QPTAS for the CVRP with a Moderate Number of Routes in a Metric Space of Any Fixed Doubling Dimension

Michael Khachay[1,2,3](✉) and Yuri Ogorodnikov[1,2](✉)

[1] Krasovsky Institute of Mathematics and Mechanics, Ekaterinburg, Russia
{mkhachay,yogorodnikov}@imm.uran.ru
[2] Ural Federal University, Ekaterinburg, Russia
[3] Omsk State Technical University, Omsk, Russia

Abstract. The Capacitated Vehicle Routing Problem (CVRP) is the well-known combinatorial optimization problem having a host of valuable practical applications in operations research. The CVRP is strongly NP-hard both in its general case and even in very specific settings (e.g., on the Euclidean plane). The problem is APX-complete for an arbitrary metric and admits Quasi-Polynomial Time Approximation Scheme (QPTAS) in the Euclidean space of any fixed dimension (and even PTAS, under additional constraints). In this paper, we significantly extend the class of metric settings of the CVRP that can be approximated efficiently. We show that the metric CVRP admits QPTAS any time, when it is formulated in a metric space of a fixed doubling dimension $d > 1$ and is restricted to have an optimal solution of at most $\operatorname{polylog} n$ routes.

1 Introduction

The Capacitated Vehicle Routing Problem (CVRP) is the well-known combinatorial optimization problem having a lot of valuable practical applications in operations research. The problem was introduced by Dantzig and Ramser in their seminal paper [8] as a mathematical model for routing the fleet of gasoline trucks servicing a network of gas stations from a bulk terminal.

Since then, the field of the algorithmic design for the CVRP is developed in a number of research directions as follows. The first direction is based on a reduction of the problem in question to some appropriate mixed integer program and finding an optimal solution of this program using some of the well-known branch-and-price methods [25]. Recently, a significant success was achieved in development such algorithms and computational hardware [11,21]. Unfortunately, due to strongly NP-hardness of the CVRP, instances of this problem that are managed to be solved efficiently within this approach still remain quit modest.

Another direction is closely related to involving a wide range of heuristic algorithms and meta-heuristics including the local search [2], VNS [22], Tabu search [23], evolutionary and bioinspired methods [19], and their combinations [7, 18].

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These algorithms often demonstrate an amazing performance finding close-to-optimal or even exact solutions to really huge instances of the CVRP coming from practice. Unfortunately, an absence of any theoretical guarantees implies additional computational expenses related to numerical evaluation of their accuracy and possible tuning during the transition to any novel class of instances. In addition, there are known cases when such a tuning is impossible at all, e.g. for the security reasons.

The third research direction is related to the design of approximation algorithms with theoretic performance guarantees and dates back to seminal papers of Haimovich and Rinnooy Kan [10], and Arora [3]. It is known that the CVRP is strongly NP-hard even on the Euclidean plane [20]. The problem is hardly approximable in general case, APX-complete for an arbitrary metric [4] and admits Quasi-Polynomial Time Approximation Schemes (QPTAS) in finite-dimensional Euclidean spaces [9]. For the planar CVRP with restricted capacity growth, there are known several Polynomial Time Approximation Schemes (PTAS), among them the PTAS proposed in [1] is the most general. The approach introduced in [10] is managed to extend to a number of modifications of the planar CVRP including the CVRP formulated in the Euclidean space of any fixed dimension [15,17], the case of multiple depots [12,16], the CVRP with Time Windows [13], and non-unit customer demand [14].

Thus, until now, the class of instances of the metric problems approximable by PTAS or QPTAS was exhausted by the Euclidean settings of the problem except maybe some special cases investigated in [6] Meanwhile, in recent papers by Talwar [24] and Bartal et al. [5] such a class for the closely related Traveling Salesman Problem (TSP) was substantially extended to include the instances of the problem in a metric space of an arbitrary fixed doubling dimension.

In this paper, we propose the first QPTAS for the CVRP formulated in such a space. Our contribution is as follows.

Theorem 1. *For the CVRP in a metric space of an arbitrary doubling dimension $d > 1$, an $(1 + O(\varepsilon))$-approximate solution can be found by a randomized approximation algorithm within time* $\operatorname{poly} n \cdot \left(m^2 n\right)^{m^2 \cdot \operatorname{polylog} n}$*, where* $m = O\left(\left(\frac{d(\log n - \log \varepsilon)}{\varepsilon}\right)^d\right)$*. The algorithm can be derandomized efficiently.*

The rest of the paper is structured as follows. In Sect. 2, we recall the statement of the CVRP. Then, in Sect. 3 we propose a short overview of the proposed approximation scheme. Finally, at Conclusion, we summarize the results obtained and overview some possible directions for the future work.

2 Problem Statement

In the classic Capacitated Vehicle Routing Problem (CVRP), we are given by a set of *customers* $X = \{x_1, \ldots, x_n\}$ having the same unit demand, which should be serviced by a *vehicle* located at some dedicated point y that is called *depot*. All vehicles have the same *capacity* q and visit the customers by cyclic routes, each of them departs from and arrives to the depot y. The goal is to provide a collection of the capacitated routes visiting each customer once and minimizing the total transportation costs.

Let $V = X \cup y$. An instance of the CVRP is specified by a complete undirected edge-weighted graph $G = (V, E, w)$ and an integer $q \geq 3$. The symmetric weighting function $w : E \to R_+$, to any edge $\{u, v\} \in E$, assigns the direct transportation cost $w(u, v)$. A simple cycle $\pi = y, x_{i_1}, x_{i_2}, \ldots, x_{i_s}, y$ in the graph G is referred to a *feasible route*, if it satisfies the capacity constraint, i.e. visits at most q customers. For the route π, its cost $w(\pi) = w(y, x_{i_1}) + w(x_{i_1}, x_{i_2}) + \cdots + w(x_{i_{s-1}}, x_{i_s}) + w(x_{i_s}, y)$. The goal is to find a family of feasible routes $\Pi = \{\pi_1, \ldots, \pi_k\}$ of the least total transportation cost that covers the total customer demand.

In this paper, we consider a restriction of the metric CVRP with the following additional constraints:

(i) for some $d > 1$, the weighting function w is a metric of doubling dimension d, i.e. for an arbitrary $v \in V$ and $R > 0$, there exist nodes $v_1, \ldots, v_M \in V$, such that the metric ball $B(v, R) \subseteq \bigcup_{j=1}^{M} B(v_j, R/2)$ and $M \leq 2^d$.
(ii) the problem is supposed to have an optimal solution, whose number of routes does not exceed $\operatorname{polylog} n$.

3 Approximation Scheme: An Overview

The main idea of our approximation scheme extends the well-known Arora's PTAS for the Euclidean TSP and its generalization proposed in [5] to the TSP in a metric space of any fixed doubling dimension. The scheme consists of the following stages.

Accuracy-Driven Rounding. At this stage, given by $\varepsilon > 0$, to the initial instance, we assign a *rounded* one, such that each s $(1 + \varepsilon)$-approximate solution of the latter instance can be transformed in polynomial time to the appropriate $(1 + O(\varepsilon))$-approximate solution of the former one.

Without loss of generality, we assume that the diameter Δ of the set V is equal to n/ε (since otherwise we can rescale the initial metric by the factor $\frac{n}{\Delta\varepsilon}$). Then, we *round* each customer $x \in X$ to the nearest node $\xi \in X'$, where X' is some metric 1-net of the set X. Finally, we consider an auxiliary instance of the CVRP, specified by the set X' and inheriting all other parameters (y, q, and w) from the initial one. As a result, in the obtained rounded instance, each 'customer' ξ is counted with a multiplicity equal to the number of $x \in X$ assigned to it and, for any distinct 'customers' ξ_1 and ξ_2, $w(\xi_1, \xi_2) > 1$.

Randomized Hierarchical Clustering. Following to [5], we fix a number $s \geq 6$ and put $L = \lceil \log_s(n/\varepsilon) \rceil$. For any level $l = 0, \ldots, L + 1$, we construct an s^{L-l}-net N_l of the set $V' = X' \cup \{y\}$. Without loss of generality, we assume that N_0 is a singleton, $N_L = N_{L+1} = V'$ and $N_l \subset N_{l+1}$ for any l. We proceed with hierarchical clustering of the set V' by induction on l. For $l = 0$, we construct the only cluster $C_1^0 = V'$. Further, let $C_1^{l-1}, \ldots, C_K^{l-1}$ be the partition constructed at level $l-1$. To proceed at level l, we partition each cluster C_j^{l-1} separately. To make such a partition, we take point by point from the net N_l in a random order σ and, to each net point $\nu_{\sigma(i)}$, we assign a random radius $\eta \in [s^{L-l}, 2s^{L-l})$ from the uniform distribution. Then, the i-th subcluster of the cluster C_j^{l-1} is

$$C_{ji}^l = B(\nu_{\sigma(i)}, \eta) \cap C_j^{l-1} \setminus \bigcup_{t<i} C_{jt}^l.$$

By construction, all clusters at level $L+1$ are singletons.

Following to [5], our scheme deals with approximate solutions of some special kind, which are referred to as *net-respecting and light*. To define this concept, we choose the number M as some degree of s, such that $M/s < d \cdot L/\varepsilon \leq M$. For any cluster C_j^l, each points from the s^{L-l}/M-net is called *portals*. As it follows from the well-known Packing Lemma (see, e.g. [24]), the number m of portals located in each cluster at an arbitrary level $l > 0$ does not exceed $(8 \cdot M)^d = O\left(\left(\frac{d(\log n - \log \varepsilon)}{\varepsilon}\right)^d\right)$. A route is called *net-respecting* if, for any its edge $\{u, v\}$ of length λ, both points u and v belong to the net N_l, where $s^{L-l} \leq \varepsilon\lambda < s^{L-l+1}$. Further, for some $r > 0$, a net-respecting route is called *r-light*, if it crosses the border of any cluster C_j^l (of any level $l > 0$) at most r times.

As it follows from the Structure Theorem [24], with high probability, for $r = m$, there exists an approximate solution of the CVRP, consisting of net-respecting r-light routes, whose total transportation cost is at most $(1+\varepsilon)\cdot\mathrm{OPT}$. Therefore, to approximate the initial instance within the given accuracy, we can restrict ourselves on such solutions.

Dynamic Programming. For a given randomized clustering, we find the minimum-cost approximate solution consisting of net-respecting r-light routes using the dynamic program as follows. Entries of the DP table are defined by *configurations* that are assigned to each cluster C_j^l. For any cluster C_j^l, an associated configuration $\mathfrak{C}$ is a list of at most polylog n tuples (p_1, p_2, q_j, dep_j), each of them specifies a route segment entering and leaving this cluster at the portals p_1 and p_2 respectively, visiting q_j customers exactly and passing through the depot y or not depending on dep_j.

The table entries are computed bottom-up. Level $L+1$ is the base case. Each configuration at this level can be computed trivially. Then, let C^l be some cluster at level l, $l = 0, \ldots, L$. To compute any configuration $\mathfrak{C}$ for the cluster C^l, we enumerate all combinations of the feasible configurations $\mathfrak{C}_1, \ldots, \mathfrak{C}_K$ associated with subclusters $C_1^{l+1}, C_2^{l+1}, \ldots, C_K^{l+1}$, $K = 2^{O(d)}$ to find such a combination that is compatible with the configuration $\mathfrak{C}$ and induces the set of route segments crossing the cluster C^l (maybe augmented by some routes contained in this cluster completely) of the minimum total cost. The required solution is obtained by minimization on the set of feasible configurations for the unique cluster at level 0.

Following to the approach proposed in [24], we can show that our algorithm admits an efficient derandomization.

4 Conclusion

In this paper we announce an approximation scheme for the CVRP in the metric space of an arbitrary doubling dimension $d > 1$. Our algorithm is a QPTAS, if the problem has an optimal solution, whose number of routes does not exceed $\operatorname{polylog} n$. It is easy to verify that this condition holds, for instance, when $q = \Omega(n/\operatorname{polylog} n)$. We postpone the proof of Theorem 1 to the forthcoming paper.

Although, to the best of our knowledge, the proposed algorithm appears to be the first approximation scheme for the metric CVRP for the spaces of fixed doubling dimension, the question: *'Can the QPTAS proposed by A.Das and C.Mathieu* [9] *for the Euclidean CVRP be extended to metric spaces of a fixed doubling dimension without any restriction on the capacity growth?'* still remains open. We'll try to bridge this gap in the future work.

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