Chapter 3
Evenly Quasiconvex Functions

It is well-known that a real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous if and only if its *graph*, gph $f := \{(x, f(x)) : x \in \mathbb{R}^n\}$, is a closed subset of $\mathbb{R}^{n+1}$. Since gph f can be written as the intersection of the sets epi $f := \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq \lambda\}$ and hypo $f := \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R} : f(x) \geq \lambda\}$, called *epigraph* and *hypograph* of f, respectively, one can split the continuity in two weaker properties: f is said to be *lower* (*upper*) *semicontinuous* whenever epi f (hypo f, resp.) is closed, so that f is continuous if and only if it is lower and upper semicontinuous.

In the same vein, the modern treatment of extended real-valued functions emphasizes the role played by the set-based approach. So, functions of the form $f : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ such that

$$f((1 - \mu)x + \mu y) \leq (1 - \mu)f(x) + \mu f(y), \forall x, y \in \mathbb{R}^n, \forall \mu \in [0, 1],$$

are called *convex* and are characterized by the convexity of their epigraph epi f while the *concave* functions f (for which $-f$ is convex) are those functions whose hypograph hypo f is convex. Similarly, functions $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ such that

$$f((1 - \mu)x + \mu y) \leq \max\{f(x), f(y)\}, \forall x, y \in \mathbb{R}^n, \forall \mu \in [0, 1],$$

which are called *quasiconvex*, are those functions whose *lower level sets* $[f \leq r] := \{x \in \mathbb{R}^n : f(x) \leq r\}$ (or equivalently, their *strict lower level sets*, $[f < r] := \{x \in \mathbb{R}^n : f(x) < r\}$) are convex for all $r \in \mathbb{R}$. The opposite of the quasiconvex functions are called *quasiconcave*, and they are characterized by the convexity of their *upper level sets* $[f \geq r] := \{x \in \mathbb{R}^n : f(x) \geq r\}$, $r \in \mathbb{R}$.

One of the ways of facing the unconstrained minimization of a given function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ whose bad behavior does not allow the application of known numerical methods consists in the replacement of f by a function $\widehat{f} \leq f$ (that is, a *minorant* of f) enjoying desirable properties and such that the *approximation gap* $\inf_{x \in \mathbb{R}^n} f(x) - \inf_{x \in \mathbb{R}^n} \widehat{f}(x) \geq 0$ is sufficiently small.

In this sense, if $\mathscr{F} = \{f_i, i \in I\}$ is a family of minorants of f, then the *supremum* of $\mathscr{F}$, $\sup \mathscr{F}$, defined as $(\sup \mathscr{F})(x) := \sup\{f_i(x), i \in I\}$, is also a minorant of f and its epigraph and lower level sets are the intersection of the epigraphs and the lower level sets, respectively, of the family members. Since the intersection of closed sets is closed, there exists a greatest lower semicontinuous minorant of f, which is called *lower semicontinuous hull* of f, denoted by cl f and defined as the supremum of the family of all lower semicontinuous minorants of f. In the same way, since the intersection of convex sets is convex, one can consider the *convex* (*quasiconvex*) *hull* of f, denoted by co f (qco f, resp.) and defined as the largest convex (quasiconvex, resp.) minorant of f. Similarly, cu f denotes the *upper semicontinuous hull* of f (defined as its smallest upper semicontinuous majorant, whose hypograph is the closure of hypo f). The reader is referred to [59, 91, 148, 191], among many other textbooks on convex analysis, for a comprehensive introduction on these notions.

The approximation gap decreases when one replaces the given minorant $\widehat{f}$ by a greater one. In Sect. 3.1 we define a class of functions, the evenly quasiconvex ones, which provide greater minorants than the smaller class of the lower semicontinuous quasiconvex functions (analogously, in the next chapter, we analyze the class of evenly convex functions, which extends the class of lower semicontinuous convex functions). Section 3.2 studies the evenly quasiconvex hull providing the largest evenly quasiconvex minorant of a given function. Section 3.3 analyzes conjugates and subdifferentials for evenly quasiconvex functions, while Sect. 3.4 provides a sketch of quasiconvex duality. Finally, Sect. 3.5 describes an application in mathematical economy.

3.1 Evenly Quasiconvex Functions

A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be *lower semicontinuous*, lsc in brief, (*upper semicontinuous*, usc in brief) at a point $\overline{x} \in \mathbb{R}^n$ if for any $\lambda \in \mathbb{R}$, $\lambda < f(\overline{x})$ ($\lambda > f(\overline{x})$, resp.), there exists a neighbourhood of $\overline{x}$, $V_{\overline{x}}$, such that $\lambda < f(x)$ ($\lambda > f(x)$, resp.) for all $x \in V_{\overline{x}}$. It is well-known that a function f is lsc at any point of $\mathbb{R}^n$ if and only if epi f is closed, or equivalently, if $[f \leq r]$ is closed for every $r \in \mathbb{R}$. As a function f is usc if and only if $-f$ is lsc, usc functions turn out to be those functions whose strict lower level sets are open.

An extended real-valued function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be *evenly quasiconvex*, e-quasiconvex in brief, (*strictly evenly quasiconvex*, resp.) if the lower level set $[f \leq r]$ (the strict lower level set $[f < r]$, resp.) is e-convex for every $r \in \mathbb{R}$. In particular, if $f : \mathbb{R} \to \overline{\mathbb{R}}$ is an univariate quasiconvex function, then all its lower level sets are e-convex (since all of them are intervals) and, therefore, f is e-quasiconvex. Since every convex set being closed or open is also e-convex, it is obvious that every lsc quasiconvex function is e-quasiconvex and every usc quasiconvex function is strictly e-quasiconvex. Moreover, since $[f \leq r] = \cap_{r<q}[f < q]$ and the intersection of e-convex sets is e-convex, every strictly

Fig. 3.1 The strict lower
level set $[f < 1]$ is not
e-convex
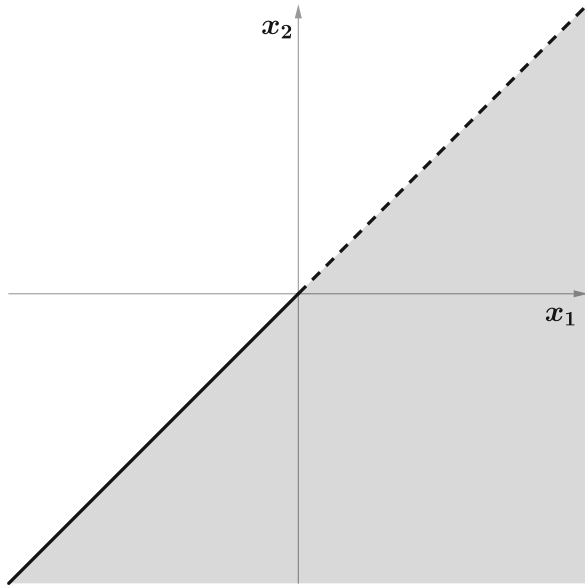
e-quasiconvex function is e-quasiconvex. The converse is not true, as the following
example shows.

Example 3.1 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined as follows:

$$f(x_1, x_2) = \begin{cases} 0, & \text{if } x_1 \geq x_2 \text{ and } x_2 \leq 0, \\ x_2/x_1, & \text{if } x_1 > x_2 > 0, \\ 1, & \text{elsewhere.} \end{cases}$$

All the lower level sets of f are closed and convex, and so e-convex, showing that
f is e-quasiconvex. However,

$$[f < 1] = \{x \in \mathbb{R}^2 : x_1 > x_2 > 0\} \cup \{x \in \mathbb{R}^2 : x_1 \geq x_2, \ x_2 \leq 0\}$$

is not e-convex (see Fig. 3.1).

The relationships between the different families of quasiconvex functions are
summarized in Diagram 3.1. In Example 3.2, we show a strictly e-quasiconvex (and
so, e-quasiconvex) function which is neither lsc nor usc.

Example 3.2 Consider the function $f : \mathbb{R} \to \overline{\mathbb{R}}$ defined by

$$f(x) = \begin{cases} x^2, & \text{if } -1 < x < 1, \\ 3, & \text{if } x = 1, \\ +\infty, & \text{otherwise,} \end{cases}$$

Diagram 3.1 Evenly
quasiconvex and related
functions

usc quasiconvex

$\Downarrow$

strictly e-quasiconvex

$\Downarrow$

e-quasiconvex
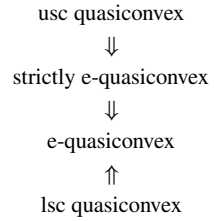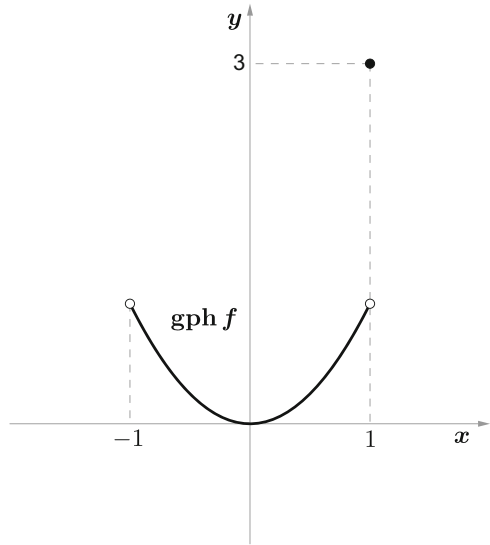
$\Uparrow$

lsc quasiconvex

Fig. 3.2 Strictly
e-quasiconvex function which
is neither usc nor lsc

whose graph is represented in Fig. 3.2. It is easy to see that all the strict lower level
sets are e-convex. However, $[f < 4] = [f \le 4] =]-1, 1]$ is neither open nor
closed, so that f is a strictly e-quasiconvex function which is not usc and it is also
an e-quasiconvex function which is not lsc.

As a straightforward consequence of the definitions given in this section, we
observe   that   the   solution   set   of   a   system   of   the   form
$\{g_t(x) \le 0, t \in W; g_t(x) < 0, t \in S\}$ with $W \cap S = \emptyset$, g_t e-quasiconvex (in
particular, lsc quasiconvex) for all $t \in W$ and g_t strictly e-quasiconvex for all
$t \in S$, is e-convex.

Proposition 3.1 (Operations with e-Quasiconvex Functions) *The following
statements hold:*

(*i*) *If* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *is an e-quasiconvex function and* $\alpha > 0$*, then* αf *is
    e-quasiconvex.*
(*ii*) *If* $\{f_i : \mathbb{R}^n \to \overline{\mathbb{R}}, \ i \in I\}$ *is a family of e-quasiconvex functions, then* $\sup_{i \in I} f_i$
    is an e-quasiconvex function.

Proof

(i) It is a direct consequence of the equalities

$$[\alpha f \le r] = \left\{x \in \mathbb{R}^n : (\alpha f)(x) \le r\right\} = \left\{x \in \mathbb{R}^n : f(x) \le \frac{r}{\alpha}\right\} = \left[f \le \frac{r}{\alpha}\right].$$

(ii) As $\left[\left(\sup_{i \in I} f_i\right) \le r\right] = \cap_{i \in I} [f_i \le r]$ and even convexity is preserved under intersections, we have that the family of e-quasiconvex functions is closed under pointwise suprema.

$\square$

The sum of e-quasiconvex functions is not, in general, e-quasiconvex as we can see in the following example.

Example 3.3 Let $f, g : \mathbb{R} \to \mathbb{R}$ be the functions defined by $f(x) = x$, for all $x \in \mathbb{R}$ and

$$g(x) = \begin{cases} x^2, & \text{if } -1 \le x \le 1, \\ 1, & \text{otherwise.} \end{cases}$$

It is easy to see that f and g are e-quasiconvex functions, whereas the function $f + g$, whose graph is represented in Fig. 3.3, is not e-quasiconvex.

Fig. 3.3 Graphical representation of $f + g$
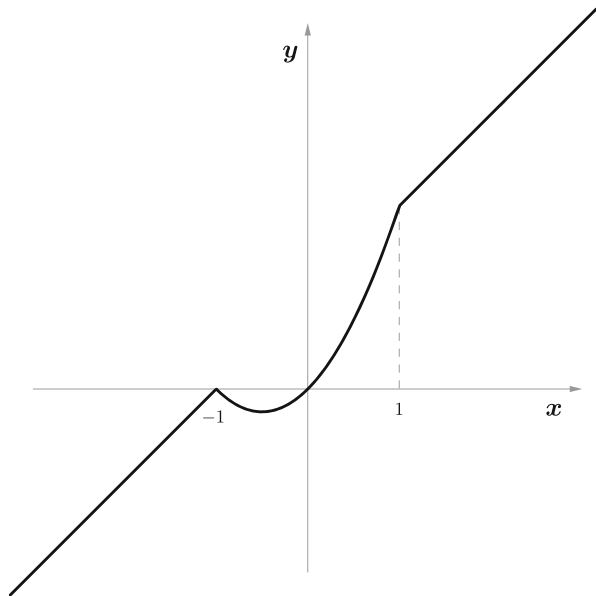
3.2 Evenly Quasiconvex Hull

As a consequence of Proposition 3.1(*ii*), every function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ has a largest e-quasiconvex minorant, which is called its *evenly quasiconvex hull* (*e-quasiconvex hull*, in short) and denoted by eqco f. Obviously,

$$\text{cl qco } f \leq \text{eqco } f \leq \text{qco } f \leq f \tag{3.1}$$

and, therefore, cl eqco $f = $ cl qco f. We shall say that a function f is *e-quasiconvex at* $\overline{x} \in \mathbb{R}^n$ if $f(\overline{x}) = (\text{eqco } f)(\overline{x})$. Clearly, f is e-quasiconvex if and only if it is e-quasiconvex at every $\overline{x} \in \mathbb{R}^n$.

The following examples show that the inequalities in (3.1) can be strict.

Example 3.4 Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x + 1, & \text{if } x \leq -1, \\ x^2 + x, & \text{if } -1 < x < 1, \\ x + 2, & \text{otherwise,} \end{cases}$$

whose graph is represented in Fig. 3.4. Obviously, f is not a quasiconvex function. Figure 3.5 represents the graphs of qco f and cl qco f and, as we can see, one has:

$$\text{cl qco } f \lneqq \text{qco } f \lneqq f.$$

In this case, as qco f is an univariate quasiconvex function, it is also e-quasiconvex and we have that eqco $f = $ qco f.

Fig. 3.4 The function f is not quasiconvex
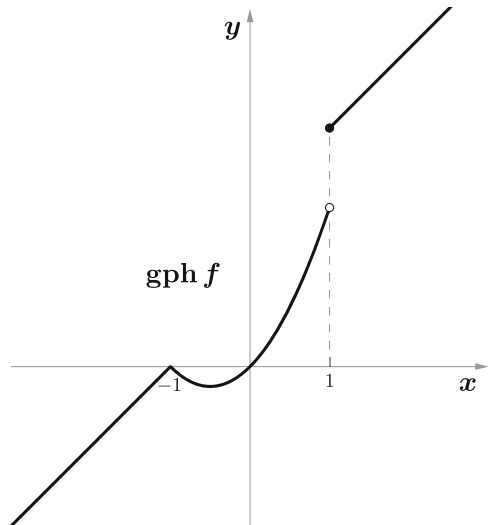
Fig. 3.5 (**a**) The graph of qco f; (**b**) The graph of cl qco f
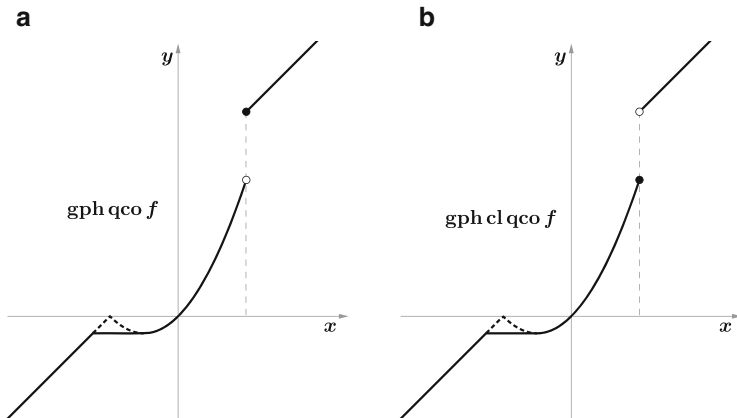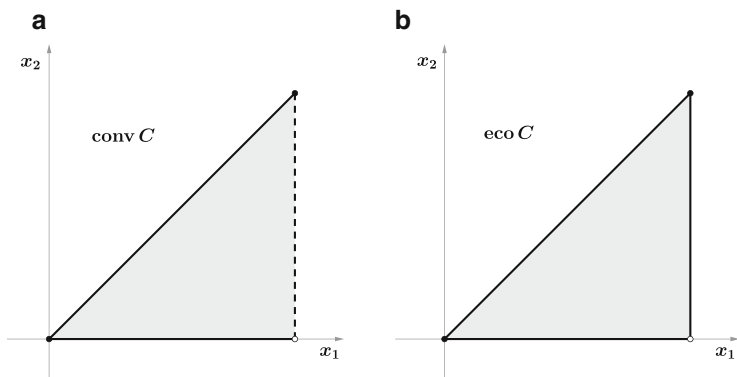

Fig. 3.6 (**a**) convex hull of C; (**b**) e-convex hull of C

Example 3.5 Consider the set $C = \{x \in \mathbb{R}^2 : 0 \leq x_1 < 1, x_2 = 0\} \cup \{(1, 1)\}$ and its *indicator function* $\delta_C : \mathbb{R}^2 \to \overline{\mathbb{R}}$ defined by $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$ if $x \in \mathbb{R}^2 \backslash C$. Since $C \subsetneq \text{conv}\, C \subsetneq \text{eco}\, C$ (see Fig. 3.6), one gets

$$\text{eqco}\, \delta_C = \delta_{\text{eco}\, C} \lneq \text{qco}\, \delta_C = \delta_{\text{conv}\, C} \lneq \delta_C.$$

It is easy to see that, for every function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and for all $r, r' \in \mathbb{R}$ with $r \leq r'$, the following inclusion holds,

$$[f \leq r] \subset [f \leq r'].$$

In general, we say that the family $\mathscr{A} := \{A_t\}_{t \in \mathbb{R}}$ of (possibly empty) subsets of $\mathbb{R}^n$ is an *ascending family* if $A_t \subset A_{t'}$ for all $t, t' \in \mathbb{R}$ with $t \leq t'$. We also associate to the ascending family $\mathscr{A}$ the function $\psi_{\mathscr{A}} : \mathbb{R}^n \to \overline{\mathbb{R}}$, which we call the *extended*

gauge of $\mathscr{A}$, defined by

$$\psi_{\mathscr{A}}(x) := \inf\{t \in \mathbb{R} : x \in A_t\}, \ \forall x \in \mathbb{R}^n, \tag{3.2}$$

with the convention $\inf \emptyset = +\infty$. It is easy to prove that, for every $r \in \mathbb{R}$,

$$A_r \subset [\psi_{\mathscr{A}} \le r] = \bigcap_{t>r} A_t. \tag{3.3}$$

The motivation of the name given to the function $\psi_{\mathscr{A}}$ relies on the following fact. For a convex set $A \subset \mathbb{R}^n$ such that $0_n \in \operatorname{int} A$, the family $\mathscr{A} = \{A_t\}_{t \in \mathbb{R}}$, defined by $A_t = \emptyset$ if $t < 0$ and $A_t = tA$ if $t \ge 0$, is ascending and the function

$$\psi_{\mathscr{A}}(x) = \inf\{t \ge 0 : x \in A_t\}, \ \forall x \in \mathbb{R}^n,$$

is nothing else but the *gauge* (or Minkowski functional) of A.

Example 3.6 Consider the ascending family $\mathscr{A} = \{A_t\}_{t \in \mathbb{R}}$ given by $A_t =]-\infty, t[$ if $t < 0$, $A_t =]-\infty, 1[$ if $t = 0$ and $A_t =]-\infty, e^t[$ if $t > 0$. The gauge function of $\mathscr{A}$ is

$$\psi_{\mathscr{A}}(x) = \begin{cases} x, & \text{if } x < 0, \\ 0, & \text{if } 0 \le x < 1, \\ \ln(x), & \text{if } x \ge 1. \end{cases}$$

Obviously, the family of all lower level sets of a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, $\mathscr{A}_f := \{[f \le t]\}_{t \in \mathbb{R}}$, is an ascending family whose extended gauge is the function f. In fact, by (3.3), one has

$$[f \le r] \subset [\psi_{\mathscr{A}_f} \le r] = \bigcap_{t>r} [f \le t] = [f \le r], \ \forall r \in \mathbb{R},$$

so $[\psi_{\mathscr{A}_f} \le r] = [f \le r]$ for all $r \in \mathbb{R}$, which implies that $\psi_{\mathscr{A}_f} = f$.

Proposition 3.2 (Even Quasiconvexity of the Extended Gauge) *Let $\mathscr{A} = \{A_t\}_{t \in \mathbb{R}}$ be an ascending family of e-convex sets of $\mathbb{R}^n$. Then, the extended gauge $\psi_{\mathscr{A}}$ is e-quasiconvex.*

Proof Taking into account (3.3) and that the intersection of a family of e-convex sets is an e-convex set, we have that $[\psi_{\mathscr{A}} \le r]$ is e-convex for all $r \in \mathbb{R}$. □

Next, for a family of sets $\mathscr{A} = \{A_t\}_{t \in \mathbb{R}}$, we shall denote $\operatorname{eco} \mathscr{A} := \{\operatorname{eco} A_t\}_{t \in \mathbb{R}}$. Observe that $\operatorname{eco} \mathscr{A}$ is an ascending family of e-convex sets whenever $\mathscr{A}$ is an ascending family.

Proposition 3.3 (Evenly Quasiconvex Hull of the Extended Gauge) *Let $\mathscr{A} =$ $\{A_t\}_{t\in\mathbb{R}}$ be an ascending family of sets in $\mathbb{R}^n$. Then,*

$$\text{eqco } \psi_\mathscr{A} = \psi_{\text{eco}\,\mathscr{A}}.$$

Proof Since $A_t \subset \text{eco}\, A_t$ for every $t \in \mathbb{R}$, one has $\psi_{\text{eco}\,\mathscr{A}} \leq \psi_\mathscr{A}$, with $\psi_{\text{eco}\,\mathscr{A}}$ being an e-quasiconvex function due to Proposition 3.2. Now, by the definition of the e-quasiconvex hull of $\psi_\mathscr{A}$, we obtain

$$\psi_{\text{eco}\,\mathscr{A}} \leq \text{eqco } \psi_\mathscr{A} \leq \psi_\mathscr{A}.$$

On the other hand, since $A_t \subset [\psi_\mathscr{A} \leq t] \subset \big[\text{eqco } \psi_\mathscr{A} \leq t\big]$ for every $t \in \mathbb{R}$, then

$$\text{eco } A_t \subset \text{eco } [\psi_\mathscr{A} \leq t] \subset \text{eco } \big[\text{eqco } \psi_\mathscr{A} \leq t\big] = \big[\text{eqco } \psi_\mathscr{A} \leq t\big], \ \forall t \in \mathbb{R}.$$

Therefore,

$$[\psi_{\text{eco}\,\mathscr{A}} \leq r] = \bigcap_{t>r} \text{eco } A_t \subset \bigcap_{t>r} \big[\text{eqco } \psi_\mathscr{A} \leq t\big] = \big[\text{eqco } \psi_\mathscr{A} \leq r\big], \ \forall r \in \mathbb{R},$$

which implies $\psi_{\text{eco}\,\mathscr{A}} \geq \text{eqco } \psi_\mathscr{A}$. □

Corollary 3.1 (Characterization of the e-Quasiconvex Hull) *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $\mathscr{A} = \{A_t\}_{t\in\mathbb{R}}$ be an ascending family such that $[f < t] \subset A_t \subset [f \leq t]$ for every $t \in \mathbb{R}$. Then,*

$$\text{eqco } f = \psi_{\text{eco}\,\mathscr{A}}.$$

Proof From the hypothesis, one obtains

$$[f \leq r] = \bigcap_{t>r} [f < t] \subset \bigcap_{t>r} A_t \subset \bigcap_{t>r} [f \leq t] = [f \leq r].$$

Hence, $[\psi_\mathscr{A} \leq r] = [f \leq r]$ for every $r \in \mathbb{R}$, and so $\psi_\mathscr{A} = f$. Now, by Proposition 3.3, we have that eqco $f = \text{eqco } \psi_\mathscr{A} = \psi_{\text{eco}\,\mathscr{A}}$. □

In the previous result, we can consider the ascending family $\mathscr{A}_f = \{[f \leq t]\}_{t\in\mathbb{R}}$, (equivalently, $\mathscr{A} = \{[f < t]\}_{t\in\mathbb{R}}$). Then, one obtains the following representation for the e-quasiconvex hull of f:

$$(\text{eqco } f)(x) = \inf\{r \in \mathbb{R} : x \in \text{eco } [f \leq r]\}, \ x \in \mathbb{R}^n \qquad (3.4)$$

(equivalently, $(\text{eqco } f)(x) = \inf\{r \in \mathbb{R} : x \in \text{eco } [f < r]\}$, for every $x \in \mathbb{R}^n$).

Proposition 3.4 (A Sufficient Condition for e-Quasiconvexity) *For a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, if there is an ascending family $\mathscr{A} = \{A_t\}_{t\in\mathbb{R}}$ of e-convex sets in $\mathbb{R}^n$ such that $[f < t] \subset A_t \subset [f \leq t]$ for all $t \in \mathbb{R}$, then f is e-quasiconvex.*

Proof As $f = \psi_{\mathscr{A}}$ and $\mathscr{A}$ is an ascending family of e-convex sets, due to Proposition 3.2, one has that f is e-quasiconvex. $\square$

A direct consequence of Proposition 3.4 is that any function whose strict lower level sets are e-convex (that is, any strictly e-quasiconvex function), is e-quasiconvex as well (as pointed out in Diagram 3.1).

Next result gathers together four characterizations of e-quasiconvexity at a point given in the literature. For illustrative purposes, we give a proof of $(i) \Longleftrightarrow (iii)$ based on the notion of ascending family, whereas precise references for the complete proof can be found in Sect. 3.6. The latter also applies to all the missing proofs in this chapter.

Theorem 3.1 (Characterizations of e-Quasiconvexity at a Point) *Consider* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *and* $\overline{x} \in \mathbb{R}^n$. *The following statements are equivalent:*

- (i) f *is e-quasiconvex at* $\overline{x}$.
- (ii) $f(\overline{x}) = \inf\{t \in \mathbb{R} : \overline{x} \in \mathrm{eco}\,[f \leq t]\}$.
- (iii) $\overline{x} \notin \mathrm{eco}\,[f \leq r]$ *for all* $r < f(\overline{x})$.
- (iv) *For every* $r < f(\overline{x})$, *there exists* $q \in \mathbb{R}^n$ *such that* $\langle q, x - \overline{x}\rangle < 0$ *for all* $x \in [f \leq r]$.
- (v) f *is quasiconvex and for every* $\overline{y} \in \mathbb{R}^n$ *such that* $f(\overline{y}) < f(\overline{x})$, *every sequence* $y^k \subset \mathbb{R}^n$ *such that* $\{y^k\} \to \overline{y}$, *and every* $\{\mu_k\} \subset (0, +\infty)$, *one has*

$$f(\overline{x}) \leq \liminf_{k \to +\infty} f(\overline{x} + \mu_k(\overline{x} - y^k)).$$

Partial Proof Consider the ascending family $\mathrm{eco}\,\mathscr{A}_f := \{\mathrm{eco}\,[f \leq t]\}_{t \in \mathbb{R}}$.

$[(i) \Rightarrow (iii)]$ Let $r \in \mathbb{R}$ be such that $\overline{x} \in \mathrm{eco}\,[f \leq r] \subset \left[\psi_{\mathrm{eco}\,\mathscr{A}_f} \leq r\right]$. By applying (i) and Corollary 3.1, one has

$$f(\overline{x}) = (\mathrm{eqco}\,f)(\overline{x}) = \psi_{\mathrm{eco}\,\mathscr{A}_f}(\overline{x}) \leq r.$$

$[(iii) \Rightarrow (i)]$ Suppose that (iii) holds. Then, $f(\overline{x}) \leq r$, for every $r \in \mathbb{R}$ such that $\overline{x} \in \mathrm{eco}\,[f \leq r]$, so that, by applying Corollary 3.1, we have

$$(\mathrm{eqco}\,f)(\overline{x}) = \psi_{\mathrm{eco}\,\mathscr{A}_f}(\overline{x}) = \inf\{t \in \mathbb{R} : \overline{x} \in \mathrm{eco}\,[f \leq t]\}$$

$$\geq \inf\{t \in \mathbb{R} : \overline{x} \in [f \leq t]\} = f(\overline{x}).   \square$$

3.3   Conjugacy and Subdifferentiability

Duality theory plays an essential role in convex optimization. Its construction is based on Fenchel conjugation, an important tool in convex analysis. Recall that the *Fenchel conjugate* of a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by $f^*(\cdot) := \sup\{\langle \cdot, x\rangle - f(x) : x \in \mathrm{dom}\,f\}$, where $\mathrm{dom}\,f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$

is the *effective domain* of f. If f is a *proper* function (that is, dom $f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in \mathbb{R}^n$), then f^* is a proper lsc convex function (since it is the pointwise supremum of a collection of affine functions). In particular, the second conjugate of f, f^{**}, is the largest proper lsc convex minorant of f. Therefore, a proper function f is lsc and convex if and only if $f = f^{**}$, result that turns out to be crucial for convex duality (see, e.g., [148, Th. 12.2]).

Another essential tool in duality theory is the notion of subdifferential due to Moreau and Rockafellar [129, 148]. Given $\varepsilon \geq 0$, a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be *ε-subdifferentiable* at a point $\overline{x} \in f^{-1}(\mathbb{R})$ if there exists $u \in \mathbb{R}^n$ such that

$$f(x) \geq f(\overline{x}) + \langle u, x - \overline{x}\rangle - \varepsilon, \quad \forall x \in \mathbb{R}^n. \tag{3.5}$$

The set of those points $u \in \mathbb{R}^n$ satisfying (3.5) is the *ε-subdifferential* of f at $\overline{x}$, denoted by $\partial_\varepsilon f(\overline{x})$. If $f(\overline{x}) \notin \mathbb{R}$, it is assumed to be empty. When $\varepsilon = 0$, we just write $\partial f(\overline{x})$ and it is called the *subdifferential* of f at $\overline{x}$. The function f is said to be ε-subdifferentiable on a subset A of $\mathbb{R}^n$ if it is ε-subdifferentiable at each point of A.

Subdifferentiability is used to obtain optimality conditions in both convex and nonconvex optimization (see, e.g., [89]). Thus, given a proper convex function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $\overline{x} \in \mathbb{R}^n$, [148, Th. 23.5] establishes that $\overline{x} \in \operatorname{argmin} f := \{x \in \mathbb{R}^n : f(x) \leq f(y), \forall y \in \mathbb{R}^n\}$ if and only if $0_n \in \partial f(\overline{x})$.

However, convex duality is not adequate for nonconvex problems. With the aim of providing a basis for duality theory in the e-quasiconvex case, some conjugation and subdifferentiability notions were developed in the literature. Next we show some of them.

3.3.1 $\mathcal{H}$-Conjugation

Let $\mathcal{H}$ be a family of univariate extended real-valued functions, closed under pointwise supremum. A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be *$\mathcal{H}$-convex* if it can be expressed as the supremum of those minorants obtained by composing some $h \in \mathcal{H}$ with some linear function, that is, if for each $\overline{x} \in \mathbb{R}^n$ one has

$$f(\overline{x}) = \sup\{h(\langle y, \overline{x}\rangle) : h \in \mathcal{H}, y \in \mathbb{R}^n, h(\langle y, x\rangle) \leq f(x), \ \forall x \in \mathbb{R}^n\}.$$

The supremum of $\mathcal{H}$-convex functions is again an $\mathcal{H}$-convex function, and the greatest $\mathcal{H}$-convex function majorized by a given function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is called the *$\mathcal{H}$-convex hull* of f.

For $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, its *$\mathcal{H}$-conjugate* is $f^{\mathcal{H}} : \mathbb{R}^n \to \mathcal{H}$ given by

$$f^{\mathcal{H}}(y) := \sup\{h : h \in \mathcal{H}, h(\langle y, x\rangle) \leq f(x) \ \forall x \in \mathbb{R}^n\}.$$

Conversely, given $g : \mathbb{R}^n \to \mathcal{H}$, its $\mathcal{H}'$-conjugate function $g^{\mathcal{H}'} : \mathbb{R}^n \to \overline{\mathbb{R}}$ is defined by

$$g^{\mathcal{H}'}(x) := \sup_{y \in \mathbb{R}^n} g(y)(\langle y, x \rangle). \tag{3.6}$$

As a consequence of these definitions the following statements hold.

Proposition 3.5 (Properties of $\mathcal{H}$-Conjugation) *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$. Then, the following statements hold:*

(i) $f^{\mathcal{H}}(y)(\langle y, x \rangle) \le f(x)$, *for all $y, x \in \mathbb{R}^n$.*
(ii) $f^{\mathcal{H}\mathcal{H}'}$ *is $\mathcal{H}$-convex.*
(iii) *The $\mathcal{H}$-convex hull of f is equal to the second conjugate $f^{\mathcal{H}\mathcal{H}'}$.*
(iv) *f is $\mathcal{H}$-convex if and only if $f = f^{\mathcal{H}\mathcal{H}'}$.*

This generalized concept of conjugation encompasses the classical Fenchel conjugation method by considering $\mathcal{H}$ the family of functions $h_b(t) = t - b$, with $b \in \overline{\mathbb{R}}$, in which case the $\mathcal{H}$-convex functions are the lsc convex functions. It also includes conjugation schemes which are suitable both for lsc quasiconvex functions and e-quasiconvex functions.

Now we give the details of the conjugation for the family of e-quasiconvex functions. For that purpose, recall that an *e-quasiaffine* function is a function which is both e-quasiconvex and quasiconcave. If $\mathcal{H}$ is assumed to be the family of all nondecreasing univariate extended real-valued functions, then the $\mathcal{H}$-conjugation method introduced above provides a characterization of the family of e-quasiconvex functions. More precisely, one gets the following results.

Proposition 3.6 (Characterization of e-Quasiaffine Functions) *Let $q : \mathbb{R}^n \to \overline{\mathbb{R}}$. Then, the following statements are equivalent:*

(i) *q is e-quasiaffine.*
(ii) *$q(x) = h(\langle y, x \rangle)$ for all $x \in \mathbb{R}^n$, for some $y \in \mathbb{R}^n$ and $h : \mathbb{R} \to \overline{\mathbb{R}}$ nondecreasing.*
(iii) *For each $r \in \mathbb{R}$, $[q \le r]$ is either a closed or open halfspace or $\emptyset$ or $\mathbb{R}^n$.*

Proposition 3.7 (Conjugation for e-Quasiconvex Functions) *For a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, if $\mathcal{H}$ is the family of all nondecreasing univariate extended real-valued functions, then the following statements hold:*

(i) *$f^{\mathcal{H}}(y)(t) = \inf\{f(x) : \langle y, x \rangle \ge t\}$ for all $y \in \mathbb{R}^n$ and $t \in \mathbb{R}$.*
(ii) *$f^{\mathcal{H}\mathcal{H}'} = \text{eqco } f$.*
(iii) *f is e-quasiconvex if and only if $f = f^{\mathcal{H}\mathcal{H}'}$.*

The conjugation theory described above (called $\mathcal{H}$-conjugation) is not symmetric, since the conjugate of an extended real-valued function is a function whose values are taken in a family of functions $\mathcal{H}$ instead of $\overline{\mathbb{R}}$. However, it provides a good geometric interpretation of e-quasiconvexity, since any e-quasiconvex function

f is the supremum of e-quasiaffine minorants. Recall from (3.6) that

$$f^{\mathcal{H}\mathcal{H}'}(\cdot) := \sup_{y \in \mathbb{R}^n} f^{\mathcal{H}}(y)(\langle y, \cdot \rangle),$$

where $f^{\mathcal{H}}(y)(\langle y, \cdot \rangle)$, for each $y \in \mathbb{R}^n$, is an e-quasiaffine minorant of f in virtue of Propositions 3.5 and 3.6.

3.3.2 Moreau's Generalized Conjugation Theory

Next we show how the conjugacy for e-quasiconvex functions can be derived from the generalized conjugation theory that was developed by Moreau [130] in an abstract framework. We begin by recalling the essentials of Moreau's conjugation theory (see, for instance, [124]). Let X and Y be two arbitrary sets and let $c : X \times Y \to \overline{\mathbb{R}}$ be a function, called *the coupling function*. For any function $f : X \to \overline{\mathbb{R}}$, its c-conjugate $f^c : Y \to \overline{\mathbb{R}}$ is defined by

$$f^c(y) = \sup_{x \in X}\{c(x, y) - f(x)\}, \quad \forall y \in Y,$$

where the conventions $+\infty + (-\infty) = -\infty + (+\infty) = +\infty - (+\infty) = -\infty - (-\infty) = -\infty$ are assumed. In the same way, for every $g : Y \to \overline{\mathbb{R}}$, its c'-conjugate is the function $g^{c'} : X \to \overline{\mathbb{R}}$ defined by

$$g^{c'}(x) = \sup_{y \in Y}\{c(x, y) - g(y)\}, \quad \forall x \in X.$$

This notation is consistent with considering the coupling function $c' : Y \times X \to \overline{\mathbb{R}}$ given by $c'(y, x) = c(x, y)$. Functions of the form $x \in X \mapsto c(x, y) - \beta \in \overline{\mathbb{R}}$, with $y \in Y$ and $\beta \in \mathbb{R}$, are called *c-elementary*. Similarly, *c'-elementary* functions are those of the form $y \in Y \mapsto c'(y, x) - \beta \in \overline{\mathbb{R}}$, with $x \in X$ and $\beta \in \mathbb{R}$. We denote by Γ_c and $\Gamma_{c'}$ the sets of c-elementary and c'-elementary functions, respectively. A function $f : X \to \overline{\mathbb{R}}$ is said to be Γ_c-*convex* if it is the pointwise supremum of a subset of Γ_c. Hence, every function $f : X \to \overline{\mathbb{R}}$ has a largest Γ_c-convex minorant, which is called its Γ_c-*convex hull*. Similarly, a function $g : Y \to \overline{\mathbb{R}}$ is said to be $\Gamma_{c'}$-*convex* if it is the pointwise supremum of a subset of $\Gamma_{c'}$, being the largest $\Gamma_{c'}$-convex minorant of g its $\Gamma_{c'}$-*convex hull*. We summarize in the following proposition the main properties of this conjugation theory.

Proposition 3.8 *Let $f : X \to \overline{\mathbb{R}}$ and $g : Y \to \overline{\mathbb{R}}$. Then,*

(i) $f^c(y) \leq c(x, y) - f(x)$ and $g^{c'}(x) \leq c(x, y) - g(y)$, for all $x \in X$, $y \in Y$.
(ii) f^c and $g^{c'}$ are $\Gamma_{c'}$-convex and Γ_c-convex, respectively.
(iii) $f = f^{cc'}$ if and only if f is Γ_c-convex, and $g = g^{c'c}$ if and only if g is $\Gamma_{c'}$-convex.

Following this generalized conjugation theory in order to get an appropriate conjugation scheme for the family of e-quasiconvex functions, we shall consider the coupling function $c : \mathbb{R}^n \times (\mathbb{R}^n \times \mathbb{R}) \to \overline{\mathbb{R}}$ defined by

$$c(x, (y, \alpha)) = \begin{cases} 0, & \text{if } \langle y, x \rangle \geq \alpha, \\ -\infty, & \text{otherwise.} \end{cases} \tag{3.7}$$

The conjugation formulas are then

$$f^c(y, \alpha) = -\inf\{f(x) : \langle y, x \rangle \geq \alpha\} \tag{3.8}$$

for $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $(y, \alpha) \in \mathbb{R}^n \times \mathbb{R}$, and

$$g^{c'}(x) = -\inf\{g(y, \alpha) : \langle y, x \rangle \geq \alpha\}$$

for $g : \mathbb{R}^n \times \mathbb{R} \to \overline{\mathbb{R}}$ and $x \in \mathbb{R}^n$. The second c-conjugate of $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is, for every $\overline{x} \in \mathbb{R}^n$,

$$f^{cc'}(\overline{x}) = \sup_{y \in \mathbb{R}^n} \inf\{f(x) : \langle y, x \rangle \geq \langle y, \overline{x} \rangle\}. \tag{3.9}$$

We illustrate these formulas with an example.

Example 3.7 Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$. By (3.8), its c-conjugate (with c the coupling function in (3.7)) is, for every $(y, \alpha) \in \mathbb{R}^2$,

$$f^c(y, \alpha) = \begin{cases} -\infty, & \text{if } y = 0, \alpha > 0, \\ -(\alpha/y)^3, & \text{if } y > 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Now, given $\overline{x} \in \mathbb{R}$, we observe that if $y > 0$ then $\inf\{f(x) : \langle y, x \rangle \geq \langle y, \overline{x} \rangle\} = \overline{x}^3$, and if $y \leq 0$ then $\inf\{f(x) : \langle y, x \rangle \geq \langle y, \overline{x} \rangle\} = -\infty$. Hence, in virtue of (3.9) we get

$$f^{cc'}(\overline{x}) = \sup_{y \in \mathbb{R}^n} \inf\{f(x) : \langle y, x \rangle \geq \langle y, \overline{x} \rangle\} = \overline{x}^3 = f(\overline{x}).$$

Therefore, as a straightforward consequence of the general theory of conjugation, one has that eqco $f = f^{cc'}$ for any $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, getting a further characterization of the even quasiconvexity at a point besides the ones given in Theorem 3.1.

Theorem 3.2 (Characterization of e-Quasiconvexity at a Point) *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $\overline{x} \in \mathbb{R}^n$. The following statements are equivalent:*

(*i*) *f is e-quasiconvex at $\overline{x}$.*
(*ii*) *$f(\overline{x}) = \sup_{y \in \mathbb{R}^n} \inf\{f(x) : \langle y, x \rangle \geq \langle y, \overline{x} \rangle\}$.*

The above result shows that every e-quasiconvex function can be expressed as a supremum of a family of e-quasiaffine functions. More precisely, if f is e-quasiconvex, then

$$f = \sup_{y \in \mathbb{R}^n} f_y,$$

with $f_y = \inf\{f(x) : \langle y, x \rangle \geq \langle y, \cdot \rangle\}$.

3.3.3 Subdifferentials

Quasiconvex functions are not (Moreau-Rockafellar) subdifferentiable in general. Because of that, several subdifferential notions have been proposed for quasiconvex functions in the literature, being the Greenberg–Pierskalla subdifferential [82] the first to be proposed and the key for subgradient methods in quasiconvex optimization problems. Next we aim to link e-quasiconvexity of a function and subdifferentiability (in the sense of Greenberg–Pierskalla) via its strict lower level sets.

Given $\varepsilon \geq 0$, a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be *ε-GP-subdifferentiable* at a point $\overline{x} \in f^{-1}(\mathbb{R})$ if there exists $u \in \mathbb{R}^n$ such that

$$\langle u, x - \overline{x} \rangle \geq 0 \Rightarrow f(x) \geq f(\overline{x}) - \varepsilon, \quad \forall x \in \mathbb{R}^n. \tag{3.10}$$

We define the ε -*GP-subdifferential* of f at $\overline{x}$, denoted by $(\partial_\varepsilon^{GP} f)(\overline{x})$ as the set of those points $u \in \mathbb{R}^n$ satisfying (3.10). If $f(\overline{x}) \notin \mathbb{R}$, it is assumed to be empty. When $\varepsilon = 0$, we just write $(\partial^{GP} f)(\overline{x})$ and it is called the *GP-subdifferential* of f at $\overline{x}$. The function f is said to be ε-GP-subdifferentiable on a subset A of $\mathbb{R}^n$ if it is ε-GP-subdifferentiable at each point of A.

Next result characterizes the ε-GP-subdifferentiability of a function at a given point in terms of the even convexity of a given strict lower level set.

Proposition 3.9 (Non-emptiness of the ε-GP-Subdifferential) *Consider $\varepsilon \geq 0$, $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $\overline{x} \in f^{-1}(\mathbb{R})$. Then, the following statements are equivalent:*

(*i*) *$(\partial_\varepsilon^{GP} f)(\overline{x}) \neq \emptyset$.*
(*ii*) *$\overline{x} \notin \text{eco}\,[f < f(\overline{x}) - \varepsilon]$.*

Proof It follows from the definition of ε-GP-subdifferential since (3.10) is equivalent to

$$\langle u, x \rangle < \langle u, \overline{x} \rangle, \quad \forall x : f(x) < f(\overline{x}) - \varepsilon,$$

and this is equivalent to (*ii*) according to (1.8). $\square$

Corollary 3.2 *Let* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *and* $\overline{x} \in f^{-1}(\mathbb{R})$. *Then:*

(*i*) $(\partial^{GP} f)(\overline{x}) \neq \emptyset$ *if and only if* $\overline{x} \notin \mathrm{eco}\,[f < f(\overline{x})]$.
(*ii*) *If* $[f < f(\overline{x})]$ *is e-convex, then* f *is GP-subdifferentiable at* $\overline{x}$.

Corollary 3.2(*ii*) provides a sufficient condition for GP-subdifferentiability based on the even convexity of an strict lower level set. The next result shows that, under certain even convexity assumptions on either the function or its domain, the even convexity of all strict lower level sets is a necessary condition for the subdifferentiability. The results in this section are in line with the fact that a given e-quasiconvex function at a point is not necessarily GP-subdifferentiable at that point (cf. [183]).

Proposition 3.10 (Necessary Condition for GP-Subdifferentiability) *Assume that a function* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *is GP-subdifferentiable on* $f^{-1}(\mathbb{R})$ *and either* f *is e-quasiconvex or* dom f *is e-convex. Then,* $[f < r]$ *is e-convex for every* $r \in \mathbb{R}$ *(i.e.,* f *is strictly e-quasiconvex).*

Proof Let $r \in \mathbb{R}$ and $\overline{x} \notin [f < r]$, that is, $f(\overline{x}) \geq r$.

Firstly, assume that $f(\overline{x}) < +\infty$ and so, $f(\overline{x}) \in \mathbb{R}$. As f is GP-subdifferentiable on $f^{-1}(\mathbb{R})$, there exists $u \in (\partial^{GP} f)(\overline{x})$ such that if $\langle u, x - \overline{x} \rangle \geq 0$, then $f(x) \geq f(\overline{x}) \geq r$. Hence, $\overline{x} \notin \mathrm{eco}\,[f < r]$ and so, $[f < r]$ is e-convex.

Now, if $f(\overline{x}) = +\infty$ and dom f is e-convex, there exists $u \in \mathbb{R}^n$ such that $\langle u, \overline{x} \rangle > \langle u, x \rangle$ for all $x \in$ dom f. Since $[f < r] \subset$ dom f, then $\overline{x} \notin \mathrm{eco}\,[f < r]$ and, so, $[f < r]$ is e-convex.

Finally, assume that $f(\overline{x}) = +\infty$ and f is e-quasiconvex. If $\overline{x} \in \mathrm{eco}\,[f < r]$, then $\overline{x} \in [f \leq r] = \mathrm{eco}\,[f \leq r]$, but this is impossible as $f(\overline{x}) = +\infty$. Consequently, $\overline{x} \notin \mathrm{eco}\,[f < r]$ and the conclusion follows. $\square$

Following Moreau's generalized conjugation theory described in Sect. 3.3.2, it is possible to define alternative notions of subdifferentials based on the coupling function employed for the conjugacy. Thus, $f : X \to \overline{\mathbb{R}}$ is said to be *c-subdifferentiable* at $\overline{x} \in X$ if $f(\overline{x}) \in \mathbb{R}$ and there exists $\overline{y} \in Y$ such that

$$c(\overline{x}, \overline{y}) \in \mathbb{R} \text{ and } f(x) - f(\overline{x}) \geq c(x, \overline{y}) - c(\overline{x}, \overline{y}), \quad \forall x \in X. \tag{3.11}$$

The set of those points $\overline{y} \in Y$ satisfying (3.11) is called the *c-subdifferential* of f at $\overline{x}$, denoted by $\partial_c f(\overline{x})$. If $f(\overline{x}) \notin \mathbb{R}$, $\partial_c f(\overline{x}) = \emptyset$ by definition. The following properties hold.

Proposition 3.11 (Properties of the c-Subdifferential) *Let $f : X \to \overline{\mathbb{R}}$, $\overline{x} \in X$ and $\overline{y} \in Y$. If $c(\overline{x}, \overline{y}) \in \mathbb{R}$, then:*

(*i*) $\overline{y} \in \partial_c f(\overline{x})$ *if and only if* $f(\overline{x}) + f^c(\overline{y}) = c(\overline{x}, \overline{y})$.
(*ii*) $\overline{y} \in \partial_c f^{cc'}(\overline{x})$ *if and only if* $\overline{x} \in \partial_{c'} f^c(\overline{y})$.
(*iii*)   *If* $\partial_c f(\overline{x}) \neq \emptyset$, *then* f *is* Γ_c-*convex at* $\overline{x}$.
(*iv*)   *If* f *is* Γ_c-*convex at* $\overline{x}$, *then* $\partial_c f^{cc'}(\overline{x}) = \partial_c f(\overline{x})$.

In particular, if we consider again the coupling function $c : \mathbb{R}^n \times (\mathbb{R}^n \times \mathbb{R}) \to \overline{\mathbb{R}}$ introduced in (3.7) for the appropriate conjugation scheme of the e-quasiconvex functions, one gets a corresponding notion of c-subdifferential, say ∂_c. The relationship between this coupling-based subdifferential and the GP-subdifferential is as follows: for $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $\overline{x} \in f^{-1}(\mathbb{R})$, one has

$$\overline{y} \in \partial^{GP} f(\overline{x}) \iff (\overline{y}, \langle \overline{y}, \overline{x} \rangle) \in \partial_c f(\overline{x}).$$

Finally, taking into account the $\mathscr{H}$-conjugation method described in Sect. 3.3.2, we can define further notions of subdifferentials based on the family $\mathscr{H}$ of univariate extended real-valued functions. Thus, $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be $\mathscr{H}$-*subdifferentiable* at $\overline{x} \in \mathbb{R}^n$ if $f(\overline{x}) \in \mathbb{R}$ and there exist $h \in \mathscr{H}$ and $y \in \mathbb{R}^n$ such that

$$h(\langle y, \overline{x} \rangle) = f(\overline{x}) \text{ and } h(\langle y, x \rangle) \leq f(x), \quad \forall x \in \mathbb{R}^n. \tag{3.12}$$

The set of those points $y \in \mathbb{R}^n$ satisfying (3.12) is called the $\mathscr{H}$-*subdifferential* of f at $\overline{x}$, denoted by $\partial_{\mathscr{H}} f(\overline{x})$. If $f(\overline{x}) \notin \mathbb{R}$, $\partial_{\mathscr{H}} f(\overline{x}) = \emptyset$ by definition. When $\mathscr{H}$ is the family of all nondecreasing functions, then $\partial_{\mathscr{H}} f(\cdot) = \partial^{GP} f(\cdot)$.

3.4   Duality in Quasiconvex Optimization

Consider an arbitrary unconstrained optimization problem

$$(GP) \quad \underset{x \in \mathbb{R}^n}{\text{Min}}\ F(x),$$

where $F : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a proper function. We now recall (see, e.g., [41, 149]) the well-known *perturbational approach to duality*, whose key is the use of a perturbation function $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$, such that $\Phi(x, 0_m) = F(x)$ for all $x \in \mathbb{R}^n$, $\mathbb{R}^m$ being the space of perturbation variables. Thus, for each $y \in \mathbb{R}^m$, we have the perturbed optimization problem

$$(GP_y) \quad \underset{x \in \mathbb{R}^n}{\text{Min}}\ \Phi(x, y),$$

and we associate to the family of perturbed problems, the *infimum value function* $p : \mathbb{R}^m \to \overline{\mathbb{R}}$ defined by $p(y) := \inf_{x \in \mathbb{R}^n} \Phi(x, y)$. It is obvious that $p(0_m) = \inf_{x \in \mathbb{R}^n} F(x)$. In order to apply the conjugation scheme for e-quasiconvex functions, we consider the coupling function $c : \mathbb{R}^m \times (\mathbb{R}^m \times \mathbb{R}) \to \overline{\mathbb{R}}$ introduced in (3.7) as

$$c(y, (y^*, \alpha)) = \begin{cases} 0, & \text{if } \langle y^*, y \rangle \geq \alpha, \\ -\infty, & \text{otherwise,} \end{cases}$$

for all $y \in \mathbb{R}^m$ and $(y^*, \alpha) \in \mathbb{R}^m \times \mathbb{R}$, and the c-conjugate of p, $p^c : \mathbb{R}^m \times \mathbb{R} \to \overline{\mathbb{R}}$, defined by

$$p^c(y^*, \alpha) = \sup_{y \in \mathbb{R}^m} \{c(y, (y^*, \alpha)) - p(y)\}, \quad \forall (y^*, \alpha) \in \mathbb{R}^m \times \mathbb{R}. \tag{3.13}$$

From (3.1), it is easy to obtain

$$p(0_m) \geq c(0_m, (y^*, \alpha)) - p^c(y^*, \alpha), \ \forall (y^*, \alpha) \in \mathbb{R}^m \times \mathbb{R},$$

so that we define the dual problem of (GP) as

$$(GD) \quad \underset{y^* \in \mathbb{R}^m, \alpha \in \mathbb{R}}{\text{Max}} c(0_m, (y^*, \alpha)) - p^c(y^*, \alpha),$$

which can be equivalently written as

$$(GD) \underset{y^* \in \mathbb{R}^m, \alpha \in \mathbb{R}}{\text{Max}} \inf\{\Phi(x, y) : x \in \mathbb{R}^n, \langle y^*, y \rangle \geq \alpha\}$$
$$\text{s.t.} \quad \alpha \leq 0.$$

Proposition 3.12 (Duality Theorem) *The following statements hold:*

(i) $v(GD) \leq v(GP)$.

(ii) $v(GD) = (\text{eqco } p)(0_m)$. *If it is finite, the optimal solution set is* $\partial_c(\text{eqco } p)(0_m)$.

(iii) $v(GD) = v(GP)$ *if and only if p is e-quasiconvex at 0_m. In this case, if the optimal value is finite, then the optimal solution set to (GD) is $\partial_c p(0_m)$.*

3.5   An Application to Consumer Theory

In economics it is rather common the following duality framework (see, e.g., [141, 142]). Given a real-valued, non-decreasing function u on the non-negative orthant of $\mathbb{R}^n$, a dual function v is defined by the relation $v(y) = \sup_{x \in \mathbb{R}^n_+} \{u(x) : \langle x, y \rangle \leq 1\}$. It is easy to see that v is non-increasing and quasiconvex. If u is quasiconcave, some additional assumptions allow to obtain a complete duality between the primal

function u and the dual function v in the sense that u can be obtained from v through the relation $u(x) = \inf_{y \in \mathbb{R}_+^n} \{v(y) : \langle x, y \rangle \leq 1\}$. A case in which this scheme is applied is the one of the duality between direct and indirect utility functions in consumer theory.

In an economy in which n different type of commodities are available, each vector $x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$, with x_i denoting the quantity of i-commodity, represents a consumption option. The preferences of a consumer in the set of commodity bundles, $\mathbb{R}_+^n$, are usually represented by a so-called *utility function u* : $\mathbb{R}_+^n \to \overline{\mathbb{R}}$, that is, for any $x, y \in \mathbb{R}_+^n$, x is preferred to y if and only if $u(x) > u(y)$. If $M > 0$ is the maximal amount of money that the consumer can spend and $p \in \mathbb{R}_+^n$ is the vector of commodity prices, the consumer chooses a commodity bundle x by maximizing $u(x)$ subject to the budget constraint $\langle p, x \rangle \leq M$. As $M > 0$, one can consider the vector of normalized prices $y = p/M$ and the consumer's utility problem may be written as

$$(P(y)) \qquad \sup \left\{ u(x) : \langle y, x \rangle \leq 1, x \in \mathbb{R}_+^n \right\}.$$

The function v that associates to y the optimal value of the parameterized problem $P(y)$, $v(y) = \sup \left\{ u(x) : \langle y, x \rangle \leq 1, x \in \mathbb{R}_+^n \right\}$, is called the *indirect utility function* associated with u and it gives the maximum utility level that the consumer can attain when he or she faces the vector y of normalized prices. A complete duality between $u(x)$ and $v(y)$ allows to obtain u from v through the relation $u(x) = \inf \left\{ v(y) : \langle x, y \rangle \leq 1, y \in \mathbb{R}_+^n \right\}$, which implies that the behavior of the consumer can be equivalently described through the indirect utility function, whose variables are the prices.

The duality between the utility function of a consumer and the corresponding indirect utility function has been studied extensively. In 1977, Crouzeix established quite symmetric conditions for the utility functions when they are continuous [34] and, later, in 1983, for the differentiable case [36]. In 1991, Martínez-Legaz [122] obtained a symmetric duality under the weakest possible assumptions.

Proposition 3.13 *Let $v : \mathbb{R}_+^n \to \overline{\mathbb{R}}$. There exists a utility function $u : \mathbb{R}_+^n \to \overline{\mathbb{R}}$ having v as its associated indirect utility function if and only if v is non-increasing, evenly quasiconvex and satisfies*

$$v(y) \leq \lim_{\alpha \to 1^-} (\mathrm{cl}\, v)(\alpha y), \quad \forall y \in \mathrm{bd}\, \mathbb{R}_+^n. \tag{3.14}$$

In this case, one can take u non-decreasing, evenly quasiconcave and satisfying

$$u(x) \geq \lim_{\alpha \to 1^-} \mathrm{cu}\, u(\alpha x), \quad \forall x \in \mathrm{bd}\, \mathbb{R}_+^n. \tag{3.15}$$

Under these conditions, u is unique, namely,

$$u(x) = \inf \left\{ v(y) : \langle x, y \rangle \leq 1, y \in \mathbb{R}_+^n \right\}, \quad \forall x \in \mathbb{R}_+^n.$$

According to this theorem, any non-increasing e-quasiconvex function $v : \mathbb{R}_+^n \to \overline{\mathbb{R}}$ satisfying (3.14) is the indirect utility function associated with a unique non-decreasing e-quasiconcave function $u : \mathbb{R}_+^n \to \overline{\mathbb{R}}$ satisfying (3.15).

3.6 Bibliographic Notes

The concept of e-quasiconvex function first appeared in the PhD thesis of J.E. Martínez-Legaz [119] (see also [120]) on generalized conjugation under the name of "normal quasiconvex function". The term "evenly quasiconvex function" was introduced by Passy and Prisman in [138], a work on conjugacy in quasiconvex programming which was followed by a sequel on the same subject [139]. After that, Martínez-Legaz [121] presented a survey on quasiconvex duality theory based on generalized conjugation methods and showed that e-quasiconvex functions constitute the class of regular functions in most of the conjugation schemes.

This chapter is based on [37, 79, 121, 123, 138]. More precisely, the concept of strictly e-quasiconvex function and the relationships reflected in Diagram 3.1 appeared in [138]. Although Passy and Prisman provide an example of an e-quasiconvex function which is not strictly e-quasiconvex, our Example 3.1 has been taken from [37, p. 64]. The e-quasiconvex hull of a function is introduced in [138, Def. 2.5] and its representation as in (3.4) is [138, Th. 2.1]. The notions regarding ascending families were previously considered in [35, Sec. 2]. The equivalences in Theorem 3.1 appear in [37] in the more general context of separable Banach spaces. In particular, $(i) \iff (ii)$ is [37, Prop. 8], $(i) \iff (iii)$ is [37, Prop. 10], $(i) \iff (iv)$ is [37, Prop. 11] and $(i) \iff (v)$ is [37, Prop. 12].

Several generalizations and particularizations of e-quasiconvex functions have been studied in the literature. Next we review two of them.

- According to Thach [179], a convex set $C \subset \mathbb{R}^n$ is R-convex if $\lambda x \in C$ for all $x \in C$ and $\lambda \geq 1$. Thus, in the same way that quasiconvex functions are defined by the convexity of its lower level sets, a function f is said to be R-quasiconvex when its lower level sets are R-convex, and it is R-evenly quasiconvex when these lower level sets are R-evenly convex, i.e., intersection of a family of open halfspaces whose closures do not contain 0_n. Martínez-Legaz [123] characterized the R-evenly quasiconvex functions as those evenly quasiconvex functions f that satisfy a certain simple relation with their lower semicontinuous hull cl f.
- Rubinov and Glover [153] defined, for a given pair of sets (X, V) with a coupling function $[,] : V \times X \to \overline{\mathbb{R}}$, the so-called evenly-$(X, V)$-convex sets as those sets $Z \subset X$ such that, for each $x \in X \backslash Z$ there exists $v \in V$ such that $[v, x] > [v, z]$ for all $z \in Z$. Then, they define the evenly-(X, V)-quasiconvex functions as those whose lower level sets are evenly-(X, V)-convex.

Regarding the conjugation schemes, the $\mathscr{H}$-conjugation method described in Sect. 3.3.1 was introduced by Martínez-Legaz in [119, 120], whereas the alternative conjugation method in Sect. 3.3.2 was pointed out in [124] and obtained as a particular case from the generalized conjugation theory developed by Moreau [130]. More precisely, Proposition 3.5 is [120, Prop. 1], [120, Prop. 2] and [120, Cor. 3] for statements (i), (iii) and (iv), respectively, Proposition 3.6 is [120, Prop. 23'], Proposition 3.7 is [120, Props. 24'–26'], Proposition 3.8 is [124, Props. 6.1 and 6.2] and the result in Theorem 3.2 is given in [124, p. 258]. Finally, the proofs of Propositions 3.11 and 3.12 can be found in [124, Secs. 6.3 and 6.5].

Independently to the above methods, a related symmetric conjugation scheme for quasiconvex functions was introduced by Passy and Prisman in [138]. In this case, however, the biconjugate function coincides with the proper homogeneous e-quasiconvex hull instead of just the e-quasiconvex hull. The motivation of this result is as follows. For $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, its *perspective function of order* 0 (the perspective function in [90] can be understood as the one of order 1) $g_f : \mathbb{R}^n \times \mathbb{R} \to \overline{\mathbb{R}}$ is defined by

$$g_f(x, \lambda) := \begin{cases} f(x/\lambda), & \text{if } \lambda > 0, \\ \sup f, & \text{otherwise.} \end{cases}$$

By construction, g_f is a *positively homogeneous* function of degree zero (that is, $g_f(tx, t\lambda) = g_f(x, \lambda)$ for all $t > 0$), and f is recoverable from g_f in the sense that $f(\cdot) = g_f(\cdot, 1)$. It holds that f is e-quasiconvex if and only if g_f is e-quasiconvex. Only quasiconvex positively homogeneous of degree zero functions (those functions whose lower level sets are convex cones) are considered in [138]. In this case, if f is quasiconvex, then g_f is *proper* in the sense that if $\alpha < \sup g_f$, then one has $0_{n+1} \notin [g_f \leq \alpha]$ (which implies $g_f(0_{n+1}) = \sup g_f(x, \lambda)$). Hence, for any function $g : \mathbb{R}^n \to \overline{\mathbb{R}}$, its *quasiconjugate* function $g^{\odot} : \mathbb{R}^n \to \overline{\mathbb{R}}$ is given by

$$g^{\odot}(y) = -\inf\{g(x) : \langle y, x \rangle \geq 0\}, \quad \forall y \in \mathbb{R}^n,$$

having that $g^{\odot}$ is a proper homogeneous e-quasiconvex function, and $g = g^{\odot\odot}$ if and only if g is a proper homogeneous e-quasiconvex function (cf. [138, Th. 3.1]). Consequently, the quasi-conjugate $^{\odot}$ induces a one-to-one mapping on the family of proper homogeneous e-quasiconvex functions. The study of those functions whose lower level sets are e-convex cones allows us to recover, with some improvements, some results of Passy and Prisman, and Martínez-Legaz. For instance, a function that attains its maximum at the origin is e-quasiconvex and homogeneous if and only if all its lower level sets are evenly convex cones (cf. [182, Th. III.1.2]).

The notion of λ-quasiconjugate ($\lambda \in \mathbb{R}$) of a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, defined by $f_\lambda^*(y) = \lambda - \inf\{f(x) : \langle y, x \rangle \geq \lambda\}$ for all $y \in \mathbb{R}$, was introduced by Greenberg and Pierskalla [82] and plays an important role in quasiconvex optimization and in the theory of surrogate duality (as well as the Fenchel conjugate does in convex

conjugation and Lagrangian duality). Thach [178, 179] established two dualities for a general quasiconvex optimization problem, restricting himself to particular classes of quasiconvex functions. For that purpose, he introduced the notions of H-quasiconjugate and R-quasiconjugate. On the one hand, the *H-quasiconjugate* of f is defined by $f^H(y) = -\inf\{f(x) : \langle y, x \rangle \geq 1\}$ if $y \neq 0_n$, and $f^H(0_n) = -\sup\{f(x) : x \in \mathbb{R}^n\}$. The fundamental theorem says that f is H-evenly quasiconvex (i.e., all its lower level sets are evenly convex containing 0_n) and $f(0_n) = \inf\{f(x) : x \in \mathbb{R}^n \setminus \{0_n\}\}$ if and only if $f = f^{HH}$. On the other hand, the *R-quasiconjugate* of f is defined by $f^R(y) = -\inf\{f(x) : \langle y, x \rangle \geq -1\}$ for all $y \in \mathbb{R}^n$. In this case, the fundamental theorem says that a function f is R-evenly quasiconvex if and only if $f = f^{RR}$. These concepts have been applied by Suzuki and Kuroiwa [171, 172, 174, 175] to provide duality theorems and set containment characterizations for quasiconvex programming. Furthermore, the paper [179] provides an application to a decentralization by prices for the von Neumann equilibrium problem.

Rubinov and Dutta [154] obtained a Hadamard type inequality for non-negative evenly quasiconvex functions that attain their minimum. The asymptotically sharp constant associated with the inequality over the unit square in the two-dimensional plane is explicitly calculated. An extension of this Hadamard type inequality to non-negative quasiconvex functions was obtained a year later by Hadjisavvas [84].

Quasiconvex analysis has always been deeply related with economic theory. Thus, the seminal work of de Finetti [57] was motivated by problems in Paretian ordinal utility and, later, research in quasiconvex duality was motivated by the dual description of preferences and technologies in microeconomics (see, e.g., [38]). The application of duality theory to problems arising in economics, provides dual problems which usually have nice interpretations that give new perspectives for analyzing the associated primal problems. That is the case of the application to consumer theory described in Sect. 3.5. Proposition 3.13, which establishes a symmetric duality between direct and indirect utility functions under the weakest possible assumptions, is [124, Th. 6.16], although the original result appeared in [122, Th. 2.2] in the more general framework of locally convex topological vector spaces.

Other similar duality schemes, involving suitable extensions of the concept of e-convex set, enjoy a number of applications in finance and economics, particularly, in the context of decision theory [29] and risk measures [30, 60]. For instance, Frittelli and Maggis [60, 115] introduced a generalization of the concept of e-convex set in the conditional framework, providing also the corresponding generalized version of the bipolar theorem. Then, they applied this notion to obtain the dual representation of conditionally evenly quasiconvex maps, which turns out to be a key tool in the study of quasiconvex dynamic risk measures.