan (2019) will be numbered as in the third arXived version of this paper.

10.3.1 The $\mathbf{F}_8$ *Problem*

There is a long and time-honoured tradition that aims at characterising subvarieties of varieties of ordered algebras in terms of “forbidden configurations”, harking back to Dedekind’s celebrated result to the effect that the distributive subvariety of the variety of lattices is the one whose members do not contain as subalgebras $\mathbf{M}_3$ or $\mathbf{N}_5$, while the modular subvariety is the one whose members do not contain $\mathbf{N}_5$. Other important results in the same vein appear in the theory of ortholattices. For example, the benzene ring $\mathbf{B}_6$:



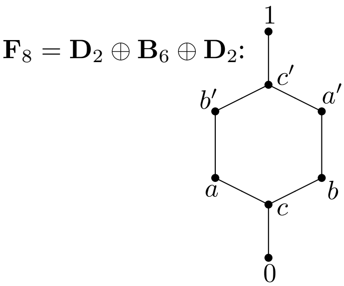
is a forbidden configuration for the orthomodular subvariety of the variety of ortholattices; more precisely,

$$\mathbb{OML} = \{\mathbf{L} \in \mathbb{OL} : \mathbf{B}_6 \notin \mathrm{S}_{\mathbb{I}}(\mathbf{L})\}\,.$$

Consequently:

Lemma 10.1 $(\mathbb{OML}, V_{\mathbb{BI}}(\mathbf{B}_6))$ *is a splitting pair in* Subvar $(\mathbb{OL})$.

In this subsection, we intend to give a first, limited application of this method, by means of a forbidden configuration consisting of a "paraorthomodular analogue" of $\mathbf{B}_6$: the antiortholattice $\mathbf{D}_2 \oplus \mathbf{B}_6 \oplus \mathbf{D}_2$, hereafter denoted by $\mathbf{F}_8$, along with any of its reducts, for the sake of brevity:



Since it has the 0 meet–irreducible, the antiortholattice $\mathbf{F}_8$ satisfies SDM, thus $\mathbf{F}_8 \in \mathbb{SAOL}$. The question arises naturally as to which subvarieties $\mathbb{V}$ of $\mathbb{PBZL}^*$ are maximal with respect to the property that $\mathbf{F}_8 \notin \mathrm{S}_{\mathbb{I}}(\mathbb{V})$, i.e., $\mathbf{F}_8 \notin \mathrm{S}_{\mathbb{I}}(\mathbf{A})$ for any $\mathbf{A} \in \mathbb{V}$. This problem will be referred to as the "$\mathbf{F}_8$ problem". Although we will not give an answer to this question, we provide a quasiequational characterisation of paraorthomodular bounded involution lattices that do not contain $\mathbf{F}_8$ as a bounded involution sublattice and we study the varieties of PBZ*–lattices that contain the antiortholattice $\mathbf{F}_8$.

Clearly, for any $\mathbf{L}, \mathbf{M} \in \mathbb{BI}$, we have: $\mathbf{D}_2 \oplus \mathbf{M} \oplus \mathbf{D}_2 \in \mathrm{S}_{\mathbb{I}}(\mathbf{L})$ if and only if $\mathbf{D}_2 \oplus \mathbf{M} \oplus \mathbf{D}_2 \in \mathrm{S}_{\mathbb{BI}}(\mathbf{L})$. The right-to-left direction is trivial, while, if $\mathbf{D}_2 \oplus \mathbf{M} \oplus \mathbf{D}_2 \in \mathrm{S}_{\mathbb{I}}(\mathbf{L})$ and $A = M \cup \{0, 1\}$, then $\mathbf{D}_2 \oplus \mathbf{M} \oplus \mathbf{D}_2 \cong_{\mathbb{BI}} \mathbf{A} \in \mathrm{S}_{\mathbb{BI}}(\mathbf{L})$. In particular, for any $\mathbf{A} \in \mathbb{BZL}$, we have that $\mathbf{F}_8 \in \mathrm{S}_{\mathbb{I}}(\mathbf{A})$ if and only if $\mathbf{F}_8 \in \mathrm{S}_{\mathbb{BI}}(\mathbf{A})$; also, if $\mathbf{F}_8 \in$

$\mathrm{S}_{\mathbb{BZL}}(\mathbf{A})$, then $\mathbf{F}_8 \in \mathrm{S}_{\mathbb{BI}}(\mathbf{A})$, while, if $\mathbf{A}$ is an antiortholattice, then $\mathbf{F}_8 \in \mathrm{S}_{\mathbb{BZL}}(\mathbf{A})$ if and only if $\mathbf{F}_8 \in \mathrm{S}_{\mathbb{BI}}(\mathbf{A})$.

Observe what follows:

- no distributive PBZ*–lattice can contain $\mathbf{B}_6$ or $\mathbf{F}_8$ as sublattices, in particular as sub-involution lattices;
- since $\mathbf{B}_6$ is a sub-involution lattice of $\mathbf{F}_8$ and $\mathbf{B}_6$ is not orthomodular, no orthomodular lattice can contain $\mathbf{F}_8$ as a sub-involution lattice;
- by the above, any subvariety $\mathbb{V}$ of $\mathbb{PBZL}^*$ such that $\mathbb{V} \subseteq \mathbb{DIST} \cup \mathbb{OML}$ satisfies $\mathbf{F}_8 \notin \mathrm{S}_{\mathbb{I}}(\mathbb{V})$;
- $\mathbf{F}_8 \in \mathbb{SAOL}$, whence any subvariety $\mathbb{V}$ of $\mathbb{PBZL}^*$ such that $\mathbb{SAOL} \subseteq \mathbb{V}$ satisfies $\mathbf{F}_8 \in \mathrm{S}_{\mathbb{I}}(\mathbb{V})$.

Let us now consider the following quasiidentities in the language of I–lattices:

$\mathbb{Q}$ $\quad x \leq y' \;\&\; x' \wedge y' \leq x \wedge y \Rightarrow x = y'$
$\mathbb{Q}'$ $\quad x' \wedge (x' \wedge u)' \leq x \wedge (x' \wedge u) \Rightarrow u \leq x'$

Note that $\mathbb{Q}$ is equivalent to $\mathbb{Q}'$.

Lemma 10.2 *If $\mathbf{A} \in \mathbb{I}$ and $a, b \in A$ are such that $a \leq b'$ and $a' \wedge b' \leq a \wedge b$, then $a \wedge a' = b \wedge b' = a' \wedge b' = a \wedge b$.*

Proof Let $c = a' \wedge b'$. Then $c \leq a \wedge b$ by the choice of a and b, therefore, since we also have $a \leq b'$ and thus $b \leq a'$: $a \wedge a' = a \wedge b' \wedge a' = a \wedge c = c$; $b \wedge b' = b \wedge a' \wedge b' = b \wedge c = c$; $a \wedge b = a \wedge b' \wedge b = a \wedge c = c$. ■

Lemma 10.3 *For any $\mathbf{A} \in \mathbb{PBI}$, we have:*

$$\mathbf{B}_6 \in \mathrm{S}_{\mathbb{I}}(\mathbf{A}) \textit{ if and only if } \mathbf{F}_8 \in \mathrm{S}_{\mathbb{BI}}(\mathbf{A}).$$

Proof The right-to-left direction is trivial. Now assume that $\mathbf{B}_6 \in \mathrm{S}_{\mathbb{I}}(\mathbf{A})$, with $B_6 = \{c, a, b, a', b', c'\} \subseteq A$, where $c = a \wedge b$ and $a < b'$. Assume ex absurdo that $c = 0$, so that $a' \wedge b' = 0$. Since $\mathbf{A}$ is paraorthomodular, it follows that $a = b'$, and we have a contradiction. Therefore $c \neq 0$, so, if we denote by $L = \{0, c, a, b, a', b', c', 1\}$, then $\mathbf{F}_8 \cong_{\mathbb{BI}} \mathbf{L} \in \mathrm{S}_{\mathbb{BI}}(\mathbf{A})$. ■

Proposition 10.4 *For any $\mathbf{A} \in \mathbb{I}$, we have:*

$$\mathbf{A} \vDash \mathbb{Q} \textit{ if and only if } \mathbf{B}_6 \notin \mathrm{S}_{\mathbb{I}}(\mathbf{A}).$$

Proof For the direct implication, assume that $\mathbf{B}_6 \in \mathrm{S}(\mathbf{A})$, with $B_6 = \{c, a, b, a', b', c'\} \subseteq A$, where $c = a \wedge b$ and $a < b'$. Then $a \leq b'$ and $a' \wedge b' = a \wedge b \leq a \wedge b$, but $a \neq b'$, hence $\mathbf{A} \nvDash \mathbb{Q}$.

For the converse, assume that $\mathbf{A} \nvDash \mathbb{Q}$, so that there exist $a, b \in A$ with $a' \wedge b' \leq a \wedge b$ and $a < b'$, so $b < a'$. Then, by Lemma 10.2, if we denote by $c = a' \wedge b'$, then $c = a \wedge b = a \wedge a' = b \wedge b'$. Since $a < b'$, $a \wedge b \leq a' \vee b'$; were it the case that $a \wedge$

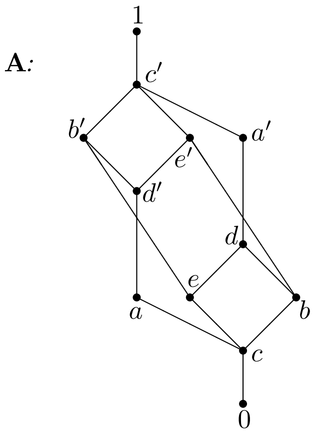
$b = a' \vee b'$, we would have that $a' \leq a' \vee b' = a \wedge b \leq b$, a contradiction. Hence $c' = (a \wedge b)' = a' \vee b' > a \wedge b = c$. Also, $a \vee b = (a' \wedge b')' = c'$, $a \vee a' = (a \wedge a')' = c'$ and $b \vee b' = (b \wedge b')' = c'$. If we had $a \leq b$, then $a \leq b \wedge b' = c = a \wedge a' \leq a$, hence $c = a \wedge a' = a < b' \leq a' \wedge b' = c$, and we have a contradiction again. Similarly, $b \nleq a$. Hence a and b are incomparable. Were it $a \leq a'$, then $c = a \wedge a' = a$, which would lead to the same contradiction as above. On the other hand, if $a' \leq a$, then $c = a \wedge a' = a' > b \geq b \wedge b' = c$, which gives us another contradiction. Hence a and a' are incomparable and so are, analogously, b and b'. Therefore, if we denote by $L = \{c, a, b, a', b', c'\}$, then $\mathbf{B}_6 \cong_{\mathbb{I}} \mathbf{L} \in \mathrm{S}_{\mathbb{I}}(\mathbf{A})$. ■

Theorem 10.5 *For any* $\mathbf{A} \in \mathbb{PBI}$, *we have:*

$$\mathbf{A} \vDash \mathbb{Q} \text{ if and only if } \mathbf{F}_8 \notin \mathrm{S}_{\mathbb{BI}}(\mathbf{A}).$$

Proof By Lemma 10.3 and Proposition 10.4. ■

Example 10.6 Here is an antiortholattice (in particular, a paraorthomodular BI–lattice) $\mathbf{A}$ such that $\mathbf{F}_8 \notin \mathrm{S}_{\mathbb{BI}}(\mathbf{A})$, but $\mathbf{F}_8 \in \mathrm{H}_{\mathbb{BZL}}(\mathbf{A})$, in particular $\mathbf{F}_8 \in \mathrm{S}_{\mathbb{BI}}((\mathrm{H}_{\mathbb{BZL}}(\mathbf{A}))) \subseteq \mathrm{S}_{\mathbb{BI}}(\mathrm{H}_{\mathbb{BI}}(\mathbf{A}))$:



The equivalence relation θ with cosets

$$\{0\}, \{a\}, \{c, e\}, \{b, d\}, \{b', d'\}, \{c', e'\}, \{a'\}, \{1\}$$

belongs to $\mathrm{Con}_{\mathbb{BI}01}(\mathbf{A}) \subset \mathrm{Con}_{\mathbb{BZL}}(\mathbf{A})$ and $\mathbf{A}/\theta \cong \mathbf{F}_8$, but, as announced above, $\mathbf{F}_8 \notin \mathrm{S}_{\mathbb{BI}}(\mathbf{A})$.

Corollary 10.7 $\mathbb{Q}$ *is not an equational condition in* $\mathbb{PBI}$ *or* $\mathbb{PBZL}^*$.

Now let us investigate the subvarieties of $\mathbb{PBZL}^*$ that contain $\mathbf{F}_8$. We consider the following identity in the language of BZ–lattices:

D2OL∨ $(x \wedge x')^{\sim} \vee (y \wedge y')^{\sim} \vee x \vee x' \approx (x \wedge x')^{\sim} \vee (y \wedge y')^{\sim} \vee y \vee y'$

By Giuntini et al. (2018), $V_{\mathbb{BZL}}(\mathbb{AOL})$ is axiomatised by J0 relative to $\mathbb{PBZL}^*$. By Mureşan (2019), $V_{\mathbb{BZL}}(\mathbf{D}_2 \oplus \mathbb{OL} \oplus \mathbf{D}_2)$ is axiomatised by D2OL∨ relative to $\mathbb{SAOL}$.

We use the following notation from Mureşan (2019): for any $k, n, p \in \mathbb{N}$ and any identity $t \approx u$, where $t(x_1, \ldots, x_k, z_1, \ldots, z_p)$ and $u(y_1, \ldots, y_n, z_1, \ldots, z_p)$ are terms in the language of $\mathbb{BI}$ having the arities $k + p$, respectively $n + p$, and p common variables $z_1, \ldots, z_p$, we denote by $m(t, u)$ the following $(k + n)$–ary term in the language of $\mathbb{BZL}$:

$$m(t, u)(x_1, \ldots, x_k, y_1, \ldots, y_n, z_1, \ldots, z_p) =$$

$$\bigvee_{i=1}^{k}(x_i \wedge x_i')^{\sim} \vee \bigvee_{j=1}^{n}(y_j \wedge y_j')^{\sim} \vee \bigvee_{h=1}^{p}(z_h \wedge z_h')^{\sim} \vee t(x_1, \ldots, x_k, z_1, \ldots, z_p).$$

Note that:

$$m(u, t)(x_1, \ldots, x_k, y_1, \ldots, y_n, z_1, \ldots, z_p) =$$

$$\bigvee_{i=1}^{k}(x_i \wedge x_i')^{\sim} \vee \bigvee_{j=1}^{n}(y_j \wedge y_j')^{\sim} \vee \bigvee_{h=1}^{p}(z_h \wedge z_h')^{\sim} \vee u(y_1, \ldots, y_n, z_1, \ldots, z_p).$$

Lemma 10.8 *(Mureşan 2019, Corollary 6.14) For any $\mathbb{C} \subseteq \mathbb{BI}$ and any $\mathbb{D} \subseteq \mathbb{PKA}$, $V_{\mathbb{BI}}(\mathbf{D}_2 \oplus \mathbb{C} \oplus \mathbf{D}_2) = V_{\mathbb{BI}}(\mathbf{D}_2 \oplus V_{\mathbb{BI}}(\mathbb{C}) \oplus \mathbf{D}_2)$ and $V_{\mathbb{BZL}}(\mathbf{D}_2 \oplus \mathbb{D} \oplus \mathbf{D}_2) = V_{\mathbb{BZL}}(\mathbf{D}_2 \oplus V_{\mathbb{BI}}(\mathbb{D}) \oplus \mathbf{D}_2)$.*

Proposition 10.9 *$V_{\mathbb{BI}}(\mathbf{F}_8) = V_{\mathbb{BI}}(\mathbf{D}_2 \oplus V_{\mathbb{BI}}(\mathbf{B}_6) \oplus \mathbf{D}_2)$ and $V_{\mathbb{BZL}}(\mathbf{F}_8) = V_{\mathbb{BZL}}(\mathbf{D}_2 \oplus V_{\mathbb{BI}}(\mathbf{B}_6) \oplus \mathbf{D}_2)$.*

Proof By Lemma 10.8 and the fact that $\mathbf{F}_8 = \mathbf{D}_2 \oplus \mathbf{B}_6 \oplus \mathbf{D}_2$. ■

The following consequence of results from Mureşan (2019) shows that we can obtain an axiomatisation for $V_{\mathbb{BZL}}(\mathbf{F}_8)$ relative to $\mathbb{PBZL}^*$ from an axiomatisation of $V_{\mathbb{BI}}(\mathbf{B}_6)$ relative to $\mathbb{OL}$; note that any such axiomatisation can be written with nonnullary terms over $\mathbb{BI}$, since $\mathbb{OL}$ satisfies the identities $x \vee x' \approx 1$ and $x \wedge x' \approx 0$.

Corollary 10.10 *$\{t_i \approx u_i : i \in I\}$ is an axiomatisation of $V_{\mathbb{BI}}(\mathbf{B}_6)$ relative to $\mathbb{OL}$ such that, for each $i \in I$, the terms t_i and u_i have nonzero arities if and only if $\{m(t_i, u_i) \approx m(u_i, t_i) : i \in I\} \cup \{J0, D2OL\vee\}$ is an axiomatisation of $V_{\mathbb{BZL}}(\mathbf{F}_8)$ relative to $\mathbb{PBZL}^*$.*

Proof By Proposition 10.9, the fact that $V_{\mathbb{BI}}(\mathbf{B}_6) \subseteq \mathbb{OL}$ and Mureşan (2019, Theorem 6.38.(ii)). ■

Theorem 10.11 *(Mureşan 2019, Theorem 6.25) The operator $\mathbb{V} \mapsto V_{\mathbb{BZL}}(\mathbf{D}_2 \oplus \mathbb{V} \oplus \mathbf{D}_2)$ is a bounded lattice embedding from the lattice of subvarieties of $\mathbb{PKA}$ to the principal filter generated by $V_{\mathbb{BZL}}(\mathbf{D}_3)$ in the lattice of subvarieties of $\mathbb{SAOL}$.*

Corollary 10.12 $(V_{\mathbb{BZL}}(\mathbf{D}_2 \oplus \mathbb{OML} \oplus \mathbf{D}_2), V_{\mathbb{BZL}}(\mathbf{F}_8))$ *is a splitting pair in the lattice of subvarieties of* $\mathbb{OL}$.

Proof By Lemma 10.1, Proposition 10.9 and Theorem 10.11. ■

Proposition 10.13 • $V_{\mathbb{BI}}(\mathbf{B}_6) \subsetneq V_{\mathbb{BI}}(\mathbf{F}_8) = V_{\mathbb{BI}}(\mathbf{D}_n \oplus \mathbf{F}_8 \oplus \mathbf{D}_n)$ *for any* $n \in \mathbb{N}^*$;
• $V_{\mathbb{BZL}}(\mathbf{F}_8) \subsetneq V_{\mathbb{BZL}}(\mathbf{D}_2 \oplus \mathbf{F}_8 \oplus \mathbf{D}_2) = V_{\mathbb{BZL}}(\mathbf{D}_n \oplus \mathbf{F}_8 \oplus \mathbf{D}_n)$ *for any* $n \in \mathbb{N} \setminus \{0, 1, 2\}$.

Proof By Proposition 10.9, the fact that $V_{\mathbb{BI}}(\mathbf{B}_6) \subseteq \mathbb{OL}$, while $\mathbf{D}_3 \in V_{\mathbb{BI}}(\mathbf{F}_8)$, and Mureşan (2019, Corollary 6.23), we get that $V_{\mathbb{BI}}(\mathbf{B}_6) \subsetneq V_{\mathbb{BI}}(\mathbf{F}_8) = V_{\mathbb{BI}}(\mathbf{D}_2 \oplus \mathbf{F}_8 \oplus \mathbf{D}_2)$ and hence $V_{\mathbb{BI}}(\mathbf{F}_8) = V_{\mathbb{BI}}(\mathbf{D}_n \oplus \mathbf{F}_8 \oplus \mathbf{D}_n)$ for any $n \in \mathbb{N}^*$. This, Theorem 10.11 and again Proposition 10.9 show that $V_{\mathbb{BZL}}(\mathbf{F}_8) \subsetneq V_{\mathbb{BZL}}(\mathbf{D}_2 \oplus \mathbf{F}_8 \oplus \mathbf{D}_2) = V_{\mathbb{BZL}}(\mathbf{D}_n \oplus \mathbf{F}_8 \oplus \mathbf{D}_n)$ for any $n \in \mathbb{N} \setminus \{0, 1, 2\}$. ■

10.3.2 Covers in the Lattice of Subvarieties of $\mathbb{PBZL}^*$

In this subsection, we continue the study of the lattice $\mathrm{Subvar}(\mathbb{PBZL}^*)$ of subvarieties of PBZ*–lattices, started in Giuntini et al. (2016, 2017, 2018, 2020), Mureşan (2019). We begin by recapitulating a few known results.

Lemma 10.14 *(i) (Giuntini et al. 2016, Subsection 5.3)* $\mathbb{BA}$ *is the unique atom of* $\mathrm{Subvar}(\mathbb{PBZL}^*)$.
(ii) (Giuntini et al. 2016, Theorem 5.4.(2)) $\mathbb{BA} = \mathbb{OML} \cap V_{\mathbb{BZL}}(\mathbb{AOL})$.
(iii) (Bruns and Harding 2000, Corollary 3.6) The unique cover of $\mathbb{BA}$ *in the ideal* $(\mathbb{OML}]$ *of* $\mathrm{Subvar}(\mathbb{PBZL}^*)$ *is* $V_{\mathbb{BZL}}(\mathbf{MO}_2)$.
(iv) (Giuntini et al. 2016, Theorem 5.5) For any $\mathbf{L} \in \mathbb{PBZL}^* \setminus \mathbb{OML}$, *we have* $\mathbf{D}_3 \in HS(\mathbf{L}) \subseteq V_{\mathbb{BZL}}((\mathbf{L}))$, *so the unique non–orthomodular cover of* $\mathbb{BA}$ *in* $\mathrm{Subvar}(\mathbb{PBZL}^*)$ *is* $V_{\mathbb{BZL}}(\mathbf{D}_3)$.

By the above, in $\mathrm{Subvar}(\mathbb{PBZL}^*)$ $V_{\mathbb{BZL}}(\mathbf{MO}_2)$ and $V_{\mathbb{BZL}}(\mathbf{D}_3)$ are the only covers of $\mathbb{BA}$, and $\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbf{D}_3)$ is the unique cover of $\mathbb{OML}$.

Lemma 10.15 *(Giuntini et al. 2017, Lemma 3.3.(1)) All subdirectly irreducible members of* $V_{\mathbb{BZL}}(\mathbb{AOL})$ *belong to* $\mathbb{AOL}$.

Lemma 10.16 *(Mureşan 2019)*

(i) $\mathbb{BA} = \mathbb{OML} \cap V_{\mathbb{BZL}}(\mathbb{AOL}) = V_{\mathbb{BZL}}(\mathbf{D}_2) \subsetneq V_{\mathbb{BZL}}(\mathbf{D}_3) \subsetneq V_{\mathbb{BZL}}(\mathbf{D}_4) \subsetneq V_{\mathbb{BZL}}(\mathbf{D}_5)$.
(ii) $Si(V_{\mathbb{BZL}}(\mathbf{D}_3)) = V_{\mathbb{BZL}}(\mathbf{D}_3) \cap \mathbb{AOL} = \mathrm{I}_{\mathbb{BZL}}(\{\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3\})$.

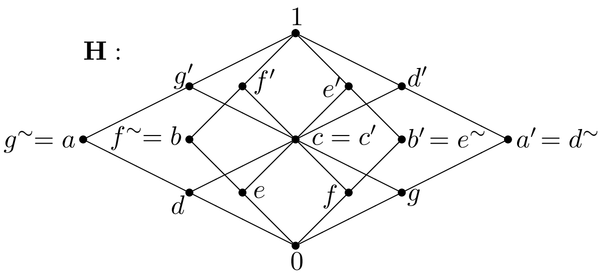
We now prove the main result of this subsection.

Theorem 10.17 *The only cover of* $V_{\mathbb{BZL}}(\mathbf{D}_3)$ *in* $\mathrm{Subvar}(\mathbb{PBZL}^*)$ *included in* $V_{\mathbb{BZL}}(\mathbb{AOL})$ *is* $V_{\mathbb{BZL}}(\mathbf{D}_4)$.

Proof For any subvariety $\mathbb{W}$ of $V_{\mathbb{BZL}}(\mathbb{AOL})$ such that $V_{\mathbb{BZL}}(\mathbf{D}_3) \subsetneq \mathbb{W}$, there exists an $\mathbf{A} \in Si(\mathbb{W}) \setminus Si(V_{\mathbb{BZL}}(\mathbf{D}_3)) = (\mathbb{W} \cap \mathbb{AOL}) \setminus \mathrm{I}_{\mathbb{BZL}}(\{\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3\})$ by Lemma 10.15 and Lemma 10.16.(ii), thus $\mathbf{A}$ is an antiortholattice with $|A| > 3$. Hence, there exists an $a \in A \setminus \{0, 1\} = A \setminus S_{\mathbb{BZL}}(\mathbf{A})$ with $a \neq a'$, so that $0 < a \wedge a' < a \vee a' < 1$. Therefore $\{0, a \wedge a', a \vee a', 1\}$ is the universe of a subalgebra of $\mathbf{A}$ isomorphic to $\mathbf{D}_4$, i.e., $\mathbf{D}_4 \in \mathrm{S}_{\mathbb{BZL}}(\mathbf{A})$, thus $V_{\mathbb{BZL}}(\mathbf{D}_4) \subseteq V_{\mathbb{BZL}}(\mathbf{A}) \subseteq \mathbb{W}$. Since $V_{\mathbb{BZL}}(\mathbf{D}_3) \subsetneq V_{\mathbb{BZL}}(\mathbf{D}_4)$ by Lemma 10.16.(i), it follows that $V_{\mathbb{BZL}}(\mathbf{D}_4)$ is the only cover of $V_{\mathbb{BZL}}(\mathbf{D}_3)$ in $\mathrm{Subvar}(\mathrm{V}_{\mathbb{BZL}}(\mathbb{AOL}))$, which is, of course, a convex sublattice of $\mathrm{Subvar}(\mathbb{PBZL}^*)$, thus $V_{\mathbb{BZL}}(\mathbf{D}_4)$ is a cover of $V_{\mathbb{BZL}}(\mathbf{D}_3)$ in $\mathrm{Subvar}(\mathbb{PBZL}^*)$. ■

It remains open to determine whether $V_{\mathbb{BZL}}(\mathbf{D}_4)$ is the only cover of $V_{\mathbb{BZL}}(\mathbf{D}_3)$ in $\mathrm{Subvar}(\mathbb{PBZL}^*)$. Recall, also, that $V_{\mathbb{BZL}}(\mathbf{D}_5) = \mathbb{SDM} \cap \mathbb{DIST}$ contains all antiortholattice chains, i.e., all PBZ*–chains.

Example 10.18 Let us consider the following example of a PBZ*–lattice from Giuntini et al. (2018):



Note that:

- $\mathbf{H} \vDash \{\mathrm{SDM}, \mathrm{SK}\}$, thus $\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbf{H}) \vDash \{\mathrm{SDM}, \mathrm{SK}\}$ since $\mathbb{OML} \vDash \{\mathrm{SDM}, \mathrm{SK}\}$;
- $\mathbf{H} \nvDash \mathrm{J2}$, thus $\mathbf{H} \notin \mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbb{AOL}) \vDash \mathrm{J2}$, in particular $\mathbf{H} \notin \mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbf{D}_3)$;
- $\mathbf{D}_3 \in \mathrm{S}(\mathbf{H})$, hence $\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbf{D}_3) \subseteq \mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbf{H})$, therefore $\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbf{D}_3) \subsetneq \mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbf{H})$ by the above;
- since $\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbf{H}) \vDash \mathrm{SK}$ and $\mathbf{D}_4 \nvDash \mathrm{SK}$, we have $\mathbf{D}_4 \notin \mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbf{H})$, hence $\mathbf{D}_4$ does not belong to every proper supervariety of $\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbf{D}_3)$.

$\mathbf{H} \vDash \{\mathrm{SDM}, \mathrm{SK}\}$, $\mathbf{H} \nvDash \mathrm{J2}$ and $\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbb{AOL}) \vDash \mathrm{J2}$, hence $\mathbf{H} \in (\mathbb{SDM} \cap \mathbb{SK}) \setminus (\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbb{AOL}))$, thus $\mathbb{SDM} \cap \mathbb{SK} \nsubseteq \mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbb{AOL})$. $\mathbb{AOL} \nvDash \mathrm{SDM}$ and $\mathbb{AOL} \nvDash \mathrm{SK}$, thus $\mathbb{AOL} \nsubseteq \mathbb{SDM}$ and $\mathbb{AOL} \nsubseteq \mathbb{SK}$, in particular $\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbb{AOL}) \nsubseteq \mathbb{SDM} \cap \mathbb{SK}$. Therefore $\mathbb{SDM} \cap \mathbb{SK} || \mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbb{AOL})$. Now let $\mathbb{V} = V_{\mathbb{BZL}}(MO_2) \vee V_{\mathbb{BZL}}(\mathbf{D}_3) \subseteq \mathbb{SDM} \cap \mathbb{SK}$. $\mathbf{D}_3 \notin \mathbb{OML}$, thus $\mathbb{V} \nsubseteq \mathbb{OML}$. $MO_2 \notin V_{\mathbb{BZL}}(\mathbb{AOL})$, thus $\mathbb{V} \nsubseteq V_{\mathbb{BZL}}(\mathbb{AOL})$. Finally, $\mathbb{V}$ satisfies the modular law, while both $\mathbb{OML}$ and $V_{\mathbb{BZL}}(\mathbb{AOL})$ fail it, hence $\mathbb{OML} \nsubseteq \mathbb{V}$ and $V_{\mathbb{BZL}}(\mathbb{AOL}) \nsubseteq \mathbb{V}$. Therefore $\mathbb{OML} || \mathbb{V} || V_{\mathbb{BZL}}(\mathbb{AOL})$.

We list hereafter a few problems that remain open at the time of writing:

- Is $\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbf{D}_4)$ a successor of $\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbf{D}_3)$ in $\mathrm{Subvar}(\mathbb{PBZL}^*)$? Is it its only successor?
- Is $\mathrm{Subvar}(\mathbb{PBZL}^*)$ strongly atomic? If so, then $\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbf{H})$ includes a successor of $\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbf{D}_3)$ which differs from $\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbf{D}_4)$.

10.3.3 Subdirect Products and Varieties of PBZ–lattices*

Let $\mathbb{V}$ and $\mathbb{W}$ be varieties of the same type. Obviously, if $\mathbb{V}$ and $\mathbb{W}$ are incomparable, then there exist $\mathbf{A} \in (\mathbb{V} \vee \mathbb{W}) \setminus \mathbb{V}$ and $\mathbf{B} \in (\mathbb{V} \vee \mathbb{W}) \setminus \mathbb{W}$, so that $\mathbf{A} \times \mathbf{B} \in (\mathbb{V} \vee \mathbb{W}) \setminus (\mathbb{V} \cup \mathbb{W})$ and thus $\mathbb{V} \cup \mathbb{W} \subsetneq \mathbb{V} \vee \mathbb{W}$. Recall that the *subdirect product* of $\mathbb{V}$ and $\mathbb{W}$ is the class, denoted by $\mathbb{V} \times_s \mathbb{W}$, whose members are isomorphic images of subdirect products of a member of $\mathbb{V}$ and a member of $\mathbb{W}$. Clearly, $\mathbb{V} \cup \mathbb{W} \subseteq \mathbb{V} \times_s \mathbb{W} \subseteq \mathbb{V} \vee \mathbb{W}$, so that

$$Si(\mathbb{V}) \cup Si(\mathbb{W}) = Si(\mathbb{V} \cup \mathbb{W}) \subseteq Si(\mathbb{V} \times_s \mathbb{W}) \subseteq Si(\mathbb{V} \vee \mathbb{W}).$$

For any $\mathbf{M} \in Si(\mathbb{V} \times_s \mathbb{W})$, $\mathbf{M}$ is a subdirect product of an $\mathbf{A} \in \mathbb{V}$ and a $\mathbf{B} \in \mathbb{W}$, so that $\mathbf{A}$ is trivial, case in which $\mathbf{M} \in Si(\mathbb{W})$, or $\mathbf{B}$ is trivial, case in which $\mathbf{M} \in Si(\mathbb{V})$. Thus $Si(\mathbb{V} \times_s \mathbb{W}) \subseteq Si(\mathbb{V}) \cup Si(\mathbb{W})$, hence $Si(\mathbb{V} \times_s \mathbb{W}) = Si(\mathbb{V}) \cup Si(\mathbb{W})$. Since $\mathbb{V} \times_s \mathbb{W} \subseteq \mathbb{V} \vee \mathbb{W}$, we get that the following equivalence holds: $\mathbb{V} \vee \mathbb{W} = \mathbb{V} \times_s \mathbb{W}$ if and only if $Si(\mathbb{V} \vee \mathbb{W}) = Si(\mathbb{V}) \cup Si(\mathbb{W})$.

Sufficient Maltsev-type conditions for the equivalence $\mathbb{V} \vee \mathbb{W} = \mathbb{V} \times_s \mathbb{W}$ to hold are available in the literature: see Płonka (1971), Kowalski and Paoli (2011), Kowalski et al. (2013). These contributions are all inspired by the celebrated result by Grätzer, Lakser and Płonka according to which two independent similar varieties $\mathbb{V}$ and $\mathbb{W}$ are such that every member of $\mathbb{V} \vee \mathbb{W}$ is isomorphic to the *direct* product of a member of $\mathbb{V}$ and a member of $\mathbb{W}$ (Grätzer et al. 1969). Of course, the notion of independence is of limited use in the context of PBZ*–lattices, since $\mathbb{BA}$ is the unique atom in $\mathrm{Subvar}(\mathbb{PBZL}^*)$ and thus there are no two nontrivial disjoint (hence, no two independent) varieties of PBZ* –lattices. The investigation of subdirect products of varieties of PBZ*–lattices, however, can be carried out with more ad hoc methods, yielding useful information on joins of specific subvarieties.

If $\mathbb{V} \vee \mathbb{W} = \mathbb{V} \times_s \mathbb{W}$ and $\mathbb{U}$ is a variety of the same type as $\mathbb{V}$ and $\mathbb{W}$, then $(\mathbb{U} \cap \mathbb{V}) \times_s (\mathbb{U} \cap \mathbb{W}) \subseteq (\mathbb{U} \cap \mathbb{V}) \vee (\mathbb{U} \cap \mathbb{W}) \subseteq \mathbb{U} \cap (\mathbb{V} \vee \mathbb{W})$ and

$$\begin{aligned} Si(\mathbb{U} \cap (\mathbb{V} \vee \mathbb{W})) &= Si(\mathbb{U}) \cap Si(\mathbb{V} \vee \mathbb{W}) \\ &= Si(\mathbb{U}) \cap (Si(\mathbb{V}) \cup Si(\mathbb{W})) \\ &= (Si(\mathbb{U}) \cap Si(\mathbb{V})) \cup (Si(\mathbb{U}) \cap Si(\mathbb{W})) \\ &= Si(\mathbb{U} \cap \mathbb{V}) \cup Si(\mathbb{U} \cap \mathbb{W}) \\ &= Si((\mathbb{U} \cap \mathbb{V}) \times_s (\mathbb{U} \cap \mathbb{W})), \end{aligned}$$

hence $\mathbb{U} \cap (\mathbb{V} \vee \mathbb{W}) = (\mathbb{U} \cap \mathbb{V}) \vee (\mathbb{U} \cap \mathbb{W}) = (\mathbb{U} \cap \mathbb{V}) \times_s (\mathbb{U} \cap \mathbb{W})$. For instance, since $\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbb{AOL}) = \mathbb{OML} \times_s V_{\mathbb{BZL}}(\mathbb{AOL})$ (see Lemma 10.21 below), it follows that

$$\begin{aligned}\mathbb{SDM} \cap (\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbb{AOL})) &= (\mathbb{SDM} \cap \mathbb{OML}) \vee (\mathbb{SDM} \cap V_{\mathbb{BZL}}(\mathbb{AOL}))\\ &= \mathbb{OML} \vee \mathbb{SAOL} = \mathbb{OML} \times_s \mathbb{SAOL}.\end{aligned}$$

As a consequence of the above, if $\mathbb{V} \vee \mathbb{W} = \mathbb{V} \times_s \mathbb{W}$ and $\mathrm{Subvar}(\mathbb{V})$ and $\mathrm{Subvar}(\mathbb{W})$ are distributive, then $\mathrm{Subvar}(\mathbb{V} \vee \mathbb{W})$ is distributive.

Problem 10.19 *If $\mathbb{V} \vee \mathbb{W} = \mathbb{V} \times_s \mathbb{W}$, $\mathbb{C}$ is a subvariety of $\mathbb{V}$ and $\mathbb{D}$ is a subvariety of $\mathbb{W}$, under what conditions does it follow that $\mathbb{C} \vee \mathbb{D} = \mathbb{C} \times_s \mathbb{D}$? Does the condition that $\mathbb{C} \cap \mathbb{D} = \mathbb{V} \cap \mathbb{W}$ suffice? A partial answer to this question is given by Lemma 10.20 below.*

If $\mathbb{V} \vee \mathbb{W} = \mathbb{V} \times_s \mathbb{W}$ and $\mathbb{U}$ is a subvariety of $\mathbb{V}$, then $\mathbb{U} \vee \mathbb{W}$ is a subvariety of $\mathbb{V} \vee \mathbb{W}$, so that

$$\begin{aligned}Si(\mathbb{U} \vee \mathbb{W}) &= (\mathbb{U} \vee \mathbb{W}) \cap Si(\mathbb{V} \vee \mathbb{W})\\ &= (\mathbb{U} \vee \mathbb{W}) \cap (Si(\mathbb{V}) \cup Si(\mathbb{W}))\\ &= ((\mathbb{U} \vee \mathbb{W}) \cap Si(\mathbb{V})) \cup ((\mathbb{U} \vee \mathbb{W}) \cap Si(\mathbb{W}))\\ &= ((\mathbb{U} \vee \mathbb{W}) \cap Si(\mathbb{V})) \cup Si(\mathbb{W}).\end{aligned}$$

Lemma 10.20 *Let $\mathbb{V}$ and $\mathbb{W}$ be varieties of a similarity type τ, $\mathbb{U}$ a subvariety of $\mathbb{V}$ and Γ a set of identities over τ such that $\mathbb{V} \vee \mathbb{W} = \mathbb{V} \times_s \mathbb{W}$, $\mathbb{W} \vDash \Gamma$ and $\mathbb{U} = \{\mathbf{A} \in \mathbb{V} : \mathbf{A} \vDash \Gamma\}$. Then:*

- $\mathbb{U} \vee \mathbb{W} = \mathbb{U} \times_s \mathbb{W} = \{\mathbf{A} \in \mathbb{V} \vee \mathbb{W} : \mathbf{A} \vDash \Gamma\}$*;*
- $\mathbb{U} = \mathbb{V}$ *if and only if* $\mathbb{U} \vee \mathbb{W} = \mathbb{V} \vee \mathbb{W}$.

Proof Of course, $Si(\mathbb{U}) \cup Si(\mathbb{W}) \subseteq Si(\mathbb{U} \vee \mathbb{W})$. For all $\mathbf{A} \in Si(\mathbb{U} \vee \mathbb{W})$, we have: $\mathbf{A} \in Si(\mathbb{V} \vee \mathbb{W}) = Si(\mathbb{V}) \cup Si(\mathbb{W})$ and $\mathbf{A} \vDash \Gamma$, so that either $\mathbf{A} \in Si(\mathbb{W})$ or $\mathbf{A} \in Si(\mathbb{V}) \subset \mathbb{V}$ and $\mathbf{A} \vDash \Gamma$, the latter of which implies that $\mathbf{A} \in Si(\mathbb{V}) \cap \mathbb{U} = Si(\mathbb{U})$. Therefore $Si(\mathbb{U} \vee \mathbb{W}) = Si(\mathbb{U}) \cup Si(\mathbb{W})$, thus $\mathbb{U} \vee \mathbb{W} = \mathbb{U} \times_s \mathbb{W}$. We have that:

$$\begin{aligned}Si(\{\mathbf{A} \in \mathbb{V} \vee \mathbb{W} : \mathbf{A} \vDash \Gamma\}) &= \{\mathbf{A} \in Si(\mathbb{V} \vee \mathbb{W}) : \mathbf{A} \vDash \Gamma\}\\ &= \{\mathbf{A} \in Si(\mathbb{V}) \cup Si(\mathbb{W}) : \mathbf{A} \vDash \Gamma\}\\ &= \{\mathbf{A} \in Si(\mathbb{V}) : \mathbf{A} \vDash \Gamma\} \cup Si(\mathbb{W})\\ &= Si(\{\mathbf{A} \in \mathbb{V} : \mathbf{A} \vDash \Gamma\}) \cup Si(\mathbb{W})\\ &= Si(\mathbb{U}) \cup Si(\mathbb{W}) = Si(\mathbb{U} \vee \mathbb{W}),\end{aligned}$$

hence $\mathbb{U} \vee \mathbb{W} = \{\mathbf{A} \in \mathbb{V} \vee \mathbb{W} : \mathbf{A} \vDash \Gamma\}$.

Trivially, $\mathbb{U} = \mathbb{V}$ implies $\mathbb{U} \vee \mathbb{W} = \mathbb{V} \vee \mathbb{W}$. Conversely, if $\mathbb{V} \vee \mathbb{W} = \mathbb{U} \vee \mathbb{W} = \{\mathbf{A} \in \mathbb{V} \vee \mathbb{W} : \mathbf{A} \vDash \Gamma\}$, then $\mathbb{V} \vee \mathbb{W} \vDash \Gamma$, thus $\mathbb{V} \vDash \Gamma$, hence $\mathbb{U} = \{\mathbf{A} \in \mathbb{V} : \mathbf{A} \vDash \Gamma\} = \mathbb{V}$. ∎

Lemma 10.21 *(Giuntini et al. 2018) All subdirectly irreducible members of $\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbb{AOL})$ belong to $\mathbb{OML} \cup \mathbb{AOL}$, in particular $\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbb{AOL}) = \mathbb{OML} \times_s V_{\mathbb{BZL}}(\mathbb{AOL})$.*

We can derive from the above the following result from Giuntini et al. (2018):

Proposition 10.22 • $\mathbb{OML} \vee \mathbb{SAOL} = \mathbb{OML} \times_s \mathbb{SAOL}$ *and* $\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbf{D}_3) = \mathbb{OML} \times_s V_{\mathbb{BZL}}(\mathbf{D}_3)$*;*
• $\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbf{D}_3) \subsetneq \mathbb{OML} \vee \mathbb{SAOL} \subsetneq \mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbb{AOL})$.

Proof Recall from Giuntini et al. (2017, Corollary 3.3) that $V_{\mathbb{BZL}}(\mathbf{D}_3) = \{\mathbf{A} \in V_{\mathbb{BZL}}(\mathbb{AOL}) : \mathbf{A} \vDash \{\text{SDM}, \text{SK}\}\}$. Now apply the fact that $\mathbb{OML} \vDash \{\text{SDM}, \text{SK}\}$ and Lemmas 10.21 and 10.20 to obtain first that $\mathbb{OML} \vee \mathbb{SAOL} = \mathbb{OML} \times_s \mathbb{SAOL}$, then that $\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbf{D}_3) = \mathbb{OML} \times_s V_{\mathbb{BZL}}(\mathbf{D}_3)$. Recall that $\mathbf{D}_5 \in \mathbb{SAOL} \setminus V_{\mathbb{BZL}}(\mathbf{D}_3)$, which is easily noticed from the fact that $\mathbf{D}_5 \nvDash \text{SK}$. The antiortholattice $\mathbf{D}_2^2 \oplus \mathbf{D}_2^2 \in V_{\mathbb{BZL}}(\mathbb{AOL}) \setminus \mathbb{SAOL}$. Hence $V_{\mathbb{BZL}}(\mathbf{D}_3) \subsetneq \mathbb{SAOL} \subsetneq V_{\mathbb{BZL}}(\mathbb{AOL})$, thus $\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbf{D}_3) \subsetneq \mathbb{OML} \vee \mathbb{SAOL} \subsetneq \mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbb{AOL})$ by Lemma 10.20 and the above. ■

Let us consider the identities:

WDSDM	$(x \wedge (y \vee z))^{\sim} \approx (x \wedge y)^{\sim} \wedge (x \wedge z)^{\sim}$
DIST$\vee^{\sim}$	$(x \vee x^{\sim}) \wedge (y \vee y^{\sim} \vee z \vee z^{\sim}) \approx$ $((x \vee x^{\sim}) \wedge (y \vee y^{\sim})) \vee ((x \vee x^{\sim}) \wedge (z \vee z^{\sim}))$
WDIST$\vee^{\sim}$	$((x \vee x^{\sim}) \wedge (y \vee y^{\sim} \vee z \vee z^{\sim}))^{\sim} \approx$ $(((x \vee x^{\sim}) \wedge (y \vee y^{\sim})) \vee ((x \vee x^{\sim}) \wedge (z \vee z^{\sim})))^{\sim}$

Note that WDSDM implies WDIST$\vee^{\sim}$ and DIST$\vee^{\sim}$ implies WDIST$\vee^{\sim}$. Also, recall from Giuntini et al. (2017), Mureşan (2019) that $V_{\mathbb{BZL}}(\mathbf{D}_5) = \mathbb{SAOL} \cap \mathbb{DIST}$.

Proposition 10.23 $V_{\mathbb{BZL}}(\mathbf{D}_5) = \mathbb{SAOL} \cap \mathbb{DIST} \subsetneq \mathbb{SAOL}, \mathbb{DIST} \subsetneq \mathbb{SAOL} \vee \mathbb{DIST} \subsetneq V_{\mathbb{BZL}}(\mathbb{AOL})$.

Proof Observe that the identity WDSDM is satisfied both in $\mathbb{SAOL}$ and in $\mathbb{DIST}$. The antiortholattice on $\mathbf{M}_3 \oplus \mathbf{M}_3$ fails WDSDM, because, if a, b, c are its three atoms, then $(a \wedge (b \vee c))^{\sim} = a^{\sim} = 0$, yet $(a \wedge b)^{\sim} \wedge (a \wedge c)^{\sim} = 0^{\sim} \wedge 0^{\sim} = 1$. Hence $\mathbf{M}_3 \oplus \mathbf{M}_3 \in \mathbb{AOL} \setminus (\mathbb{SAOL} \vee \mathbb{DIST}) \subseteq V_{\mathbb{BZL}}(\mathbb{AOL}) \setminus (\mathbb{SAOL} \vee \mathbb{DIST})$. The antiortholattice $\mathbf{D}_2 \oplus \mathbf{M}_3 \oplus \mathbf{D}_2 \in \mathbb{SAOL} \setminus \mathbb{DIST}$, while the antiortholattice $\mathbf{D}_2^2 \oplus \mathbf{D}_2^2 \in \mathbb{DIST} \setminus \mathbb{SAOL}$, hence $\mathbb{SAOL}$ and $\mathbb{DIST}$ are incomparable, thus $\mathbb{SAOL} \cap \mathbb{DIST} \subsetneq \mathbb{SAOL}, \mathbb{DIST} \subsetneq \mathbb{SAOL} \vee \mathbb{DIST}$. ■

Proposition 10.24 • $\mathbb{OML} \vee \mathbb{DIST} = \mathbb{OML} \times_s \mathbb{DIST}$ *and* $\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbf{D}_5) = \mathbb{OML} \times_s V_{\mathbb{BZL}}(\mathbf{D}_5)$*;*
• $\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbf{D}_3) \subsetneq \mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbf{D}_5) = \mathbb{OML} \vee (\mathbb{SAOL} \cap \mathbb{DIST}) = (\mathbb{OML} \vee \mathbb{SAOL}) \cap (\mathbb{OML} \vee \mathbb{DIST}) \subsetneq \mathbb{OML} \vee \mathbb{SAOL}, \mathbb{OML} \vee \mathbb{DIST} \subsetneq \mathbb{OML} \vee \mathbb{SAOL} \vee \mathbb{DIST} \subsetneq \mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbb{AOL})$, *in particular the varieties* $\mathbb{OML} \vee \mathbb{SAOL}$ *and* $\mathbb{OML} \vee \mathbb{DIST}$ *are incomparable.*

Proof Note that $\mathbb{OML} \vDash \text{DIST}\vee^{\sim}$ and that, in $\mathbb{AOL}$, DIST$\vee^{\sim}$ is equivalent to DIST, that is $\mathbb{DIST} \cap \mathbb{AOL} = \{\mathbf{A} \in \mathbb{AOL} : \mathbf{A} \vDash \text{DIST}\vee^{\sim}\}$. The latter, along with the fact that $\mathbb{DIST}$ is a subvariety of $V_{\mathbb{BZL}}(\mathbb{AOL})$ and Lemma 10.15, give us:

$$\begin{aligned} Si(\mathbb{DIST}) &= \mathbb{DIST} \cap Si(V_{\mathbb{BZL}}(\mathbb{AOL})) \\ &= \mathbb{DIST} \cap Si(\mathbb{AOL}) \\ &= Si(\mathbb{DIST} \cap \mathbb{AOL}) \\ &= Si(\{\mathbf{A} \in \mathbb{AOL} : \mathbf{A} \vDash \mathrm{DIST}\vee^{\sim}\}) \\ &= Si(\{\mathbf{A} \in V_{\mathbb{BZL}}(\mathbb{AOL}) : \mathbf{A} \vDash \mathrm{DIST}\vee^{\sim}\}), \end{aligned}$$

therefore $\mathbb{DIST} = \{\mathbf{A} \in V_{\mathbb{BZL}}(\mathbb{AOL}) : \mathbf{A} \vDash \mathrm{DIST}\vee^{\sim}\}$. By Lemmas 10.21 and 10.20, it follows that $\mathbb{OML} \vee \mathbb{DIST} = \mathbb{OML} \times_s \mathbb{DIST}$. By the above, $\mathbb{OML} \vDash \{\mathrm{SDM}, \mathrm{DIST}\vee^{\sim}\}$ and $V_{\mathbb{BZL}}(\mathbf{D}_5) = \mathbb{SAOL} \cap \mathbb{DIST} = \{\mathbf{A} \in V_{\mathbb{BZL}}(\mathbb{AOL}) : \mathbf{A} \vDash \{\mathrm{SDM}, \mathrm{DIST}\vee^{\sim}\}\}$, hence $\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbf{D}_5) = \mathbb{OML} \times_s V_{\mathbb{BZL}}(\mathbf{D}_5)$ by Lemmas 10.21 and 10.20. By the above, Propositions 10.22 and 10.23 and again Lemma 10.20, it follows that:

$$\begin{aligned} \mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbf{D}_3) &\subsetneq \mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbf{D}_5) \\ &\subsetneq \mathbb{OML} \vee \mathbb{SAOL}, \mathbb{OML} \vee \mathbb{DIST} \\ &\subsetneq \mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbb{AOL}). \end{aligned}$$

By Lemma 10.20, the above and Proposition 10.22,

$$\begin{aligned} \mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbf{D}_5) &= \mathbb{OML} \vee (\mathbb{SAOL} \cap \mathbb{DIST}) \\ &= \mathbb{OML} \vee \{\mathbf{A} \in V_{\mathbb{BZL}}(\mathbb{AOL}) : \mathbf{A} \vDash \{\mathrm{SDM}, \mathrm{DIST}\}\} \\ &= \{\mathbf{A} \in \mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbb{AOL}) : \mathbf{A} \vDash \{\mathrm{SDM}, \mathrm{DIST}\}\} \\ &= \{\mathbf{A} \in \mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbb{AOL}) : \mathbf{A} \vDash \mathrm{SDM}\} \\ &\cap \{\mathbf{A} \in \mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbb{AOL}) : \mathbf{A} \vDash \mathrm{DIST}\} \\ &= (\mathbb{OML} \vee \mathbb{SAOL}) \cap (\mathbb{OML} \vee \mathbb{DIST}), \end{aligned}$$

hence $(\mathbb{OML} \vee \mathbb{SAOL}) \cap (\mathbb{OML} \vee \mathbb{DIST}) \subsetneq \mathbb{OML} \vee \mathbb{SAOL}, \mathbb{OML} \vee \mathbb{DIST}$, so that $\mathbb{OML} \vee \mathbb{SAOL}$ and $\mathbb{OML} \vee \mathbb{DIST}$ are incomparable. Therefore

$$\begin{aligned} \mathbb{OML} \vee \mathbb{SAOL}, \mathbb{OML} \vee \mathbb{DIST} &\subsetneq \mathbb{OML} \vee \mathbb{SAOL} \vee \mathbb{OML} \vee \mathbb{DIST} \\ &= \mathbb{OML} \vee \mathbb{SAOL} \vee \mathbb{DIST}. \end{aligned}$$

Since $\mathbb{OML} \vDash \mathrm{DIST}\vee^{\sim}$ and $\mathbb{SAOL} \vee \mathbb{DIST} \vDash \mathrm{WDSDM}$, it follows that $\mathbb{OML} \vee \mathbb{SAOL} \vee \mathbb{DIST} \vDash \mathrm{WDIST}\vee^{\sim}$. Note that, in $\mathbb{AOL}$, $\mathrm{WDIST}\vee^{\sim}$ is equivalent to WDSDM, hence, by the proof of Proposition 10.23, the antiortholattice $\mathbf{M}_3 \oplus \mathbf{M}_3$ fails $\mathrm{WDIST}\vee^{\sim}$. It follows that $\mathbf{M}_3 \oplus \mathbf{M}_3 \in \mathbb{AOL} \setminus (\mathbb{OML} \vee \mathbb{SAOL} \vee \mathbb{DIST}) \subseteq (\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbb{AOL})) \setminus (\mathbb{OML} \vee \mathbb{SAOL} \vee \mathbb{DIST})$, therefore $\mathbb{OML} \vee \mathbb{SAOL} \vee \mathbb{DIST} \subsetneq \mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbb{AOL})$. ■

Lemma 10.25 *For any subvariety* $\mathbb{V}$ *of* $\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbb{AOL})$, $Si(\mathbb{V}) = \mathbb{V} \cap Si(\mathbb{OML} \cup \mathbb{AOL})$.

Proof By Lemma 10.21. ■

Note that, if a PBZ*–lattice $\mathbf{L}$ satisfies the SDM, then 0 is meet–irreducible in the join–subsemilattice $T(\mathbf{L})$ of $\mathbf{L}$, but the converse does not hold.

Lemma 10.26 *Let* $\mathbf{A}$ *be an antiortholattice without SDM and* $(\mathbf{A}_i)_{i\in I}$ *be a non–empty family of antiortholattices. Then:*

- *if* $\mathbf{A} \in \mathbb{S}_{\mathbb{BZL}}(\prod_{i\in I} \mathbf{A}_i)$*, then the family* $(\mathbf{A}_i)_{i\in I}$ *contains no nontrivial antiortholattice with SDM;*
- $\mathbf{A} \in \mathbb{S}_{\mathbb{BZL}}(\prod_{i\in I} \mathbf{A}_i)$ *if and only if* $\mathbf{A} \in \mathbb{S}_{\mathbb{BZL}}(\prod_{i\in I, \mathbf{A}_i \nvDash SDM} \mathbf{A}_i)$.

Proof The second statement obviously follows from the first. Now assume that $\mathbf{A} \in \mathbb{S}_{\mathbb{BZL}}(\prod_{i\in I} \mathbf{A}_i)$, let $J = \{j \in I : \mathbf{A}_j \vDash \text{SDM}\}$ and assume ex absurdo that there exists a $k \in J$ such that $\mathbf{A}_k$ is nontrivial. We may consider $A \subseteq \prod_{i\in I} A_i$. $\mathbf{A}$ is an antiortholattice that fails SDM, in particular a nontrivial antiortholattice, hence there exist $a = (a_i)_{i\in I}, b = (b_i)_{i\in I} \in A \setminus \{0\} = D(\mathbf{A}) = D(\prod_{i\in I} \mathbf{A}_i) = \prod_{i\in I} D(\mathbf{A}_i) = \prod_{i\in I} ((A_i \setminus \{0\}) \cup \{1\})$ such that $a \wedge b = 0$, so that $a_k \wedge b_k = 0$ and $a_k, b_k \in D(\mathbf{A}_k) = A_k \setminus \{0\}$, which contradicts the fact that $\mathbf{A}_k$ satisfies the SDM. ■

Proposition 10.27 *If* $\mathbb{V}$ *is a subvariety of* $V_{\mathbb{BZL}}(\mathbb{AOL})$*, then:* $\mathbb{V} \vee \mathbb{SAOL} = \mathbb{V} \times_s \mathbb{SAOL}$ *if and only if* $(\mathbb{V} \vee \mathbb{SAOL}) \cap \mathbb{AOL} = (\mathbb{V} \cup \mathbb{SAOL}) \cap \mathbb{AOL}$.

Proof By the above, $\mathbb{V} \vee \mathbb{SAOL} = \mathbb{V} \times_s \mathbb{SAOL}$ if and only if $Si(\mathbb{V} \vee \mathbb{SAOL}) = Si(\mathbb{V} \cup \mathbb{SAOL})$. Since $Si(V_{\mathbb{BZL}}(\mathbb{AOL})) \subset \mathbb{AOL}$, the right-to-left implication holds. Now assume that $Si(\mathbb{V} \vee \mathbb{SAOL}) = Si(\mathbb{V} \cup \mathbb{SAOL})$, and assume ex absurdo that there exists an $\mathbf{L} \in ((\mathbb{V} \vee \mathbb{SAOL}) \cap \mathbb{AOL}) \setminus (\mathbb{V} \cup \mathbb{SAOL})$. Then $\mathbf{L} \in \mathbb{V} \times_s \mathbb{SAOL}$, hence $\mathbf{L} \in \mathbb{S}_{\mathbb{BZL}}(\mathbf{A} \times \mathbf{B})$ for some $\mathbf{A} \in \mathbb{V}$ and some $\mathbf{B} \in \mathbb{SAOL}$, therefore $\mathbf{L} \in \mathbb{S}_{\mathbb{BZL}}(\mathbf{A} \times \prod_{j\in J} \mathbf{B}_j)$ for some family $(\mathbf{B}_j)_{j\in J} \subseteq \mathbb{SAOL} \cap \mathbb{AOL}$. Thus $\mathbf{L} \in \mathbb{S}_{\mathbb{BZL}}(\mathbf{A})$ by Lemma 10.26, so that $\mathbf{L} \in \mathbb{V}$, a contradiction. Hence $(\mathbb{V} \vee \mathbb{SAOL}) \cap \mathbb{AOL} \subseteq (\mathbb{V} \cup \mathbb{SAOL}) \cap \mathbb{AOL}$. ■

10.4 Comparison with Other Structures

10.4.1 Distributive Lattices with Two Unary Operations

Bounded distributive lattices expanded both by a De Morgan complementation and a unary operation with Stone-like properties have been the object of rather intensive investigations over the past decades. In particular, Blyth et al. (2015) have studied, under the label of *quasi-Stone De Morgan algebras*, bounded distributive lattices with two unary operations that make their appropriate reducts, at the same time, De Morgan algebras and *quasi-Stone algebras* (Sankappanavar and Sankappanavar (1993), Gaitàn (2000), Celani (2011)). Quasi-Stone De Morgan algebras that are simultaneously *Stone* algebras and *Kleene* algebras are known under the name of *Kleene-Stone algebras*; they have been studied in Guzmàn (1994) and, more recently, in the already quoted Blyth et al. (2015). We begin this section by showing that the variety of antiortholattices generated by the algebra $\mathbf{D}_5$ coincides with the variety of Kleene-Stone algebras. This fact explains the similarity of some results independently obtained in Blyth et al. (2015), Giuntini et al. (2017), Mureşan (2019).

Definition 10.28 A *quasi-Stone algebra* is an algebra $\mathbf{A} = (A, \wedge, \vee, ^\sim, 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and the unary operation $^\sim$ satisfies the following conditions for all $a, b \in A$:

QS1 $0^\sim = 1$ and $1^\sim = 0$; ***QS4*** $a \leq a^{\sim\sim}$;
QS2 $(a \vee b)^\sim = a^\sim \wedge b^\sim$; ***QS5*** $a^\sim \vee a^{\sim\sim} = 1$
QS3 $(a \wedge b^\sim)^\sim = a^\sim \vee b^{\sim\sim}$;

A quasi-Stone algebra $\mathbf{A}$ is a *Stone algebra* if it additionally satisfies SDM.

The following useful lemma contains results to be found in Sankappanavar and Sankappanavar (1993) and Blyth et al. (2015):

Lemma 10.29 *Let $\mathbf{A} = (A, \wedge, \vee, ^\sim, 0, 1)$ be a quasi-Stone algebra. Then:*

(i) $\mathbf{A}$ *satisfies the following conditions for all $a, b \in A$:*

QS6 *if $a \leq b$, then $b^\sim \leq a^\sim$;* ***QS8*** $a^{\sim\sim\sim} = a^\sim$;
QS7 $a \wedge a^\sim = 0$; ***QS9*** $a \wedge b^\sim = 0$ *if and only if* $a \leq b^{\sim\sim}$.

(ii) The set $B(\mathbf{A}) = \{a^\sim : a \in A\} = \{a \in A : a = a^{\sim\sim}\}$ is a Boolean subuniverse of $\mathbf{A}$.

Clearly, in case $\mathbf{A}$ is a Stone algebra, the condition QS9 can be strengthened to the pseudocomplementation equivalence:

S1 $a \wedge b = 0$ if and only if $a \leq b^\sim$ for all $a, b \in A$.

Definition 10.30 A *quasi-Stone De Morgan algebra* is an algebra $\mathbf{A} = (A, \wedge, \vee, ', ^\sim, 0, 1)$ of type $(2, 2, 1, 1, 0, 0)$ such that $(A, \wedge, \vee, ', 0, 1)$ is a De Morgan algebra, $(A, \wedge, \vee, ^\sim, 0, 1)$ is a quasi-Stone algebra, and $a' \in B(\mathbf{A})$ whenever $a \in B(\mathbf{A})$. If $(A, \wedge, \vee, ', 0, 1)$ is a Kleene algebra and $(A, \wedge, \vee, ^\sim, 0, 1)$ is a (quasi-)Stone algebra, then $\mathbf{A}$ is said to be a *Kleene-(quasi-)Stone algebra*.

Lemma 10.31 *(Blyth et al. 2015) If $\mathbf{A}$ is a quasi-Stone De Morgan algebra, then for all $a \in A$ we have that $a^{\sim\sim} = a^{\sim\prime\sim\prime}$.*

Recall from Proposition 10.23 that the variety generated by the 5-element antiortholattice chain $\mathbf{D}_5$ is axiomatised relative to $\mathbb{PBZL}^*$ by the lattice distribution axiom DIST and the Strong De Morgan law SDM (J0 easily follows from these assumptions in the context of $\mathbb{PBZL}^*$). We now show that:

Theorem 10.32 $V_{\mathbb{BZL}}(\mathbf{D}_5)$ *coincides with the variety of Kleene-Stone algebras.*

Proof It is readily seen that $\mathbf{D}_5$ satisfies all the defining conditions of Kleene-Stone algebras. Conversely, by the above remark, it will be sufficient to show that Kleene-Stone algebras satisfy all the axioms of PBZ*–lattices, since they are clearly distributive as lattices and satisfy SDM by definition. We confine ourselves to the sole nontrivial items. (i) The condition $(*)$, $\left(x \wedge x'\right)^{\sim} = x^{\sim} \vee x'^{\sim}$ directly follows from SDM. (ii) We show that $a^{\sim\sim} = a^{\sim\prime}$. By QS5, $a^{\sim} \vee a^{\sim\sim} = 1$, whence $a^{\sim\prime} \wedge a^{\sim\sim\prime} = 0$. By S1, $a^{\sim\sim\prime} \leq a^{\sim\prime\sim}$, whence, given the fact that $a^{\sim\sim} \in B\left(\mathbf{A}\right)$,

$$a^{\sim\prime} \leq_{(QS4)} a^{\sim\prime\sim\sim} \leq_{(QS6)} a^{\sim\sim\prime\sim} = a^{\sim\sim}.$$

From this inequality, QS6 and QS8 we obtain that $a^{\sim} = a^{\sim\sim\sim} \leq a^{\sim\prime\sim}$ and thus, by Lemma 10.31, $a^{\sim\sim} = a^{\sim\prime\sim\prime} \leq a^{\sim\prime}$. The converse inequality follows from S1 and the fact that $a^{\sim} \in B\left(\mathbf{A}\right)$. (iii) To round up our proof, it will suffice to show that any Kleene algebra is paraorthomodular. Thus, let $a \leq b$ and $a' \wedge b = 0$. Then $a' \wedge a \leq a' \wedge b = 0$, whence a is sharp and thus $a \vee a' = 1$. As $a \wedge b = a$ and $a' \wedge b = 0$, distributivity implies that

$$a = (a \wedge b) \vee \left(a' \wedge b\right) = \left(a \vee a'\right) \wedge b = 1 \wedge b = b.$$

■

The question as to whether the distributive subvariety $\mathbb{DIST}$ of $V_{\mathbb{BZL}}\left(\mathbb{AOL}\right)$ coincides with the variety of Kleene-quasi-Stone algebras is of a certain interest. The next Example answers this problem in the negative.

Example 10.33 The BZ-lattice $\mathbf{BZ}_4$ (see Giuntini et al. (2016, Fig. 5)) is a Kleene-quasi-Stone algebra, yet it is not even a member of $\mathbb{PBZL}^*$. In fact, call a and a' its two atoms. We have that:

$$\left(a \wedge a'\right)^{\sim} = 0^{\sim} = 1 \neq 0 = a^{\sim} \vee a'^{\sim}.$$

Finally, we prove that the variety generated by the 3-element antiortholattice chain $\mathbf{D}_3$ is a discriminator variety (Werner 1978).

Proposition 10.34 *$V_{\mathbb{BZL}}\left(\mathbf{D}_3\right)$ is a discriminator variety.*

Proof Clearly, it suffices to find a ternary term that realises the discriminator function on $\mathbf{D}_3$. Let first

$$e\left(x, y\right) = \left(x^{\sim} \wedge \Diamond y\right) \vee \left(y^{\sim} \wedge \Diamond x\right) \vee \left(\Box x \wedge \left(\Box y\right)^{\sim}\right) \vee \left(\Box y \wedge \left(\Box x\right)^{\sim}\right).$$

It is a routinary matter to check that for all $a, b \in D_3$, $e^{\mathbf{D}_3}\left(a, a\right) = 0$ and $e^{\mathbf{D}_3}\left(a, b\right) = 1$ if $a \neq b$. It follows that

$$t\left(x, y, z\right) = \left(e\left(x, y\right) \vee z\right) \wedge \left(e\left(x, y\right)' \vee x\right)$$

realises the discriminator function on $\mathbf{D}_3$. ■

Observe that the algebra $\mathbf{D}_3$ fails to be primal, because it has the nontrivial proper subuniverse $\{0, 1\}$. Nonetheless, upon identifying D_3 with the set of rational numbers $\left\{0, \frac{1}{2}, 1\right\}$, the truncated sum operation is definable as follows:

$$x \oplus y = \min(1, x + y) = (x \vee \Diamond y) \wedge (y \vee \Diamond x).$$

It is easy to check that, upon expanding its signature by this binary operation, $\mathbf{D}_3$ becomes an instance of a *De Morgan Brouwer-Zadeh MV-algebra* (Cattaneo et al. 1998, 1999) and, therefore, generates a subvariety of such. The interest of this remark lies in the fact that the variety of De Morgan Brouwer-Zadeh MV-algebras is known to be term-equivalent to other well-known varieties of algebras of logic, including Heyting-Wajsberg algebras, Stonean MV-algebras and MV algebras with Baaz delta (Cattaneo et al. 2004). In the next section, we will see that $V_{\mathbb{BZL}}(\mathbf{D}_3)$ is term-equivalent to another well-known variety of algebras of logic.

10.4.2 Modal Algebras

The standard examples of modal algebras (monadic algebras or interior algebras, to name a few examples) were devised as the algebraic counterparts of normal modal logics, which are extensions of classical propositional logic—therefore, they all have a Boolean algebra reduct. There is a thriving literature, however, on “nonstandard” modal algebras based on generic De Morgan algebras: see below for the appropriate references. The aim of this section is to chart this area of research and locate term-equivalent counterparts of some distributive subvarieties of PBZ*–lattices on this map. We consider algebras $\mathbf{M} = \left(M, \wedge, \vee, ', \Diamond, 0, 1\right)$ of type (2, 2, 1, 1, 0, 0), where $\left(M, \wedge, \vee, ', 0, 1\right)$ is a De Morgan algebra. We assume that $'$ binds stronger than $\Diamond$, to reduce the number of parentheses. The following list of identities will be crucial for defining the varieties that follow; henceforth, $\Box x$ is short for $\left(\Diamond x'\right)'$.

M1 $\Diamond 0 \approx 0$
M2 $\Diamond(x \vee y) \approx \Diamond x \vee \Diamond y$
M3 $x \leq \Diamond x$
M4 $\Diamond x \approx \Diamond\Diamond x$
M5 $\Diamond x \wedge (\Diamond x)' \approx 0$
M6 $\Diamond x \approx \Box\Diamond x$
M7 $\Diamond\left(x \wedge x'\right) \approx \Diamond x \wedge \Diamond x'$
M8 $x' \vee \Diamond x \approx 1$
M9 $\Diamond(x \wedge y) \approx \Diamond x \wedge \Diamond y$
M10 $x \wedge x' \approx \Diamond x \wedge x'$

Definition 10.35 (i) A $\Diamond$-*De Morgan algebra* is an algebra $\mathbf{M} = (M, \wedge, \vee, ', \Diamond, 0, 1)$ of type (2, 2, 1, 1, 0, 0), where $\left(M, \wedge, \vee, ', 0, 1\right)$ is a De Morgan algebra and the identities M1 and M2 are satisfied.

(ii) A *topological quasi-Boolean algebra* is a $\Diamond$-De Morgan algebra satisfying the identities M3 and M4.
(iii) A *classical $\Diamond$-De Morgan algebra* is a topological quasi-Boolean algebra satisfying the identity M5.
(iv) A *monadic De Morgan algebra* is a classical $\Diamond$-De Morgan algebra satisfying the identity M6.

$\Diamond$-De Morgan algebras and classical $\Diamond$-De Morgan algebras were introduced in dual form by Celani (2011, pp. 253–254). Topological quasi-Boolean algebras were first investigated by Banerjee and Chakraborty in the context of the theory of rough sets (Banerjee and Chakraborty 1993). The authors of Saha et al. (2014) also introduce, under the label of *topological quasi-Boolean algebras 5*, a subvariety of topological quasi-Boolean algebras that satisfy M6 but not M5. Clearly, topological quasi-Boolean algebras are meant to be a nonclassical counterpart of interior algebras, while monadic De Morgan algebras can be viewed as a nonclassical counterpart of monadic algebras. Condition M5, which is of course trivial once our algebras have a Boolean nonmodal reduct, is there to restore the Boolean behaviour of the nonmodal operators, when applied to arguments of the form $\Diamond x$. Observe that all classical $\Diamond$-De Morgan algebras satisfy the identity M8 (Celani 2011, Lemma 2.3).

There are several ways to strengthen the defining conditions of classical $\Diamond$-De Morgan algebras with an eye to obtaining varieties with more interesting properties.

(i) A possible avenue is to impose on the possibility operator properties that would determine a collapse of modality when the underlying structures are Boolean algebras. For example, *tetravalent modal algebras* (Monteiro 1963; Loureiro 1983) are classical $\Diamond$-De Morgan algebras that satisfy M10, although they are usually presented in a streamlined axiomatisation containing only the axioms for De Morgan algebras plus M8 and M10. They form a discriminator variety, generated by a quasiprimal four-element algebra (see item (iv) of the proof of Theorem 10.40 below).
(ii) On the other hand, one can enforce what Cattaneo et al. (2011) call a "deviant" behaviour of the possibility operator, requesting that it distribute not only over joins, but over meets as well. *Involutive Stone algebras* ((Cignoli and Gallego 1983); cp. also Cattaneo et al. (2011), where these structures are called *MDS5-algebras*), thus, are classical $\Diamond$-De Morgan algebras satisfying M9. It is known that both involutive Stone algebras and tetravalent modal algebras are monadic De Morgan algebras: see Cignoli and Gallego (1983) and Font and Rius (2000, Proposition 1.2), respectively.

We now introduce the modal analogue of distributive PBZ*–lattices.

Definition 10.36 A *weak Łukasiewicz algebra* is a classical $\Diamond$-De Morgan algebra $\mathbf{M} = \left(M, \wedge, \vee, ', \Diamond, 0, 1\right)$ such that its $\Diamond$-free reduct is a Kleene algebra and the identity M7 is satisfied.

Theorem 10.37 *(i) Every weak Łukasiewicz algebra* **M** *is a monadic De Morgan algebra.*

(ii) The variety of weak Łukasiewicz algebras is term-equivalent to $\mathbb{DIST}$.

Proof (i) Let $a \in M$. Using M1, M5, M7 and M4, we have that

$$0 = \Diamond 0 = \Diamond\left(\Diamond a \wedge (\Diamond a)'\right) = \Diamond\Diamond a \wedge \Diamond\left((\Diamond a)'\right) = \Diamond a \wedge \Diamond\left((\Diamond a)'\right).$$

Thus $(\Diamond a)' \vee \Box\Diamond a = 1$, whence, by M5,

$$\Diamond a = \Diamond a \wedge \left((\Diamond a)' \vee \Box\Diamond a\right) = \left(\Diamond a \wedge (\Diamond a)'\right) \vee (\Diamond a \wedge \Box\Diamond a) = \Diamond a \wedge \Box\Diamond a.$$

Consequently, $\Diamond a \leq \Box\Diamond a$. The converse inequality follows from M3.

(ii) Let $\mathbf{M} = \left(M, \wedge, \vee, ', \Diamond^{\mathbf{M}}, 0, 1\right)$ be a weak Łukasiewicz algebra. We define $f(\mathbf{M})$ as the algebra $\left(M, \wedge, \vee, ', {}^{\sim f(\mathbf{M})}, 0, 1\right)$, where for all $a \in M$, $a^{\sim f(\mathbf{M})} = \left(\Diamond^{\mathbf{M}} a\right)'$. Conversely, given a distributive PBZ*–lattice $\mathbf{L} = \left(L, \wedge, \vee, ', {}^{\sim\mathbf{L}}, 0, 1\right)$, we define $g(\mathbf{L})$ as the algebra $\left(L, \wedge, \vee, ', \Diamond^{g(\mathbf{L})}, 0, 1\right)$, where for all $a \in L$, $\Diamond^{g(\mathbf{L})} a = a^{\sim\mathbf{L}\sim\mathbf{L}}$. Clearly, $f(\mathbf{M})$ has a Kleene lattice reduct. If $a \in M$, then $a \wedge a^{\sim f(\mathbf{M})} = a \wedge \left(\Diamond^{\mathbf{M}} a\right)' \leq \Diamond^{\mathbf{M}} a \wedge \left(\Diamond^{\mathbf{M}} a\right)' = 0$, by M3 and M5. Moreover,

$$a^{\sim f(\mathbf{M})\sim f(\mathbf{M})} = \left(\Diamond^{\mathbf{M}}\left(\Diamond^{\mathbf{M}} a\right)'\right)' = \Diamond^{\mathbf{M}} a \geq a,$$

by M3 and item (1). For the same reason, $a^{\sim f(\mathbf{M})'} = \left(\Diamond^{\mathbf{M}} a\right)'' = \Diamond^{\mathbf{M}} a = a^{\sim f(\mathbf{M})\sim f(\mathbf{M})}$. Finally, by M2, whenever $a \leq b$,

$$\Diamond^{\mathbf{M}} b = \Diamond^{\mathbf{M}}(a \vee b) = \Diamond^{\mathbf{M}} a \vee \Diamond^{\mathbf{M}} b,$$

i.e., $\Diamond^{\mathbf{M}} a \leq \Diamond^{\mathbf{M}} b$, whence $b^{\sim f(\mathbf{M})} = \left(\Diamond^{\mathbf{M}} b\right)' \leq \left(\Diamond^{\mathbf{M}} a\right)' \leq a^{\sim f(\mathbf{M})}$. In sum, $f(\mathbf{M})$ is a distributive BZ-lattice. Condition $(*)$ holds because of M7. Similarly, by reverse-engineering $g(\mathbf{L})$, it is not hard to show that it is a weak Łukasiewicz algebra. To round off the proof, observe that for $a \in L$,

$$a^{\sim f(g(\mathbf{L}))} = \left(\Diamond^{g(\mathbf{L})} a\right)' = a^{\sim\mathbf{L}\sim\mathbf{L}'} = a^{\sim\mathbf{L}\sim\mathbf{L}\sim\mathbf{L}} = a^{\sim\mathbf{L}},$$
$$\Diamond^{g(f(\mathbf{M}))} a = a^{\sim f(\mathbf{M})\sim f(\mathbf{M})} = \left(\Diamond^{\mathbf{M}}\left(\Diamond^{\mathbf{M}} a\right)'\right)' = \Diamond^{\mathbf{M}} a.$$

Thus, f and g are mutually inverse functions. ∎

Similar term-equivalence results with subvarieties of $\mathbb{PBZL}^*$ are obtained in Cattaneo et al. (1998) and Cattaneo and Nisticò (1989) for two special subvarieties of weak Łukasiewicz algebras.

Definition 10.38 (i) (Cattaneo et al. 1998, Definition 4.2) A *Łukasiewicz algebra* is a weak Łukasiewicz algebra that satisfies the identity M9.

(ii) (Monteiro 1963) A *three-valued Łukasiewicz algebra* is a Łukasiewicz algebra that satisfies the identity M10.

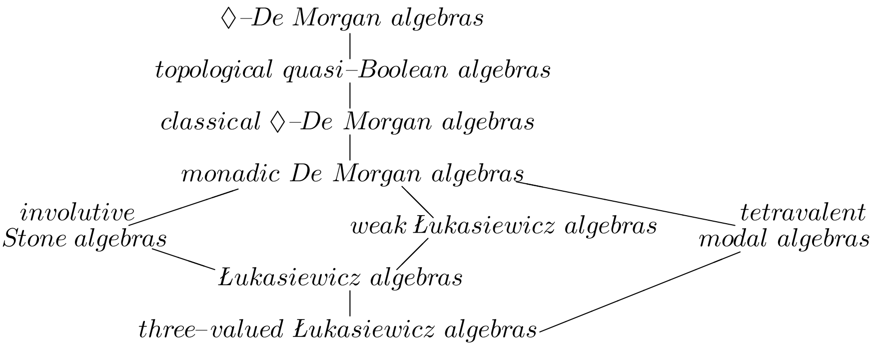
Clearly, Łukasiewicz algebras are exactly the involutive Stone algebras whose ♢-free reduct is a Kleene lattice. There is a burgeoning literature on three-valued Łukasiewicz algebras, see e.g., Abad and Figallo (1985), Monteiro (1963), Moisil (1963). Three-valued Łukasiewicz algebras can be equivalently characterised as tetravalent modal algebras satisfying M9, in which case, the Kleene identity follows from the axioms. They are also called *pre-rough algebras* in the literature (Saha et al. 2014).

Theorem 10.39 *(Cattaneo et al. 1998, Theorems 4.3 and 5.7)*

(i) The variety of Łukasiewicz algebras is term-equivalent to $V_{\mathbb{BZL}}(\mathbf{D}_5)$.
(ii) The variety of three-valued Łukasiewicz algebras is term-equivalent to $V_{\mathbb{BZL}}(\mathbf{D}_3)$.

Taking into account the remarks at the end of last section, it is evident that $V_{\mathbb{BZL}}(\mathbf{D}_5)$ and $V_{\mathbb{BZL}}(\mathbf{D}_3)$ have repeatedly resurfaced in many different incarnations, with different choices of primitives or with different axiomatisations. We collect many of the observations made thus far in the following result.

Theorem 10.40 *The strict inclusions and incomparabilities depicted in the following diagram all hold:*



Proof All that remains to be proved is that the inclusions are strict and that the varieties not connected by upward chains are incomparable.

(i) Consider the algebra $\mathbf{D}_2$ as a De Morgan algebra, and let $\Diamond 0 = \Diamond 1 = 0$. This algebra is a ♢-De Morgan algebra which is not a topological quasi-Boolean algebra.
(ii) Consider the algebra $\mathbf{D}_3$ as a De Morgan algebra, and let $\Diamond x = x$ for all $x \in D_3 = \{0, a, 1\}$. This algebra is a topological quasi-Boolean algebra which is not a classical ♢-De Morgan algebra. In fact, $\Diamond a \wedge (\Diamond a)' = a \neq 0$.
(iii) Consider the algebra $\mathbf{D}_2^2$ as a De Morgan algebra with universe $\{0, a, a', 1\}$, and let $\Diamond x = x$ for all $x \in \{0, a, 1\}$, and $\Diamond a' = 1$. This algebra is a topological quasi-Boolean algebra which is not a monadic De Morgan algebra. In fact, $\Box\Diamond a = 0 \neq a = \Diamond a$.

(iv) Let $\mathbf{B}_4$ be the four-element algebra on $\{0, a, b, 1\}$ that generates De Morgan algebras, with $a = a'$ and $b = b'$. Let $\Diamond 0 = 0$ and $\Diamond x = 1$ for all $x \neq 0$. This is a tetravalent modal algebra (actually, it generates this variety), hence a monadic De Morgan algebra, but not an involutive Stone algebra. In fact, $\Diamond (a \wedge b) = 0 \neq 1 = \Diamond a \wedge \Diamond b$. Having two fixpoints for the involution, it also fails to be a weak Łukasiewicz algebra, hence a Łukasiewicz algebra or a three-valued Łukasiewicz algebra.

(v) Consider the algebra $\mathbf{D}_2^2$ as a De Morgan algebra with universe $\{0, a, a', 1\}$, and let $\Diamond 0 = 0$, and $\Diamond x = 1$ for all $x \neq 0$. This algebra is a monadic De Morgan algebra which is not a tetravalent modal algebra. In fact, $\Diamond a \wedge a' = a' \neq 0 = a \wedge a'$.

(vi) Consider the ordinal sum $\mathbf{D}_2 \oplus \mathbf{B}_4 \oplus \mathbf{D}_2$ as a De Morgan algebra with universe $\{0, a, b, c, a', 1\}$, with $b = b'$ and $c = c'$, and let $\Diamond 0 = 0$, and $\Diamond x = 1$ for all $x \neq 0$. This algebra is an involutive Stone algebra which is not a weak Łukasiewicz algebra (or a Łukasiewicz algebra) since it has two fixpoints for the involution.

(vii) Consider the ordinal sum $\mathbf{D}_2^2 \oplus \mathbf{D}_2^2$ as a De Morgan algebra on $\{0, a, b, c, b', a', 1\}$, with $c = c'$, and let $\Diamond 0 = 0$, and $\Diamond x = 1$ for all $x \neq 0$. This is a weak Łukasiewicz algebra which is not an involutive Stone algebra, for $\Diamond (a \wedge b) = 0 \neq 1 = \Diamond a \wedge \Diamond b$. A fortiori, it fails to be a Łukasiewicz algebra.

(viii) Finally, consider the algebra $\mathbf{D}_4$ as a De Morgan algebra on $\{0, a, a', 1\}$, and let $\Diamond 0 = 0$, and $\Diamond x = 1$ for all $x \neq 0$. This is a Łukasiewicz algebra, hence both an involutive Stone algebra and a weak Łukasiewicz algebra. However, it fails to be a tetravalent modal algebra (hence a three-valued Łukasiewicz algebra), for $\Diamond a \wedge a' = a' \neq a = a \wedge a'$. ∎

Acknowledgements This work was supported by the research grants "Proprietà d'ordine nella semantica algebrica delle logiche non classiche", Regione Autonoma della Sardegna, L. R. 7/2007, n. 7, 2015, CUP: F72F16002920002; "Theory and applications of resource sensitive logics", PRIN 2017, Prot. 20173WKCM5, CUP: F74I19000720001; "Per un'estensione semantica della Logica Computazionale Quantistica- Impatto teorico e ricadute implementative", Regione Autonoma della Sardegna, (RAS: RASSR40341), L.R. 7/2007, 2017- FSC 2014-2020. The authors thank Davide Fazio for the insightful discussions on the topics of the present paper.

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Chapter 11
From Intuitionism to Many-Valued Logics Through Kripke Models

Saeed Salehi

Abstract Intuitionistic Propositional Logic is proved to be an infinitely many valued logic by Gödel (Kurt Gödel collected works (Volume I) Publications 1929–1936, Oxford University Press, pp 222–225, 1932), and it is proved by Jaśkowski (Actes du Congrés International de Philosophie Scientifique, VI. Philosophie des Mathématiques, Actualités Scientifiques et Industrielles 393:58–61, 1936) to be a countably many valued logic. In this paper, we provide alternative proofs for these theorems by using models of Kripke (J Symbol Logic 24(1):1–14, 1959). Gödel's proof gave rise to an intermediate propositional logic (between intuitionistic and classical), that is known nowadays as Gödel or the Gödel-Dummett Logic, and is studied by fuzzy logicians as well. We also provide some results on the inter-definability of propositional connectives in this logic.

Keywords Intuitionistic propositional logic · Many-Valued logics · Kripke models · Gödel-Dummett logic · Inter-definability of propositional connectives

11.1 Introduction and Preliminaries

Intuitionism grew out of some of the philosophical ideas of its founding father, Luitzen Egbertus Jan Brouwer (see e.g. Brouwer 1913); what is known nowadays as intuitionistic logic is a formalization given by his student Heyting (1930). Kripke models (originating from Kripke 1959) provided an interesting mathematical interpretation for this formalization. Let us review some preliminaries about these models:

Dedicated to Professor Mohammad Ardeshir with high appreciation and admiration.

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M. Mojtahedi et al. (eds.), *Mathematics, Logic, and their Philosophies*, Logic, Epistemology, and the Unity of Science 49,
https://doi.org/10.1007/978-3-030-53654-1_11

Definition 1 (*Kripke Frames*)
A *Kripke frame* is a partially ordered set; i.e., an ordered pair $\langle K, \succcurlyeq\rangle$ where $\succcurlyeq\, \subseteq K^2$ is a reflexive, transitive and anti-symmetric binary relation on K. ✧

Definition 2 (*Atoms, Formulas, Languages*)
Let $\mathtt{At}$ be the set of all the propositional atoms; atoms are usually denoted by letters $\mathfrak{p}$ or $\mathfrak{q}$. Let $\top$ denote the verum (truth) constant.
The language of propositional logics studied here is $\mathscr{L} = \{\neg, \wedge, \vee, \rightarrow, \top\}$.
For any $A \subseteq \mathtt{At}$ and $B \subseteq \mathscr{L}$, the set of all the formulas constructed from A by means of B is denoted by $\mathcal{L}(B, A)$.
Let $\mathtt{Fm}$ denote the set of all the formulas; i.e., $\mathcal{L}(\mathscr{L}, \mathtt{At})$. ✧

Definition 3 (*Kripke Models*)
A *Kripke model* is a triple $\mathcal{K} = \langle K, \succcurlyeq, \Vdash\rangle$, where $\langle K, \succcurlyeq\rangle$ is a Kripke frame equipped with a persistent binary (satisfaction) relation $\Vdash\, \subseteq K \times \mathtt{At}$; persistency (of the relation $\Vdash$ with respect to $\succcurlyeq$) means that for all $k, k' \in K$ and $\mathfrak{p} \in \mathtt{At}$, if $k' \succcurlyeq k \Vdash \mathfrak{p}$ then $k' \Vdash \mathfrak{p}$.

The satisfaction relation can be extended to all the (propositional) formulas, i.e., to $\Vdash\, \subseteq K \times \mathtt{Fm}$, as follows:

- $k \Vdash \top$.
- $k \Vdash (\varphi \wedge \psi) \iff k \Vdash \varphi$ and $k \Vdash \psi$.
- $k \Vdash (\varphi \vee \psi) \iff k \Vdash \varphi$ or $k \Vdash \psi$.
- $k \Vdash (\neg\varphi) \iff \forall k' \succcurlyeq k (k' \nVdash \varphi)$.
- $k \Vdash (\varphi \rightarrow \psi) \iff \forall k' \succcurlyeq k (k' \Vdash \varphi \Rightarrow k' \Vdash \psi)$. ✧

Remark 1 (*On Persistency and its Converse*)
It can be shown that the persistency conditions is inherited by the formulas; i.e., for any $k, k' \in K$ in any Kripke model $\mathcal{K} = \langle K, \succcurlyeq, \Vdash\rangle$ and for any formula φ, if $k' \succcurlyeq k \Vdash \varphi$ then $k' \Vdash \varphi$.
Obviously, the converse may not hold ($k' \Vdash \psi$ and $k' \succcurlyeq k$ do not necessarily imply that $k \Vdash \psi$); however, a partial converse holds for negated formulas:
if $k' \succcurlyeq k$ and $k' \Vdash \neg\varphi$, then $k \nVdash \varphi$. ✧

By the soundness and completeness of the intuitionistic propositional logic (IPL) with respect to finite Kripke models, the tautologies of IPL are the formulas (in $\mathtt{Fm}$) that are satisfied in all the elements of any finite Kripke model. A super-intuitionistic and sub-classical logic is the so-called Gödel-Dummett logic (see Dummett 1959), whose tautologies are the formulas that are satisfied in all the elements of all the connected finite Kripke models. A kind of Kripke model theoretic characterization for this logic is given in Safari and Salehi (2018).

Definition 4 (*Connectivity*)
A binary relation $R \subseteq K \times K$ is called *connected*, when for any $k, k', k'' \in K$, if $k' \succcurlyeq k$ and $k'' \succcurlyeq k$, then we have either $k' \succcurlyeq k''$ or $k'' \succcurlyeq k'$ (cf. Švejdar and Bendová 2000). ✧

The logic IPL is perhaps the most famous non-classical logic. A natural question (that according to Gödel 1932 was asked by his supervisor Hans Hahn) was whether IPL is a finitely many valued logic or not. Gödel (1932) showed that IPL is not finitely many valued. Jaśkowski (1936) showed that IPL is indeed a countably (infinite) many valued logic. In Sect. 11.2 we give alternative proofs for these theorems by using models of Kripke (1959). Gödel's proof gave birth to an intermediate logic, that today is called the Gödel-Dummett logic (GDL). Finally, in Sect. 11.3 we study the problem of inter-definability of propositional connectives in GDL and IPL.

11.2 ω-Many Values for Intuitionistic Propositional Logic

Let us begin with a formal definition of a many-valued logic. Throughout the paper, we are dealing with propositional logics only.

Definition 5 (*Many-Valued Logics*)
A *many-valued logic* is $\langle \mathscr{V}, \boldsymbol{\tau}, \backsim, \Lambda, \mathrm{V}, \Rrightarrow \rangle$, where $\mathscr{V}$ is a set of values with a *designated* element $\boldsymbol{\tau} \in \mathscr{V}$ (interpreted as the *truth*) and the functions $\backsim\colon \mathscr{V} \to \mathscr{V}$, $\Lambda\colon \mathscr{V}^2 \to \mathscr{V}$, $\mathrm{V}\colon \mathscr{V}^2 \to \mathscr{V}$, and $\Rrightarrow\colon \mathscr{V}^2 \to \mathscr{V}$ constitute a *truth table* on $\mathscr{V}$.
A *valuation function* is any mapping $\boldsymbol{\nu}\colon \mathrm{At} \to \mathscr{V}$, which can be extended to all the formulas, denoted also by $\boldsymbol{\nu}\colon \mathrm{Fm} \to \mathscr{V}$, as follows:

- $\boldsymbol{\nu}(\neg\varphi) = \backsim \boldsymbol{\nu}(\varphi)$.
- $\boldsymbol{\nu}(\varphi\wedge\psi) = \boldsymbol{\nu}(\varphi) \ \Lambda \ \boldsymbol{\nu}(\psi)$.
- $\boldsymbol{\nu}(\varphi\vee\psi) = \boldsymbol{\nu}(\varphi) \ \mathrm{V} \ \boldsymbol{\nu}(\psi)$.
- $\boldsymbol{\nu}(\varphi\rightarrow\psi) = \boldsymbol{\nu}(\varphi) \Rrightarrow \boldsymbol{\nu}(\psi)$.

A formula θ is called *tautology*, when it is mapped to the designated value under any valuation function; i.e., $\boldsymbol{\nu}(\theta) = \boldsymbol{\tau}$ for any valuation $\boldsymbol{\nu}$. ✧

Theorem 1 appears in Safari (2017) and Safari and Salehi (2019). In the following, the disjunction operation ($\vee$) is assumed to be commutative and associative.

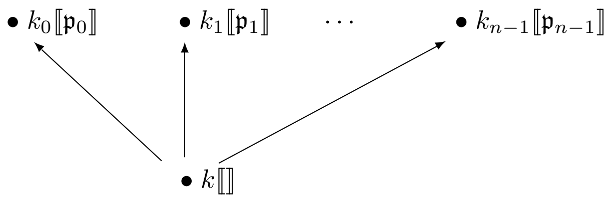
Lemma 1 (A Tautology in n-Valued Logics)
For any $n > 1$, the formula $\bigvee\!\!\!\bigvee_{i<j\leqslant n}(\mathfrak{p}_i \rightarrow \mathfrak{p}_j)$ is a tautology in any n-valued logic in which the formula $(\mathfrak{p} \rightarrow \mathfrak{p})\vee\mathfrak{q}$ is a tautology.

Proof In an n-valued logic, the $n+1$ atoms $\{\mathfrak{p}_0, \mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ can take n values. So, under a valuation function, there should exist some $i < j \leqslant n$ such that $\mathfrak{p}_i$ and $\mathfrak{p}_j$ take the same value, by the Pigeonhole Principle. Since $(\mathfrak{p} \rightarrow \mathfrak{p})\vee\mathfrak{q}$ is a tautology, then the formula $\bigvee\!\!\!\bigvee_{i<j\leqslant n}(\mathfrak{p}_i \rightarrow \mathfrak{p}_j)$ should be mapped to the designated value by all the valuation functions. ❑

The lemma implies that the formula $(A \rightarrow B) \vee (A \rightarrow C) \vee (B \rightarrow C)$ is a tautology in the classical propositional logic; this formula is not a tautology in the intuitionistic (or even Gödel-Dummett) propositional logic.

Theorem 1 (Gödel 1932: IPL Is Not Finitely Many Valued)
Intuitionistic propositional logic is not finitely many valued.

Proof By Lemma 1 it suffices to show that for any $n > 1$, $\bigvee\!\!\!\bigvee_{i<j\leqslant n}(\mathfrak{p}_i \to \mathfrak{p}_j)$ is not a tautology in IPL. Consider the Kripke model $\mathcal{K} = \langle K, \succcurlyeq, \Vdash\rangle$ with
$K = \{k, k_0, k_1, \ldots, k_{n-1}\}$,
$\succcurlyeq = \{(k_i, k) \mid i < n\} \cup \{(k_i, k_i) \mid i < n\} \cup \{(k, k)\}$, and
$\Vdash = \{(k_0, \mathfrak{p}_0), (k_1, \mathfrak{p}_1), \ldots, (k_{n-1}, \mathfrak{p}_{n-1})\}$.



For any $i < n$ we have $k_i \Vdash \mathfrak{p}_i$, and also $k_i \nVdash \mathfrak{p}_j$ for any $j > i$. So, $k_i \nVdash \mathfrak{p}_i \to \mathfrak{p}_j$ for any $i < j \leqslant n$; which implies that $k \nVdash \bigvee\!\!\!\bigvee_{i<j\leqslant n}(\mathfrak{p}_i \to \mathfrak{p}_j)$. ❑

The rest of this section is devoted to proving Jaśkowski's result (Theorem 2) that IPL is a countably infinite many valued logic.

Definition 6 (*Monotone Functions*)
For a Kripke frame $(K, \succcurlyeq)$, a function $f : K \to \{0, 1\}$ is called *monotone*, when for any $k, k' \in K$, if $k' \succcurlyeq k$, then $f(k') \geqslant f(k)$. We indicate the monotonicity of f by writing $f : (K, \succcurlyeq) \to \{0, 1\}$. ✧

Example 1 ($\mathfrak{f}^{\psi}_{\mathcal{K}}$)
For any Kripke model $\mathcal{K} = (K, \succcurlyeq, \Vdash)$ and any formula ψ, the function

$$\mathfrak{f}^{\psi}_{\mathcal{K}} : K \to \{0, 1\}, \qquad \mathfrak{f}^{\psi}_{\mathcal{K}}(k) = \begin{cases} 1 & \text{if } k \Vdash \psi \\ 0 & \text{if } k \nVdash \psi \end{cases}$$

is monotone. ✧

Definition 7 ($\backsim, \curlywedge, \curlyvee$ *and* $\Rrightarrow$)
For a Kripke frame $(K, \succcurlyeq)$ and monotone functions $f, g : (K, \succcurlyeq) \to \{0, 1\}$, let

$\backsim f : K \to \{0, 1\}$ be defined by $(\backsim f)(k) = \begin{cases} 1 & \text{if } \forall k' \succcurlyeq k (f(k') = 0) \\ 0 & \text{if } \exists k' \succcurlyeq k (f(k') = 1) \end{cases}$,
$f \curlywedge g : K \to \{0, 1\}$ be defined by $(f \curlywedge g)(k) = \min\{f(k), g(k)\}$,
$f \curlyvee g : K \to \{0, 1\}$ be defined by $(f \curlyvee g)(k) = \max\{f(k), g(k)\}$,
$f \Rrightarrow g : K \to \{0, 1\}$ be defined by

$$(f \Rrightarrow g)(k) = \begin{cases} 1 & \text{if } \forall k' \succcurlyeq k (f(k') = 1 \Rightarrow g(k') = 1) \\ 0 & \text{if } \exists k' \succcurlyeq k (f(k') = 1 \;\&\; g(k') = 0) \end{cases},$$

for all $k \in K$. ✧

Definition 8 (*Constant Functions*)
Let $\mathbf{1}_K : K \to \{0, 1\}$ be the constant 1 function, i.e., $\mathbf{1}_K(k) = 1$ for all $k \in K$; and let $\mathbf{0}_K : K \to \{0, 1\}$ be the constant 0 function: $\mathbf{0}_K(k) = 0$ for all $k \in K$. ✧

It is easy to see that the functions $\mathbf{1}_K$ and $\mathbf{0}_K$ obey the rules of the classical propositional logic with the operations $\backsim$, $\land$, $\lor$ and $\Rightarrow$. For example, $(\backsim\mathbf{1}_K) = \mathbf{0}_K$, $(\mathbf{1}_K \land \mathbf{1}_K) = \mathbf{1}_K$, $(\mathbf{0}_K \lor \mathbf{1}_K) = \mathbf{1}_K$ and $(\mathbf{1}_K \Rightarrow \mathbf{0}_K) = \mathbf{0}_K$. We omit the proof of the following straightforward observation.

Lemma 2 (Monotonicity of $\mathbf{1}_K, \mathbf{0}_K, \backsim f$, $f \land g$, $f \lor g$ and $f \Rightarrow g$)
For any Kripke frame $(K, \succcurlyeq)$, the constant functions $\mathbf{1}_K$ and $\mathbf{0}_K$ are monotone, and if $f, g\colon (K, \succcurlyeq) \to \{0, 1\}$ are monotone, then so are $\backsim f$, $f \land g$, $f \lor g$ and $f \Rightarrow g$. ❑

Finally, we can provide the following countably many values for IPL:

Definition 9 (*Countably Many Values for IPL*)
Enumerate all the finite Kripke frames as $(K_0, \succcurlyeq_0), (K_1, \succcurlyeq_1), (K_2, \succcurlyeq_2), \ldots$, where $K_n \subset \mathbb{N}$ for all $n \in \mathbb{N}$. Let
$\mathscr{V} = \{\langle f_0, f_1, f_2, \ldots\rangle \mid \forall n[f_n : (K_n, \succcurlyeq_n) \to \{0, 1\}]$ &
$\exists N \in \mathbb{N}[(\forall n \geqslant N f_n = \mathbf{1}_{K_n}) \text{ or } (\forall n \geqslant N f_n = \mathbf{0}_{K_n})]\}$.
In the other words, the set of values $\mathscr{V}$ consists of all the sequences $\langle f_0, f_1, f_2, \ldots\rangle$ such that for each n, f_n is a monotone function on $(K_n, \succcurlyeq_n)$, and the sequences are ultimately constant (from a step onward, f_n's are either all $\mathbf{1}_{K_n}$ or all $\mathbf{0}_{K_n}$).
Let $\boldsymbol{\tau} = \langle \mathbf{1}_{K_0}, \mathbf{1}_{K_1}, \mathbf{1}_{K_2}, \ldots\rangle$ be the designated element (for truth).
For $\mathfrak{f} = \langle f_0, f_1, f_2, \ldots\rangle \in \mathscr{V}$ and $\mathfrak{g} = \langle g_0, g_1, g_2, \ldots\rangle \in \mathscr{V}$, let (cf. Definition 7)
$\backsim\mathfrak{f} = \langle \backsim f_0, \backsim f_1, \backsim f_2, \ldots\rangle$,
$\mathfrak{f} \land \mathfrak{g} = \langle f_0 \land g_0, f_1 \land g_1, f_2 \land g_2, \ldots\rangle$,
$\mathfrak{f} \lor \mathfrak{g} = \langle f_0 \lor g_0, f_1 \lor g_1, f_2 \lor g_2, \ldots\rangle$, and
$\mathfrak{f} \Rightarrow \mathfrak{g} = \langle f_0 \Rightarrow g_0, f_1 \Rightarrow g_1, f_2 \Rightarrow g_2, \ldots\rangle$. ✧

It can be immediately seen that $\mathscr{V}$ is a countable set, and Lemma 2 implies that $\mathscr{V}$ is closed under the operations $\backsim$, $\land$, $\lor$ and $\Rightarrow$. Before proving the main theorem, we make a further definition and prove an auxiliary lemma.

Definition 10 ($\langle\!\langle\alpha\rangle\!\rangle_n$, $\Vdash_n^{\boldsymbol{\nu}}$ *and* $\boldsymbol{\nu}_m^{\Vdash}$)
For a sequence α, let $\langle\!\langle\alpha\rangle\!\rangle_n$ denote its n-th element (if any), for any $n \in \mathbb{N}$.
(1) Let a valuation $\boldsymbol{\nu}\colon \mathtt{At} \to \mathscr{V}$ be given. The satisfaction relation $\Vdash_n^{\boldsymbol{\nu}}$ is defined on any finite Kripke frame $(K_n, \succcurlyeq_n)$, with $K_n \subset \mathbb{N}$ (see Definition 9), by the following for any atom $\mathfrak{p} \in \mathtt{At}$ and any $k \in K_n$: $k \Vdash_n^{\boldsymbol{\nu}} \mathfrak{p} \iff \langle\!\langle\boldsymbol{\nu}(\mathfrak{p})\rangle\!\rangle_n(k) = 1$.
(2) Let a Kripke model $\mathcal{K} = (K_m, \succcurlyeq_m, \Vdash)$ on the Kripke frame $(K_m, \succcurlyeq_m)$ be given (see Definition 9). Define the valuation $\boldsymbol{\nu}_m^{\Vdash}$ by

$$\boldsymbol{\nu}_m^{\Vdash}(\mathfrak{p}) = \langle \mathbf{1}_{K_0}, \ldots, \mathbf{1}_{K_{m-1}}, \mathfrak{f}_{\mathcal{K}}^{\mathfrak{p}}, \mathbf{1}_{K_{m+1}}, \ldots\rangle$$

for any $\mathfrak{p} \in \mathtt{At}$, where $\mathfrak{f}_{\mathcal{K}}^{\mathfrak{p}}\colon K_m \to \{0, 1\}$ is the function that was defined in Example 1: $\mathfrak{f}_{\mathcal{K}}^{\mathfrak{p}}(k) = 1$ if $k \Vdash \mathfrak{p}$, and $\mathfrak{f}_{\mathcal{K}}^{\mathfrak{p}}(k) = 0$ if $k \nVdash \mathfrak{p}$, for any $k \in K_m$. ✧

It is clear that the relation $\Vdash_n^{\boldsymbol{\nu}} \subseteq K_n \times \mathtt{At}$ is persistent.

Lemma 3 (On $\Vdash_n^{\boldsymbol{\nu}}$ *and* $\boldsymbol{\nu}_m^{\Vdash}$)
(1) Let a valuation $\boldsymbol{\nu}\colon \mathtt{At} \to \mathscr{V}$ be given, and the satisfaction relation $\Vdash_n^{\boldsymbol{\nu}}$ be defined on $(K_n, \succcurlyeq_n)$ as in Definition 10. Then for any formula $\varphi \in \mathtt{Fm}$ and any $k \in K_n$, we have $k \Vdash_n^{\boldsymbol{\nu}} \varphi \iff \langle\!\langle \boldsymbol{\nu}(\varphi) \rangle\!\rangle_n(k) = 1$.
(2) Let a Kripke model $\mathcal{K} = (K_m, \succcurlyeq_m, \Vdash)$ be given on the frame $(K_m, \succcurlyeq_m)$, and the valuation $\boldsymbol{\nu}_m^{\Vdash}$ be defined as in Definition 10. Then for any formula $\varphi \in \mathtt{Fm}$ and any $k \in K_m$, we have $k \nVdash \varphi \iff \langle\!\langle \boldsymbol{\nu}_m^{\Vdash}(\varphi) \rangle\!\rangle_m(k) = 0$.

Proof Both assertions can be proved by induction on φ. They are clear for $\varphi = \top$ and hold for atomic $\varphi \in \mathtt{At}$ by Definition 10. The inductive cases follow immediately from Definitions 3, 5, 7, and 9. ❑

Theorem 2 (Jaśkowski 1936: IPL Is Countably Many Valued)
Intuitionistic propositional logic is countably infinite many valued.

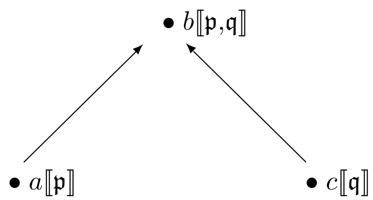
Proof We show that a formula $\varphi \in \mathtt{Fm}$ is satisfied in all the elements of all the finite Kripke models if and only if it is mapped to the designated element under all the valuation functions:
(1) If φ is satisfied in any element of any finite Kripke model, then for any valuation $\boldsymbol{\nu}$ by Lemma 3(1) we have $\langle\!\langle \boldsymbol{\nu}(\varphi) \rangle\!\rangle_n = \mathbf{1}_{K_n}$ for any $n \in \mathbb{N}$, so $\boldsymbol{\nu}(\varphi) = \boldsymbol{\tau}$.
(2) If φ is not satisfied in some element of some finite Kripke model, then for some $m \in \mathbb{N}$ there is a Kripke model $\mathcal{K} = (K_m, \succcurlyeq_m, \Vdash)$ such that $\Bbbk \nVdash \varphi$ for some $\Bbbk \in K_m$. So, by Lemma 3(2) we have $\langle\!\langle \boldsymbol{\nu}_m^{\Vdash}(\varphi) \rangle\!\rangle_m(\Bbbk) = 0$, thus $\boldsymbol{\nu}_m^{\Vdash}(\varphi) \neq \boldsymbol{\tau}$. ❑

11.3 Propositional Connectives Inside Gödel-Dummett Logic

In classical propositional logic (which is a two-valued logic), all the connectives can be defined by (the so-called complete set of connectives) $\{\neg, \wedge\}$, $\{\neg, \vee\}$ or $\{\neg, \rightarrow\}$ only. In this last section we will see that no propositional connective is definable from the others in IPL, and in GDL only the disjunction operation ($\vee$) can be defined by the conjunction ($\wedge$) and implication ($\rightarrow$) operations. Most of these facts are already known (they appear in e.g. Safari and Salehi 2019 and Švejdar and Bendová 2000). Theorem 3 is from Švejdar and Bendová (2000) with a slightly different proof; Theorem 4 is from Švejdar and Bendová (2000) with the same proof. All of our proofs are Kripke model theoretic, as usual.

Theorem 3 ($\wedge$ Is Not Definable From the Others in GDL)
In Gödel-Dummett Logic, the conjunction connective ($\wedge$) is not definable from the other propositional connectives.

Proof Consider the Kripke model $\mathcal{K} = \langle K, \succcurlyeq, \Vdash \rangle$ where $K = \{a, b, c\}$, $\succcurlyeq$ is the reflexive closure of $\{(a, b), (c, b)\}$, and $\Vdash = \{(a, \mathfrak{p}), (b, \mathfrak{p}), (b, \mathfrak{q}), (c, \mathfrak{q})\}$, for atoms $\mathfrak{p}, \mathfrak{q} \in \mathtt{At}$.



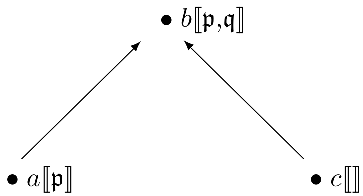
We show that for all formulas $\theta \in \mathcal{L}(\neg, \vee, \rightarrow, \top, \mathfrak{p}, \mathfrak{q})$ we have:

(∗) $b \Vdash \theta \Longrightarrow a \Vdash \theta$ or $c \Vdash \theta$.

This will prove the desired conclusion, since $b \Vdash \mathfrak{p} \wedge \mathfrak{q}$ but $a, c \nVdash \mathfrak{p} \wedge \mathfrak{q}$, and so $\mathfrak{p} \wedge \mathfrak{q}$ cannot belong to $\mathcal{L}(\neg, \vee, \rightarrow, \top, \mathfrak{p}, \mathfrak{q})$. We prove (∗) by induction on θ. The cases of $\theta = \top, \mathfrak{p}, \mathfrak{q}$ are trivial, and the induction step of $\neg\varphi$ follows from Remark 1, and the case of $\varphi \vee \psi$ is rather easy. So, only the non-trivial case of $\theta = \varphi \rightarrow \psi$ remains. Suppose that (∗) holds for φ and ψ, and assume (for the sake of a contradiction) that $b \Vdash \varphi \rightarrow \psi$ but $a, c \nVdash \varphi \rightarrow \psi$. So, $a \Vdash \varphi$ and $a \nVdash \psi$; and also $c \Vdash \varphi$ and $c \nVdash \psi$. Whence, by persistency, we should have also $b \Vdash \varphi$, thus $b \Vdash \psi$. So, by the induction hypothesis (∗ for $\theta = \psi$) we should have either $a \Vdash \psi$ or $c \Vdash \psi$; a contradiction. ❑

Theorem 4 (→ Is Not Definable From the Others in GDL)
In Gödel-Dummett Logic, the implication connective (→) *is not definable from the other propositional connectives.*

Proof For the Kripke model $\mathcal{K} = \langle K, \succcurlyeq, \Vdash \rangle$, where $K = \{a, b, c\}$, $\succcurlyeq$ is the reflexive closure of $\{(a, b), (c, b)\}$, and $\Vdash = \{(a, \mathfrak{p}), (b, \mathfrak{p}), (b, \mathfrak{q})\}$, for $\mathfrak{p}, \mathfrak{q} \in \mathtt{At}$,



we show that for all the formulas $\theta \in \mathcal{L}(\neg, \vee, \wedge, \top, \mathfrak{p}, \mathfrak{q})$, the following holds:

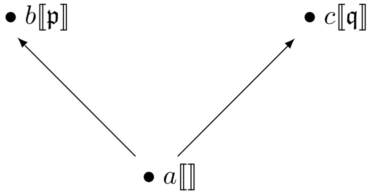
(∗) $b, c \Vdash \theta \Longrightarrow a \Vdash \theta$.

This completes the proof since $b, c \Vdash \mathfrak{p} \rightarrow \mathfrak{q}$ but $a \nVdash \mathfrak{p} \rightarrow \mathfrak{q}$ (by $a \Vdash \mathfrak{p}$, $a \nVdash \mathfrak{q}$); thus we have $(\mathfrak{p} \rightarrow \mathfrak{q}) \notin \mathcal{L}(\neg, \vee, \wedge, \top, \mathfrak{p}, \mathfrak{q})$. The proof of (∗) is by induction on θ; the only non-trivial cases to consider are $\theta = \varphi \vee \psi$ and $\theta = \varphi \wedge \psi$. Suppose that (∗) holds for φ and ψ; and that $b, c \Vdash \varphi \vee \psi$. Then we have either $c \Vdash \varphi$ or $c \Vdash \psi$; by the persistency, the former implies $b \Vdash \varphi$ and the latter $b \Vdash \psi$. So, in either case by the induction hypothesis we have $a \Vdash \varphi \vee \psi$. The case of $\theta = \varphi \wedge \psi$ is even simpler. ❑

The following has been known for a long time; see e.g. Dummett (1959).

Theorem 5 (∨ *Is* Definable From ∧, → in GDL)
In Gödel-Dummett Logic, the disjunction connective (∨) is *definable from some other propositional connectives.*

Proof It is rather easy to see that IPL $\Vdash (\mathfrak{p}\vee\mathfrak{q}) \longrightarrow [(\mathfrak{p}\to\mathfrak{q})\to\mathfrak{q}]\wedge[(\mathfrak{q}\to\mathfrak{p})\to\mathfrak{p}]$. Now, we show that GDL $\Vdash [(\mathfrak{p}\to\mathfrak{q})\to\mathfrak{q}]\wedge[(\mathfrak{q}\to\mathfrak{p})\to\mathfrak{p}] \longrightarrow (\mathfrak{p}\vee\mathfrak{q})$ holds. Take an arbitrary *connected* Kripke model $\mathcal{K} = \langle K, \succcurlyeq, \Vdash\rangle$, and suppose that for an arbitrary $a \in K$ we have $a \Vdash [(\mathfrak{p}\to\mathfrak{q})\to\mathfrak{q}]\wedge[(\mathfrak{q}\to\mathfrak{p})\to\mathfrak{p}]$. We show that $a \Vdash \mathfrak{p}\vee\mathfrak{q}$. Assume not; then $a \nVdash \mathfrak{p},\mathfrak{q}$. Therefore, $a \nVdash (\mathfrak{p}\to\mathfrak{q})$ and $a \nVdash (\mathfrak{q}\to\mathfrak{p})$, by $a \Vdash [(\mathfrak{p}\to\mathfrak{q})\to\mathfrak{q}]$ and $a \Vdash [(\mathfrak{q}\to\mathfrak{p})\to\mathfrak{p}]$, respectively. So, there are some $b, c \in K$ with $b, c \succcurlyeq a$ such that $b \Vdash \mathfrak{p}$, $b \nVdash \mathfrak{q}$, $c \Vdash \mathfrak{q}$, and $c \nVdash \mathfrak{p}$.



By the connectivity of $\succcurlyeq$, we should have either $b \succcurlyeq c$ or $c \succcurlyeq b$. Both cases lead to a contradiction, by the persistency condition. So, the following equivalence

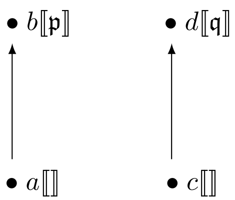
$$(\mathfrak{p}\vee\mathfrak{q}) \;\equiv\; [(\mathfrak{p}\to\mathfrak{q})\to\mathfrak{q}]\wedge[(\mathfrak{q}\to\mathfrak{p})\to\mathfrak{p}]$$

holds in GDL. ❑

The fact of the matter is that $(\mathfrak{p}\vee\mathfrak{q}) \equiv [(\mathfrak{p}\to\mathfrak{q})\to\mathfrak{q}]\wedge[(\mathfrak{q}\to\mathfrak{p})\to\mathfrak{p}]$ is the only non-trivial equivalence relation between the propositional connectives in GDL. The first half of the following theorem was proved in Safari and Salehi (2019).

Theorem 6 (In GDL $\vee$ Is Not Definable Without Both $\wedge, \to$)
In Gödel-Dummett Logic, disjunction ($\vee$) *is not definable from the other propositional connectives, unless both the conjunction and the implication connectives are present. In the other words, $\vee$ is definable* neither *from* $\{\neg, \to, \top\}$ nor *from* $\{\neg, \wedge, \top\}$.

Proof Take the Kripke model $\mathcal{K} = \langle K, \succcurlyeq, \Vdash\rangle$ with $K = \{a, b, c, d\}$, $\succcurlyeq\,=$ the reflexive closure of $\{(a, b), (c, d)\}$, and $\Vdash\, = \{(b, \mathfrak{p}), (d, \mathfrak{q})\}$, for $\mathfrak{p}, \mathfrak{q} \in \mathtt{At}$.



We show that for all $\theta \in \mathcal{L}(\neg, \to, \top, \mathfrak{p}, \mathfrak{q})$ we have
(∗) $b, d \Vdash \theta \Longrightarrow a \Vdash \theta$ or $c \Vdash \theta$.
Since $b, d \Vdash \mathfrak{p}\vee\mathfrak{q}$ but $a, c \nVdash \mathfrak{p}\vee\mathfrak{q}$, then it follows that $\mathfrak{p}\vee\mathfrak{q} \notin \mathcal{L}(\neg, \to, \top, \mathfrak{p}, \mathfrak{q})$.
Now, (∗) can be proved by induction on θ; the only non-trivial case is $\theta = \varphi\to\psi$. If (∗) holds for φ and ψ, then if $b, d \Vdash \varphi\to\psi$ but $a \nVdash \varphi\to\psi$ and $c \nVdash \varphi\to\psi$, then we

should have $a \Vdash \varphi$ and $a \nVdash \psi$, and also $c \Vdash \varphi$ and $c \nVdash \psi$. So, by persistency, $b \Vdash \varphi$ and $d \Vdash \varphi$; thus $b \Vdash \psi$ and $d \Vdash \psi$. So, by the induction hypothesis ($*$ for $\theta = \psi$) we should have either $a \Vdash \psi$ or $c \Vdash \psi$; a contradiction.
Now, for proving $\mathfrak{p} \vee \mathfrak{q} \notin \mathcal{L}(\neg, \wedge, \top, \mathfrak{p}, \mathfrak{q})$, we show that for all the formulas θ in $\mathcal{L}(\neg, \wedge, \top, \mathfrak{p}, \mathfrak{q})$ we have

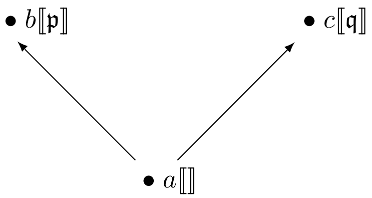
(‡) $\quad b, d \Vdash \theta \Longrightarrow a, c \Vdash \theta$.

Trivially, (‡) holds for $\theta = \top, \mathfrak{p}, \mathfrak{q}$; so by Remark 1 it only suffices to show that (‡) holds for $\theta = \varphi \wedge \psi$, when it holds for φ and ψ. Now, if $b, d \Vdash \varphi \wedge \psi$ then $b, d \Vdash \varphi$ and $b, d \Vdash \psi$; so the induction hypothesis (‡ for $\theta = \varphi, \psi$) implies that $a, c \Vdash \varphi$ and $a, c \Vdash \psi$, therefore $a, c \Vdash \varphi \wedge \psi$. ❑

We end the paper with a Kripke model theoretic proof of a known fact.

Proposition 1 (No Connective Is Definable From the Others in IPL)
In IPL, no propositional connective is definable from the others.

Proof By Theorems 3 and 4, $\wedge$ and $\rightarrow$ are not definable from the other connectives even in GDL. The statement $\neg\mathfrak{p} \notin \mathcal{L}(\wedge, \vee, \rightarrow, \top, \mathfrak{p})$ can be easily verified by noting that all the operations on the righthand side are positive. So, it only remains to show that we have $\mathfrak{p} \vee \mathfrak{q} \notin \mathcal{L}(\neg, \wedge, \rightarrow, \top, \mathfrak{p}, \mathfrak{q})$ in IPL (cf. Theorem 5). Consider the Kripke model $\mathcal{K} = \langle K, \succcurlyeq, \Vdash \rangle$ with $K = \{a, b, c\}$, $\succcurlyeq =$ the reflexive closure of $\{(a, b), (a, c)\}$, and $\Vdash = \{(b, \mathfrak{p}), (c, \mathfrak{q})\}$, for $\mathfrak{p}, \mathfrak{q} \in \mathtt{At}$.



We show that for all formulas $\theta \in \mathcal{L}(\neg, \wedge, \rightarrow, \top, \mathfrak{p}, \mathfrak{q})$ we have:

($*$) $\quad b, c \Vdash \theta \Longrightarrow a \Vdash \theta$.

This will prove the theorem, since $b, c \Vdash \mathfrak{p} \vee \mathfrak{q}$ but $a \nVdash \mathfrak{p} \vee \mathfrak{q}$, and so $\mathfrak{p} \vee \mathfrak{q}$ is not in $\mathcal{L}(\neg, \wedge, \rightarrow, \top, \mathfrak{p}, \mathfrak{q})$ in IPL. Indeed, ($*$) can be proved by induction on θ; for which we consider the case of $\theta = \varphi \rightarrow \psi$ only. So, suppose that ($*$) holds for φ and ψ and that $b, c \Vdash \varphi \rightarrow \psi$ but $a \nVdash \varphi \rightarrow \psi$. Then we should have $a \Vdash \varphi$ and $a \nVdash \psi$; but by persistency we should have that $b, c \Vdash \varphi$, and so $b, c \Vdash \psi$ holds. Now, the induction hypothesis ($*$ for $\theta = \psi$) implies that $a \Vdash \psi$, a contradiction. ❑

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Chapter 12
Non-conditional Contracting Connectives

Luis Estrada-González and Elisángela Ramírez-Cámara

Abstract It has been claimed that contracting connectives are conditionals. Our modest aim here is to show that the conditional-like features of a contracting connective depend on the defining features of the conditional in a particular logic, yes, but they also depend on the underlying notion of logical consequence and the structure of the collection of truth values. More concretely, we will show that under P-consequence and suitable satisfiability conditions for the conditional, conjunctions are contracting connectives for some logics without thereby being conditional-ish.

Keywords Contracting connective · Detachable connective · Conditional · Conjunction · P-consequence

[1]We do not intend to endorse (Detachment) as a good definition, much less the *right* one, of a detachable connective. If a close connection between conditionals and detachable connectives was intended, the definition is too broad, as conjunction satisfies (Detachment) in most logics. The problem is not about specific connectives, though, but rather that as it stands, the definition does not distinguish between 'self-detachable' connectives, like conjunction, for which the other premise plays no role, and other connectives for which the additional premise is essential. Proof-theoretically, it might be demanded that both A and $A \rhd B$ are effectively used to prove (in L) B or, model-theoretically, that $A \rhd B$ alone has none of its proper sub-formulas as logical consequences (in L). These changes would affect some details of the discussion below, for sure—for instance, whether conditionals are necessarily detachable, and if not, whether what is needed in the definition of a contracting connective is a detachable connective or a conditional suffices—, but we stick to the definition in the literature and leave the discussion of a better definition for another occasion.

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M. Mojtahedi et al. (eds.), *Mathematics, Logic, and their Philosophies*,
Logic, Epistemology, and the Unity of Science 49,
https://doi.org/10.1007/978-3-030-53654-1_12

12.1 Introduction

Following Beall (2015b), we call a binary connective $\rhd$ in a logic L *detachable (in L)* if and only if it satisfies the following condition[1]:

(Detachment)

$$A, A \rhd B \Vdash_L B$$

A detachable connective typically serves to define a connective $\bowtie$ with the following properties:

$A \rhd B, B \rhd A \Vdash_L A \bowtie B$
$A \rhd B \nVdash_L A \bowtie B$
$B \rhd A \nVdash_L A \bowtie B$
$A \bowtie B \Vdash_L A \rhd B$
$A \bowtie B \Vdash_L B \rhd A$

Such a connective is used to formulate naïve principles of the form $F \bowtie \copyright(\ldots S(\bar{F})\ldots)$, where $S(x)$ is a semantic predicate, '$\bar{x}$' is a suitable name of x (e.g. its number in a Gödelization) and $\copyright$ an n-ary connective.

Now, let $\rhd$ be a detachable connective in L. A binary connective $\odot$ in a logic L is called *contracting (in L)* if and only if the following hold:

(C1) $A \rhd B \Vdash_L A \odot B$
(C2) $A, A \odot B \Vdash_L B$
(C3) $A \odot (A \odot B) \Vdash_L A \odot B$

Contracting connectives are notorious for trivializing theories that include naïve principles quite easily (self-referential vocabulary is omitted for simplicity):

1. $A \bowtie (A \odot \bot)$	[Naïve principle]
2. $A \rhd (A \odot \bot)$	[1, definition of '$\bowtie$']
3. $(A \odot \bot) \rhd A$	[1, definition of '$\bowtie$']
4. $(A \odot (A \odot \bot))$	[2, C1]
5. $A \odot \bot$	[4, C3]
6. $(A \odot \bot) \odot A$	[3, C1]
7. A	[5, 6, C2]
8. $\bot$	[5, 7, C2]

It has been claimed that contracting connectives are conditionals. Restall (1993) says: "An operator satisfying the three conditions [(C1)–(C3)] is said to be a *contracting implication*." This is repeated by Rogerson and Butchart (2002). Beall and Murzi say that "any such $\odot$ exhibiting C1–C3 is near enough to being *conditional-ish*" Beall and Murzi (2013).

Our modest aim in this short note is to show that the conditional-like features of a contracting connective depend on the defining features of the conditional in a particular L, yes, but they also depend on the underlying notion of logical consequence and the structure of the collection of truth values. More concretely, we will show that under P-consequence, conjunctions are contracting connectives for some logics without thereby being conditional-ish.

The plan for the paper is as follows. In Sect. 12.2, we set some preliminaries for the discussion. Then, Sects. 12.3–12.6 cover a couple of cases in which, with the appropriate modifications, conjunctions are also contracting connectives in the examined logics. Finally, Sect. 12.7 rounds up the discussion with some concluding remarks.

12.2 Setting the Stage

For our purposes, the only detachable connectives we are going to consider are among those that have been proposed, for one reason or another, as suitable conditionals. What makes them "suitable" will be discussed later. For simplicity, we are going to use '$\rightarrow$' to designate any generic conditional, and when in presence of conditionals that need to be distinguished or that should be somehow highlighted, we will use subscripts. Furthermore, we will also use the same collection of truth values in every case, namely $V = \{0, *, 1\}$ with the ordering $0 < * < 1$. Moreover, it will always be the case that $1 \in D^+$ and $0 \in D^-$, where 'D^+' and 'D^-' stand for a collection of designated and antidesignated values, respectively, certain subsets of V that, as we will see, are typically understood as generalizations of truth and falsity, and are thus used to define logical consequence in many-valued settings.

Additionally, we will adopt a very common satisfiability condition for the conjunction, namely,
$\sigma(A \wedge B) = 1$ if and only if $\sigma(A) = \sigma(B) = 1$;
$\sigma(A \wedge B) = 0$ if and only if either $\sigma(A) = 0$ or $\sigma(B) = 0$;
$\sigma(A \wedge B) = *$ otherwise.
throughout.[2]

Finally, let us recall the Tarskian notion of logical consequence:

(T-consequence) $\Gamma \Vdash A$ if and only if $\sigma(A) \in D^+$ whenever for every B such that $B \in \Gamma, \sigma(B) \in D^+$

This notion underlies most well-known logics. For classical and some many-valued logics in which 1 is the only designated value, one can replace "$= 1$" for "$\in D^+$" to easily obtain the even more familiar notion that says that $\Gamma \Vdash A$ if and only if, whenever the premisses are true, the conclusion is also true.[3] In any case, (T-consequence) is meant to encode our understanding of logical consequence as *necessary forward*

[2] Although very widespread, this satisfiability condition for conjunction does not encompass all conjunctions. For example, Bochvar's (internal) conjunction does not fall into its scope.

[3] This notion can also be made compatible with logics that take more than one designated value. However, this requires explanations that are unnecessary and even distracting for our purposes. The only point we want to get across is that (T-consequence) is a very natural generalization of our usual definition of validity.

truth preservation. That (T-consequence) is *the* (correct/only possible) way of encoding a legitimate notion of logical consequence is seldom questioned, notwithstanding that this stems from very specific constraints placed on those relations that are to be counted as logical consequence relations. We will discuss some of these constraints in more detail in Sect. 12.3, below.

By the same token, taking classical logic as starting point, the truth-functional account of conditionality arises quite naturally. One can think of the truth-functional account of conditionality as a sentential forward truth preservation. Because the unquestioned adoption of classical logic comes with its own set of constraints, our best approximation at encoding this idea is the truth table for material implication. This is not to say that we are settling for less, because, as many have already noted and defended, the truth-functional account of conditionality does get it right in many important cases. As happens with validity, what we take issue with is the lack of recognition that this is the best approximation to conditionality, given the features of the underlying logical framework.

With all those elements in place, one can easily see why conditions (C2) and (C3), but not (C1), are valid for conjunction in classical and many other logics. In those logics we have in mind, the elements at play are classical enough, so $A \to B \Vdash_L A \wedge B$ fails to preserve designatedness, as there is a valuation such that $\sigma(A \to B) \in D^+$ when $\sigma(A) \in D^-$, and consequently, $\sigma(A \wedge B) \in D^-$. Even without fully specifying what those elements are, it becomes clear that the usual satisfiability condition for the conditional might be responsible for the invalidity of (C1).

Thus, a straightforward way of verifying whether conjunctions can be contracting connectives consists in replacing the detachable (conditional) connective in (C1) for another (conditional-like) one. Of course, if the use of the same connective on both sides of the turnstile in (C1) is allowed, then, given that conjunctions technically are detachable connectives, that they are contracting connectives becomes quite an uninteresting matter. However, as mentioned above, a connection between conditionality and Detachment might be intended. Because of this, we find that it might be reasonable to restrict those connectives on (C1)'s left side of the turnstile to detachable conditional(-like) connectives while leaving the right side open to the placement of conjunctions.

12.3 Blamey's Transplication

To test our hypothesis that the satisfiability conditions for the conditional is what does most of the work to keep conjunctions from being contracting connectives, we will substitute the left-hand connective in (C1) for Blamey's *transplication* (Blamey 1986). The connective is defined through the following satisfiability conditions:
$\sigma(A \to_B B) = 1$ if and only if $\sigma(A) = \sigma(B) = 1$;
$\sigma(A \to_B B) = 0$ if and only if $\sigma(A) = 1$ and $\sigma(B) = 0$;
$\sigma(A \to_B B) = *$ otherwise.

Transplication is detachable, so it can be used to define contracting connectives. Moreover, it is not difficult to verify that transplication itself is a contracting connective. We include the truth tables for more visibility:

$A \to_B B$	$\Vdash$	$A \to_B B$	A	$A \to_B B$	$\Vdash$	B	$A \to_B (A \to_B B)$	$\Vdash$	$A \to_B B$
1		1	1	1		1	1		1
$*$		$*$	1	$*$		$*$	$*$		$*$
0		0	1	0		0	0		0
$*$		$*$	$*$	$*$		1	$*$		$*$
$*$		$*$	$*$	$*$		$*$	$*$		$*$
$*$		$*$	$*$	$*$		0	$*$		$*$
$*$		$*$	0	$*$		1	$*$		$*$
$*$		$*$	0	$*$		$*$	$*$		$*$
$*$		$*$	0	$*$		0	$*$		$*$

The interesting fact is that in a logic L that includes conjunction and transplication and is such that its logical consequence is Tarskian over a distribution of truth values $D^+ = \{1\}$ and $D^- = \{*, 0\}$, $\wedge$ is a contracting connective. In particular, (C1) is valid because there is no σ such that $\sigma(A \to B) \in D^+$ when $\sigma(A) \notin D^+$. The corresponding truth tables provide more visibility:

A	B	$A \to_B B$	$\Vdash$	$A \wedge B$	A	$A \wedge B$	$\Vdash$	B	$A \wedge (A \wedge B)$	$\Vdash$	$A \wedge B$
1	1	1		1	1	1		1	1		1
1	$*$	$*$		$*$	1	$*$		$*$	$*$		$*$
1	0	0		0	1	0		0	0		0
$*$	1	$*$		$*$	$*$	$*$		1	$*$		1
$*$	$*$	$*$		$*$	$*$	$*$		$*$	$*$		$*$
$*$	0	$*$		0	$*$	0		0	0		0
0	1	$*$		0	0	0		1	0		1
0	$*$	$*$		0	0	0		$*$	0		$*$
0	0	$*$		0	0	0		0	0		0

An immediate concern is that conjunction, despite the appearances, is now a conditional, or that transplication is closer to a conjunction than a conditional. This worry is unfounded, though. Maybe it would suffice to say that a conditional $\to$ is not a conjunction (nor vice versa) if there is a σ such that $\sigma(A \to B) \neq \sigma(A \wedge B)$, but we think that a more conceptually principled distinction can be offered, but in order to do so let us present the objection in a rather strong way.

Given T-consequence and the satisfiability conditions for conjunction, $A \to_B B \dashv\Vdash A \wedge B$, and this seems evidence enough to hold that transplication is a conjunction and not a conditional.[4] Moreover, both $(A \to_B B) \Vdash A$ and $(A \to_B B) \Vdash B$, on the one hand, and $(A \wedge B) \Vdash A$ and $(A \wedge B) \Vdash B$ hold; but $(A \to_B B) \to_B A$ and $(A \to_B B) \to_B B$, and $(A \wedge B) \to_B A$ and $(A \wedge B) \to_B B$ do not hold. This strongly suggests that transplication is not a conditional at all, and the alleged failure of Simplification is simply due to the fact that the transplication arrow is just another conjunction.

[4] A quick note on notation: so far, we have been using $\Vdash$ as a sort of generic turnstile, meant to be read from left to right, as usual. We will be using $A \dashv\Vdash B$ as shorthand for '$A \Vdash B$ and $B \Vdash A$'.

In fact, Égré et al. (forthcoming) have shown that transplication plus (T-consequence) will fail at least one of the following:

- Detachment, $A, A \to B \Vdash B$
- Self-identity, $\Vdash A \to A$
- Non-symmetry $(A \to B \nVdash B \to A)$ or Non-entailment of conjunction $(A \to B \nVdash A \wedge B)$.

This seems devastating enough for the conditional nature of transplication. Nonetheless, they have also shown that transplication plus *(TT-consequence)*, that is

(TT-consequence) $\Gamma \Vdash A$ if and only if $0 \notin \sigma(A)$ whenever for every B such that $B \in \Gamma$, $0 \notin \sigma(B)$

satisfies Self-identity, Non-symmetry and Non-entailment of conjunction, although it still fails Detachment. Let us grant for the sake of the argument that Detachment is so central to conditionality that if a connective satisfies a bunch of other conditional-ish properties but fails Detachment, it is not really a conditional.

This result for transplication resembles the situation of the conditional in González-Asenjo's/Priest's **LP**. Beall has stressed several times (see for example Beall 2011, 2015a) that even if Detachment is invalid for the conditional in **LP**, it is *default valid* in the sense that $A, A \to B \Vdash B \vee (A \wedge \sim A)$ holds, that is, either Detachment holds or the antecedent is a formula with the value $*$—understood in that context as "both true and false" because it is taken as designated—which arguably is not the case in most situations. The second disjunct internalizes in the conclusion, so to speak, the structure of truth values into the object language. In the case of transplication, the underlying structure of truth values is such that $*$ is understood as "neither true nor false", because it is not designated, and one actually has that $A, A \to B \Vdash B \vee \sim(A \vee \sim A)$ holds, that is, either Detachment holds or the antecedent has the value $*$ (again, understood as neither true nor false).

Thus, the moral is that if one wants to judge a connective and tell it apart from others using not only its truth and falsity conditions, but also appealing to some pre-theoretical properties that the connective is supposed to meet, the underlying notion of logical consequence plays a pivotal role.[5]

So, in order to tell a conditional apart from a conjunction, one could show that each satisfies enough of the traditional properties associated to them, but the other fails them while satisfying its own set of traditional properties. This is exactly what happens with transplication and conjunction here. Although the view is not uncontentious, what one in general expects from a conditional connective is that it be detach-

[5] And this is a recurrent lesson in many semantic projects. For one of its more recent appearances in proof-theoretic semantics, see Dicher and Paoli (2021).

able, contraposable,[6] non-symmetric[7] and that it never happens that $\sigma(A \rightarrow A) = 0$, while a conjunction is typically expected to fail these conditions, with the only exception being Detachment as defined above. The truth tables below show the invalidity of Non-symmetry and Contraposition for transplication. Moreover, the validity of Detachment was shown above and it is very easy to verify that $A \rightarrow_B A$ is never 0:

$A \rightarrow_B B$	$\dashv\vdash$	$B \rightarrow_B A$	$A \rightarrow_B B$	$\dashv\vdash$	$\sim B \rightarrow_B \sim A$
1		$*$	1		$*$
$*$		$*$	$*$		0
0		0	0		0
$*$		$*$	$*$		$*$
$*$		$*$	$*$		$*$
$*$		$*$	$*$		$*$
$*$		$*$	$*$		$*$
$*$		$*$	$*$		1
$*$		1	$*$		1

The connective $\wedge$ is not contraposable but is symmetric, as is shown with the tables below. Furthermore, it is clear that $\sigma(A \wedge A) = 0$ when $\sigma(A) = 0$.

$A \wedge B$	$\dashv\vdash$	$B \wedge A$	$A \wedge B$	$\dashv\vdash$	$\sim B \wedge \sim A$
1		1	1		0
–		–	–		0
0		0	0		0
–		–	–		–
–		–	–		–
0		0	0		0
0		0	0		1
0		0	0		–
0		0	0		0

Thus, this example shows that the contracting character of a connective depends not only on its own satisfiability conditions, but also on those of the detachable connective used in condition (C1) and the underlying notion of logical consequence. In the following sections, we will consider some other cases where conjunction contracts to show that there are more elements, such as the underlying notion of logical consequence and the structure of truth values, which determine whether a connective is contracting or not.

[6] A connective k is *contraposable (in L)* if and only if, $A\ k\ B \dashv_L\vdash \sim B\ k\ \sim A$, where $\sim$ is a generic negation. In order to evaluate contraposition, we make use of the usual generalized satisfiability condition for negation: $\sigma(\sim A) = 1$ if and only if $\sigma(A) = 0$, $\sigma(\sim A) = 0$ if and only if $\sigma(A) = 1$, and $\sigma(\sim A) = *$ otherwise. Our choice reflects nothing beyond the decision to stick with basic, not too deviant, many-valued logical vocabulary. The inclusion of other negation connectives would certainly make for interesting discussion; however, we are also aware that matters might already be complicated enough as they stand. Consequently, we think that the introduction of other negation connectives deserves its own treatment elsewhere.

[7] A connective c is *non-symmetric (in L)* if and only if, either $A\ c\ B \nVdash_L B\ c\ A$ or $B\ c\ A \nVdash_L A\ c\ B$.

12.4 The OCO Conditional and P-Logical Consequence

Let us now consider a different logic, one in which $D^+ = \{1, *\}$ and $D^- = \{0\}$, the underlying notion of consequence is Tarskian and the conditional in this logic is defined as follows:

$\sigma(A \rightarrow_{OCO} B) = *$ if and only if $\sigma(A) = 0$;

$\sigma(A \rightarrow_{OCO} B) = \sigma(B)$ otherwise.[8]

It is not difficult to verify that the OCO conditional is itself a contracting connective under the assumptions just given. The truth tables allow a quick verification of the claim:

$A \rightarrow_{OCO} B$	$\Vdash$	$A \rightarrow_{OCO} B$	A	$A \rightarrow_{OCO} B$	$\Vdash$	B	$A \rightarrow_{OCO} (A \rightarrow_{OCO} B)$	$\Vdash$	$A \rightarrow_{OCO} B$
1		1	1	1		1	1		1
$*$		$*$	1	$*$		$*$	$*$		$*$
0		0	1	0		0	0		0
1		1	$*$	1		1	1		1
$*$		$*$	$*$	$*$		$*$	$*$		$*$
0		0	$*$	0		0	0		0
$*$		$*$	0	$*$		1	$*$		$*$
$*$		$*$	0	$*$		$*$	$*$		$*$
$*$		$*$	0	$*$		0	$*$		$*$

Because there is a σ such that $\sigma(A \rightarrow_{OCO} B) \in D^+$ when $\sigma(A) \in D^-$ (for the first clause defining the conditional), (C1) fails for conjunction. Considering a different distribution of truth values—like $D^+ = \{1\}$ and $D^- = \{*, 0\}$, or even $D^+ = \{1\}$ and $D^- = \{0\}$, with $*$ as neither designated nor antidesignated—, will not do the job by itself. The assumption of (T-consequence) ensures that for each alternative distribution considered there would still be a σ such that $\sigma(A \rightarrow_{OCO} B) \in D^+$ and $\sigma(A \wedge B) \notin D^+$, namely $\sigma(A) = *$ and $\sigma(B) = 1$. This is not to say that we will not need an alternative distribution *at all*. What we do mean, is that in addition to the adoption of the non-exhaustive distribution of values we will also need to accommodate an alternative notion of logical consequence (and one of the reasons for choosing the relevant value distribution is because it lets us do just that).

Notice that the latter partition is not bivalent, as it is not exhaustive with respect to the total collection of values given. This sets the logic we are proposing apart from most classical and many-valued logics, as those typically require that the partition on V be both collectively exhaustive and mutually exclusive. This means that here, unlike what happens in those cases, a value's not being designated does not entail its being antidesignated, just like its not being antidesignated does not always entail its being designated.[9]

This fact makes an important difference in the definition of logical consequence. Just like the equivalences between being designated and being not antidesignated, and vice versa, break down, another rupture of this sort happens when defining

[8] This conditional was first presented, with different primitives though, by Olkhovikov (2001) and then independently by Cantwell (2008) and Omori (2016).

[9] For more on this discussion, see Wansing and Shramko (2008).

logical consequence. This is because demanding that (T-consequence) be satisfied stops being equivalent to demanding that the following be satisfied:

(P-consequence) $\Gamma \Vdash A$ if and only if $\sigma(A) \notin D^-$ whenever for every B such that $B \in \Gamma, \sigma(B) \in D^+$

The most salient feature of (P-consequence) is that it clearly states something weaker than (T-consequence). This feature, some might argue, may bring about undesired consequences that (T-consequence) lacks. For instance, we are aware that construing the intermediate value as neither designated nor antidesignated leads us to another of P-consequence's most striking features. Suppose that $\sigma(p) = 1$, $\sigma(q) = *$ and $\sigma(r) = 0$. Thus, even if q were a P-logical consequence of p and r were a P-logical consequence of q, r would not be a P-logical consequence of p, because P-logical consequence requires that if premises are designated, conclusions must be not antidesignated, which is not the case in this example: P-logical consequence is not transitive.

Is the non-transitivity of P-consequence enough to argue against its standing as a logical consequence relation? We think it is not. But instead of attempting to show that there are good reasons to disregard (P-consequence) in favor of (T-consequence), one could go back to the origin of the breakdown and argue that allowing the partition on V to fail collective exhaustiveness is somehow illicit. We find this line of argument uncompelling, though, as there seems to be motivation for our view.

Intermediate truth values are often interpreted as gaps, and are meant to be something that is different in kind from truth and falsity. More precisely, gaps can be read as both a lack of truth that does not imply falsity and a lack of falsity that does not imply truth. In those cases, leaving values intended to be understood as gappy outside the sets of designated and antidesignated values should provide a better interpretation than forcing them to belong to any of the collections that readily lend themselves to be interpreted as generalizations of truth and falsity. Even if one is unwilling to see designatedness and antidesignatedness like this, we find it hard to defend the view that placing an intermediate value in any of the collection does not amount to at least considering that it is of the same kind as either truth or falsity.

Thus, the breakdown between (T-consequence) and (P-consequence) caused by the newfound independence of designatedness from antidesignatedness (and vice versa) merely shows that "forward truth preservation"—as embodied by (T-consequence) in a classical enough setting—can be broken into two *prima facie* equally important, yet non-equivalent components: the avoidance of loss of truth (T-consequence), and the avoidance of falsity introduction (P-consequence), that are nonetheless equivalent when further conditions on the structure of truth values, like collective exhaustiveness of the partitions, are imposed. Therefore, at least technically, logics with P-logical consequence as a basis are as legitimate *qua* logics as non-classical logics are.

Of course, more elaborate motivations could be given along the lines of non-monotonic logics. For example, P-consequence would allow to logically "jump" to conclusions less certain than the premises. Something in this spirit is already found in the works of Cobreros, Egré, Ripley and van Rooij. In Cobreros et al. (2012) they tie vagueness to non-transitivity, and show how a consequence relation that allows the step from strict to tolerant truth allows for a very classical framework that provides a semantics for vagueness. Then, (Cobreros et al. 2013) builds on the idea of adopting a non-transitive consequence relation in order to accommodate a transparent truth predicate within classical logic while avoiding many of the semantic paradoxes.[10]

Returning to our discussion, assuming P-logical consequence together with the distribution $D^+ = \{1\}$ and $D^- = \{0\}$, with $*$ as neither designated nor antidesignated, and the OCO conditional standing as the detachable connective in the left hand side of (C1) results in conjunction being a contracting connective. The problematic interpretation above, $\sigma(A) = *$ and $\sigma(B) = 1$, is not a counterexample to the P-logical validity of (C1), because even if $\sigma\ (A \rightarrow_{OCO} B) = 1$ and $\sigma(A \wedge B) = *$, this is not a case in which premise is designated and the conclusion is antidesignated.

A	B	$A \rightarrow_{OCO} B$	$\Vdash^P$	$A \wedge B$	A	$A \wedge B$	$\Vdash^P$	B	$A \wedge (A \wedge B)$	$\Vdash^P$	$A \wedge B$
1	1	1		1	1	1		1	1		1
1	$*$	$*$		$*$	1	$*$		$*$	$*$		$*$
1	0	0		0	1	0		0	0		0
$*$	1	1		$*$	$*$	$*$		1	$*$		1
$*$	$*$	$*$		$*$	$*$	$*$		$*$	$*$		$*$
$*$	0	0		0	$*$	0		0	0		0
0	1	$*$		0	0	0		1	0		0
0	$*$	$*$		0	0	0		$*$	0		0
0	0	$*$		0	0	0		0	0		0

Again, the worry that conjunction has become conditional-ish, or that the conditional has turned into a conjunction, might surface. And again, this is an ill-founded worry. The following truth tables show that, although this conditional is symmetric, it remains contraposable (it was already shown that it is detachable and it is very easy to verify that there is no σ such that $\sigma(A \rightarrow A) = 0$):

$A \rightarrow_{OCO} B$	$\dashv^P\vdash$	$B \rightarrow_{OCO} A$	$A \rightarrow_{OCO} B$	$\dashv^P\vdash$	$\sim B \rightarrow_{OCO} \sim A$
1		1	1		$*$
$*$		1	$*$		0
0		$*$	0		0
1		$*$	1		$*$
$*$		$*$	$*$		$*$
0		$*$	0		$*$
$*$		0	$*$		$*$
$*$		0	$*$		1
$*$		$*$	$*$		1

[10]For an even more elaborate defense of the logicality of P-logical consequence and other non-Tarskian notions of logical consequence, see Estrada-González (2015).

On the other hand, conjunction is symmetric but not contraposable, as expected (and, as in every case, there is a σ such that $\sigma(A \wedge A) = 0$):

$A \wedge B$	$\dashv^P\vdash$	$B \wedge A$	$A \wedge B$	$\dashv^P\vdash$	$\sim B \wedge \sim A$
1		1	1		0
$*$		$*$	$*$		0
0		0	0		0
$*$		$*$	$*$		$*$
$*$		$*$	$*$		$*$
0		0	0		0
0		0	0		1
0		0	0		$*$
0		0	0		0

Thus, although the conditional has some conjunction-like features, it is still different enough from a conjunction.

12.5 Rogerson and Butchart's Conditional

Rogerson and Butchart's (henceforth, RB) conditional (Rogerson and Butchart 2002) is defined by the truth table below:

A	B	$A \rightarrow_{RB} B$
1	1	1
1	$*$	0
1	0	$*$
$*$	1	$*$
$*$	$*$	1
$*$	0	0
0	1	0
0	$*$	$*$
0	0	1

What makes RB interesting is that it is not a contracting connective—it actually was used to generate counterexamples to Restall's conjecture that robust contraction freedom is enough to have non-trivial theories based on naïve principles—, as (C3) is invalid for it: consider the interpretation $\sigma(A) = *$ and $\sigma(B) = 1$.

However, that result is obtained under the assumption of Tarskian logical consequence and the distribution $D^+ = \{1\}$ and $D^- = \{*, 0\}$. With P-logical consequence and the distribution $D^+ = \{1\}$ and $D^- = \{0\}$, with $*$ as neither designated nor antidesignated, things are different. In that case, the RB conditional is a contracting connective; here are the tables:

$A \to_{RB} B$	$\Vdash^P$	$A \to_{RB} B$	A	$A \to_{RB} B$	$\Vdash^P$	B
1		1	1	1		1
0		0	1	0		$*$
$*$		$*$	1	$*$		0
$*$		$*$	$*$	$*$		1
1		1	$*$	1		$*$
0		0	$*$	0		0
0		0	0	0		1
0		0	0	0		0

$A \to_{RB} (A \to_{RB} B)$	$\Vdash^P$	$A \to_{RB} B$
1		1
$*$		0
0		$*$
1		$*$
$*$		1
0		0
0		0
$*$		$*$
0		0

Nonetheless, this does not make conjunction contracting, as there is a σ such that $\sigma(A \to B) \in D^+$ when $\sigma(A) \in D^-$, namely $\sigma(A) = \sigma(B) = 0$.

12.6 A Variant of Rogerson and Butchart's Conditional

However, one could be dissatisfied with Rogerson and Butchart's conditional because of the counterintuitiveness of some of its valuations. In particular, the value of $A \to B$ is not 0 when A is 1 and B is 0, but it is when A is 0 and B is 1. By mimicking from Blamey's transplication the idea of making the value of a conditional $*$ when the antecedent is 0 and fixing the valuations mentioned at the end of the previous section, one get the following conditional:

A	B	$A \to_{RB^+} B$
1	1	1
1	$*$	0
1	0	0
$*$	1	$*$
$*$	$*$	1
$*$	0	0
0	1	$*$
0	$*$	$*$
0	0	$*$

With a distribution of values as in the previous cases and using P-logical consequence, this conditional is contracting; this is shown by the following truth tables:

$A \to_{RB^+} B$	$\Vdash^P$	$A \to_{RB^+} B$	A	$A \to_{RB^+} B$	$\Vdash^P$	B
1		1	1	1		1
0		0	1	0		∗
0		0	1	0		0
∗		∗	∗	∗		1
1		1	∗	1		∗
0		0	∗	0		0
∗		∗	0	∗		1
∗		∗	0	∗		∗
∗		∗	0	∗		0

$A \to_{RB^+} (A \to_{RB^+} B)$	$\Vdash^P$	$A \to_{RB^+} B$
1		1
0		0
0		0
1		∗
∗		1
0		0
∗		∗
∗		∗
∗		∗

Under these assumptions, conjunction is a contracting connective too. Here are the truth tables for more visibility:

A	B	$A \to_{RB^+} B$	$\Vdash^P$	$A \wedge B$	A	$A \wedge B$	$\Vdash^P$	B	$A \wedge (A \wedge B)$	$\Vdash^P$	$A \wedge B$
1	1	1		1	1	1		1	1		1
1	∗	0		∗	1	∗		∗	∗		∗
1	0	0		0	1	0		0	0		0
∗	1	∗		∗	∗	∗		1	∗		∗
∗	∗	1		∗	∗	∗		∗	∗		∗
∗	0	0		0	∗	0		0	0		0
0	1	∗		0	0	0		1	0		0
0	∗	∗		0	0	0		∗	0		0
0	0	∗		0	0	0		0	0		0

Once again, it seems that conjunctions and conditionals are different enough to avoid considering that the conditional has turned into a conjunction or vice versa. The following truth tables show the features of each connective:

$A \rightarrow_{RB^+} B$	$\dashv^P\vdash$	$B \rightarrow_{RB^+} A$	$A \rightarrow_{RB^+} B$	$\dashv^P\vdash$	$\sim B \rightarrow_{RB^+} \sim A$
1		1	1		$*$
0		$*$	0		0
0		$*$	0		0
$*$		0	$*$		$*$
1		1	1		1
0		$*$	0		0
$*$		0	$*$		$*$
$*$		0	$*$		$*$
$*$		$*$	$*$		1

$A \wedge B$	$\dashv^P\vdash$	$B \wedge A$	$A \wedge B$	$\dashv^P\vdash$	$\sim B \wedge \sim A$
1		1	1		0
$*$		$*$	$*$		0
0		0	0		0
$*$		$*$	$*$		$*$
$*$		$*$	$*$		$*$
0		0	0		0
0		0	0		1
0		0	0		$*$
0		0	0		0

Contraposition fails for conjunction but symmetry holds. (And, once more, there is a case in which $A \wedge A$ is 0.) On the other hand, as in the case of the OCO conditional, the variant of RB satisfies symmetry, which is a conjunction-like feature, but it is still different enough from a conjunction. We go further and, just as we did with the OCO conditional, claim that this connective is still quite conditional-like, as we still get contraposition and no σ such that $\sigma(A \rightarrow_{RB^+} A) = 0$.

Overall, we have settled with defending that, to conclude that a conjunction is not a conditional or vice versa, it suffices to show that (a) there is at least one σ such that $\sigma(A \rightarrow B) \neq \sigma(A \wedge B)$; and (b) that each connective has some of the features that one would expect of each kind of connective. There are many ways that these criteria could be strengthened. For instance, one could demand the satisfaction of a specific subset of the features as a minimal condition to be met for a connective to be considered as belonging to a certain kind. We find this discussion interesting, and deserving of a much more detailed exposition than we can manage here. Consequently, taking those practical concerns alone into account, we stand by our decision to defend the weakest claim possible.

12.7 Conclusions

In this paper, we have shown that some authors are inclined to say that only conditionals are contracting connectives because they are assuming a very specific set of features that the background logic has to have. Indeed, if we restrict ourselves to classical logic and some non-classical logics in its neighborhood—all of which share the supposition that the consequence relation is Tarskian, that the partition on the collection of truth values is both collectively exhaustive and mutually exclusive, and include a detachable conditional—then conditionals seem to be the only kind of contracting connectives. By modifying some of those features, namely, the detachable connective that appears on the left side of (C1), the logical consequence relation and the structure on the collection of truth values, we have here developed three cases in which conjunctions are contracting connectives.

Throughout, the main concern has been that our proposed modifications are always at risk of coming across as too radical. This is important, because a pervasive worry is that a radical enough modification will be a step too far outside the realm of logic. Should that happen, our project would become quite uninteresting, as it would be easy to argue that while we are discussing some very peculiar abstract structures, they are of no philosophical or logical interest. Our reply to this concern consisted in showing that none of the proposed modifications lack motivation in the existing literature.

In so doing, we revisited some important questions, many of which cannot be answered in this paper, about the nature of certain connectives, the logicality of non-transitive consequence relations and the possible need of logically non-bivalent semantics for many-valued logics. Thus, we are inclined to say that besides succeeding in exhibiting some sufficient conditions for restricting contraction to conditionals and showing how conjunctions can be contracting connectives, we have also succeeded in providing good motivation to study all those questions for their own sake.

Acknowledgements The present work was funded by UNAM's PAPIIT project IN403719 "Intensionalidad hasta el final: un nuevo plan para la relevancia lógica".

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Chapter 13
Deflationary Reference and Referential Indeterminacy

Bahram Assadian

Abstract Indeterminacy of reference appears to be incompatible with the deflationist conceptions of reference: in deflationism, the singular term 'a' refers to a, if it exists, and to nothing else, whereas if the term is referentially indeterminate, it has a variety of equally permissible reference-candidates: referential indeterminacy and deflationism cannot both be maintained. In this paper, I discuss the incompatibility thesis, critically examine the arguments leading to it, and thereby point towards ways in which the deflationist can explain referential indeterminacy.

Keywords Referential indeterminacy · Deflationism · Singular reference · Vagueness · Kilimanjaro

13.1 The Incompatibility Thesis

What does 'Kilimanjaro' refer to? It seems that, setting lexical ambiguities aside, it refers to a particular object: Kilimanjaro. Inflationists about reference think that, if 'Kilimanjaro' refers, there has to be some causal or metaphysical relationship that connects the term to the particular object it refers to. Deflationists declare that 'Kilimanjaro' refers to Kilimanjaro, but they explain the reference relation by regarding sentences such as "If Kilimanjaro exists, then 'Kilimanjaro' refers to Kilimanjaro and to nothing else" as constituting an implicit definition of the term 'refers'; and after noting that Kilimanjaro exists, they conclude that 'Kilimanjaro' refers to Kilimanjaro. In their view, thus, 'Kilimanjaro' refers to Kilimanjaro not in virtue of some causal or metaphysical relationship between the term and the dormant volcano in Africa, but rather, because of how we use 'refers' in our language. Referential indeterminists, on the other hand, declare that 'Kilimanjaro' does not refer: since it is a singular term, it refers, if it does at all, to a uniquely determined object; but nothing in our use of the term determines which object it refers to; so, it does not refer.

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M. Mojtahedi et al. (eds.), *Mathematics, Logic, and their Philosophies*,
Logic, Epistemology, and the Unity of Science 49,
https://doi.org/10.1007/978-3-030-53654-1_13

'Kilimanjaro', despite appearances, is not a genuine singular term whose semantic job is to effect reference to a particular object.

This is the battlefield. The tension between the deflationists and the inflationists is well known. But the dialectical situation between the deflationists and the indeterminists has not attracted the attention it deserves. Are the two positions incompatible? It seems so: one view says that 'Kilimanjaro' refers; the other says it doesn't—or if you like, one view says that 'Kilimanjaro' has a unique referent; the other says that it has many equally admissible reference-candidates. How could they possibly be compatible? My aim in this paper is to examine what it means to say that referential indeterminacy and deflationism are incompatible, what the arguments for the alleged incompatibility are, and thereby show the extent to which the deflationist can explain referential indeterminacy.

Let me say some words about each view. The proponent of indeterminacy of reference holds that facts about our linguistic usage, together with all relevant non-semantic background facts—for example, facts about our dispositions, intentions, causal relations with our environment, or whatever we take semantic facts to supervene on—do not attach a uniquely determinate semantic-value to singular terms and predicates of our language. For example, to say that 'Kilimanjaro' is a referentially indeterminate singular term is to say that there are many slightly different entities, $k_1, k_2, \ldots, k_n$, whose difference is just one tiny atom, such that each one of these precisely demarcated mountainous regions has an equally good claim to be the referent of 'Kilimanjaro', since there are no linguistic, causal, psychological, geographical, or sociological features that can pick out one of them as *the* referent of the word. In this sense, 'Kilimanjaro' is an expression which seems to refer, but for which there seems to be no fact as to which of these several things it refers to.[1]

If we endorse referential indeterminacy, there remains a choice about how to use the word 'refers' in our language: one possibility is to say that there is no such thing as the reference relation. Since there are no facts that could single out a unique referent for 'Kilimanjaro', the assumption that it refers to a particular object is phantasmal. But the deflationist thinks that this eliminativist stance towards reference is unnecessarily harsh. She would prefer to use the notion of reference deflationarily, so that, on her view, 'Kilimanjaro' refers to Kilimanjaro (and to nothing else), but not in virtue of the obtaining of a substantial causal or metaphysical relation between 'Kilimanjaro' and

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[1]This is a familiar reading of Unger's (1980) Problem of the Many as an argument for the referential indeterminacy of the terms purporting to refer to ordinary concrete objects. See McGee (1997) and McGee and McLaughlin (2000) for a powerful support of this reading, which is anticipated by Lewis (1993). In this paper, we shall focus only on the referential indeterminacy of singular terms, and will not discuss indeterminacy in vague expressions that admit borderline cases—for example, indeterminacy as to whether a given person is in the extension of 'bald' or 'rich'—even though the central claims to be discussed below also arise for indeterminacy of vague expressions. Also, if you are not happy with the example of Kilimanjaro, you can instead talk about mathematical singular terms such as 'The number 2'. In order for the terms of our arithmetical theory to have determinate reference, one model of the theory has to be privileged over the others. Yet, as it has been argued by Benacerraf (1965), there is nothing in our use of arithmetical terms that can privilege one model over any other. In this sense, 'The number 2' is an expression which seems to refer, but for which there seems to be no fact as to which of things in different models it refers to.

the dormant volcano in Africa, but rather, in virtue of the meaning of the word 'refers'. (More on this below.) To the deflationist, "'Kilimanjaro' refers to Kilimanjaro" is analytically or definitionally equivalent to 'Kilimanjaro is identical to Kilimanjaro'.

Deflationism about reference has received different forms and formulations. However, despite important differences between them, there is a thesis that they all have in common: there is no more to reference than is encoded in the disquotation schemas of this form: for any object a, if a exists, 'a' refers to a and to nothing else; and for any predicate F, if there are Fs, then 'F' applies to the Fs and to nothing else. I shall talk in more detail about the disquotation schemas, but it is important to note here that the arguments that will be examined in this paper are independent of any particular version of deflationism.[2]

It thus seems that referential indeterminacy and deflationism are *incompatible*. How could the terms of our language be referentially indeterminate if reference is understood in accordance with the disquotation schema that encodes, somehow, determinate reference? Although the incompatibility thesis raises an important challenge for deflationism and has been in the wind for a while, it has not, curiously, garnered the attention it deserves.[3] There are two main arguments supporting the incompatibility thesis, which I will label 'the argument from disquotation' and 'the argument from explanation'. The first says that the deflationist's disquotation principles rule out referential indeterminacy. The second has it that the only way a deflationist could *explain* what referential indeterminacy would have to involve—that is, what it would take for terms to be referentially indeterminate—rules out the possibility of any term being referentially indeterminate.

Thus, the deflationist should either argue that deflationism indeed provides a successful argument against referential indeterminacy, or else, respecting ubiquitous indeterminacy of reference, try to defuse the arguments for the incompatibility thesis, and thereby establish the compatibility of her standpoint with referential indeterminacy. Since I think that there is no hope, and even no need, to purify our language from indeterminacy, I will explore the prospects of the deflationist's standpoint along the latter route.

13.2 The Argument from Disquotation

According to the argument from disquotation, the deflationist's characterization of reference in terms of the disquotation schema rules out the possibility of referential indeterminacy. The argument proceeds from the plausible claim that the deflationist takes reference to be exhausted by the disquotation schema:

(1) $(\exists! x)(`\tau\text{' refers to } x) \leftrightarrow (\exists! x)(\tau = x)$,

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[2]For different versions of deflationism about reference, see Field (1994a, b, 2000), McGee (2005), and McGee and McLaughlin (2000). And for a useful survey of the space of possibilities, see Armour-Garb and Beall (2005).

[3]But see Field (1994a, 1994b, 2000), McGee (2016), and in particular, Taylor (2017).

which states that for a singular term 'τ', "There is a unique x such that 'τ' refers to it" is equivalent to 'There is a unique x such that τ is identical to it'. According to the deflationist, (1) is valid for all meaningful terms of our language. It is, in effect, true by stipulation that 'Kilimanjaro' refers to Kilimanjaro, and only to Kilimanjaro. Since, as it is encoded in (1), there is a unique referent for 'Kilimanjaro', there can be no other equally admissible reference-rivals for the term, and so, 'Kilimanjaro' is not referentially indeterminate. In sum: (1) would preclude there being anything else that 'Kilimanjaro' is indeterminate between referring to. So 'Kilimanjaro' cannot be referentially indeterminate.[4]

Let us present a general formulation of the argument as follows:

(P1) If one is committed to (1), there can be no referential indeterminacy;
(P2) Deflationism is committed to (1);
(C) Therefore, there can be no referential indeterminacy.

The deflationist, however, is in a position to offer ways past this challenge by rejecting (P1). She admits that (1) indeed ensures that it is determinate that 'Kilimanjaro' refers to Kilimanjaro, but, in her view, that does not settle under what conditions statements containing it are determinately true: the fact that (1) tells us that it is determinate that 'Kilimanjaro' refers to Kilimanjaro does not preclude there being things 'Kilimanjaro' is indeterminate between referring to. According to the deflationist, (1) allows us to assert that 'Kilimanjaro' refers to Kilimanjaro, but, because of indeterminacy, it does not allow us to assert that it refers to k_1 as opposed to k_2, where each is a precisely demarcated mountainous region. (I am assuming that there are only two reference-candidates; nothing hangs on this other than ease of presentation.) This just means that (P1) is to be rejected. That is, since

(D) It is determinate that 'Kilimanjaro' only refers to Kilimanjaro

is compatible with

(I) It is indeterminate as to whether 'Kilimanjaro' refers to k_1 or to k_2

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[4]Various forms of this argument have been defended by Leeds (1978, 2000, p. 107), Akiba (2002), and Hill (2014, p. 70). For a detailed critical discussion of this argument, see Taylor (2017). Leeds argues that in order to explain what it is for a term to be referentially indeterminate, one must inevitably say that it has more than one equally acceptable referent; but a deflationist can "make no sense" of the notion of *being equally acceptable* when the referent of the term is "given uniquely" by the disquotation schema. Hill also argues that the deflationist can easily explain why a proposition such as *The concept of rabbit denotes rabbits* can be seen to be true *a priori*, for the proposition is "either a component of a definition or a trivial consequence of a definition [i.e. (1)]" (2014, p. 70). In his view, this suffices to rule out indeterminacy. It should be pointed out that in Hill's view, deflationism, first and foremost, rules out indeterminacy concerning *concepts*, but he mentions that his argument can be applied to resist *linguistic* indeterminacy, too. Akiba proposes that since deflationism is incompatible with the sort of indeterminacy that is induced by vagueness, the deflationist can make sense of indeterminacy only by embracing the worldly or ontic view of vagueness.

the deflationist will not be committed to denying that ‘Kilimanjaro’ is referentially indeterminate. She thereby locates the diagnosis of the failure of the argument from disquotation in the compatibility between (D) and (I).[5]

The supporter of the argument from disquotation may try to bolster her argument by noting that (1) entails that “There is something that ‘Kilimanjaro’ determinately refers to”. The fact that it is determinate that ‘Kilimanjaro’ refers to Kilimanjaro and to nothing else entails that there is a unique individual to which ‘Kilimanjaro’ refers; and this is exactly what rules out the referential indeterminacy of ‘Kilimanjaro’.

The deflationist, again, has a convincing rejoinder.[6] In fact, in her view, there is no denying that if Kilimanjaro exists, then ‘Kilimanjaro’ only refers to Kilimanjaro, but this does not entail that ‘Kilimanjaro’ has a uniquely determinate reference; and this is exactly where the argument from disquotation fails. The fallacious move from “It is determinate that ‘Kilimanjaro’ refers to Kilimanjaro” to “There is something that ‘Kilimanjaro’ determinately refers to” is familiar from the behavior of the necessity operator in certain modal contexts: the inference from ‘It is necessary that someone wins the match’ to ‘There is someone of whom it is necessary that he wins the match’ is valid only when the singular term that the variable bound by the existential quantifier replaces refers to the same object in every possible world in which it exists. Likewise, as Lewis (1988) has argued, the step from “It is determinate that ‘Kilimanjaro’ refers to Kilimanjaro” to “There is an object of which it is determinate that ‘Kilimanjaro’ refers to it” is not valid: it is valid only when the singular term that the variable replaces is referentially determinate: it refers to the same object in every interpretation of ‘Kilimanjaro’. But this is exactly what the referential indeterminacy of ‘Kilimanjaro’ rules out.[7]

To summarize: there are two ways for the deflationist to resist the argument from disquotation: she can argue for the compatibility between “It is determinate that ‘Kilimanjaro’ refers to Kilimanjaro” and “It is indeterminate as to whether ‘Kilimanjaro’ refers to k_1 or to k_2”. She can also point to the fallacious move from “It is determinate that ‘Kilimanjaro’ refers to Kilimanjaro” to “There is something that

[5] This argument is due to Field (2000, pp. 4–5).

[6] I have borrowed the argument of this paragraph from Taylor (2017). See also McGee (2005, pp. 416–417, 2016, pp. 3164–3165).

[7] This is connected to another argument for the incompatibility thesis, which runs as follows: since according to deflationism “‘Kilimanjaro’ refers to k iff Kilimanjaro $= k$”, ‘Kilimanjaro’ could be taken to be referentially indeterminate only if the right-hand side of the biconditional is an indeterminate identity statement, construed *de re*. That is, indeterminacy of reference is possible as long as there can be an indeterminacy in identity. However, since, as Evans (1978) demonstrated, indeterminate identity is incoherent, then so is referential indeterminacy from the deflationary standpoint. Hence, deflationism fails to make any room for referential indeterminacy. As Lewis (1988) convincingly shows, though, the main point of Evans’ s argument is that referential indeterminacy cannot be extended to indeterminate identity statements. For Evans’s argument to have any force against indeterminate identity, we should assume the referential determinacy of the terms involved. This does not mean that ‘$k =$ Kilimanjaro’ is not an indeterminate identity statement. It indeed is. However, for Lewis—and also for our deflationist—to say that it is indeterminate is not to say that the identity relation is indeterminate, or that the objects themselves are indeterminate. All it could mean is that ‘Kilimanjaro’ is referentially indeterminate.

'Kilimanjaro' determinately refers to". There does not seem to be a way to argue from the disquotation schema to the incompatibility between referential indeterminacy and deflationism. The most one could say is that deflationism is *neutral* as to whether singular terms are afflicted by referential indeterminacy: since, in the deflationist's view, the disquotation schema has the status of a definitional or stipulative truth, it is determinately true and hence, it is determinate that there is a unique object to which 'Kilimanjaro' refers. But that does not mean that the term has a uniquely determinate referent; nor does it mean that it does not have a uniquely determinate referent. The deflationist standpoint is just neutral about referential indeterminacy.

All the same, the above rejoinders to the argument from disquotation are silent as to how the deflationist is supposed to *explain* the involved notion of indeterminacy. We will pursue this theme in the context of the next argument for the incompatibility thesis.

13.3 The Argument from Explanation

The second argument, the argument from explanation, directs at the conceptual resources that are available to the deflationist for explaining referential indeterminacy. The argument plausibly proceeds from the thesis that referential indeterminacy is not a *primitive* phenomenon, resisting any sort of explanation. This should not be surprising if we plausibly hold that neither reference nor indeterminacy is primitive. As Taylor, who has recently defended the argument from explanation, writes

> [referential indeterminacy] is not a brute, inexplicable fixture of reality. *Something*, after all, must account for the difference between terms that are referentially indeterminate and those which are not. (Taylor 2017, pp. 62–63)

Given this assumption, the deflationist, just like the non-deflationist, owes us an explanation as to what referential indeterminacy is, or what it takes for a term to be referentially indeterminate. However, the only way she could explain what it would take for terms to be referentially indeterminate rules out the very possibility of any term being referentially indeterminate.

13.3.1 The Formulation of the Argument

Let us discuss the argument in more detail. Taylor first puts forward the following thesis to capture how the deflationist explains facts about reference:

> *The Insubstantiality of Reference*. The fact that 'Barack' refers to Barry holds in virtue of (at most) the fact that Barry = Barack and the meaning of 'refers'. It thus requires no additional explanation or grounding in terms of facts about causation, dispositions, intentions,

> associated descriptions, relative naturalness, or anything else, except to the extent that these facts go into explaining the fact that Barry = Barack or the fact that 'refers' means what it does. (2017, p. 54)

Taylor's own example illustrates the point. Suppose we implicitly define a new two-place predicate, 'eviates', in English according to the following schema:

For all x, 'a' eviates x iff x is a proper part of a.

Now, let us ask what explains the fact that 'England' eviates London. The answer is: the fact that London is a proper part of England and the fact that 'eviates' means what it does. There is no further explanation in terms of causal or metaphysical relations between 'England' and London; nor is there any explanation in terms of our intentions to use 'England' to refer to an entity. The suggestion is that the deflationist's explanation of the fact that 'Kilimanjaro' refers to Kilimanjaro is analogous to the above explanation of the fact that 'England' eviates London.

Taylor goes on to argue that precisely because of her commitment to Insubstantiality of Reference, the deflationist cannot explain what referential indeterminacy consists in:

> And what I'd like to show is that deflationists, given their commitment to [the Insubstantiality of Reference], simply cannot offer such an explanation – at least not without making RI [referential indeterminacy] out to be derivative of some other type of indeterminacy. That is, deflationists have no way of explaining what it would be for there to be direct RI. Hence they have no way of making sense of such RI. (ibid, p. 63)

But what is Taylor's argument? As said above, in the deflationist's view, the disquotation schema (1) holds in virtue of the meaning of 'refers'. This is because she views (1) as an implicit definition of 'refers'. Assuming that $\tau = x$, she holds, on the basis of Insubstantiality of Reference, that the fact that 'τ' refers to x holds in virtue of the meaning of 'refers' and the fact that $\tau = x$. Let us then encapsulate Insubstantiality of Reference in the following thesis:

(2) The fact that 'τ' refers to x holds in virtue of the fact that $\tau = x$, and of the fact that 'refers' means what it does in our language.

Taylor's claim is that it is precisely the deflationist's commitment to (2) that makes an explanation of referential indeterminacy impossible: the fact that τ is identical to x and facts about the meaning of 'refers' do not leave any room for referential indeterminacy, since they always determine, for any given object x and any singular term 'τ', whether or not x is the referent of 'τ'. As a result, the deflationist has no way of explaining what it would take for 'τ' to be referentially indeterminate.

More precisely, the proponent of the argument from explanation claims that if referential indeterminacy and deflationism were compatible, the deflationist should have been able to appeal to (2) in order to explain referential indeterminacy, for (2) is the only explanation of reference that is available to her. But the deflationist cannot—for she does not have the resources to—explain referential indeterminacy in this way.

To appreciate this point, let us assume, with a view to a *reductio*, that the deflationist can indeed supply the required explanation: she appeals to (2) and puts forward the following claim:

(3) Facts about the meaning of 'refers' and the identity of Kilimanjaro do not determine a unique referent for 'Kilimanjaro': they neither determine k_1 nor k_2 as the referent of 'Kilimanjaro'.

If (3) is accepted, then both of the following statements must be compatible with the facts about identity:

(4) 'Kilimanjaro' refers to k_1.
(5) 'Kilimanjaro' refers to k_2.

However, given (1), the above two statements, respectively, entail:

(6) Kilimanjaro $= k_1$.
(7) Kilimanjaro $= k_2$.

These latter statements must likewise be compatible with the facts about the identity of Kilimanjaro. But they are not, since we have assumed that k_1 and k_2 are numerically distinct; and yet (6) and (7) jointly entail that k_1 is identical to k_2. Hence, (3) should be rejected, and since (3) is supposed to explain referential indeterminacy from the deflationist's standpoint, its rejection is tantamount to the endorsement of the thesis that the deflationist cannot explain the referential indeterminacy of 'Kilimanjaro'. Let me put the point equivalently as follows: if (6) and (7) are to be compatible with the identity facts, then it must be the case that k_1 be identical to k_2. But this contradicts (3), according to which no particular object can be picked out as the referent of 'Kilimanjaro'. So, the deflationist has to conclude that there is a particular object that is picked out as the referent of 'Kilimanjaro'. This just means that deflationism is incompatible with referential indeterminacy.

In the following two sections, I will discuss two difficulties for the argument from explanation; one focuses on the notion of incompatibility; the other on explanation.

13.3.2 Incompatibility

Powerful as it may seem, Taylor's argument from explanation can be resisted. The crucial point to note is that the arguments for referential indeterminacy work only when we assume that the background notion of reference is *not* deflationary. For example, the explanation of the claim that 'Kilimanjaro' is referentially indeterminate is grounded in the thesis that facts about the way we use 'Kilimanjaro' and all the relevant non-linguistic facts fail to single out a unique referent for the term. The referential indeterminacy, in this sense, is the result of the fact, if it is one, that the way the speakers in a community use their language, and also the relevant non-semantic facts—such as the arrangement of atoms around Kilimanjaro—fail to pick out a uniquely determined referent for ‘Kilimanjaro’. The argument for the referential indeterminacy of ‘Kilimanjaro’ would not work if the background notion of reference was deflationary reference whose explanation is exhausted by the disquotation schema that implicitly defines the word ‘refers’, and therefore does not involve any fact about the way in which ‘Kilimanjaro’ is used by the speakers of the community. So seen, referential indeterminacy arises only when a non-deflationary conception of reference is already in play.

The argument from explanation demands that the deflationist should be able to explain referential indeterminacy in terms of the conceptual resources built into the deflationary conception of reference—for how could she explain it in any other way if she recognizes no notion of reference beyond the deflationary one? This is an important question, and we shall come back to it in the next section, but I do not think that it poses a challenge to the deflationist. For she could say that if we had used ‘refer’ non-deflationarily, reference would have been vulnerable to indeterminacy. If reference had been taken as an inflationary relation, then referential indeterminacy could have been explained in the standard way: there is nothing in our thoughts, linguistic practices, and non-linguistic facts that could single out a unique object as the referents of our words. And if a deflationary notion of reference were employed from the outset, referential indeterminacy would not arise.

But doesn’t that *just* mean that there would be no place for referential indeterminacy from the deflationist’s standpoint? The previous paragraph seems to restate the incompatibility thesis by saying that referential indeterminacy arises only when a non-deflationary conception of reference is employed.[8]

As I have set up the stage above, deflationism is *motivated* by the thesis that the non-deflationary reference is indeterminate. That is, since reference, non-deflationarily construed, is indeterminate, the deflationist invites us to use the notion of reference deflationarily if we still want to keep ‘refers’ in our language. What various arguments for referential indeterminacy show is that ‘refers’, as is used in our ordinary thought and talk, cannot express the non-deflationary reference relation. And since non-deflationary reference is vulnerable to indeterminacy, the deflationist proposes the deflationary reference as a way of understanding ‘refers’ in ways that vindicate our ordinary thought and talk.[9] In this sense, it will be a truism to say that referential indeterminacy is incompatible with deflationism. They are incompatible not because one rules out the other, but because they do not meet each other.

To see the point more closely, let us consider the following formulation of the argument from explanation:

(i) The deflationist’s putative explanation of referential indeterminacy has to go through the resources available to her; i.e. facts about identity and the meaning of ‘refers’.

[8] See Taylor (2017, footnotes, 26 and 45).

[9] See Soames (1999) who argues that this was Quine’s (1969) own strategy: the lesson of Quine’s “inscrutability of reference” is that since the word-world inflationary reference is vulnerable to indeterminacy, the best we can do is to replace it with a deflationary reference relation.

(ii) But these facts rule out the possibility of any terms being referentially indeterminate.
(iii) Therefore, the deflationist cannot explain referential indeterminacy.

I argued that (i) should be rejected, not because the deflationist can explain referential indeterminacy in terms of a conception of reference which is not available to her; but rather, because she is just not committed to the thesis that referential indeterminacy is to be explained in terms of deflationary reference: indeterminacy is the byproduct of non-deflationary reference. Why should we expect the deflationist to explain something that does not arise within the realm of her deflationary reference? (i), in effect, states that the deflationist's explanation of referential indeterminacy (which is the byproduct of inflationary reference) must go through resources involving deflationary reference; and this premise can be resisted.

Deflationism, so seen, is the right conclusion to draw from the argument for referential indeterminacy: if reference is non-deflationary, it is indeterminate. But there is a sense in which reference is not indeterminate: when we use 'Kilimanjaro', we intend to refer to exactly one object; i.e. Kilimanjaro. Thus, deflationary reference should be adopted in order to account for our everyday talk about reference. Here is how McGee has expressed this tension about reference:

> There isn't anything in my linguistic usage or in the linguistic usage of my speech community that picks out a unique individual as the thing I refer to by "Kilimanjaro" ; this is so even if we allow the thoughts we express by employing a word to count as part of its "usage," and even if we allow "usage" to take account of causal connections between our words and our environment. Yet, quite unmistakably, there is one and only one individual I refer to when I use the name "Kilimanjaro," namely Kilimanjaro itself. (McGee 2005, pp. 409–410)

And the solution he offers is that compatibly with the inflationary notion of reference suffering from indeterminacy, there is a way of understanding 'refer' that accommodates our ordinary thought and talk. This is captured by the deflationary notion of reference.

Let us then summarize the deflationist's position: she would say that on the basis of the disquotation schema, 'Kilimanjaro' refers to Kilimanjaro and to nothing else, but that does not entail that 'Kilimanjaro' is referentially determinate; nor does it entail that it is not referentially determinate: whether or not it is referentially determinate depends on what reference-determining facts might constraint our use of 'Kilimanjaro'. If none of these determines a unique referent for 'Kilimanjaro', deflationism cannot narrow down the range of the candidates, either.

13.3.3 Explanation

The argument from explanation faces another challenge. As we said above, according to the customary explanation of referential indeterminacy, reference-determining facts, generously understood, fail to pick out a unique referent for a given term. However, to say that the customary explanation is not available to the deflationist does not mean that there are no other explanations that she may appeal to. If referential indeterminacy could be explained independently of the ingredients of a non-deflationary theory of reference, then its explanation would be available to the deflationist.

A proposal along this line has been developed by Field (2000, 2003). He seems to admit that the deflationist cannot explain referential indeterminacy in terms of our linguistic practices having failed to pick out a unique referent for the terms of our language. For how can she explain it in terms of the ingredients of a theory of reference if she recognizes no notion of reference beyond the deflationary one? As a result, she must explain referential indeterminacy independently of a theory of reference.

Field's proposal is to understand indeterminacy in terms of non-standard probability distributions in the speaker's belief system. On this account, indeterminacy is, fundamentally, a psychological phenomenon, such that the degree of belief in the indeterminacy of a sentence is measured by the extent to which the probability of it and its negation sum to less than 1. That is, for an agent to believe that a sentence is indeterminate is for her to have degrees of belief in it and its negation that sum to less than 1; and to believe that a sentence is determinate is to have degrees of belief in it and its negation that sum to 1.

This account of indeterminacy does not rely on any theory of reference, and is thus available to anyone who thinks that reference is to be explained only in terms of the deflationary conception. If indeterminacy of sentences (and derivatively, the referential indeterminacy of sub-sentential expressions) can be explained along this line, the deflationist does not need to give up her belief in referential indeterminacy: she can ground it not in a theory of reference, but in the probability distributions in our belief system. On this view, there can still fail to be a determinate referent for 'Kilimanjaro', or a determinate extension for 'bald', but the indeterminacy in question is not to be explained in semantic terms. The argument from explanation, therefore, fails to supply a knock-down argument against the deflationist's ability to explain referential indeterminacy.[10]

The questions of whether Field's framework is defensible and whether there is another way to find referential indeterminacy at the ground level independently of semantics are large and largely unexplored questions. What it is clear, though, is that any proponent of the argument from explanation has to examine such arguments before establishing the incompatibility claim.

It should also be noted that by referring to Field's account, I do not aim to emphasize on the familiar point that deflationism can explain vagueness as a non-referential phenomenon. For example, Akiba (2002) argues, from a deflationist standpoint, for

[10]There are also other theories of indeterminacy that do not explain referential indeterminacy in the standard way. For example, in his (2018), Andrew Bacon defends a *propositional* account, according to which it can be propositionally indeterminate whether a term 'a' refers to x—i.e. the proposition that 'a' refers to x is indeterminate—and this indeterminacy is not explained by the way people use particular words (e.g. 'refers', 'a', or 'x', etc) but in terms of the way they think about reference, 'a', x, etc.

a metaphysical account of vagueness. The deflationist Horwich (1998) concedes that vagueness cannot be taken as a referential phenomenon, and defends a version of epistemicism. Field also treats vagueness as a psychological phenomenon. My point against the argument from explanation does not rest on the thesis that deflationism can treat vagueness as a non-referential phenomenon, but rather, is based on the claim that where referential indeterminacy arises, the deflationist will be in a position to explain it independently of semantic resources; and so her explanation would not rest on semantic notions such as inflationary reference, which are not available to her.

13.4 Concluding Remarks

The standard account of referential indeterminacy maintains that indeterminacy arises from the use of terms, and the relevant non-linguistic facts, failing to latch on to a unique referent. So it seems that deflationism about reference does not make much room for referential indeterminacy. If it is a definitional—and so, a determinate—truth that 'Kilimanjaro' refers to Kilimanjaro, then it would not be possible for there being anything else that 'Kilimanjaro' is indeterminate between referring to. So there cannot be indeterminacy of reference.

In Sect. 13.2, we have argued that the deflationist can resist this challenge, and in Sect. 13.3, we have shown where the argument from explanation fails: the argument tells us that the deflationist cannot explain referential indeterminacy in terms of the components of a deflationary conception of reference. But my point is that the deflationist is just not committed to the thesis that referential indeterminacy is to be explained in terms of deflationary reference.

In the end, we have stressed on the fact that there are other theories of indeterminacy—in particular, Field's—which do not explain referential indeterminacy in terms of our linguistic practices having failed to pick out a unique referent for 'Kilimanjaro', or—in Field's case—a unique extension for vague predicates such as 'bald'. Since the indeterminacy in question is not explained in semantic terms, the argument from explanation does not put forward an adequate argument against the deflationist's ability to explain referential indeterminacy.

Acknowledgements I am honored to be able to contribute to this volume in honour of Mohammad Ardeshir. Ever since I met him, before starting my postgraduate studies, he has been a great source of inspiration and encouragement. I learned a lot from him over these years; in conversation, from his teachings, and through his written work. I am very grateful to the editors of this volume for the invitation; and also to Mahrad Almotahari, Thomas Schindler, and anonymous referees for their very useful comments.

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Chapter 14
The Curious Neglect of Geometry in Modern Philosophies of Mathematics

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Abstract From ancient times to 19th century geometry symbolized the essence of mathematical thinking and method, but modern philosophy of mathematics seems to have marginalized the philosophical status of geometry. The roots of this transformation will be sought in the ascendance of logical foundations in place of intuitive primacy as the cornerstone of mathematical certainty in the late 19th century. Nevertheless, geometry and geometrical thinking, in multiple manifestations, have continued to occupy a central place in the practice of mathematics proper. We argue that this, together with advances in the neuroscience of mathematical processes, calls for an expansion of the present limited remit of the philosophy of mathematics.

Keywords Geometry · Arithmetic · Set · Manifold · Riemann · Arithmetization · Ontology

The term 'modern' refers here to philosophical discussions about mathematics beginning with the 'Big Three' (1) philosophies of early twentieth century, namely Logicism, Intuitionism and Formalism, up to the current discourse. Whereas in earlier times, Geometry appeared as one of the two founding pillars of mathematics, often representing the purported true character of the field, it now tends to occupy a marginal place philosophically in comparison to such areas as set theory or arithmetic. Notwithstanding the merits of the rather maverick representation of set theory as the 'geometrization of mathematics' by mathematician Yuri Manin (2), the consensus among philosophers of mathematics does not appear to favor a primary role for geometry or geometric thought. A more indicative sentiment "… a disturbing secret fear that geometry may ultimately turn out to be no more than the glittering intuitional trappings of analysis" was expressed by G. D. Birkhoff in 1938 (3). In this paper we try to show how the same historical forces that engendered the modern

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M. Mojtahedi et al. (eds.), *Mathematics, Logic, and their Philosophies*,
Logic, Epistemology, and the Unity of Science 49,
https://doi.org/10.1007/978-3-030-53654-1_14

discourse in the philosophy of mathematics contributed to the change in the philosophical standing of geometry. The mathematical developments of the second half of 19th century will necessarily play a prominent role in this brief historical account.

14.1 Origins of Paradigm Change

Classically, mathematics was conceived as the science of magnitudes: arithmetic dealing with the discrete, and geometry with the continuous. This is explicitly enunciated, e.g., by Aristotle in the books comprising the *Organon*. In *Categoriae* he describes the two categories, and in *Analytica Posteriora* he warns against mixing of the two on account of their separate ontological underpinnings (4). Al-Khwarizmi, the inventor of algebra, writes in the preamble to his treatise, *al-Jabr wa al-Muqabala*, that he has come up with a method that is equally effective in dealing with problems of both natures (5). Nevertheless, medieval philosophical writers, in both Arabic and Latin traditions, were dismissive of algebra as a proper science, perhaps because of the suspicious ontological essence of the '*unknown x*'. Even later, Descartes did not regard his wholesale reduction of geometric problems to the computation of numerical quantities as a demotion of the status of geometry, but simply as the introduction of the method of 'analysis', a tool for discovery and proof (Descartes 1954). Geometry retained its standing as the science of extant physical space and a subject of philosophical speculation at least until the middle of the 19th century. Kant's transcendental idealism did not detract from the empirical reality of geometry, it only redefined geometry as the framework of intuitive spatial construction (Kant 1998, B37-40, B120).

Several mathematical developments of the second half of 19th century provided the bases for paradigmatic change in the position of geometry. One is best exemplified by B. Riemann's celebrated *Habilitation* lecture of 1854 (6) in which he declares, among other things, that the investigation of the geometry of existing universe is not a purely mathematical question, but one that belongs to experimental science. Riemann showed how a multidimensional continuum (*manifold* or '*Mannigfaltigkeit*') is capable of carrying an infinite number of possible geometries, the choice to be determined by contingent criteria. Thus, the purely mathematical study of geometry changed focus from a natural science of extant space (either a priori as in Kant, or ultimately experience-based) to the wide-open mathematical exploration of possible geometries.

Earlier, Riemann had pioneered a geometric approach to complex analysis through the introduction of what are known today as Riemann surfaces. In that approach one witnesses the first effective appearance of *analysis situs* (7), a non-metric form of geometry, which became the forerunner of algebraic topology in the hands of Poincaré, Brouwer and their followers. Because the proper language for the rigorous treatment of the subject was not available then, Riemann's intuitive approach did not find universal approval, and a well-known controversy ensued between the followers of Riemann and those of the more rigorous analytic school of Weierstrass. Among other difficulties, Riemann surfaces were beset by the ontological problem of the absence of a natural habitat in physical space; they were among the early members of the proliferating list of ‘ideal mathematical objects’, as they were referred to in German mathematics of the 19th century. The likely interaction with German idealism has been noted by several authors (see e.g., Laugwitz (1999), Scholz (1982) and Wagner (2017)).

Another development was the ongoing logical analysis of the deductive structure of Euclidean geometry. This had a long history going back to attempts at proving Euclid’s Fifth Postulate and the explorations of non-Euclidean geometries. In the 19th century, several mathematicians including Bolzano and Pasch initiated a rigorous approach to Euclidean geometry marked by replacing intuitive and sensible plausibility with logical primacy and simplicity. This movement culminated in Hilbert’s *Grundlagen der Geometrie* of 1899, and especially the more complete 1903 edition of the work. Therein Hilbert demonstrated that elementary Euclidean geometry, far from being a simple body of propositions encapsulated by five Euclidean axioms, possesses a rich and complex logical structure which can be broken up into simpler comprising axiomatic systems (Hilbert 1990). This marks a turning point in the conception of axiomatics in mathematical practice. As the early advocate Bolzano had prescribed (8), Hilbert supplanted perceptual and intuitive simplicity by logical purity and clarity as first principles. The ontological implication was that mathematical objects would no longer be regarded as perceptible space-time entities, but as new kinds of objects whose mode of existence stirred up discussion and controversy.

A by-product of these developments was that any rigorous investigation of continuous magnitudes would hinge on the exact formulation of the concept of continuum, i.e., the system of real numbers. But delving into the nature of continuous magnitudes, already a challenging task for ancients in view of apparent paradoxes it entailed, had become even more daunting since the introduction of calculus and infinitesimals. A succession of efforts, notably by Bolzano, Cauchy, Dedekind, Weierstrass and Peano, finally led to the so-called ‘arithmetization’ of real-number system, i.e., the ultimate construction of real numbers, hence of continuous magnitudes, from natural numbers. A consensus was being reached that natural numbers provided the incontrovertible safe haven on which all mathematics could be based. This went hand-in-hand with the growing foundationalist tendencies of the late 19th and early 20th centuries. Later endeavor of basing the natural number system on set theory further distanced geometry from the core of what was taking shape as the foundational infrastructure of mathematics.

14.2 Foundationalist Schools

We begin our brief account of the treatment of geometry by the three major schools of the philosophy of mathematics in early 20th century by noting that the three shared a remarkable attitude of abeyance toward geometry, a fact that undoubtedly contributed to subsequent further marginalization and neglect. While the forces of foundationalism and arithmetization played a major role in this tendency for all three, separate examination of the matter in each case may be of some merit. Starting with the logicists, it is doubtful whether this label befits Dedekind as is sometimes claimed (9). Without getting into the controversy, Dedekind's relevance to the present discussion is that his two foundational essays *Stetigkeit und irrationale Zahlen* and *Was sind und was sollen die Zahlen* (10) may be cast as *mathematical* manifestos of arithmetization (for which Dedekind gives original credit to Dirichlet), rather than attempts at *philosophical* system making. His declaration in the Preface of the latter tract that arithmetic (i.e., algebra and analysis) is part of logic is immediately qualified to mean that "numbers are independent of the notions and intuitions space and time", as against Kant. Instead, natural numbers are "free creations of the human mind" and give rise through a logical mental process to the construction of the continuum. It is also worth emphasizing that unlike Frege and Russell, Dedekind (the *structuralist*) does not commit himself to any firm ontology for numbers. Example: A 'Dedekind cut' is not itself a real number by Dedekind's reckoning, but a faithful representative of the 'idea' of a real number. This juxtaposition of subjective "free creation" and objective "logical process", so prone to charges of 'psychologism' by the arch-logicist Frege, was not foreign to German idealism and post-Kantianism of the 19th century, as mentioned earlier. It was probably shared by Dedekind's close mathematical companion, Bernhard Riemann. Dedekind did for algebra and analysis what Riemann had done for geometry: together they liberated mathematics from the ontological yoke of sensible and physical referents. For Dedekind, the word 'logic' seems to be synonymous with the common norms of 'pure rational thought', without any commitments to a formal system of logic. Philosophical pronouncements by Dedekind or Riemann are best regarded as thoughtful musings of philosophically minded mathematicians about their daily activity, a strong tradition of German science throughout the 19th century and the first half of the 20th. There is no direct inclusion of geometry in Dedekind's framework of arithmetization, and there seems to be no major philosophical reference to geometry in his work. It is known that he attended Riemann's various lectures of geometric flavor while they were both *privatdozents* at Göttingen, and he played a major role in publicizing Riemann's work after the latter's death. Even if Dedekind's representation of the continuum leads one to believe that he considered geometric discourse logically reducible to numerical domain, it is doubtful that he considered such an endeavor worthwhile in view of an explicit statement that "…I see nothing meritorious…in actually performing this wearisome circumlocution…," (11) referring here, in general, to re-writing products of mathematics research in terms of integer arithmetic.

Moving along the logicist thread, although Frege was a realist about the existence of mathematical objects, he held arithmetical and geometric objects to totally different modes of existence. He ventured on a project to embed arithmetic in logic by proposing the first formal definition of natural numbers, an undertaking that came to an end with the revelation of Russell's paradox. On the other hand, Frege accepted Kant's account of geometry as a synthetic a priori science about existing space and never seemed to have abandoned the idea of unique space-time referents for geometric concepts. This is clearly borne out in his correspondence with Hilbert (Frege 1980) regarding the latter's work on the foundations of geometry. The disagreements in the exchanges about the status of definitions and axioms demonstrate that Frege was out of step with the emerging *Zeitgeist* in the field. Tappenden has tried valiantly to invalidate this opinion, see Tappenden (1995, 2006), by pointing out Frege's Göttingen background, his extensive teaching of complex analysis at Jena in Riemannian style and even his library borrowing record. But irrespective of any sympathetic leanings toward Riemann's geometric treatment of complex analysis, Frege, the mathematics philosopher, was completely silent about the nature of existence of geometric objects such as Riemann surfaces or non-Euclidean manifolds. The disconnect may be understandable in view of Frege's insistence that what distinguishes mathematics from games is its applicability to science. Some, but not all, of Riemann's geometric work had connections with science at the time, but the ontological status of the geometric objects in the work remained uncomfortably outside Frege's domains of reference. This said, however, one can still confidently assert that there is no evidence that at any time during his career, Frege intended arithmetization to extend to geometry.

In Russell's full-blooded logicism, all of mathematics, including geometry to the extent that it was not the applied geometry of physical space, had to be subdued under the banner of arithmetic, which in turn was meant to be logicized. In his 1903 *The Principles of Mathematics*, which was to serve as a philosophical prelude to the later joint work with A. N. Whitehead, *Principia Mathematica*, Russell writes:

> There is thus no mystery to the continuity of space, and no need of any notions not definable in Arithmetic. (Russell 1996, paragraph 419)

In particular, he adopted Cantor's approach to the realization of the continuum of real numbers and went on to discuss how all known geometries could be constructed on the basis of this definition. It is true that in an earlier work, *An Essay on the Foundations of Geometry* (1897) (Russell 1956), he accepted the Kantian conception of geometry as synthetic and a priori, only replacing Kant's Euclidean geometry by Projective Geometry. In that treatise, however, Russell's focus was on the *applied* geometry of existing physical universe, not the geometries of pure mathematics.

The cases of Brouwer and Hilbert, representing respectively the Intuitionist and Formalist philosophies of mathematics, are especially puzzling. Brouwer's proper mathematical work, outside the development of Intuitionist mathematics, was almost entirely geometric in nature. Not only is his best-known mathematical work in the geometric realm of *analysis situs*; his approach is a virtuoso display of exceptional geometric intuition. Surprisingly, then, he rejects the Kantian notion of a priori spatial intuition but retains the intuition of time, which Kant associated with arithmetic. Brouwer considers the transition from 'one-ness' to 'two-ity' to be the source of all mathematics, as shown by the following excerpt:

> Must it be concluded that there is no a priori form of perception of at all for the world of experience? There is, but only in so far as any experience is perceived as spatial or non-spatial *change*, whose intellectual abstraction is the *intuition of time* or *the intuition of two-in-one*. From this intuition of time, independent of experience, all the mathematical systems, including spaces with their geometries, have been built up … (12)

Although Brouwer's intuitionistically constructed continuum is entirely distinct from that of Dedekind-Cantor-Weierstrass, a version of arithmetization again lurks in the background. Both Brouwer and Russell base their rejection of Kantian spatial intuition on a narrow interpretation of Kant to the effect that his conception of space is decidedly Euclidean. That such an interpretation is founded on an uncharitable reading of Kant has been pointed out, among others, by Cassirer, see e.g., Biagioli (2019).

On the face of it, arithmetization is more understandable in connection with Hilbert's Formalist program. Hilbert set out to defend the ontology-liberated and sometimes non-constructive mathematics that had emerged with epicenter at Göttingen, as well as Cantor's set theory, against the critics he considered overly traditional, if not outright reactionary. He envisaged a solid core of contention-free mathematics surrounded by a large nebula of what was being dubbed as 'idealistic mathematics'. The latter contained abstract mathematical constructs not immediately representable in space-time, propositions dependent on completed infinities and non-constructive existence proofs. Hilbert was fond to point out that many concepts, such as imaginary numbers, which had initially met resistance as legitimate mathematical entities, had over time achieved respectability by finding acceptable representation and/or providing facility of discourse in traditional mathematics. He aspired to show, through the use of non-controversial 'finitary' methods, that this surrounding envelope will not cause any inconsistency that would blemish the certainty so characteristic of the core mathematics. An important task was to decide what to include in the core. A minimal inventory seemed to include finite combinatorial mathematics, as well as the arithmetic of common integers at least to the extent that no strong form of mathematical induction, relying on the existence of the completed infinity of integers, would be used. The very act of delimiting the core beyond contention seemed to involve seeking a 'foundation' for a part of mathematics. A rough consensus among Hilbert's followers identified Skolem's Primitive Recursive Arithmetic as the core. Although Hilbert did not object to the inclusion of PRA, and formalist efforts followed this arithmetical thread, he does not seem to have explicitly equated the core with PRA (13). One could speculate that Hilbert may have had the inclination to include experience-based elementary geometry in the intuitively certain core of mathematics just as he included common integers, although it is hard to conceive how such a project would have been assembled. In fact, there is plenty of evidence that Hilbert's 'world view' of mathematics extended well beyond what became known as the Hilbert Program. In connection with geometry, the following excerpt from the Preface of his *Anschauliche Geometrie* (14) is telling:

> … the common superstition that mathematics is but a continuation, a further development, of the fine art of arithmetic, of juggling with numbers. Our book aims to combat that superstition, …

It should be pointed out that this book, published in 1932, was based on lectures Hilbert had given in 1920–21. These were the times Hilbert was in fact engaged in his foundationalist thoughts. In addition to his earlier work on the foundations of geometry, Hilbert made important contributions to differential geometry and General

Relativity, although his work in algebra and analysis is better known. As has been pointed out (see, e.g., Franks (2009)), Hilbert was primarily a mathematician, and his forays into the philosophy of mathematics were motivated by putting an end to philosophical misgivings about the legitimacy of the paradigm change that was taking shape in mathematics. Although his specific project met disappointment in Gödel's incompleteness results, the style of mathematics he was advocating thrived in the 20th century, and his Program ushered in new areas of research in mathematics.

14.3 Geometry as a Mode of Mathematical Thought

Notwithstanding the dwindling attention paid to geometry in the emerging foundational philosophies of mathematics, geometrical thinking figured prominently in the mathematical developments of the same period. As pointed out earlier, Riemann could be identified as the source of two of the strongest currents of 20th century mathematics, both of geometric flavor. One flowed from the aforementioned *Habilitation* tract and its further mathematical elaboration in a paper of 1861 (both unpublished in his lifetime). These gave rise to tensor calculus and modern differential geometry. His earlier work in complex analysis was the first effective appearance of *analysis situs*, which became the forerunner of algebraic topology in the hands of Poincaré, Brouwer and others. The ontological vacuum that afflicted Riemann's original conception of Riemann surfaces was remedied through the use of the language of point-set topology by Hermann Weyl. Algebraic topology methods became one of the most influential tools in 20th century mathematics, among the off-shoots of which one should mention category theory, a field that has been regarded as rival to set theory for the foundation of mathematics. Riemann surface theory itself, through fusion with other fields and generalizations, continues to be a centerpiece of mathematical research today.

One could mention as aside here a possible justification for Manin's earlier-mentioned characterization of set-theoretic framework for mathematics as a geometrization. It is well-known that Cantor's interest in sets initiated with his attempt to generalize the work of Dirichlet and Riemann on the possible sets of discontinuities of functions representable by trigonometric series (Grattan-Guinness 1980). Thus, complicated point sets of the real line were Cantor's early targets of study; an undertaking of obvious geometric flavor. In fact as mentioned by various authors, e.g., Ferreiros (2007, p. 72), Cantor's choice of terminology for a *point set* in his early papers was not the German *Menge* (=set), but *Mannigfaltigkeit* (=manifold), a term adopted by Riemann (and in fact of earlier use) to describe abstract habitats of geometric discourse in mathematics. The visual image of a point set survives in the mind and the informal practice of mathematicians, especially in fields such as geometric topology. Moreover, the set language is often presented visually to beginners through Venn diagrams and similar devices. However, such geometric semantics has long been usurped, no doubt through necessity, by the logical syntax of axiomatic set theory.

In contrast to pre-Cartesian geometry that stood apart from arithmetic in subject and method, most of modern geometry/topology is so heavily infused with the powerful tools of analysis and abstract algebra that it is sometimes difficult to judge where the 'real geometry' resides. No wonder Birkhoff's lament in the opening paragraph of this essay, uttered by a mathematician whose initial fame owed much to proving the so-called 'Poincaré's last *geometric* theorem.' In fact, this inter-mixing of disciplines has been the hallmark of mathematics for more than half a century. The 'foundational fields', i.e., set theory and mathematical logic, have come to occupy their separate corner while there is intensive interaction between other fields of mathematics to the extent that most of the current cutting-edge research cannot be assigned one pure lineage. This estrangement between the foundations and the traditional body of mathematics has contributed to the alienation of the bulk of mathematical community from what is currently dubbed as the philosophy of mathematics. Current discourse in this field, especially that carried out by academics of analytic tradition, equates mathematics with set theory and is essentially oblivious to the philosophical concerns of most practicing mathematicians. This is in stark contrast to the heydays of early twentieth century when mathematicians of all persuasions were involved in the then burgeoning debates about the nature and the methodology of mathematics.

Getting back to the subject of the paper, we can now identify three inter-related factors that have affected the change in the philosophical status of geometry:

1. Geometry started out as a natural science of extant physical space, a physics stripped of motion. As such it provided a static, stable and perceptually simple background upon which physics could be based. Developments in mathematics and science have expanded this into a study of diverse geometries many of which are not immediately perceptible as space-time entities.
2. The search for logically simple foundations for pure mathematics focused attention on purportedly primitive notions such as natural numbers and pure sets. As Hilbert showed, perceptual simplicity of geometry, even of Euclidean variety, concealed considerable logical complexity. This, and the proliferation of geometries, disqualified geometry as a logical foundation for pure mathematics.
3. The changing nature of mathematics, especially the synthesis of once disparate fields, has expanded geometric inquiry, from the study of specific areas historically identified as geometry, to a style and approach for comprehending complex mathematical phenomena. Broadly stated, arithmetical mathematics, emanating from a one-dimensional configuration, is best suited for the syntactic and algorithmic treatment of mathematics and lends itself more naturally to analysis within existing logical systems. On the other hand, geometric approach to mathematical content is a more holistic and semantic form of comprehension tied ultimately to human perception and intuition. This duality was highlighted in Poincaré's semi-philosophical popular trilogy (15), Hilbert's *Anschauliche Geometrie*, and continues to be a subject of discussion in contemporary mathematical community (see Jaffe and Quinn (1994)). While there is no static demarcation between the two modes of comprehension, extremes of intuitive approach have raised alarm flags of subjectivism and psychologism, a predicament that did

not fare well in the positivist philosophical milieu where the current philosophy of mathematics originally took shape. It can be argued that the proper medium for the study of mathematical intuition is cognitive science or neuroscience. Such studies are being attended to, (16), but are still in the stage of infancy. Future philosophy of mathematics would have to expand its remit to deal with the fruits of such research if it is to have relevance to the practice of mathematics.

Notes

(1) The appellation 'Big Three' was coined by Stewart Shapiro as the heading of a part of his book *Thinking about Mathematics: The Philosophy of Mathematics,* Oxford 2000.

(2) See Yuri Manin's discussion of Sets in his book *Mathematics and Physics*, reproduced as Part II of Manin (2000).

(3) The quote is from G. D. Birkhoff's address 'Fifty Years of American Mathematics' in *Amer. Math. Soc., Semicentennial Addresses*, *Vol. 2*, pp. 270–315, reproduced in Birkhoff (1968).

(4) In *Organon* of Aristotle (2001) see *Categoriae* 6, 20–35 and *Analytica Posteriora* I7, 40.

(5) See Rashed (1994, Chap. I, pp. 9–10).

(6) *Über die Hypothesen welche der Geometrie zu Grunde liegen.* First English translation by W. K. Clifford, a more modern translation can be found in Spivak (1979).

(7) *Analysis situs* was originally used by Leibniz apparently as a general term for *transformation geometry*; see e.g., V. De Risi's *Geometry and Monadology: Leibniz's Analysis Situs and Philosophy of Space* (Birkhäuser 2007). In the 19th century the term came to be used for what is now *topology*.

(8) See excerpts A and B of the work of Bolzano in Ewald (1996, Chap. 6).

(9) For a discussion, see Reck (2013) and Demopoulos and Clark (2007).

(10) English translations of Dedekind's two essays are now available in one volume (Dedekind 1963).

(11) See the Preface to the First Edition of the second part of Dedekind (1963, p. 35).

(12) See Brouwer's article 'The Nature of Geometry' in Brouwer (1975).

(13) Accounts of Hilbert's philosophy can be found in Ewald (1996, Chap. 24 excerpts), and in his 'On the Infinite', reproduced in *Philosophy of Mathematics: Selected Readings* (2nd ed.), ed. by P. Benacerraf and H. Putnam, Cambridge 1983.

(14) The quote is from the English translation by P. Nemenyi of the book under the title *Geometry and Imagination* (Hilbert and Cohn-Vossen 1952).

(15) English translations of Poincaré's three popular works *Science and Hypothesis* (1903), *The Value of Science* (1905) and *Science and Method* (1908) appear as a single volume (Poincaré 2001).

(16) Pioneering work in this area is carried out, among others, by Stanislas Dehaene and co-workers. It is reported in Amalric and Dehaene (2016) that "high-level mathematical reasoning rests on a set of brain areas that do not overlap with the classical left-hemisphere regions involved in language processing or verbal semantics." Perhaps surprisingly, they found no appreciable difference between geometers and non-geometers tested in the experiments.

Acknowledgement The author thanks anonymous referee for helpful comments.

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Chapter 15
De-Modalizing the Language

The Case of Physics

Kaave Lajevardi

Abstract With the aim of providing an empiricist-friendly rational reconstruction of scientists' modal talk, I represent and defend the following unoriginal idea of relative modalities, focused on natural ones: the assertion of physical necessity of φ can be understood as the logical provability of φ from the background theory of the context of assertion. I elaborate on my conception of the background theory, and reply to a number of objections, among which an objection concerning the failure of factivity.

Keywords Natural modalities · Modal talk in physics · Physical necessity · Background theory · Laws of nature · Factivity

Some empiricists, me included, abhor some alethic modalities. Not that I am in any way opposed to logico-algebraic studies of modal logics (modal logic*s*, in the plural); rather, what I do dislike is speaking of "modal facts" as if there are facts which, in any significant and not purely logical sense, transcend experience. For all I have gathered from friendly and informal chats with an eminent intuitionist, I surmise that the dislike is shared by some intuitionists.[1]

My task in this article is not to launch any argument against the metaphysics of the modern Kripkean orthodoxy. What I intend to do is this. *Suppose* that an empiricist has triumphantly fought a metaphysical battle against the realist, and *suppose* that our empiricist is now challenged to make sense of *modal talk in sciences*, given that

[1] The intuitionist in question is my former teacher, once-colleague, and invaluable friend and mentor of more than a decade now, logician Mohammad Ardeshir, to whom this article is dedicated and this volume is presented. Having first encountered Ardeshir's name through logic when, as a high-school student, I discovered his mid-1980s translation of Ernest Nagel and James R. Newman's *Gödel's Proof*, with its gaudy yellow cover, I would have loved my contribution to this volume to be something in logic; as that didn't materialize, I chose to re-present a chapter of my doctoral dissertation. Embellished with a number of comments on the works of several logicians on relative necessity, I hope a minimal logical flavour can be tasted now.

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M. Mojtahedi et al. (eds.), *Mathematics, Logic, and their Philosophies*,
Logic, Epistemology, and the Unity of Science 49,
https://doi.org/10.1007/978-3-030-53654-1_15

she has supposedly cast serious doubts on the coherence of some modalities. This article is my attempt to meet the challenge in the realm of physics.

More specifically, I will defend a relativistic interpretation of physicists' talk of 'it is necessary that' and 'it is possible that', an interpretation which is both empiricist-friendly and captures the actual use of assertions such as 'it is impossible to move faster than light'. To repeat: it should be kept in mind that my discussion is *not* a metaphysical one—I am not concerned with the metaphysical status *of* physical statements or the feasibility of a reduction of physical necessity to logical or any other kind of necessity; my business is to give an account—a 'rational reconstruction', if you will—of the working physicists' modal talk *in* physics.

This article is organized as follows. After this introduction, I will (Sect. 15.1) formulate my thesis of relativistic reading of physical modalities as they occur in the everyday life of physicists, and I will elaborate on my notion of background theory to which I reduce the talk of physical modalities. I will then (Sect. 15.2) defend the thesis against a number of objections. Finally (Sect. 15.3, which is really an appendix), I will present a very brief history of relativizing modalities.

15.1 Relativizing to the Background Theory

What does a working physicist mean when she says that moving faster than light is impossible? Enquiring into what someone *means* by something might, in part, be a psychological enquiry which I cannot possibly undertake here; what I am really looking for is a plausible rational reconstruction of physicists' modal talk. I aim at providing a way of paraphrasing modal talk in physics, in such a way that paraphrasing a sentence results in a modality-free sentence which is of the same truth-value as the paraphrased one. It is my contention that modal claims in physics can always be understood as dependent on the background theory of the context of the utterance: in my rational reconstruction, scientific talk of physical possibility is talking of compatibility with the background theory, and scientific talk of physical necessity is talking of provability from the background theory (see the Thesis below). But first I have to clarify the notion of background theory which is in use here.

In modern mathematics today, we may almost always present portions of standard set theory as our background theory, but things may not always be that clear-cut in the case of physics. In any given context, by the **physical background theory** (or just the **physics**, for short) I mean all the explicitly formulated physical principles, plus all the assumptions about initial conditions, plus all the needed mathematics, that are used in arguments and derivations.[2]

[2]Thus what I call a 'background theory' is a set of *sentences*. I am not unaware of some philosophical debates on how to understand the notion of a scientific theory (as contested by those who favour the so-called semantic and syntactic views of theories), but such discussions are orthogonal to my concern here, which is understanding *modal talk in the sciences*, not *theorizing about the notion of a physical theory or physical modalities*.

Thus in a given context, our physical background theory—as I stipulatively define it—is the totality of whatever assumptions we use in that context. This may include some statements of common-sense intuitions, as well as “a good deal of unformulated general opinion”, as Monton and van Fraassen (2003, p. 410) put it in their discussion of counterfactuals.

The background theory may vary from context to context: some general statements, initial conditions, or particular facts might be part of the background theory in one context and not part of another. To get a clearer image of this notion of background theory, it might not be amiss to give a non-scientific example. Here is a case of a shift in context, which is based upon an example given by Russell (1905, p. 519). Consider these two events: (*A*) the awarding of the Fields Medal to Paul J. Cohen, August 1966, and (*B*) the death of L.E.J. Brouwer, December 1966. Was it necessary that *A* happened before *B*? I think we ordinarily answer this question in the negative: for aught we know about the relevant facts (including facts about the driver of the vehicle who ran over Brouwer in Bralicum), there is no significant connexion between *A* and *B*. Yet in a context wherein you and I both know about the times these events occurred, if we hear someone saying that such and such happened before Cohen won the medal but after Brouwer died, we may quite reasonably say that that is *impossible* because Cohen became a Fields Medalist before Brouwer's death. The point is that here we have a context wherein we take the particular fact into account that *A* occurred before *B*—this, together with many other facts such as the linear ordering of moments of time, is part of the background theory of this little conversation.

Having such a broad notion of background theory, my notions of provability and compatibility are strictly and classically logical: *p* is **provable** from *T* iff there is a proof of *p* from *T* in the standard sense of classical first-order logic, and *p* is **compatible** with *T* iff *p* is consistent with *T* in the same sense.

A realist may contend that, independent of what the consequences of our current theories are and independent of what is or is not logically compatible with our physical theories, moving faster than light is, *as a matter of (modal) fact*, either physically possible or physically impossible. Now whether or not there are objective, genuine modal facts about the world is an issue I do not deal with in this article. What I am arguing for is that what a scientist does when she examines a physical modal claim can be rationally reconstructed as her examination of the logical relationship between a corresponding non-modal claim and her (non-modal) background theory. Let me display my thesis before arguing for it and considering several objections to it.

THESIS. In any given context, physical modal statements can be understood as true or false relative to a background physical theory (in short: they can be understood relative to a physics). With respect to a physics *T*, the assertion of the physical possibility of φ can be understood—insofar as truth-conditions are concerned—as the assertion of φ's logical compatibility with *T*; so far as truth-conditions are concerned, the assertion of the physical necessity of φ can be understood as the assertion of the provability of φ from *T*.[3]

Normally, the reference to *T* is dropped when the context is clear. Also, here φ is assumed to be non-modal; if φ is itself modal, paraphrasing takes several steps.

The Thesis obviously satisfies a quite minimal requirement: in their paraphrased forms, to say that φ is physically necessary is, insofar as truth-value is concerned, equivalent to say that not-φ is not physically possible. Also, logical necessities (i.e., theorems of logic) come necessary relative to any background theory whatsoever, and I am happy with this.

The major objective of this article is to defend the above thesis (call it the **relativistic thesis**) against the realist's objections. I will concentrate on the case of necessity and possibility in physics, though I think the relativistic reading is a plausible rational reconstruction of modal talk in all sciences.[4]

Disclaimers. Note what the Thesis is not. It is *not* a thesis to the effect that modal talk in physics is nonsense; it is not a non-cognitivist thesis either: it actually presents truth-conditions for modal physical statements. Nor does it say that modal physical claims are useless or unrelated to the physical world. We certainly are interested in the logical consequences of our physical theories, because, obviously, if a theory is approximately true, then so are its consequences (even though the corresponding approximations may be different). If special relativity holds in the actual world, then so is its particular consequence that no object moves faster than light; thus, if we believe in special relativity, then we should believe that any attempt to move faster than light will be unsuccessful—at least, we should so believe insofar as we *know* that this is a logical consequence of that theory.

The Thesis does *not* say that physical modality is just a matter of what we believe. For every physics *T* and for every statement φ, either φ follows from *T* or not, and either φ is compatible with *T* or not; and *these facts* are independent of the way we think of them—they are even independent of the fact that anyone ever formulated *T* or thought of φ. All that the Thesis says is that statements of modalities can be understood derivatively: their truth-conditions, at least in physicists's parlance, are determined by logical relations between theories and sentences. The choice of the background theory is ours; yet, given any background theory, what is possible or necessary relative to that theory is independent of our will or wish. As a matter of mathematical fact, moving faster than light is incompatible with the theory of special

[3]One last time: the Thesis, and this article as a whole, is about modal *talk*, not about the nature of modalities themselves. I may occasionally write as if my topic is modality per se; yet all the way I mean *stating* modal facts, *asserting* them, *judging* them, etc. Please forgive me if I occasionally (appear to) slip into discussing modalities themselves.

[4]A version of the Thesis designed for de-modalizing mathematical discourse works pretty well, or so I claim. I cannot cover the issue here (see Chap. 4 of my dissertation), and at any rate I prefer to deal with the more challenging case of modal talk in physics. Wilfrid Hodges (2007) offers a detailed study of modal talk in mathematics.

relativity, and *this* fact was true even before Einstein, even if no intelligent creature ever lived in the universe.[5]

No presumption of truth. When a physicist says that such and such is physically necessary, the Thesis rationally reconstructs her as saying that such and such follows from her background theory T. In conversations, it might be understood here that the physicist's talk *presumes* her belief that T is true; but this need not be the case. In fact, perhaps every working physicist who has ever attended a lecture in philosophy of science, or has ever went to a real-life laboratory, believes that her own (or everybody else's) physical background assumptions are less than wholly and completely true. For all I am interested in here, the physicist may believe that T is (approximately) true, or just that T is empirically adequate in the sense championed by van Fraassen (van Fraassen 1980). And there are more options. Fixing T as the background theory, the physicist may believe that T is *false* or even empirically *inadequate*: for some reason, she might be interested to see how the world would look like if T were true or if T were empirically adequate. Or she may just give it as an exercise to her students to show that such and such follows from, say, a patently false assumption in Aristotelian physics, hence physically necessary relative to Aristotelian physics.

Arguing for the Thesis. In the next section, I will indirectly defend the Thesis via rejecting a number of objections to it. My only direct or positive argument for it is by looking at what physicists actually do when they assert that an actually false sentence φ is possible or that a sentence ψ is necessary. And it seems obvious that all they do is showing that φ is compatible with the background theory of the context, and showing that ψ is derivable from the background theory of the context.[6] I will say a bit more about this in 2.1 and 2.2 below.

15.2 Objections and Replies

Before considering a number of objections to the Thesis, let me examine a case which, though one may hear it presented as an objection, I think actually supports the relativistic reading of physical modalities. If the way I explain this particular case

[5] Ontological honesty requires me to be explicit here: I am a realist about the logical implication (in the classical sense).

[6] As for possibility statements, one may discern a weak sense and a strong one. The **weak** sense is when the physicist has not yet discovered any incompatibility between φ and the background theory. In the **strong** sense, which I gather to be possibility *simpliciter* (and not possibility *for-all-we-know*), the compatibility is demonstrated, normally via constructing a model. Thus one may say that, in the context of relativity-informed cosmology, the possibility of time travelling could be asserted in the weak sense before 1949, and was first demonstrated in the strong sense by Gödel (1949).

Incidentally, Hawking's (1990) introductory note to Gödel's paper illustrates my Thesis, when he writes of a solution provided by Gödel to certain Einstein field equations, that this solution "was the first to be discovered that had the curious property that in it it was *possible* to travel into the past … Gödel was the first to show that it was *not forbidden* by the Einstein equations" (1990, p. 189, my italics).

is sound, then my use of it is somehow ironic, because, in a different context (laws of nature) the kind of story described in this case is oftentimes suggested by a realist to argue *against an empiricist view*.

15.2.1 The Case of "Un-Actualized Physical Possibilities"

In a famous example, David Armstrong (1983, pp. 17–18) asks us to suppose that nowhere in the universe has there ever been a solid lump of gold with a volume greater than a cubic mile. (Let us call any such huge lump of gold a **Hugold**.) Suppose, moreover, that there will never be any Hugold at any place in the future. Nevertheless, says Armstrong, the existence of a Hugold is not a physical impossibility, as opposed to the existence of a piece of uranium-235 of the same size (a **Huranium**), which *is* a physical impossibility, as critical-mass considerations show. There is a manifold of such examples in the literature on laws of nature.

So, let us assume that nowhere at any moment in the whole history of the universe—past, present, and future—is there either a Hugold or a Huranium. Still, a difference might be felt: in principle, if we cared to (and if we had enough gold), we *could* make a Hugold; on the contrary, no matter how hard we try and how much uranium we possess, there really *cannot* be a Huranium.

I think the relativistic reading of physical modalities nicely explains the feeling. The existence of a huge lump of ^{235}U is ruled out by our accepted physics: critical-mass considerations, which are incorporated in our physics, are incompatible with the existence of a Huranium and we know this; hence the corresponding impossibility judgement. On the other hand, so far as we are aware of the consequences of our physics, no such considerations are applicable to a huge lump of gold—the existence of a Hugold is, to the best of our knowledge, compatible with our physical principles; hence the corresponding possibility judgement. If we believe our physical principles to be true, then we have good reason to think that there will be no Huraniums, no matter what; we do not have such a reason in case of Hugolds.

The realist's intuition is that even if we know that there is no Hugold in the whole history of the universe, there still *could* be one. However, it is not clear to me how we can make scientific sense of this 'could', if it is not to be understood relativistically. Our reason for asserting the physical impossibility of the existence of a Huranium is grounded in critical-mass considerations, which are parts of our physics. We cannot hold both (1) critical-mass considerations, and (2) the statement that a Huranium exists at some point in the history of the universe, for we know that they are incompatible. As we have good reasons to keep (1), we reject (2). But if this is the way we discover, or argue for, *im*possibility claims (and there seems to be no other way), it is odd to think that for possibility claims we should seek something over and above compatibility with the background theory.

The realist may admit that the way physicists *prove* or *discover* physical modalities is via investigating logical relations between statements and theories, but add that this does not show that, e.g., *to be physically necessary* is to be deducible from the background theory. Here I am not denying that there might be irreducible physical modalities; my point is that even if there are such things, *scientists* do not deal with them as such. In rationally reconstructing modal talk in science we need not talk about irreducible modalities, even if there are such things as irreducible modalities.

One move the realist might try at this point is to accept the conclusion of the theory-dependence of physical modalities but insist that some of the principles of our physics (or some laws of nature) are irreducibly modal, and some irreducible modality is thereby inherited by physical statements like the impossibility of the existence of a Huranium. However, again, I do not deny that there might be (irreducibly) modal features of the physical world; but it seems straightforward to argue that even if there are such features, the scientist cannot discover them and include them in his background physics.[7] Hence, if my argument is sound, then irreducible physical modalities are scientifically irrelevant, and irreducible modalities cannot be found in the principles of our physics either. Apart from that, below in my discussion of actual physicists' use I will explain away the appearance of modalities in some of the principles of physics.

15.2.2 Objection: The Open-Minded Physicist

Suppose a physicist—call him **Ramin**—says that moving faster than light is physically impossible. According to the relativistic thesis, this is, so far as truth-conditions are concerned, nothing but saying that it is a theorem of Ramin's background physics that no moving object moves faster than light. Let us also assume that Ramin has recently checked the details of the relevant argument again, and he is absolutely certain about its validity: he knows it for a (logical) fact that his physics rules out speeds greater than c. Now suppose that today he hears news about a recent achievement of a joint group of physicists and engineers in Berlin with respect to faster-than-light rockets. The source having been reliable in the past and the Berliners being world-class scientists and rocket experts, Ramin takes the news seriously—he seriously considers this: moving faster than light is possible. But, according to the Thesis (the objection concludes), he just can't: as we stipulated, he still believes that moving faster than light is incompatible with his physics, hence a physical impossibility. So there is more to a possibility claim than just consistency with the background theory.

In reply, I think one should distinguish two cases: (α) Ramin hears that the Berliners have actually *observed* the phenomenon he considers impossible [allow me not to be worried about how one could observe that!], or (β) he hears that they have just *theoretically proved* that moving faster than light is possible. As the story has it, in both cases Ramin takes the news seriously; but the implications for the relativistic thesis might be different. For suppose that the Berliners had just claimed that moving faster than light was possible (β), without claiming that they had observed—or had brought into existence—an instance of it. Then it seems clear what Ramin would

[7]This is what I think I have done in Chap. 5 of my dissertation (2008).

do. He would ask for their argument and he would peruse it. Given that he is certain of the correctness of his own proof of the incompatibility between his background theory and the statement that something moves faster than light, he would enquire into the Berliners' background theory. Perhaps their physics does not include all of his principles or particular assumptions? If that doesn't explain the tension, he will try to find mistakes in their argument for the possibility claim. If none of these settle the disagreement, he might think that perhaps a subset of the union of his and the Berliners' physics is inconsistent.[8] Another option, still further from the "edge of the system" (as Quine would put it), is to blame mathematics and logic. But all these are questions of what follows, or does not follow, from principles and extra assumptions. Hence if in the story the Berliners are just said to have purely theoretically argued for the possibility, then this is no threat to the relativistic thesis. It seems that whatever the ontological status of physical modalities might be, theoretical *arguments* about impossibilities and non-actual possibilities are just arguments about incompatibility and compatibility with our background principles.

Now let us assume, as per (α), that Ramin, our physicist, thinks that a statement φ is physically impossible, and he takes it seriously that φ has actually been *observed* to be the case. What is going on here? As the objection stipulates, Ramin still thinks that, as a matter of logical fact, φ is incompatible with his background theory T. If T were a correct description of what is the case in the world, φ would not be the case; now that he has good reason to think that φ is the case, he has good reason to think that part of T is false. If it really turns out to be the case that φ has been observed to be true, then our rational physicist will say that his T is not true. However, he still retains his belief that φ is ruled out by T—despite the falsification of T, *this* fact remains true (though, of course, it loses much of its importance). And if, after the rejection of T, Ramin now wonders about the possibility of another statement ψ as part of a significant research programme and not just as an exercise in theoretical physics, we have a change of context, a change of background theory: he is now thinking of the compatibility, or lack thereof, of ψ with a different, perhaps yet to be developed, physics T_1.

Perhaps the point of the objection is that just looking through the logical consequences of theories is not a good way of finding real possibilities: if things happen to be as in the objector's α-story, then moving faster than light is *really possible*, no matter what consequences of our physics are. However, it should be clear that a *tu-quoque* reply to the realist is available here: the realist himself has no way other than a compatibility argument to say that a *non-actual* φ is physically possible. And as for actual φs, there is no disagreement about the fact that they are possible.

[8]It is *not* a well-kept secret that not only there are pairs of mutually inconsistent physical theories, but there are also physical theories which are inconsistent in themselves (see, for instance, Monton (2011) and Costa and Vickers (2002)). Hence a background theory, as I have defined the notion, may be inconsistent in more than one way. Whenever the background theory of a context is inconsistent, my account allows assertions of the form 'it is necessary that φ' *and* 'it is necessary that not-φ' relative to that theory. I find this in no way worse than the inconsistency of the background theory itself.

15.2.3 Objection: Practical Versus in-Principle

Consider Armstrong's case again. It is *in principle* impossible to fabricate a huge lump of uranium-235, as we know from critical-mass considerations; whereas for a huge lump of gold, there is no such in-principle impossibility. But suppose we know that there is not enough gold in the universe to make a lump of gold of the specified size, and suppose that we augment our background theory with this particular fact. Now the relativistic thesis would announce that the existence of a Hugold can be asserted as a physical impossibility with respect to this physics. But surely (the realist objects) there is no "deep" reason for this—the impossibility of a Hugold would be just *practical*. There is an intuitive distinction between the in-principle and the practical impossibility; but the relativistic thesis is too coarse to make this distinction—if enough particular facts are included in a background physics, then the practical versus in-principle distinction is lost. So says the objection.

Concerning impossibilities, the realist has an intuition about the difference between the "practical" and the "in-principle", a difference which is presumably a difference of kind. But how can he, the realist himself, demarcate the two? One way might be to say that an in-principle impossibility is one that is ruled out by laws of nature (whatever they are), while a practical impossibility is one which is ruled out by laws of nature plus some other true assumptions (e.g., assumptions about how much gold we have). But if *this* is thought to be what distinguishes the two notions of physical impossibility, how can we ever know that a logically possible situation is an in-principle physical impossibility? Well, the realist might say: if we know the laws of nature, then we know the in-principle impossibilities.

However, I reply, even waiving worries about the antecedent of this conditional,[9] the problem is that now the realist's account of impossibility is really not different from the relativistic account. Certainly the set of verbalized laws of nature, should there be such a set, is an excellent candidate for a background physics, and, with respect to this background theory, the relativistic physical impossibilities are the same as the realist's in-principle impossibilities. *If* we know the laws of nature, then we can measure impossibilities against them and reserve the unqualified 'impossible', or the qualified 'impossible in principle', for whatever that is excluded by laws of nature; and if a statement is ruled out only by laws of natures plus some particular facts, we may call it 'merely practically impossible'. But now the two accounts—i.e., the realist and the relativistic—are really not different in this case, and the realist has admitted that, with respect to truth-conditions of impossibility claims, modality is a matter of a logical relationship of a statement with the background theory—only he thinks that the background theory is something very special. But even in this case there are no irreducible modalities. (Again, the realist might think that some laws of nature are themselves modal. Here the realist owes us some examples. I will consider two putative cases of such laws below; and in a longer version of this article, I would argue that even if there are modal features of the world, presumably described by some

[9] See the next footnote for an elaboration of a closely related point.

genuinely modal laws of nature, such features cannot be discovered scientifically. A defence of such a claim can be found in Chap. 5 of my (2008).)

So I think the realist owes us an explanation of his intuition, an account of the in-principle physical impossibility. Not that if he cannot offer a good explanation we *have to* quine the intuition; but here the intuition, which is perhaps not backed by a good theory, need not be shared by the empiricist—to appeal to an intuition about the difference between the in-principle and the practical is perhaps begging the question against the empiricist.[10]

But perhaps one can explain the difference between in-principle and practical impossibilities in an empiricist-friendly way, without appealing to irreducible modalities. I think the difference is not a difference in kind, and I think determining it is a pragmatic issue. Quine (1951) argues that nothing in science is absolutely immune to revision. With no claim of having a worked-out theory of this, I want to suggest that perhaps the in-principle versus practical distinction is just a matter of degree: the more central and the less susceptible to revision a background theory T is, the more of an in-principle character the T-impossible statements are. If it is not that hard to revise a background T, if the costs of such a revision are not very high, then we think of what is ruled out by T as not *in-principle* impossible. Thus revising Einstein's special relativity, or totally setting it aside, will be a great change in science; so moving faster than light is considered to be an in-principle impossibility. On the contrary, it will not be a big deal if we realize that our estimation of the total amount of gold in the universe was mistaken; hence the merely *practical* impossibility of a Hugold (given that it is part of our physics that there is not enough gold to build a Hugold).

15.2.4 Objection: Physicists' Use of Modalities

Let me distance myself for a while from speculation and armchair philosophy, and look at the actual working physicists. Here is an objection. If the Thesis is true, then, in a discussion in physics and insofar as truth-conditions are concerned, to say that φ is necessary is just to say that φ follows from the background theory of the discussion.

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[10] Another thing that the realist owes us here is an argument to the effect that the practical versus in-principle distinction is *scientifically* significant. If the distinction is supposed to be based on the notion of laws of nature, then we should note that there are reasons to think that it is scientifically irrelevant—as Nagel puts it (1961, p. 49), "The label 'law of nature' (or similar labels such as 'scientific law,' 'natural law,' or simply 'law') is not a technical term defined in any empirical science; and it is often used, especially in common discourse, with a strong honorific intent but without a precise import." Chap. 8 of Mumford (2004) presents a defence of Nagel's view.

While on the topic of laws of nature, let me quickly add that one may suspect that to the extent that physicists talk about laws of nature, they are at least tacitly speaking a modal language, for, presumably, one characteristic feature of laws of nature is their counterfactual support, which is presumably a modal notion. In my (2011), I offer an empiricist-friendly account of the the issue of counterfactual support.

Therefore, in particular, to say of a principle of physics[11] that *it* is necessary, is to add nothing to the mere assertion of that principle. Given that physicists are no dumbs and at least some of them have at least an implicit and intuitive knowledge of the Thesis if it is true, then the occurrence of the necessity operator in the statement of a principle looks odd, for it adds nothing to the content of the principle. Yet we have several cases of modal talk in physics texts, *specially* in the expositions of principles. Therefore somethings is wrong with the Thesis. [End of the objection.]

It is undeniable that oftentimes we see modal statements in physics texts. For example, a standard undergraduate textbook states Newton's first law in a modal language—this is from Halliday et al. (2005, p. 88, my italics):

> If no force acts on a body, the body's velocity *cannot* change; that is, the body *cannot* accelerate.

However, there are always—so I claim—modality-free formulations of a modally formulated scientific proposition. Regarding Newton's second law for instance, here is the required version, formulated by Isaac Newton (1726, p. 416, my non-italics):

> Every body preserves in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by forces impressed.

I take it for granted that the authors of the two versions are expressing the same law.

Here is another example, the second law of thermodynamics. In this case the original formulations are modal. I quote from Bailyn (1994, p. 88), to which I add emphasis. Thus spoke Rudolf Clausius in 1850:

> No process is *possible* whose sole effect is to transfer heat from a cold body to a hot body. By sole effect is meant without the rest of the universe changing, or changing in a cycle of operations.

And William Thomson (Lord Kelvin of Largs), 1851:

> It is *impossible* by means of inanimate material agency to derive mechanical effect from any portion of matter by cooling it below the temperature of the coldest of the surrounding objects.

Now it is well known that the second law of thermodynamics admits of many equivalent formulations. The following is from Baierlein (1999, p. 29, my italics):

> If a system with many molecules is permitted to change, then—with overwhelming probability—the system *will evolve* to the macrostate of largest multiplicity and will subsequently remain in that macrostate. Stipulation: allow the system to evolve in isolation. (The stipulation includes the injunction, do not transfer energy to or from the system.)

A quick review of the concepts involved here might be in order, to make sure that Baierlein's formulation is not modal. As one would expect, a **macrostate**—an abbreviation for 'macroscopic state of affairs'—is a state described by "a few

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[11] A principle, properly so called—that is to say, an *axiom* of the background theory in my sense (which need not be what, in the previous footnote, I reported Nagel's qualms about).

gross, large-scale properties", such as pressure, volume, temperature, and total mass (p. 27). A **microstate** is one described by "specifying in great detail the location and momentum of each molecule and atom" (p. 25). The **multiplicity** of a macrostate is the number of microstates that correctly describe it. Thus (Baierlein's example, p. 27) suppose there are four balls—call them A, B, C, D—and two bowls. The macrostate *all balls are in the left-hand bowl* has minimum multiplicity, viz. 1 (the location of balls within a bowl doesn't matter). The macrostate *the balls are evenly distributed in the two bowls* has the largest multiplicity, viz. 6: each of the microstates (AB, CD), (AC, BD), (AD, CB), (BC, AD), (BD, AC), and (CD, AB) corresponds to it. Finally, though Baierlein does not explicitly define it, it is clear that his notion of the probability of an event is the familiar, purely combinatorial one—thus (p. 26) if you toss a "fair" coin a million times, the probability that the number of heads is within 1 per cent of 500,000 is $1 - 2.7 \times 10^{-23}$, which is "overwhelmingly" close to 1. There is no modal notion here.

Or, to put the second law more succinctly, let us talk about **entropy**, which is basically defined as the logarithm of multiplicity. Now, "for all practical purposes, the one-line version of the Second Law is this: An isolated macroscopic system will evolve to the macrostate of largest entropy and will then remain there" (p. 46). Again, no modality is involved.

It may be instructive to examine a large number of physics texts and try to find non-modal versions of the statements that are occasionally formulated in a modal language. It may also be interesting to see if there is a correlation between the presence of modal discourse, or lack thereof, in a physics text on the one hand, and the extent to which its author is considered a rigorous author, on the other.[12] This, however, is not what I wanted to do here. I hope I have provided enough empirical data (about what one can find in physics texts) to confirm my a posteriori claim that for each physical statement one finds in a physics text, there is a non-modal version of it. This is of course not unexpected, given that we see no modal operators in the formulas we find in physics texts.

The advocate of genuine, irreducible physical modalities may object, echoing Bressan (1974, p. 299), that "the use of modalities (possibility concepts) and the use of a modal language are not equivalent". I agree. Yet, if the formulas of physics texts do not contain boxes and diamonds, then I think it is incumbent on the realist to argue that, *nevertheless*, physics deals with irreducible modalities.

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[12]The case of an elementary text in philosophy might be of some interest. While explaining the difference between necessity and certainty, Elliott Sober gives the example of the second law of thermodynamics: "Finally, in the nineteenth century, physicists working in the area called thermodynamics proved that perpetual motion machines are *impossible*" (2005, p. 49, my italics). In his next chapter, Sober argues against the creationists' confused application of the second law: "They claim that this law makes it impossible for order to arise from disorder by natural process" (p. 62). And what is the second law, again? Now that precision matters more, his formulation is basically the modality-free one I quoted from Baierlein: Sober writes, "What the Second Law actually says is that a closed system *will (with high probability) move from states of greater order to states of lesser order*" (ibid., with change of emphasis).

Also note that for the Thesis to be a good rational reconstruction of physical modal talk, it need not be the case that for every modally formulated physical statement there is already a textbook non-modal version. The Thesis is not a sociological claim about the way physicists actually talk (though I think I have provided some evidence that it is not alien to actual physicists' talk); rather, the Thesis presents a way to make sense of their modal talk. Even if I had failed to find Baierlein's formulation of the second law of thermodynamics, I still could offer a routine reformulation of Clausius's: *there is no process, nor will there be one, whose sole effect is to transfer heat from a cold body to a hot body.*

15.2.5 The Failure of Factivity

Following Hale and Leech (2017, p. 4), let us say a necessity operator $\Box_R$ is **factive** iff for every formula φ, the conditional $\Box_R\, \varphi \rightarrow \varphi$ is true in the actual world.[13] Given that our background theories are not supposed to be true (see the last paragraph of Sect. 15.1), it follows that there is no guarantee that whatever is necessary relative to a background physics is actually true. This may raise eyebrows: is necessity not supposed to be stronger than truth? Though I have been emphatic that I am concerned only with physicists' modal *talk* and not with physical modalities per se, the rhetorical question still has some force—wouldn't it be odd for someone, in a given context, to assert the necessity of φ without also committing herself to the truth of φ?

There are a couple of things I may say in reply. First, from a merely formal point of view, the failure of the law $\Box\varphi \rightarrow \varphi$ is, in itself, not a defect of any modal system. This is evidenced by one of the most famous and well-studied modal systems, the Gödel-Löb provability logic GL. In fact, a simple theorem has it that *no* instance of $\Box\varphi \rightarrow \varphi$ is a theorem of GL unless its subsequent is a tautology.[14]

Secondly, take a physicist who says that a certain φ is necessary. My thesis paraphrases her assertion as asserting that φ is a theorem of the background theory T of her context. Now consider two levels. *Level 1* is the reporter's third-person point of view—say Galileo is reporting what is considered necessary in an Aristotelean physics. Surely here the report need not (in fact: must not) be factive—Galileo says that the Aristotelean physicist thinks that heavier objects fall faster by necessity, while *this* is not, in Galileo's point of view, true. *Level 2* is the physicist's first-person report, and here, too, the account need not be factive. According to van Fraassen's constructive empiricism (1980), to which I am very sympathetic indeed on other grounds, a scientist's acceptance of a theory does not require her to think of the theory as *true*—it suffices that she thinks of it as empirically adequate. However, as I understand that

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[13] Hale and Leech's exact wording is "[…] factive, in the sense that, where $\Box_C$ is our relative necessity operator, $\Box_C\, p \rightarrow p$, for every p. In other words, the characteristic axiom of the quite weak modal logic T holds for $\Box_C$."

[14] Boolos (1993:13). Even some instances of the weaker axiom $\Box\varphi \rightarrow \Diamond\varphi$ are not provable in GL.

not many philosophers are very fond of constructive empiricism, let me put forward the following idea:

Thirdly, I am of the opinion that normally, or at least ideally, scientists do *not* take their theories to be true—or at least, they are not openly dogmatic enough to announce their theories to be true. The theories are, in my opinion, merely hypotheses, and one usual way of testing a theory is to put on trial what is necessitated by it. Thus even one hundred years after the advent of relativity, physicists still make experiments to see whether what *must* be the case if relativity holds, is in fact true. Another aspect of not believing the background theory to be true is manifested in an old practice of logicians—a practice continued well through 1940s—to have a theorem whose official statement would begin by "**Theorem (AC)**. *Suppose*…", with the bracketed acronym making it explicit that the Axiom of Choice was exploited in the proof of the theorem. Strictly speaking, AC was part of the background theory here, while our wary logicians used to make it explicit that they are not committed to its truth.

Finally, let me use a picturesque language (which I think is very apt to engender deep confusions) to justify the lack of factivity. Consider a background theory T to which the modal talk of our physicist is relativized. Now we may think of $\Box_T \varphi$ as saying that φ is true in all possible worlds wherein T holds. If that is the picture, then why—or *why on earth*!—should we expect that if φ is true in all *those* worlds then φ is true in the actual world? You may, for whatever reason, think that one particular type of necessity is not relativisable to another,[15] and you may criticize a given relativisation for just that reason; yet to bring lack of factivity as a further criticism of a given relativisation is neither fair nor even relevant, in my view.

Appendix to 2.4.: why a simple remedy does not work

She who thinks her account of physical necessity should be factive, might try the obvious and re-formulate the Thesis by saying that a physicist's assertion that 'φ is necessary' is replaceable, salva veritate, by the assertion that φ *is true and* is a theorem of the background physics T. As a definition of relative necessity, Hale and Leech (2017, Sect. 6.1.) criticize this by reference to some technical work of Lloyd Humberstone's. Yet I think a simpler criticism is available. A formulation of the obvious remedy is presented (but not endorsed) in a less technical piece by Hale (2017, p. 808):

> Clearly, whenever we have a more or less definite body of propositions constituting a discipline D, there can be introduced a relative notion of necessity—expressed by 'It is D-ly necessary that'—according to which a proposition will be D-ly necessary just in case it is true and a consequence of D.

So let us consider a view according to which

(H) the sentence φ is T-ly necessary iff φ is true and φ is a (logical) consequence of T.

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[15] Thus Kit Fine (2002) holds that metaphysical, natural, and moral necessities are mutually irrelativisable, and that none of them can be relativized to any other type of necessity. I need not take positions on Fine's view here, for I am merely theorizing about physicists' *talk* of natural necessity, not natural necessity itself.

Given that the possibility of φ is logically equivalent to the non-necessity of not-φ, it follows that the advocate of the above account must be committed to

(H*) the sentence φ is T-ly possible iff either φ is true or φ is compatible with T.

Now let T be a theory with at least one false axiom φ_1. Then not-φ_1 is true, hence, by (H*), T-ly possible. Therefore, the above account is committed to saying that the very negation of each false axiom of T is *possible relative to* T. This I find very odd indeed. Thus let P be a theory an axiom of which is that the Earth is the centre of the universe. This axiom is false, and (H*) implies that the proposition that the Earth is *not* the centre of the universe is P-ly possible. Which is bizarre.

15.2.6 Objection: All Laws Are Necessary?

Suppose the background theory is the Newtonian physics, which includes the principle (the second law of motion) that $F = ma$. Let us put the following question to our physicist: *Is it possible that* $F \neq ma$? And here is an objection: whereas any reasonable physicist would answer affirmatively (since, conceivably, F could be equal to $13ma$, or there could have been no non-trivial connexion between force and acceleration), according to Thesis the answer is negative. Therefore the Thesis is false. A reply is in order.

First, let me note (rather nitpickingly), that the Thesis, as I stated it, is a rational reconstruction of physicists' modal talk, a translation of their modal talk into a non-modal language, *not* vice versa. When a physicist says something which contains 'possibly' or the like, the Thesis rationally reconstructs her as expressing a non-modal proposition. I am in no way committed to saying that the physicist should, or is in fact inclined to, go the reverse direction and put a box before the statement of whatever she infers from her physics, or a diamond before the statement of whatever she knows that is consistent with her physics. Indeed, I think physicists—*qua* physicist—are not very fond of modal language when they write technical papers or give technical talks.[16]

But this is just quibbling. For suppose *we ask* our physicist whether Newton's second law is necessary, and suppose that she, as expected by the common sense, answers that it is not. Then, according to the Thesis, our physicist is saying that the second law is not a theorem of her physics, and *this* is patently false. Nor is it reasonable to say that we have a change of context—it seems that, even with the Newtonian physics in the background, she is just saying that, intuitively speaking, the world could have been otherwise, that F could have been unequal to ma. A better defence is required.

Secondly, there are philosophers of realist persuasion who maintain that true laws of nature—whatever these laws are, and whether or not Newton's second law be

[16]The case of popular, *Scientific American* type of physics talk needs a separate discussion.

among them—*are* in fact metaphysically necessary. But this is a substantial philosophical view, which should not come out as the result of a conceptual analysis of the notion of physical necessity.

Thirdly, let us delve more deeply into the problem. I ask my physicist friend if it is possible that $F \neq ma$. Apparently contrary to the Thesis, she says that it is. What is going on here?

Qua physicist, she is being asked whether it is *physically* possible that $F \neq ma$, and the correct answer to the literal question is 'no'. (Recall that we are assuming that she is a Newtonian physicist.) However, since the verbatim question put to her is truly trivial if she knows that I know her to be Newtonian, she thinks that I must have something else—something "deeper"—in mind, something with a metaphysical import, namely 'Could it be the case that the world is governed by a different law (or by no laws) concerning the relationship between force and acceleration?', to which, following common sense, she answers 'yes'.[17]

The Thesis is about physical necessity, while the objection has some force only to the extent that it concerns *metaphysical* necessity.

15.3 Ways of Relativizing Modalities

15.3.1 À la Montague et Anderson

So far as I know, the idea of relativizing modalities goes back to Anderson (1958) and Montague (1960). Smiley (1963) is perhaps the earliest in-detail logical analysis of the idea, for an old review of which I suggest Anderson (1967). In Montague's paper, first presented in 1955, the main ideas "are based on the following unoriginal considerations" (1960, p. 71):

> Let Φ be a sentence. Then ***it is logically necessary that Φ*** is true if and only if Φ is a theorem of logic; ***it is physically necessary that Φ*** is true if and only if Φ is deducible from a certain class of physical laws which is specified in advance; ***it is obligatory that Φ*** is true if and only if Φ is deducible from a certain class of ethical laws which is again specified in advance.[18]

Now I have not counted citations, but I am under the impression that Montague's paper is not referred to very frequently in the literature. What is very influential

[17]She might have added that that is not, strictly speaking, a scientific question—rather, it belongs either to theology (asking what reason was behind the creator's actualizing *this* world, not another one), or else it is a meaningless question.

[18]Due to my TeXnical disability, I have replaced Quine corners of the original with boldface italics.
Montague does not attribute the idea to anyone in particular. This is similar to the case of Boolos, who wrote a book (1993) on the interpretation of 'necessarily' as provability from a certain background theory (portions of Peano Arithmetic). Scrupulous as he is about giving credit for ideas, he does not tell us where the main idea came from. It seems safe to say that the idea is just part of the logico-philosophical folklore.

(though not very easy to read) is van Fraassen's (1977) which opens with something about physical necessity:

> Are there necessities in nature? The nominalists, and subsequently the empiricists, answered that all necessities are reducible to logical necessity [...]. What is physically necessary is the same, on this view, as what is logically implied by some tacit antecedent—say, the laws of physics.

I gather that the idea expressed by Montague and van Fraassen is just what I incorporated in the Thesis, specially because neither Montague's Φ nor van Fraassen's "tacit antecedent" is supposed to be true.[19]

Given that the idea of reducing the physical necessity to the logical one is not original with Montague or van Fraassen, I think their contributions were intended to solve some techno-logical problems in *implementing* the basic idea, not to defend the idea that necessity can be understood as provability from a background theory. Likewise, if there is any novelty in my attempts above, it consists in working out this familiar idea in the realm of the philosophy of science and scientists' practice, and defending it against some realistic intuitions—plus, of course, arguing that the account need not be factive. Also, as explicitly mentioned by Montague (1960, p. 71), his meta-theory contains no modality; while this can simply be assumed for a constructed formal language, in a scientific context one has to *argue* that there is no modality in the background theory, which I would have done had I had more space available here.

15.3.2 *À la Hale-Leech*

However, this is not the only way of relativizing modalities in the literature. Hale and Leech (2017), which is a detailed analysis of the notion of relative necessity partly drawn on the critical and constructive works of Lloyd Humberstone, culminates in a reformulation of the idea in the following sophisticated way (2017, p. 22), where the un-indexed box denotes logical necessity

$$\Box_{\Phi} A =_{df.} \exists q_1 \ldots \exists q_n (\Phi(q_1) \wedge \ldots \wedge \Phi(q_n) \wedge \Box(q_1 \wedge \ldots \wedge q_n \rightarrow A)).$$

The special case of physical necessity is discussed on pages 12–14 of their article, where on page 14 we have

> It is physically necessary that p iff $\exists q(\varphi(q) \wedge \Box(q \rightarrow p))$,

[19] I acknowledge that the word 'law' (which occurs in both passages) may, to some ears, mean something which has a certain recognized status *and is true*. In the case of physical necessity, the assumption of truth is explicit in Hale and Leech (2017, p. 13), though somehow buried in the middle of a page. Not a native English speaker, I myself do not hear any sense of truth in 'law', but that might be a moot point. Also note that, obviously, truth cannot be assumed in non-alethic laws such as the normative ones mentioned by Montague.

where φ(q) abbreviates the statement that q is a law of nature. (There should be no worries that here p is said to be logically implied by a *single* law of nature: take the conjunction of those laws which are required in the deduction of p, and note (2017, p. 13*n*20) that any conjunction of the laws of nature may be reasonably thought to be a law itself.)

I cannot do justice to all they say in their elegant paper, and I will close with (re-) expressing a qualm about their formulation. Hale and Leech assume that laws of physics are true,[20] and I am not happy with this. My reason, again, is that I think it is quite natural to relativize to false theories. Thus, I find it quite natural to say that such and such is necessary relative to Aristotelean physics—or relative to Ptolemy's cosmology, or Christian theology, etc.

To summarize, I have presented and defended a way of paraphrasing physicists' modal talk which gets rid of modalities. (Salient in my replies to the objections was that my account of a physicist's modal talk need not be factive.) If my argument goes through, we have some reason to think that modalities are not indispensable to sciences.[21]

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[20] Thus "being true is necessary, but not sufficient, for being a law of physics", and "we are taking φ to be factive" on pages 13 and 16 respectively. In general, they do *not* require instances of Φ to be true.

[21] This article is, for the most part, based on Chap. 5 of my dissertation (2008), supervised by Anjan Chakravartty. For the present version, I am grateful to Mojtaba Mojtahedi and Sajed Tayebi.

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Chapter 16
On Descriptional Propositions in Ibn Sīnā: Elements for a Logical Analysis

Shahid Rahman and Mohammad Saleh Zarepour

Abstract Employing Constructive Type Theory (CTT), we provide a logical analysis of Ibn Sīnā's descriptional propositions. Compared to its rivals, our analysis is more faithful to the grammatical subject-predicate structure of propositions and can better reflect the morphological features of the verbs (and descriptions) that extend time to intervals (or spans of times). We also study briefly the logical structure of some fallacious inferences that are discussed by Ibn Sīnā. The CTT-framework makes the fallacious nature of these inferences apparent.

Keywords Ibn sīnā (Avicenna) · Modal syllogistic · Descriptional (*waṣfī*) propositions · Substantial (*ḏātī*) propositions · Constructive type theory · Temporal logic · Logical fallacies

16.1 Introduction

In his discussions of the various readings of modal propositions, Ibn Sīnā's focus is mostly on a distinction which was later labelled the distinction between descriptional (*waṣfī*) and substantial (*ḏātī*) readings of a modal proposition.[1] Given that for Ibn Sīnā all categorical propositions are either implicitly or explicitly modal, the substantial–

[1] Hasnawi and Hodges (2017, p. 61) have correctly pointed out that 'substantial' is not Ibn Sīnā's own term. Indeed, as Strobino and Thom (2017, p. 345) have mentioned, it is only in the later stage of the tradition of Arabic logic that the terminology of 'substantial' and 'descriptional' became mainstream. Some of the other names which have been employed to refer to the distinction under discussion will be mentioned later in the chapter.

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M. Mojtahedi et al. (eds.), *Mathematics, Logic, and their Philosophies*,
Logic, Epistemology, and the Unity of Science 49,
https://doi.org/10.1007/978-3-030-53654-1_16

descriptional distinction is in some sense applicable to the readings of all categorical propositions.[2] This distinction is based on how (i.e., under which conditions) the predicate of a categorical proposition is true of its subject. According to the substantial reading, the predicate is true of the subject (perhaps with a certain alethic or temporal modality) as long as the substance of the subject exists. On the other hand, according to the descriptional reading, the predicate is true of the subject (again, perhaps with a certain modality) as long as the substance of the subject is truly described by the subject. To be clearer, consider the following proposition:

(1) Every *S* is *P*.[3]

The difference between the substantial and descriptional readings of (1) can be articulated as follows:

Substantial Reading of (1): Every *S*, as long as it exists, is *P*.
Descriptional Reading of (1): Every *S*, as long as it is *S*, is *P*.

It is in principle possible that a proposition is true on one of these readings and false on the other. It is only the context which determines how a proposition must be read to be true.[4] To give an example, consider the following proposition:

(2) Every bachelor is unmarried.

The substantial and descriptional readings of (2) are respectively as follows:

(3) Every bachelor, as long as he exists, is unmarried.
(4) Every bachelor, as long as he is bachelor, is unmarried.

These two propositions have different truth values. Contrary to (3)—which is false—(4) is true. This is because a bachelor is unmarried only insofar as he is described as a bachelor. So (4) is true. By contrast, it is in principle possible for a person who is a bachelor in some period(s) of time to be married in some other period(s) of time; this is so at least if we assume that 'as long as' has a temporal meaning. In other words, it is not necessary for such a person to be always unmarried. The mere existence of the substance of this person does not guarantee his being unmarried. Thus (3) is false. There are, however, other propositions that are true on the substantial reading. For example, consider the following proposition:

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[2] Street (2002, Sect. 1.1) and Strobino and Thom (2017, Sect. 14.2.1) emphasize that for Ibn Sīnā all propositions have either temporal or alethic modality. Absolute propositions are implicitly modal and all other propositions are explicitly modal. Lagerlund (2009, p. 233) highlights that even the absolute propositions can be taken to be descriptional.

[3] Strictly speaking, there is an important difference between a *sentence* and the *proposition* expressed by it. Accordingly, it is a sentence (rather than a proposition) which can be read in different ways. So what a substantial (respectively, descriptional) reading of a sentence expresses is a substantial (respectively, descriptional) proposition. Nonetheless, such a clear difference between sentence and proposition cannot be detected either in Ibn Sīnā's own discussion of the substantial–descriptional distinction or in the secondary literature on this issue. So to remain more focused on the main points we would like to make—and of course for the sake of simplicity—we do not make the sentence–proposition distinction bold.

[4] See Hodges and Johnston (2017, p. 1057).

(5) Every human is animal.

The predicate *Animal* is true of every human as long as s/he exists. Put otherwise, what makes it true to say that every human is animal is the mere existence of human substances. This means that not only the descriptional but also the substantial reading of (5) is true.[5] Indeed, since every human exists if and only if s/he is human, the substantial and descriptional readings of (5) express one and the same fact.

As Ibn Sīnā himself insists, he is the first logician to have focused on the above distinction and pondered on its fruitfulness for removing some difficulties with Aristotle's syllogistic.[6] Since the distinction plays a crucial role in Ibn Sīnā's syllogistic, it is discussed in several places in his logical oeuvre.[7] Moreover, the distinction was subject to continually heated discussions in Arabic logic after Ibn Sīnā. For instance, the distinction was accepted by Rāzī and Khūnağī, on the one hand, and was seen as redundant by Ibn Rushd.[8] The substantial sense of propositions corresponds to what is called the 'divided' sense of propositions in the Latin tradition. However, although the descriptional sense of propositions plays an important role in Arabic syllogistic, it has no widely discussed counterpart in the Latin tradition.[9] These observations strongly suggest that a comprehensive picture of Arabic syllogistic from Ibn Sīnā onwards cannot be achieved unless we have a clear logical analysis of the aforementioned distinction. An effective and popular strategy for providing such an analysis is to look at the different readings of a proposition through the lens of modern formal logic. Therefore, it is important to find out which formal language has the best capacity to capture various aspects of this distinction and the insights behind it. In the literature, several attempts have been made to formalize the different readings of propositions in the languages of classical predicate or temporal logics.[10] In this chapter, we put forward an alternative based on Martin-Löf's constructive type theory (CTT).[11] Compared to its rivals, our analysis is more faithful to the

[5] These examples are adopted from El-Rouayheb (2019, p. 24).

[6] See *al-Qiyās* (1964, Chapter III.1, p. 126) in which Ibn Sīnā complains that previous philosophers have not paid enough attention to this distinction.

[7] A famous passage in which Ibn Sīnā discusses this distinction can be found in the logic part of *al-Išārāt* (1983, Chap. 4.2, pp. 264–266). For translations of this passage see Street (2005, pp. 259–260) and Ibn Sīnā (1984, Chap. 4.2, p. 92). In the logic part of *al-Nağāt* (1985, pp. 34–37)—whose translation can be found in Ahmed (2011, Sect. 48)—Ibn Sīnā proposes six different readings of necessary propositions. The second and the third readings include respectively substantial and descriptional necessities. This distinction is discussed also in *al-Qiyās* (1964) and *Manṭiq al-Mašriqīyīn* (1910). Translations of some relevant passages from these two works are provided by Hodges and Johnston (2017, Appendix A.2). They discuss a distinction between *ḍarūrī* and *lāzim* propositions in passages from *Manṭiq al-Mašriqīyīn* that is tantamount to the distinction between substantial and descriptional readings of propositions.

[8] See El-Rouayheb (2017, pp. 72 & 81).

[9] See Street (2002, p. 133).

[10] See, among others, Rescher and vander Nat (1974), Hodges and Johnston, and Chatti (2019a, 2019b).

[11] See Martin-Löf (1984). In what follows, a basic familiarity with CTT is assumed. All the background requirements can be found in Rahman et al. (2018, Chap. 2).

grammatical subject-predicate structure of propositions and can better reflect the morphological features of the verbs (and descriptions) that extend time to intervals (or spans of times). It is worth noting that our focus will mostly be on the analysis of the descriptional reading of propositions (which can also be called 'the descriptional propositions' for the sake of brevity).

16.2 On *What* and *How*

The distinction between substantial and descriptional propositions seems to be related to Joseph Almog's famous distinction between *what* and *how* a thing is.[12] More precisely, it is relevant to how the subject term of a categorical proposition is true of its objects in the two different readings we introduced above. The expression '*a* is *S*' can in principle encode two basic forms of predication. The expression encodes what *a* is, if *S* represents an essential feature of *a*. For instance, if *S* is a genus of *a* or a category to which *a* belongs, then '*a* is *S*' encodes (at least partially) what *a* is.[13] On the other hand, the expression '*a* is *S*' encodes how *a* is, if *S* represents an accidental feature of *a*. For instance, if *S* is a description which can be sometimes but not always true of *a*, then '*a* is *S*' encodes (again, at least partially) how *a* is.

Returning to the distinction between substantial and descriptional propositions, it seems that the subject term of a true substantial proposition establishes what its

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[12] See Almog (1991, 1996).

[13] As pointed out by Ranta (1994, p. 55), "the most serious criticism against the type-theoretical analysis of everyday language comes from intuitionistic thinking" (i.e., from the very same framework within which CTT is developed). The concern is that although intuitionistic logic is an appropriate tool for mathematical reasoning, its application outside mathematics is inappropriate. This is mainly because, by contrast with mathematical reasoning in which objects are almost always fully presented, everyday reasoning is usually based on an incomplete presentation of objects. For example, although a natural number can be fully presented by its canonical expression, giving a full presentation of a continent seems to be extremely difficult, if not impossible. Stated differently, the presentations of continents (like many other things) in the natural language is usually incomplete in the sense that they are usually referred to by expressions which only *partially* determine what a continent is. There seems to be no canonical expression of the non-mathematical objects like continents, humans, trees, etc. One possible way to deal with this concern, as Ranta (1994, pp. 55–56) suggests, is "to study delimited *models* of language use, 'language games'. Such a 'game' shows, in an isolated form, some particular aspect of the use of language, without any pretention to covering all aspects." For example, a term like 'human', depending on the context, can be partially modelled by the set of canonical names of the people who are referred to by the term 'human' in that specific context. Accordingly, a set like {John, Mary, Jones, Madeline} can be considered as the interpretation of the term 'human' in a certain context. The elements of such a set are fully represented by the canonical names 'John', 'Mary', etc. Although we are still far from the full presentation of humans in flesh and blood, we have a model which enables us to formalize certain fragments of language in which talking about humans is nothing but talking about those four persons. By developing such models, we can formalize larger fragments of language. An alternative dialogical approach for dealing with this concern is put forward by Rahman et al. (2018, Sect. 10.4). This dialogical alternative is inspired by Martin-Löf (2014).

objects are. By contrast, the subject term of a true descriptional proposition establishes how its objects are. For example, '*a* is human' expresses what *a* is. But '*a* is bachelor' expresses how *a* is.[14] So a reasonable expectation of an accurate analysis of substantial and descriptional propositions is that it must capture the difference between *what* and *how* things are. This shows that classical logic cannot be an eligible candidate for the frameworks in which such an analysis is supposed to be provided. This is because those different forms of predication cannot be distinguished in classical logic, where there is only one way to analyse the expression '*a* is *S*', namely as the propositional function $S(a)$. It is, therefore, an advantage of CTT over classical logic that the language of the former is sensitive to the difference between these two kinds of predication.

Suppose that '*a* is *A*' expresses what *a* is. This can be captured in the framework of CTT as *a*'s being a member of the category (or domain or type) *A*. The latter notion can be represented in the language of CTT as follows:

$a: A$

In this expression ':' can be read as 'is' (in the sense of expressing the *what*) or, equivalently, as 'belongs to'. Moreover, suppose that '*a* is *B*' expresses how *a* is. In other words, the expression describes *a* as having the property *B* and this expression constitutes a proposition. This can be captured in the language of CTT as follows:

$B(a): prop$

In this expression '*prop*' represents the category of propositions. Accordingly, that *a* is an object of the type *A* which bears the description *B* can be expressed by combining the two previous expressions as follows:

$B(a): prop$, given $a: A$.

More generally, $B(x)$ constitutes a proposition when an *x* that is of the category *A*, bear the description *B*. Formally,

$B(x): prop\ (x: A)$

In fact, this expresses the well-formedness of the predicate $B(x)$.[15] It determines the domain upon which the predicate is defined. More explicitly, it clarifies that *B* is predicated upon the objects which belong to the set *A*. In general, the CTT formation rules for predicates are in accordance with Plato's observation that how something is cannot be asserted without presupposing what that thing is.[16] Once

[14]This picture needs to be refined. As we will shortly see, even in the descriptional reading the whatness of the objects of the subject term is mentioned, albeit only implicitly.

[15]In CTT, the well-formation is not only syntactic but also semantic. Consider, for example, the predicate *Hungry*. The well-formedness of this predicate can be expressed by '*Hungry*(*x*): *prop* (*x*: *Animal*)', which reveals not only the correct syntactical use of that predicate but also the semantic domain of the objects of which that predicate can be true.

[16]In their thorough and meticulous discussion of Plato's *Cratylus*, Lorenz and Mittelstrass (1967) highlight the distinction between *naming* (ὀνομάζειν)—as establishing *what* something is—and *stating* (λέγειν)—as establishing *how* something is. They (1967, p. 6) point out that "[t]he subject

the well-formedness of a predicate has been established, we can produce formal structures expressing that the predicate is true of some objects. For example, that B is true of a, which is an arbitrary but fixed element of A, can be expressed by:

$B(a)$

Similarly, that some or all of the elements of A are B can be expressed, respectively, by the following expressions:

$(\exists x: A)\ B(x)$
$(\forall x: A)\ B(x)$

In all of the latter three expressions the formation rule $B(x)$: *prop* $(x: A)$ is presupposed.[17]

By employing this machinery, the grammatical structure of categorical propositions can comprehensively be reflected in the language of CTT and this can be counted as a significant advantage of CTT over classical logic. Consider the proposition 'some students are good'. An oversimplified analysis of this proposition within the classical logic with unrestricted quantification would be as follows:

$(\exists x)\ [Student(x) \wedge Good(\mathrm{x})]$

By contrast, in the language of CTT, 'some students are good' could be formalized by this notation:

$(\exists x: Student)\ Good(x)$

The latter translation restricts *Good* to the grammatical subject of the proposition i.e., *Student*. It singles out the set of those students that are good insofar as they are students. Quite the contrary, the former translation does not distinguish the subject and predicate. It coarsely refers to persons who are both good and student, no matter whether or not that those persons are good at being a student or at something else. To discuss one of Ibn Sīnā's own examples, consider the following propositions:

(6) Imraʾa al-Qays is good.
(7) Imraʾa al-Qays is a poet.
(8) Imraʾa al-Qays is a good poet.

Ibn Sīnā argues that (8) cannot be concluded from the conjunction of (6) and (7).[18] For him, such an argument is fallacious. But if we translate these propositions in the language of classical predicate logic, we cannot see why this argument is fallacious.

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has to be effectively determined, i.e., it must be a thing correctly named, before one is going to state something about it".

[17] In CTT, the judgment that the proposition $B(a)$ is true is usually represented by '$B(a)$ *true*'. But as long as we are considering a proposition itself (without making any judgment that it is true) we do not really need to add '*true*'.

[18] Ibn Sīnā proposes this example in the logic part of *al-Išārāt* (1983, Chap. 10.1, pp. 501–502). We are grateful to Alexander Lamprakis for drawing our attention to this example.

Suppose that '*a*' refers to Imraʾa al-Qays, and '*Poet*' and '*Good*' represents, respectively, being a poet and being good. An oversimplified translation of (6)–(8) in the framework of classical logic (with unrestricted domain of quantification) yields:

(6-CL) $Good(a)$
(7-CL) $Poet(a)$
(8-CL) $Poet(a) \wedge Good(a)$

In such a framework, concluding (8-CL) from (6-CL) and (7-CL) is unproblematic. But this is an undesirable result for Ibn Sīnā. This shows that such a framework is not suitable for formalizing Ibn Sīnā's logic. By contrast, in CTT the propositions (6)–(8) would be translated as follows:

(6-CTT) $Good(a)$: *prop*, given a: *Human*.[19]
(7-CTT) $Poet(a)$: *prop*, given a: *Human*.
(8-CTT) $Good(a)$: *prop*, given a: *Poet*.

(8-CTT) cannot be concluded from (6-CTT) and (7-CTT); and this is exactly what Ibn Sīnā expects.[20]

16.3 Substantial and Descriptional Propositions

Consider the following proposition:

(9) Every moving is changing.[21]

The substantial and descriptional readings of (9) can be stated respectively as follows:

(10) Every moving, as long as it exists, is changing.
(11) Every moving, as long as it is moving, is changing.

It is not the case that every moving object, as long as its essence exists, is changing. Rather, it is changing as long as its essence can be described as moving. A moving object can in principle stop moving at some time without ceasing to exist. So (9) is true only if it is read in the descriptional sense as (11), rather than in the substantial sense as (10). To see how the difference between (10) and (11) can be mirrored in their formal constructions in the language of CTT, a deeper investigation about the subject and predicate of these propositions needs to be carried out.

At first sight the subject of (9) is *Moving* and the predicate is *Changing*. However, *Moving* is a description whose bearer is concealed (*muḍmar*). Thus it is legitimate

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[19]That Imraʾa al-Qays belongs to the category *Human* is not explicitly mentioned in (6) and (7). But it is necessary be added to the picture. See the next section for more details on this issue.

[20]Notice that if we simply take *Good-Poet*(x) as a predicate, then from *Good-Poet*(a) we cannot infer either that a is good or that a is a poet.

[21]The example is borrowed from the logic part of *al-Išārāt* (1983, Chap. 4.2, p. 265).

to ask what the category of moving objects is. Put otherwise, what is the type of the things which are supposed in (9) to be *Moving*?[22] (9) itself does not determine whether moving things are supposed to be, for example, humans, animals, or bodies (*ağsām*) in general. This can be established only by the context in which (9) is stated. But in any case it is undoubtable that moving things must be considered to be of a specific category, even if this category is not explicitly mentioned. To preserve the generality of our analysis we can assume that this category is *O*. It means that (9) expresses a fact about those objects of the type *O* which are moving. Depending on the context, *O* can be replaced with the categories like *Human*, *Animal*, or, more generally, *Body*.

According to this understanding, every element which lies in the scope of the universal quantifier of (9) has two different aspects. One aspect reveals *what* it is (i.e., it belongs to *O*) and the other reveals *how* it is (i.e., it is moving). In other word, if z is an element in the scope of the universal quantifier of (9)—i.e., if z is one of those objects that are moving—it can be represented as having the canonical form $< x, b(x) >$ in which x is of the type *O* and $b(x)$ is a method evidencing that x can be described as being *Moving*. $b(x)$ can be seen as a truth-maker or a proof for the proposition that x is moving.[23] The difference between the substantial and descriptional readings of (6) is rooted in how these two different components are combined with each other and in the roles each of them plays in the predication. The descriptional reading of (9)—i.e., (11)—can be formalized in the language of CTT as follows:

(11-CTT) $\{\forall z : (\exists x : O)\ Moving(x)\}\ Changing(left(z))$

Subject	***Predicate***
$(\exists x : O)\ Moving(x)$	$Changing(left(z))$

In this translation, *left* can be interpreted as a projection function which extracts the left-side element of every z. Similarly, *right* can be defined as the projection function which extracts the right-side element of every z. So if z is considered to be a *compositum* of the form $<x, b(x)>$ in which x is of the type *O* and $b(x)$ is a method evidencing that x can be described as moving, then $left(z) = left(<x, b(x)>) = x: O$ and $right(z) = right(<x, b(x)>) = b(x): Moving(x)\ (x: O)$. The above construction can be seen as involving an anaphora whose head is *Moving-O* (i.e., it is an object of the type *O* that is *Moving*). The tail of the anaphora is constituted by the projection function $left(z)$ which picks out those objects of the type *O* that are described as moving in the grammatical subject.

Philosophically speaking, in the descriptional reading of (9), the two different aspects of the subject (i.e., the one which reveals *what* it is and the one which reveals

[22] On how and why the bearer of the subject of descriptional propositions is concealed see Schöck (2008, pp. 350–351).

[23] A *truth-maker* is in fact a rudimentary form of what is called proof-object in CTT. See Ranta (1994, p. 54). However, in the context of this chapter, we assume 'truth-maker' and 'proof-object' to be synonymous terms. It is also worth mentioning that in the CTT-framework one and the same true proposition has according to the rule more than one object which makes it true.

how it is) are merged into a compound unity, so that each constituent carries the information about the other. Now it is this compound unity upon which *Changing* is predicated. More precisely, the subject is assumed to have a *whatness* (i.e., its belonging to *O* which can also be seen as the *substantial* component of the subject) and a *howness* (i.e., its being moving which can also be seen as the *descriptional* component of the subject). In the descriptional reading these two aspects are combined with each other to make a new compound *whatness* (i.e., *Moving-O*) upon which another *howness* (i.e., *Changing*) is predicated. Such a compound whatness plays no role in the substantial reading of (9). In (10), the objects upon which *Changing* is going to be predicated are still selected by the description *Moving* from the domain of the objects of type *O*. Nonetheless, the truth of such a substantial predication is not supposed to be dependent on whether or not those objects preserve the description *Moving*. In general, in the substantial predication the objects of predication are selected by a description but the truth of the predication does not depend on whether or not those objects preserve the description. By contrast, in the descriptional reading not only are the objects of predication selected by the description but also the predication is true only as long as those objects preserve the description. Given these observations, our proposal for the translation of the substantial reading of (9)—i.e., (10)—goes as follows:

(10-CTT) $\{\forall z{:}\ (\exists x{:}\ O \mid Moving(x))\}\ Changing(first(z))$

Subject	***Predicate***
$(\exists x{:}\ O \mid Moving(x))$	$Changing(first(z))$

Here the description *Moving* is characterizing the domain of the objects of the type *O* upon which *Changing* is predicated. Nonetheless, the description is not a component of the unified *whatness* upon which *Changing* is predicated. The subject is not considered as a compound entity of which the description *Moving* is an irremovable component. Moreover, to highlight the distinction between the substantial and descriptional reading, instead of *left* and *right*, here we use the projection functions *first* and *second*. The main difference between these two couples of the projection functions is that when one of the functions of the former couple extracts an element of the pair $<x, b(x)>$, the selected element carries some piece of information about the other element. By contrast, what is selected by one of the functions of the latter couple does not contain any piece of information about the the other element. So *first*(z) selects one instantiation of *O* and forgets about the second component of z.[24]

To generalize our formalizations, reconsider the proposition (1)—i.e., 'every *S* is *P*'—and suppose that *S* is a description whose bearers are of the type *D*. The descriptional and substantial reading of this proposition can be formalized as follows:

Substantial Reading of (1): $\{\forall z{:}\ (x{:}\ D \mid S(x))\}\ P(first(z))$

[24]To put it in more technical language, if in the proposition 'every *B*, as long as it exists, is *C*', the bearers of the description *B* are of the type *A*, then *first*(z): *A* must be understood as what Sundholm (1989, p. 10) calls '*A-injection*'.

Descriptional Reading of (1): $\{\forall z: (\exists x: D)\ S(x)\}\ P(left(z))$

Now we can easily see the advantage of this analysis over one of its rivals which is proposed in the framework of classical predicate logic. Saula Chatti formalizes the descriptional reading of (1) as follows:

$(\forall x)[S(x) \supset (S(x) \supset P(x))] \wedge (\exists \mathrm{x})\mathrm{S}(\mathrm{x})$

As she herself pointed out, the above proposition is equivalent to:

$(\forall x)(S(x) \supset P(x)) \wedge (\exists \mathrm{x})\mathrm{S}(\mathrm{x})$

But this is exactly what classical predicate logic proposes for the formalization of all *A*-form absolute propositions. So it cannot reflect how the descriptional reading of a proposition differs from the other possible readings.[25]

These formalizations enable us to see better how some *seemingly* contradictory propositions, like the two below, can be both true at the same time:

(12) Every sitting dog, as long as it exists, can walk.

(13) Every sitting dog, as long as it is sitting, cannot walk.

In (13) the subject must be taken as a compound entity to which the predicate *Cannot-walk* applies. Here the projection function *left* selects only those dogs that are sitting. In (12), the function *first* takes the subject in its *substantial* sense. This means that although the subject is analysed into *Dog* and *Sitting*, when *first* selects a dog, its selection does not carry information on whether or not the dog is sitting. So the formal translations of (12) and (13) go respectively as follows:

(12-CTT) $\{\forall z\ (x: Dog \mid Sitting(x))\}\ Can\text{-}walk(first(z))$
(13-CTT) $\{\forall z: (\exists x: Dog)\ Sitting(x)\}\ Cannot\text{-}walk(left(z))$

So far so good. But our conception of the roles Ibn Sīnā considers for the substantial–descriptional distinction in different contexts will not be comprehensive until we understand how the time-parameter and the existence predicate can be added to the picture.

16.4 Time Parameters

Time parameters add more complexities to the structure of descriptional propositions. But, fortunately, CTT has the capacity to handle them. Reconsider the following descriptional proposition propositions:

(11) Every moving, as long as it is moving, is changing.

Which is equivalent to:

[25] See Chatti (2019b, pp. 113–114). Since Ibn Sīnā considers existential import for *A*-form propositions, Chatti emphasizes that the above formulas must include the conjunct ‘$(\exists x)S(x)$’.

(14) Every moving is changing while it is moving.

Now if we add the time parameter, (14) can be read as follows:

(15) Every moving is changing all the time it is moving.

This proposition is usually rendered as an equivalent of the following proposition:

(16) Every (sometime-)moving is changing all the time it is moving.[26]

Accordingly, (16) is usually formalized as follows:

(16-CL) $(\forall x)[(\exists \mathrm{t})Moving(t,x) \supset (\forall \mathrm{t})(Moving(t,x) \supset Changing(t,x))] \wedge (\exists t)(\exists x)Moving(t,x)$

In this analysis '*x*' and '*t*' are variables for respectively the moments of time and the bearers of the description *Moving*. '*Moving*(*t,x*)' can be naturally read as '*x* is moving at *t*'.[27] Our analysis is, however, quite different. But before presenting our proposal, we should first discuss some preliminaries on how temporality can be dealt with in the framework of CTT.

16.4.1 Preliminaries on Temporal Reference in CTT

16.4.1.1 Time Scales

Usually when we are talking about time, we are talking about a specific time scale. Depending on the length of the temporal units, we can introduce different time scales. For instance, we can talk about either years, or months, or days, or hours, etc. These time scales can be represented as, respectively, *Year*, *Month*, *Day*, etc.[28] Each of these time scales is a temporal category. The time scale we are talking about naturally depends on the context in which the proposition is stated. For example, when someone is talking about waking up early in the morning, the time scale such a person considers is probably *Day*. But when the president of a university is presenting statistics about their graduates, her/his time scale is likely to be *Year*. In general, we can represent the time scale we are talking about as *T*.

16.4.1.2 Time Spans

An advantage of the CTT-framework as implemented for time reference is that it provides the opportunity of considering not only moments of time, but also time spans

[26] Hasnawi and Hodges (2017, p. 61) label such propositions as '(*a*-ℓ)' which can be considered as an abbreviation for '*A*-form *lāzim*' propositions.

[27] This formalization is in accordance with what Hodges and Johnston (2017, p. 1061) put forward following Rescher and vander Nat (1974). The conjunct '$(\exists t)(\exists x)Moving(t,x)$' is added to guarantee the existential import of the proposition.

[28] For a detailed technical definition of time scales, see Ranta (1994, Sect. 5.1).

and intervals with a beginning and an end. This is particularly important because actions like moving, running, etc. do not happen in a moment. Rather they should be considered as extended events which happen in temporal intervals. Indeed, one of the main shortcomings of the aforementioned analysis of (16) is that it does not consider moving and changing as extended events. So it is important to have tools to express the occurrence of events not only in singular moments of time but also in temporal intervals. This helps us to formally describe how an object that bears a specific description in a specific span of time can also bear some other descriptions in some specific sub-spans of the former span. It is also possible to express how an object can have the same description with different qualifications in different spans of time. For example, an object that is moving in a span of time might be slow-moving in some parts (or sub-spans) of that span and fast-moving in some others. So it seems to be crucial to see how a span of time can be defined in the framework of CTT.

The category of the spans of a time scale T can be defined as the Cartesian product of T and the set of natural numbers N.[29] More precisely:

$span(T) = T \times N$

To make it clearer, a span of the time scale T is a pair whose first element refers to the beginning point of that span in T and whose second element refers to the number of temporal units (of the scale T) which must be added to the beginning point to form the span under discussion. Stated differently, the second element determines the length of the span. So if $d = <t_0, n> : span(T)$, d is a span of the time scale T which begins at t_0 and ends at $t_0 + n$. The span d can also be represented as $[t_0, t_0 + n]: span(T)$. As we will shortly see, the following functions are also useful:

$left(d) = begin(d) = t_0: T$
$end(\mathrm{d}) = t_0 + n: T$
$right(d) = length(d) = n: N$

As an example of the spans of time in the time scale *Day*, consider the following span:

<14 June 2018, 31> : span(*Day*)

This span of time begins on 14 July 2018 and extends for 31 days. This is exactly the interval in which Football World Cup 2018 took place.

It is noteworthy that since 0 is a member of N, every singular moment of the time scale T can be considered as a span of the length 0. In other word, every t of the time scale T corresponds to $<t, 0>$ which is a member of $span(T)$. This shows that everything expressible by the terminology of singular moments of time is also expressible by the terminology of time spans, though the other way around does not hold.

[29] For a detailed technical definition of time spans, see Ranta (1994, p. 115).

16.4.1.3 Saturation Versus Enrichment

There are at least two different approaches for dealing with temporal reference in the CTT-framework.[30] More clearly, a proposition which expresses the occurrence of an event (or fact) in a span of time can be seen in at least two different ways. Such a proposition can be seen either as an incomplete propositional function that can be saturated by that specific span of time or as an event (or a fact) that can be timed by a timing function. These two formal terminologies are translatable into each other. This means that everything expressible by one of these two approaches is also expressible by the other. Nonetheless, there is a significant philosophical difference between these two approaches. In the first approach time is primitive. Temporal entities (i.e., singular moments of times or time spans) are independent entities which can be put as the arguments of propositional functions. So, ontologically speaking, complete propositions in some sense depend on these temporal entities. By contrast, in the second approach, events (or facts, or truth-makers of the propositions which express those events) are primitive individuals which can be put as the arguments of the timing functions. Thus, in a sense, time is dependent on events. Inspired by François Recanati's terminology, we call these two approaches, respectively, 'saturation' and 'enrichment'.[31]

According to the saturation approach, 'A occurs at the span d of the time scale T' can be formalized as a propositional function A that is saturated by d. So:

$$A(d): prop\ (d: span(T))$$

By contrast, according to the enrichment approach A itself is a fully saturated proposition which is made true by different events (or facts) at different time spans. Equivalently, it has different truth-makers or proofs at different time spans. These truth-makers can be timed by a timing function. Informally speaking, the timing function operates upon the set of truth-makers (or justifications, or proofs) of A and determines the time span in which such a truth-maker is obtained.[32] So if x is a truth-maker of the proposition A (i.e., if x is an event or fact whose occurrence makes A true), then the timing function τ would determine the span of time in which x is obtained. So the role of τ can be defined as follows[33]:

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[30] See Ranta (1994, Sect. 5.4).

[31] This terminology is borrowed from Recanati (2007a, 2007b).

[32] Recall that as pointed out before, it is assumed that a proposition has different truth-makers (or proofs or justifications). In the present context this amounts to the assumption that a proposition has different truth-makers during different time spans. That a proposition is true in a specific time span is equivalent to that one of its truth-makers is obtained in that time span.

[33] See Ranta (1994, p. 108).

A: *prop*
$\tau(x)$: *span*(T) (x: A)

For example, that a human x is running in the time span d can be expressed by the saturation approach as follows:

Running(x,d): *prop* (x: *Human*, d: *span*(T))

Quite differently, the same proposition can be formalized by the enrichment approach as follows:

Running(x): *prop* (x: *Human*)
$\tau(b(x)) = d$: *span*(T) (x: *Human, b*(x): *Running*(x)).[34]

In this formalization $b(x)$ is a truth-maker or evidence for the proposition *Running*(x); and τ is a timing function which determines the time span in which $b(x)$ is obtained. So in a sense the time span d is eventually defined by that specific truth-maker of *Running*(x) that is obtained in that span. In other words, the time span d is given by the operation of the timing function τ upon the event which makes *Running*(x) true. Borrowing Aristotelian terminology, we can say that in the enrichment approach time elements are measurements—i.e., timing operations—of (and, consequently, dependent on) events.

After explaining these preliminary points, we are now well equipped to analyse the temporal interpretation of descriptional propositions through both the saturation and the enrichment approaches.

16.4.2 Descriptional Propositions Relativized by Saturation

Reconsider the proposition (15):

(15) Every moving is changing all the time it is moving.

As we mentioned, moving and changing are extended events which happen in time spans, rather than in singular moments of time. So it is plausible to restate (15) in the language of time spans. If we do so, the result would be something like the following:

(17) Every moving is changing in all the spans in which it is moving.

[34]In order to avoid notational complexity we omitted one variable within the timing function. Indeed, strictly speaking, the correct formalization must be $\tau(x, b(x)) = d$: *span*(T) (x: *Human, b*(x): *Running*(x)).

If we suppose again that the bearers of the description *Moving* are of the type O and that our time scale is T, then our proposal for the logical analysis of (17), in the saturation approach, goes as follows:

(17-CTT-S) $\{\forall z{:}\ (\exists d{:}\ \mathit{span}(T))\ ((\exists x{:}\ O)\ \mathit{Moving}(d,x))\}$ $\mathit{Changing}(\mathit{left}(z),\mathit{left}(\mathit{right}(z))$

This can be glossed as:

(17-CTT-S*) Every z that is an element of the set of those objects that are moving at some time span d is subject to change at the time span in which it is moving.

More precisely, here z is a variable for those time spans d at which some x of the type O is moving. So z can be considered as a pair of the canonical form $<d, <x, b(x) \gg$. Thus, $\mathit{left}(z)$ gives the first constituent of z which is some time span d at which the moving thing is moving. The right constituent of z is the pair of the moving object x and the evidence $b(x)$ which shows that x bears the description *Moving* at d. In other words, in the time span d, $b(x)$ is the truth-maker of the proposition that 'x is moving'. Hence, while $\mathit{left}(z)$ yields some time span d, $\mathit{left}(\mathit{right}(z))$ provides the object that is moving at that time span. This is the object of which the grammatical predicate *Changing* is true. To generalize this approach, consider the following proposition:

(18) Every S is P in all the spans in which it is S.

If the bearers of S are of the type D and our time scale is T, then the logical analysis of (18) in the language of CTT and based on the saturation approach goes as follows:

(18-CTT-S) $\{\forall z{:}\ (\exists d{:}\ \mathit{span}(T))\ ((\exists x{:}\ D)\ S(d,x))\}\ P(\mathit{left}(z),\mathit{left}(\mathit{right}(z)))$

As we previously mentioned, objects that have a description in a specific span can be described as having other properties in some specific sub-spans of the former span. Now we are well equipped to formalize some of the propositions which express such situations. Consider the following example:

(19) Everyone who studies mathematics as an undergraduate spends the first year studying calculus.

To formalize this proposition we can take our time scale to be *Year*. We can also take '$\mathit{Math}(d,x)$' to represent that x studies mathematics as an undergraduate in the span d (where d: $\mathit{span}(\mathit{Year})$ and x: *Human*). More precisely, we assume that x starts studying mathematics at $\mathit{begin}(d)$ and graduates at $\mathit{end}(d)$. If so, the first year of d can be referred to by the following function:

$first\text{-}year(d) = < begin(d), 1 > : span(Year)$

So *first-year*(*d*) refers to the span of time which begins at *begin*(*d*) and extends for 1 year. Now if '*Calculus*(*d*,*x*)' expresses that *x* studies calculus in *d*, then the formal interpretation of (19) would be as follows:

(19-CTT-S) {∀*z*: (∃*d*: *span*(*Year*) ((∃*x*: *Human*) *Math*(*d*,*x*))} *Calculus*(*first-year*(*left*(*z*)),*left*(*right*(*z*))

Here again *z* is the pair $< d, < x, b(x) > >$ in which *d* is a span of time scale *Year*, *x* is a human, and *b*(*x*) is the evidence that *x* can be described as studying mathematics at the undergraduate level. To generalize this example, suppose that for every time span *d*, the function *s-period*(*d*) determines a specific period of *d*. So *first-year* is an instance of this kind of functions. But *s-period(d)* can be defined to determine, for example, the first quarter, the second third, or any other specific part of *d*. Now consider a proposition of the following general form:

(20) Every *S* is *P* in a s-period of the time span in which it is *S*.

This can be formalized as follows:

(20-CTT-S) {∀*z*: (∃*d*: *span*(*T*) ((∃*x*: *D*) *S*(*d*,*x*))} *P*(s-*period*(*left*(*z*)),*left*(*right*(*z*))

Developing this approach would help us to formalize some other types of temporal propositions which play a crucial role in the temporal logic of Ibn Sīnā. (21) is one such proposition:

(21) Every (sometime-)*S* is *P* sometime while it is *S*.[35]

The proposition has been usually formalized as follows:

(21-CL) $(\forall x)[(\exists t)S(t,x) \supset (\exists \mathrm{t})(S(t,x) \wedge P(t,x))] \wedge (\exists t)(\exists x)S(t,x)$[36]

To analyse (21) using the saturation approach of CTT, we need to add a time span quantifier on the predicate side. Accordingly, (21) can be formalized as:

(21-CTT-S) {∀*z*: (∃*d1*: *span*(*T*) ((∃*x*: *D*) $S(d_1,x)$)} ($\exists d_2$: *span*(*T*)) [$S(d_2, left(right(z)) \wedge P(d_2, left(right(z))$]

Here *z* must still be considered as a pair of the canonical form $<d_1, <x, b(x) \gg$. Informally, (21-CTT-S) says that for every object *x* of the type *D* that is *S* in a time span d_1, there is a time span d_2 in which that object is *P* while it is *S*. Now we can turn to the enrichment approach.

[35] Hasnawi and Hodges (2017, p. 61) label such propositions as '(*a-m*)', which can be considered as an abbreviation for '*A*-form *muwāfiq*' propositions.

[36] This formalization is suggested by Hodges and Johnston (2017, p. 1061), following Rescher and vander Nat(1974). Again, the conjunct '$(\exists t)(\exists x)S(t,x)$' is added to preserve the existential import.

16.4.3 Descriptional Propositions Relativized by Enrichment

As we previously mentioned, in the enrichment approach time elements are not primitive and have no independent existence. They are dependent on events which make propositions true. In other words, they are dependent objects—i.e., functions. Since this philosophical conception of time is closer to how Ibn Sīnā understands this notion, it is more plausible to analyse his temporal propositions based on the enrichment approach (rather than based on the saturation approach in which time elements are primitive and have independent existent).[37] To see how temporal propositions can be formalized by the enrichment approach, reconsider the proposition (17):

(17) Every moving is changing in all the spans in which it is moving.

In the enrichment approach, this proposition can be formalized as:

(17-CTT-E) $\{\forall z\colon (\exists x\colon O)(Moving(x)\}\ Changing(left(z))$ AT$(\tau(right(z)))$

Here, z is a pair of the canonical form $< x, b(x) >$. So $left(z) = x$ and $right(z) = b(x)$. As a result, $\tau(right(z))$ amounts to $\tau(b(x))$. Since $b(x)$ is a truth-maker of 'x is moving', $\tau(b(x))$ yields a time span within which x is moving. Finally AT is an operator that operates upon propositions. Informally speaking, for every span d and every proposition A, A AT(d) indicates that A is the case within the time span d. So $Changing(left(z))$ AT(d) means x is changing within the time span d. Putting together all of these observations, what (17-CTT-E) says is that every x of the type O is changing in all the time spans in which it is moving. In other words, if x is moving in d, it would also be changing in this span. To generalize this formalization, reconsider (18):

(18) Every S is P in all the spans in which it is S.

If we suppose that the bearers of the description S are of the type D, then (18) can be analysed as:

(18-CTT-E) $\{\forall z\colon (\exists x\colon D)(S(x)\}\ P(left(z))$ AT$(\tau(right(z)))$

Finally, reconsider (21):

(21) Every (sometime-)S is P sometime while it is S.

To formalize (21) in the enrichment approach we need to add a temporal quantifier on the side of predicate. In this respect, there is no difference between this approach and the saturation approach. (21) can be formally analysed as:

(21-CTT-E) $\{\forall z\colon (\exists x\colon D)(S(x)\}\ (\exists d\colon span(T))[\tau(right(z)) = d \wedge P(left(z))$ AT$(d))]$

Here again z is a pair of the canonical form $< x, b(x) >$ in which x is an object of the type D and $b(x)$ is a witness (or proof) for that x is S. So $\tau(right(z))$ gives a time

[37]For Ibn Sīnā time is the number or magnitude of motion. Although he does not explicitly talk about events, his definition of time shows that he does not consider an independent existence for it. This suffices to convince us that the enrichment approach is preferable to the saturation approach. For a detailed discussion on Ibn Sīnā's view regarding time, see Lammer (2018, Chap. 6).

span within which x is S. Accordingly, what (21-CTT-E) says is that for every x of the type D that is S, there are some spans d in which x is both S and and P.

16.5 Existence With and Without Existence Predicate

According to the ontological system underlying CTT, the fact that a type has been instantiated amounts to showing that the type is not empty. In other words, we should understand the instances of types as witnessing the existence. Therefore, we do not really need to capture the existential import of propositions by adding conjuncts which guarantee the existence of the subject. This can be counted as another advantage of our analysis over some of the earlier studies.[38] If the import is automatically guaranteed, then *a fortiori* there is no need to the existence predicate. It is so, at least, unless the existence predicate is defined in some way that allows us to distinguish those instantiations that witness existence from those that do not.

It is worth mentioning that associating the instantiation of a type to the existence of its elements does not prevent us from considering different sorts of existence. Indeed, each instantiation can be understood as representing the kind of existence associated to the type they instantiate. For example, the members of the type N (i.e., natural numbers) do not have the same form of existence as the members of the type *Man*. Each type has its own form of existence depending upon how the process of its instantiation is defined. Technically speaking, such a process is defined by the rules that introduce the canonical elements of the type under discussion. For example, the notion of existence associated with the set of natural numbers is construction by mathematical induction. By contrast, members of the type *Human* have a completely different mode of existence.[39]

Having made these points, if someone still insists on adding the existence predicate to the picture, there seems to be no technical difficulty in the way of fulfilling this desire. To give examples we can reconsider the logical analysis provided in (18-CTT-S):

(18-CTT-S) $\{\forall z\text{: } (\exists d\text{: } span(T))\ ((\exists x\text{: } D)\ S(d,x))\}\ P(left(z),left(right(z)))$

The existence predicate can be added to the side of subject as follows:

(18-CTT-S*)$\{\forall z\text{: } (\exists d\text{: } span(T))\ ((\exists x\text{: } D)\ S(d,x) \wedge Exists(d,x))\}\ P(left(z),left(right(z)))$

If desired, the existence can also be added to the side of the predicate as follows:

(18-CTT-S**) $\{\forall z\text{: } (\exists d\text{: } span(T))\ ((\exists x\text{: } D)\ S(d,x) \wedge Exists(d,x))\}\ P(left(z),left(right(z))) \wedge Exists(left(z),left(right(z)))$

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[38] See notes 25, 27, and 36.

[39] The existence of the subject matter of non-mathematical propositions can be presented either by 'logical games' or by dialogical verification procedures. See note 13.

By following the same approach, the existence predicate can be added to the propositions relativized by enrichment. However, it should be borne in mind that the existence predicate cannot be introduced rigorously unless we clarify exactly how it is formed. For example, we can say that the objects of the category O exist if and only if they are temporally and spatially indexed.[40] So if the time scale and the category of location are represented by, respectively 'T' and 'L', the existence predicate can be defined as follows:

$Exists(x, y, z)$: $prop\ (x: O, y: L, z: T)$.

Obviously, such a definition of this predicate needs to be embedded into the logical structure of the preceding sentences. We will leave the task of modifying the formal structure to the reader.

16.6 Conclusion

In this chapter we provided a new logical analysis of Ibn Sīnā's descriptional propositions in the framework of CTT. Assuming an anaphoric structure for propositions, we showed that the grammatical predicate of a descriptional proposition is true of an anaphoric tail that encodes not only *what* the object of predication is, but also *how* it is. By contrast, in the case of a non-descriptional propositions the anaphoric tail only encodes *what* the object of predication is.

Our analysis has at least three advantages over its rivals. First, it better reflects the grammatical structure of propositions. Second, it is quite flexible for capturing different temporal features of propositions. In particular, it can enable us to formalize not only sentences which talk about singular moments of time, but also sentences which include actions extended in time spans. Third, the existential import of the universal propositions is automatically guaranteed by the instantiation of the types about which those propositions talk. So we do not need to consider an additional conjunct to our translations just to make sure that the existential import is preserved. Nonetheless, as we saw, there is no obstacle in the way of adding the existence predicate either to the side of subject or to that of the predicate.

Our main focus in this chapter was on how some propositions can be formalized in the framework of CTT. We did not touch on how these formal constructions can be put into the syllogism. This should be postponed to a future project. Nonetheless, there is an important insight about the theory of the syllogism which can be seen from here. We showed that the anaphoric structure is quite general and is applicable even when the temporal dimensions of propositions are completely put aside. But if we analyse the propositions of a syllogism as such anaphoric structures, then it would be obvious that the subject-term must always contain a descriptional element in relation to the individual of a domain shared by the premises. So, in a sense, it is

[40] An alternative approach is based on the introduction of the notion of ontological dependence. See Rahman and Redmond (2015).

assumed by the premises that the kind of the involved object in the inference to be drawn is known. This is one of the general insights on the theory of the syllogism which can be acquired from Ibn Sīnā.

Acknowledgements We are thankful to Leone Gazziero (STL), Laurent Cesalli (Genève), and Tony Street (Cambridge), leaders of the ERC-Generator project "Logic in Reverse: Fallacies in the Latin and the Islamic traditions," and to Claudio Majolino (STL), associated researcher to that project, for fostering the research leading to the present study. We should also thank Vincent Wistrand (UMR: 8163, STL) and Alexis Lamprakis (München) for many fruitful discussions from which we have benefited a lot. The present paper was written while Mohammad Saleh Zarepour was a Humboldt Research Fellow at LMU Munich. We are thankful to Alexander von Humboldt Foundation for their support.

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Chapter 17
Avicenna on Syllogisms Composed of Opposite Premises

Behnam Zolghadr

Abstract This article is about Avicenna's account of syllogisms comprising opposite premises. We examine the applications and the truth conditions of these syllogisms. Finally, we discuss the relation between these syllogisms and the principle of non-contradiction.

Keywords Avicenna · Syllogism · Opposition · Contradiction · Paraconsistency

17.1 Introduction

In his *Prior Analytics*, Aristotle explains syllogisms comprising opposite premises, i.e. contraries or contradictories. That some syllogisms can be composed of opposite premises might be of no surprise, considering that Aristotelian syllogistic is usually considered as a paraconsistent logic, in the sense that not everything follows from a contradiction. In another words, *ex contradictionequodlibet*- or explosion, as it is called in contemporary logic- is not a valid argument in Aristotle's syllogistic.[1] Here is an example:

All men are animals.
<u>Some men are not animals.</u>
All animals are men.

This argument which is valid in classical logic, i.e. in the logic developed by Fregeand Russell, is not valid in Aristotelian syllogistic. Thus, within Aristotelian syllogistic, inconsistencies do not make the theory trivial. However, on the other hand, considering Aristotle's extensive defense of the principle of non-contradiction, it might be surprising that in Aristotelian syllogistic some valid deductions are composed of opposite premises. Aristotle does not explicitly say much about

[1] For more detail see (Priest 2007, p. 120).

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M. Mojtahedi et al. (eds.), *Mathematics, Logic, and their Philosophies*,
Logic, Epistemology, and the Unity of Science 49,
https://doi.org/10.1007/978-3-030-53654-1_17

the applications and metaphysical implications of such syllogisms. On the other hand, Avicenna in a corresponding section, under the title 'syllogisms of opposite premises', says more about these syllogisms. Avicenna's account of these syllogisms is the main concern of this article. For not only does he tell us more about these syllogisms, but also after Avicenna, Arabic logicians mostly read his works on logic and not those of Aristotle. For this reason, Avicenna's account of syllogisms from opposite premises is also important from a historical point of view, i.e. on the way later Arabic logicians received syllogistic. I have already discussed Avicenna's defense of the principle of non-contradiction in another paper.[2] Avicenna argues that from some specific contradictions, e.g. in the form of everything is P and not P, everything follows, but he does not argue that from an arbitrary contradiction everything follows. This argument and a comparison between Avicenna's defense of the principle of non-contradiction and that of Aristotle's were discussed in that paper. Here, we are concerned with another aspect of Avicenna's view on contradictions, i.e. what follows from opposite premises in a valid syllogism and the way in which we should understand it in the context of Aristotelianism. Moreover, we are considering the following questions: What are the truth values of the opposite premises, and their conclusions? And are these syllogisms in contrast with the principle of non-contradiction? This latter question is important, because that some syllogisms are made of opposite premises is in contrast with *ex contradictionequodlibet*. However, as we will see, in these syllogisms, only contradictions follow from opposite premises. In what follows, we will discuss these syllogisms and the truth conditions of their premises.

Thus, the next section is about Aristotle's account of syllogisms which are composed of opposites. Aristotle explains to us in what figures and in what moods such syllogisms can be established. Then, in the third section, we will discuss Avicenna's interpretation of Aristotle's account, as well as his own account. We will examine the corresponding section from three of his works: *Middle Summary on Logic* (*al-Muẖtaṣar al-awsaṭ fil-Manṭiq*), *The Cure* (*Šifā*) and *The Deliverance* (*Naǧāt*). In this section, we will firstly explore the truth conditions of opposite premises and, then, the validity of syllogisms from such premises as well as the cases in which, according to Avicenna, such syllogisms are established by mistake. We will conclude with some remarks on the paraconsistency of Aristotelian syllogistic.

17.2 Aristotle on Syllogisms Composed of Opposite Premises

As Aristotle explains,[3] opposite premises include contraries and contradictories. Contraries are corresponding A-propositions and E-propositions and contradictories are either corresponding A-propositions and O-propositions or corresponding

[2] (Zolghadr 2019).

[3] *Prior Analytics*, 63b31–63b38. All quotations of Aristotle are from (Barnes 1991).

E-propositions and I-propositions. Syllogisms from such contrary or contradictory premises can be made only in the second and the third figures, and these syllogisms cannot be made in the first figure. Aristotle explains this in two steps. In the first figure, no affirmative deduction from opposite premises can be made, because one of the opposite premises must be affirmative and the other must be negative. Otherwise, they are not opposite. But affirmative deductions are made of only affirmative premises. Thus, no affirmative deduction can be made from opposite premises. This is true of other figures as well. In no figure are syllogisms from opposite premises affirmative. Thus, the conclusion of a syllogism from opposite premises, if any, is only negative. Aristotle's second step is to show that in the first figure no negative deduction can be made from opposite premises. Opposites affirm and deny the same predicate of the same subject. But in the first figure the middle term which occurs in both premises is the subject of one and the predicate of the other. As Aristotle put it[4]:

> And the middle term in the first figure is not predicated of both extremes, but one thing is denied of it, and it is affirmed of something else and such propositions are not opposed.

However, one might ask whether a similar point can be made of the other figures as well. In the second figure, the middle term is the predicate of both premises, and thus the premises have the same predicate. But the subjects are different. So, to have opposite premises, the subjects of the premises should be one. Similarly, in the third figure, the subjects of premises are the same, but the predicates are not. Thus, to have opposite premises, the predicates of the premises should be one. Therefore, syllogisms from opposite premises are special kinds of syllogisms. In the second figure, these syllogisms are those in which the major term and the minor term which are the subjects of premises are the same. In the third figure, these deductions are those in which the major term and the minor term, which are the predicates of the premises, are the same. The conclusion of these syllogisms, in both figures, have a distinctive feature. Since the minor and major terms are the same, the subject and predicate of the conclusion are the same. We already know that these syllogisms are negative. Thus, the consequent of syllogisms which are made of opposite premises is either a universal or affirmative proposition denying something of itself.

As already mentioned, accordingto Aristotle, there is no negative deduction of opposite premises in the first figure, because the subjects as well as the predicates of the premises are not the same. In the second and the third figure both subjects and predicates are not the same. Valid syllogisms in these figures, according to Aristotle, are special cases in which subjects are the same in the second figure and predicates are the same in the third figure. Thus, one might consider a similar approach to the first figure, i.e. syllogisms from opposite premises can be made in the first figure only if all the terms, including the middle, major and minor, are the same. Yet, it can be only made in the moods with negative conclusions, i.e. Clarent and Ferio. However, Aristotle, as we saw, excludes this possibility. The reason might depend on his theory of predication, according to which, something must be predicated of

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[4] Ibid.

something else, and thus self-predication is excluded.[5] In the first figure, to have opposite premises, all four terms should be one, and consequently both premises will be self-predications. If self-predication is excluded from syllogistic, there will be no syllogism from opposite premises in the first figure. Let us move on to the second and the third figures.

In the second figure, syllogisms can be made of opposite premises in all four moods (63b39–64a19). In Cesare and Camestres, premises are contraries, and in Festino and Baroco, the premises are contradictories. For Cesare, consider:

No C is A.
All Bs are As.
No Bs are Cs.

Let A stands for good and let B and C stand for science. The conclusion will be 'No science is a science'. Exchanging the major premise for the minor premise we will have Camestres and the conclusion will be: No Cs are Bs. For Festino, consider:

No Bs are As.
Some Cs are As.
Some Cs are not Bs.

Aristotle's examples are: Supposition for A and Science for B and C. The conclusion is: 'some sciences are not science'.

The last mood of the second figure is Baroco:

All Bs are As.
Some Cs are not A.
Some Cs are not Bs.

Aristotle's examples are good for A and science for B and C. The conclusion is: 'some sciences are not science'.

In all these four examples, premises share the same subjects and the same predicates. Moreover, none of the deductions of the second figure is affirmative. As Aristotle put it (63b39–64a19):

> Consequently it is possible that opposites may lead to a conclusion, though not always or in every mood, but only if the terms subordinate to the middle are such that they are either identical or related as whole to part. Otherwise it is impossible; for the propositions cannot anyhow be either contraries or opposites.

In the previous examples, minor and major terms are identical. However, here, Aristotle mentions that they can be related as whole to part. In Sect. 3.2, we will discuss Avicenna's commentary on this point, where he offers additional explanation.

Now the third figure. In three moods, i.e. Darapti, Disamis and Datisi in which both premises are affirmative, syllogisms cannot be made of opposite premises for the same reason we discussed regarding the first figure. But in the other three moods of the

[5] See (Gomes and D'Ottaviano 2010) and *Categories*, 1b9–15, 2b19–22.

third figure, syllogisms can be made of opposite premises. In Felapton, these syllogisms are made of contrary premises, and in Bocardo and Frison, these syllogisms are made of contradictory premises.

For Felapton:

All As are Bs.
No As are Cs.

Some Cs are not Bs.

Aristotle examples are: Medicine for A and Science for B and C. The conclusion will be 'Some sciences are not science'. For Bocardo and Ferison, respectively:

Some Bs are not As.
All Cs are As.

Some Cs are not Bs.

No Bs are As.
Some Cs are As.

Some Cs are not Bs.

The examples are the same as in the case of Felapton. In both moods, the conclusions are 'some sciences are not science'.

Thus, according to Aristotle, syllogisms of opposite premises can be made in seven moods. The conclusion is always denying something of itself, whether universally or particularly, which is unacceptable due to the principle of non-contradiction.

A question still remains: How can some syllogisms be made from opposite premises without violating the principle of non-contradiction? Perhaps the answer can be found in a distinction Aristotle makes between different kinds of syllogisms based on the truth of their premises. We will discuss this question in the next section through Avicenna's exposition of the subject.

17.3 Avicenna on Opposite Premises

17.3.1 Truth and Opposite Premises

We will examine Avicenna's view on the syllogisms which are composed of opposite premises, from three of his works: *The Cure* (*Šifā*), *The Deliverance* (*Nağāt*) and *Middle Summary on Logic* (*al-Muḫtaṣar al-awsaṭ fil-Manṭiq*). We find a section entitled 'Syllogisms of Opposite Premises' in every one of these works. The section from *The Cure* is in the book of Syllogism (*Qīyās*) which is almost a repetition of the same section in *Middle Summary on Logic* (hereafter, *MSL*).[6]

[6]For a comparison between the two sections see editor's introduction to *MSL*.

The section on opposite premises in Deliverance is much shorter than the ones in *The Cure* and *MSL*. There, Avicennaexplains syllogisms composed of opposite premises as follows[7]:

> A syllogism comprising opposite premises is one composed of two premises that share terms, but are different with respect to quantity. [Such syllogisms] are widespread only due to the fact that, in some of the terms, the nouns are substituted, so that [the opposition of the premises] is not detected. [Thus,] for example, it is not said, ‘Man is a laugher; and man is not a laugher’. Rather, after saying, ‘Man is a laugher’, they say, ‘Humans are not laughers’. The result of this syllogism is that a thing is not itself. For example, ‘Man is not human’.

Thus,these syllogisms are widespread when there is a substitution of a noun with its synonyms, which probably makes the opposition hidden. In this case, such syllogisms are made because it is not explicit that although one treats them as two different terms, the two terms have the same meaning. In fact, the terms are one and make two opposite propositions.[8] However, such syllogisms, as Avicenna explains, are useful and do have some applications. Avicenna continues[9]:

> The sophists (mughālaṭūn) use it only as a way of overcoming [their opponents]. And it is sometimes [also] used in the service of dialectics when an opponent disagrees [with another] regarding his point of departure. Thus he gets him to concede a premise; and then he gets him to concede some others, which yields the contradictory of the one [originally] conceded. Then the conclusion and its contradictory, [the premise] that was first conceded, are taken up and a syllogism out of opposite [premises] is constructed. This will conclude that a thing is not [itself].

Not only sophists use such arguments for their own purposes, these arguments can also be used by an Aristotelian logician in dialectics and as an *ad hominem* argument.[10] Our concern, here, is the latter. Avicenna, following Aristotle, advocates the principle of non-contradiction. One of the implications of this principle is that no two opposite propositions are true. Thus, one might ask whether this is at odds with establishing syllogisms which are composed of opposite premises. Avicenna defines a syllogism as a statement composed of other statements, namely premises, which, when they are posited, another statement, namely the conclusion, follows necessarily from them.[11] ‘To follow’, as Avicenna explains, means that when the truth of the premises is granted the truth of the conclusion is inferred. Thus, in order to establish a syllogism, the premises do not have to be true. Some syllogisms have true premises but not all syllogisms need to have true premises. In the latter case, the conclusion follows from the premises only for one who admits the truth of premises while the premises may not be true in reality (*bi hasab al-ˀamr*).[12] It is only in demonstration

[7](Ahmed 2010, p. 80).

[8]See also (Avicenna 1964, p. 524; Avicenna 2017, p. 216).

[9](Ahmed 2010, p. 80).

[10](Avicenna 1964, p. 524; Avicenna 2017, p. 216).

[11](Ahmed 2010, p. 42).

[12](Avicenna 2005, p. 40).

(*burhān*) that the premises of syllogisms need to be true, while in dialectics, rhetoric and poetics they can be false.[13]

Therefore, syllogisms from opposite premises do not necessarily commit us to accepting the truth of the premises in reality. These syllogisms show only that if one admits the truth of the premises, the conclusion follows, without any commitment to the truth of the premises in reality.

17.3.2 Syllogisms from Opposite Premises

Avicenna's explanation of there being no syllogism from opposite premises in the first figure is similar to Aristotle's. Opposite propositions share the same subject and predicate, which is not so in the first figure.[14] As we discussed above, since the middle term is the subject of one premise and the predicate of the other, we cannot get opposite premises only by having the minor and major term to be one. The only exception in the first figure would be the case in which all three terms, i.e. minor, major and middle, are the same.

In the second figure,[15]a syllogism obtains from contradictories only if the major premise is universal. As we saw in Sect. 17.2, these are the two moods Festino and Baroco. such condition does not apply to the contraries, and they can be obtained in the moods Cesare and Camestres. Avicenna explains that the two major and minor terms should be the same, either actually or potentially. Thus, one term can be the species or the particular of the other term. However, in this case, the syllogism is not truly one and truly composed of opposite premises. In fact, there are two syllogisms, one of which is hidden and is truly a syllogism of opposite premises. It is hidden, because, as Avicenna explains, a judgment about a universal implies that judgment about the particular under the aforementioned universal, and there is no need to state it. Consider: 'all animals are bodies' and 'no men are bodies'. One can conclude that 'no men are men'.

These are the two syllogisms:

All animals are bodies.
All men are animal.
―――――――――――
All men are bodies.

No men are bodies.
All men are bodies.
―――――――――――
No men are men.

Moreover, depending on how one forms the hidden syllogism, one can also get the conclusion 'Some animals are not animals'.

[13](Avicenna 1964, p. 4).

[14](Avicenna 2017, p. 217).

[15]The Discussion of the second figure is from (Avicenna 2017, 217).

Let us move on to the third figure.[16] In this figure, syllogisms from opposite premises can be made only in the moods which have negative conclusions. For in the other moods the premises are both affirmative, but two opposite premises cannot be both affirmative. Thus, in Felapton, Bocardo and Ferison such syllogisms can be made. In all these moods the major premise is the negative one.

Avicenna warns the reader that although it is possible to have true conclusions from false premises, it is not possible to get true conclusion from opposite premises[17]:

> One should not think that since truth can follow from false [premises], true [premises] can also follow from opposite [premises]. For it implies that the object is not itself. And why it is possible that this can occur in the mind is because it is possible for the man to have false syllogisms entailing false conclusions and it is common in his rational soul.[18]

Avicenna, then, explains to us why one might think that true conclusions can follow from opposite premises and in what cases such mistakes are made.

In some cases, one has some true premise, but via some invalid(*fāsid*) syllogisms she concludes the opposite of the true premise, or she concludes some other conclusions which via some other syllogisms entails the opposite of the true premise. Avicenna's example is as follows.[19] Suppose we have the true premise 'some numbers are odd'. We conclude via some false syllogisms that 'all numbers are divisible by two'. This implies that 'no number is odd', and then with this conclusion and our true premise, we conclude 'some numbers are not numbers' or 'some odd numbers are not odd numbers'. But there are other cases in which the syllogism is valid but premises are false. Considering these combinations, Avicenna divides such mistakes into three categories: (1) The premise is true but its opposite follows from invalid (*fāsid*) syllogisms. (2) The premise is false but its opposite is obtained from some syllogisms which can be true or not. (3) There are valid (*ṣaḥīḥ*) syllogism and there are invalid syllogisms. From the valid one the true conclusion follows and from the invalid one the false conclusion, which is the opposite of the true conclusion, follows.

Yet, all these occur by means of some deception (*hīla*). Avicenna explains these deceptions which cause the aforementioned mistakes as follows[20]:

> What causes these [mistakes] is not possible without the occurrence of a deception. Among those deceptions is accepting the particular which is the contradictory of the universal, such as accepting "all sciences are suppositions" and "no medicine is supposition", or accepting "the complex is not any of its parts" and then negating one of the parts from the complex and take the complex as the middle term, such as "the white animal is not white".[21]

The first shows the case in which the opposition is not explicit. We already discussed this case in the section on the second figure. The second example, 'the

[16]The Discussion of the third figure is from (Avicenna 2017, 218).

[17](Avicenna 2017, p. 219).

[18] و لا يجب أن يظنّ أنّا لمّا جوّزنا أن يكون صدق أنتج عن كاذب أن يكون أيضاً عن متقابلتين نتيجة صادقة البتّة، لأنّ هذا ينتج أنّ الشيء ليس هو. و أمّا أنّه كيف يمكن أن يعرض هذا في جامعها عند نفسه.

[19](Avicenna 2017, p. 219).

[20](Avicenna 2017, p. 220).

[21] و أمّا إذا وقع ذلک ابتداءً فلا يمكن أن يستعمل من غير حيلة، فمن تلك الحيل أن نتسلّم جزئيّة مناقضة لكلّيّة، كما نتسلّم «أنّ كلّ علم ظنّ» و أن «لا شيء من الطبّ ظنّ» أويوهم «أنّ المركّب ليس أحد الجزأين» فيسلب أحد الجزأين عن المركّب و يجعل المركّب حدًّا أوسط، فيقال «إنّ حيوان الأبيض ليس بأبيض»

white animal is not white', means that the white animal is not the singular abstract (*muğarradan waḥda*) white, but this condition (of being singular and abstract) has not been made explicit. The conclusion has been drawn from the non-identity of white animal with white (as a singular abstract thing), i.e. as Avicenna explained, the non-identity of a complex with one of its parts.

For another example, consider 'an alive man is white', and the conclusion from it that 'that man is not white', which means the man in reality is not white as the singular abstract. Suppose that man is Zayd. Then, someone might conclude, 'that man is not white', and make a syllogism in the second figure as follows: 'that man is not white' and 'Zayd is white', therefore, 'that man is not Zayd'. But that man is, in reality, Zayd himself. The contradiction is the result of, on the one hand, referring to the same subject by two different terms, i.e. 'that man' and 'Zayd', and, on the other hand, using one term, i.e. 'white', in two different meaning as the predicates of the two premises.

For a universal example consider[22]:

All men are rational animals.
No rational animal is rational.
No man is rational.

All these cases in which a syllogism from opposite premises is established show that opposite premises are not true in reality. These syllogisms might be established by mistake or for dialectical reasons. As Aristotle mentioned, syllogisms can be made from opposite premises, however not in every mood and every figure. This lead us to the paraconsistency of Aristotelian syllogistic.

17.4 Conclusion: Paraconsistency and Syllogistic

Paraconsistent logics are those logics that are not explosive, i.e. those logics in which explosion is not a valid argument.[23] By this criterion, as we saw, Aristotelian syllogistic is paraconsistent. No paraconsistent logic commits one to accepting the truth of any one contradiction.[24] By this understanding of paraconsistency, that Aristotelian syllogistic is paraconsistent is not at odds with Aristotle's defense of the principle of non-contradiction. Avicenna argued that opposite premises of a valid syllogism and their conclusion cannot be true, and, moreover, that this does not refute the validity

[22](Avicenna 2017, p. 220).

[23]The first argument for explosion was given by William Soissons in the twelfth century. Aristotle never argued for explosion. For Aristotle not arguing for explosion see (Gottlieb 2015, Sect. 9) and (Priest 2006, p. 12).

[24]There is a distinction between dialetheism and paraconsistency. Dialetheism is the view that some contradictions are true. paraconsistent logics, on the other hand, do not commit one to accept the truth of any contradiction. If one is a dialetheist, she has to use a paraconsistent logic, but one can use a paraconsistent logic without being a dialetheist.

of the syllogism. It is only in demonstration that syllogisms have true premises and true conclusions. However, there can be valid syllogisms with false premises and conclusions in dialectics, rhetoric and poetics. In these cases, the truth of the opposite premises is only supposed without being true in reality. Thus, that some valid syllogisms can be made of opposite premises is not at odds with the principle of non-contradiction. As we saw, a syllogism from opposite premises always results in a contradictory conclusion. Therefore, according to syllogistic, only contradiction follows from contradiction. In other words, if a syllogism can be established from opposite premises, the conclusion is a contradiction. These contradictory conclusions are always self-negating, which is necessarily false in the context of Aristotelianism.

Acknowledgements I would like to thank the anonymous reviewer of this paper and Bethany Somma. They both improved this paper by their very useful comments. I am also thankful to Saleh Zarepour for a discussion on the subject of this paper. Finally, thanks to Fritz-Thyssen Stiftung for their support of a project part of which is this paper.

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Chapter 18
Is Avicenna an Empiricist?

Seyed N. Mousavian

Abstract I will focus on the following question: "Is Avicenna an empiricist?". I will introduce Avicenna's language of "signification", "understood content", "mental impression" and "conception". Then, following Kenneth P. Winkler, I will distinguish between origin-empiricism (OE) and content-empiricism (CE) and reinterpret the distinction in Avicenna's language as OEA and CEA. I will show that Avicenna's analysis of the relationship between knowledge, on the one hand, and sensation and imagination, on the other hand, includes three empiricist themes. I use these themes to argue that that Avicenna is committed to OEA. Then, I will consider three "possible" limitations to Avicenna's origin-empiricism. I will show that a common empiricist solution, that relies on the compositionality of the "understood content", *quia* demonstration and relative conceptions, has significant limitations. I will argue that a careful examination of these limitations, and the epistemology of the primary conceptions, show that Avicenna is *not* committed to CEA. I will conclude that "Is Avicenna an empiricist?" has no simple yes-or-no answer. This raises a parallel open question: "Is Avicenna a rationalist?", *in some sense*. After briefly commenting on this question, I will consider a related, but generally dissociated, question on the reality of abstraction, namely "Is Avicenna an abstractionist?" I will explain how the common replies to both questions rely on different incompatibility principles, according to which emanation from the active intellect is incompatible with apprehension by or abstraction from sense-perception. I will end by outlining the elements of a reading of Avicenna that assumes neither of these incompatibility principles.

This paper springs from previous work with Mohammad Ardeshir and Mohammad Saleh Zarepour, pursuing our common interest in Avicenna's philosophy of mathematics and epistemology. I would like to thank both for long discussions. I am also thankful to Mohammad Saleh Zarepour for bringing Winkler (2010) to my attention. I am also indebted to the comments of two anonymous referees. I discussed an earlier version of this paper in the reading group on Medieval Islamic/Arabic Philosophy I organised at IPM, Tehran, in 2015. I am grateful to my audience therein, particularly Mahmoud Morvarid, Amir Saemi, Asadollah Fallahi, and Davood Hosseini.

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M. Mojtahedi et al. (eds.), *Mathematics, Logic, and their Philosophies*,
Logic, Epistemology, and the Unity of Science 49,
https://doi.org/10.1007/978-3-030-53654-1_18

Keywords Avicenna · Empiricism · Rationalism · Concept · Content · Mental

18.1 Introduction

In his illuminating paper *The Empiricism of Avicenna*, Dimitri Gutas concludes that "it is clear from the foregoing discussion, first, that if Locke's 'philosophy of mind and cognition' is empiricist, so is Avicenna's."[1] He puts Avicenna and Locke in the same category by characterizing their philosophies as being similarly based on "our immediate experience of ourselves, an empiricism, so to speak, of the self."[2] Analogously, emphasizing that "Avicenna's appeal to the Active Intellect is part and parcel of his naturalism,"[3] Jon McGinnis concludes: "I believe Avicenna would happily endorse W. V. O. Quine's position, 'Epistemology, or something like it, simply falls into place as a chapter of psychology and hence of natural science. It studies a natural phenomenon, viz., a physical human subject' (Quine 1994, 25)."[4] This "empiricist" reading primarily aims at defeating a previously standard reading of Avicenna according to which "Avicenna is a rationalist,"[5] believes in "a priori concepts" and uses them to construct "a priori proofs" for the existence of God, for example.[6]

In this paper, I will focus on one question, namely "Is Avicenna an empiricist?". First, in Sect. 18.2, I will introduce Avicenna's language of "signification", "understood content", "mental impression" and "conception". I will illustrate this language, in Sect. 18.3, by two examples. In Sect. 18.4, I will borrow, and explain, Kenneth P. Winkler's distinction between origin-empiricism (OE) and content-empiricism (CE). In Sect. 18.5, I reinterpret the distinction in Avicenna's language, and introduce OEA (origin-empiricism reinterpreted in Avicenna's language) and CEA (content-empiricism reinterpreted in Avicenna's language) . I will show, in Sect. 18.6, that Avicenna's analysis of the relationship between knowledge, on the one hand, and sensation and imagination, on the other hand, includes three empiricist themes, namely Sects. 18.6.1, 18.6.2 and 18.6.3. I use these themes to argue that that Avicenna is committed to OEA. In Sect. 18.7, I will quickly consider three "possible" limitations to Avicenna's origin-empiricism. I will show that a common solution to such

[1] Dimitri Gutas, "The Empiricism of Avicenna," *Oriens* 40, 2012, 424.

[2] Dimitri Gutas, *Avicenna and the Aristotelian Tradition: Introduction to Reading Avicenna's Philosophical Works*, 2nd edition, Leiden, The Netherlands: Brill, 2014, 375.

[3] Jon McGinnis, "Avicenna's Naturalized Epistemology and Scientific Methods," in S. Rahman, T. Street, H. Tahiri (eds.), *The Unity of Science in the Arabic Tradition*, Dordrecht: Kluwer Academic Publishers, 2008, 143.

[4] McGinnis, Avicenna's Naturalized Epistemology, 147.

[5] Michael Marmura, "Some Aspects of Avicenna's Theory of God's Knowledge of Particulars," *Journal of the American Oriental Society* 82/3, 1962, 303.

[6] Michael Marmura, "Avicenna's Proof from Contingency for God's Existence in the Metaphysics of the *Shifā*," *Mediaeval Studies* 42, 1980, 343). Along the same lines, Goodman claims that according to "young Ibn Sīnā" the undefined elements" signified by *being*, *necessity*, *simplicity*, *goodness*, *oneness* are "a priori givens" (Lenn E. Goodman, *Avicenna*, New York, NY: Routledge, 2005, 132).

cases, that relies on the compositionality of the understood content, *quia* demonstration and relative conceptions, has significant limitations. In Sect. 18.8, I will argue that a careful examination of these limitations, and the epistemology of the primary conceptions, show that Avicenna is not committed to CEA. Therefore, I conclude, "Is Avicenna an empiricist?" has no simple yes-or-no answer. This raises an open question: "Is Avicenna a rationalist?", *in some sense*. At the end, in Sect. 18.9, after briefly commenting on this question, I will consider a related, but commonly dissociated, question on the reality of abstraction, namely "Is Avicenna an abstractionist?" I will explain how the common replies to these questions rely on two incompatibility principles, namely INC1 and INC2. I will end by outlining the elements of a reading of Avicenna that assumes neither INC1 nor INC2.

18.2 Avicenna's Language

Let us begin by explicating how I translate and interpret Avicenna's key semantic and epistemological terms in this context.[7] Elsewhere,[8] I have argued that *maʿnā* has a technical use according to which it can be translated as *signifcandum*, given some proper signification relations. Consider, for example, the following text:

> [Text 1] The meaning (*maʿnā*) of signification (*dalāla*) of a vocal expression (*lafẓ*) is this: when what is heard from the name (*masmūʿu ismin*) is imprinted (*irtasama*)[9] in the imagination (*al-khayāl*), then the *maʿnā* is imprinted in the soul (*an-nafs*) and the soul recognises/realises (*taʿarrafu*) that this heard [expression] belongs to this understood [content] (*al-mafhūm*). Then, whenever the sense brings it [i.e. what is heard from the name] to the soul, then it [i.e. the soul] turns/attends (*iltafatata ilā*) to its *maʿnā*.[10]

A *maʿnā*, as imprinted in the soul, can be associated with a vocal expression such that the soul recognizes/realizes that the impression resulting from hearing the utterance of the expression belongs to this understood content (*al-mafhūm*). In Text 1, Avicenna uses "understood [content]" in a descriptive sense, as "what is conceived". He identifies the "*maʿnā* imprinted in the soul" with "this understood [content]" or "such perceiving." This, I suggest below, is a case of identifying the *existence* of

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[7]My examples in the next section should clarify my summary of Avicenna's semantics of *ma'nā*.

[8]Seyed N. Mousavian, "Avicenna on the Semantics of *Ma'nā*," in Sten Ebbesen, Christina Thomsen Thörnqvist and Juhana Toivanen (eds.), *Forms of Representation in the Aristotelian Tradition: Concept formation*, forthcoming.

[9]I am following Gutas, *Avicenna and the Aristotelian Tradition*, 218, in translating *irtisām* as *imprinting*.

[10]Ibn Sīnā, *aš-Šifāʾ, al-Manṭiq, al-ʿIbāra* [*The Healing, The Logic, The Interpretation*], ed. M. al-Khuḍayrī (Cairo: al-Hayʾa al-miṣrīyya al-ʿāmma li-t-taʾlīf wa-n-našr, 1970, 4. All translations are mine unless otherwise specified. In my language, I use small-caps to refer to *maʿānī,* e.g. HUMAN refers to the *maʿnā* of "human." In my translation of Avicenna's texts, I use italics to refer to *maʿānī.* I also use italics to refer to conceptions and names. I hope that on each occasion, the context of use disambiguates my use of italics.

something with the *thing itself*. Otherwise put, the mental impression is the existence of the understood content. I will return to this point shortly.

Next, consider the following notorious paragraph from *aš-Šifāʾ, al-Manṭiq, al-ʿIbāra*:

> [Text 2] What is emitted vocally (*bi-al-ṣawt*) signifies what is in the soul, and these are what are called "[[affections/]]impressions" (*āthāran*),[11] whereas what is in the soul signifies things (*al-umūr*), and these are what are called "meanings" (*maʿānī*), that is, the things intended by the soul (*maqāṣida li-nafs*). In the same way, the impressions too, by analogy to [[/in relation to]] the expressions (*bi-l-qīyās ila-l-alfāẓ*),[12] are intentions [*maʿānin*].[13]

According to Text 2, the vocal expression signifies an impression (*athar*) in the soul. Given that generally by hearing the utterance of an expression, one primarily conceives its meaning, not its pronunciation for example, it follows that the "impression in the soul" signified by the expression is the *maʿnā* as imprinted in the soul, introduced in Text 1. This impression, in turn, signifies the thing (*al-amr*) intended by the soul, through some "intentional" act, e.g. the act of using the expression to signify something. The thing (*al-amr*), signified through the above signification relations, is called the "*maʿnā*" itself.

I take *maʿnā* and *amr* to be conceptually linked. *Maʿnā* stand for any *significandum*, properly signified or intended. 'Properly' is supposed to qualify the signification relation(s) involved. An expression accomplishes its semantic function through two signification relations: First, the relation between the expression and the '*maʿnā* as imprinted in the soul': the *maʿnā* of the utterance is "essentially intended" (*al-maqṣūd biḏ-ḏāt*), not accidentally, by that name (*fi ḏālik al-ism*). Second, the relation between the '*maʿnā* as imprinted in the soul' and the thing (*al-amr*) itself: the "signification of what is in the soul vis-à-vis the *umūr* is a natural signification (*dalālatun ṭabīʿīyya*)".[14] The thing (*al-amr*) is called "*maʿnā*", if the resulting signification relation is a proper combination of the above two signification relations. This picture can schematically be represented as follows (Diagram 18.1, the signification relations are signified as arrows):

There is another term, namely "conception" (*taṣawwur*), used both as an infinitive and a noun, which is primally epistemological and can be explained in terms of being cognizant/aware (*wāqif*) of *maʿnā*:

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[11] I may use double brackets to introduce my revisions/insertions to others' translations. The mental affections/impressions may be understood as accidents of a nonmaterial substance, namely the human soul, and/or of a power of the human soul, e.g. the human intellect or the human imagination. Thus, in this use, "impression" or "affection" have no connotation of *being material*, though some impressions, e.g. imaginable forms, are material.

[12] The phrase "*bi-l-qīyās ilā*", in this context, should be translated as "in relation to" instead of "by analogy to". Likewise, "*imtina' bi-l-qīyās ila-l-ghayr*" should be translated as "impossibility in relation to something else", not "impossibility by analogy to something else." Thanks to Stephen Menn for this point (Mousavian, "Avicenna on the Semantics of Ma'nā").

[13] Ibn Sīnā, *aš-Šifāʾ, al-Manṭiq, al-ʿIbāra*, 2–3. This is a slightly revised version of Deborah Black's translation: Deborah Black, "Intentionality in Medieval Arabic Philosophy," *Quaestio* 10 (2010): 68.

[14] Ibn Sīnā, *aš-Šifāʾ, al-Manṭiq, al-ʿIbāra* [*The Interpretation*], 5.

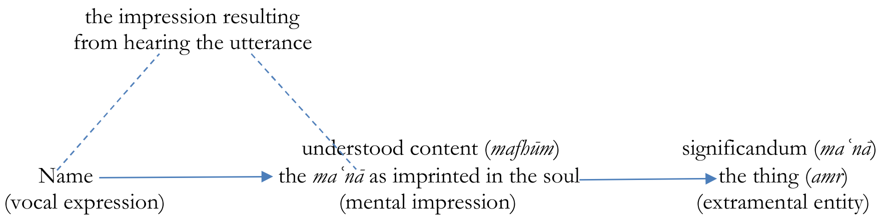

Diagram 18.1 Signification: understood content and significandum

> [Text 3] […] something can be known in two ways: first it can be only conceptualized such that, given that it has a name, if it [i.e. its name] is uttered (*nuṭiqa bih*) its *maʿnā* is exemplified (*tamathala*) in the mind, even though there is no truth or falsehood, like when it is said "human" or it is said "do such and so", so when you become cognizant/aware of the *maʿnā* you are conversed with, you have conceptualize it. And the second [way in which something can be known] is [through the] assent (*taṣdīq*) accompanying the conception […].[15]

This is a well-known distinction.[16] One can relate these pieces of terminology together assuming that Avicenna links his philosophy of language to his philosophy of mind, on a metaphysical background, by means of *maʿānī*. Let's begin from the extramental realm. A thing (*amr*) has two aspects: the *significandum* (*al-maʿnā*), if there are linguistic and mental means to reach the thing (*amr*) via proper signification relations, and the existence of this *significandum*, which is an extramental entity. (Diagram 18.1, the rightmost column.) *Maʿānī*, in turn, are identified by a set of semantic functions. *Maʿānī* have different modes of existence: namely mental and extramental. In each mode, they have a specific name, namely, the *understood content* (*al-mafhūm*) and the *significandum* (*al-maʿnā*). The *existence* of an understood content is a mental impression (an accident in the mind). Otherwise put, an understood content exists *as* a mental entity; or, the mental entity is the ontological "bearer" of the understood content. (Diagram 18.1, the middle column.) From an epistemological point of view, a conception is acquired by attentively forming the *maʿnā* in the mind. Therefore, a conception has two aspects: the understood content, or what is understood, and the existence of this content, which is an accident in the mind, namely a mental impression. The mind[17] and extramental realm are linked together through the sameness of *maʿānī*. The understood content is the same *significandum* (*al-maʿnā*) as imprinted in the mind. The *existence* of a *significandum*,

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[15] Ibn Sīnā, *aš-Šifāʾ, al-Manṭiq, al-Madkhal* [*The Healing, The Logic, The Isagoge*]. Ed. Ǧ. Š. Qanawātī, M. al-Khuḍayrī, A. F. al-Ahwānī. Cairo: al-Maṭbaʿa al-amīriyya, 1952, 17.

[16] For an old, but still useful, research on conception and assent in the Arabic philosophical tradition see: Harry Austryn Wolfson, "The Terms *Taṣawwur* and *Taṣdīq* in Arabic Philosophy and Their Greek, Latin, and Hebrew Equivalents," *The Muslim World* 33:2, April 1943, 114–128. For a recent study on the later developments of this doctrine in Mulla Sadra and his commentators, see Joep Lameer (editor and translator), *Conception and Belief in Sadr al-Din Shirazi* (Ca 1571–1635), Tehran: Iranian Institute of Philosophy.

[17] I use 'mind' for the cognitive basis or a power/faculty, in a broad sense, of the human soul, not necessarily for the human intellect.

in normal circumstances in which one intends to and succeeds in talking about an existing extramental entity, is an extramental entity. In other words, a *significandum*, in normal circumstances, exists as an extramental entity; or, the extramental entity is the ontological "bearer" of the *significandum*.[18] Some examples may help.

18.3 Two Examples

First, consider a proper name, e.g. "Zayd", as a vocal expression. Let's start from language and mind. Given that the meaning (I use "meaning" in our language) of "Zayd" is fixed, an utterance of "Zayd" *primarily* signifies the '*maʿnā* of "Zayd" as imprinted in the soul', namely **ZAYD**.[19] **ZAYD** is essentially intended by a proper use of "Zayd". The *existence* of **ZAYD** is a mental impression, that is an accident in the mind. **ZAYD**, in turn, *naturally* signifies the *maʿnā* of "Zayd" itself, namely ZAYD. The *existence* of ZAYD is an extramental individual, that is a particular substance among concrete particulars or external things (*fīl-aʿy ān*). ZAYD, the *maʿnā* itself, is the individual essence of Zayd properly intended by the mind.[20] Now let's return from the extramental realm to the mind and language. Zayd, the individual, is the thing (*amr*). If this *thing* is brought into a semantic network, via proper signification relations, then there will be a *significandum* (*maʿnā*), i.e. ZAYD. If this *maʿnā* is

[18]Two points are worth emphasizing. First, in the above circumstances, the existence of the *significandum* of a name, not the *significandum* itself, is the existence of the extramental object. In some circumstances, the extramental object may cease to exist when the name still signifies its *significandum*. Second, I will avoid using the term *suppositum* in my interpretation of *maʿnā* for three reasons: Firstly, syntactically speaking, I do not find the translation appropriate. Secondly, semantically speaking, signification is prior to supposition and *maʿnā* has a similar status in Avicenna's view: "Signification concerns the first imposition of what a term shall be used to talk about, whereas supposition concerns uses of a term that already has a signification, and the semantic variations that may be prompted by the propositional context" (Catarina Dutilh Novaes, "Supposition Theory" in Henrik Lagerlund (ed.) *Encyclopedia of Medieval Philosophy: Philosophy Between 500 and 1500*, Dordrecht, Springer, 1231). Thirdly, historically speaking, in early medieval theories "suppositum" is understood as the "bearer of form." (For a helpful discussion of early supposition theories see Sten Ebbesen, "Early supposition theory (12th–13th century)", *Histoire Épistémologie Langage*, tome 3, fascicule 1, 1981. Sémantiques médiévales: Cinq études sur la logique et la grammaire au Moyen Âge. pp. 35–48.) I may add that, generally, theories of supposition in medieval philosophy are interpreted as theories of reference. This interpretation has recently been criticized. (See, for example, Catarina Dutilh Novaes, *Formalizing Medieval Logical Theories—Suppositio, Consequentia and Obligationes*, Berlin, Springer, 2007, Ch. 1.) The debate, however, does not provide support for translating *maʿnā* as suppositum. Whether Avicenna has a theory of supposition, or something to the same effect, is a separate issue (see Allan Bäck, "Avicenna's Theory of supposition," *Vivarium* 51, 2013: 81–115).

[19]"The" in "the *maʿnā* of 'Zayd' as imprinted in the soul" is not intended to imply uniqueness. An expression may be associated with different *maʿānī*, as an extramental individual may be signified in different ways.

[20]"The *maʿnā* of "Zayd"—if taken as a unique *maʿnā*—is the unique essence (*ḏāt*) of Zayd" (Ibn Sīnā, *An-Najāt* (*al-Manṭiq*), trans. A. Q. Ahmed, in *The Deliverance: Logic* (Karachi: Oxford University Press, 2011, 6), slightly revised.

imprinted in the soul, then there will be an understood content i.e. **ZAYD**.[21] **ZAYD** and ZAYD are the same *maʿnā*, with two modes of existence. This can be formulated as follows: **ZAYD** or ZAYD, the *maʿnā*, has no fixed ontology. (This formulation, however, is not necessary for my thesis on Avicenna's epistemology.)[22] If the *maʿnā* exists in external "reality", then the existence of ZAYD is an extramental individual (a substance). If the *maʿnā* exists in the mind, then the existence of **ZAYD** is a mental impression (an accident). Avicenna's epistemology is built on *maʿānī* as conceived and mental states and attitudes. If one is cognizant/aware of the *maʿnā* as imprinted in the soul, one has formed the/a conception (*taṣawwur*) of Zayd, that is a piece of knowledge.[23] Conceptions (and assents) are building blocks of Avicenna's epistemology.

Here is the second example. In Chapter 8 of Book 3 of *The Metaphysics of the Healing*, Avicenna addresses the following objection to his theory of knowledge: Knowledge of something (existent) is acquiring its form without the matter. Therefore, knowledge of a(n existent) substance is acquiring the form of the substance without the matter. The form of a substance is a substance. Knowledge of a substance, however, is not a substance, rather it is an accident in the mind. But one and the same thing cannot be both a substance and an accident. Avicenna's reply goes like this:

> [Text 4] We say: The quiddity of the substance is substance in the sense that it is the existent in concrete particulars/external things (*fīl-aʿy ān*) not in a subject. This description exists for the quiddity of the intelligible substances for it is a quiddity with the disposition to exist, not in a subject, in external things. That is, this quiddity is intellectually apprehended of a thing (*amr*) whose existence in external things is not in a subject. As for its existence in the intellect with this description, this is not in its definition inasmuch as it is a substance. That is, it is not the definition of the substance that it is not in a subject in the intellect; rather, its definition is that, regardless of whether it is or is not in the intellect, its existence in the concrete is not in a subject.[24]

Avicenna appeals to what Stephen Menn calls "neutral quiddity,"[25] which can be generalized to "neutral *maʿnā*", assuming that the quiddities under discussion are

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[21] One might object that Zayd is a particular/individual and thus has no intelligible *understood content* (*mafhūm*). In reply, two points are worth mentioning. First, *maʿānī* (and *mafāhīm*) need not necessarily be intelligible or universal; they may have different epistemological profiles. Second, for Avicenna, an individual/particular can be understood in a *universal way*. Thus understood, an individual/particular like Zayd can have an intelligible *understood content* (*mafhūm*).

[22] In contemporary language, this can be explained in different ways. I believe that Geach's theory of relative identity and Meinong's principle of independence can provide alternative frameworks for reconstructing Avicenna's idea. For a short discussion on this point see Mousavian, "Avicenna on the Semantics of Ma'nā", footnotes 13–15. Stephen Menn explains the idea as "Avicenna's theory of neutral quiddity" (Stephen Menn, "Avicenna's Metaphysics" in Peter Adamson (ed.) *Interpreting Avicenna; Critical Essays*, Cambridge, Cambridge University Press, 2013, 165–169). This directly relates to the next example above.

[23] Again, I suppose there are more than one conception of Zayd. Here by "the conception of Zayd" I mean the conception that is based on the individual essence of Zayd.

[24] Ibn Sīnā, *The Metaphysics of The Healing* [*aš-Šifāʾ, al-Ilāhiyyāt*], ed. Michael Marmura, Provo: Brigham Young University (Islamic Translation Series), 107–108, slightly revised.

[25] Menn, "Avicenna's Metaphysics", 169.

quidditative *maʿānī*. If a substance itself, e.g. a human, exists among concrete particulars, it does not exist *in* a subject. After all, it is a substance, by assumption. If a substance is *properly* signified, it is semantically characterized as a *maʿnā*, namely HUMAN. This *maʿnā* can be quidditative in the following sense: it is *all* that is necessary for its exemplification instance (*šakhṣ*) and no other *maʿnā* is such.[26] HUMAN is a quiddity. The *existence* of HUMAN is an extramental substance. If HUMAN is intellectually apprehended, then the *maʿnā* as imprinted in the soul is the understood content, namely **HUMAN**. The existence of **HUMAN** is a mental impression (an accident in the mind). **HUMAN**, as a quiddity in the mind, is a substance in the sense that "it is intellectually apprehended of a thing (*amr*) whose existence in external things is not in a subject". In other words, this *maʿnā* has a "dispositional" character: when it exists among concrete particulars/external things (*fīl-aʿy ān*), it does not exist in a subject.[27] **HUMAN** and HUMAN are the same *maʿnā* with two modes of existence. This can be formulated as follows: **HUMAN** or HUMAN, the *maʿnā*, has no fixed ontology. (Again, this formulation is not necessary for my thesis on Avicenna's epistemology.) If the *maʿnā* exists in external "reality", then the existence of HUMAN is an extramental individual (a substance). If the *maʿnā* exists in the mind, then the existence of **HUMAN** is a mental impression (an accident). To repeat, Avicenna's epistemology is built on *maʿānī* as conceived and mental states and attitudes. The *maʿnā* of "human" as imprinted in the soul, i.e. **HUMAN**, is an understood content. If one is cognizant/aware of this *maʿnā*, one has formed the conception (*taṣawwur*) of human, that is a piece of knowledge.

18.4 What Is Empiricism?

Now we may move to our main question, namely "Is Avicenna an empiricist?" By 'empiricism' I mean concept empiricism. Following Winkler,[28] let us distinguish origin-empiricism from content-empiricism as follows:

(OE) Origin-empiricism: For every concept C, the origin of C is experience.

(CE) Content-empiricism: For every concept C, the content of C is characterized by some experiential conditions.[29]

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[26] Seyed H. Mousavian, "Mahīyyat az Dīdgāhe Ibn Sīnā" [Avicenna on the Quiddity], *Maqālāt wa Barrisīhā* 71, 1381/2002, 251.

[27] Menn, "Avicenna's Metaphysics", 166.

[28] Kenneth P. Winkler, "Kant, the empiricists, and the enterprise of deduction," in Paul Guyer (ed.), *The Cambridge Companion to Kant's Critique of Pure Reason*, Cambridge University Press, 2010, 48–60.

[29] This formulation is slightly different from Winkler's ("Kant, the empiricists, and the enterprise of deduction," 50). He uses the notion of *experiential terms*, in one occasion, and the notion of *implications for experience*, in another, and moves between the modal and non-modal characterizations. I have avoided using 'experiential terms' hoping to sidestep the philosophy of language issues raised by this terminology.

OE and CE are formulated in our language. For simplicity, I will primarily confine myself to simple or atomic concepts. By "the origin" I mean "the causal origin" and by "the origin of C is experience" I mean that some experience of a specific type essentially contributes to the formation of C.[30] For an experience to be the origin of C, the concept and the experience should be linked properly such that the content of C applies to the object that is the salient cause of the corresponding experience, namely an instance (or a reference) of C.[31] For example, if seeing Zayd, the individual, contributes to the experience which leads to the formation of one's concept of human, then the content of the concept of human applies to Zayd, who is the salient cause of this experience.[32] It follows, then, that "Zayd is human" is true. Depending on whether "the" in "the origin of a concept" implies uniqueness or not (in which case, other causal factors may also contribute to the formation of C), OE finds strong or weak readings.

The so-called "empiricist" philosophers, e.g. John Locke, can accept OE; accordingly, all concepts are causally based on experience, directly or indirectly.[33] OE, in some form, can also be held by many philosophers who are *not* categorized as "empiricist." For example, Immanuel Kant, as a transcendental idealist, approves that "As far as time is concerned, then, no cognition in us precedes experience, and with experience every cognition begins."[34] In contrast, the so-called "rationalist" philosophers, e.g. Descartes, can reject OE. A rationalist may hold that there are some "innate" concepts. The origin of an innate concept is the human mind, in the sense that the concept is "present in the mind", in some form, from the origination of the human mind.[35] Descartes believes in Cartesian innate ideas, whereas Locke does not.

Back to CE, by "experiential conditions" I mean some "information", that essentially contain reference to (actual or counterfactual) experience. By "essentially contain reference to" I mean that reference to experience is necessary for the characterization of the "information" in question. Depending on whether the content of a concept is fully or partially characterizable by experiential conditions, CE finds strong or weak readings. Nonetheless, a rationalist like Descartes, who has a broad

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[30]Thus, Avicenna's theory of causation is directly relevant to his theory of knowledge. I will return to this point in Sect. 18.9.3.

[31]Here by "experience," I shall mean *veridical experience*.

[32]If one is externalist with regard to the content of the concept of human, this content can be the very property of *being a human*.

[33]See, for instance, Locke: "*All ideas come from sensation or reflection*. [...] Whence has it [i.e. the mind] all the materials of reason and knowledge? To this I answer, in one word, from *experience*. In that all our knowledge is founded" (John Locke, *An Essay Concerning Human Understanding* (abridged edition), edited by Kenneth P. Winkler. Hackett Publishing Company, 1996, 33). Italics are in the original.

[34]Immanuel Kant, *Critique of Pure Reason*, translated and edited by Paul Guyer and Allen W. Wood. Cambridge University Press, 1998, 136. The reference was originally found in Winkler, "Kant, the empiricists, and the enterprise of deduction," 48.

[35]René Descartes, *The Philosophical Writings of Descartes*, vol. 1, translated [with a] general introduction by John Cottingham, Robert Stoothoff, and Dugald Murdoch, Cambridge University Press, 1985, 306.

notion of "experience" which means "whatever reaches the intellect from external sources or from its own reflexive contemplation,"[36] may find CE objectionable. For him, there are some innate "idea", e.g. *God*, whose content is not characterized by experience; "the content of this idea [i.e. *God*] is so great as to exceed the capacity of my mind to construct it."[37] Descartes' solution for the problem of acquiring this idea is that "God, in creating me, should have placed this idea in me to be, as it were, the mark of the craftsman stamped on the work— not that the mark need be anything distinct from the work itself."[38]

18.5 Empiricism Reinterpreted

I depart from the use of "concept" in contemporary ancient and medieval studies according to which a concept is necessarily a stable, universal, public intelligible entity.[39] A concept may be tolerably unstable and undergo some changes without losing its identity, or at least I'd like to keep this possibility open. Concepts needn't be necessarily universal either: A particular concept, namely a concept that is semantically characterized to purport to refer to a particular individual, is not conceptually incoherent. Finally, not all concepts need to be public: the possibility of having essentially private concepts should not be eliminated by our way of using the term "concept." Therefore, I shall opt for a minimal conception of "concept". A concept is a relatively stable mental entity that has about-ness and is not truth-evaluable; a concept may be "true of" something (not true simpliciter), in the sense of being about it.[40]

By the "content" of a concept I mean the epistemologically significant feature of the concept. The content of a concept explains the meaning of the expression associated with the concept. A concept, as a mental entity, is part of a mental state. Mental states are states of the human mind and necessarily so. The content of a concept is what one understands (comes to know)[41] when one possesses the concept. This is a rudimentary explanation of "content" and intentionally so.[42] A short comparison

[36] John Cottingham, *A Descartes Dictionary*. The Blackwell Philosopher Dictionaries. Blackwell, 1993, 59.

[37] John Cottingham, *A Descartes Dictionary*, 90.

[38] René Descartes, *The Philosophical Writings of Descartes*, vol. 2, translated [with a] general introduction by John Cottingham, Robert Stoothoff, and Dugald Murdoch. Cambridge University Press, 1984, 35. John Cottingham, *A Descartes Dictionary*, 165.

[39] For such a use of the term see, for example, Christoph Helmig, *Forms and Concepts: Concept Formation in the Platonic Tradition*. Göttingen; De Gruyter, 2012, chapter 1.

[40] Here, in having "relatively stable", I am following Helmig, *Forms and Concepts*, 16.

[41] Here I use "knowledge" in a broad sense.

[42] Among contemporary analytic philosophers, it's fair to say, there is no consensus on what the content of a concept is. For a naturalistic account of the content of a concept in terms of its reference, see Jerry A. Fodor and Zenon W. Pylyshyn, *Minds without Meanings: An Essay on the Content of Concepts*, The MIT Press, 2014. For a contrary view, according to which the content of a concept

between my use of "content" and Fregean senses may be helpful.[43] Tyler Burge, for example, distinguishes between three functions that the Fregean sense of an expression is supposed to perform[44]: (1) the "mode of presentation to the thinker" that accounts for the information value of the expression, (2) what determines the reference of the expression, and (3) the reference in "oblique contexts" (for instance, in "Hammurabi believes that Hesperus is not Phosphorus", "Hesperus" refers to the *sense* of "Hesperus", not Hesperus itself). In my usage, the content of a concept, as its epistemologically significant aspect, *may* perform some semantic functions but it is not necessary to perform all of the above functions.[45] More particularly, for my purposes, "content" needs to perform the first function. Thus, I borrow Burge's term "information value" and apply it to concepts, as mental entities, to explain "content". I assume that what "accounts" for the information value of a concept is some information.

To study Avicenna's epistemology from the perspective of empiricism/rationalism debate, a translation between our language and Avicenna's is necessary. As I read Avicenna, he does not have a term that can literally be translated as "concept." Instead, one finds at least four technical terms in the vicinity, namely: *significandum* (*maʿānī*), conception (*taṣawwur*), understood content (*al-mafhūm*) and mental impressions (*athar ḏihnī*).[46] The closest term, in meaning, to "concept" is *conception* (*taṣawwur*). Assuming that a *maʿnā* is not necessarily mental but a "concept," in our language, is necessarily *mental*, it follows that *maʿānī* are not necessarily identical with concepts. Conceptions, for Avicenna, have two aspects: understood content (*al-mafhūm*) and mental impression (*athar ḏihnī*). Here are two working hypotheses of mine. First, our use of "content" can be translated as "understood content" (*al-mafhūm*), in Avicenna, and "what is conceived" or the "*maʿnā* as imprinted in the soul", in Avicenna, can be

(intension) is "wholly determined by the intrinsic state of the thinker" see, for example, David Chalmers, *Constructing the World*, Oxford: Oxford University Press, 2012.

[43]For an introduction to a systematic comparison between Frege's logic and Avicenna's, see Wilfrid Hodges, "Ibn Sina, Frege, and the Grammar of Meaning," *Al-Mukhatabat* 2013: 5/2, 29–60.

[44]Tylor Burge, "Belief *De re*", in *Foundations of Mind*; *Philosophical Essays Volume 2*, Oxford; Oxford University Press, 2007, 59.

[45]Hodges, in his comparative study between Frege and Avicenna, concludes that "ultimately both Ibn Sina and Frege rely on a rather chemical notion of meanings that allows us to split meanings into their elements and build them up into compound structures by putting bonds between them" (Hodges, "Ibn Sina, Frege, and the Grammar of Meaning," 60).

[46]One may disagree with my claim and hold that one of the above terms, or a closely associated expression in Avicenna's language, is literally translatable as "concept", as we use the term. I hope that this dispute does not have a significant bearing on the philosophical side of the problem, unless perhaps forcing a new translation of my argument. I tried to broaden the meaning of "concept", to cover different kinds of mental representation, and then single out the epistemological aspect of "concept" as "content." I should add that *at-taṣawwur* and *al-mafhūm*, in Avicenna, are close to "concept" and "content", correspondingly. Also, translating *at-taṣawwur* and *al-mafhūm* as "conception" and "concept", respectively, may work if one keeps in mind the wide use of *al-mafhūm*, not being restricted to universals, and its primary infinitive sense, as *what is conceived*. To be more faithful to the literal meaning of *al-mafhūm*, however, I translated it as "understood [content]", putting "content" in brackets to indicate that this part is not syntactically present in the original word.

interpreted as "content" in our language. Second, when we talk about "concept", as a mental entity, it can be translated as "mental impression" (*athar ḏihnī*) in Avicenna, and "accident in the mind", in Avicenna, can be interpreted as "mental entity" in our language. Recall that the *existence* of the '*maʿnā* as imprinted in the soul' is an accident in the mind. It is also noteworthy that mental impressions can have different ontological, as well as epistemological, profiles: intelligible (*maʿqūl*), imaginable (*mutakhayyal*) and sensible (*maḥsūs*) *maʿānī*. For our purposes, we may only need to focus on the *intelligible maʿānī*.[47] Having these translations handy, OE and CE can be reinterpreted in Avicenna's language as follows:

(OEA) Origin-empiricism (in Avicenna's language): For every mental impression *M*, the origin of *M* is experience.

(CEA) Content-empiricism (in Avicenna's language): For every understood content *U*, U is characterized by some experiential conditions.

18.6 Is Avicenna an Origin-Empiricist?

As Gutas explains, Locke and Avicenna are committed to the thesis that "all the materials of Reason and Knowledge" come "from Experience":

> In the passage from the *Išārāt* where Avicenna introduces sensation and reflection as the two aspects of *mušāhada*, there is no doubt that he is referring to Experience, our direct apprehension of the outside world and internal self. It is precisely what John Locke referred to by the term "experience" in his Essay Concerning Human Understanding, Bk. II, Ch. I, §2: "Whence has [the Mind] all the materials of Reason and Knowledge? To this I answer, in one word, From Experience: … Our Observation employ'd either about *external, sensible Objects; or about the internal Operations of our Minds, perceived and reflected on by ourselves* […]."[48]

By "Experience", here, Gutas refers to the perceptual faculty of experience. Gutas makes his point clearer when he explains the role of sensation in cognition according to *An-Najāt* [*The Salvation*]: "Sensation then does provide the intellect the raw data out of which it can form the concepts which lead to definitions and then the primary propositions" (Gutas 2012, 406).[49]

Avicenna's main works include three origin-empiricist themes: (i) sense perception and imagination assist the intellect in perception (ii) perceiving the intelligibles needs the mediation of the sensible forms, and (iii) lack of sensation implies lack of knowledge. Below I will quickly examine each theme and its possible interpretations.

[47] The human intellect, for Avicenna, has no physical organ. The mental impressions associated with the intelligible *maʿānī* are not physical either. They are accidents in the human intellect, which is an immaterial substance.

[48] Dimitri Gutas, "The Empiricism of Avicenna," *Oriens* 40; 2012, 429.

[49] For Avicenna, the primary propositions, e.g. the whole is greater than the part, are the most foundational premises, epistemologically speaking.

18.6.1 Sense Perception and Imagination Assist the Intellect

Let's begin with textual evidence:

> [Text 5] In forming concepts, sense perception and imagination (*takhayyul*) assist the intellect because sense perception presents to the [internal sense of] imagery (*khayāl*) things in a mixture, and imagery [presents them] to the intellect. The intellect then discriminates among them, breaks them down into parts [i.e., categories], takes up each one of the concepts individually,[50] and arranges [in order] the most particular and the most general, and the essential and the accidental. Thereupon there are impressed on the intellect, in [a process of] concept formation, the primary notions (*al-maʿānī al-ūlā*), and then definitions are composed out of them.[51]

The expression "things in mixture" is a translation of *umūran mukhtaliṭatan*, which most likely refers to sensible and imaginable "forms" with their particularizing accidents. In the second sentence, Avicenna says that "the intellect then discriminates among them, breaks them down into parts", and Gutas adds in brackets "i.e. categories". This may be questionable if by "categories" Gutas means "Aristotelian categories" and thus intends to conclude that the human intellect at this stage comes to possess the conceptions of Aristotelian categories. Avicenna, in Text 5, attempts to explain one of the most foundational mental acts of the human intellect, namely "forming conceptions" (*taṣawwur*) and it is not clear how the intellect possesses Aristotelian categories before forming *al-maʿānī al-ūlā*. This reading, in fact, may be used to justify a rationalist reading of Avicenna, and thus undermine Gutas's empiricist construal. One may note that in "…takes up each one of the concepts individually", Gutas uses "concept" as a translation of *al-maʿānī*. However, *al-maʿānī*, in general, are not "concepts" or conceptions, as was explained above.[52] The middle part of Text 5 can be read as explaining the more complex mental operations, performed by the human intellect through the power of estimation, on the sensible and imaginable *maʿānī* as imprinted in the soul. The key sentence in Text 5, I suggest, is the following: "Thereupon there are impressed on the intellect, in [a process of] concept formation, the primary notions (*al-maʿānī al-ūlā*), and then definitions are composed out of them." Let me discuss this sentence in more detail.

"Thereupon there are impressed on the intellect in [a process of] concept formation", as a conclusion of the paragraph, suggests that the *impressions* of the conceptions, at this stage, are causally linked to the sensible and imaginable forms already discriminated and broken down to parts by the intellect at the earlier stages of cognition.[53] This provides evidence for Avicenna's commitment to OEA. Nothing here,

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[50]The term "concept" is Gutas's translation of *al-maʿānī*. I will discuss this point shortly. The translation can be acceptable if by *al-maʿānī* here Avicenna means *al-maʿānī* as imprinted in the mind.

[51]Ibn Sīnā, an-Najāt, ed. M. T. Dānešpažūh, Tehran: Entešārāt-e Dānešgāh-e Tehrān, 1364/1985–1986, 170. The translation is Gutas's (2012, 406).

[52]Also see Mousavian, "Avicenna on the Semantics of *Ma'nā*."

[53]This, of course, does not eliminate the possibility of *conceptual* link between sensible and imaginable *maʿānī*, on the one hand, and intelligible *maʿānī*, on the other hand.

however, excludes the possibility of contribution of other non-empirical/material causes, e.g. the active intellect, to the origination of the impressions of *al-maʿānī al-ūlā*.[54] As I have already explained,[55] there are reasons to believe that by *al-maʿānī al-ūlā* Avicenna does *not* necessarily, and only, mean the primary conceptions (e.g. *existent*, *thing*, and *necessary*). Here are two reasons to distinguish between "first" and "primary" intelligibles (and allow both to cover conceptions as well as assents (whose "objects" are propositions, or premises).

First, *al-ūlā* is the feminine form of *al-awwal* (the first) whose irregular plural form is *al-awāʾil*. *Al-awwal* has a relative adjective as *al-awwalī* (primary), and *al-awwalīyya* (feminine), whose regular plural form is *al-awwalīyyāt*. On some occasions, as in relation to propositions, Avicenna does not use *al-ūlā* (first) and *al-awwalī* (primary) interchangeably:

> [Text 6] The first thing originated in the material intellect (*ʿaqli hayūlānī*) by the active intellect (*ʿaqli faʿāl*) is the dispositional intellect (*ʿaqli bi-malaki*). And that is the forms of the first intelligibles (*al-maʿqūlāt al-ūlā*) some of which occur (*ḥaṣala*) [in the intellect] by no experience, no syllogism, and no induction at all, like "the whole is greater than the part", and some of which occur [in the intellect] by experience, like "every [chunk of] earth is heavy".[56]

In Text 6, the *first* intelligible premises or propositions, include, and are not identical to, the primary intelligible premises such as "the whole is greater than the part". Note that a proposition like "every chunk of earth is heavy", which can only be known by experience, is also categorized as a "first intelligible."

Second, the first intelligibles, or the firsts (*al-awāʾil*) are introduced as being obtained (*tahṣulu*) in the human intellect not by acquisition (*iktisāb*). This does not necessarily imply that the firsts are indefinable (if they are conceptions) or indemonstrable (if they are premises or propositions). Consider, for example:

> [Text 7] 'The firsts [or the first intelligibles] are obtained (*tahṣulu*) in the human intellect not by acquisition (*iktisāb*) and [the intellect/the soul] does not know [/is not aware of] wherefrom [they] are obtained and how [they] are obtained in it [i.e. in the intellect/the soul]'.[57]

It may well be the case that the first intelligibles, as conceptions, include the primary conceptions such as *existent*, *thing* and *necessary*, as well as some "empirical" conceptions, such as *body* and *motion*.[58] It is worth emphasizing that, at the end of Text 5, Avicenna says that "definitions are composed out of" *al-maʿānī al-ūlā*. Given that one cannot compose definitions *only* by means of the primary conceptions such as *existent*, *thing*, and *necessary*, it follows that a wider range of conceptions

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[54] See Sect. 18.9 below.

[55] Mousavian and Ardeshir (2018, 222–223).

[56] Ibn Sīnā, *Al-Mabdaʾ wa-l-maʿād* [*The Provenance and Destination*], ed. Nūrānī, Abdallāh, Tehran: The Institute of Islamic Studies, 1363/1984, 99.

[57] Ibn Sīnā, *At–Taʿlīqāt* [*The Annotations*]. Ed. Seyyed Hossein Mousavian. Tehran: Iranian Institute of Philosophy Press, 2013, 46. See Mousavian and Ardeshir (2018, 215).

[58] I assume that, for Avicenna, conceptions like body and motion originate in experience.

are included under "the first intelligibles." The first intelligibles are called "first" in the sense of initially being obtained, not acquired through discursive knowledge.[59] Let me tentatively conclude that widening the scope of "first intelligibles" bolsters Avicenna's origin-empiricist theme expressed in Text 5.

18.6.2 Perceiving the Intelligibles Needs the Mediation of the Sensible Forms

A clear formulation of this theme can be found in *at-Taʿlīqāt*:

> [Text 8] It is not possible for man to perceive the intelligible-ness of things without the mediation of their sensible-ness on account of the deficiency of his [rational] soul and his need for the mediation of the sensible forms in order to perceive the intelligible forms.[60]

This reference, used by Gutas among others, should be taken with extra care. Two points are worth emphasizing: First, "things", in the first sentence, in the above context refers back to "sensible things", things that are capable of producing sensible forms in the human soul, as the previous sentence witnesses: "The soul perceives the sensible forms by means of the senses and it perceives their intelligible forms through the mediation of their sensible forms."[61] So, dependency on sensible forms, here, applies to conceptions whose referents are sensible, not necessarily to *all* conceptions. Second, the passage is part of a longer section on the distinction between humans' perceiving and separate (or immaterial) substances'. What is most relevant for explaining this distinction, after Avicenna's emphasis that "perceiving belongs to the soul" and is not sensation, is the phenomenon of intellection of the conceptions of sensible objects, not that of all conceptions. Particularly, conceptions that do not require the epistemic process of abstraction, in which the particularizing accidents are separated from sensible and imaginable forms, are not at stake.

The very same point has also been discussed at the first part of the 6th chapter of the fifth treatise of the psychology of *The Healing* where Avicenna attempts to explain the difference between two ways of conceiving intelligibles depending on whether the form of the intelligible is abstracted by the intellect's act of abstraction or the form is abstract in itself.[62]

To conclude, Text 8 does not aim at explaining the origin of *all* conceptions perceivable by the human intellect. And in this regard, it falls short of establishing Avicenna's full-fledged commitment to OEA.

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[59] "Discursive knowledge", here, primarily refers to definitions and demonstrations.

[60] Ibn Sīnā, *at-Taʿlīqāt*, ed. Mousavian, 31. Translation is Gutas's (2012, 412).

[61] Gutas, "The Empiricism of Avicenna," 412.

[62] Ibn Sīnā, *Avicenna's De anima* [*aš-Šifāʾ, aṭ-Ṭabīʿīyāt, an-Nafs*], London, Oxford University Press, 1959, 239.

18.6.3 Lack of Sensation Implies Lack of Knowledge

Here is the most succinct formulation of this theme:

> [Text 9] Thus, everyone who lacks some sensation (*ḥiss-um-mā*) also lacks some knowledge (*li-ʿilm-im-mā*), though sensation is not knowledge.[63]

Text 9 has two claims: First, a bottom-up principle: if some sensation is missing, then some knowledge is missing. And second, sensation is *not* knowledge; they belong to two different categories. A rationalist may also accept these claims. For example, she can accept the first claim and explain it as follows. There are two kinds of intelligibles: some originate in sensation and some do not. With regard to the first category, if some sensation is missing, then some knowledge is missing. This, however, does not imply that there is no mental impression whose origin is not experience. In fact, Text 9 does not imply the following top-down principle: if some knowledge is missing, then some sensation is missing. The second claim is part of a rationalist view as well: sensation and knowledge are distinct.

Avicenna discusses the issue in more detail some pages earlier:

> [Text 10] It is necessary to know that nothing from intelligibles is sensible and nothing from sensibles, inasmuch as it is brought before sensation, is intelligible, namely [it] is brought before intellect's perception (*li-idrāk-il-ʿaql*), though sensation is some origin (*mabdaʾ-um-mā*) for obtaining (*ḥuṣūl*) most of the intelligibles (*kathīrun min-al-maʿqūlāt*).[64]

Text 10 suggests an empiricist reading of Avicenna: "sensation is some origin (*mabdaʾ-um-mā*) for obtaining [not acquiring] most of the intelligibles". This can be read as follows: sensation, at *some* level of explanation, contributes to the origination of intelligible conceptions in the human mind. This phrase implicates that sensation can contribute in different modes, such as immediate vs. mediate, to obtaining intelligibles. The same sentence can also be read as suggesting that there are other contributing factors, beside sensation, to the cognitive process of obtaining intelligibles. Note that the disconnection between sensibles and intelligibles, as mutually exclusive objects of perception, is not explained symmetrically: when no intelligible is sensible, unqualifiedly, a sensible is not an intelligible, in a qualified way: the sensible inasmuch as it is brought before sensation is not intelligible. A sensible is not intelligible in the sense that it has particularizing accidents and particularizing accidents, as particulars, are not intelligible. Therefore, that which is immediately presented to sensation is not intelligible. This leaves room for the "sensible" to be described as "intelligible" under a different mode of consideration, by the intellect. One point, however, clearly circumscribes the empiricist interpretation. The word "most of" (*kathīr*), also translatable as "many", implicates that "the" origin of some intelligibles may not be sensation. Therefore, it is perfectly consistent with Text 10

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[63] Ibn Sīnā, *Aš-Šifāʾ, al-Manṭiq, al-Burhān*, ed. Abū-l-ʿAlā ʿAfīfī, Cairo, al-Maṭbaʿa alamīriyya, 1937/1956, 224. Or, in its Latin formulation, *nihil est in intellectu quod non prius fuerit in sensu* ("there is nothing in the intellect which was not first in the senses").

[64] Ibn Sīnā, *Aš-Šifāʾ, al-Manṭiq, al-Burhān*, ed. ʿAfīfī, 220.

to suppose that there are two kinds of intelligibles: some that originate in experience (and these may be described, in some sense, as intelligible) and some that do not. Alternatively, it may be that by "some origin (*mabdaʾ-um-mā*)", Avicenna means some form of "immediate origin" which does not include involvement of other intelligibles or arguments for obtaining (*ḥuṣūl*) most of the intelligibles. These observations necessitate a careful look at possible limitations to Avicenna's origin-empiricism.

18.7 Limitations to Avicenna's Origin-Empiricism

Let me quickly consider three prima facie problematic cases for Avicenna's origin-empiricism: celestial bodies, unseen things and immaterial substances. (I will examine the primary conceptions such as *existence*, *thing*, and *necessary* in a separate section.)

18.7.1 Celestial Bodies

Gutas introduces "two concessions" to Avicenna's "empiricism":

> The first is his admission that in the formation of concepts (*taṣawwur*) through definitions, it is possible to arrive at knowledge of concepts of some real existents not through sensation but intellectively: these are the celestial bodies which can be grasped individually by the intellect alone because they are the only member of their particular species. However, the intellective reasoning that establishes the existence of the celestial bodies is ultimately based on the existence of the necessary existent, which is itself based on the empirical sense that we exist and that there is existence.[65]

The problem that celestial bodies raise for Avicenna's "empiricism" is that one may come to "grasp them" individually, with no immediate help of sense perception, and solely by "intellective reasoning", given Avicenna's epistemology and metaphysics. This seems to make "experience" unnecessary for the origination of such conceptions. Gutas replies that "knowledge" of the existence of the celestial bodies is based on the knowledge of the necessary existent and that, in turn, is based on experience. This reply, in the above formulation, however, does not exclude a significant alternative possibility. For a Cartesian, for instance, the idea of God is innate in at least two senses: its origin is not experience and it is present in the human mind from its origination.[66] Thus, experience is unnecessary for obtaining the idea of God. An argument for the existence of God, nonetheless, may contain other ideas that originate in experience (or are not present in the human mind at its origination). One might envisage a similar possibility for Avicenna's conception of a celestial body. Accordingly, the conception of a celestial body does not originate in experience,

[65] Gutas, "The Empiricism of Avicenna," 418–419.
[66] Cottingham, *A Descartes Dictionary*, 92.

though there is some argument for its existence that employs empirical conceptions. This consideration in itself, however, is not tantamount to a criticism of Gutas's *claim.*

One may try to accommodate the conception of a celestial body into Avicenna's "empiricism" not through the analysis of an argument for the existence of a celestial body, rather through an analysis of its very conception. The conception of a celestial body is a composite conception. Note that a celestial body itself is a composite material object. It is composed of a celestial form and some matter. Like other material objects, therefore, it can be known through composition of other conceptions that, mediately, are based on experience. Thus, the conception of a celestial body can be acquired through composition of other conceptions which themselves are based on experience.[67]

18.7.2 Unseen Things

Gutas's "second concession to Avicenna's empiricism" is "knowledge that humans can have of what Avicenna calls the "unseen" (*al-ghayb*)", including truths from the future:

> Human souls acquire this knowledge directly from their congeners, the souls of the celestial spheres, through the mediation of the practical intellect of the humans, which transmits it, and of the internal sense of imagination, which represents it. […] The rational soul, or the intellect, has the vanguard role both in investigating these phenomena and, once it has found their cause, resting in the certainty of the knowledge. […] Thus, even in this case of knowledge of the "unseen" (*al-ghayb*) whose source is manifestly non-sensory, Avicenna treats it as knowledge acquired through testing […].[68]

Experience seems unnecessary for receiving knowledge of the "unseen", including future events, from celestial spheres. Gutas, defending Avicenna's "empiricism", explains this case by appealing to "testing", categorized as "experience" by Avicenna: one "should observe them", namely the knowledge-claims about unseen matters, "several times successively in others" to the point that all this becomes experience (*tajriba*)."[69] Obtaining or receiving some piece of knowledge through "testing" does not imply that the concepts involved in the process of testing originate in experience, even if the knowledge itself is certain. Testing and approving may confer *certainty* on a piece of knowledge via establishing the reliability of the underlying processes involved. For example, a mathematical proposition, say Goldbach conjecture namely "every even integer greater than two is the sum of two prime numbers", can empirically be tested and approved by different numerical methods. This testing, however, hardly provides evidence against a form of mathematical Platonism according to which mathematical concepts do not originate in experience.

[67] This proposal may not save Gutas's reasoning but can support his claim.

[68] Gutas, "The Empiricism of Avicenna," 419–420.

[69] Gutas, "The Empiricism of Avicenna," 420.

Again, Avicenna's commitment to OEA can be defended by appealing to the compositionality of (the understood contents of) the conceptions of unseen things. A future event, for example, is presently nonexistent *in re*. Nonetheless, its conception, like the conception of an existent event *in re*, is a composite conception that contains conceptions of other things that have already been known and are mediately based on past experience. What is not known about future events, and can only be known through intellect and testing, is their "existence in future", not their conceptions.

18.7.3 Immaterial Substances

The immaterial substances, including human intellects/souls, celestial intellects/souls, and the First or God, may also be prima facie problematic cases for Avicenna's "empiricism". The conception of a human soul as *self* has a distinguished status if the phenomenon of self-awareness, in some sense, is non-empirical conceptual knowledge.[70] According to Gutas, "Black would seem to be denying the empirical basis of the phenomenon ("it can hardly be an empirical inference")."[71] Gutas's response, in this particular case, is to appeal to a "reflective" sense of self-awareness in which "reflection […] is another way of knowing through Experience."[72] This reply, however, cannot save the foundational explanatory role of *primitive* self-awareness as always in, or more precisely identical to, the human intellect, "even in sleep or drunkenness".

As with celestial intellects/souls, Gutas's strategy is to show that apparently non-empirical knowledge of such objects is based on the knowledge of the existence of the necessary existent or God. Consequently, the problem boils down to problem of reconciling knowledge of God with Avicenna's "empiricism". Given Avicenna's emphasis on the modal argument for the existence of God and the fact that this argument assumes the contingency of material objects, Gutas concludes that this series of "intellective reasoning" is ultimately based on "Experience" (assuming that knowledge of the contingency of material objects is obtained through experience).

A celestial intellect and God are not composite objects, like celestial bodies, nor can their conceptions be derived from the present or past experiences, like unseen things,[73] nor are they immediately knowable, like one's self. To accommodate such

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[70] If primitive awareness of the self contains no *conception* of the self, and thus is a form of non-conceptual knowledge, e.g. knowledge by presence (*ʿilm ḥuḍūrī*), then primitive self-awareness and Avicenna's "empiricism" may vacuously be consistent. See Deborah Black, "Avicenna on Self-Awareness and Knowing that One Knows," in Shahid Rahman et al. (eds.), *Arabic Logic, Epistemology and Metaphysics: The Interconnections between Logic, Science and Philosophy in the Arabic Tradition*, Dordrecht, Springer, 2008, 65–70.

[71] Gutas, "The Empiricism of Avicenna," 404 n. 32.

[72] Gutas, "The Empiricism of Avicenna," 404.

[73] It might be objected that (i) the modal argument for the existence of God is derived from the present experience of existence, as one of its premises. Therefore, (ii) the conception of God can be derived from present experience. It might further be claimed that this is Gutas's point. The move

cases, Gutas relies on the assumption that "humans know these things by going from the effect to the cause."[74] It might be tempting to reconstruct this reply in terms of compositionality of (the understood contents of) the conceptions involved. On such a proposal, the way to know a celestial intellect or God is through "*quia* demonstration". One perceives the (understood content of the) conception of a celestial intellect, for example, through (the understood content of) a relative conception, e.g. *the cause of*. (In general, the derivative conceptions may involve other relations such as *similarity* or *negation*.) Therefore, a celestial intellect which itself cannot be perceived by (internal or external) senses, is conceivable through a relative conception whose constituting conceptions are ultimately based on experience. This strategy, however, faces different problems. To discuss the matter in more detail, we need to turn to Avicenna's view on "content-empiricism."

18.8 Is Avicenna a Content-Empiricist?

The first problem for the above reply is that Avicenna does not introduce conceptions like *intellect* or *soul* as complex/composite conceptions whose understood contents contain the understood contents of other conceptions:

> [Text 11] A simple (*basīṭ*) *maʿnā* is one such that it is not possible for the intellect (*al-ʿaql*) to consider in it (*yaʿtabiru fīh*) any combination (*at-taʾalluf*) or composition (*at-tarakkub*) of some [other] *maʿānī*. Hence, it is not possible to genuinely define it (*taḥdiduh*). And this is like the intellect (*al-ʿaql*) or the soul (*an-nafs*).[75]

If the *maʿnā* of "intellect", i.e. INTELLECT, is *simple*, then this *maʿnā* as conceived in the mind, i.e. its understood content, does not contain the understood contents of other conceptions, including conceptions like *cause* or *effect*. This is consistent with what Avicenna says on how we conceive separate/immaterial substances:

> [Text 12] Hence, [in the case of] these things [namely, separate/immaterial substances], it is only the *maʿānī* of their quiddities, not [these things] themselves, that are realized in the human *intellects*. These [*maʿānī*] are governed by the same rules governing the rest of what is intellectually apprehended of substances, except in one thing – namely, in that these [latter] require interpretations (*tafsirāt*) so as to abstract from them a *maʿnā* that is intellectually apprehended, whereas [the former] require nothing other than the existence of the *maʿnā* as it is, the soul becoming imprinted by it.[76]

Thus, simple *maʿānī* referred to in Text 11 and Text 12, can be conceived without combining or composing other *maʿānī* or performing the epistemic processes of

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from (i) to (ii), however, is not justified. That there is an argument, with an empirical premise, for the existence of an entity does not imply that the conception of the entity is empirical. See Sect. 9.7.1 above.

[74] Gutas, "The Empiricism of Avicenna," 415.

[75] Ibn Sīnā, *at-Taʿlīqāt*, ed. Mousavian, 41.

[76] Ibn Sīnā, *The Metaphysics of the Healing*, ed. Marmura, 110. I have slightly modified Marmura's translation. I have substituted 'intellect' (in italics) for 'mind' (Avicenna's term is *ʿuqūl al-bašarīyya*) and *maʿnā* and *maʿānī* for *idea* and *ideas* respectively.

abstraction or interpretation (*tafsir*). Elsewhere,[77] Avicenna explains that the only barrier to knowing such *maʿānī* is the occupation of the human soul with the human body. These pieces of evidence suggest that the understood content of a simple *maʿnā* is not complex and hence should not be identified with the understood content of some complex, e.g. relative or privative, conception.

Furthermore, the semantics of a different class of *simple maʿānī*, namely the primary conceptions, show that Avicenna is not committed to content-empiricism (CEA):

> [Text 13] We say: The *maʿānī* of "the existent," "the thing," and "the necessary" are impressed in the soul in a primary way. This impression does not require better-known things to bring it about. [This is similar] to what obtains in the category of assent, where there are primary principles, found to be true in themselves [...]. Similarly, in conceptions of things, there are things which are principles for conception that are conceived in themselves. If one desires to indicate them, [such indication] would not, in reality, constitute making an unknown thing known but would merely consist in drawing attention to them to bring them to mind through the use of a name or a sign which, in itself, may be less known than [the principles] but which for some cause or circumstance, happens to be more obvious in its signification. [...] If every conception were to require that [another] conception should precede it, then [such a] state of affairs would lead either to an infinite regress or to circularity. The things that have the highest claim to be conceived in themselves are those common to all matters – as, for example, "the existent," "the one thing," and others. For this reason, none of these things can be shown by a proof totally devoid of circularity or by the exposition of better-known things.[78]

The primary conceptions in Avicenna's philosophy have been subject to many studies and I cannot even provide a review here.[79] For my purposes, I will focus on one of their most significant features, namely indefinability. In Text 13, Avicenna introduces the *maʿnā* of "existent", for instance, as being "conceivable in itself"; the

[77] Ibn Sīnā, *Avicenna's De anima*, 237.

[78] Ibn Sīnā, *The Metaphysics of the Healing*, ed. Marmura, 22–23, slightly revised.

[79] Here are a few references. For a historically insightful study of Avicenna's primary conceptions see Jan A. Aertsen, "Avicenna's Doctrine of the Primary Notions and its Impact on Medieval Philosophy," in Anna Akasoy and Wim Raven (eds.), *Islamic Thought in the Middle Ages*, Leiden, Brill, 2008, 21–43. For the influence of Avicenna's theory of primary conceptions on Aquinas, see Daniel D. De Haan, "A Mereological Construal of the Primary Notions *Being* and *Thing* in Avicenna and Aquinas," *American Catholic Philosophical Quarterly*, Vol. 88, No. 2 (2014), 335–360. There is a scholarly debate on the conceptual priority of the conception of "being" over the conception of "thing" in Avicenna. Amos Bertolacci, criticising Aertsen and Robert Wisnovsky, *Avicenna's Metaphysics in Context*, Ithaca, NY: Cornell University Press, 2003, defends this claim (see Amos Bertolacci, "The Distinction of Essence and Existence in Avicenna's Metaphysics: The Text and Its Context," in Felicitas Opwis and David Reisman (eds.), *Islamic Philosophy, Science, and Religion: Studies in Honor of Dimitri Gutas*, Leiden: Brill, 2012, 257–88.) For Avicenna's conception of "thing", see Thérèse-Anne Druart, "'Shay' or 'Res' as Concomitant of 'Being' in Avicenna," *Documenti e Studi sulla Tradizione Filosofica Medievale* 12 (2001): 125–42 and Wisnovsky, *Avicenna's Metaphysics in Context*. Some other primary conceptions, e.g. necessity, call for a different treatment. See, for example, Amos Bertolacci, "'Necessary' as Primary Concept in Avicenna's Metaphysics," in Stefano Perfetti (ed.), *Conoscenza e contingenza nella tradizione aristotelica*, Pisa: Edizioni ETS, 2008), 31–50.

mind does not require "better-known things" to conceive it. Therefore, the understood contents of empirical conceptions, and of the relative conceptions constructed out of them, are not required for apprehending the understood content of the primary conceptions. This shows that what is understood from the *maʿnā* of "existent" i.e. its understood content, does not contain the understood contents of empirical conceptions (recall that "understood content" is the epistemologically significant aspect of a conception).

In the latter part of Text 13, Avicenna argues, in two steps, that conceptions like *existent* and *thing* are primary: First, not all conceptions need some other conceptions to precede them in understanding because if it were so, one would end up either in an infinite regress or in a vicious circle. Either way, "understanding" would be impossible. This step clarifies the meaning of "primacy" in the argument. Second, the conceptions that have the highest claim to be primary are those "common to all matters", namely the conceptions whose extensions include everything.[80] The conceptions *existent* (*mawjud*) and *thing* (*shayʾ*) are among the most common conceptions.[81]

In the middle part of Text 13, Avicenna discusses an objection and a reply. One might object that *thing* can be defined as follows, for example: "The thing is that about which it is valid/correct [to give] an informative/indicative statement".[82] Hence, the primary conceptions can be known by some other conceptions and therefore they are not primary in understanding. Avicenna's reply goes like this: the conception of "is valid/correct" (*ṣaḥiḥ*) and that of "informative/indicative statement" (*khabar*) in themselves are less-known than the conception of "thing" and, therefore, cannot make a genuine definition of "thing" (in fact "thing" is used in the definition of both conceptions). In general, conceptions that are in themselves less-known than the primary conceptions may be used to draw one's attention to the primary conception and bring them to the mind because the less-known conceptions can be associated with expressions which are more obvious in signification. Avicenna, here, distinguishes between the conceptual primacy of a conception and the immediacy of the signification of its associated expression. This allows Avicenna to explain how some expressions can, in some sense, clarify or explicate, the expressions associated with the primary conceptions, though the primary conceptions themselves are, epistemologically speaking, the most fundamental conceptions.

In the last line of Text 13, Avicenna concludes that "none of these things", namely the primary conceptions, "can be shown by a proof (*bayān*) totally devoid of circularity or by the exposition (*bayān*) of better-known things". Something has a definition, only if it can be known by better-known things. If nothing is better-known than the primary conceptions, then they are neither definable nor explicable. There is no

[80] The argument is not applicable to the conception *necessary*, given that it is not "common to all matters." For further discussion see Aretsen "Avicenna's Doctrine of the Primary Notions and its Impact on Medieval Philosophy," and Bertolacci, "'Necessary' as Primary Concept in Avicenna's Metaphysics."

[81] See footnote 53. Avicenna, in the same chapter, argues for this latter claim against some early Muslim theologians (*mutikallimun*).

[82] Ibn Sīnā, *The Metaphysics of the Healing*, ed. Marmura, 23, slightly revised.

argument, including *quia* demonstration, for them either. Therefore, the understood contents of the primary conceptions need not be characterized by some experiential conditions, namely by conceptions whose understood contents are characterized by the contents of some experiential terms. Therefore, Avicenna is not committed to CEA.[83]

18.9 Open Questions

To recap, I introduced two empiricist theses, namely OE (origin-empiricism) and CE (content-empiricism). I *re*interpreted these theses in Avicenna's language as OEA and CEA. Avicenna's analysis of the relationship between knowledge, on the one hand, and sensation and imagination, on the other hand, includes three empiricist themes, namely Sects. 18.6.1, 18.6.2 and 18.6.3. I used these themes to argue that that Avicenna is committed to OEA. There are, however, some *prima facie* problematic cases, namely Sects. 18.7.1, 18.7.2 and 18.7.3. I showed that a common solution to such cases, that relies on the compositionality of the understood content, *quia* demonstration and relative conceptions, has significant limitations. A careful examination of these limitations, and the epistemology of the primary conceptions, show that Avicenna is not committed to CEA.

As an alternative explanation, for the limitations to Avicenna's origin-empiricism, one may appeal to the empirical nature of the development of the human intellect. It may be argued that the human mind, as an abstract or immaterial entity, is temporally originated and empirically developed through time.[84] Hence, origination of all conceptions, immediately or mediately, requires experience as a necessary condition for the development of the intellect to come to the stage of the actual intellect. Accordingly, experience, via causal relations, contributes to the origination of the "material" necessary for intellection, though the understood contents of all intelligible conceptions cannot be exhausted by the understood contents of empirical conceptions.

This alternative explanation for Avicenna's commitment to "empiricism", in the sense of OEA, raises a similar open question: Is Avicenna a rationalist, *in some sense*? After briefly commenting on this question, I will consider a related, but commonly dissociated, question in the literature on the reality of abstraction, as a cognitive process in acquiring (some) universals. Finally, I will question two presuppositions of the common replies to both questions.

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[83] Avicenna's philosophy of mind and language may be reconstructed along the same lines with some auxiliary hypotheses. Here is one such hypothesis: if the existence of a *maʿnā* among concrete particulars/external things does not need *matter*, then the understood content, namely the *maʿnā* as imprinted in the mind, does not "essentially contain reference to" sensory experience because being material is a prerequisite of being perceivable by external senses.

[84] Seyed N. Mousavian and Seyed H. Saadat Mostafavi, "Avicenna on the Origination of the Human Soul," *Oxford Studies in Medieval Philosophy* 5, 2017, 41–86.

18.9.1 Is Avicenna a Rationalist?

Some Avicenna scholars, e.g. Marmura and Goodman, based on evidence such as Text 13 have attempted to argue that Avicenna is a "rationalist". Consider, for instance:

> Thus, according to Avicenna, there are "primary concepts" – concepts of the widest generality that are epistemically prior to the "acquired" concepts, forming a necessary condition for the latter's acquisition. They are, moreover, "rational," not only by the very fact that they are concepts, but in that they do not require for their apprehension perception of the material world.[85]

Marmura claims that the primary conceptions are the necessary condition for the acquisition of *all* other conceptions, including empirical conceptions. He, then, justifies his reasoning by appealing to Text 13:

> The sense in which these concepts [namely, the primary conceptions] are "impressed" (*tartasim*) in the soul is not explained [in Text 13]. If their analogy with the logical self-evident truths is pressed, these in the final analysis would have to be direct emanations from the Active Intellect, the last of the series of intelligences emanating from God.[86]

Here is my formulation of the above reasoning. The primary conceptions are the principles for conceiving other conceptions as the logical self-evident truths, namely the primary propositions, are the principles for giving assent to other truths. The logical self-evident truths are direct emanations from the Active intellect. Therefore, by analogy, the primary conceptions are direct emanations from the Active intellect. Marmura concludes that "they [i.e. primary conceptions] do not require for their apprehension perception of the material world."

Goodman expresses a similar view:

> Like all rationalists, Ibn Sīnā sees that no pure concept can be derived empirically. Thus, he posits that the ideas of being and necessity are primitives, given to the mind by the hypostatic Active Intellect. No process can discover them. They cannot be learned. For they are presupposed in any mental process that would lead to them, and any effort to build them out of simpler constituents would come to ground rapidly in circularity.[87]

In contrast, Gutas rejects the above rationalist reading of Avicenna. He does not accept the assumption that the primary conceptions are direct emanations from the active intellect:

> To our understanding of the way in which the primary notions come about in the dispositional intellect, this passage adds emphatically that they do so without the child intending to attain them and while he is unaware that in fact he is attaining them. It is important to realize that

[85] Michael Marmura, "Avicenna on Primary Concepts in the Metaphysics of the *Shifā*," in R. Savory and D. A. Aguis (eds.), *Logos Islamikos: Studia Islamica in honorem Georgii Michaelis Wickens,* Toronto: Pontifical Institute of Mediaeval Studies, 1984, 220.

[86] Marmura, "Avicenna on Primary Concepts in the Metaphysics of the *Shifā*," 222 (the brackets are mine). The criterion "concepts with widest generality" may be used to identify semantic "categories" and/or logical "notions" (see Tarski (1966/1986), for example), depending on how one construes "widest generality".

[87] Goodman, *Avicenna*, 124.

> these primary notions do not come from the active intellect, nor do they appear as a result of search and demand on the part of the child.[88]

To do justice to this debate, different "rationalist" claims should be disambiguated and the nature of human reason, the epistemic role of the active intellect in human cognition and the notion of "innateness" for Avicenna should be examined.[89] Space limitation does not allow me to do so here. Therefore, I leave this question open. There is one point, however, that I would like to add. Both sides of the debate presuppose the following incompatibility principle:

> INC1. If a primary conception is direct emanation from the active intellect, then its apprehension does not require sense-perception.[90]

Marmura and Goodman attempt to argue that the antecedent of INC1 is true and thus by modus ponens, the consequent of INC1 follows, that is the primary conceptions "do not require for their apprehension perception of the material world."[91] In reply, Gutas in effect suggests that the antecedent of INC1 is not true, that is "these primary notions do not come from the active intellect,"[92] and hence there is no modus ponens to derive the consequent of INC1. This style of reasoning, centered around the question of whether some intelligible conceptions are "directly" emanated from the active intellect, is sometimes extended to a different problem, namely "How do we acquire universals?". For example, Davidson claims that the "intelligible thoughts, he [Avicenna] has maintained, flow directly from the active intellect and are not abstracted at all" (Davidson 1992, 93). The reality of abstraction in Avicenna's epistemology, nonetheless, is a different problem. Let me explain.

18.9.2 Is Avicenna an Abstractionist?

Dag Hasse summarizes the problem of acquiring universals as follows:

> The theory of abstraction is one of the most puzzling parts of Avicenna's philosophy. What Avicenna says in many passages about the human intellect's capacity to derive universal knowledge from sense-data seems to plainly contradict passages in the same works about the emanation of knowledge from the active intellect, a separately existing substance. When he maintains that "considering the particulars [stored in imagination] disposes the soul for something abstracted to follow upon it from the active intellect", he appears to combine two incompatible concepts in one doctrine: either the intelligible forms emanate from above or they are abstracted from the data collected by the senses, but not both.[93]

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[88] Gutas, "The Empiricism of Avicenna," 413.

[89] The latter question has been investigated, in part, in Seyed N. Mousavian and Mohammad Ardeshir, "Avicenna on Primary Propositions," *History and Philosophy of Logic* 29.3, 2018, 201–231.

[90] Let us assume that perception of material world is sense-perception.

[91] Marmura, "Avicenna on Primary Concepts in the Metaphysics of the *Shifā*," 220.

[92] Gutas, "The Empiricism of Avicenna," 413.

[93] Dag Nikolaus Hasse, "Avicenna on Abstraction," in Wisnovsky (ed.), *Aspects of Avicenna*, 2001, 39.

The issue, in this formulation, is a dichotomy. As a solution, Guts and Hasse hold that Avicenna is an abstractionist in the sense that "there is only one active power in the process, the human intellect: it turns towards the imaginable forms and acts upon them."[94] Davidson and Black, in contrast, hold that "Avicenna explicitly denies any causal influence of the imagination upon the intellect, that is, he denies the reality of abstraction as a cognitive process."[95]

The question of the reality of abstraction is how to explain the role of the human soul, particularly sensation and imagination, on the one hand, and that of the active intellect, as a separate intellect, on the other hand, in acquiring (quidditative) *universals*, first and foremost, by the human intellect. This is not identical to the problem of the origin of the *primary* conceptions. Both sides of this debate, nonetheless, presuppose an incompatibility principle that can be formulated as follows:

> INC2. If an intelligible form (i.e. a universal) emanates from the active intellect, then it is not abstracted from sense-perception.

Rahman and Black, among others, have attempted to argue that the antecedent of INC2 is true and thus, by modus ponens, the consequent of INC2, or something to the same effect, follows, namely "The 'abstraction' of the form, therefore, for Avicenna is only a *façon de parler*."[96] Gutas and Hasse, by contrast, have attempted to show that the antecedent of INC2 is not true, or has no epistemological significance, and thus there is no modus ponens to derive the consequent of INC2. If Avicenna's "empiricism" is a complex and nuanced view, as I have attempted to argue, one may wonder if there is a "compatibilist" reading of Avicenna according to which neither INC1 nor INC2 is presupposed.

18.9.3 Is Avicenna a Compatibilist?

Here, I will try to outline the elements of a positive answer to the last question. Given the distinction between understood content (*al-mafhūm*) and mental impression (*athar ḏihnī*), as two aspects of a conception, it can be argued that "apprehension of a conception" (used by Marmura in Sect. 9.9.1) is ambiguous. If "apprehension of a conception" means *apprehending the understood content of a conception*, then apprehending the understood content of *existent*, as a primary conception, does not

[94] Hasse, "Avicenna on Abstraction," 63. Gutas sees no epistemological role played by the active intellect: "What has to be kept in mind is that for Avicenna the concept of the emanation of the intelligibles from the active intellect has its place in his cosmology and it serves to solve essentially an ontological problem, not an epistemological one, which is the location of the intelligibles" (Gutas, "The Empiricism of Avicenna," 411).

[95] Deborah Black, "Avicenna and the Ontological and Epistemic Status of Fictional Beings," *Documenti e Studi sulla Tradizione Filosofica Medievale,* 8, 1997, 445. The references are recovered from Hasse, "Avicenna on Abstraction," 39.

[96] Fazlur Rahman, *Prophecy in Islam: Philosophy and Orthodoxy*, George Allen & Unwin, 1958, reprinted by Routledge, 2008, 15.

require perception of the material world in the sense that this understood content is not explicable/definable by better-known conceptions, including the conceptions obtained through sense-perception. (Recall that the primary conceptions cannot be *acquired* through discursive knowledge, that is, they cannot be known through genuine definitions, though they may be signified by lesser-known expressions.) However, if “apprehension of a conception” means *obtaining the mental impression of a conception*, then obtaining the mental impression of *existent* is a mental event that (normally) requires *obtaining* some other mental impressions, including the mental impressions of some empirical conceptions. Therefore, INC1 is ambiguous. INC1 has a reading according to which the understood content of a primary conception is not conceivable through the understood contents of the conceptions obtained through sense-perception. I hold that this reading is true and reinforces the view that Avicenna is not committed to CEA. INC1 also has a reading according to which obtaining a primary conception as a mental impression, in the human soul, generically does not require obtaining some other conceptions, particularly sensible and imaginable forms. Given that Avicenna is committed to OEA, this reading is false. Recall that the human soul at its origination, which is called the “material intellect” or “potential intellect,” does not contain any intelligible form, including the primary conceptions and only at the stage of “dispositional intellect” it comes to possess the first intelligibles. Therefore, I conclude, INC1 does not hold in general.

INC2 is about the acquisition of the (quidditative) universals, e.g. *human*, whose exemplification instances (*aškhās*) exist amongst concrete particulars (*aʿyān*) or external things.[97] INC2 may have two readings both of which can be resisted. First, if “emanation of an intelligible form from the active intellect” implicates origination of the mental impression of an intelligible form in the human soul, and “it is not abstracted from sense-perception” means that the mental impression has no causal origin in sense-perception, then a compatibilist account of the causal origin of the mental impression, according to which both the active intellect and sense-perception contribute to the origination o the mental impression, undermines INC2. Consequently, sense-perception is *some* origin, as a necessary but insufficient part, of a sufficient complex cause, which incorporates the active intellect, for the acquisition of the mental impression of an intelligible conception, say *human*, by the human intellect.[98] This view can be defended by a compatibilist account of Avicenna’s theory of causation in the extramental realm according to which both the active intellect, as the form-giver, and corporal causes, as the agents educing form from the potency of

[97]Neither the generic, e.g. *matter*, nor the most common, e.g. *existent*, *maʿānī* are quidditative.

[98]I am *not* attributing Mackie’s INUS theory of causation (John L. Mackie, *The Cement of the Universe: A Study of Causation*, Ney York, NY, Oxford University Press, 2002) to Avicenna. The gist of my suggestion is to identify different elements in Avicenna’s theory of causation such that the active intellect and sense perception both can fit in with Avicenna’s theory of abstraction.

matter, can contribute to the process of causation.[99] Kara Richardson has developed and defended such an account:

> The distinction between causes of species and causes of individuals suggests that a superior principle – in my view, Avicenna's Agent Intellect – is the cause of a type replicated by tokens of the type. This view does not entail an Infusion Model of generation. [...] If Avicenna's claim in Physics 1.10 that 'that which gives the constitutive form belonging to natural species is extrinsic to natural things' refers to an incorporeal cause of the species of natural things, then this claim is compatible with the view that corporeal agents educe form from the potency of matter in generation.[100]

It will remain to be seen if this proposal can be developed to a full-fledged theory of causation within the mental realm.

Now let's turn to the second reading of INC2. If "emanation of an intelligible form from the active intellect" implicates making the content of an intelligible form by the active intellect, and "it is not abstracted from sense-perception" means that the understood content of the intelligible form is not derived from the understood content of sense-perception, then a compatibilist account of abstraction, as an epistemological process, according to which both the active intellect and sense-perception contribute to the understood content of the intelligible form, undermines INC2. McGinnis (2007, 2013), for instance, developing and defending such an account[101]:

> Once the rational soul has abstracted away the concomitants of matter, the emanation from the active intellect, that is, its intellectualizing forms, radiates upon and mixes with the potentially intelligible object making it actually intelligible, analogous to the way that radiant light mixes with potential color to make it actual color and form rays.[102]

The "intellectualizing forms" are intelligible accidents such as universality or essentiality (in predication). According to McGinnis, emanation from the active intellect as well as a form of abstraction from sense perception both contribute to characterization of the understood content of an intelligible form. The human intellect abstracts away the concomitants of matter from sense perception and then the intellectualizing forms are bestowed upon the outcome of this process. Therefore, without the emanation of the active intellect, an intelligible form is not actually intelligible.[103]

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[99] Black's recent view may also be categorized as compatibilist in holding that abstraction is a necessary but not sufficient condition for acquiring a wide range of concepts (see Deborah Black, "How Do We Acquire Concepts?", in Jeffrey Hause (ed.), *Debates in Medieval Philosophy*, Routledge, 2014.

[100] Kara Richardson, "Avicenna and Aquinas on Form and Generation," in Dag Nikolaus Hasse and Amos Bertolacci (eds.), *The Arabic, Hebrew and Latin Reception of Avicenna's Metaphysics*, Göttingen, De Gruyter, 2012, 264.

[101] I said "may be read" because to fully develop such a compatibilist view one needs to work out a clear account of the "derivation of understood content."

[102] Jon McGinnis, "New Light on Avicenna: Optics and its role in Avicennan Theories of Vision, Cognition and Emanation," in Luis Xavier Lopez-Farjeat and Jörg Alejandro Tellkamp (eds.), *Philosophical Psychology in Arabic Thought and the Latin Aristotelianism of the 13th Century*, Paris, Librairie Philosophique J Vrin, 2013, 55.

[103] A similar view is developed in Jon McGinnis, "Making Abstraction Less Abstract: The Logical, Psychological, and Metaphysical Dimensions in Avicenna's Theory of Abstraction," *Proceedings*

I should add that I agree with Hasse that "the distinction between abstract forms (or essences) and intellectualizing forms (or accidents)", on which McGinnis's view is based, "does not have a textual basis in Avicenna's psychological works."[104] However, and this is against Hasse's earlier view,[105] I find McGinnis's move to allow divine emanation and real abstraction play epistemologically significant roles in Avicenna's theory of "abstraction" on the right track. I differ from both McGinnis and Hasse (2013) on how to make "emanation" and "abstraction" compatible.[106]

18.10 Conclusion

I argued that Avicenna is an empiricist in one sense, namely OEA, and not an empiricist in another sense, namely CEA. I first introduced Avicenna's terminology of "*signifcandum*" (*maʿnā*), "understood content" (*al-mafhūm*), and "mental impression" (*athar ḏihnī*). "Conceptions" (*taṣawwurāt*) were interpreted as having two aspects: understood content and menta impression. Then, borrowing Winkler's (2010) distinction between origin-empiricism (OE) and content-empiricism (CE) and reinterpreting that distinction in Avicenna's language, as OEA and CEA, I provided textual evidence for Avicenna's commitment to OEA and his lack of commitment to CEA. Therefore, "Is Avicenna an empiricist?" has no simple yes-or-no answer. This raises a similar open question: "Is Avicenna a rationalist?". After briefly commenting on this question, I considered a related, but commonly dissociated, question on the reality of abstraction, namely "Is Avicenna an abstractionist?" I explained how the common replies to these questions rely on two incompatibility principles, namely INC1 and INC2. I ended by outlining the elements of a reading of Avicenna that assumes neither INC1 nor INC2.

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of the American Catholic Philosophical Association, 80, 2007, 169–83, and subsequently defended by Thérèse-Anne Druart, "Avicennan Troubles: The Mysteries of Heptagonal House and of the Phoenix," *Tópicos* 42, 2012: 51–73.

104 Dag Nikolaus Hasse, "Avicenna's epistemological optimism," in Peter Adamson (ed.), *Interpreting Avicenna: Critical Essays*, Cambridge, Cambridge University Press, 2013, 113.

105 Hasse, "Avicenna on Abstraction."

106 In the recent literature, on both sides of the debate, some, e.g. Black, "How Do We Acquire Concepts?" and Hasse "Avicenna's epistemological optimism," are sympathetic with a compatibilist reading. A thorough compatibilist interpretation, however, needs to spell out the *epistemological* role of the active intellect in the process of abstraction, or so I believe. For two studies along this line, in addition to McGinnis's work, see Tommaso Alpina, "Intellectual Knowledge, Active Intellect and Intellectual Memory in Avicenna's *Kitāb al-Nafs* and Its Aristotelian Background," *Documenti e studi sulla tradizione filosofica medievale* XXV, 2014, 131–183, and Stephen Ogden, "Avicenna's Emanated Abstraction," *Philosophers' Imprint*, forthcoming, 1–39.

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Correction to: Binary Modal Companions for Subintuitionistic Logics

Dick de Jongh and Fatemeh Shirmohammadzadeh Maleki

Correction to:
Chapter 2 in: M. Mojtahedi et al. (eds.), *Mathematics, Logic, and their Philosophies*, Logic, Epistemology, and the Unity of Science 49, https://doi.org/10.1007/978-3-030-53654-1_2

The book was inadvertently published with chapter author's incorrect family name. This information has been updated from "First Name: Dick de, Family Name: Jongh" to "First Name: Dick, Family Name: de Jongh" in the initially published version of chapter "2".

The chapter and book have been updated with the change.

The updated version of this chapter can be found at
https://doi.org/10.1007/978-3-030-53654-1_2

M. Mojtahedi et al. (eds.), *Mathematics, Logic, and their Philosophies*,
Logic, Epistemology, and the Unity of Science 49,
https://doi.org/10.1007/978-3-030-53654-1_19

Author Index

A
Abad, M., 334
Abramsky, S., 164, 177
Adamson, P., 453, 475
Aertsen, J.~A., 467
Aghaei, M., v, xiii, xiv
Aguis, D.~A., 470
Ahmed, A.~Q., 417, 442, 452
Akasoy, A., 467
Akbar Tabatabai, A., 161, 201, 204
Akiba, K., 368, 375
Alizadeh, M., 93, 112–114, 165, 179, 210, 217, 317, 214
Almog, J., 414
Alpina, T., 475
Amalric, M., 388
Anderson, A. ~ R., 406
Anel, M., 164
Apt, K. ~ R., 118, 125, 127, 404
Ardeshir, M., 36, 53, 112–114, 132, 137, 161, 165, 179, 210–212, 214, 217, 253, 255, 257–259, 261, 266, 268–270, 272, 273, 278, 281, 283, 285, 288, 295, 298, 308, 312–315, 317, 319, 343, 376, 380, 395, 447, 460, 471
Aristotle, 380, 387, 413, 433–439, 441
Armour-Garb, B. P., 367
Armstrong, D. ~ M., 396, 399
Artemov, S.~N., 254
Assadian, B., 365
Attamah, M., 118, 125, 131
Avicenna, 443–445, 447, 449, 450, 453–460, 462–469, 471

B
Bacon, A., 375
Bäck, A., 448
Baierlein, R., 401–403
Bailyn, M., 401
Baker, B., 118
Balbes, R., 219, 226
Banerjee, M., 332
Bankova, D., 136
Barnes, J., 434
Baroni, M., 136
Barr, A, 147
Barwise, J., 136, 137, 140
Battilotti, G., vi
Beall, J., 350, 354, 367
Beklemishev, L.~D., 254
Benacerraf, P., 366, 387
Bendová, K., 340, 344
Bennet, C., 54
Berarducci, A., 262
Bernardi, R, 136
Bertolacci, A., 463, 464
Bezdek, J.~C., 137
Bezhanishvili, N., 217, 220
Biagioli, F., 384
Birkhoff, G. D., 110, 379, 386
Black, D., 366, 367, 373, 375, 461, 468
Blamey, S., 352, 360
Blyth, T.S., 328
Bodenhofer, U., 137
Bonchi, F., 136
Boolos, G., 107, 110
Borceux, F., 165, 167, 168
Bressan, A., 402
Brouwer, L. E. J., 339, 383, 393
Bruns, G., 322

M. Mojtahedi et al. (eds.), *Mathematics, Logic, and their Philosophies*,
Logic, Epistemology, and the Unity of Science 49,
https://doi.org/10.1007/978-3-030-53654-1

Burge, T., 453
Burris, S, 110
Buss, S.~R., 56
Butchart, S., 350, 359, 360

C
Cantwell, J., 356
Castro, J., 217
Cattaneo, G., 332
Celani, S., 103–105, 217, 218, 332
Cesalli, L., 430
Chakraborty, M. K., 332
Chalmers, D., 453
Chatti, S., 420
Chellas, B., 41
Cignoli, R., 332
Ciucci, D., 331–334
Clark, P., 136
Clark, S., 136
Clark, P., 387
Cobreros, P., 357
Cohn-Vossen, S., 387
Cooper, R., 136, 137, 140
Corsi, G., 35

D
Dagan, I., 136
Dalla Chiara, M.~L., 331, 333, 334
Davey, B. .~A., 95, 219, 229
Dawson, J.~W.
De Bouvère, K.~L., 85
Dedekind, R., 317, 381, 382, 384, 387
Dehaene, S., 388
de Groote, H.~F., 314
De Haan, D.~D., 463
De Jongh, D., 35–41, 46, 51, 60, 254
de Lavalette, G.~R.
Demopoulos, W., 387
de Myers, D., 54
Descartes, R., 380, 451, 452, 459
Dicher, B., 354
Dimov, G., 218, 220
Dinu, G., 136
Doerr, B., 129
Došen, K., 165, 208
Dostál, M., 135, 137
D'Ottaviano, I., 435
Druart, T., 463, 471
Dubois, D., 332
Dumais, S., 136
Dummett, M., 138, 339, 340, 341, 344–346
Dunn, J.~M., 103
Düntsch, I., 218, 220
Dutilh Novaes, C., 448
Dwinger, P., 219, 226
Dyckhoff, 138

E
Ebbesen, S., 445, 448
Egré, P., 357
El-Rouayheb, K., 413
Enderton, H.~B., 71
Esakia, L., 233–235
Estrada-González, L., 349, 358
Evans, G., 369
Ewald, W., 387

F
Fang, J., 328, 329
Farulewski, M., 111
Feferman, S., 261, 262
Feigenbaum, E.~W., 147
Ferreiros, J., 385
Field, H., 367, 369, 375, 376, 393
Figallo, M, 334
Fine, K., 404
Firth, J., 135
Fischer-Servi, G., 202
Fodor, J., 452
Font, J.~M., 332
Franks, C., 51, 385
Frege, G., 136, 382, 383, 433, 453
Friedman, H., 54, 73, 75, 76, 254, 255
Friedrich, T., 129

G
Gaitàn, H., 328
Galatos, N., 94, 95, 97, 101, 191
Gallego, M.~S., 332
Gattinger, M., 117
Gazziero, L., 430
Geffet, M., 136
Genovese, F., 142
Ghafouri, Z., vii, xi, xv
Giuntini, R., 313–315, 321–323, 325, 326, 328, 330
Gödel, K., 138, 143, 254, 339, 389, 399
Gomes, E, 435
Goodenough, J., 135, 136
Goodman, L.~E., 444, 466, 467
Gottlieb, P., 441
Grattan-Guinness, I., 385

Grätzer, G., 219, 226, 324
Gray, J.~J.
Grefenstette, E., 136
Grossi, D., 118, 125, 127
Gutas, D. ~ N., 444, 445, 454, 455, 457, 459–462, 466–468
Guzmàn, F., 338

H
Hájek, P., 56, 259
Hale, B., 403, 404, 407, 408
Halliday, D., 401
Hanf, W., 54
Haniková, Z., 111
Hardegree, G.M., 103
Harding, J., 322
Harris, J.~D., 137
Hasnawi, A., 411, 421, 426
Hasse, D. ~ N., 467, 468, 470, 471
Hasuo, I., 165
Hawking, S.~W., 395
Hedetniemi, S.~M., 118
Hedetniemi, S.~T., 118
Hedges, J., 136, 137, 142, 157, 158
Helmig, C., 452
Herzig, A., 118, 119, 120
Hesaam, B., 165, 208, 210
Heunen, C., 165
Heyting, A., 93–95, 98, 102, 103, 107, 112, 138, 143, 161, 165, 170, 177–179, 182, 183, 254, 255, 313, 331, 339
Hilbert, D., 314, 381, 383–387
Hill, C.~S., 368
Hodges, W., 3, 65, 86, 394, 411–413, 421, 426, 453
Horčík, R., 111
Horwich, P., 376
Hughes, J., 165

I
Ibn Sīnā, 453
Iemhoff, R., 165, 254, 255, 266, 268, 287, 296, 308
Ilik, D., xi
Ivan, C, xi

J
Jacobs, B., 165
Jaffe, A., 386
Jansana, R., 102, 107, 108, 112, 114, 165, 178, 201, 208–210, 217
Japaridze, G., 51, 60
Jaśkowski, S., 339, 341, 342, 344
Jipsen, P., 94, 97, 101, 191
Johnston, S., 412, 413, 421, 426
Johnstone, P.~T.
Joyal, A., 161, 164

K
Kant, E., 380, 382–384, 450, 455
Kant, I., 380, 382–384, 450, 455
Kartsaklis, D., 136
Katriňák, T., 226
Kelly, G., 136, 140
Kermarrec, A.~M., 118
Kerre, E.~E., 137
Khaniki, E., xv
Kleene, C., 313–317, 328–330, 332–334
Kock, A., 140
Kotlerman, L., 136
Kowalski, T., 97, 324
Krabbe, E.~C.~W., vi
Krajíček, J., 75, 76
Kremer, P., 181
Kreyszig, E., 314
Kripke, S., 35, 36, 47, 49–51, 161, 167, 181, 183, 187, 190, 200–204, 206, 207, 209–212, 214, 261, 262, 267–270, 273–275, 277, 279, 280, 286–291, 293, 295–305, 306, 339–347
Krishnamurthy, J., 136
Kuijer, L.~B., 118, 121, 122, 124, 126, 129, 131, 132

L
Lagerlund, H., 412, 448
Lajevardi, Kaave, 391
Lakser, H., 324
Lameer, J., 447
Lammer, A., 427
Landauer, T., 136
Laplaza, M., 136, 140
Laugwitz, D., 381
Ledda, A., 313–315, 321–323, 325, 326, 328, 330
Leech, J., 403, 404, 407, 408
Leeds, S., 368
Leivant, D., 254–257
Lewis, D.~K., 51, 366, 369
Lewis, M., 136
Liestman, A.~L., 118
Lin, D., 136
Lindley, S., 165

Lindström, P., 54, 60
Lin, Z., 110
Litak, T., 165, 200
Löb, M., 254, 257, 274, 295, 403
Locke, J., 444, 451, 454
Lopez-Farjeat, L.~X., 470
Lorenz, K., 415
Loureiro, I., 332

M
Ma, M., 110, 112
Mac Lane, S., 165
Mackie, J.~L., 469
Maffre, F., 118–120
Maillard, J., 136
Majolino, C., 430
Maksimova, L.~L., 113
Manin, Yu., 379, 385
Marion, M., 138
Marmura, M., 448, 453, 466–468, 470–472
Marsden, D., 142
Martin-Löf, P., 413, 414
McCarthy, D., 136
McConaughey, Z., 413, 414
McCurdy, M., 136, 140, 141
McGee, V., 366, 367, 369, 374
McGinnis, J., 444, 470, 471
McKinsey, J.C.C., 163
McLaughlin, B. P., 366, 367
Memarzadeh, M., 214
Menn, S., 446, 449, 450
Meyer, R. K., 187
Mints, G., 181
Mitchell, T.~M., 136
Mittelstrass, J., 415
Moerdijk, I., 161, 162
Moerdijk, I., 162
Moggi, E., 165
Mohammadian, M., xi
Moisil, G.~C., 334
Mojtahedi, M., vi, vii, 1, 35, 53, 93, 117, 135, 161, 221, 253, 255, 257, 264–266, 268, 269, 277, 279, 281, 284, 288, 294, 298, 308, 313, 339, 349, 365, 379, 391, 408, 411, 433, 443
Moniri, M., xiii
Montague, R., 406, 407
Monteiro, L., 333, 334
Monton, B., 393, 398
Moore, G.H.
Mousavian, Seyed N., 443, 445, 449, 450, 453, 454, 459, 460, 462, 465, 467, 469, 471
Mulder, H., vi
Mulvey, C.~J., 117
Mumford, S., 400
Mureşan, C., 313, 315, 316, 321, 322, 326, 328
Murzi, J., 350
Mycielski, J., 54, 75, 89
Myhill, J., 254, 255

N
Nabavi, F., xiv
Nagel, E., xiii, 391, 400, 401
Newman, J., xiii, 391
Newton, I., 401, 405
Nisticò, G., 333
Novák, V., 137, 158
Nute, D., 51

O
Ogden, S., xi, 471
Okada, M., 208
Olkhovikov, G. K., 356
Olson, M.~P., 314
Omori, H., 356
Ono, H., xi, 93, 95, 97
Opwis, F., 463

P
Pacuit, E., 37, 51
Paoli, F., 313, 324, 354
Paoli, P., 313, 324, 354
Pardo, P., 118, 121, 122, 124, 126, 129, 131, 132
Paterson, R., 165
Perfetti, S., 463
Pilla, R., 331–334
Płonka, J., 324
Poincaré, H., 380, 385–387
Priest, G., xi, 95, 437, 445
Priestley, H.~A., 115, 218, 219, 229–233, 235–237, 239–246, 248–251
Pudlák, P., 56, 75, 89, 259
Pulman, S., 136
Pylyshyn, Z. ~ W., 452

Q
Quine, W.~V.~O., 373, 398, 400, 444
Quinn, F., 386

R

Rahman, F., 413, 414, 461, 468
Rahman, S., 411, 413, 414, 461
Ramezanian, R., 132
Ramírez-Cámara, E., 349
Ranta, A., 414, 418, 421–423
Rashed, R., 387
Raven, W., 463
Recanati, F., 423
Reck, E.~H., 387
Redmond, J., 429
Reidel, D.
Reisman, D., 463
Rescher, N., 413, 421, 426
Resnick, R, 401
Restall, G., 35, 51, 115, 165, 178, 208, 350, 359
Richardson, K., 470
Ripley, D., 356
Rius, M., 357
Rogerson, S., 350, 359, 360
Rosenthal, K.~I., 167
Rossi, L., 354
Routley, R., 350, 359, 360
Rubenstein, H., 136
Ruitenburg, W., 36, 165, 178, 191, 208, 210, 313
Russell, B., 382–384, 393, 433

S

Saadat Mostafavi, Seyed H., 465
Sadrzadeh, M., 135, 136, 138, 142, 157, 158
Safari, P., 341
Saha, A., 332, 334
Salehi, S., vi, xiv, 339–341, 343, 346
Salton, G., 136
Sambin, G., vi, xiv
Sankappanavar, H.~P., 110, 115, 327, 329
Sankappanavar, N.~H., 110, 327, 329
Sano, K., 50
Sasaki, K., 165, 208
Sauerwald, T., 129
Schaffner
Schindler, T., 376
Schöck, C., 376, 418
Scholz, E., 381
Schuetze, H., 136
Schwarzentruber, F., 129
Sen, J., 336, 338
Shahriari, M., xv
Shahshahani, Siavash, 376
Shapiro, S., 391
Shipley, J.
Shirmohammadzadeh Maleki, F., 35, 36, 38, 39–41, 46, 51
Shkatov, D., 103, 115
Shostak, R., 118
Shramko, Y., 356
Simpson, A.~K., 200
Smiley, T., 406
Smoryński, C., 75, 76, 262, 270, 277, 279, 280, 284
Soames, S., 373
Sober, E., 402
Sobocinski, P., 136
Solovay, R.~M., 254, 255, 277, 283, 289, 310
Sourabh, S., 255
Spada, L., xi
Spivak, M., 387
Sprenger, J., 358
Squier, C.~C.
Steedman, M., 136
Stern, A.~S., 75, 89
Street, T., 430
Strobino, R., 411, 412
Stroock, D.~W., 314
Sundholm, G., 419
Suzuki, Y., vi, 165, 208
Švejdar, V., 54, 340, 344
Szczerba, L.~W., 65, 78
Szpektor, I., 136

T

Tahiri, H., 448
Tappenden, J., 383
Tarski, A., 3, 163
Taylor, D.~E.., 367–373
Tellkamp, J.~A., 470
Thom, P., 367–373
Tierney, M., 164
Tijdeman, R., 118, 128
Troelstra, A.~S.., 19, 33, 51, 259, 261, 280
Truffaut, J., 138
Turney, P.~D., 136

U

Unger, P., 366

V

Vaezian, V., vi, xiv
Vakarelov, D., 218, 220, 251
Van Alten, C., 103, 115
van Atten, M., xiii

Van Benthem, J., 78
Van Dalen, D., 19, 33, 51, 161, 162, 261, 280
Van Dantzig, D., 19, 33
van der Hoek, W., 118, 125, 127
vander Nat, A., 413, 421, 426
Van Ditmarsch, H., 117, 118, 121, 122, 124, 126, 129, 131, 132
van Eijck, J., 129
van Fraassen, B.~C., 393, 395, 403, 407
van Heijenoort, J.
Van Rooij, R., 357
van Steen, M., 118
Veldman, W., 2, 5, 6, 12, 17, 18, 20, 23, 25, 26, 30, 31
Veltman, F., 51
Venema, Y., 217, 218, 220, 251
Verbrugge, R., 258
Vickers, P., 164, 167, 177
Vickers, S., 164, 167, 177
Visser, A., 35, 51, 95, 107–110, 112, 114, 115, 254, 255, 257, 259, 260, 262, 263, 265, 268, 283, 310, 313

W
Wadler, P., 165
Wagner, R., 381
Walker, J., 405
Wang, L., 332, 333
Wansing, H., 358
Weeds, J., 136
Wehmeier, K.~F., vi
Weir, D., 136
Werner, H., 330
Whitman, A.
Wijnholds, G., 135
Winkler, K. ~ P., 443, 444, 450, 451, 471
Wisnovsky, R., 463
Wolfson, H.~A., 443, 444, 450, 471
Wolter, F., vi
Wong, A., 136

Y
Yallop, J., 165
Yang, C.~S., 136

Z
Zadeh, L.~A., 135, 137, 138, 146–149, 151, 153, 157
Zakharyaschev, M., vi
Zamparelli, R., 136
Zanasi, F., 136
Zare, M., xi
Zarepour, M.~S., 411, 430, 443
Zhang, Y., 136
Zhao, Z., 101, 103
Zhitomirsky-Geffet, M., 136
Zhou, C., 165
Zoethout, J., 255, 310
Zolghadr, Behnam, 433

Subject Index

A
Almost-enumerable spread, 12
Apartness, 7, 19, 20
Arithmetic, 54, 70, 72, 73, 146, 254–256, 258, 259, 261, 288, 379, 380, 382–384, 386
Arithmetic, sequentiality, 72
Arithmetization, 381–384
Avicenna, 433, 434, 436–441, 443–446, 448–451, 453–457, 459–471

B
Background theory, 391–400, 403–405, 407
Brouwer's continuity principle, 2, 5, 6, 10, 380, 383–385, 393
Brouwer-Zadeh lattices, 316, 331

C
Category theory, 385
Classical modal logic, 35–37, 267, 391
Common knowledge, 117, 119–122, 124, 129, 130, 132
Concept, 54, 55, 60, 135, 218, 219, 221, 251, 368, 381, 383, 384, 401, 402, 444, 450–455, 459, 460, 466, 467
Conditional, 36, 51, 147, 177, 349–354, 356, 358–363, 399, 403
Conjunction, 1, 43, 45, 47, 114, 136, 158, 163, 164, 169, 199, 263, 275, 278, 297, 298, 344, 346, 349, 351–356, 358–363, 408, 416
Constructive Type Theory (CTT), 165, 411, 413, 415–418, 420–423, 425–429
Contact relations, 221, 222
Content, 51, 137, 162, 175, 181, 386, 401, 443–445, 447, 449–454, 461–465, 468–471
Contracting connective, 349–353, 356, 358, 359, 361–363
Contradiction, 5, 11, 19–21, 25, 26, 30, 64, 132, 201, 203–205, 224, 228, 233, 236–238, 240–242, 245, 247, 249, 274, 287, 289, 292, 295, 296, 319, 320, 328, 345–347, 433, 434, 441

D
Decidable point of a spread, 2
Deflationism, 365–376
Degrees of interpretability, 54
Descriptional (waṣfī) propositions, 411, 415–419, 421–425, 428, 431, 433
Detachable connective, 349–352, 355, 358, 363
Distributional semantics, 135–137, 157
Distributive lattices, 93, 94, 103–1085, 111, 113, 115, 221–230, 232–236, 242, 244, 332, 333
Distributive lattices with operators, 93, 94, 113, 115, 217, 218, 223, 226, 229, 230, 328
Dynamics, 118, 138, 161, 175, 176, 181, 401

E
Empiricism, 403, 404, 443, 444, 450, 452, 453, 459–461, 465, 468

F
Factivity, 391, 403, 404

M. Mojtahedi et al. (eds.), *Mathematics, Logic, and their Philosophies*, Logic, Epistemology, and the Unity of Science 49,
https://doi.org/10.1007/978-3-030-53654-1

Fuzzy sets, 137, 138, 146–149, 151, 153, 154, 156–158

G
Generalised quantifiers, 135–139, 149, 150, 157
Geometry, 161, 162, 164, 379–386
Gödel-Dummet Logic, 339–341, 344–346, 345
Gossip protocol, 117–121, 123, 130

H
Heyting Arithmetic (HA), 255
Higher-order epistemic goals, 118, 119

I
Ibn Sīnā (Avicenna), 411, 413, 416, 417, 420, 426, 427, 429, 430, 453
Inter-definability of propositional connectives, 339, 341
Intermediate logic, 341
Intuitionistic logic, 137, 138, 161, 191, 207, 254, 256, 339, 414
Intuitionistic propositional logic, 37, 39, 43, 163, 192, 339–342, 344

K
Kilimanjaro, 365–369, 371–376
Kripke models, 50, 200, 206, 261, 262, 268, 275, 280, 286–289, 291, 294, 295, 297–306, 340, 342–344, 346, 347

L
Lattice-ordered groupoid, 93–100, 102, 104–107, 110, 115
Lattice with implication, 94, 98–100, 102, 103, 105–107
Laws of nature, 396, 397, 399, 400, 405, 408
Law of residuation, The, 94, 98, 102, 103, 110
Lindenbaum algebras, 198, 205–207, 210
Logical fallacies, 415

M
Manifold, 380, 383, 385, 396
Many valued logics, 138, 339, 341, 342
Many valued relations, 135, 137, 142, 143, 150, 151, 154, 157
Mental, 162, 163, 165–167, 175, 176, 181, 382, 443–445, 447–450, 452–455, 458, 464, 466, 468–471
Modal companion, 35, 36, 40, 43, 49–51
Modal logic, 35–37, 39, 40, 47, 49, 50, 132, 137, 255, 257, 267, 305, 331, 391
Modal syllogistic, 417
Modal talk *in* physics, 392, 394, 401
Multi-agent knowledge, 117

N
Natural language data, 135–138, 140, 157, 158, 418
Natural modalities, 395
Nonclassical modal algebras, 313, 315

O
Ontology, 382, 449, 450
Opposition, 437, 438, 440
Orthomodular lattices, 314, 316, 317, 319

P
Paraconsistency, 434, 441
PBZ*-lattices, 313–316
P-consequence, 349, 351, 357
Peano Arithmetic (PA), 162, 259, 406
(Perhapsive) extensions, 23, 31
Physical necessity, 391, 392, 394, 404, 406, 407
Provability logic, 165, 253–255, 260, 272, 273, 283–285, 293, 308, 310, 403

Q
Quantum logic, 317
Quasi-modal operators, 218, 220–223, 226, 227, 238

R
Rationalism, 453
Reduction, 253, 255, 270–273, 280–282, 284–286, 293, 308, 309, 380, 392
Referential indeterminacy, 365–376
Relative provability logic, 253, 255, 256, 283, 293
Residuated expansion, 93–95, 98–100, 103, 105–110, 112, 115
Riemann, 314, 380–383, 385

S
Set, 379, 381, 384–386, 392, 399, 415, 416, 422, 423, 425, 428, 447
Singular reference, 369, 370, 373–375
Standard model, 54, 256, 259
Strict implication, 35, 36, 40–43, 47, 49–51, 115, 161
Sub-intuitionistic logics, 35–37, 39, 40, 50, 51, 93, 115, 167, 178, 198, 208–210, 212
Subordination relations on distributive lattices, 224
Substantial (d̲ātī) propositions, 91, 370, 410–419, 421–424
Sub-structural logics, 94, 95, 167
Syllogism, 429, 430, 433–441, 456

T
Telephone problem, 117, 118, 122
Temporal logic, 147, 413, 426
Topological semantics, 167, 195, 208
Toy spread, 13, 17

V
Vagueness, 358, 368, 375, 376
Vector space models, 135, 136
Visser algebra, 107–110, 112, 114, 115
Vitali equivalence relation, 1–3, 18–20, 23–25

W
Weak Heyting algebra, 93–95, 102, 107
Weak implications, 104, 165, 167, 191, 208