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Even Convexity and Optimization

Handling Strict Inequalities

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Preface

This is the first book devoted to *linear systems* of the form:

$$\sigma = \{\langle a_t, x\rangle \leq b_t,\ t \in W;\ \langle a_t, x\rangle < b_t,\ t \in S\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^n$, W and S are arbitrary disjoint index sets, $a_t \in \mathbb{R}^n$ and $b_t \in \mathbb{R}$ for all $t \in T := W \cup S \neq \emptyset$ (a possibly infinite set), to their solution sets (called *evenly convex* by Fenchel in 1952), and to those extended real-valued functions whose epigraphs (or at least their lower level sets) are evenly convex. The necessity of this book comes from the fact that only a few available monographs pay some attention to the existence theorems for linear systems containing strict inequalities, as those of Schrijver (on linear programming) and Mangasarian (on nonlinear programming), Owen (on game theory), being the three of them for T finite, and Goberna and López (on semi-infinite programming) for an arbitrary T. The situation is even worst regarding evenly convex sets and related functions, which have been only mentioned up to now in two monographs: the second edition of Soltan's book on convex sets [166] (which devotes the last two notes for Chapter 9 to evenly convex sets and evenly convex hulls) and the PhD thesis of Maggis [115] (where evenly convex and evenly quasiconvex functions are used in finance and economics). The consequence of this almost null presence of linear systems containing strict inequalities, evenly convex sets and related functions in monographs and textbooks, has been decades of stagnation for this interesting research field and the frequent rediscovery of known results on mathematical objects that have received different names over the years.

We say that σ is *ordinary* whenever $S = \emptyset$ (i.e., σ is exclusively formed by weak linear inequalities), in which case the solution set is closed and convex (a well-known type of evenly convex set). We also say that the above linear system σ is *finite* whenever T is finite and *semi-infinite* otherwise (this name being motivated by the fact that they have finitely many unknowns but infinitely many inequalities). Finite ordinary linear systems were firstly considered by Fourier in 1826 and secondly by Farkas around 1900, to characterize equilibrium points in mechanical structures. They have been intensively analyzed due to the crucial role they play in linear programming, as this widely used optimization model is computationally equivalent to the feasibility problem for finite ordinary linear systems, thanks to the duality theorem and the Fourier–Motzkin elimination method. Semi-infinite ordinary linear systems were first considered by Haar in 1824, in a paper that remained unnoticed until the publication of a free translation by Charnes, Cooper, and Kortanek in 1963, in their seminal paper on linear semi-infinite programming. We only consider in this book ordinary linear systems with comparative purposes as they are treated in the abovementioned monographs, while their nonordinary counterparts have been systematically ignored up to now.

Finite nonordinary linear systems were first considered by Gordan in 1873. They were used sporadically in the twentieth century, for example, by Kuhn in 1956, Walkup and Wets in 1969, and Kannan in 1992, while they are intensively used in the twenty-first century in operational research (e.g., in optimization and games), computational sciences, and other fields. Their solution sets, which are called *evenly convex polyhedra* in this book, have been rediscovered and studied again and again under different names as wholefaced polyhedra, copolyhedra, not necessarily closed (NNC) convex polyhedra, G-polyhedra, and semiclosed polyhedra.

Regarding functions, in the same way that quasiconvex functions and convex functions are defined as those whose lower level sets and epigraphs are convex, respectively, evenly quasiconvex functions and evenly convex functions are those whose lower level sets and epigraphs are evenly convex, respectively. Evenly quasiconvex functions, which have been applied in economics, were introduced by Martínez-Legaz under the name of normal quasiconvex and, independently, by Passy and Prisman, in the early 1980s, while evenly convex functions were introduced by Rodríguez and Vicente-Pérez quite recently, in 2011.

Taking into account the objective of the book, the main concepts and basic results are illustrated with suitable examples, figures, tables, and diagrams. The penultimate section of each chapter is devoted to describe a selection of applications to different fields, with the unique exception of Chap. 4, whose content is so recent that no application is yet known. The last section of each chapter gives precise references for all known results published after a reliable peer review, where the reader can find the corresponding proofs. So, we only include in this book (detailed or sketched) proofs of new results and some classical ones whose original proofs are hardly readable today due to the use of obsolete terms and notations. Accordingly, in the case of results that contain several statements, we provide partial proofs in the sense that we only prove those items that are not proved in the literature.

This book is primarily intended as a guide for further readings addressed to graduate and post-graduate students of mathematics, economics, operational research, and computing and also to researchers specialized in those topics where strict inequalities arise in a natural way. They could have difficulties in accessing to the sparse literature on evenly convex sets and related functions, and their applications to systems and optimization problems involving strict inequalities, due to the lack of a unified terminology in this field. For the sake of simplicity, we decided to work in a finite-dimensional setting, which forced us to consider even optimization problems with finitely many constraints (as each constraint has an associated dual variable). However, the book can also be useful as a source of open problems for researchers interested in functional analysis. To this aim, we include an appendix with an exhaustive list of those results (sometimes whole subsections or sections), which have been shown to be valid in Banach or even in general locally convex Hausdorff topological vector spaces.

The book is organized as follows. Chapter 1 treats in parallel linear systems and evenly convex sets, providing characterizations, operations rules, and separation theorems for evenly convex sets, and existence and Farkas-type results for nonordinary systems. Chapter 2 follows the same scheme as Chap. 1 but now applied to finite linear systems and evenly convex polyhedra; the main difference is that, in this particular setting, it is possible to extend the Motzkin representation theorem for polyhedra. Chapters 3 and 4 also follow similar schemes for evenly quasiconvex functions and evenly convex functions, respectively: characterizations, properties, conjugacy, and duality statements for optimization problems involving such functions, extending in this way well-known results on lower semicontinuous convex functions.

Alicante, Spain
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Contents

Nomenclature

- $\mathbb{R}_+$ the set of nonnegative real numbers
- $\mathbb{R}_{++}$ the set of positive real numbers
- $\mathbb{R}_-$ the set of nonpositive real numbers
- $\mathbb{R}_{--}$ the set of negative real numbers
- $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ the extended real line
- $\sup X$ the smallest upper bound of $X \subset \mathbb{R}$, with $\sup \emptyset = -\infty$
- $\inf X$ the greatest lower bound of $X \subset \mathbb{R}$, with $\inf \emptyset = +\infty$
- 0_n the zero vector in $\mathbb{R}^n$
- $\{e_1, \ldots, e_n\}$ the canonical basis of $\mathbb{R}^n$
- $A^\top$ the transpose of a matrix A
- $\langle x, y\rangle := x^\top y$ the standard inner product of two vectors $x, y \in \mathbb{R}^n$
- $\|x\| := \sqrt{\langle x, x\rangle}$ the Euclidean norm of $x \in \mathbb{R}^n$
- $d(x, y) = \|x - y\|$ the Euclidean distance from x to y, $x, y \in \mathbb{R}^n$
- $d(X, Y)$ the Euclidean distance from X to Y, $\emptyset \neq X, Y \subset \mathbb{R}^n$
- $\mathbb{R}^n_+$ the nonnegative orthant in $\mathbb{R}^n$
- $\mathbb{R}^n_{++}$ the positive orthant in $\mathbb{R}^n$
- $\mathbb{R}^T$ the (product) linear space formed by all functions $\lambda : T \to \mathbb{R}$ with T any index set
- $\operatorname{supp} \lambda := \{t \in T : \lambda_t \neq 0\}$ the support of $\lambda \in \mathbb{R}^T$
- $\mathbb{R}^{(T)}$ the subspace of $\mathbb{R}^T$ formed by those $\lambda \in \mathbb{R}^T$ such that $\operatorname{supp} \lambda$ is finite
- $\mathbb{R}^{(T)}_+$ the convex cone, in $\mathbb{R}^{(T)}$, of the nonnegative finite sequences
- $[x, y] := \{(1 - \lambda)x + \lambda y : \lambda \in [0, 1]\}$ the closed segment joining $x, y \in \mathbb{R}^n$
- The definitions of $]x, y[$, $[x, y[$, and $]x, y]$ are similar, with $]0, 1[$, $[0, 1[$, and $]0, 1]$ instead of $[0, 1]$, respectively
- $\operatorname{span} X$ the linear subspace of $\mathbb{R}^n$ spanned by X, with $\operatorname{span} \emptyset = \{0_n\}$
- $\operatorname{aff} X$ the affine hull of X
- $\operatorname{conv} X$ the convex hull of X
- $\operatorname{cone} X$ the convex cone generated by $X \cup \{0_n\}$
- $\operatorname{eco} X$ the evenly convex hull of X
- $\mathrm{e}' \operatorname{co} X$ the e′-convex hull of X

- cl X the closure of X
- int X the interior of X
- rbd X the relative boundary of X
- rint X the relative interior of X
- extr X the set of extreme points of a convex set X
- 0^+X the recession cone of X
- $X^\circ := \{v \in \mathbb{R}^n : \langle v, x \rangle \leq 1, \ \forall x \in X\}$ the (negative) polar set of a nonempty set X
- $K^\circ := \{v \in \mathbb{R}^n : \langle v, x \rangle \leq 0, \ \forall x \in K\}$ the (negative) polar cone of a convex cone K
- lin K the lineality space of a convex cone K
- dim X the dimension of X
- $|X|$ the cardinality of X
- $D(X; x)$ the cone of feasible directions at $x \in X$
- $T_X(x)$ the tangent cone to X at $x \in \operatorname{cl} X$
- $AX := \{A(x) : x \in X\}$ for a linear mapping $A : \mathbb{R}^n \to \mathbb{R}^m$ and $X \subset \mathbb{R}^n$
- $A^{-1}Y := \{x \in \mathbb{R}^n : A(x) \in Y\}$ for a linear mapping $A : \mathbb{R}^n \to \mathbb{R}^m$ and $Y \subset \mathbb{R}^m$
- $A^\top : \mathbb{R}^n \to \mathbb{R}^n$ the adjoint operator of the linear mapping $A : \mathbb{R}^n \to \mathbb{R}^n$
- $X + Y$ the Minkowski sum of $X, Y \subset \mathbb{R}^n$
- $\sigma = \{\langle a_t, x \rangle \leq b_t, \ t \in W; \langle a_t, x \rangle < b_t, \ t \in S\}$ a linear system
- F the solution set of the system σ
- $C(\sigma)$ the moment set of σ
- $D(\sigma)$ the characteristic set of σ
- $\overline{\sigma}$ the relaxed system of σ
- $\overline{F}$ the solution set of the relaxed system $\overline{\sigma}$
- $N(\overline{\sigma})$ the second order moment cone of $\overline{\sigma}$
- $K(\overline{\sigma})$ the characteristic cone of $\overline{\sigma}$
- $\mathscr{K}(\sigma)$ the representative cone of σ (when it is finite)
- $\operatorname{proj}_p^J : \mathbb{R}^p \to \mathbb{R}^{|J|}$, with $p \in \mathbb{N}$, $\emptyset \neq J \subsetneq \{1, \ldots, p\}$ and $\operatorname{proj}_p^J(x) := \left(x_j\right)_{j \in J}$, the projection of $\mathbb{R}^p$ onto $\mathbb{R}^{|J|}$
- υ_X the valley function of X
- δ_X the indicator function of X
- σ_X the support function of X
- $\operatorname{dom} f = \{x \in \mathbb{R}^n : f(x) < +\infty\}$ the effective domain of $f : \mathbb{R}^n \to \overline{\mathbb{R}}$
- $\operatorname{gph} f = \{(x, f(x)) : x \in \mathbb{R}^n, f(x) \in \mathbb{R}\}$ the graph of f
- $\operatorname{epi} f = \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R} : x \in \operatorname{dom} f, f(x) \leq \lambda\}$ the epigraph of f
- $\operatorname{epi}_s f = \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R} : x \in \operatorname{dom} f, f(x) < \lambda\}$ the strict epigraph of f
- $\operatorname{hypo} f = \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R} : x \in \operatorname{dom} f, f(x) \geq \lambda\}$ the hypograph of f
- $\operatorname{Im} f = \{f(x) : x \in \operatorname{dom} f\}$ the image set of f
- argmin f the set of points in which f reaches its minimum value
- $[f \leq r]$ the lower level set of f for $r \in \mathbb{R}$
- $[f < r]$ the strict lower level set of f for $r \in \mathbb{R}$
- $[f \geq r]$ the upper level set of f for $r \in \mathbb{R}$
- cl f the lower semicontinuous hull of f

- cu f the upper semicontinuous hull of f
- co f the convex hull of f
- qco f the quasiconvex hull of f
- eqco f the evenly quasiconvex hull of f
- eco f the e-convex hull of f
- e$'$ co f the e$'$-convex hull of f
- f^* the Fenchel conjugate of f
- f^c the c-conjugate of f
- $f^{c'}$ the c'-conjugate of f
- $\partial_\varepsilon f(\overline{x})$ the Moreau–Rockafellar ε-subdifferential of f at $\overline{x}$, for $\varepsilon \geq 0$
- $(\partial_\varepsilon^{GP} f)(\overline{x})$ the Greenberg–Pierskalla ε-subdifferential of f at $\overline{x}$, for $\varepsilon \geq 0$
- $\partial_c f(\overline{x})$ the c-subdifferential of f at $\overline{x}$
- $f \square g$ the infimal convolution of f with g
- sup $\mathscr{F}$ the supremum of the family $\mathscr{F}$ of extended real-valued functions

Chapter 1
Evenly Convex Sets: Linear Systems Containing Strict Inequalities

This chapter deals with linear systems with an arbitrary (possibly infinite) number of weak and/or strict inequalities and their solution sets, the so-called evenly convex sets, which can be seen as the two faces of a same coin. Section 1.1 provides different characterizations of evenly convex sets and shows that this class of sets enjoys most of the well-known properties of the subclass of closed convex sets. Since the intersection of evenly convex sets belongs to the same family, any set has an evenly convex hull. Section 1.2 is focussed on the operations with evenly convex hulls and their relationships with other hulls. Section 1.3 reviews different types of separation theorems involving evenly convex sets. Section 1.4 provides existence theorems for linear systems with strict inequalities and characterizations of the linear inequalities which are consequence of consistent systems (those systems with nonempty solution set), which allows us to tackle set containment problems involving evenly convex sets. Section 1.5 is aimed to study the so-called evenly linear semi-infinite programming problems (i.e., linear semi-infinite programming problems with strict inequalities). Finally, Sect. 1.6 describes applications to polarity (treated in a detailed way as it was the problem which inspired the concept of evenly convex set), semi-infinite games, approximate reasoning, optimality conditions in mathematical programming, and strict separation of families of sets.

1.1 Evenly Convex Sets

Since any equation can be replaced by two inequalities, we shall consider (*linear*) *systems* in $\mathbb{R}^n$ of the form

$$\sigma = \{\langle a_t, x\rangle \leq b_t,\ \ t \in W;\ \langle a_t, x\rangle < b_t,\ t \in S\}, \tag{1.1}$$

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where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^n$, W and S are disjoint index sets, $a_t \in \mathbb{R}^n$ and $b_t \in \mathbb{R}$ for all $t \in T := W \cup S \neq \emptyset$ (a possibly infinite set). The solution set of σ, say

$$F = \left\{ x \in \mathbb{R}^n : \langle a_t, x \rangle \leq b_t, \;\; t \in W; \; \langle a_t, x \rangle < b_t, \;\; t \in S \right\},$$

is the intersection of halfspaces and, so, it is a *convex set*, i.e., any segment $[x, y]$ determined by $x, y \in F$ is contained in F. Since $[x, y]$ is an arc joining x and y, any convex set is arc-connected and, so, it is *connected*, i.e., it cannot be represented as the union of two or more disjoint nonempty open subsets for the topology induced by the Euclidean norm in F.

The above system σ is said to be *ordinary* when $S = \emptyset$, *finite* when T is finite and *semi-infinite* otherwise. Moreover, it is said *homogeneous* when $b_t = 0$ for all $t \in T$.

The solution sets of ordinary systems are intersections of closed halfspaces and, so, they are closed and convex. The converse holds as a consequence of the basic separation theorem which asserts that, if $\emptyset \neq C \subsetneq \mathbb{R}^n$ is a closed convex set and $y \notin C$, then, there exist a vector $a \neq 0_n$ and a scalar $b \in \mathbb{R}$ such that $\langle a, y \rangle > b$ and $\langle a, x \rangle \leq b$, for all $x \in C$ (see, e.g., [91, Ch. A: Th. 4.1.1]). Since we can write $\mathbb{R}^n = \{x \in \mathbb{R}^n : \langle 0_n, x \rangle \leq 0\}$ and $\emptyset = \{x \in \mathbb{R}^n : \langle 0_n, x \rangle \leq -1\}$, any closed convex set is the solution set of some ordinary system.

The following concept is the counterpart of closed convex set for non-ordinary systems: a set $C \subset \mathbb{R}^n$ is *evenly convex* (*e-convex*, in brief) if it is the intersection of some family, possibly empty, of open halfspaces. Clearly, any e-convex set is convex, and the converse only holds for sets in $\mathbb{R}$. From the definition, any e-convex set is the solution set of a system as the one in (1.1). Conversely, since any weak inequality $\langle a, x \rangle \leq b$, with $a \neq 0_n$ and $b \in \mathbb{R}$, has the same solutions as the system of strict inequalities $\left\{ \langle a, x \rangle < b + \frac{1}{k}, k \in \mathbb{N} \right\}$, the solution set of the linear system σ in (1.1) is an e-convex set. So, ordinary systems and closed convex sets are particular types of linear systems and e-convex sets, respectively, as Diagram 1.1 shows:

Linear systems $\sigma = \left\{ \begin{array}{l} \langle a_t, x \rangle \leq b_t, \; t \in W \\ \langle a_t, x \rangle < b_t, \; t \in S \end{array} \right\}$	$\Leftarrow$	Ordinary linear systems $\sigma = \{\langle a_t, x \rangle \leq b_t, \; t \in W\}$
$\Updownarrow$		$\Updownarrow$
Evenly convex sets	$\Leftarrow$	**Closed convex sets**

Diagram 1.1 Linear representations of convex sets

In this book we mainly use the standard notation of convex analysis and optimization. So, given a set $X \subset \mathbb{R}^n$, we denote by $\operatorname{int} X$, $\operatorname{rint} X$, $\operatorname{cl} X$, $\operatorname{bd} X$, and $\operatorname{rbd} X$ the interior, the relative interior, the closure, the boundary, and the relative boundary of X, respectively. The set X is said to be *relatively open* if $\operatorname{rint} X = X$ (so, $\emptyset$ and $\mathbb{R}^n$ are relatively open). Moreover, $\operatorname{conv} X$ stands for the convex hull of X, whereas $\operatorname{cone} X := \mathbb{R}_+ \operatorname{conv} X$ means the convex conical hull of $X \cup \{0_n\}$, where 0_n denotes the null vector of $\mathbb{R}^n$. Additionally, if X is a nonempty finite set, we say that $\operatorname{conv} X$ is a *polytope* and $\operatorname{cone} X$ is a *finitely generated cone*. When $\emptyset \neq X \subset \mathbb{R}^n$, we denote by $\operatorname{span} X$ and $\operatorname{aff} X$ the linear span and the affine span of X, respectively, and by

$$0^+ X := \{d \in \mathbb{R}^n : x + td \in X, \forall t \geq 0, \forall x \in X\}$$

the *recession cone* of X. The *Minkowski sum* of X, $Y \subset \mathbb{R}^n$ is the set $X + Y := \{x + y : x \in X,\ y \in Y\}$. Additionally, if C is a nonempty convex subset of $\mathbb{R}^n$, $\dim C$ denotes the dimension of C (defined as the dimension of $\operatorname{aff} C$) and, for $\overline{x} \in C$, the *cone of feasible directions* of C at $\overline{x}$ is

$$D(C, \overline{x}) := \left\{v \in \mathbb{R}^n : \overline{x} + \alpha v \in C \text{ for some } \alpha > 0\right\} = \mathbb{R}_+ \left(C - \overline{x}\right).$$

If $\overline{x} \in \operatorname{cl} C$, the *tangent cone* to C at $\overline{x}$ is $T_C\left(\overline{x}\right) := \operatorname{cl} D(\operatorname{cl} C, \overline{x})$. A convex subset $D \subset C$ is said to be a *face* of C if, for all pairs $v_1 \neq v_2$ of C such that $D \cap]v_1, v_2[\neq \emptyset$, one has $[v_1, v_2] \subset D$. The *extreme points* and *edges* of C are the zero- and one-dimensional faces of C, respectively. We say that a hyperplane H is a *supporting hyperplane* of C at $\overline{x} \in C$ if $\overline{x} \in H$ and C lies in one of the closed halfspaces determined by H. In such a case, we say that H *supports* C at $\overline{x}$. The supporting hyperplane theorem establishes that, if C is a nonempty convex set and $\overline{x} \in C \cap \operatorname{bd} C$, then there is a supporting hyperplane of C at $\overline{x}$. The intersections of C with its supporting hyperplanes are called *exposed faces* of C. Given $X \subset C$, the intersection of exposed faces of C containing X is an exposed face of C. So, there exists a minimal exposed face of C containing X.

Given a convex cone K, the *lineality space* of K is the greatest linear subspace contained in K. We denote it by $\operatorname{lin} K$. Obviously, $\operatorname{lin} K = K \cap (-K)$.

The next result provides eleven characterizations of e-convex sets. One of them involves the following concept: the halfline $\{x + \lambda y : \lambda \geq 0\}$ is a *tangent ray* for the convex set C if $x \in \operatorname{rbd} C$, $y \in \operatorname{cl} D(\operatorname{cl} C, x)$, and $\{x + \lambda y : \lambda \geq 0\} \cap \operatorname{rint} C = \emptyset$.

Theorem 1.1 (Characterization of e-Convex Sets) *Let $C \subset \mathbb{R}^n$ be such that $\emptyset \neq C \neq \mathbb{R}^n$. Then, the following statements are equivalent to each other:*

- (i) *C is e-convex;*
- (ii) *C is the result of eliminating from a closed convex set (precisely,* $\operatorname{cl} C$*) the union of a certain family of its exposed faces;*
- (iii) *C is a convex set and for each $x \in \mathbb{R}^n \setminus C$ there exists a hyperplane H such that $x \in H$ and $H \cap C = \emptyset$;*

(iv) *C is connected and through every point not in C there is some hyperplane H such that $H \cap C = \emptyset$;*
(v) *C is a convex set and $x \in C$ for all $x \in \operatorname{rbd} C$ such that $\{x - \lambda y : \lambda \geq 0\} \cap C \neq \emptyset$ for some tangent ray $\{x + \lambda y : \lambda \geq 0\}$;*
(vi) *C is a convex set and $(x + \operatorname{lin} T_C(x)) \cap C = \emptyset$, for any $x \in (\operatorname{cl} C) \setminus C$;*
(vii) *C is the intersection of a nonempty collection of nonempty open convex sets;*
$(viii)$ *C is a convex set and is the intersection of a collection of complements of hyperplanes;*
(ix) *C is a convex set and for any convex set D contained in $(\operatorname{cl} C) \setminus C$, there exists a hyperplane containing D and not intersecting C;*
(x) *C is a convex set and for any convex set $D \subset (\operatorname{cl} C) \setminus C$, the minimal exposed face (in $\operatorname{cl} C$) containing D does not intersect C;*
(xi) *C is a convex set and for any $x \in (\operatorname{cl} C) \setminus C$, the minimal exposed face (in $\operatorname{cl} C$) containing x does not intersect C; and*
(xii) *C is a convex set and for any $x \in (\operatorname{cl} C) \setminus C$, there exists a supporting hyperplane of $\operatorname{cl} C$ at x not intersecting C.*

The equivalence of the statements (i)–(vi) has been established in different published works (precise references can be found in Sect. 1.7), while the possibility of enlarging this list with statements (vii)–(xii) was conjectured by J.E. Martínez-Legaz in a private communication to one of the authors. So, we limit ourselves to prove the equivalence of each of the statements from (vii) to (xii) with those from (i) to (vi) by turning to the following consequence of $(i) \Longleftrightarrow (iii)$: any relatively open convex set is e-convex. In fact, according to [148, Th. 11.2], given a nonempty relatively open convex set C and an affine manifold M such that $C \cap M = \emptyset$, there exists a hyperplane H such that $M \subset H$ and C is contained in one of the two open halfspaces determined by H. Applying this result to the zero dimensional affine manifolds, i.e., the singleton sets, it is easy to see that condition (iii) holds. So, any relatively open convex set (in particular, any open convex set) is e-convex.

Partial Proof of Theorem 1.1 $[(i) \Rightarrow (vii) \Rightarrow (i)]$ By the definition, any e-convex set C such that $\emptyset \neq C \neq \mathbb{R}^n$ is the intersection of some nonempty family of open halfspaces, so C satisfies (vii). Conversely, it is obvious that the intersection of e-convex sets is an e-convex set and, since each nonempty open convex set is e-convex, the intersection of a collection of nonempty open convex sets is e-convex.

$[(iii) \Rightarrow (viii) \Rightarrow (i)]$ If C satisfies condition (iii), then, given $t \in T := \mathbb{R}^n \setminus C$, there exists a hyperplane H_t such that $t \in H_t$ and $H_t \cap C = \emptyset$. Therefore,

$$C \subset \bigcap_{t \in T} \left(\mathbb{R}^n \setminus H_t\right) \tag{1.2}$$

and

$$\mathbb{R}^n \setminus C = T \subset \bigcup_{t \in T} H_t. \tag{1.3}$$

By applying De Morgan's laws to (1.3), we obtain the equality in (1.2), so C satisfies (*viii*).

Now, suppose that C satisfies (*viii*) and let $C = \underset{t\in T}{\cap} (\mathbb{R}^n \backslash H_t)$, with $H_t = \{x \in \mathbb{R}^n : \langle a_t, x\rangle = b_t\}$, $a_t \in \mathbb{R}^n \backslash \{0_n\}$ and $b_t \in \mathbb{R}$, for all $t \in T$. Since C is a convex set and, for each $t \in T$, $C \subset \mathbb{R}^n \backslash H_t$, we have that C is contained in one of the two open halfspaces determined by H_t. Then, we can suppose, without loss of generality, that

$$C \subset \underset{t\in T}{\cap} \left\{x \in \mathbb{R}^n : \langle a_t, x\rangle > b_t\right\}. \tag{1.4}$$

On the other hand, if $\overline{x} \notin C$, there exists $s \in T$ such that $\overline{x} \notin \mathbb{R}^n \backslash H_s$ or, equivalently, $\langle a_s, \overline{x}\rangle = b_s$. Therefore, $\overline{x} \notin \{x \in \mathbb{R}^n : \langle a_s, x\rangle > b_s\}$ and we obtain the equality in (1.4). So, C is e-convex.

Finally, we shall prove $(ii) \Rightarrow (ix) \Rightarrow (x) \Rightarrow (xi) \Rightarrow (xii) \Rightarrow (iii)$.

[$(ii) \Rightarrow (ix)$] Let $\{X_t,\ t \in T\}$ be a family of exposed faces of $\operatorname{cl} C$ such that

$$C = (\operatorname{cl} C) \setminus \left[\underset{t\in T}{\cup} X_t\right].$$

Let $D \subset (\operatorname{cl} C) \setminus C = \underset{t\in T}{\cup} X_t$ be a nonempty convex set (if $D = \emptyset$, any hyperplane not intersecting C contains D) and let $\overline{x} \in \operatorname{rint} D$. Then, there exists $t \in T$ such that $\overline{x} \in X_t$, so that X_t is a face of $\operatorname{cl} C$ intersecting $\operatorname{rint} D$ and, by [148, Th. 18.1], $D \subset X_t$. Since X_t is an exposed face of $\operatorname{cl} C$, there exists a hyperplane H such that $X_t = H \cap \operatorname{cl} C$ and, therefore, $D \subset H$ and

$$H \cap C = H \cap [(\operatorname{cl} C) \cap C] = X_t \cap C = \emptyset.$$

[$(ix) \Rightarrow (x)$] Let $D \subset (\operatorname{cl} C) \setminus C$ be a convex set and let X be the minimal exposed face (in $\operatorname{cl} C$) such that $D \subset X$. By (*ix*), there exists a hyperplane H such that $D \subset H$ and $H \cap C = \emptyset$. If we take $Y := H \cap \operatorname{cl} C \neq \emptyset$ (since $D \subset Y$), then Y is an exposed face containing D such that

$$Y \cap C = (H \cap \operatorname{cl} C) \cap C = H \cap C = \emptyset. \tag{1.5}$$

Since X is the minimal exposed face containing D, we have $X \subset Y$ and, by (1.5), $X \cap C = \emptyset$.

[$(x) \Rightarrow (xi)$] It is trivial because (xi) is a particular case of (x).

[$(xi) \Rightarrow (xii)$] Let $x \in (\operatorname{cl} C) \setminus C$ and let X be the minimal exposed face (in $\operatorname{cl} C$) containing x. Since X is an exposed face of $\operatorname{cl} C$, there exists a hyperplane H such that $\operatorname{cl} C$ is contained in one of the closed halfspaces determined by H and $X = H \cap \operatorname{cl} C$, so that $x \in H$ and H supports $\operatorname{cl} C$ at x. Moreover, since $X \cap C = \emptyset$, we have

$$H \cap C = H \cap [(\operatorname{cl} C) \cap C] = X \cap C = \emptyset.$$

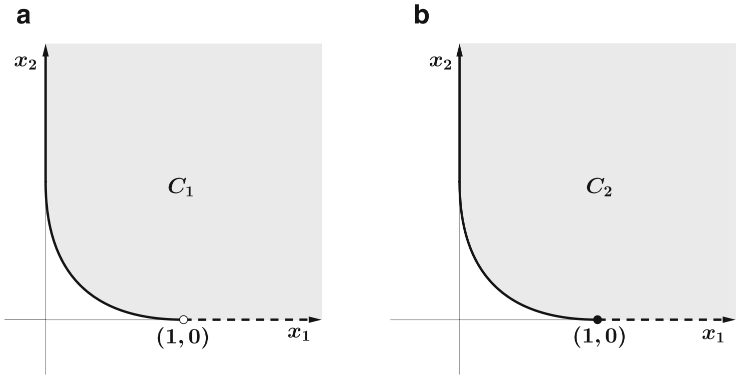


Fig. 1.1 (**a**) The e-convex set C_1; (**b**) The non e-convex set C_2

[(*xii*) $\Rightarrow$ (*iii*)] Let $x \in \mathbb{R}^n \backslash C$. We obtain a hyperplane H such that $x \in H$ and $H \cap C = \emptyset$ as a consequence of (*xii*), if $x \in (\operatorname{cl} C) \backslash C$, and as a consequence of $\operatorname{cl} C$ being an e-convex set, if $x \notin \operatorname{cl} C$. □

Example 1.1 Consider the closed convex set

$$C = \left\{x \in \mathbb{R}^2 : -tx_1 + (t-1)x_2 \leq t^2 - t,\ t \in [0,1]\right\}. \tag{1.6}$$

The set $C_1 := C \backslash ([1, +\infty[\times \{0\})$ is e-convex, whereas $C_2 := C \backslash (]1, +\infty[\times \{0\})$ is not, even though C_2 is convex, and so connected (see Fig. 1.1). In fact, one has:

1. The elimination of the unique exposed face of $C = \operatorname{cl} C_2$ containing $(2, 0)$, $[1, +\infty[\times \{0\}$, yields C_1 instead of C_2 (so (*ii*) fails).
2. C_2 is convex, $(2, 0) \notin C_2$, but $H \cap C_2 \neq \emptyset$ for any hyperplane H such that $(2, 0) \in H$, so (*iii*), (*iv*) and (*ix*) fail (taking $D = \{(2, 0)\} \subset (\operatorname{cl} C_2) \backslash C_2$ in the latter case).
3. C_2 is convex and $[2, +\infty[\times \{0\}$ is a tangent ray emanating from $(2, 0) \in \operatorname{bd} C_2$ such that $([-\infty, 2[\times \{0\}) \cap C_2 = \{(1, 0)\}$, but $(2, 0) \notin C_2$ (so (v) fails).
4. C_2 is convex, $\overline{x} = (2, 0) \in (\operatorname{cl} C_2) \backslash C_2$, $T_{C_2}(\overline{x}) = \mathbb{R} \times [0, +\infty[$, $\overline{x} + \operatorname{lin} T_{C_2}(\overline{x}) = \mathbb{R} \times \{0\}$ and $\left(\overline{x} + \operatorname{lin} T_{C_2}(\overline{x})\right) \cap C_2 = \{(1, 0)\}$ (so, (*vi*) fails).
5. C_2 is convex and $[1, +\infty[\times \{0\}$ is the minimal exposed face in $\operatorname{cl} C_2$ containing $(2, 0)$, but $([1, +\infty[\times \{0\}) \cap C_2 = \{(1, 0)\}$ (so, (x) and (xi) fail).
6. C_2 is convex and the unique supporting hyperplane of $\operatorname{cl} C_2$ at $(2, 0)$ is $H = \left\{x \in \mathbb{R}^2 : x_2 = 0\right\}$, but $H \cap C_2 = \{(1, 0)\}$ (so, (*xii*) fails).

From the comment prior to the proof of Theorem 1.1, since $\operatorname{rint} C$ is relatively open, any convex set $C \neq \emptyset$ can be fitted from inside by its relative interior $\operatorname{rint} C$ and from outside by its closure $\operatorname{cl} C$, both approximating sets being e-convex. Analogously, any *strictly convex set* C (i.e., a convex set C such that its boundary, $\operatorname{bd} C$, does not contain segments) is e-convex since the exposed faces of $\operatorname{cl} C$ are the singleton sets determined by its boundary points. On the other hand, any convex set $C \neq \emptyset$ in the real line $\mathbb{R}$ is an interval and, by Theorem 1.1(*ii*), it is always an e-convex set.

The next result allows to compare the cone of feasible directions at x, $D(C, x)$, the set $\operatorname{extr} C$ of extreme points, and the recession cone 0^+C of an e-convex set C with those of its closure $\operatorname{cl} C$. This comparison shows that e-convex sets enjoy many known properties of closed convex sets (see, e.g., [148, Ths. 8.3, 8.4 and Cor. 8.4.1]).

Proposition 1.1 (Properties of e-Convex Sets) *If $C \subset \mathbb{R}^n$ is a nonempty e-convex set, then the following statements hold:*

(i) $D(C, x) = D(\operatorname{cl} C, x)$ *for all* $x \in C$.
(ii) $\operatorname{extr} C = C \cap \operatorname{extr} \operatorname{cl} C$.
(iii) $[x, y[\subset C$ *for any* $x \in C$ *and* $y \in \operatorname{cl} C$.
(iv) $0^+C = 0^+(\operatorname{cl} C)$. *Consequently, C is bounded if and only if* $0^+C = \{0_n\}$.
(v) *If $y \neq 0_n$ and there exists $x \in C$ such that $\{x + \lambda y : \lambda \geq 0\} \subset C$, then $y \in 0^+C$.*
(vi) *If M is an affine manifold such that $C \cap M$ is a nonempty bounded set, then $M' \cap C$ is also bounded for each affine manifold M' which is parallel to M.*

The convex sets satisfying property (*iii*) are said to be *wholefaced* in the sense of Motzkin [131]. The well-known accessibility lemma asserts that, for any convex set C, $[x, y[\subset \operatorname{rint} C$ for any $x \in \operatorname{rint} C$ and $y \in \operatorname{cl} C$ (see, e.g., [148, Th. 6.1]). Since $\operatorname{cl} \operatorname{rint} C = \operatorname{cl} C$, this lemma means that the relatively open convex sets are wholefaced and (*iii*) is nothing but the extension of this property from relatively open convex sets to e-convex sets.

The next example shows that all statements of Proposition 1.1 may hold, and also fail, simultaneously for convex sets which are not e-convex.

Example 1.2 Let C be as in Example 1.1. Neither $C_3 := C \setminus \{(1, 0)\}$ nor $C_4 = C \setminus (]2, +\infty[\times \{0\})$ is e-convex; however, C_3 satisfies statements from (i) to (vi) of Proposition 1.1; in particular, C_3 is wholefaced even though it is not e-convex. In the contrary, C_4 violates the six statements in Proposition 1.1. In fact, taking $x = (2, 0)$, $y = (3, 0)$, $M = \mathbb{R} \times \{0\}$ and $M' = \mathbb{R} \times \{1\}$ in Fig. 1.2b, we can see that (i), (iii) and (vi) fail. Moreover, $x = (2, 0) \in (\operatorname{extr} C_4) \setminus (\operatorname{extr} \operatorname{cl} C_4)$ (so, (ii) fails) and $\{(0, 1) + \lambda (1, 0) : \lambda \geq 0\} \subset C_4$ with $(0, 1) \in C_4$ and $(1, 0) \in \left(0^+ (\operatorname{cl} C_4)\right) \setminus \left(0^+C_4\right)$ (so (iv) and (v) fail).

The class of e-convex sets is closed for the same operations as the class of closed convex sets, except for the sum. Sufficient conditions for the sum of two e-convex sets to be e-convex will be given in Corollary 1.1.

Proposition 1.2 (Operations with e-Convex Sets) *The following statements hold:*

(i) *If $C \subset \mathbb{R}^n$ is an e-convex set, then αC (resp., $C + v$) is e-convex for all $\alpha \in \mathbb{R}$ (resp., $v \in \mathbb{R}^n$).*

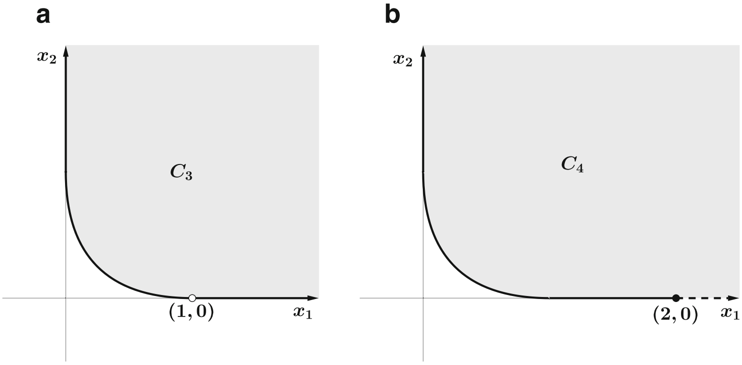


Fig. 1.2 (**a**) C_3 is wholefaced but not e-convex; (**b**) C_4 violates all the statements in Proposition 1.1

(*ii*) *If* $C \subset \mathbb{R}^n$ *is an e-convex set and* $A : \mathbb{R}^m \to \mathbb{R}^n$ *is a linear transformation such that* $A^{-1}C \neq \emptyset$, *then* $A^{-1}C$ *is e-convex and* $0^+(A^{-1}C) = A^{-1}\left(0^+C\right)$.

(*iii*) *If* $C_1 \subset \mathbb{R}^n$ *and* $C_2 \subset \mathbb{R}^m$ *are nonempty sets, then* $C_1 \times C_2$ *is e-convex if and only if* C_1 *and* C_2 *are e-convex.*

(*iv*) *If* $C_1, C_2 \subset \mathbb{R}^n$ *are nonempty e-convex sets such that* $\left(0^+C_1\right) \cap \left(-0^+C_2\right) = \{0_n\}$, *then*

$$0^+\left(C_1 + C_2\right) = 0^+C_1 + 0^+C_2. \tag{1.7}$$

(*v*) *If* $\{C_i, i \in I\}$ *is a family of e-convex sets in* $\mathbb{R}^n$ *such that* $\bigcap_{i\in I} C_i \neq \emptyset$, *then* $\bigcap_{i\in I} C_i$ *is e-convex and*

$$0^+\left(\bigcap_{i\in I} C_i\right) = \bigcap_{i\in I} 0^+C_i.$$

(*vi*) *Let* $C \subset \mathbb{R}^n$ *be a nonempty convex set with* $\dim C = n$, $x \in \mathbb{R}^n$, *and* $k \in \mathbb{Z}$ *such that* $1 \leq k \leq n$. *If* $C \cap M$ *is e-convex for each* k*-dimensional affine manifold* M *containing* x, *with* $k \geq 3$, *or* $x \in \operatorname{int} C$ *and* $k \geq 2$, *then* C *is e-convex.*

In statement (*vi*), conditions over k can be weakened when we replace "e-convex" by "open" or "closed". So, C is open if $C \cap M$ is relatively open and $1 \leq k \leq n$, and C is closed if $C \cap M$ is closed and $k \geq 2$ or $x \in \operatorname{int} C$ [101, Prop. 2.1]. However, with even convexity, statement (*vi*) fails when $k = 2$ and $x \notin \operatorname{int} C$, as the following example shows.

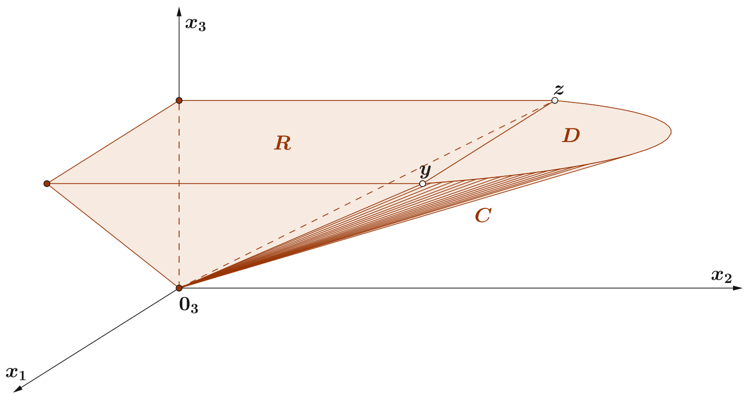


Fig. 1.3 The non e-convex set $C = \operatorname{conv}(G \cup [0, y[\cup [0, z[)$

Example 1.3 Consider in the plane of $\mathbb{R}^3$ given by $x_3 = 1$, a closed rectangle R and a closed half-disk D that is disjoint with rint R and whose diameter coincides with one of the sides of R, say the segment $[y, z]$, and let $G = (R \cup D) \setminus \{y, z\}$. The set $C = \operatorname{conv}(G \cup [0, y[\cup [0, z[)$ is not e-convex (see Fig. 1.3). However, for each 2-dimensional affine manifold M containing $0_3 \notin \operatorname{int} C$, $C \cap M$ consists of a single point, a segment, a closed triangle, or a triangle with one or two missing vertices, and each one of these sets is e-convex.

Concerning the sum of closed convex sets, it is well-known that the condition $\left(0^+ C_1\right) \cap \left(-0^+ C_2\right) = \{0_n\}$ guarantees that $C_1 + C_2$ is closed convex too (see, e.g., [148, Cor. 9.1.2]). The next example shows that this is not true for e-convex sets (even though one of the two sets is bounded).

Example 1.4 Consider the e-convex set C in (1.6). The compact convex set $C_5 := \{x \in C : x_1 + x_2 \leq 1\}$ and the set

$$C_6 := \left\{x \in \mathbb{R}^2 : x_1 \geq 0;\;\; x_2 \geq 0;\;\; x_1 + x_2 > 0\right\}$$

(see Fig. 1.4) are obviously e-convex and satisfy (1.7). Nevertheless, $C_5 + C_6$ is not e-convex (see Fig. 1.5).

Observe also that $C_5 + C_6 = A(C_5 \times C_6)$ if we define $A : \mathbb{R}^{2n} \to \mathbb{R}^n$ as $A(x, z) = x + z$. This shows that the image of an e-convex set through a linear transformation may fail to be e-convex (as it happens with the closed convex sets). In contrast, the linear transformation of a relatively open convex set is another relatively open convex set [148, Th. 6.6].

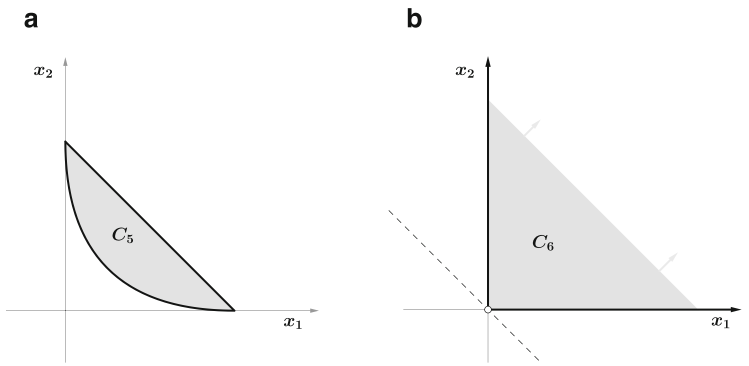


Fig. 1.4 (**a**) The compact convex set C_5; (**b**) The e-convex set C_6

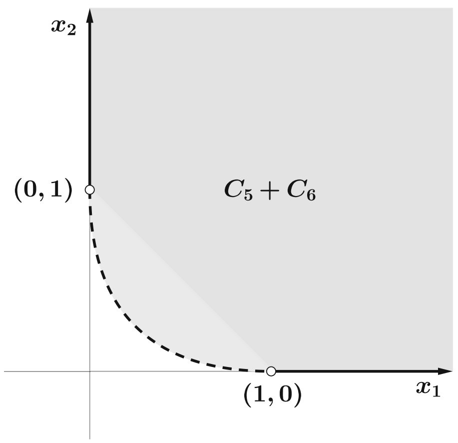


Fig. 1.5 $C_5 + C_6$ is not e-convex

1.2 Evenly Convex Hull

Given $X \subset \mathbb{R}^n$, if $\operatorname{conv} X \subsetneq \mathbb{R}^n$, the intersection of all open halfspaces containing X is the minimal e-convex set which contains X, i.e., it is the *e-convex hull* of X, denoted by $\operatorname{eco} X$. Alternatively, if $\operatorname{conv} X = \mathbb{R}^n$ (i.e., if it does not exist a halfspace containing X), then $\operatorname{eco} X = \mathbb{R}^n$. Obviously, X is e-convex if and only if $\operatorname{eco} X = X$. This happens, for instance, if X is either a closed or a relatively open convex set. Consequently, if X is a compact (open) set, then $\operatorname{conv} X$ is a compact (open) convex set and $\operatorname{eco} X = \operatorname{conv} X$. This is the case, in particular, if $|X| < \infty$, where $|X|$ denotes the cardinality of X. From the properties of e-convex sets, for each $\overline{x} \in \mathbb{R}^n$ one has

$$\overline{x} \notin \operatorname{eco} X \iff \exists\, z \in \mathbb{R}^n : \langle z, x \rangle < \langle z, \overline{x} \rangle\,,\ \forall x \in X. \tag{1.8}$$

For any $X \subset \mathbb{R}^n$, since $\operatorname{cl} \operatorname{conv} X$ is e-convex and $\operatorname{eco} X$ is convex, we have

$$\operatorname{conv} X \subset \operatorname{eco} X \subset \operatorname{cl} \operatorname{conv} X. \tag{1.9}$$

For $\emptyset \neq X \subset \mathbb{R}^n$, since $\operatorname{aff} \operatorname{conv} X = \operatorname{aff} \operatorname{cl} \operatorname{conv} X$ [148, Th. 6.2], we also have that $\operatorname{aff} \operatorname{eco} X = \operatorname{aff} \operatorname{conv} X$ and $\dim \operatorname{eco} X = \dim \operatorname{conv} X$.

The next result establishes the existing relation between the two latter sets in (1.9).

Proposition 1.3 (Characterization of e-Convex Hulls) *For any $X \subset \mathbb{R}^n$, $\operatorname{eco} X$ is the result of eliminating from $\operatorname{cl} \operatorname{conv} X$ the union of all its exposed faces which do not intersect X.*

Example 1.5 Given the set

$$X := \left\{ x \in \mathbb{R}^2 : x_2 = \frac{1}{1 + x_1^2} \right\},$$

we have that $\operatorname{conv} X = (\mathbb{R} \times]0, 1[) \cup \{(0, 1)\}$ (see Fig. 1.6), $\operatorname{eco} X = \mathbb{R} \times]0, 1]$ and $\operatorname{cl} \operatorname{conv} X = \mathbb{R} \times [0, 1]$. Therefore, the inclusions in (1.9) are strict in this case. Observe that $\operatorname{eco} X$ is obtained by eliminating from $\operatorname{cl} \operatorname{conv} X$ its unique exposed face which does not intersect X (the line $\mathbb{R} \times \{0\}$).

The next result describes how e-convex hulls behave under different operators as closures, relative interiors and convex or conical hulls.

Proposition 1.4 (Relationships Between eco and Other Hulls) *Let $X \subset \mathbb{R}^n$. Then, the following statements hold:*

(i) $\operatorname{cl} \operatorname{eco} X = \operatorname{cl} \operatorname{conv} X$.
(ii) $\operatorname{rint} \operatorname{eco} X = \operatorname{rint} \operatorname{conv} X$.
(iii) $\operatorname{eco} \operatorname{conv} X = \operatorname{eco} X = \operatorname{conv} \operatorname{eco} X$.
(iv) $\operatorname{cone} \operatorname{eco} X \subset \operatorname{eco} \operatorname{cone} X = \operatorname{cl} \operatorname{cone} X$.
(v) *If X is a nonempty bounded set, then* $\operatorname{cl} \operatorname{eco} X = \operatorname{eco} \operatorname{cl} X = \operatorname{conv} \operatorname{cl} X$.

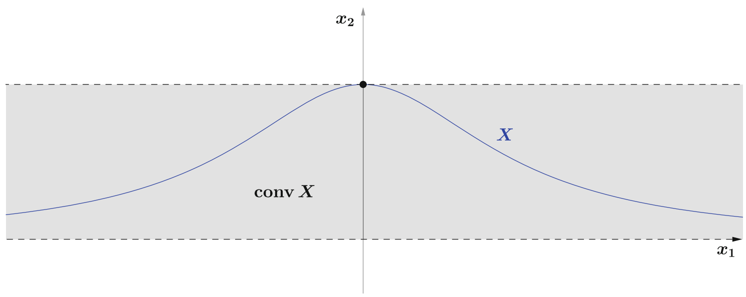


Fig. 1.6 The convex hull of X

Statements (i) and (ii) are easily obtained by taking closures and relative interiors, respectively, in (1.9). An immediate consequence of the equality in statement (iv) is that a translated convex cone containing its apex is e-convex if and only if it is closed [101, Prop. 3.3].

The inclusion in statement (iv) can be strict and the boundedness assumption in statement (v) cannot be eliminated, as we can see in the next example.

Example 1.6 (Example 1.5 Revisited) We have that

$$\operatorname{cone}\operatorname{eco} X = \operatorname{cone} X = (\mathbb{R} \times]0, +\infty[) \cup \{0_2\},$$

whereas $\operatorname{eco}\operatorname{cone} X = \operatorname{cl}\operatorname{cone} X = \mathbb{R}\times [0, +\infty[$, so that the inclusion in statement (iv) is strict.

On the other hand, X is an unbounded set for which $\operatorname{conv}\operatorname{cl} X = \operatorname{conv} X$ and $\operatorname{eco}\operatorname{cl} X = \operatorname{eco} X$ (since X is closed). Moreover, by statement (i), $\operatorname{cl}\operatorname{eco} X = \operatorname{cl}\operatorname{conv} X$. Therefore, as we have already seen in Example 1.5,

$$\operatorname{conv}\operatorname{cl} X \subsetneq \operatorname{eco}\operatorname{cl} X \subsetneq \operatorname{cl}\operatorname{eco} X$$

and so, the boundedness assumption in statement (v) cannot be removed.

Proposition 1.5 (Operations with e-Convex Hulls) *The following statements hold:*

(i) *If* $X, Y \subset \mathbb{R}^n$ *and* $X \subset Y$, *then* $\operatorname{eco} X \subset \operatorname{eco} Y$.
(ii) *If* $X \subset \mathbb{R}^n$ *and* $Y \subset \mathbb{R}^m$, *then* $\operatorname{eco}(X \times Y) = (\operatorname{eco} X) \times (\operatorname{eco} Y)$.
(iii) *If* X *is a nonempty set in* $\mathbb{R}^m$ *and* $A : \mathbb{R}^m \to \mathbb{R}^n$ *is a linear transformation, then* $A(\operatorname{eco} X) \subset \operatorname{eco}(AX)$.
(iv) *If* $X, Y \subset \mathbb{R}^n$, *then* $\operatorname{eco} X + \operatorname{eco} Y \subset \operatorname{eco}(X + Y)$.
(v) *If* X *is a nonempty set in* $\mathbb{R}^n$ *and* $A : \mathbb{R}^n \to \mathbb{R}^n$ *is a bijective linear transformation, then* $A(\operatorname{eco} X) = \operatorname{eco}(AX)$.
(vi) *If* X *is a nonempty set in* $\mathbb{R}^n$ *and* $A : \mathbb{R}^m \to \mathbb{R}^n$ *is a linear transformation such that* $A^{-1}X \neq \emptyset$, *then* $\operatorname{eco}(A^{-1}X) \subset A^{-1}(\operatorname{eco} X)$.
(vii) *If* $\{X_i, i \in I\}$ *is a family of nonempty sets in* $\mathbb{R}^n$, *then*

$$\operatorname{eco}\left(\bigcap_{i\in I} X_i\right) \subset \bigcap_{i\in I} (\operatorname{eco} X_i).$$

As pointed out in Sect. 1.7, where the reader can find precise references, the above statements are already known with the unique exception of statement (v).

Partial Proof of Proposition 1.5 (v) By definition of e-convex hull, if one has $\overline{y} \in \operatorname{eco}(AX)$, then $\overline{y}$ belongs to any open halfspace containing AX. As A is bijective, we can consider $\overline{x} := A^{-1}\overline{y}$. We shall prove that $\overline{x} \in \operatorname{eco} X$.

If $\overline{x} \notin \operatorname{eco} X$, then by (1.8), there exists $z \in \mathbb{R}^n \setminus \{0_n\}$ such that $\langle z, x - \overline{x}\rangle < 0$ for all $x \in X$ and, therefore,

$$0 > \langle z, x - \overline{x} \rangle = \left\langle z, A^{-1}A\,(x - \overline{x}) \right\rangle = \left\langle (A^{-1})^{\top} z, Ax - A\overline{x} \right\rangle, \ \forall x \in X,$$

where $(A^{-1})^{\top}$ represents the adjoint operator of A^{-1} (i.e., the unique linear transformation such that $\left\langle x, A^{-1}y \right\rangle = \left\langle (A^{-1})^{\top}x, y \right\rangle$ for all $x, y \in \mathbb{R}^n$). Taking $d := (A^{-1})^{\top}z$ and $y := Ax$, we have that $\langle d, y - \overline{y} \rangle < 0$ for all $y \in AX$ and so, $\overline{y} \notin \operatorname{eco}(AX)$. We have shown that $\operatorname{eco}(AX) \subset A\,(\operatorname{eco} X)$ and the conclusion follows from (*iii*). □

Taking the e-convex sets $X = C_5$ and $Y = C_6$ as in Example 1.4, we have that $\operatorname{eco} X = X$ and $\operatorname{eco} Y = Y$ whereas $\operatorname{eco}(X + Y) \neq X + Y$ (since $X + Y$ is not an e-convex set; see Fig. 1.5). So, the inclusions in statements (*iii*) and (*iv*) cannot be replaced by equalities.

In the same way, the inclusions in statements (*vi*) and (*vii*) can be strict as we can see in the following examples.

Example 1.7 Let $X := (]0, +\infty[\times]0, +\infty[) \cup \{0_2\} \subset \mathbb{R}^2$ and let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined as $A\,(x_1, x_2) = (x_1, 0)$. Then, $A^{-1}X = \{0\} \times \mathbb{R}$ is an e-convex set whereas $A^{-1}\,(\operatorname{eco} X) = \mathbb{R}_+ \times \mathbb{R}$. So, $\operatorname{eco}(A^{-1}X) = A^{-1}X \subsetneq A^{-1}\,(\operatorname{eco} X)$.

Example 1.8 Let $X_1 := \mathbb{R}^2 \times \{0\}$ and

$$X_2 := \operatorname{conv}\left[\left\{\begin{pmatrix} -\cos t \\ -\sin t \\ -1 \end{pmatrix},\ t \in]0, 2\pi[\right\} + \mathbb{R}_+\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}\right] \cup \left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\},$$

(represented in Fig. 1.7).

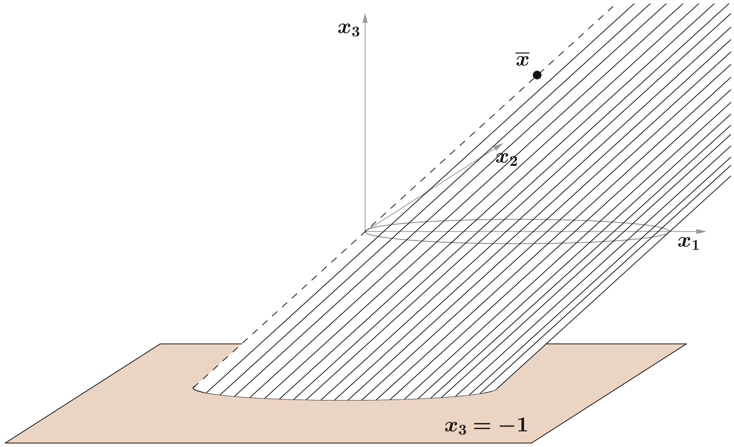


Fig. 1.7 The set X_2, where $\overline{x} = (1, 0, 1)$

Since $\operatorname{eco} X_1 = X_1$ and $\operatorname{eco} X_2 = \operatorname{cl} X_2$, we have $(\operatorname{eco} X_1) \cap (\operatorname{eco} X_2) = \{(x_1, x_2, 0) \in \mathbb{R}^3 : (x_1 - 1)^2 + x_2^2 \leq 1\}$ whereas

$$\operatorname{eco}(X_1 \cap X_2) = X_1 \cap X_2 = \left\{(x_1, x_2, 0) \in \mathbb{R}^3 : (x_1 - 1)^2 + x_2^2 \leq 1\right\} \setminus \{0_3\}.$$

1.3 Separation Theorems

The standard separation theorem for convex sets asserts that any two nonempty disjoint convex sets $X, Y \subset \mathbb{R}^n$ are *weakly separated* by a hyperplane, that is, there exists a hyperplane H such that one of the closed halfspaces determined by H contains X and the other one contains Y. This type of separation is so weak that it does not require X and Y be disjoint. Stronger types of separation are valid for pairs X, Y of convex sets satisfying suitable topological assumptions as openness, closedness and compactness of some of the two sets. The separation theorems for pairs of closed convex sets are useful tools in the study of ordinary systems and those optimization problems whose constraint system is ordinary. Analogously, separation theorems for pairs of e-convex sets are useful in the study of non-ordinary systems and optimization problems with strict inequality constraints. This is the type of separation theorems provided by Victor Klee in 1968 in his attempt to obtain maximal separation theorems, that is, sufficient conditions for certain type of separation of X from Y under minimal hypotheses on these sets. Following Klee [99], given two disjoint sets $X, Y \subset \mathbb{R}^n$, we say that a hyperplane H separates X from Y:

- *Nicely* provided that H is disjoint from X or from Y (without specifying which).
- *Openly* provided that H is disjoint from X.
- *Strictly* provided that H is disjoint from both X and Y.
- *Strongly* provided that H is at positive distance from both X and Y.

Diagram 1.2 Types of separation

$$\begin{array}{c}
\text{Strong separation} \\
\Downarrow \\
\text{Strict separation} \\
\Downarrow \\
\text{Open separation} \\
\Downarrow \\
\text{Nice separation} \\
\Downarrow \\
\text{Weak separation}
\end{array}$$

It is easy to prove that, given a closed convex set X and $x \in \mathbb{R}^n \backslash X$, the hyperplane H orthogonal to the midpoint of the segment joining x with its projection on X separates strongly X from $\{x\}$. So, a nonempty set X is closed and convex if and only if it is strongly separated from any singleton set contained in $\mathbb{R}^n \backslash X$. Analogously, from the equivalence $(i) \Longleftrightarrow (iii)$ in Theorem 1.1, a set X is e-convex if and only if it is openly separated from any singleton set contained in $\mathbb{R}^n \backslash X$ (cf. (1.8)).

It is easy to separate by suitable examples the concepts involved in Diagram 1.2 for any $n \geq 2$. For instance, given a hyperplane H determining two open halfspaces, H_+ and H_-, and two different points, $x, y \in H$, defining the disjoint convex sets $X := H_+ \cup \{x\}$ and $Y := H_- \cup \{y\}$, H separates weakly X from Y, but not nicely. If one aggregates the condition that X and Y should be closed, the counterexample must be built in dimension at least 3, as the following one.

Example 1.9 Consider the line $X := \{(0, x_2, 1) : x_2 \in \mathbb{R}\}$ and the closed convex cone

$$Y := \left\{ y \in \mathbb{R}^3_+ : y_3^2 \leq y_1 y_2 \right\}.$$

The hyperplane $H = \left\{ x \in \mathbb{R}^3 : x_1 = 0 \right\}$ contains X, while Y lies in the halfspace $\left\{ x \in \mathbb{R}^3 : x_1 \geq 0 \right\}$. In fact, H is the unique hyperplane separating weakly X from Y, but the separation is not nice, as Fig. 1.8 shows.

Maximal strong and strict separation theorems involving closedness or openness (among other) assumptions can be found in [99, Ths. 2 and 3]. In particular, [99, Th. 3(d)] states that the openness of two nonempty disjoint convex sets is a minimal condition for strict separation, so that weaker conditions as even convexity

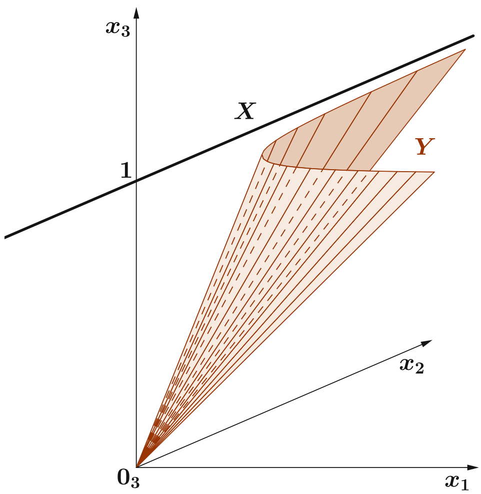


Fig. 1.8 X is (just) weakly separated from Y

cannot imply this kind of separation. Before stating Klee's (maximal) nice and open separation theorems we must introduce some concepts.

Given a set X such that $\emptyset \neq X \subsetneq \mathbb{R}^n$, if $Y \subset \mathbb{R}^n \setminus X$ is a j-dimensional affine manifold such that $d(X, Y) := \inf \{d(x, y) : x \in X, y \in Y\} = 0$, then Y is called a *j-asymptote* of X.

A convex set X is called *continuous* provided that X is closed and its support function $\sigma_X := \sup \{\langle \cdot, x\rangle : x \in X\}$ is continuous; this is equivalent to say that there is no halfline contained in bd X and no 1-asymptote. Given a supporting hyperplane of X, we say that X is *continuous relative to H* if $H \cap X$ is closed and convex but it has neither ray contained in its relative boundary nor 1-asymptote relative to H.

A convex set $\emptyset \neq X \subsetneq \mathbb{R}^n$ is called a *strip* provided that it is a union of translates of a given hyperplane. Equivalently, a strip is a hyperplane, an open or closed halfspace, or a set of the form S or $H_1 \cup S$, or $H_1 \cup S \cup H_2$, where H_1 and H_2 are parallel hyperplanes and S is the set of all points of $\mathbb{R}^n$ lying between H_1 and H_2. All strips are e-convex.

A set $X \subset \mathbb{R}^n$ is said to be *quasi-polyhedral* (or *boundedly polyhedral*) provided that its intersection with any polytope is a polytope and to be *polyhedral* at $x \in X$ provided that X contains a polytope which is a neighborhood of x relative to X. A set is quasi-polyhedral if and only if it is closed, convex, and polyhedral at each of its points.

The next two results collect six open and six nice separation theorems, respectively.

Theorem 1.2 (Open Separation Theorems) *For $X, Y \subset \mathbb{R}^n$ disjoint nonempty convex sets, each of the following conditions implies X is openly separated from Y.*

(i) *X is open; Y is arbitrary.*
(ii) *X is e-convex and its intersection with any supporting hyperplane is compact; Y is closed.*
(iii) *X admits no asymptote in any supporting hyperplane intersecting X; Y admits no asymptote.*
(iv) *X is e-convex and its intersection with any supporting hyperplane is closed; Y is e-convex, Y admits no hyperplane asymptote, and Y is continuous relative to every supporting hyperplane.*
(v) *X's projections are all e-convex; Y admits no asymptote and is quasi-polyhedral.*
(vi) *X is e-convex; Y is singleton or a closed strip.*

Concerning Theorem 1.2, each statement "(i) (respectively, (ii), ..., (vi)) implies X is openly separated from Y" is an open separation theorem, and all of them are maximal in Klee's sense [99], except that (vi) does not when $n = 2$.

Theorem 1.3 (Nice Separation Theorems) *For $X, Y \subset \mathbb{R}^n$ disjoint nonempty convex sets, each of the following conditions implies X is nicely separated from Y.*

- (I) *X is open or a strip; Y is arbitrary.*
- (II) *X is e-convex and is continuous relative to any supporting hyperplane; Y is e-convex and its intersection with any supporting hyperplane is closed.*
- (III) *X admits no asymptote in any supporting hyperplane; Y admits no asymptote in any supporting hyperplane.*
- (IV) *X's projections are all e-convex and X is polyhedral at each of its points; Y's projections are all e-convex and Y is polyhedral at each of its points.*
- (V) *X's projections are all e-convex; Y admits no asymptote in any supporting hyperplane and Y is polyhedral at each of its points.*
- (VI) *X is e-convex; Y is singleton or open or a strip.*

Regarding Theorem 1.3, each statement "(I) (respectively, $(II), \ldots, (VI)$) implies X is nicely separated from Y" is a nice separation theorem, and all of them are maximal in Klee's sense [99], except that (VI) does not when $n = 2$.

Since Klee's paper was published before the standard terminology and notation of convex analysis was established by Rockafellar in his celebrated book [148], the original proofs of the two previous theorems are hardly readable for today's readers. Because of this, we give a sketch of them, which precludes cumbersome arguments based on induction. The keys are Proposition 1.1(iii) and the following lemma.

Lemma 1.1 *Let $X, Y \subset \mathbb{R}^n$ be disjoint nonempty convex sets. Then, X is openly separated from Y if and only if there is no point $p \in X$ which lies in every hyperplane separating X from Y. Any such point p satisfies at least one of the following conditions:*

- (a) $p \in \operatorname{cl} Y$.
- (b) *There is a point $w \in \operatorname{cl} Y$ such that $[p, w] \subset (\operatorname{cl} X) \cap H$ for every hyperplane H separating X from Y.*
- (c) *There are sequences $\{p^k\} \subset \mathbb{R}^n$, $\{x^k\} \subset X$, and $\{y^k\} \subset Y$ such that $y^k \in [p^k, x^k]$ for all k, $\lim p^k = p$, $\lim x^k = x$, and $[p, x]$ is contained in some ray which lies in $(\operatorname{cl} X) \cap H$ for every hyperplane H separating X from Y.*

If X and Y are e-convex, then condition (c) is satisfied for each point $p \in X$ which lies in every hyperplane separating X from Y and each separating hyperplane H such that $X \cap H$ and $Y \cap H$ are both closed and nonempty.

Sketch of the Proofs of Theorems 1.2 and 1.3. By the standard separation theorem, there is a hyperplane H separating X from Y.

If X is open, then $X \cap H = \emptyset$. If X is a strip, it can happen that H supports X and, therefore, $H \subset X$ and $Y \cap H = \emptyset$, or that $H \cap X = \emptyset$. So, statements (i) and (I) imply that X is openly and nicely separated from Y, respectively.

The proofs for statements (vi) and (VI) are trivial.

The separation theorems corresponding to statements (ii), (iv) and (II) are proved by contradiction. Supposing that X is not openly separated from Y, by Lemma 1.1, there is a point $p \in X$ which lies in every hyperplane H separating X from Y. Therefore, $p \in X \cap H$ and H is a supporting hyperplane of X, which, under statements (*ii*), (*iv*) or (*II*), implies that X is e-convex and $X \cap H$ is nonempty and closed.

Regarding the set Y, if $Y \cap H \neq \emptyset$, any of the three conditions implies that Y is e-convex and $Y \cap H$ is nonempty and closed, and then, by the last assertion in Lemma 1.1, condition (c) is satisfied for p, X, Y and H.

If $Y \cap H = \emptyset$, under (*ii*), conditions (a) and (b) in Lemma 1.1 are excluded by the fact that Y is closed, so (c) is satisfied; under (*iv*), the fact that Y admits no hyperplane asymptote yields to a contradiction; and finally, under (*II*), $Y \cap H = \emptyset$ implies that X is nicely separated from Y and there is nothing to prove.

Condition (c) in Lemma 1.1 claims the existence of a ray

$$r := \{p + \lambda u : \lambda \geq 0\} \subset (\operatorname{cl} X) \cap H$$

and, since $p \in X$ and X is e-convex, by Proposition 1.1(*iii*), $r \subset X \cap H$, which is a contradiction under statements (*ii*) and (*II*). Finally, under (*iv*), [63, Lems. 1.1 and 1.2] assert the existence of a parallel ray to r which is a boundary ray or an asymptote of $Y \cap H$ and we obtain a contradiction again.

The remaining statements are proved by induction on n. □

Corollary 1.1 (Even Convexity of the Sum of Convex Sets) *If X and Y are two proper convex sets in $\mathbb{R}^n$, not necessarily disjoint, and they satisfy any of the conditions of Theorems 1.2 and 1.3, then the set $X + Y$ is e-convex.*

Proof The conditions on Y in Theorems 1.2 and 1.3 are symmetric in the sense that they hold for Y if and only if they hold for $-Y$ and they are also preserved under translations. If we take $z \notin X + Y$, then the sets X and $-Y + z$ are disjoint (otherwise, there exist $x \in X$ and $y \in Y$ such that $x = -y + z$ and $z = x + y \in X + Y$). Then, since X and $-Y + z$ are disjoint, any of the conditions of Theorems 1.2 and 1.3 implies the existence of $a \in \mathbb{R}^n \setminus \{0_n\}$ such that $\langle a, x\rangle < \langle a, -y + z\rangle$ for all $x \in X$ and $y \in Y$, whence, $\langle a, x + y\rangle < \langle a, z\rangle$ and we have that $H = \{x \in \mathbb{R}^n : \langle a, x\rangle = \langle a, z\rangle\}$ is a hyperplane which contains z and misses $X + Y$ and, since $X + Y$ is convex, by Proposition 1.1(*iii*), $X + Y$ is e-convex. □

Observe that X and Y are simultaneously e-convex under conditions (*ii*), (*iv*), (*vi*), (*II*) and (*IV*), so that Corollary 1.1 can be interpreted, in those cases, as providing sufficient conditions for the sum of two e-convex sets to be e-convex.

1.4 Linear Systems Containing Strict Inequalities

This section provides characterizations of the existence of solutions of linear systems (Sect. 1.4.1), of the linear inequalities defining half-spaces which include their solution sets (Sect. 1.4.2), and of those pairs of systems such that the solution set of one of them is contained in the solution set of the other one (Sect. 1.4.3). The common feature of all these characterizations is that they are checkable in the sense that they involve different hulls of sets which are expressed in terms of the data (that is, the coefficients of the inequalities).

We associate with the linear system

$$\sigma = \{\langle a_t, x\rangle \leq b_t,\ t \in W;\ \langle a_t, x\rangle < b_t,\ t \in S\} \tag{1.10}$$

its *relaxed system*

$$\overline{\sigma} = \{\langle a_t, x\rangle \leq b_t,\ t \in T\},$$

obtained by replacing $\langle a_t, x\rangle < b_t$ with $\langle a_t, x\rangle \leq b_t$ for all $t \in S$. Obviously, the consistency of $\overline{\sigma}$ does not entail the consistency of σ (consider, e.g., the system $\sigma = \{0 < x < 0\}$ in $\mathbb{R}$). The next simple result, on the relationships between the respective solution sets, F and $\overline{F}$, is fundamental along this section.

Proposition 1.6 (Relationships Between F and $\overline{F}$) *Let F and $\overline{F}$ be the solution sets of σ and $\overline{\sigma}$, respectively. Then, the following statements hold:*

(i) *If $F \neq \emptyset$, then $\overline{F} = \operatorname{cl} F$.*
(ii) *If $F = \emptyset$ and σ does not contain the trivial inequality $\langle 0_n, x\rangle \leq 0$, then either $\overline{F} = \emptyset$ or $\dim \overline{F} < n$.*

1.4.1 Existence of Solutions

We recall that σ is *consistent* if $F \neq \emptyset$ and *inconsistent* otherwise. We next show that the consistency of a linear system σ as in (1.10), with strict inequalities (i.e., $S \neq \emptyset$), can be characterized in terms of the membership, or not, of two particular vectors to the closed convex hull and the e-convex hull of suitable sets involving the data.

It is well known that the consistency of $\overline{\sigma}$ can be characterized by means of the cones

$$N(\overline{\sigma}) := \operatorname{cone}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T\right\} \text{ and } K(\overline{\sigma}) := N(\overline{\sigma}) + \mathbb{R}_+\begin{pmatrix} 0_n \\ 1 \end{pmatrix},$$

which are called, in the linear semi-infinite programming literature, *second order moment cone* and *characteristic cone* of $\overline{\sigma}$, respectively. Indeed, $\overline{\sigma}$ is consistent if and only if

$$\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \notin \operatorname{cl} N(\overline{\sigma}), \tag{1.11}$$

if and only if

$$\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \notin \operatorname{cl} K(\overline{\sigma}). \tag{1.12}$$

Analogously, we define the *moment set* of σ as

$$C(\sigma) := \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, \ t \in S \right\} + \mathbb{R}_+ \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, \ t \in W \right\},$$

and the *characteristic set* of σ as

$$D(\sigma) := C(\sigma) \cup \left\{ \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\}.$$

Observe that $\operatorname{cone} C(\sigma) \subset N(\overline{\sigma}) \subset \operatorname{cl}\operatorname{cone} C(\sigma)$ and, therefore, $\operatorname{cl}\operatorname{cone} C(\sigma) = \operatorname{cl} N(\overline{\sigma})$. Similarly, $\operatorname{cone} D(\sigma) \subset K(\overline{\sigma}) \subset \operatorname{cl}\operatorname{cone} D(\sigma)$ and $\operatorname{cl}\operatorname{cone} D(\sigma) = \operatorname{cl} K(\overline{\sigma})$. So, conditions (1.15) and (1.17) below express the consistency of the relaxed system $\overline{\sigma}$, which is necessary but not sufficient for the consistency of σ. We now show that, assuming the consistency of $\overline{\sigma}$, the additional conditions, (1.16) and (1.18), at statements (*ii*) and (*iii*) , are equivalent, that is, the non-trivial implication, $[\Rightarrow]$, in

$$0_{n+1} \notin \operatorname{eco} C(\sigma) \Longleftrightarrow 0_{n+1} \notin \operatorname{eco} D(\sigma)$$

holds. Since $\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \notin \operatorname{cl}\operatorname{cone} C(\sigma)$, the separation theorem for closed convex cones allows to assert the existence of some vector $\begin{pmatrix} u \\ u_{n+1} \end{pmatrix} \in \mathbb{R}^{n+1} \setminus \{0_{n+1}\}$ such that

$$-u_{n+1} = \left\langle \begin{pmatrix} u \\ u_{n+1} \end{pmatrix}, \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\rangle < 0$$

and

$$\left\langle \begin{pmatrix} u \\ u_{n+1} \end{pmatrix}, \begin{pmatrix} x \\ x_{n+1} \end{pmatrix} \right\rangle \geq 0, \qquad \forall \begin{pmatrix} x \\ x_{n+1} \end{pmatrix} \in C(\sigma). \tag{1.13}$$

Assume that $0_{n+1} \notin \operatorname{eco} C(\sigma)$. Then, by Theorem 1.1(*iii*), there exist $\begin{pmatrix} v \\ v_{n+1} \end{pmatrix} \in \mathbb{R}^{n+1} \setminus \{0_{n+1}\}$ and $v_{n+2} \in \mathbb{R}$ such that

$$v_{n+2} = \left\langle \begin{pmatrix} v \\ v_{n+1} \end{pmatrix}, \begin{pmatrix} 0_n \\ 0 \end{pmatrix} \right\rangle = 0$$

and

$$\left\langle \begin{pmatrix} v \\ v_{n+1} \end{pmatrix}, \begin{pmatrix} x \\ x_{n+1} \end{pmatrix} \right\rangle > v_{n+2} = 0, \qquad \forall \begin{pmatrix} x \\ x_{n+1} \end{pmatrix} \in C(\sigma). \tag{1.14}$$

Since $u_{n+1} > 0$, $v_{n+1} + \alpha u_{n+1} > 0$ for some $\alpha > 0$ sufficiently large. For such a large scalar α one has, from (1.13) and (1.14), that

$$\left\langle \begin{pmatrix} v + \alpha u \\ v_{n+1} + \alpha u_{n+1} \end{pmatrix}, \begin{pmatrix} x \\ x_{n+1} \end{pmatrix} \right\rangle > 0, \qquad \forall \begin{pmatrix} x \\ x_{n+1} \end{pmatrix} \in C(\sigma)$$

while

$$\left\langle \begin{pmatrix} v + \alpha u \\ v_{n+1} + \alpha u_{n+1} \end{pmatrix}, \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\rangle = v_{n+1} + \alpha u_{n+1} > 0$$

as well. So, the hyperplane

$$\left\langle \begin{pmatrix} v + \alpha u \\ v_{n+1} + \alpha u_{n+1} \end{pmatrix}, \begin{pmatrix} x \\ x_{n+1} \end{pmatrix} \right\rangle = 0$$

contains 0_{n+1} while $D(\sigma)$ lies in one of the two open halfspaces it determines, proving that

$$0_{n+1} \notin \operatorname{eco} D(\sigma).$$

Theorem 1.4 (Existence Theorem) *Let $\sigma = \{\langle a_t, x\rangle \leq b_t,\ t \in W;\ \langle a_t, x\rangle < b_t,\ t \in S\}$ with $S \neq \emptyset$. Then, the following statements are equivalent:*

(*i*) *σ is consistent.*
(*ii*)

$$\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \notin \operatorname{cl}\operatorname{cone}\left[\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in S\right\} + \mathbb{R}_+ \left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in W\right\}\right] \tag{1.15}$$

and

$$0_{n+1} \notin \operatorname{eco}\left[\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in S\right\} + \mathbb{R}_+ \left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in W\right\}\right]. \tag{1.16}$$

(*iii*)

$$\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \notin \operatorname{cl}\operatorname{cone}\left[\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in S\right\} + \mathbb{R}_+ \left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in W\right\} \cup \left\{\begin{pmatrix} 0_n \\ 1 \end{pmatrix}\right\}\right] \tag{1.17}$$

and

$$0_{n+1} \notin \operatorname{eco}\left[\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix}, \ t \in S\right\} + \mathbb{R}_+ \left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix}, \ t \in W\right\} \cup \left\{\begin{pmatrix} 0_n \\ 1 \end{pmatrix}\right\}\right]. \tag{1.18}$$

The equivalence $[(ii) \Longleftrightarrow (iii)]$ holds because conditions (1.15) and (1.16) are equivalent to (1.17) and (1.18), respectively. Nevertheless, it is easy to prove that condition (1.18) implies (1.17), so that we obtain the following result as a consequence of $[(i) \Longleftrightarrow (iii)]$ in Theorem 1.4.

Corollary 1.2 *Let σ be as in Theorem 1.4. Then, σ is consistent if and only if (1.18) holds.*

Proof We only prove that (1.18) implies (1.17). Suppose that (1.18) holds, that is, $0_{n+1} \notin \operatorname{eco} D(\sigma)$. Then, by (1.8), there exists $c \in \mathbb{R}^{n+1}$ such that $\langle c, x\rangle < 0$, for all $x \in D(\sigma)$. So, $\left\langle c, \begin{pmatrix} 0_n \\ 1 \end{pmatrix}\right\rangle < 0$ or, equivalently, $\left\langle c, \begin{pmatrix} 0_n \\ -1 \end{pmatrix}\right\rangle > 0$. Denoting $X := \left\{x \in \mathbb{R}^{n+1} : \langle c, x\rangle \leq 0\right\}$, we have that X is an homogeneous closed halfspace such that $D(\sigma) \subset X$ and $\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \notin X$. Then, by [148, Cor. 11.7.2], $\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \notin \operatorname{cl}\operatorname{cone} D(\sigma)$. □

The next corollaries are straightforward consequences of the equivalence $(i) \Longleftrightarrow (ii)$ in Theorem 1.4. Similar results could be obtained from the equivalence $(i) \Longleftrightarrow (iii)$.

Corollary 1.3 *Let σ be as in Theorem 1.4. Then:*

(i) *If σ is consistent, then* $\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \notin \operatorname{cl}\operatorname{cone} C(\sigma)$ *and*

$$0_{n+1} \notin \operatorname{conv}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix}, \ t \in S\right\} + \operatorname{cone}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix}, \ t \in W\right\}. \tag{1.19}$$

(ii) *If $\overline{\sigma}$ is consistent, (1.19) holds and the set in (1.19) is closed, then σ is consistent.*

In [78, Lem. 2.1], it is proved that $\operatorname{conv}(A + \mathbb{R}_+ B) = \operatorname{conv} A + \operatorname{cone} B$ for any nonempty sets A and B in $\mathbb{R}^n$, so that, taking into account the definition of $C(\sigma)$, we have

$$\operatorname{conv} C(\sigma) = \operatorname{conv}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix}, \ t \in S\right\} + \operatorname{cone}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix}, \ t \in W\right\}. \tag{1.20}$$

Since $\operatorname{eco} C(\sigma) = \operatorname{eco} \operatorname{conv} C(\sigma)$, one has

$$\operatorname{eco} C(\sigma) = \operatorname{eco}\left[\operatorname{conv}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix}, \ t \in S\right\} + \operatorname{cone}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix}, \ t \in W\right\}\right], \tag{1.21}$$

so (1.19) is an immediate consequence of (1.16). On the other hand, if the set in (1.19) is closed, then $\operatorname{eco} C(\sigma) = \operatorname{conv} C(\sigma)$ and conditions (1.19) and (1.16) coincide.

The following example shows that the closedness assumption in statement (*ii*) of Corollary 1.3 is not superfluous. It also illustrates the use of Theorem 1.4 in the dubious case that the mentioned closedness fails.

Example 1.10 Let $\sigma := \left\{-tx_1 - x_2 < t^2, \ t \in [-1, 1] \setminus \{0\}; \ x_2 < 0\right\}$. The set in Fig. 1.9a,

$$\left\{\begin{pmatrix} -t \\ -1 \\ t^2 \end{pmatrix}, \ t \in [-1, 1] \setminus \{0\}\right\} \cup \left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\},$$

generates the (non-closed) characteristic cone

$$K(\overline{\sigma}) = \left(\mathbb{R}^2 \times \left]0, +\infty\right[\right) \cup \mathbb{R}_+ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

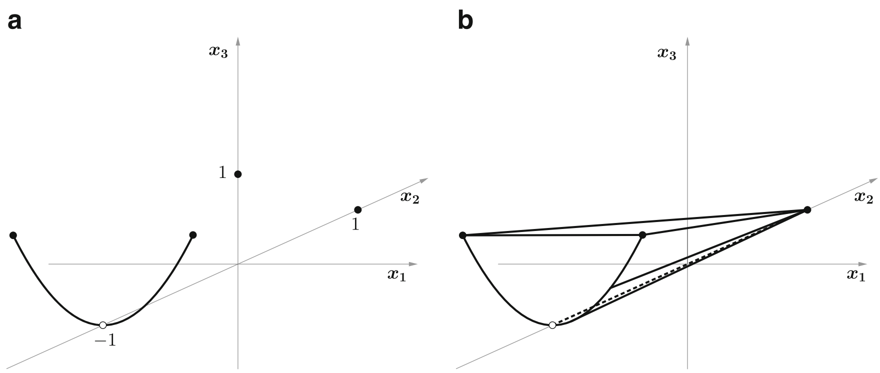


Fig. 1.9 (**a**) Generators of $K(\overline{\sigma})$; (**b**) The set Y

Since (1.15) holds, $\overline{\sigma}$ is consistent. The set in (1.19) is

$$\begin{aligned} Y &:= \operatorname{conv}\left(\left\{\begin{pmatrix}-t\\-1\\t^2\end{pmatrix}, t\in[-1,1]\setminus\{0\}\right\}\cup\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}\right) \\ &= \operatorname{conv}\left\{\begin{pmatrix}-t\\-1\\t^2\end{pmatrix}, t\in[-1,1];\ \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}\setminus(\{0\}\times[-1,1[\times\{0\}) \end{aligned}$$

which does not contain 0_3. So, σ satisfies the necessary condition (i) in Corollary 1.3 but not the sufficient one, (ii). We finally observe that $\operatorname{eco} C(\sigma)$ is the closure of the set Y in Fig. 1.9b as all its exposed faces contain points of Y. Hence,

$$0_3 \in \operatorname{eco} C(\sigma) = \operatorname{cl} Y = \operatorname{conv}\left(\left\{\begin{pmatrix}-t\\-1\\t^2\end{pmatrix}, t\in[-1,1]\right\}\cup\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}\right),$$

and σ turns out to be inconsistent by Theorem 1.4.

Corollary 1.4 (Motzkin-Like Existence Theorem) *Let*

$$\sigma = \{\langle a_t, x\rangle < 0,\ t\in S;\ \langle a_t, x\rangle \le 0,\ t\in W;\ \langle a_t, x\rangle = 0,\ t\in E\} \tag{1.22}$$

be an homogeneous system such that the index sets are pairwise disjoint and $S \neq \emptyset$. Then, σ is consistent if and only if

$$0_n \notin \operatorname{eco}\left[\{a_t,\ t\in S\} + \mathbb{R}_+\{a_t,\ t\in W\} + \mathbb{R}\{a_t,\ t\in E\}\right]. \tag{1.23}$$

In the particular case that the set

$$\operatorname{conv}\{a_t,\ t\in S\} + \operatorname{cone}\{a_t,\ t\in W\} + \operatorname{span}\{a_t,\ t\in E\}$$

is closed, σ is consistent if and only if

$$0_n \notin \operatorname{conv}\{a_t,\ t\in S\} + \operatorname{cone}\{a_t,\ t\in W\} + \operatorname{span}\{a_t,\ t\in E\}.$$

It is customary in mathematical programming to express the existence theorems in the equivalent form of alternative theorems asserting that exactly one of two statements holds, being one of them relative to the consistency of some system, and the other relative to the membership of certain vector in a suitable set. For instance, the reformulation of Corollary 1.4 as an alternative theorem asserts that either σ in (1.22) is consistent or

$$0_n \in \operatorname{eco}\left[\{a_t,\ t\in S\} + \mathbb{R}_+\{a_t,\ t\in W\} + \mathbb{R}\{a_t,\ t\in E\}\right],$$

but not both.

Corollary 1.5 (Gordan-Like Existence Theorem) *An homogeneous system* $\sigma = \{\langle a_t, x\rangle < 0,\ t \in S\}$ *is consistent if and only if*

$$0_n \notin \operatorname{eco}\{a_t,\ t \in S\}. \tag{1.24}$$

In the particular case that $\operatorname{conv}\{a_t,\ t \in S\}$ *is closed,* σ *is consistent if and only if* $0_n \notin \operatorname{conv}\{a_t,\ t \in S\}$.

Obviously, Corollary 1.5 can be expressed as an alternative theorem by saying that either $\sigma = \{\langle a_t, x\rangle < 0,\ t \in S\}$ is consistent or $0_n \in \operatorname{eco}\{a_t,\ t \in S\}$, but not both.

The next alternative theorem is another immediate consequence of Theorem 1.4. There, $\mathbb{R}^{(T)}$ denotes the *space of generalized finite sequences*, that is, the linear space of those functions $\lambda : T \to \mathbb{R}$ whose support, $\operatorname{supp}\lambda := \{t \in T : \lambda_t \neq 0\}$, is finite. We denote by $\mathbb{R}_+^{(T)}$ the positive cone in $\mathbb{R}^{(T)}$. This notation allows us to characterize the existence of solutions in terms of the inexistence of certain multipliers.

Corollary 1.6 (Rockafellar-Like Alternative Theorem) *Let* σ *be as in Theorem 1.4. Assume that* $\{\langle a_t, x\rangle \le b_t,\ t \in W\}$ *is consistent and that*

$$\operatorname{cone}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in T\right\} \text{ and } \operatorname{conv}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in S\right\} + \operatorname{cone}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in W\right\}$$

are closed sets. Then one and only one of the following alternatives holds:

(i) σ *is consistent.*
(ii) *There exists* $\lambda \in \mathbb{R}_+^{(T)}$ *such that at least one of the numbers* λ_t , $t \in S$, *is nonzero, and*

$$\sum_{t\in T}\lambda_t a_t = 0_n \text{ and } \sum_{t\in T}\lambda_t b_t \le 0.$$

If $\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in S\right\}$ is compact and $\operatorname{cone}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in W\right\}$ is closed, the closedness assumptions in Corollary 1.6 hold. In particular, if S and W are finite, then Proposition 1.6 becomes [148, Th. 22.2] (see Corollary 2.4).

The following alternative theorem is an immediate consequence of Corollary 1.6 for systems of strict linear inequalities.

Corollary 1.7 (Carver-Like Alternative Theorem) *Let* $\sigma = \{\langle a_t, x\rangle < b_t,\ t \in S\}$ *be such that* $\operatorname{conv}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in S\right\}$ *is closed. Then one and only one of the following alternatives holds:*

(i) σ *is consistent.*

(*ii*) *There exists* $\lambda \in \mathbb{R}_+^{(T)}$ *such that at least one of the numbers* $\lambda_t, t \in S$, *is nonzero, and*

$$\sum_{t\in S}\lambda_t a_t = 0_n \ \ \textit{and} \ \sum_{t\in S}\lambda_t b_t \leq 0.$$

We finish this section showing the natural way to decide whether σ is consistent or not, and to compute a solution of σ in the first case. To do this we associate with σ the ordinary linear semi-infinite programming (LSIP in short) problem

$$\begin{array}{lll} (P_\sigma) & \underset{(x,x_{n+1})\in\mathbb{R}^{n+1}}{\operatorname{Min}} & x_{n+1} \\ & \text{s.t.} & \langle a_t, x\rangle - x_{n+1} \leq b_t, t \in S, \\ & & \langle a_t, x\rangle \leq b_t, t \in W, \end{array}$$

whose optimal value is denoted by $v(P_\sigma)$.

Proposition 1.7 (Checking the Consistency of σ via LSIP) *The following statements hold:*

(*i*) *If* $v(P_\sigma) < 0$, *then* σ *is consistent.*
(*ii*) *If* $v(P_\sigma) > 0$, *then* σ *is inconsistent.*
(*iii*) *If* $v(P_\sigma) = 0$ *and* (P_σ) *is not solvable, then* σ *is inconsistent.*

If $v(P_\sigma) = 0$ and (P_σ) is solvable, there exists an optimal solution of (P_σ) which can be written as $\binom{\overline{x}}{0}$. Then $\overline{x}$ is solution of $\overline{\sigma}$. Nevertheless, σ is not necessarily consistent as the next example shows.

Example 1.11 Consider the inconsistent system in Example 1.10. We claim that $v(P_\sigma) = 0$ with (P_σ) solvable. In fact, taking limits as $t \to 0$ in $-tx_1 - x_2 - x_3 \leq t^2$, $t \neq 0$, gives $-x_2 - x_3 \leq 0$. The remaining constraint is $x_2 - x_3 \leq 0$, so that $x_3 \geq 0$ for all feasible solution of (P_σ). Since 0_3 is a feasible solution, $v\,(P_\sigma) = 0$ and 0_3 is an optimal solution of (P_σ).

Observe that, given $\varepsilon < 0$, if $\binom{\overline{x}}{\overline{x}_{n+1}}$ is a solution of

$$\sigma_\varepsilon := \{\langle a_t, x\rangle + b_t x_{n+1} \leq \varepsilon,\ t \in S;\ x_{n+1} \leq \varepsilon;\ \langle a_t, x\rangle + b_t x_{n+1} \leq 0,\ t \in W\},$$

then $(-\overline{x}_{n+1})^{-1}\binom{\overline{x}}{\varepsilon}$ is a feasible solution of (P_σ), so that (as observed in [42]) the consistency of σ_ε entails the consistency of σ, according to Proposition 1.7. The converse statement holds if $|S| < \infty$ (since, given $\overline{x} \in F$, then $\varepsilon\delta^{-1}\binom{\overline{x}}{-1}$ is solution of σ_ε for $\delta := \max\{-1;\ \langle a_t, \overline{x}\rangle - b_t,\ t \in S\}$), but it may fail for infinite systems. In fact, for the system in $\mathbb{R}$ $\sigma = \left\{-tx < t^2,\ t \neq 0\right\}$, $F = \{0\}$ whereas σ_ε is inconsistent for all $\varepsilon < 0$.

1.4.2 Consequent Inequalities

An inequality $\langle a, x\rangle \leq b$ (respectively, $\langle a, x\rangle < b$) is *consequence* of

$$\sigma = \{\langle a_t, x\rangle < b_t,\ t \in S;\ \langle a_t, x\rangle \leq b_t,\ t \in W\}$$

if $\langle a, \overline{x}\rangle \leq b$ (respectively, $\langle a, \overline{x}\rangle < b$) holds for every $\overline{x} \in \mathbb{R}^n$ solution of σ. If σ is inconsistent, then any linear inequality is consequence of σ. So we assume throughout this section that σ is consistent.

One way of generating consequent weak inequalities of the ordinary system $\overline{\sigma} = \{\langle a_t, x\rangle \leq b_t, t \in T\}$ consists in picking some $\lambda \in \mathbb{R}_+^{(T)}$, multiplying the inequality $\langle a_t, x\rangle \leq b_t$ by λ_t for each $t \in \operatorname{supp}\lambda$ and summing up these inequalities. The resulting inequality,

$$\left\langle \sum_{t\in T}\lambda_t a_t, x\right\rangle \leq \sum_{t\in T}\lambda_t b_t, \tag{1.25}$$

is of course a consequence of $\overline{\sigma}$, as well as those obtained by strengthening the constant term in (1.25), i.e., the inequalities of the form

$$\left\langle \sum_{t\in T}\lambda_t a_t, x\right\rangle \leq \sum_{t\in T}\lambda_t b_t + \mu, \text{ with } \mu \geq 0.$$

In other words, if

$$\begin{pmatrix} a \\ b \end{pmatrix} \in \operatorname{cone}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T; \begin{pmatrix} 0_n \\ 1 \end{pmatrix}\right\} = K(\overline{\sigma}),$$

then $\langle a, x\rangle \leq b$ is a consequence of $\overline{\sigma}$. These are all the consequences of $\overline{\sigma}$ when T is finite, by the non-homogeneous Farkas lemma proved by Minkowski in 1911, but we can use limits to get more consequences whenever T is infinite. Indeed, if $\left\{\begin{pmatrix} a^k \\ b^k \end{pmatrix}\right\}$ is a sequence in $\mathbb{R}^{n+1}$ converging to $\begin{pmatrix} a \\ b \end{pmatrix}$ such that

$$\begin{pmatrix} a^k \\ b^k \end{pmatrix} \in K(\overline{\sigma}), k = 1, 2, \ldots$$

then $\langle a, x\rangle \leq b$ is consequence of $\overline{\sigma}$. Even more, the weak inequalities which are consequence of $\overline{\sigma}$ are characterized by the generalized non-homogeneous Farkas lemma for ordinary systems [72, Th. 3.1] as follows:

$$\langle a, x\rangle \leq b, \forall x \in \overline{F} \Longleftrightarrow \begin{pmatrix} a \\ b \end{pmatrix} \in \operatorname{cl} K(\overline{\sigma}). \tag{1.26}$$

Regarding σ, where we now assume that $S \neq \emptyset$, one can obtain a strict consequent inequality $\langle a, x\rangle < b$ by picking some $\lambda \in \mathbb{R}_+^{(T)}$ such that $\lambda_t > 0$ for at least one index $t \in S$, in which case

$$\left\langle \sum_{t\in T} \lambda_t a_t, x \right\rangle < \sum_{t\in T} \lambda_t b_t \tag{1.27}$$

is a consequence of σ. A given $\lambda \in \mathbb{R}_+^{(T)}$ is said to be *legal* if $\lambda_t > 0$ for at least one index $t \in S$, while its corresponding strict inequality (1.27) is called a *legal linear combination* of σ. Obviously,

$$\left\langle \sum_{t\in T} \lambda_t a_t, x \right\rangle < \sum_{t\in T} \lambda_t b_t + \mu, \text{ with } \mu \geq 0,$$

is also consequence of σ, but the limiting process mentioned above provides consequent inequalities of the form $\langle a, x\rangle \leq b$, with

$$\begin{pmatrix} a \\ b \end{pmatrix} \in \operatorname{cl} K(\overline{\sigma}) = \operatorname{cl}\operatorname{cone} D(\sigma),$$

whose corresponding strict inequality $\langle a, x\rangle < b$ is not necessarily a consequence of σ. The next result replaces this limiting process by a stronger dual condition involving its characteristic and moment sets and legal linear combinations.

Theorem 1.5 (Characterization of Consequent Inequalities) *For any consistent system* $\sigma = \{\langle a_t, x\rangle \leq b_t,\ t \in W;\ \langle a_t, x\rangle < b_t,\ t \in S\}$ *the following statements hold:*

(i) $\langle a, x\rangle \leq b$ *is consequence of* σ *if and only if*

$$\begin{pmatrix} a \\ b \end{pmatrix} \in \operatorname{cl}\operatorname{cone}\left[\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in S\right\} + \mathbb{R}_+ \left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in W\right\} \cup \left\{\begin{pmatrix} 0_n \\ 1 \end{pmatrix}\right\}\right].$$

(ii) $\langle a, x\rangle < b$ *is consequence of* σ *if and only if either*

$$\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \in \operatorname{cl}\operatorname{cone}\left(\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in S\right\} + \mathbb{R}_+ \left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in W\right\} - \mathbb{R}_+ \begin{pmatrix} a \\ b \end{pmatrix}\right) \tag{1.28}$$

or

$$0_{n+1} \in \operatorname{eco}\left(\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in S\right\} + \mathbb{R}_+ \left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in W\right\} - \mathbb{R}_+ \begin{pmatrix} a \\ b \end{pmatrix}\right). \tag{1.29}$$

(*iii*) *If* $\langle a, x\rangle < b$ *is a legal linear combination of the system* $\{\langle a_t, x\rangle \leq b_t, t \in W \cup S; \langle 0_n, x\rangle < 1\}$, *then (1.28) holds and* $\langle a, x\rangle < b$ *is consequence of* σ. *The converse holds whenever the cone*

$$\operatorname{cone}\left(\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in S\right\} + \mathbb{R}_+\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in W\right\} - \mathbb{R}_+\begin{pmatrix} a \\ b \end{pmatrix}\right)$$

is closed.

(*iv*) *If* $\langle a, x\rangle < b$ *is a legal linear combination of* σ, *then (1.29) holds and the inequality* $\langle a, x\rangle < b$ *is consequence of* σ. *The converse holds whenever the convex set*

$$\operatorname{conv}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in S\right\} + \mathbb{R}_+\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in W\right\} - \mathbb{R}_+\begin{pmatrix} a \\ b \end{pmatrix} \tag{1.30}$$

is e-convex.

Statement (i) above means that $\langle a, x\rangle \leq b$ is consequence of σ if and only if it is consequence of $\overline{\sigma}$. Regarding (ii) and (iii), taking into account that

$$\operatorname{cone}\left(C(\sigma) - \mathbb{R}_+\begin{pmatrix} a \\ b \end{pmatrix}\right) \subset N(\overline{\sigma}) - \mathbb{R}_+\begin{pmatrix} a \\ b \end{pmatrix} \subset \operatorname{cl}\operatorname{cone}\left(C(\sigma) - \mathbb{R}_+\begin{pmatrix} a \\ b \end{pmatrix}\right),$$

condition (1.28) is equivalent to

$$\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \in \operatorname{cl}\left(N(\overline{\sigma}) - \mathbb{R}_+\begin{pmatrix} a \\ b \end{pmatrix}\right)$$

and, whenever $\operatorname{cone}\left(C(\sigma) - \mathbb{R}_+\begin{pmatrix} a \\ b \end{pmatrix}\right)$ is closed, we have

$$\operatorname{cone}\left(C(\sigma) - \mathbb{R}_+\begin{pmatrix} a \\ b \end{pmatrix}\right) = N(\overline{\sigma}) - \mathbb{R}_+\begin{pmatrix} a \\ b \end{pmatrix}.$$

In that case, condition (1.28) obviously yields $\langle a, x\rangle < b$ is a legal linear combination of $\overline{\sigma} \cup \{\langle 0_n, x\rangle < 1\}$.

Regarding (iv), by (1.20), we have

$$\begin{aligned}\operatorname{conv}\left(C(\sigma) - \mathbb{R}_+\begin{pmatrix} a \\ b \end{pmatrix}\right) &= \operatorname{conv}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in S\right\} + \operatorname{cone}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in W; -\begin{pmatrix} a \\ b \end{pmatrix}\right\} \\ &= \operatorname{conv} C(\sigma) - \mathbb{R}_+\begin{pmatrix} a \\ b \end{pmatrix},\end{aligned}$$

so that

$$\operatorname{eco}\left(C(\sigma) - \mathbb{R}_+\begin{pmatrix} a \\ b \end{pmatrix}\right) = \operatorname{eco}\left(\operatorname{conv} C(\sigma) - \mathbb{R}_+\begin{pmatrix} a \\ b \end{pmatrix}\right)$$

and, whenever $\operatorname{conv} C(\sigma) - \mathbb{R}_+ \binom{a}{b}$ is e-convex, we have

$$\operatorname{eco}\left(C(\sigma) - \mathbb{R}_+ \binom{a}{b}\right) = \operatorname{conv} C(\sigma) - \mathbb{R}_+ \binom{a}{b}.$$

The closedness assumption at (*iii*) and even convexity at (*iv*) are not superfluous for the validity of the converse statements, as the next example shows.

Example 1.12 The inequality $-x_2 < 0$, in $\mathbb{R}^2$, is a consequence of

$$\sigma := \left\{2tx_1 - x_2 < t^2,\ t \in U;\ x_1 - x_2 < 0\right\},$$

where $U =]-1, 0[$, as far as the solution set F of σ is the interior of the convex hull of the graph of the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x_1) = \begin{cases} -2x_1 - 1, & \text{if } x_1 < -1, \\ x_1^2, & \text{if } -1 \leq x_1 \leq 0, \\ x_1, & \text{if } x_1 > 0, \end{cases}$$

(see Fig. 1.10). Nevertheless, $-x_2 < 0$ fails to be a legal linear combination of $\overline{\sigma} \cup \{\langle 0_n, x\rangle < 1\}$ or σ, since the following two systems are inconsistent:

$$\left\{ \begin{array}{c} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \sum_{t \in U} \lambda_t \begin{pmatrix} 2t \\ -1 \\ t^2 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ \lambda \in \mathbb{R}_+^{(U)},\ \gamma \geq 0,\ \mu > 0 \end{array} \right\}$$

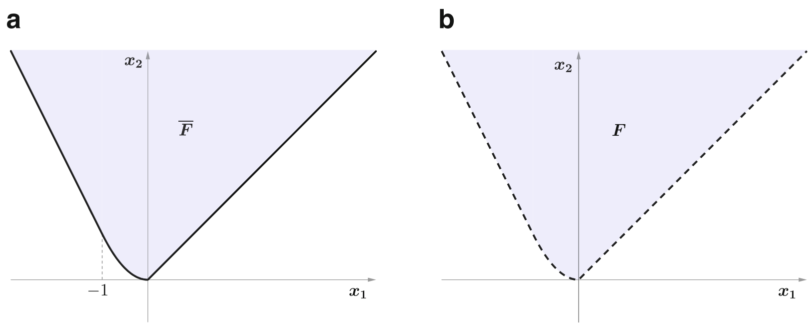


Fig. 1.10 (**a**) The closure of F; (**b**) The solution set F

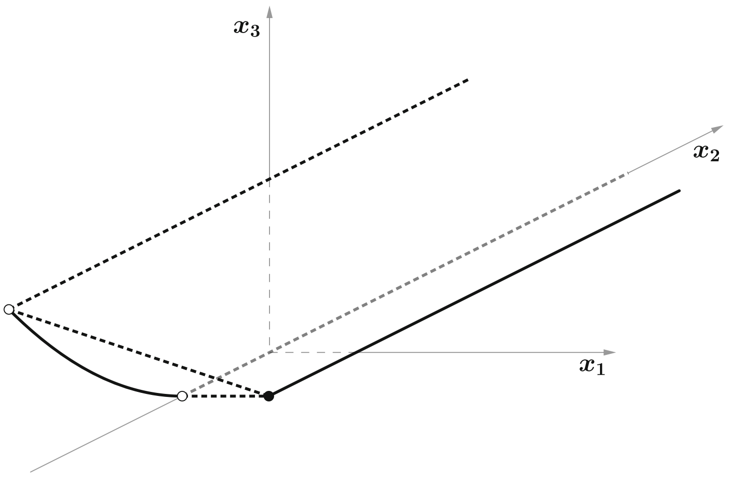


Fig. 1.11 The set $\operatorname{conv} C(\sigma) - \mathbb{R}_+ \binom{a}{b}$

and

$$\left\{ \begin{array}{c} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \sum_{t \in U} \lambda_t \begin{pmatrix} 2t \\ -1 \\ t^2 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \\ \sum_{t \in U} \lambda_t + \gamma > 0 \\ \lambda \in \mathbb{R}_+^{(U)}, \ \gamma \geq 0 \end{array} \right\}.$$

Actually, (1.29) holds, but $\operatorname{conv} C(\sigma) - \mathbb{R}_+ \binom{a}{b}$ is not e-convex (see Fig. 1.11).

Regarding the fulfilment of the additional assumption for the converse statement in (*iv*), i.e., the even convexity of $\operatorname{conv} C(\sigma) - \mathbb{R}_+ \binom{a}{b}$, this set is open whenever $\operatorname{conv} C(\sigma)$ is open, and it is closed whenever $\left\{ \binom{a_t}{b_t}, t \in S \right\}$ is compact and $W = \emptyset$ (and so e-convex in both cases). Moreover, according to Corollary 1.1, $\operatorname{conv} C(\sigma) - \mathbb{R}_+ \binom{a}{b}$ is e-convex if the pair of sets $X := \operatorname{conv} C(\sigma)$ and $Y := \mathbb{R}_+ \binom{a}{b}$ satisfies any of the conditions of Theorems 1.2 and 1.3, for instance, at least one of the conditions (*ii*), (*iii*), (*v*), and (I), which here collapse to:

- $\operatorname{conv} C(\sigma)$ is e-convex and its intersection with any supporting hyperplane is compact.
- $\operatorname{conv} C(\sigma)$ admits no asymptote in any supporting hyperplane.
- The projections of $\operatorname{conv} C(\sigma)$ are all e-convex.
- $\operatorname{conv} C(\sigma)$ is a strip.

1.4.3 Set Containment of Evenly Convex Sets

One of the basic tools in machine learning are the so-called *linear classifiers*, which are affine functions allowing to incorporate prior knowledge obtained from sets (the so-called learning sets), which are usually formed by individuals which have been previously classified (e.g., as either healthy or ill in medical tests). Mangasarian considered infinite learning sets, starting with the simple ones, the polyhedral sets [117, 118]. The main question was how to model the condition that a given *polyhedral knowledge set* F satisfies $F \subset \{x \in \mathbb{R}^n : \langle a, x\rangle - b \leq 0\}$, where $a \in \mathbb{R}^n \setminus \{0_n\}$ and $b \in \mathbb{R}$ are the decision variables determining the best linear classifier $\langle a, \cdot\rangle - b$ for certain criterion.

Since, by the separation theorem, any closed convex set F is the intersection of all closed halfspaces containing F, we associate with F the so-called *weak dual cone* of F,

$$K_F^{\leqq} := \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^{n+1} : \langle a, x\rangle \leq b, \forall x \in F \right\}. \tag{1.31}$$

Then,

$$F \subset \left\{x \in \mathbb{R}^n : \langle a, x\rangle \leq b\right\} \Longleftrightarrow \begin{pmatrix} a \\ b \end{pmatrix} \in K_F^{\leqq}. \tag{1.32}$$

If F is expressed as the solution set of a finite system $\{\langle a_t, x\rangle \leq b_t,\ t \in W\}$, the classical non-homogeneous Farkas lemma asserts that

$$K_F^{\leqq} := \operatorname{cone}\left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in W;\ \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\},$$

providing the aimed characterization,

$$F \subset \left\{x \in \mathbb{R}^n : \langle a, x\rangle \leq b\right\} \Longleftrightarrow \begin{pmatrix} a \\ b \end{pmatrix} \in \operatorname{cone}\left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in W;\ \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\},$$

for the containment of F in a halfspace in terms of the data.

More generally, the *containment problem*, which consists of deciding, for a given couple of subsets of $\mathbb{R}^n$, the *inbody* F and the *circumbody* G, whether $F \subset G$ or not, was first posed in 2000 by Mangasarian [117]. The sets F and G are usually given as solution sets of inequality systems, and the aim is the characterization of the inclusion $F \subset G$ in terms of the data (usually, the constraints describing these sets). Mangasarian [117] solved the containment problem for polyhedral convex sets (the situation illustrated by Fig. 1.12) via linear programming, by exploiting the

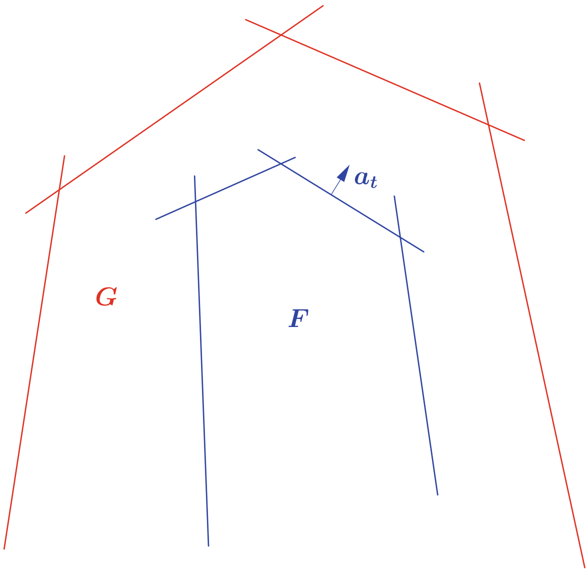


Fig. 1.12 Inclusion of a polyhedron F in another one G

following consequence of (1.32): given two polyhedra F and G,

$$F \subset G \Longleftrightarrow K_G^{\leqq} \subset K_F^{\leqq}, \tag{1.33}$$

where $K_F^{\leqq}$ and $K_G^{\leqq}$ are the weak dual cones of F and G defined as in (1.31).

Consequently,

$$F = G \Longleftrightarrow K_F^{\leqq} = K_G^{\leqq}.$$

If F is the solution set of $\{\langle a_t, x\rangle \leq b_t,\ t \in W\}$, from the definition of $K_F^{\leqq}$ and the non-homogeneous Farkas lemma for linear semi-infinite systems [72, Cor. 3.1.2], one gets

$$K_F^{\leqq} = \operatorname{cl}\operatorname{cone}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in W; \begin{pmatrix} 0_n \\ 1 \end{pmatrix}\right\}.$$

As shown in [68, Prop. 4.1], if F and G are the sets of solutions of two given systems of weak inequalities, the preservation of the inclusion $F \subset G$ under sufficiently small perturbations of the respective linear representations is related with the condition that $F \subset \operatorname{int} G$. Here, both sets F and $\operatorname{int} G$ are e-convex as the inbody F is a closed convex set while the circumbody $\operatorname{int} G$ is an open convex set. In order to extend (1.33) to e-convex sets we must associate with F the cone which results of replacing "$\leq$" with "$<$" in (1.31).

One can define the weak dual cone $K_F^{\leqq}$ of an arbitrary set F by (1.31). Then, given a couple F and G of proper subsets of $\mathbb{R}^n$, one has $\operatorname{cl}\operatorname{conv} F \subset \operatorname{cl}\operatorname{conv} G$ if and only if $K_G^{\leqq} \subset K_F^{\leqq}$ and $\operatorname{cl}\operatorname{conv} F = \operatorname{cl}\operatorname{conv} G$ if and only if $K_G^{\leqq} = K_F^{\leqq}$.

The *strict dual cone* of a set F such that $\emptyset \neq F \subset \mathbb{R}^n$ is

$$K_F^{<} := \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^{n+1} : \langle a, x\rangle < b, \forall x \in F \right\}.$$

Observe that $K_F^{<}$ and $K_F^{\leqq}$ can be seen as solution sets of homogeneous linear systems of strict and weak inequalities in $\mathbb{R}^{n+1}$ indexed by F, respectively. So, they are an e-convex cone not containing 0_{n+1} and a closed convex cone, respectively. From their definitions, we have that $\operatorname{cl} K_F^{<} \subset K_F^{\leqq}$. Moreover, the reverse inclusion holds as any $\begin{pmatrix} a \\ b \end{pmatrix} \in K_F^{\leqq}$ is the limit of the sequence $\left\{ \begin{pmatrix} a \\ b + \frac{1}{k} \end{pmatrix} \right\}$ contained in $K_F^{<}$. So,

$$\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in \operatorname{cl} K_F^{<} = K_F^{\leqq}.$$

Proposition 1.8 (Characterization of the Strict Dual Cone) *Let F be the solution set of $\sigma = \{\langle a_t, x\rangle < b_t,\ t \in S;\ \langle a_t, x\rangle \leq b_t,\ t \in W\}$. Then the following statements hold:*

(i) *$K_F^{<}$ is formed by all vectors $\begin{pmatrix} a \\ b \end{pmatrix}$ such that $\langle a, x\rangle < b$ is a legal linear combination of σ provided the set*

$$C(\sigma) = \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in S \right\} + \mathbb{R}_+ \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in W \right\}$$

satisfies at least one of the following conditions:

- (a) $\operatorname{conv} C(\sigma)$ *is open.*
- (b) $\left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in S \right\}$ *is compact and $W = \emptyset$.*
- (c) $\operatorname{conv} C(\sigma)$ *is e-convex and its intersection with any supporting hyperplane is compact.*
- (d) $\operatorname{conv} C(\sigma)$ *admits no asymptote in any supporting hyperplane.*
- (e) *The projections of* $\operatorname{conv} C(\sigma)$ *are all e-convex.*
- (f) $\operatorname{conv} C(\sigma)$ *is a strip.*

(ii) *If $W = \emptyset$, then*

$$K_F^{<} = \operatorname{eco} \mathbb{R}_{++} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in S;\ \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\}. \tag{1.34}$$

Partial Proof Since statement (ii) is known, we only prove here statement (i). Any of the conditions from $(i.a)$ to $(i.f)$ guarantees that $\operatorname{conv} C(\sigma) - \mathbb{R}_+ \begin{pmatrix} a \\ b \end{pmatrix}$ is e-convex for all $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^{n+1}$ by Corollary 1.1. The conclusion follows from Theorem 1.5(iv). □

The strict dual cone of F, $K_F^<$, gathers useful geometric information on F; for instance, the recession cone of F is

$$0^+ F = \left\{ y \in \mathbb{R}^n : \langle a, y \rangle \leq 0,\ \forall \begin{pmatrix} a \\ b \end{pmatrix} \in K_F^< \right\},$$

and F is bounded if and only if $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in \operatorname{int} K_F^<$.

Proposition 1.9 (Dual Characterization of the Containment of e-Convex Sets) *Given two proper subsets of* $\mathbb{R}^n$, F *and* G, *one has*

$$\operatorname{eco} F \subset \operatorname{eco} G \Longleftrightarrow K_G^< \subset K_F^<.$$

Moreover, if both sets are e-convex, then

$$F \subset G \Longleftrightarrow K_G^< \subset K_F^<.$$

Consequently,

$$F = G \Longleftrightarrow K_F^< = K_G^<.$$

Proof The first statement comes from the definitions of strict dual cone and e-convex hull, and the second from the obvious equations $\operatorname{eco} F = F$ and $\operatorname{eco} G = G$ when these two sets are e-convex. □

The next result suggests the existence of an intriguing topological duality between e-convex sets and their strict dual cones.

Proposition 1.10 (On the Topology of e-Convex Sets and Their Dual Cones) *Let* $F \neq \emptyset$ *be an e-convex set. Then the following statements hold:*

(i) *F is open if and only if $K_F^< \cup \{0_{n+1}\}$ is closed.*
(ii) *If $K_F^<$ is relatively open, then F is closed.*
(iii) *If F is compact, then $K_F^<$ is open.*

Example 1.13 Consider the closed convex set $F = \mathbb{R}^2_+$. From the definition of strict dual cone one gets $K_F^< = (-\mathbb{R}_+)^2 \times \mathbb{R}_{++}$ (it can also be obtained from (1.34) by observing that $F = \left\{ x \in \mathbb{R}^2 : -x_i < \alpha, \alpha \in \mathbb{R}_{++}, i = 1, 2 \right\}$). Since $K_F^<$ is neither closed nor open, the converse statement of Proposition 1.10(ii) does not hold, and one cannot replace "compact" with "closed" in Proposition 1.10(*iii*). Finally, let us observe that $\begin{pmatrix} 0_2 \\ 1 \end{pmatrix} \in \operatorname{bd} K_F^<$ as F is unbounded.

1.5 Evenly Linear Semi-Infinite Programming

Strict inequality constraints naturally arise in many real situations, even though they are usually replaced in optimization models by their corresponding weak inequalities. For instance, in production planning problems, where the decision variable x_j represents the production level of the j-th good (or commodity), $j \in J$, sign constraints of the type $x_j > 0$ are usually replaced by their relaxation $x_j \geq 0$, which provides relaxed optimization problems of the accurate ones. In the same context, when the j-th good contains a percentage p_j of some obnoxious component and the percentage of this component at the whole production is required to be less or equal than some given percentage threshold P, this condition can be formulated as $\frac{\sum_{j\in J} p_j x_j}{\sum_{j\in J} x_j} \leq P$ provided that $x_j > 0$ for some $j \in J$, that is, the following pair of linear constraints hold:

$$\sum\nolimits_{j\in J} \left(p_j - P\right) x_j \leq 0 \text{ and } \sum\nolimits_{j\in J} x_j > 0.$$

In this section we obtain information on the accurate model from its relaxation.

To this aim, we associate with the given *linear semi-infinite programming problem* with strict inequalities (in short, *e-LSIP problem*)

$$\begin{aligned} (P)\ \operatorname*{Min}_{x\in\mathbb{R}^n}\ & \langle c, x\rangle \\ \text{s.t.}\ & \langle a_t, x\rangle \leq b_t, t \in W, \\ & \langle a_t, x\rangle < b_t, t \in S \neq \emptyset, \end{aligned} \tag{1.35}$$

where $c \neq 0_n$ and W and S are arbitrary disjoint index sets, its *relaxed problem*

$$\begin{aligned} (\overline{P})\ \operatorname*{Min}_{x\in\mathbb{R}^n}\ & \langle c, x\rangle \\ \text{s.t.}\ & \langle a_t, x\rangle \leq b_t, t \in T = W \cup S. \end{aligned} \tag{1.36}$$

We denote by σ and $\overline{\sigma}$ the linear systems given by the constraints of (P) and $\left(\overline{P}\right)$, respectively, and by F and $\overline{F}$ their corresponding solution sets, which are the feasible sets of (P) and $\left(\overline{P}\right)$, respectively. Observe that, since $\overline{\sigma}$ corresponds to the relaxed system of σ, by Proposition 1.6, we have that $\overline{F} = \operatorname{cl} F$ when (P) is consistent (that is, $F \neq \emptyset$).

In order to obtain geometrical information on the feasible set F and to decide whether a given $x^* \in F$ is an optimal solution of (P) by means of a condition involving the data (that is, the coefficients of the constraints), we must introduce assumptions on σ which are generically called *constraint qualifications*.

1.5.1 Constraint Qualifications

We first introduce five global constraint qualifications. We say that σ is *continuous* if T is a compact topological space and the coefficient function $t \longmapsto (a_t, b_t)$ is continuous on T. In particular, σ is said to be *analytical* (respectively, *polynomial*) if T is a compact interval in $\mathbb{R}$ and the $n+1$ projections of the vector-valued function $t \longmapsto (a_t, b_t)$ are analytical (respectively, polynomial). We say that σ satisfies the *Slater constraint qualification* (SCQ in short) if there exists some $\widehat{x} \in \mathbb{R}^n$ such that $\langle a_t, \widehat{x}\rangle < b_t$ for all $t \in T$. Finally, we say that σ satisfies the *Farkas-Minkowski constraint qualification* (FMCQ in brief) if it is consistent and each weak inequality $\langle a, x\rangle \leq b$ which is consequence of σ is also consequence of a finite subsystem of σ.

Assume that FMCQ holds and $\langle a, x\rangle \leq b$ is consequence of σ. Then there exist two finite sets $W_1 \subset W$ and $S_1 \subset S$ such that $\langle a, x\rangle \leq b$ is consequence of $\sigma_1 = \{\langle a_t, x\rangle \leq b_t,\ t \in W_1;\ \langle a_t, x\rangle < b_t,\ t \in S_1\}$ and, according to Theorem 1.5,

$$\begin{pmatrix} a \\ b \end{pmatrix} \in \operatorname{cl}\operatorname{cone}\left(\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in S_1\right\} + \mathbb{R}_+\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in W_1\right\} \cup \begin{pmatrix} 0_n \\ 1 \end{pmatrix}\right).$$

Since the convex cone generated by the union of finitely many halflines together with one element of the vertical axis is not necessarily closed, we cannot assert that

$$\begin{pmatrix} a \\ b \end{pmatrix} \in \operatorname{cone}\left(\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in S_1\right\} + \mathbb{R}_+\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in W_1\right\} \cup \begin{pmatrix} 0_n \\ 1 \end{pmatrix}\right).$$

Thus, the closedness of cone $D(\sigma)$ is a sufficient, but not necessary, condition for the fulfillment of FMCQ. Actually, σ satisfies FMCQ if and only if the characteristic cone of the relaxed system $\overline{\sigma}$,

$$K(\overline{\sigma}) = \operatorname{cone}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T; \begin{pmatrix} 0_n \\ 1 \end{pmatrix}\right\},$$

is closed [72, Th. 5.3(i)]. The definition of the next two constraint qualifications involves $\overline{\sigma}$.

We say that σ satisfies the *locally Farkas-Minkowski constraint qualification* (LFMCQ in short) at $\overline{x} \in \overline{F}$ (not necessarily in F) if it is consistent and each inequality $\langle a, x\rangle \leq b$ which is consequence of $\overline{\sigma}$ and binding at $\overline{x}$ (i.e., $\langle a, \overline{x}\rangle = b$) is also consequence of a finite subsystem of σ. Obviously, FMCQ implies LFMCQ at any $\overline{x} \in \overline{F}$, but the converse statement does not hold (see Example 1.14).

Finally, we say that σ satisfies the *locally polyhedral constraint qualification* (LOPCQ) at $\overline{x} \in \overline{F}$ if $A(\overline{x})^{\circ} = D(\overline{F}; \overline{x})$, where

$$A(\overline{x}) := \operatorname{cone}\{a_t : \langle a_t, \overline{x}\rangle = b_t,\ t \in T\}$$

is the so-called *active cone* at $\overline{x}$ and $A(\overline{x})^{\circ}$ denotes its *(negative) polar cone*, i.e.,

$$A(\overline{x})^{\circ} = \{y \in \mathbb{R}^n : \langle a_t, y\rangle \leq 0, \forall t \in T \text{ such that } \langle a_t, \overline{x}\rangle = b_t\}$$

(see Sect. 1.6.1). A necessary condition for LOPCQ is $\overline{F}$ be quasipolyhedral and its set of extreme points be isolated [72, Th. 5.6(*ii*)]. Any consistent finite system satisfies LOPCQ at any point of $\overline{F}$, which in turn implies LFMCQ at any point of $\overline{F}$. Diagram 1.3, where $\overline{x}$ denotes an arbitrary element of $\overline{F}$, shows the relationships between the above constraint qualifications.

When σ satisfies one of the above constraint qualifications, its corresponding relaxed system $\overline{\sigma}$ also satisfies the same condition (recall that, according to Theorem 1.5(i), an inequality $\langle a, x\rangle \leq b$ is consequence of σ if and only if it is consequence of $\overline{\sigma}$). This allows us to adapt the known results on ordinary semi-infinite linear systems and ordinary semi-infinite linear programming to systems and linear problems containing strict inequalities.

Example 1.14 Let $\sigma := \left\{-x_1 - t^2 x_2 < -2t,\ t \in \mathbb{R}_{++}\right\}$. It satisfies LFMCQ at any $\overline{x} \in \operatorname{bd} F = \operatorname{bd} \overline{F}$ as each inequality of $\overline{\sigma}$ describes the halfplane determined by a different supporting hyperplane to $\overline{F}$ (the convex hull of the graph of the function $x \longmapsto \frac{1}{x}$ restricted to $\mathbb{R}_{++}$). However, FMCQ fails as $-x_1 \leq 0$ and $-x_2 \leq 0$ are linear consequences of $\overline{\sigma}$ which are not consequences of finite subsystems of $\overline{\sigma}$ (Fig. 1.13 shows that the halfplanes $-x_1 \leq 0$ and $-x_2 \leq 0$ do not contain the solution set of the subsystem obtained by replacing T by $\{t_1, t_2, t_3\}$ in $\overline{\sigma}$).

Example 1.15 The system $\sigma_1 := \left\{-tx_1 + (t-1)x_2 < t^2 - t, t \in [0, 1]\right\}$ is polynomial and satisfies SCQ as $\widehat{x} = (1, 1)$ is a Slater point. Thus, σ_1 satisfies FMCQ (Fig. 1.14 shows $K(\sigma_1)$) and, so, LFMCQ everywhere. However, LFMCQ is lost by the elimination of at least one of the redundant inequalities corresponding to the indices $t = 0, 1$. For instance, $\sigma_2 = \left\{-tx_1 + (t-1)x_2 < t^2 - t,\ t \in\,]0, 1[\right\}$

Diagram 1.3 Types of ordinary linear systems

$$\begin{array}{ccccc}
\sigma \text{ polynomial} & & & & \\
\Downarrow & & & & \\
\sigma \text{ analytical} & & & & \\
\Downarrow & & & & \\
\sigma \text{ continuous} & & \text{and} & & \text{SCQ} \\
\Uparrow & & \Downarrow & & \\
\sigma \text{ finite} & \Longrightarrow & \text{FMCQ} & & \\
 & & \Downarrow & & \\
 & & \text{LFMCQ at } \overline{x} & \Longleftarrow & \text{LOPCQ at } \overline{x}
\end{array}$$

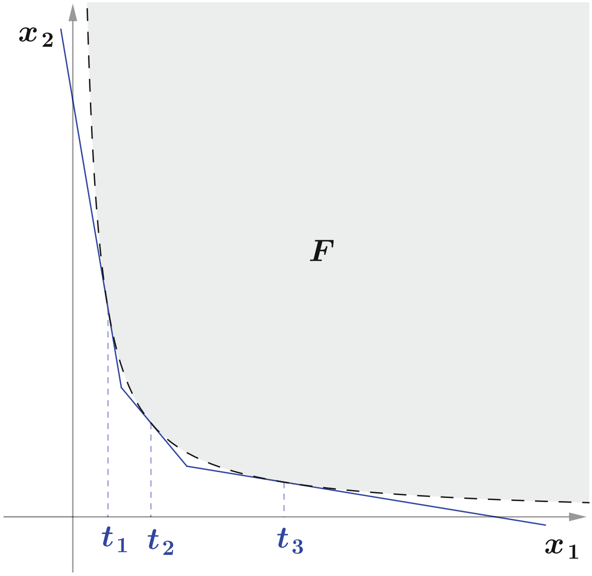


Fig. 1.13 σ satisfies LFMCQ everywhere but not FMCQ

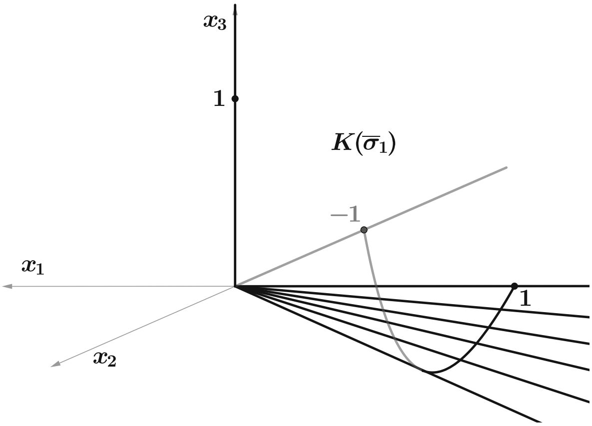


Fig. 1.14 Characteristic cone of $\overline{\sigma}_1$

does not satisfy LFMCQ at $(1, 0)$ and $(0, 1)$ as $-x_1 \leq 0$ and $-x_2 \leq 0$ are linear consequences of $\overline{\sigma}_2$ defining supporting halfspace to its feasible set $\overline{F}_2 = \overline{F}_1$, but they are not consequences of any finite subsystems of $\overline{\sigma}_2$. Figure 1.15 shows that the halfplanes $-x_1 \leq 0$ and $-x_2 \leq 0$ do not contain the solution set of the subsystem corresponding to the index set $\{t_1, t_2, t_3\}$. Observe that $\overline{F}_1 = \operatorname{int} \overline{F}_2$ is the result of eliminating from $\overline{F}_2$ its two facets.

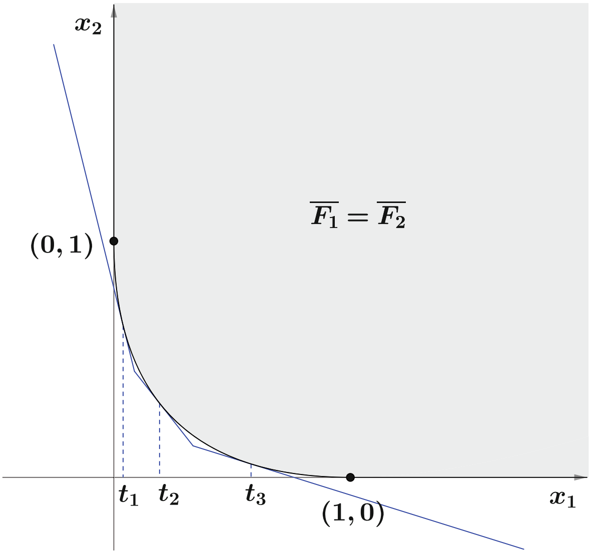


Fig. 1.15 σ_2 does not satisfy LFMCQ at two points

1.5.2 Feasible Set

Since we are assuming that $F \neq \emptyset$ and its closure is $\mathrm{cl}\, F = \overline{F}$, by [148, Ths. 6.2 and 6.3], F and $\overline{F}$ have the same relative interiors and, so, the same relative boundaries, affine hulls, and dimensions. Moreover, if $\mathrm{int}\, F \neq \emptyset$, the interiors and the boundaries of F and $\overline{F}$ are also the same.

The next result is an immediate consequence of [72, Th. 9.3] taking into account that

$$0^+F = 0^+\overline{F} = \left\{x \in \mathbb{R}^n : \langle a_t, x\rangle \leq 0,\ t \in T\right\}.$$

Proposition 1.11 (Boundedness of the Solution Set) *The following statements are equivalent:*

(i) *F is bounded.*
(ii) *0_n is the unique solution of* $\{\langle a_t, x\rangle \leq 0, t \in T\}$.
(iii) $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in \mathrm{int}\, K\, (\overline{\sigma})$.
(iv) $\mathrm{cone}\,\{a_t, t \in T\} = \mathbb{R}^n$.

To get a formula for $\dim F$ in terms of the data we must introduce another concept. Since

$$\langle a_t, x\rangle = b_t, \forall x \in F \Longleftrightarrow \langle a_t, x\rangle = b_t, \forall x \in \overline{F},$$

the *set of carrier indices* of σ, defined as

$$T^{=} := \{t \in T : \langle a_t, x\rangle = b_t, \forall x \in F\} \subset W,$$

coincides with the set of carrier indices of $\overline{\sigma}$ (as defined in [72, p. 101] for ordinary systems), SCQ holds if and only if $T^{=} = \emptyset$ [72, Cor. 5.1.1] and, by [72, Th. 5.1],

$$\operatorname{rint} F \subset \left\{x \in \mathbb{R}^n : \langle a_t, x\rangle < b_t,\ t \in T \backslash T^{=}; \langle a_t, x\rangle = b_t,\ t \in T^{=}\right\}.$$

The next result involves the lineality of a convex cone K (recall that $\operatorname{lin} K = K \cap (-K)$).

Proposition 1.12 (Dimension Formulas) *The following statements hold true:*

(i) $\dim F = n - \dim \operatorname{lin} \operatorname{cl} K(\overline{\sigma})$. *Moreover, if LFMCQ holds at each* $\overline{x} \in \overline{F}$ *(e.g., if FMCQ holds), then*

$$\dim F = n - \dim \operatorname{span}\{a_t : t \in T^{=}\}$$

and

$$\operatorname{aff} F = \left\{x \in \mathbb{R}^n : \langle a_t, x\rangle = b_t,\ t \in T^{=}\right\}.$$

(ii) $\dim 0^+F = n - \dim \operatorname{lin} \operatorname{cl} \operatorname{cone}\{a_t : t \in T\}$.
(iii) $\dim \operatorname{lin} 0^+F = n - \dim \operatorname{cone}\{a_t : t \in T\}$.

Proof (i) comes from [72, Ths. 5.8 and 5.9], as $\dim F = \dim \overline{F}$, while ($ii$) and ($iii$) follow from [72, Remark after Th. 5.8] recalling that $0^+F = 0^+\overline{F}$. □

To get information on the boundary points of F we denote by

$$F_t := \left\{x \in \overline{F} : \langle a_t, x\rangle = b_t\right\}$$

the (possibly empty) *exposed face* of $\overline{F}$ *associated with* index $t \in T$. Obviously, $F = \overline{F} \backslash \bigcup_{t \in S} F_t$.

If σ is analytical, we associate with each $\overline{x} \in F$ a linear subspace $L(\overline{x})$ of $\mathbb{R}^n$ defined as the linear span of the union of the sets of successive derivatives $\left\{a_t, a_t^{(1)}, \ldots, a_t^{(d(t))}\right\}$ of the *slack function* at $\overline{x}$, $t \longmapsto \langle a_t, \overline{x}\rangle - b_t$ at those indices $t \in T$ which are roots (with order of multiplicity $d(t) + 1$).

Proposition 1.13 (Boundary and Extreme Points) *The following statements hold true:*

(*i*) *If LFMCQ holds at each* $\overline{x} \in \overline{F}$ *(e.g., if FMCQ holds), then*

$$\operatorname{rint} F = \left\{x \in \mathbb{R}^n : \langle a_t, x\rangle = b_t,\ t \in T^{=};\ \langle a_t, x\rangle < b_t,\ t \in T \setminus T^{=}\right\},$$

$$\operatorname{rbd} F = \cup \{F_t : t \in T \setminus T^{=}\}$$

and

$$\operatorname{bd} F = \cup \{F_t : a_t \neq 0_n,\ t \in T\}.$$

(*ii*) *Given* $\overline{x} \in F$, *if* $\dim A(\overline{x}) = n$, *then* $\overline{x}$ *is an extreme point of* F. *The converse statement holds under LOPCQ at* $\overline{x}$.

(*iii*) *Let* σ *be an analytical system and let* $\overline{x} \in F$ *be such that the slack function at* $\overline{x}$ *is not the null function on* T. *Then,* $\overline{x}$ *is an extreme point of* F *if and only if* $\dim L(\overline{x}) = n$.

Proof (i) comes from [72, Th. 5.9], as $\operatorname{rint} F = \operatorname{rint} \overline{F}$, and ($ii$) and ($iii$) follow from [72, Th. 9.1], recalling that $\operatorname{extr} F = F \cap \operatorname{extr} \overline{F}$ by Proposition 1.1(ii). □

Example 1.16 (Example 1.15 Revisited) The solution set F_2 of the system

$$\sigma_2 := \left\{-tx_1 + (t-1)x_2 < t^2 - t,\ t \in]0, 1[\right\} \tag{1.37}$$

is the result of eliminating from the set $\overline{F}_1$ in Fig. 1.15 the points of the arc $\sqrt{x_1} + \sqrt{x_2} = 1$, $-x_1 < 0$, $-x_2 < 0$ (see [72, Ex 1.1]). Since

$$\operatorname{cone}\{a_t,\ t \in]0, 1[\} = \operatorname{cone}\left\{\begin{pmatrix} -t \\ t-1 \end{pmatrix},\ t \in]0, 1[\right\} \neq \mathbb{R}^2,$$

by Proposition 1.11, F is unbounded. Moreover, $\operatorname{aff} F_2 = \mathbb{R}^2$ and $\dim F_2 = 2$, as prescribed by Proposition 1.12(i) even though the additional assumption does not hold. Regarding Proposition 1.13, since $T^{=} = \emptyset$, the three equations in (i) fail, showing the necessity of LFMCQ. Regarding statement (ii), observe that $A(x) = \{0_2\}$ for all $x \in F_2$, even at the extreme points of F_2, $(1, 0)$ and $(0, 1)$, which shows that LOPCQ is not superfluous for the converse. Finally, regarding (iii), if we take the analytical system

$$\sigma_3 := \left\{-tx_1 + (t-1)x_2 < t^2 - t,\ t \in]0, 1];\ -tx_1 + (t-1)x_2 \leq t^2 - t,\ t = 0\right\},$$

whose solution set F_3 is the result of eliminating from F_2 the exposed face $\{0\} \times [1, +\infty[$, the slack function at $\overline{x} = (1, 0) \in F_3$ is $t \longmapsto -t^2$, whose unique zero is 0, with multiplicity $2 = d(0) + 1$, i.e., $d(0) = 1$. Since $a_t = \begin{pmatrix} -t \\ t-1 \end{pmatrix}$ and $a_t^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$,

$$L(\overline{x}) = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

and $\dim L(\overline{x}) = 2$, as expected.

1.5.3 Optimality and Duality

Now, we consider the e-LSIP problem (P) and its relaxed problem $(\overline{P})$, defined in (1.35) and (1.36), respectively. We adopt the standard convention that the optimal value of a minimization problem is $+\infty$ (respectively, $-\infty$) when the problem is inconsistent (respectively, unbounded). We denote by $v(P)$, F, and F^* the optimal value, the feasible set, and the optimal set of (P), and by $v(\overline{P})$, $\overline{F}$, and $\overline{F}^*$ the optimal value, the feasible set, and the optimal set of $(\overline{P})$.

Observe that F^* is the intersection of the e-convex set F with the hyperplane $\{x \in \mathbb{R}^n : \langle c, x \rangle = v(P)\}$, so it is an e-convex set too. Moreover, since $F \subset \overline{F}$, $v(\overline{P}) \leq v(P)$ (even in the case that $F = \emptyset$ due to the above convention). Observe also that, for the e-LSIP problem given by

$$\begin{aligned} (P)\ & \operatorname*{Min}_{x \in \mathbb{R}} x \\ & \text{s.t. } \ zx < 0, z \in \mathbb{Z} \setminus \{0\}, \end{aligned}$$

one has $F = F^* = \emptyset$ and $v(P) = +\infty$ while $\overline{F} = \overline{F}^* = \{0\}$ and $v(\overline{P}) = 0$. So, if $F = \emptyset$, we may have $v(\overline{P}) < v(P) = +\infty$ and $F \cap \overline{F}^* \subsetneq F^*$.

Proposition 1.14 (Optimal Value and Optimal Set in e-LSIP) *If the e-LSIP problem (P) is consistent, then $v(P) = v(\overline{P})$ and $F^* = F \cap \overline{F}^*$.*

Proof On the one hand, since $\emptyset \neq F \subset \overline{F}$, there exists a sequence $\{x^k\}$ contained in $\overline{F}$ such that

$$\lim_{k \to \infty} \langle c, x^k \rangle = v(\overline{P}). \tag{1.38}$$

On the other hand, since $\overline{F} = \operatorname{cl} F$, for each $k \in \mathbb{N}$ there exists $z^k \in F$ such that

$$\left\| z^k - x^k \right\| < \frac{1}{k}, \forall k \in \mathbb{N}. \tag{1.39}$$

From (1.38) and (1.39) one gets

$$v(P) \leq \lim_{k\to\infty} \langle c, z^k \rangle = \lim_{k\to\infty} \langle c, x^k \rangle = v(\overline{P}),$$

which combined with $v(\overline{P}) \leq v(P)$ yields $v(\overline{P}) = v(P)$.

It remains to prove the non-trivial inclusion $F^* \subset F \cap \overline{F}^*$. Take an arbitrary $x^* \in F^* \subset F$. Then $\langle c, x^* \rangle \leq \langle c, x \rangle$ for all $x \in F$. Given $\overline{x} \in \overline{F}$ there exists a sequence $\{x^k\}$ contained in F such that $\lim_k x^k = \overline{x}$. Then,

$$\langle c, \overline{x} \rangle = \lim_{k\to\infty} \langle c, x^k \rangle \geq \left\langle c, x^* \right\rangle,$$

showing that $x^* \in F \cap \overline{F}^*$. □

From Proposition 1.14, any outer approximation method for $(\overline{P})$, as grid and cutting plane discretization methods, is also an outer approximation method for (P). Similarly, any inner approximation method for $(\overline{P})$, like, for instance, feasible directions methods (in particular, simplex-like methods) and interior-point methods, provides a sequence of feasible solutions for (P) approaching $v(P)$ even though (P) is usually unsolvable.

We now discuss how to associate a suitable dual problem for (P) when $F \neq \emptyset$. To this aim we must construct lower bounds for $\{\langle c, x \rangle : x \in F\}$. Since we are assuming $F \neq \emptyset$, by Proposition 1.14,

$$v(P) = \inf\{\langle c, x \rangle : x \in F\} = \inf\left\{\langle c, \overline{x} \rangle : \overline{x} \in \overline{F}\right\} = v(\overline{P}),$$

which means that we can replace the strict inequalities with weak ones in order to get lower bounds. If $-c \in \operatorname{cone}\{a_t : t \in T\}$ there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that $\sum_{t\in T} \lambda_t a_t = -c$. Then, for any $x \in F$, one has

$$\langle c, x \rangle = -\sum\nolimits_{t\in T} \lambda_t \langle a_t, x \rangle \geq -\sum\nolimits_{t\in T} \lambda_t b_t. \tag{1.40}$$

Observe that, if λ is a legal element of $\mathbb{R}_+^{(T)}$, there exists a $t \in T$ such that $\lambda_t (\langle a_t, x \rangle - b_t) < 0$ and the inequality in (1.40) is strict, but this is not an advantage when one looks for conditions guaranteeing a zero duality gap. So, we associate with (P) the *Haar dual problem* of $(\overline{P})$, which consists in maximizing the lower bound for $\langle c, x \rangle$ provided by (1.40), that is

$$\begin{aligned} (D) \quad \operatorname*{Max}_{\lambda\in\mathbb{R}_+^{(T)}} \quad & -\sum_{t\in T} \lambda_t b_t \\ \text{s.t.} \quad & -\sum_{t\in T} \lambda_t a_t = c. \end{aligned}$$

Defining the *Lagrange function* of $(\overline{P})$ as

$$L(x, \lambda) := \langle c, x \rangle + \sum_{t \in T} \lambda_t (\langle a_t, x \rangle - b_t), \tag{1.41}$$

one has

$$\begin{aligned} \inf_{x \in \mathbb{R}^n} L(x, \lambda) &= \inf_{x \in \mathbb{R}^n} \left(-\sum_{t \in T} \lambda_t b_t + \left\langle c + \sum_{t \in T} \lambda_t a_t, x \right\rangle \right) \\ &= \begin{cases} -\sum_{t \in T} \lambda_t b_t, & \text{if } \lambda \in \mathbb{R}_+^{(T)} \text{ and } \sum_{t \in T} \lambda_t a_t = -c, \\ -\infty, & \text{otherwise,} \end{cases} \end{aligned}$$

so that (D) can be reformulated as the classical *Lagrangian dual problem* of $(\overline{P})$:

$$\operatorname*{Max}_{\lambda \in \mathbb{R}_+^{(T)}} \inf_{x \in \mathbb{R}^n} L(x, \lambda).$$

Theorem 1.6 (Duality in e-LSIP) *Let* (P) *and* (D) *be consistent. If either FMCQ holds or* $-c \in \operatorname{rint} \operatorname{cone} \{a_t : t \in T\}$, *then* $v(P) = v(D) \in \mathbb{R}$. *In the first case,* (D) *is solvable.*

Proof From the LSIP duality theorems in [72, Th. 8.4] and [155] (see also [72, Ths. 8.1 and 8.2]) one gets that $v(\overline{P}) = v(D) \in \mathbb{R}$, with (D) being solvable under FMCQ, and that $v(\overline{P}) = v(D) \in \mathbb{R}$ with $\overline{F}^*$ being the sum of a nonempty compact convex set with a linear subspace (see [75, Remark 4.16] for this last statement), respectively. We get the conclusion observing that $v(\overline{P})$ coincides with $v(P)$. □

The assumptions of Theorem 1.6 do not guarantee, however, the solvability of (P). Proposition 1.14 also allows us to obtain optimality conditions in e-LSIP.

Theorem 1.7 (Optimality and Strong Uniqueness in e-LSIP) *Each one of the following statements is sufficient for the optimality of* $x^* \in F$ *regarding* (P), *and they are also necessary when LFMCQ holds at* x^**:*

(i) $-c \in \operatorname{cl} A(x^*)$.
(ii) $-c \in A(x^*)$ (KKT condition).
(iii) *There exists a feasible solution* $\overline{\lambda}$ *of* (D) *such that* $\overline{\lambda}_t (b_t - \langle a_t, x^* \rangle) = 0$ *for all* $t \in T$ (complementarity condition).
(iv) *There exists* $\overline{\lambda} \in \mathbb{R}_+^{(T)}$ *such that* $L(x^*, \lambda) \leq L(x^*, \overline{\lambda}) \leq L(x, \overline{\lambda})$ *for all* $x \in \mathbb{R}^n$, *with* $L(\cdot, \cdot)$ *defined as in (1.41), and for all* $\lambda \in \mathbb{R}_+^{(T)}$ (Lagrange saddle point condition).

If, additionally, $-c \in \operatorname{int} A(x^*)$, *then* x^* *is a strongly unique optimal solution of* (P), *i.e., there exists* $\kappa > 0$ *such that*

$$\langle c, x \rangle \geq \left\langle c, x^* \right\rangle + \kappa \left\| x - x^* \right\| \text{ for all } x \in F. \tag{1.42}$$

The converse also holds under LFMCQ at x^*.

Proof According to [72, Th. 7.1], any of the conditions from (i) to (iv) implies that $x^* \in \overline{F}^*$. So, $x^* \in F \cap \overline{F}^* = F^*$, by Proposition 1.14. Under the LFMCQ at x^* these conditions are also sufficient, again by [72, Th. 7.1].

Regarding the strong uniqueness, $-c \in \operatorname{int} A(x^*)$ guarantees [72, Th. 10.6] the existence of $\kappa > 0$ such that

$$\langle c, x\rangle \geq \left\langle c, x^*\right\rangle + \kappa \left\| x - x^* \right\| \text{ for all } x \in \overline{F}, \tag{1.43}$$

which obviously entails (1.42). Since $\overline{F} = \operatorname{cl} F$, (1.42) is actually equivalent to (1.43) and one can apply the converse statement in [72, Th. 10.6] thanks to the LFMCQ assumption. □

Example 1.17 (Example 1.16 Revisited) Consider the e-LSIP problem

$$\begin{aligned} (P_1) \ &\operatorname*{Min}_{x\in\mathbb{R}^2} \ \langle c^1, x\rangle := x_1 \\ &\text{s.t.} \ \ -tx_1 + (t-1)x_2 < t^2 - t, \ \ t \in \,]0,1[\,. \end{aligned}$$

We have

$$A_1(x) = \{0_2\}, \forall x \in F_1^* = \overline{F}_1^* = \left\{ \begin{pmatrix} 0 \\ x_2 \end{pmatrix} : x_2 \geq 1 \right\},$$

which shows the necessity of LFMCQ at Theorem 1.7 (see Fig. 1.16). Moreover, since

$$-c^1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \notin \operatorname{cone}\{a_t, \ t \in \,]0,1[\} = \operatorname{cone}\left\{ \begin{pmatrix} -t \\ t-1 \end{pmatrix}, \ t \in \,]0,1[\right\},$$

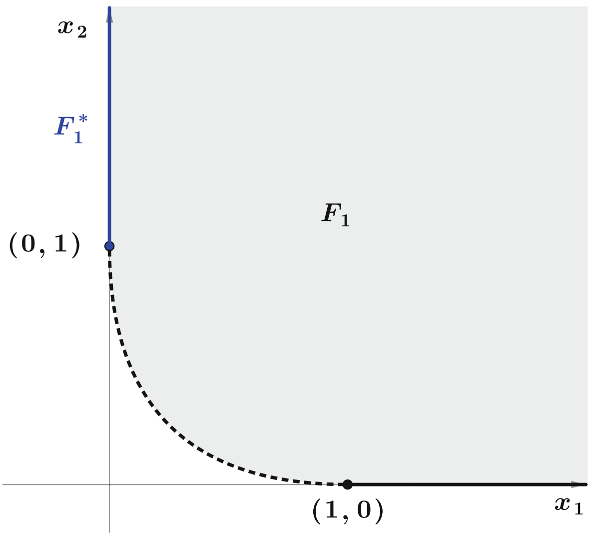


Fig. 1.16 The sets F_1 and F_1^*

its dual problem (D_1) is inconsistent and $v(D_1) = -\infty < v(P_1) = 0$, abnormality due to the violation of three conditions of Theorem 1.6, namely, the consistency of (D), FMCQ, and $-c \in \operatorname{rint} \operatorname{cone} \{a_t : t \in T\}$.

In contrast with this abnormality, for the problem

$$\begin{aligned}(P_2)\ \operatorname*{Min}_{x \in \mathbb{R}^2}\ & \langle c^2, x\rangle := x_1 + x_2 \\ \text{s.t.}\ & -tx_1 + (t-1)x_2 < t^2 - t,\ t \in \left]0, 1\right[,\end{aligned}$$

we have $F_2^* = \emptyset$ while $\overline{F}_2^* = \left\{\left(\frac{1}{4}, \frac{1}{4}\right)\right\}$, $v(P_2) = v(\overline{P}_2) = \frac{1}{2}$, and $-c^2 \in \operatorname{rint} \operatorname{cone} \{a_t : t \in T\}$, so that we must have $v(P_2) = v(D_2) \in \mathbb{R}$ by Theorem 1.6. In fact, for $\lambda^* \in \mathbb{R}_+^{(]0,1[)}$ such that

$$\lambda_t^* = \begin{cases} 2, & t = \frac{1}{2}, \\ 0, & \text{otherwise}, \end{cases}$$

one has $-\sum_{t \in]0,1[} \lambda_t^* b_t = \frac{1}{2} = v(P_2)$, so λ^* is an optimal solution of (D_2) and $v(D_2) = \frac{1}{2}$ too.

1.6 Selected Applications

Semi-infinite systems containing strict inequalities (i.e., with $S \neq \emptyset$) naturally arise in polarity, strict separation of sets, stability analysis, linear optimization, and other fields.

1.6.1 Polarity

H. Minkowski [128] defined in 1911 the (negative) *polar* of a closed convex set X such that $0_n \in X$ as

$$X^\circ := \{y \in \mathbb{R}^n : \langle y, x\rangle \leq 1,\ \forall x \in X\}.$$

Of course, X° enjoys the same properties as X.

This definition was later extended by H. Rådström [13] to any set X (not necessarily closed and convex), with X° being still closed and convex with $0_n \in X^\circ$ and

$$(\operatorname{cl} X)^\circ = X^\circ. \tag{1.44}$$

Moreover, $X \subset X^{\circ\circ}$ holds as any $y \in X^\circ$ satisfies $\langle y, x\rangle \leq 1$, for all $x \in X$, so $\operatorname{cl}\operatorname{conv} X \subset X^{\circ\circ}$. To prove the reverse inclusion, take $y \notin \operatorname{cl}\operatorname{conv} X$. By the strict separation theorem, there exist $a \in \mathbb{R}^n$, $a \neq 0_n$, and $b \in \mathbb{R}$ such that $\langle a, x\rangle < b$ for all $x \in X$ and $\langle a, y\rangle > b$. Since $0_n \in X$, $b > 0$. Then, $\frac{a}{b} \in X^\circ$ with $\left\langle \frac{a}{b}, y\right\rangle > 1$, showing that $y \notin X^{\circ\circ}$. Hence, $X^{\circ\circ} = \operatorname{cl}\operatorname{conv} X$ and the unique sets X containing 0_n and satisfying $X^{\circ\circ} = X$ are the closed convex ones.

A straightforward consequence of the last statement is the extended Farkas lemma for cones asserting that the involution formula $X^{\circ\circ} = X$ holds true for closed convex cones, for which, as observed by E. Steinitz [168],

$$X^\circ = \{y \in \mathbb{R}^n : \langle y, x\rangle \leq 0, \forall x \in X\}. \tag{1.45}$$

In fact, if X is a cone, given $y \in X^\circ$, one has $\langle \lambda y, x\rangle = \langle y, \lambda x\rangle \leq 1$, for all $x \in X$ and $\lambda > 0$. Since $\langle y, x\rangle \leq \frac{1}{\lambda}$ for all $x \in X$, taking limits when $\lambda \to +\infty$ one gets $\langle y, x\rangle \leq 0$, so that $X^\circ \subset \{y \in \mathbb{R}^n : \langle y, x\rangle \leq 0, \forall x \in X\}$, while the reverse inclusion holds trivially by observing that $\langle y, x\rangle \leq 0$ entails $\langle y, x\rangle \leq 1$.

W. Fenchel wondered in 1952 whether it was possible to define a wider class of convex sets than that of closed cones which is reproduced by a suitably defined polarity. Actually, he showed [55] that the answer is affirmative if one considers e-convex sets (concept introduced in that seminal paper) and defined the *e-polar* of a set X as

$$X^e := \{y \in \mathbb{R}^n : \langle y, x\rangle < 1, \forall x \in X\},$$

which is, obviously, an e-convex set which contains 0_n. Since $X^e \subset X^\circ$ and X° is closed, $\operatorname{cl} X^e \subset X^\circ$. Conversely, given $y \in X^\circ$, by the definitions of X° and X^e, $\left(1 - \frac{1}{k}\right) y \in X^e$ for all $k \in \mathbb{N}$, so that $y = \lim_{k\to\infty} \left(1 - \frac{1}{k}\right) y \in \operatorname{cl} X^e$. Thus, one has

$$\operatorname{cl} X^e = X^\circ. \tag{1.46}$$

We now prove that $X^{ee} = X$ characterizes the e-convex sets which contain 0_n from the characterization of the evenly convex hull in Sect. 1.2. It is obvious that if $X^{ee} = X$, then X is an e-convex set containing 0_n. Assume now that X is an e-convex set containing 0_n. The inclusion $\operatorname{eco} X = X \subset X^{ee}$ holds as any $y \in X^e$ satisfies $\langle y, x\rangle < 1$ for all $x \in X$. To prove the reverse inclusion, let $\overline{x} \notin \operatorname{eco} X$. According to (1.8), there exists $z \in \mathbb{R}^n$ such that $\langle z, x\rangle < \langle z, \overline{x}\rangle$ for all $x \in X$. Since $0_n \in X$, then $z \neq 0_n$ and $b := \langle z, \overline{x}\rangle > 0$. Letting $\widetilde{z} := \frac{z}{b}$ one has $\langle \widetilde{z}, x\rangle < 1 = \langle \widetilde{z}, \overline{x}\rangle$ for all $x \in X$. Thus, $\widetilde{z} \in X^e$ and $\langle \widetilde{z}, \overline{x}\rangle = 1$, which shows that $\overline{x} \notin X^{ee}$.

This involutory formula for the e-polars of e-convex sets implies Minkowski's one, $X^{\circ\circ} = X$, for the polars of closed convex sets. In fact, given an arbitrary closed convex set X containing 0_n, since X is e-convex, we get the nontrivial inclusion

$X^{\circ\circ} \subset X$ from (1.44) and from the fact that $X^e \subset X^\circ$ implies $X^{\circ e} \subset X^{ee}$ as follows:

$$X^{\circ\circ} = \operatorname{cl} X^{\circ e} \subset \operatorname{cl} X^{ee} = \operatorname{cl} X = X.$$

The term *e-polar* was first used to refer to X^e in [101]. In this paper, Klee and his coauthors consider the following binary operations for $X, Y \subset \mathbb{R}^n$:

$$\begin{aligned}
X \wedge_1 Y &= (\operatorname{int} X) \cap (\operatorname{int} Y) \\
X \wedge_2 Y &= X \cap Y \\
X \wedge_3 Y &= (\operatorname{cl} X) \cap (\operatorname{cl} Y) \\
X \wedge_4 Y &= (X \cap \operatorname{cl} Y) \cup (Y \cap \operatorname{cl} X) \\
X \vee_1 Y &= \operatorname{cl} \operatorname{conv} (X \cup Y) \\
X \vee_2 Y &= \operatorname{eco} (X \cup Y) \\
X \vee_3 Y &= \operatorname{conv} ((\operatorname{int} X) \cup (\operatorname{int} Y)) \\
X \vee_4 Y &= \operatorname{eco} (X \cup \operatorname{int} Y) \cap \operatorname{eco} (Y \cup \operatorname{int} X).
\end{aligned}$$

It is obvious that, if X and Y are e-convex sets, $X \wedge_i Y$ and $X \vee_i Y$ are also e-convex for $1 \leq i \leq 3$. The even convexity of $X \wedge_4 Y$ and $X \vee_4 Y$ is proved in [101, Prop. 4.1]. Moreover, in [101, Prop. 4.3], it is showed that, for $1 \leq i \leq 4$, the operations $\wedge_i$ and $\vee_i$ are dual in the sense that $(X \wedge_i Y)^e = X^e \vee_i Y^e$ and $(X \vee_i Y)^e = X^e \wedge_i Y^e$, whenever X and Y are e-convex sets with $0_n \in X \cap Y$.

1.6.2 Semi-Infinite Zero-Sum Two Person Games

A semi-infinite game is a two-person zero-sum game determined by a real matrix with finitely many columns and infinitely many rows. The first papers on semi-infinite games [74, 167, 180] considered countable many rows. They assumed that the only admissible strategies of player I are those which have a finite support so that the reward can be expressed as a finite sum. If the mentioned matrix is $\{a_t, t \in T\}$, with T countable, the mixed strategies of players I and II are

$$\Lambda := \left\{ \lambda \in \mathbb{R}_+^{(T)} : \sum_{t \in T} \lambda_t = 1 \right\} \text{ and } Y := \left\{ y \in \mathbb{R}_+^n : \sum_{i=1}^n y_i = 1 \right\},$$

respectively, whose elements are discrete probability distributions over the corresponding sets of pure strategies. The *payoff function* is $P(\lambda, y) = \sum_{t \in T} \lambda_t \langle a_t, y \rangle$, which represents the expected outcome of player II when players I and II choose the mixed strategies λ and y, respectively. The maximin and minimax values are

$$v_I := \sup_{\lambda \in \Lambda} \min_{j=1,\ldots,n} P\left(\lambda, e^j\right) \text{ and } v_{II} := \inf_{y \in Y} \sup_{t \in T} \langle a_t, y \rangle,$$

respectively. Obviously, $-\infty < v_I \leq v_{II}$. The *duality gap* is $v_{II} - v_I \geq 0$. Denote by $\widetilde{\Lambda}$ and $\widetilde{Y}$ the *sets of optimal strategies* of players I and II, respectively.

A.L. Soyster proved [167] that, if either player's problem is consistent and bounded, so is that of his adversary, and the minimax theorem ($v_{II} = v_I$) holds. Moreover, $\widetilde{\Lambda} \neq \emptyset$ while player II needs not have an optimal strategy. S.H. Tijs [180], in turn, provided sufficient conditions guaranteeing that $v_{II} = v_I$ and gave also a new proof that $\widetilde{\Lambda} \neq \emptyset$. Few years later, in [74], falling back on alternative theorems for ordinary semi-infinite systems, it was shown that $v_{II} = v_I$ and $\widetilde{Y} \neq \emptyset$; the main novelty in this paper was the consideration of semi-infinite games with an arbitrary set T (not necessarily countable) and the analysis of an important type of strategies. A pure strategy t of player I is called *essential* if, assuming $\widetilde{\Lambda} \neq \emptyset$, there exists some $\widetilde{\lambda} \in \widetilde{\Lambda}$ such that $\widetilde{\lambda}_t > 0$. Similarly, a pure strategy e^j of player II is called *essential* if, assuming $\widetilde{Y} \neq \emptyset$, there exists some $\widetilde{y} \in \widetilde{Y}$ such that $\widetilde{y}_j > 0$. Theorems 2.4 and 2.6 in [74] provide sufficient conditions for the existence of essential strategies of player II and I, respectively, whose proofs are based on Corollary 1.4. A similar methodology was used in [111] to study more general semi-infinite games, where either the pure strategies for player I are picked, instead of homogeneous linear functions, from an infinite family of convex functions, or the set of mixed strategies available to player II, instead of being the unit simplex is a given closed convex set.

The above results were used in [110, 157, 158] to analyze the so-called semi-infinite transportation and assignment games. In transportation games, an economic agent aims at maximizing her/his profit from transporting an infinitely divisible good from a finite number of suppliers to an infinite number of demanders. The assignment games arise when a finite set of agents of one type is assigned to a countably infinite set of agents of another type; this has to be done in such a way that the total profit arising from these assignments is as large as possible. Finally, it is worth mentioning that nonzero-sum semi-infinite games with arbitrary sets of pure strategies and bounded payoffs have been studied in [143].

1.6.3 Approximate Reasoning

Approximate reasoning is a subdiscipline of artificial intelligence which is focused on the treatment of imprecision and uncertainty. It covers both the foundations of uncertainty theories and the design of intelligent systems for scientific and engineering applications. The generic term *imprecise probabilities* encompasses several mathematical models to deal with ignorance and uncertainty such as upper and lower probabilities, upper and lower previsions, classes of additive probability measures and partial preference orderings, among other qualitative models (see [187]). Methods of approximate reasoning and statistical inference using imprecise probabilities are based on a behavioral interpretation of probability and principles of coherence. We now introduce some basic notions in approximate reasoning theory.

Assume that the set of outcomes of an experiment is finite, say $\Omega = \{\omega_1, \ldots, \omega_n\}$. A *gamble* is a function $x : \Omega \to \mathbb{R}$ that can be viewed as a vector in $\mathbb{R}^n$. A *preference ordering* $\succ$ is a strict partial order over pairs of gambles (i.e., a binary relation on $\mathbb{R}^n$ that is irreflexive and transitive). A gamble $x \in \mathbb{R}^n$ is called *desirable* if $x \succ 0$. The preference ordering $\succ$ is *monotone* whenever, for all $x, y \in \mathbb{R}^n$,

$$[x_i > y_i, \forall i = 1, \ldots, n] \Longrightarrow x \succ y,$$

and it satisfies the *cancellation* rule whenever, for any $x, y, z \in \mathbb{R}^n$, $\alpha \in]0, 1]$, one has

$$x \succ y \Longleftrightarrow \alpha x + (1 - \alpha) z \succ \alpha y + (1 - \alpha) z.$$

The literature on sets of desirable gambles employs cones of gambles to model preferences. More precisely, a preference ordering can be captured by focusing on preferences with respect to the zero gamble, or, equivalently, by focusing on a convex cone of gambles. In this sense, the representation in [33, Prop. 2] establishes that, if a preference ordering $\succ$ satisfies monotonicity and cancellation, then there exists a convex cone D of gambles, not containing the origin but containing the interior of the positive orthant, such that, for every $x, y \in \mathbb{R}^n$,

$$x \succ y \Longleftrightarrow x - y \in D. \tag{1.47}$$

In spite of its simplicity, the theory of desirable gambles encompasses not only the Bayesian theory of probability, but also other important mathematical models like credal sets of probabilities. Credal sets is the term used in Bayesian approximate reasoning for the probability distributions which allow to express uncertainty or doubt about the probability model that should be used, or to convey the beliefs of a Bayesian agent about the possible states of the world. They are usually assumed to be compact and convex in order to express them as the closure of the convex hull of their extreme points by the Krein–Milman theorem. More precisely, a *credal set* is any closed convex subset of the unit simplex

$$\mathbb{P}_n := \left\{ p \in \mathbb{R}^n : \sum_{i=1}^{n} p_i = 1, 0 \le p_i \le 1, \forall i = 1, \ldots, n \right\},$$

the set of all *probability measures* over Ω, where p_i stands for the probability of ω_i for each $i = 1, \ldots, n$. The duality between credal sets and (coherent) sets of almost desirable gambles (sets of gambles satisfying specific properties and representing rational choices) is well-known in the literature. Given a vector $p \in \mathbb{P}_n$ inducing a probability measure, and a gamble $x \in \mathbb{R}^n$, the *expected value of* x, denoted by $\mathbb{E}_{\mathbb{P}}[x]$, is simply the inner product $\langle p, x \rangle$.

Similarly to (1.47), F. Cozman [33] considered recently preference orderings that can be represented by evenly convex sets of desirable gambles, which indeed can also be represented by evenly convex credal sets. An evenly convex credal set can for instance encode preference judgements through strict and weak inequalities. By introducing the property of *even continuity* of a preference ordering $\succ$ whenever, for any $y \in \mathbb{R}^n$, sequence of gambles $\{x^k\}$, and sequence of positive scalars $\{\lambda_k\}$ such that $\{\lambda_k y - x^k\}$ is convergent, one has

$$x^k \succ 0 \ \forall k \in \mathbb{N}, \text{ and } y \succeq 0 \text{ is false} \Longrightarrow \lim_{k \to \infty} \left(\lambda_k y - x^k\right) \succ 0 \text{ is false},$$

and calling $\succ$ *coherent* if it is monotone, satisfies the cancellation rule, and it is even continuous, the representation in [33, Prop. 7] establishes that, if a preference ordering $\succ$ is coherent, then there exists an evenly convex cone D of gambles, not containing the origin but containing the interior of the positive orthant, such that, for every $x, y \in \mathbb{R}^n$, (1.47) holds.

In addition to that, evenly convex sets of desirable gambles can be represented by evenly convex sets of probability measures. This is shown in the representation theorem [33, Th. 9] which asserts that, if $\succ$ is a coherent preference ordering, then there exists a unique maximal evenly convex credal set C such that

$$x \succ y \Longleftrightarrow \mathbb{E}_{\mathbb{P}}[x] > \mathbb{E}_{\mathbb{P}}[y]$$

for all probability measure $p \in C$. Moreover, according to [33, Th. 10], if $\succ$ is a coherent preference ordering and C is the set built in the proof of the above representation theorem, then a subset C' of the unit simplex of $\mathbb{R}^n$ represents $\succ$ if and only if $C = \operatorname{eco} C'$. Thus, different evenly convex sets represent different preference orderings. Finally, the rest of the paper [33] analyzes the duality between preference orderings and credal sets and discusses regular conditioning, a concept which is also closely related to evenly convex sets.

1.6.4 Slater Condition in Mathematical Programming

In almost any branch of optimization there exists a so-called *Slater condition* allowing to get necessary optimality conditions, duality theorems, and stability results guaranteeing the continuity, in some sense, of the feasible set, the optimal set, and the optimal value under perturbations of the data. These Slater-type conditions can frequently be formulated in terms of the existence of solutions, called *Slater points*, for suitable non-ordinary systems. We just mention three cases.

1. Consider a *continuous linear semi-infinite programming problem* of the form

$$\begin{aligned} (P_{LSIP})\ \operatorname*{Min}_{x \in \mathbb{R}^n}\ & \langle c, x\rangle \\ \text{s.t.}\ & \langle a_t, x\rangle \leq b_t,\ t \in W, \end{aligned} \tag{1.48}$$

where W is a compact topological space and the functions $t \longmapsto a_t$ and $t \longmapsto b_t$ are continuous on W. The Slater elements of (P_{LSIP}) are the solutions of the non-ordinary continuous semi-infinite system $\{\langle a_t, x\rangle < b_t,\ t \in W\}$, so the set of Slater elements is e-convex. The *Haar dual problem* of (P_{LSIP}) is

$$\begin{array}{lll}(D^1_{LSIP}) & \underset{\lambda\in\mathbb{R}^{(W)}_+}{\text{Max}} & -\sum\limits_{t\in W}\lambda_t b_t \\ & \text{s.t.} & -\sum\limits_{t\in W}\lambda_t a_t = c,\end{array}$$

which is equivalent to the *Lagrangian dual problem* of (P_{LSIP}), that is, the unconstrained optimization problem

$$(D^2_{LSIP})\ \underset{\lambda\in\mathbb{R}^{(W)}_+}{\text{Max}}\ \inf_{x\in\mathbb{R}^n} L(x,\lambda).$$

A third dual problem for (P_{LSIP}) is the so-called *continuous dual problem*

$$\begin{array}{lll}(D^3_{LSIP}) & \underset{\mu\in\mathscr{C}'_+(W)}{\text{Max}} & -\int_W b_t\, d\mu(t) \\ & \text{s.t.} & -\int_W a_t d\mu(t) = c,\end{array}$$

where $\mathscr{C}'_+(W)$ represents the cone of non-negative regular Borel measures on W. The optimal value $v(D^j_{LSIP})$ of (D^j_{LSIP}), $j = 1, 2, 3$, is not greater than the one of (P_{LSIP}), say $v(P_{LSIP})$, i.e., weak duality always holds. Characterizations of optimality, strong duality theorems guaranteeing zero duality gap with dual solvability for the three dual pairs $(P_{LSIP}) - (D^j_{LSIP})$, $j = 1, 2, 3$, and stability theorems, all of them under the Slater condition can be found in [73, Ths. 1 and 3] and [72, Ths. 6.9 and 10.4], respectively (recall that SCQ implies, for continuous problems, the FMCQ and, so, the weaker LFMCQ at any point).

2. We now consider a linear conic optimization problem of the form

$$\begin{array}{ll}(P_K)\ \underset{x\in\mathbb{R}^n}{\text{Min}} & \langle c, x\rangle \\ \text{s.t.} & Ax + b \in K,\end{array}$$

where $c \in \mathbb{R}^n$, A is an $m \times n$ matrix, $b \in \mathbb{R}^m$, and K is a pointed closed convex cone in $\mathbb{R}^m$ such that $\operatorname{int} K \neq \emptyset$. The assumptions on K guarantee that K° satisfies the same properties and the existence of a *compact base* of K°, that is, a compact convex set W such that $0_m \notin W$ and $K = \operatorname{cone} W$ [65, p. 447]. For instance, a compact base of $\left(\mathbb{R}^m_+\right)^\circ = -\mathbb{R}^m_+$ is $W = -\operatorname{conv}\{e_1, \ldots, e_m\}$, where $\{e_1, \ldots, e_m\}$ is the *canonical basis* of $\mathbb{R}^m$. Since $K^\circ = \operatorname{cone} W$,

$$Ax + b \in K \Longleftrightarrow \langle t, (Ax + b)\rangle \leq 0, \forall t \in W. \tag{1.49}$$

Thus, (P_K) is equivalent to the continuous linear semi-infinite problem (P_{LSIP}) in (1.48) just taking $a_t = A^\top t$ and $b_t = -\langle t, b\rangle$ for all $t \in W$. As shown in [40, Rem. 2], the dual problem for (P_K) in the sense of conic optimization is equivalent to the Haar dual problem (D^1_{LSIP}). Hence, the strong duality holds, for the corresponding versions of the dual problems (D^j_{LSIP}), $j = 1, 2, 3$, under the *Slater condition* that the non-ordinary system $\left\{\left\langle A^\top t, x\right\rangle < -\langle t, b\rangle, t \in W\right\}$ is consistent.

3. We finally consider a convex (possibly semi-infinite) problem

$$(P_{CSIP}) \ \operatorname*{Min}_{x\in\mathbb{R}^n} \ f(x)$$
$$\text{s.t.} \ \ f_t(x) \leq 0, \ t \in W,$$

where $f, f_t : \mathbb{R}^n \to \mathbb{R}$ are continuous for all $t \in W$. The problem (P_{CSIP}) satisfies the Slater condition when the non-ordinary convex system $\{f_t(x) < 0, t \in W\}$ is consistent (this condition implies strong duality and allows to obtain optimality conditions). For any $t \in W$, as f_t is a convex real-valued function, it is subdifferentiable on $\mathbb{R}^n$. Thus, the set of Slater points of $\{f_t(x) \leq 0, t \in W\}$ is the solution set of the non-ordinary semi-infinite system

$$\{f_t(y) + \langle u, (x - y)\rangle < 0, (y, u) \in \operatorname{gph} \partial f_t, t \in W\}, \tag{1.50}$$

where

$$\operatorname{gph} \partial f_t := \left\{(u, y) \in \mathbb{R}^{2n} : u \in \partial f_t(y)\right\}$$

is the *graph* of the *subdifferential mapping* $\partial f_t : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by

$$\partial f_t(y) = \left\{u \in \mathbb{R}^n : f_t(x) \geq f_t(y) + \langle u, (x - y)\rangle, x \in \mathbb{R}^n\right\}.$$

In the particular case of convex quadratic systems, we can write $f_t(x) = \frac{1}{2}\langle x, Q_t x\rangle + \langle c_t, x\rangle + d_t$, where Q_t is a symmetric $n \times n$ positive semidefinite matrix, $c_t \in \mathbb{R}^n$ and $d_t \in \mathbb{R}$, for all $t \in W$. Then, (1.50) becomes

$$\left\{\langle (Q_t y + c_t), x\rangle < \frac{1}{2}\langle y, Q_t y\rangle - d_t, (y, t) \in \mathbb{R}^n \times W\right\}.$$

1.6.5 Strict Separation of Finite Families of Sets

The search of a hyperplane separating strictly a pair of disjoint sets in $\mathbb{R}^n$, say Y and Z, can be formulated as the feasibility problem for the non-ordinary system of strict inequalities (with $W = \emptyset$)

$$\{\langle y, x\rangle < x_{n+1}, \ y \in Y; \ -\langle z, x\rangle < -x_{n+1}, \ z \in Z\}, \tag{1.51}$$

where the unknown $\begin{pmatrix} x \\ x_{n+1} \end{pmatrix} \in \mathbb{R}^{n+1}$ determines the vector of coefficients of the separating hyperplane. Observe that $x \neq 0_n$ for any solution $\begin{pmatrix} x \\ x_{n+1} \end{pmatrix}$ of the system in (1.51).

The concept of strict separation of sets was extended in [12] to families of more than two sets as follows. A family of $m \geq 2$ nonempty sets in $\mathbb{R}^n$, $A_1, \ldots, A_m$ is said to *be strictly separable* if there exist m closed halfspaces $S_1, \ldots, S_m$ such that $A_j \subset \operatorname{int} S_j$, $j = 1, \ldots, m$, and $\bigcap_{j=1}^{m} \operatorname{int} S_j = \emptyset$, i.e., if, for each $j = 1, \ldots, m$, there exists a solution $\begin{pmatrix} c_j \\ d_j \end{pmatrix} \in \mathbb{R}^{n+1}$ of $\sigma_j := \left\{ \langle a, x \rangle - x_{n+1} < 0,\ a \in A_j \right\}$, with $c_j \neq 0_n$, such that the system $\sigma_0 := \left\{ \langle c_j, x \rangle < d_j,\ \ j = 1, \ldots, m \right\}$ is inconsistent.

According to [12, Th. 2], if $\bigcap_{\substack{k=1 \\ k \neq j}}^{m} A_k \neq \emptyset$, $j = 1, \ldots, m$, then the inconsistency of σ_0 can be replaced by the simple condition that $\sum_{j=1}^{m} \begin{pmatrix} c_j \\ d_j \end{pmatrix} = 0_{n+1}$.

1.7 Bibliographic Notes

Many mathematicians have contributed to the convexity theory in finite dimensional spaces, but the most influential of them, thanks to the popularity attained by their respective books on the subject, have been, in chronological order:

- Hermann Minkowski: His celebrated book [128], published in 1911 (even though the first 240 pages were a reprint of his 1896 monograph [127]), introduced the concepts of extreme point, convex hull, convex body (i.e., fully dimensional compact convex set), supporting hyperplane, support function, sum of two sets, etc. Minkowski provided a conjoint theory of ordinary linear inequality systems and polyhedra, including the proof of the finiteness of the set of extreme points of the latter sets and the characterization of the linear inequalities which are the consequence of a given consistent linear inequality system (i.e., the so-called non-homogeneous Farkas lemma).
- Werner Fenchel: His book [54], where he collected his lectures at Princeton University during his sabbatical academic year 1950/51, introduced the key concept of conjugate function, the notion of strict polar of a closed convex set, etc. A year later, in 1952, he introduced the evenly convex sets in order to extend the polarity theory to nonclosed convex sets [55].
- Ralph Tyrrell Rockafellar: The preface of his classical book [148], published in 1970, where he virtually fixed the present notation and basic results of modern convex analysis, recognizes the influence of Fenchel's view of convexity (not by chance, [148] was dedicated to Fenchel "as honorary coauthor"), in

particular the crucial role played by the notion of conjugate function. Before that, Rockafellar introduced in 1963, together with Jean-Jacques Moreau, the concepts of subgradient and subdifferential and provided results involving linear systems containing strict inequalities. Victor Klee recognized that his work [99] on maximal separation theorems for e-convex sets was inspired by the mentioned paper by Fenchel on polarity together with an unpublished separation theorem, proved by Rockafellar [146], for the so-called partially polyhedral sets (a particular class of e-convex sets defined in [150, p. 510]).

After the above mentioned seminal contributions to the theory of evenly convex sets, this type of convex sets appeared sporadically in the literature for almost 30 years. Thus, in 1970 and 1972, Schröder used evenly convex sets to obtain his linear range-domain implications [161, 162]. In the eighties and early nineties, the evenly convex sets were applied in quasiconvex programming [120, 121, 138–140] and mathematical economy [122]. Linear systems containing strict inequalities, in turn, naturally arose in convex optimization [150, 190], separation problems [12, 99, 100], and stability analysis [72, Th. 6.9], among other fields of mathematics and computer sciences.

The resurgence of the even convexity theory in the 2000s is also motivated by the mentioned stream of new applications of linear strict inequality systems and the potential applications of evenly convex and quasiconvex functions in economy. New characterizations of e-convex sets have been obtained by Daniilidis and Martínez-Legaz [37] and by Goberna, Jornet and Rodríguez [71] (in infinite and finite dimensions, respectively), showing that the class of e-convex sets enjoys most of the well-known properties of the subclasses of open and closed convex sets. Even convexity has been used to characterize the consistency of linear semi-infinite systems containing strict inequalities in [78], whose results remain valid in Banach spaces [133], and to obtain dual characterizations of set containments with strict convex inequalities [68] (results which have been extended to systems of strict cone-convex inequalities [39] and to systems of quasiconvex inequalities, see [171, 174]). Furthermore, Klee, Maluta and Zanco have studied the behavior of e-convex sets under sections and projections [101], which unfortunately became the last publication in the fruitful research career of the first author. A suitable extension of the concept of e-convex set is used in [60] to study quasi-convex dynamic risk measures.

This chapter is mostly based on [37, 55, 68, 71, 78, 99]. More precisely, concerning Sect. 1.1, recently reviewed in [79], the proof of the equivalence between the first six conditions in Theorem 1.1 can be found in [71, Prop. 3.1], for $(i) \Longleftrightarrow (ii) \Longleftrightarrow (iii)$, [55, Items 3.2 and 3.4], for $(i) \Longleftrightarrow (iv) \Longleftrightarrow (v)$, and [37, Th. 5], for $(i) \Longleftrightarrow (vi)$. The properties of e-convex sets gathered in Proposition 1.1 come from [55, Item 3.5], for *(iii)*, [71, Cor. 3.2], for *(vi)*, and [71, Prop. 3.2-3.4], for the remaining properties. The statements on operations with e-convex sets in Proposition 1.2 have been proved in [71, Prop. 3.5], for *(ii)*, [71, Prop. 3.6], for the "if" part of *(iii)*, [151, Prop. 1.2], for the "only if" part of *(iii)*, [71, Prop. 3.7], for (*iv*), [71, Prop. 3.8], for (*v*) and [101, Cor. 2.3], for (*vi*). Example 1.3 appeared in [101, Ex. 2.5].

Section 1.2, also reviewed in [79], provides the notion of evenly convex hull that was introduced by W. Fenchel in [55, Items 4.1 and 4.2]. Its characterization in Proposition 1.3 is proved in [78, Prop. 2.1]. The relationships between eco and other hulls in Proposition 1.4 are [78, (2.2) and (2.4)], for (*iii*) and (*iv*), respectively, and [78, Prop. 2.7], for (*v*). The results involving operations with e-convex hulls in Proposition 1.5 are proved in [78, Props. 2.3-2.6], for (*ii*), (*iii*), (*vi*), and (*vii*), respectively, and [78, Cor. 2.1], for (*iv*).

Section 1.3 presents a selection of the separation theorems for convex sets collected in Klee's paper [99], which was published only 2 years before the coining of the basic concepts and notations of convex analysis in the celebrated Rockafellar's book [148]. So, we have adapted here most Klee's separation theorems involving e-convex sets to the modern convex analysis language, the main difficulty being the intuitive style of some arguments and the misleading use of different concepts of supporting hyperplane in [99]. Example 1.9 is attributed to T.A. Botts by the same Klee [99, p. 134], Theorems 1.2 and 1.3 are based on [99, Th. 4] and [99, Th. 5], respectively, Lemma 1.1 is [99, Lemma 1] and Corollary 1.1 is [99, Cor. p. 138].

The e-convex sets are the solution sets of the linear inequality systems possibly containing strict inequalities whose study is the objective of Sect. 1.4. The relationships between the solution sets of a system σ containing strict inequalities and its relaxed system $\overline{\sigma}$ (Proposition 1.6) appeared in [71, Prop. 1.1]. The rest of this section is devoted, firstly, to existence theorems, secondly, to Farkas-type lemmas, and, thirdly, to the containment problem for e-convex sets.

Section 1.4.1 starts recalling the known characterizations of the consistent relaxed system. We summarize in Table 1.1 the relevant information on the most outstanding existence theorems, most of them expressed as alternative theorems, i.e., theorems which have the following form: exactly one of the two formulated propositions holds true. The 11 sources in Column 1 appear chronologically

Table 1.1 Existence theorems

<table>
<tr><th>Ref.</th><th>Year</th><th>S</th><th>W</th><th>E</th><th>b.</th><th>Cond.</th></tr>
<tr><td>[80]</td><td>1873</td><td>fin.</td><td>∅</td><td>∅</td><td>0</td><td>Not</td></tr>
<tr><td>[28]</td><td>1921–1922</td><td>fin.</td><td>∅</td><td>∅</td><td>arb.</td><td>Not</td></tr>
<tr><td>[131]</td><td>1936</td><td>fin.</td><td>fin.</td><td>fin.</td><td>0</td><td>Not</td></tr>
<tr><td>[62]</td><td>1960</td><td>∅</td><td>fin.</td><td>∅</td><td>arb.</td><td>Not</td></tr>
<tr><td>[193]</td><td>1966</td><td rowspan="2">∅</td><td rowspan="2">arb.</td><td rowspan="2">∅</td><td rowspan="2">arb.</td><td rowspan="2">Not</td></tr>
<tr><td>[50]</td><td>1968</td></tr>
<tr><td>[170]</td><td>1970</td><td>fin.</td><td>fin.</td><td>fin.</td><td>arb.</td><td>Not</td></tr>
<tr><td>[74]</td><td>1984</td><td>arb.</td><td>arb.</td><td>arb.</td><td>0</td><td>Yes</td></tr>
<tr><td>[74]</td><td>1984</td><td>arb.</td><td>∅</td><td>∅</td><td>0</td><td>Yes</td></tr>
<tr><td>[176]</td><td>1999</td><td>fin.</td><td>fin.</td><td>fin.</td><td>arb.</td><td>Not</td></tr>
<tr><td>[71]</td><td>2003</td><td>arb.</td><td>arb.</td><td>∅</td><td>arb.</td><td>Yes</td></tr>
</table>

ordered, as Column 2 shows. The Columns 3, 4 and 5, in turn, inform on the cardinality of the index sets, which can be empty, finite or arbitrary (abbreviated as "∅", "fin." and "arb.", respectively). Column 6 informs about the kind of right-hand side scalars b_t the theorems deal with (0, in the case of homogeneous systems, and arbitrary, otherwise). Finally, Column 7 informs about the full generality or not of the corresponding existence theorem, i.e., whether the result always holds or it does just under certain assumptions.

Observe that all the known existence theorems for systems containing an arbitrary number of strict inequalities are only valid provided that a suitable closedness assumption holds. Cor. 3.1.1 and Th. 3.1 in [72] provide simple proofs of the characterization of the consistency of $\overline{\sigma}$ by means of conditions (1.11) and (1.12), results which also follow from existence theorems due to K. Fan [50, Th. 1] and Y. J. Zhu [193, Th. 1], respectively. These authors considered linear systems of weak inequalities of the form $\overline{\sigma} = \left\{\left\langle a_t^*, x\right\rangle \leq b_t,\ t \in T\right\}$, where x lives in a given locally convex separated (i.e., Hausdorff) topological vector space X with topological dual X^*, $a_t^* \in X^*$ and $b_t \in \mathbb{R}$ for all $t \in T$. The mentioned characterizations of the consistency of $\overline{\sigma}$, (1.11) and (1.12), are still valid in the infinite-dimensional setting by replacing 0_n with the null linear functional of 0_{X^*}. In this infinite-dimensional setting, $N(\overline{\sigma})$, $K(\overline{\sigma}) \subset X^* \times \mathbb{R}$, so that conditions (1.11) and (1.12) can be seen as dual characterizations of the consistency of $\overline{\sigma}$. Consequently, the same adjective, dual, applies to any condition involving subsets of either X^* or $X^* \times \mathbb{R}$, which have the advantage of being expressed in terms of the data (the coefficients of $\overline{\sigma}$). The existence Theorem 1.4 for linear systems involving strict inequalities was proved in [78, Th. 3.1], Corollary 1.3 in [71, Prop. 2.1], and Corollary 1.6 in [78, Cor. 3.2], while Corollaries 1.4 (proved in [78, Cor. 3.3]) and 1.5 are improved versions of the Generalized Gordan's alternative theorem [72, Th. 3.2] and the Extended Motzkin's alternative theorem [72, Th. 3.5], respectively. The "like" in the names given to these results means that they extend to semi-infinite systems the corresponding results for finite systems. Proposition 1.7 is [71, Prop. 2.2].

Section 1.4.2 deals with the extension to systems of an arbitrary number of constraints possibly containing strict inequalities of the generalized non-homogeneous Farkas lemma (1.26) characterizing the weak linear inequalities which are consequence of a finite consistent system of inequalities of the same type [72, Th. 3.1], which remains valid in locally convex spaces [193, Th. 2]. The concept of legal linear combination was introduced by H.V. Kuhn [108] and by J. Stoer and C. Witzgall [170] for ordinary finite linear systems. The term legal for the non-null elements of $\mathbb{R}_+^{(T)}$ and for the corresponding linear combinations of σ were introduced by the same Kuhn [108]. The main result of this subsection, Theorem 1.5, subsumes [78, Props. 4.1 and 4.2].

Regarding Sect. 1.4.3, Proposition 1.10 comes from [68, Props. 5.1-5.3] and Proposition 1.8 from [68, Prop. 5.4]. The containment problem has been solved for different pairs of sets usually represented by inequality systems. The simplest and most studied version is the polytope containment problem, as it has important applications such as computational geometry [64], machine learning [61], and control theory [87]. Computational issues on this problem are discussed in [156]. The polyhedral containment problem was posed by Mangasarian [117], who also introduced non-polyhedral extensions. In fact, he characterized in [118], via nonlinear programing, the containment of a polyhedral convex set F in a *reverse convex set* (i.e., the complement of a finite union of convex sets) described by convex quadratic functions, as

$$G = \left\{ x \in \mathbb{R}^n : \frac{1}{2} \langle x, Q_i x \rangle + \langle a_i, x \rangle \geq b_i, i = 1, \ldots, m \right\},$$

with Q_i symmetric and positive semi-definite, $a_i \in \mathbb{R}^n$, and $b_i \in \mathbb{R}$ for $i = 1, \ldots, m$, that is, the situation illustrated in Fig. 1.17. The same Mangasarian [118] characterized the containment of a convex set in a reverse convex set (the situation

Fig. 1.17 Inclusion of a polyhedron F in a quadratic reverse set G

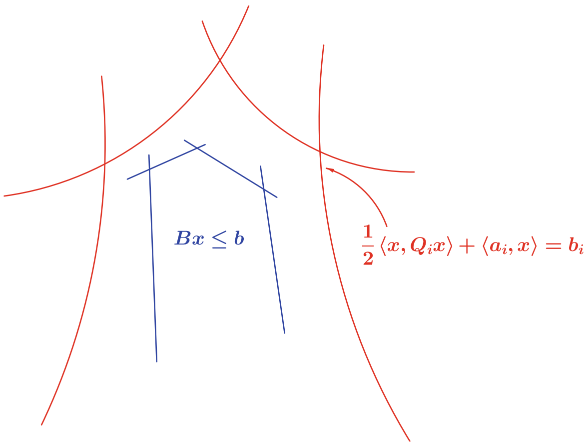
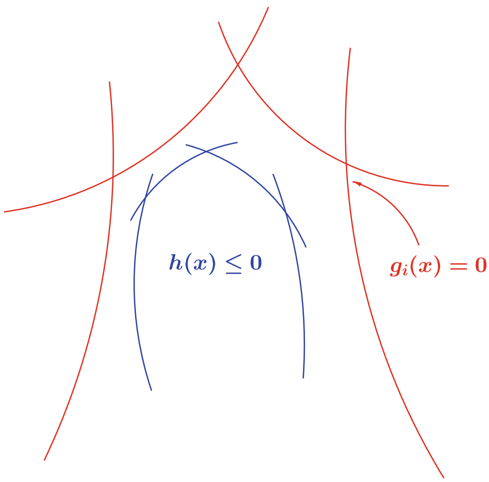


Fig. 1.18 Inclusion of a convex set F in a quadratic reverse convex set G



shown in Fig. 1.18), both represented by differentiable functions, while Jeyakumar [93] proposed a non-smooth counterpart for this containment via epigraphs of conjugate functions.

The containment problem has also been solved for pairs of sets represented by systems of cone-convex inequalities [39], by quasiconvex systems [173, 174], by quasiconvex systems containing strict inequalities [171], by sum-of-squares convex polynomial systems [96], etc.

Section 1.5 is basically the transcription of known results on ordinary linear semi-infinite systems and linear semi-infinite programming to systems and problems containing strict inequalities where the concept of carrier index plays a crucial role. As pointed out in [104], the term carrier index was used for the first time in [72] (in the context of linear SIP), but it has received different names in the literature: *always binding constraint* in [1], *implicit equality constraint* in [83], and *immobile index* in [102, 103], as well as in different papers of Kostyukova and Tchemisova (e.g., [105, 106]). The set of carrier indices was called *equality set of constraints* in [189]. The existence of carrier indexes is incompatible with the SCQ, which was introduced by M. Slater in the framework of convex programming [164], the FMCQ by Charnes, Cooper and Kortanek in [31] to guarantee zero-duality gap in LSIP, and the LFMCQ by Puente and Vera de Serio in [144] as the weakest constraint qualification allowing to characterize optimality in LSIP. The term "continuous linear system" was first used in the framework of stability while analytical systems and polynomial systems appeared in [3] (in order to give a simplex-like method for LSIP problems) and [70] (in a geometric setting), respectively. The LSIP problems satisfying the LOPCQ may also be treated by means of some simplex-like method [2].

The natural extension to the infinite-dimensional setting of LSIP problems are the linear infinite programming (LIP) problems of the form

$$\begin{aligned}(P)\ \operatorname*{Min}_{x\in X}\ & \langle c, x\rangle \\ \text{s.t.}\ & \langle a_t, x\rangle \leq b_t, t \in T,\end{aligned}$$

where the decision space X is locally convex and $c, a_t \in X^*$ for all $t \in T$. Duality theorems for different types of dual problems for (P), among them the LIP Haar dual defined in the same way as in LSIP, can be found in [75, Rem. 3.10, 4.16 and 5.6, and Cor. 4.15] and [76, Section 6]. From these results it is possible to obtain e-LIP versions of Theorem 1.7.

Chapter 2
Evenly Convex Polyhedra: Finite Linear Systems Containing Strict Inequalities

This chapter is about linear systems containing finitely many weak and/or strict inequalities, whose solution sets, provided they are nonempty, are called evenly convex polyhedral sets (e-polyhedra, in brief). Of course, all results in Chap. 1 on e-convex sets and their linear representations are valid here, but the finiteness of the linear representations of e-polyhedra allows to obtain specific results and methods.

Many families of convex sets have an internal as well as an external representation. For instance, any closed convex set C is the (Minkowski) sum of the lineality subspace of C with the set of all extreme points and extreme directions of C, as well as the intersection of all the closed halfspaces containing C. Moreover, any compact convex set is the convex hull of its set of extreme points, as well as the intersection of all the closed halfspaces determined by its supporting hyperplanes. In turn, any polyhedron C is the sum of the convex hull of the set of all extreme points of C with the convex cone generated by the set of extreme directions of C (a polyhedral convex cone), as well as the intersection of some finite family of closed halfspaces. But the outstanding advantage of the polyhedra against the other two families of convex sets is the availability of double description methods allowing to get an external representation from the internal one and vice versa. One of these methods is based on the Fourier–Motzkin elimination method and the constructive proofs of the well-known Weyl and Motzkin theorems. Extending this double description method from polyhedra to e-polyhedra is the main theme of Chap. 2.

We first describe, in Sect. 2.1, the Fourier–Motzkin elimination method to project a given e-polyhedron on the coordinate planes which, iteratively applied, allows to obtain solutions for finite linear systems containing strict inequalities. Then, in Sect. 2.2, we associate with each finite non-ordinary system its so-called representative cone, which contains all the relevant information on these systems. It allows to simplify the existence theorems and the characterizations of the consequent inequalities provided in Sect. 1.4 for systems of an arbitrary number of constraints. Section 2.3 provides, thanks to the properties of the mentioned representative cone, the aimed double description method for e-polyhedra. Finally,

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in Sect. 2.4, we consider the minimization of linear functions under weak and strict linear inequalities, that is, the so-called evenly linear programming problems.

2.1 Evenly Convex Polyhedra

A (convex) *polyhedron* is the solution set of a consistent finite ordinary linear system. Obviously, every polyhedron is a closed convex set. Moreover, a polyhedron which is also a cone is said to be a *polyhedral cone* and it is the solution set of an homogeneous finite ordinary system. In 1896, Minkowski [127] showed that every polyhedral cone is finitely generated, that is, its elements are the nonnegative linear combinations of a finite set of vectors. Later, in 1935, Weyl [188] proved that polytopes, finitely generated cones and Minkowski sums of a polytope and a finitely generated cone are polyhedra. The converse of the last statement was proved by Motzkin in 1936 [131]. In order to simplify the reference to these results, we merge them in the next lemma:

Lemma 2.1 (Motzkin–Weyl–Minkowski Theorem) *Given a set $F \subset \mathbb{R}^n$, the following statements hold:*

(i) *F is a polyhedron if and only if it is the sum of a polytope and a finitely generated cone.*
(ii) *F is a bounded polyhedron if and only if it is a polytope.*
(iii) *F is a polyhedral cone if and only if it is a finitely generated cone.*

A nonempty set $F \subset \mathbb{R}^n$ is an *evenly convex polyhedron* (*e-polyhedron*, in brief) if it is the solution set of a finite linear system

$$\sigma = \{\langle a_t, x\rangle \leq b_t,\ t \in W;\ \langle a_t, x\rangle < b_t,\ t \in S\}, \tag{2.1}$$

where $T := S \cup W$, $S \cap W = \emptyset$ and $|T| < \infty$. In such a case, we shall say that F is represented by the system σ or that σ is an *external representation* of F.

Clearly, every polyhedron (whenever $S = \emptyset$) is an e-polyhedron, and its interior is either empty or an e-polyhedron, too. Furthermore, every e-polyhedron is obviously an e-convex set.

E-polyhedra arise in a natural way in different settings, including ordinary linear and nonlinear programming. For instance, when solving a linear programming problem

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \ & \langle c, x\rangle \\ \text{s.t.} \ & \langle a_i, x\rangle \leq b_i, i = 1, \ldots, m, \end{aligned}$$

by means of the logarithm barrier method of Fiacco and McCormick [56], one minimizes at step k the barrier function

$$f_k(x) := \frac{\langle c, x\rangle}{\mu_k} - \sum_{i=1}^{m} \ln\left[b_i - \langle a_i, x\rangle\right],$$

where $\{\mu_k\}$ is a given sequence of barrier parameters such that $\mu_k \searrow 0$. So, the domain of the convex function f_k,

$$\operatorname{dom} f_k = \left\{x \in \mathbb{R}^n : \langle a_i, x\rangle < b_i, i = 1, \ldots, m\right\},$$

is an e-polyhedron.

We shall need the description of the closure of F, $\operatorname{cl} F$, and the relative interior of F, $\operatorname{rint} F$, which are e-polyhedra, by means of linear systems.

We denote by $\overline{F}$ the polyhedron which is the solution set of the relaxed system associated to σ,

$$\overline{\sigma} = \{\langle a_t, x\rangle \leq b_t,\ t \in T\}. \tag{2.2}$$

According to Proposition 1.6, if $F \neq \emptyset$, then $\operatorname{cl} F = \overline{F}$ and, by Proposition 1.1(iv),

$$0^+F = 0^+\overline{F} = \{d \in \mathbb{R}^n : \langle a_t, d\rangle \leq 0, t \in T\},$$

which is a polyhedral cone. As a consequence of Lemma 2.1(iii), 0^+F is a finitely generated cone and so, $0^+F = \operatorname{cone}\{d_1, \ldots, d_r\}$ for some *extreme rays* $d_1, \ldots, d_r \in \mathbb{R}^n$. Moreover, if $F \neq \emptyset$, then F is bounded if and only if $0^+F = \{0_n\}$.

Regarding the relative interior of F, we have $\operatorname{rint} F = \operatorname{rint}\operatorname{cl} F = \operatorname{rint} \overline{F}$. Recall that an index $t \in T$ is said to be *carrier* in $\overline{\sigma}$ whenever $\overline{F}$ is contained in the hyperplane $\{x \in \mathbb{R}^n : \langle a_t, x\rangle = b_t\}$, where $\overline{\sigma}$ and $\overline{F}$ can be replaced by σ and F as $\operatorname{aff} F = \operatorname{aff} \overline{F}$. Recall that we denote by $T^=$ the *set of carrier indices* of either $\overline{\sigma}$ or σ. Due to the finiteness of T, σ satisfies FMCQ and, by Proposition 1.13(i),

$$\operatorname{rint} F = \left\{x \in \mathbb{R}^n : \langle a_t, x\rangle < b_t,\ t \in T\backslash T^=; \langle a_t, x\rangle = b_t, t \in T^=\right\}. \tag{2.3}$$

A numerical method for the computation of $T^=$ in $\overline{\sigma}$ can be found in [107].

Kuhn [108] employed a generalization of the classical elimination procedure for ordinary systems of linear inequalities first conceived by Fourier [58] to get solutions of finite linear systems as in (2.1). Actually, the idea of Kuhn's generalization was already contained in the book of Motzkin [131] (published in German by a rather unknown publishing house), based on his Ph.D. Thesis. For this reason, the method that we describe next is usually known as the Fourier–Motzkin elimination method.

Let $F \subset \mathbb{R}^n$ be the solution set of the linear system

$$\sigma = \{\langle a_t, x\rangle \leq b_t,\ t \in W;\ \langle a_t, x\rangle < b_t,\ t \in S\}, \tag{2.4}$$

with $T := S \cup W$, $S \cap W = \emptyset$ and $|T| < \infty$. We denote $a_t = (a_{t1}, \ldots, a_{tn}) \in \mathbb{R}^n$,

$$W_+ := \{t \in W : a_{tn} > 0\},\ W_- := \{t \in W : a_{tn} < 0\},\ W_0 := \{t \in W : a_{tn} = 0\},$$

$$S_+ := \{t \in S : a_{tn} > 0\},\ S_- := \{t \in S : a_{tn} < 0\},\ S_0 := \{t \in S : a_{tn} = 0\},$$

$$T_+ := W_+ \cup S_+,\ T_- := W_- \cup S_-,\ T_0 := W_0 \cup S_0.$$

It is easy to see that the system σ is equivalent to

$$\widehat{\sigma} := \{\langle c_t, x'\rangle + x_n \leq d_t, t \in W_+;\ \langle c_t, x'\rangle - x_n \leq d_t, t \in W_-;$$

$$\langle c_t, x'\rangle + x_n < d_t, t \in S_+;\ \langle c_t, x'\rangle - x_n < d_t, t \in S_-;$$

$$\langle c_t, x'\rangle \leq d_t, t \in W_0;\ \langle c_t, x'\rangle < d_t, t \in S_0\}$$

where $x' = (x_1, \ldots, x_{n-1})$, $d_t := \frac{b_t}{|a_{tn}|}$ and $c_t := \frac{1}{|a_{tn}|}(a_{t1}, \ldots, a_{tn-1})$ for all $t \in T_+ \cup T_-$, and $d_t := b_t$ and $c_t := (a_{t1}, \ldots, a_{tn-1})$ for all $t \in T_0$. Denoting by $\mathrm{proj}_n^J : \mathbb{R}^n \to \mathbb{R}^{|J|}$, $\emptyset \neq J \subsetneq \{1, \ldots, n\}$ the projection mapping consisting in the elimination from each $x \in \mathbb{R}^n$ of the i-th coordinate for $i \notin J$, that is, $\mathrm{proj}_n^J(x) = (x_j)_{j \in J}$, we can write $x' = \mathrm{proj}_n^{\{1,\ldots,n-1\}}(x) \in \mathbb{R}^{n-1}$. Obviously, $x' \in \mathbb{R}^{n-1}$ can be identified with the projection of $x \in \mathbb{R}^n$ onto the hyperplane $x_n = 0$.

We now introduce the *reduced system* σ' associated to σ, whose solution set is denoted by $F' \subset \mathbb{R}^{n-1}$. Consider the following groups of inequalities:

$$\langle (c_t + c_s), x'\rangle \leq d_t + d_s, \quad (t, s) \in W_+ \times W_-, \tag{2.5}$$

$$\langle (c_t + c_s), x'\rangle < d_t + d_s, \quad (t, s) \in (S_- \times T_+) \cup (T_- \times S_+), \tag{2.6}$$

$$\langle c_t, x'\rangle \leq d_t,\ t \in W_0; \quad \langle c_t, x'\rangle < d_t,\ t \in S_0. \tag{2.7}$$

Table 2.1 defines σ' for the 16 different cases (according to the emptiness or not of the sets W_+, S_+, W_- and S_-) to be discussed in the proof of Theorem 2.1.

Theorem 2.1 (Fourier–Motzkin Elimination) *Given an e-polyhedron F in $\mathbb{R}^n$, represented by the system*

$$\sigma = \{\langle a_t, x\rangle \leq b_t,\ t \in W;\ \langle a_t, x\rangle < b_t,\ t \in S\},$$

$\mathrm{proj}_n^{\{1,\ldots,n-1\}}(F)$ *is the e-polyhedron F' in $\mathbb{R}^{n-1}$ represented by the system σ' in Table 2.1.*

Table 2.1 In columns 2–5, 0 and 1 stand for "$= \emptyset$" and "$\neq \emptyset$", respectively

Case	W_+	S_+	W_-	S_-	σ'
1	1	0	1	0	{(2.5), (2.7)}
2	1	1	1	1	{(2.5), (2.6), (2.7)}
3	1	1	1	0	{(2.5), (2.6), (2.7)}
4	1	0	1	1	{(2.5), (2.6), (2.7)}
5	1	1	0	1	{(2.6), (2.7)}
6	1	0	0	1	{(2.6), (2.7)}
7	0	1	1	1	{(2.6), (2.7)}
8	0	1	1	0	{(2.6), (2.7)}
9	0	1	0	1	{(2.6), (2.7)}
10	1	1	0	0	{(2.7)} if $T_0 \neq \emptyset$ and $\{\langle 0_{n-1}, x'\rangle \leq 0\}$ else
11	1	0	0	0	{(2.7)} if $T_0 \neq \emptyset$ and $\{\langle 0_{n-1}, x'\rangle \leq 0\}$ else
12	0	1	0	0	{(2.7)} if $T_0 \neq \emptyset$ and $\{\langle 0_{n-1}, x'\rangle \leq 0\}$ else
13	0	0	1	1	{(2.7)} if $T_0 \neq \emptyset$ and $\{\langle 0_{n-1}, x'\rangle \leq 0\}$ else
14	0	0	1	0	{(2.7)} if $T_0 \neq \emptyset$ and $\{\langle 0_{n-1}, x'\rangle \leq 0\}$ else
15	0	0	0	1	{(2.7)} if $T_0 \neq \emptyset$ and $\{\langle 0_{n-1}, x'\rangle \leq 0\}$ else
16	0	0	0	0	{(2.7)}

Proof Look at Table 2.1 for the description of the cases.

- Case 1. Observe that the system σ is equivalent to

$$\left\{ \begin{array}{c} \max_{t \in W_-}\{\langle c_t, x'\rangle - d_t\} \leq x_n \leq \min_{t \in W_+}\left\{d_t - \langle c_t, x'\rangle\right\} \\ \langle c_t, x'\rangle \leq d_t,\ t \in W_0 \\ \langle c_t, x'\rangle < d_t,\ t \in S_0 \end{array} \right\}.$$

  Obviously, if $\overline{x} \in F$ then $\overline{x}' \in F'$. Conversely, if $\overline{x}' \in F'$, one has from (2.5) that

$$\max_{t \in W_-}\{\langle c_t, \overline{x}'\rangle - d_t\} \leq \min_{t \in W_+}\{d_t - \langle c_t, \overline{x}'\rangle\}.$$

  By taking $\overline{x}_n \in \left[\max_{t \in W_-}\{\langle c_t, \overline{x}'\rangle - d_t\}, \min_{t \in W_+}\{d_t - \langle c_t, \overline{x}'\rangle\}\right]$, one has that $\overline{x} := \left(\overline{x}', \overline{x}_n\right) \in F$.
- Case 2. Observe that the system σ is equivalent to

$$\left\{ \begin{array}{c} \max_{t \in W_-}\{\langle c_t, x'\rangle - d_t\} \leq x_n \leq \min_{t \in W_+}\{d_t - \langle c_t, x'\rangle\} \\ \langle c_t, x'\rangle \leq d_t,\ t \in W_0 \\ \max_{t \in S_-}\{\langle c_t, x'\rangle - d_t\} < x_n < \min_{t \in S_+}\{d_t - \langle c_t, x'\rangle\} \\ \langle c_t, x'\rangle < d_t,\ t \in S_0 \end{array} \right\}.$$

  Assume that $\overline{x} \in F$, i.e., $\overline{x}$ is a solution of the system above. Obviously, the inequalities in (2.7) hold for $\overline{x}'$. By following the same reasoning as in Case 1, one has that $\overline{x}'$ satisfies the inequalities in (2.5). On the other hand, as

$$\begin{aligned}\max\nolimits_{t\in S_-}\{\langle c_t, x'\rangle - d_t\} &< \min\nolimits_{t\in T_+}\{d_t - \langle c_t, x'\rangle\},\\ \max\nolimits_{t\in T_-}\{\langle c_t, x'\rangle - d_t\} &< \min\nolimits_{t\in S_+}\{d_t - \langle c_t, x'\rangle\},\end{aligned}$$

one easily gets that $\overline{x}'$ satisfies the inequalities in (2.6). Therefore, $\overline{x}' \in F'$.

Assume now that $\overline{x}' \in F'$, i.e., $\overline{x}'$ satisfies the inequalities in (2.5), (2.6) and (2.7), which implies

$$\begin{aligned}\max\nolimits_{t\in W_-}\{\langle c_t, \overline{x}'\rangle - d_t\} &\le \min\nolimits_{t\in W_+}\{d_t - \langle c_t, \overline{x}'\rangle\},\\ \max\nolimits_{t\in S_-}\{\langle c_t, \overline{x}'\rangle - d_t\} &< \min\nolimits_{t\in T_+}\{d_t - \langle c_t, \overline{x}'\rangle\},\\ \max\nolimits_{t\in T_-}\{\langle c_t, \overline{x}'\rangle - d_t\} &< \min\nolimits_{t\in S_+}\{d_t - \langle c_t, \overline{x}'\rangle\}.\end{aligned}$$

As $\max_{t\in S_-}\{\langle c_t, \overline{x}'\rangle - d_t\} \le \max_{t\in T_-}\{\langle c_t, \overline{x}'\rangle - d_t\}$, one has

$$\max_{t\in S_-}\{\langle c_t, \overline{x}'\rangle - d_t\} < \min_{t\in S_+}\{d_t - \langle c_t, \overline{x}'\rangle\}.$$

On the other hand, one easily gets that

$$\begin{aligned}\max\nolimits_{t\in W_-}\{\langle c_t, \overline{x}'\rangle - d_t\} &< \min\nolimits_{t\in S_+}\{d_t - \langle c_t, \overline{x}'\rangle\},\\ \max\nolimits_{t\in S_-}\{\langle c_t, \overline{x}'\rangle - d_t\} &< \min\nolimits_{t\in W_+}\{d_t - \langle c_t, \overline{x}'\rangle\},\end{aligned}$$

which guarantees the nonemptyness of the interval

$$\begin{aligned}\Omega := &\left[\max\nolimits_{t\in W_-}\{\langle c_t, \overline{x}'\rangle - d_t\}, \min\nolimits_{t\in W_+}\{d_t - \langle c_t, \overline{x}'\rangle\}\right]\\ &\bigcap \left]\max\nolimits_{t\in S_-}\{\langle c_t, \overline{x}'\rangle - d_t\}, \min\nolimits_{t\in S_+}\{d_t - \langle c_t, \overline{x}'\rangle\}\right[.\end{aligned}$$

In fact, let $\alpha := \max_{t\in W_-}\{\langle c_t, \overline{x}'\rangle - d_t\}$, $\beta := \min_{t\in W_+}\{d_t - \langle c_t, \overline{x}'\rangle\}$, $\gamma := \max_{t\in S_-}\{\langle c_t, \overline{x}'\rangle - d_t\}$, and $\delta := \min_{t\in S_+}\{d_t - \langle c_t, \overline{x}'\rangle\}$. Since $\alpha < \delta$ and $\alpha \le \beta$, $\alpha \le \min\{\delta, \beta\}$. Similarly, $\gamma < \beta$ and $\gamma < \delta$, so that $\gamma < \min\{\delta, \beta\}$. If $\alpha = \beta$, then $\gamma < \beta = \alpha < \delta$ and $\alpha \in \Omega$. So, we can assume $\alpha < \beta$, in which case $\alpha < \min\{\delta, \beta\}$, $\max\{\alpha, \gamma\} < \min\{\delta, \beta\}$, and

$$\frac{\max\{\alpha, \gamma\} + \min\{\delta, \beta\}}{2} \in \left]\alpha, \beta\right[\cap \left]\gamma, \delta\right[\subset \Omega.$$

By taking $\overline{x}_n \in \Omega$ one has that $\overline{x} := (\overline{x}', \overline{x}_n) \in F$.

- Case 3. It follows by using the same reasoning as in Case 2. Given $\overline{x}' \in F'$, a solution $\overline{x} \in F$ is completed by taking $\overline{x}_n \in \Omega$ with

$$\begin{aligned}\Omega := &\left[\max\nolimits_{t\in W_-}\{\langle c_t, \overline{x}'\rangle - d_t\}, \min\nolimits_{t\in W_+}\{d_t - \langle c_t, \overline{x}'\rangle\}\right]\\ &\bigcap \left]-\infty, \min\nolimits_{t\in S_+}\{d_t - \langle c_t, \overline{x}'\rangle\}\right[.\end{aligned}$$

- Case 4. It follows by using the same reasoning as in Case 2. Given $\overline{x}' \in F'$, a solution $\overline{x} \in F$ is completed by taking $\overline{x}_n \in \Omega$ with

$$\Omega := \left[\max_{t\in W_-}\{\langle c_t, \overline{x}'\rangle - d_t\}, \min_{t\in W_+}\{d_t - \langle c_t, \overline{x}'\rangle\}\right] \\ \bigcap \left]\max_{t\in S_-}\{\langle c_t, \overline{x}'\rangle - d_t\}, +\infty\right[.$$

- Case 5. Observe that the system σ is equivalent to

$$\left\{\begin{array}{c} x_n \leq \min_{t\in W_+}\{d_t - \langle c_t, x'\rangle\} \\ \langle c_t, x'\rangle \leq d_t, t \in W_0 \\ \max_{t\in S_-}\{\langle c_t, x'\rangle - d_t\} < x_n < \min_{t\in S_+}\{d_t - \langle c_t, x'\rangle\} \\ \langle c_t, x'\rangle < d_t, t \in S_0 \end{array}\right\}.$$

  It is easy to see that if $\overline{x} \in F$, then $\overline{x}' \in F'$.

  Assume now that $\overline{x}' \in F'$, i.e., $\overline{x}'$ satisfies the inequalities in (2.6) and (2.7), which implies

$$\max_{t\in S_-}\{\langle c_t, \overline{x}'\rangle - d_t\} < \min_{t\in S_+}\{d_t - \langle c_t, \overline{x}'\rangle\}, \\ \max_{t\in S_-}\{\langle c_t, \overline{x}'\rangle - d_t\} < \min_{t\in T_+}\{d_t - \langle c_t, \overline{x}'\rangle\}.$$

  Consequently, $\max_{t\in S_-}\{\langle c_t, \overline{x}'\rangle - d_t\} < \min_{t\in W_+}\{d_t - \langle c_t, \overline{x}'\rangle\}$, which guarantees that

$$\Omega := \left]-\infty, \min_{t\in W_+}\{d_t - \langle c_t, \overline{x}'\rangle\}\right] \\ \bigcap \left]\max_{t\in S_-}\{\langle c_t, \overline{x}'\rangle - d_t\}, \min_{t\in S_+}\{d_t - \langle c_t, \overline{x}'\rangle\}\right[$$

  is nonempty. By taking $\overline{x}_n \in \Omega$, one has that $\overline{x} \in F$.

- Case 6. It follows by using the same reasoning as in Case 5. Given $\overline{x}' \in F'$, a solution $\overline{x} \in F$ is completed by taking $\overline{x}_n \in \Omega$ with

$$\Omega := \left]-\infty, \min_{t\in W_+}\{d_t - \langle c_t, \overline{x}'\rangle\}\right] \bigcap \left]\max_{t\in S_-}\{\langle c_t, \overline{x}'\rangle - d_t\}, +\infty\right[.$$

- Case 7. It follows by using the same reasoning as in Case 5. Given $\overline{x}' \in F'$, a solution $\overline{x} \in F$ is completed by taking $\overline{x}_n \in \Omega$ with

$$\Omega := \left[\max_{t\in W_-}\{\langle c_t, \overline{x}'\rangle - d_t\}, +\infty\right[\\ \bigcap \left]\max_{t\in S_-}\{\langle c_t, \overline{x}'\rangle - d_t\}, \min_{t\in S_+}\{d_t - \langle c_t, \overline{x}'\rangle\}\right[.$$

- Case 8. It follows by using the same reasoning as in Case 5. Given $\overline{x}' \in F'$, a solution $\overline{x} \in F$ is completed by taking $\overline{x}_n \in \Omega$ with

$$\Omega := \left[\max_{t\in W_-}\{\langle c_t, \overline{x}'\rangle - d_t\}, +\infty\right[\bigcap \left]-\infty, \min_{t\in S_+}\{d_t - \langle c_t, \overline{x}'\rangle\}\right[.$$

- Case 9. It follows by using the same reasoning as in Case 5. Given $\overline{x}' \in F'$, a solution $\overline{x} \in F$ is completed by taking $\overline{x}_n \in \Omega$ with

$$\Omega := \left]\max_{t\in S_-}\{\langle c_t, \overline{x}'\rangle - d_t\}, \min_{t\in S_+}\{d_t - \langle c_t, \overline{x}'\rangle\}\right[.$$

- Case 10. Let $\overline{x} \in F$. If $T_0 \neq \emptyset$ then $\overline{x}' \in F' = \{x' \in \mathbb{R}^{n-1} : \langle c_t, x'\rangle \leq d_t, t \in W_0; \langle c_t, x'\rangle < d_t, t \in S_0\}$, and if $T_0 = \emptyset$ then it is clear that $\overline{x}' \in F' = \mathbb{R}^{n-1}$. Conversely, given $\overline{x}' \in F'$, a solution $\overline{x} \in F$ is completed by taking $\overline{x}_n \in \Omega$ with

$$\Omega := \left]-\infty, \min_{t\in W_+}\{d_t - \langle c_t, \overline{x}'\rangle\}\right] \bigcap \left]-\infty, \min_{t\in S_+}\{d_t - \langle c_t, \overline{x}'\rangle\}\right[.$$

- Case 11. It follows by using the same reasoning as in Case 10. Given $\overline{x}' \in F'$, a solution $\overline{x} \in F$ is completed by taking $\overline{x}_n \in \Omega$ with

$$\Omega := \left]-\infty, \min_{t\in W_+}\{d_t - \langle c_t, \overline{x}'\rangle\}\right].$$

- Case 12. It follows by using the same reasoning as in Case 10. Given $\overline{x}' \in F'$, a solution $\overline{x} \in F$ is completed by taking $\overline{x}_n \in \Omega$ with

$$\Omega := \left]-\infty, \min_{t\in S_+}\{d_t - \langle c_t, \overline{x}'\rangle\}\right[.$$

- Case 13. It follows by using the same reasoning as in Case 10. Given $\overline{x}' \in F'$, a solution $\overline{x} \in F$ is completed by taking $\overline{x}_n \in \Omega$ with

$$\Omega := \left[\max_{t\in W_-}\{\langle c_t, \overline{x}'\rangle - d_t\}, +\infty\right[\bigcap \left]\max_{t\in S_-}\{\langle c_t, \overline{x}'\rangle - d_t\}, +\infty\right[.$$

- Case 14. It follows by using the same reasoning as in Case 10. Given $\overline{x}' \in F'$, a solution $\overline{x} \in F$ is completed by taking $\overline{x}_n \in \Omega$ with

$$\Omega := \left[\max_{t\in W_-}\{\langle c_t, \overline{x}'\rangle - d_t\}, +\infty\right[.$$

- Case 15. It follows by using the same reasoning as in Case 10. Given $\overline{x}' \in F'$, a solution $\overline{x} \in F$ is completed by taking $\overline{x}_n \in \Omega$ with

$$\Omega := \left]\max_{t\in S_-}\{\langle c_t, \overline{x}'\rangle - d_t\}, +\infty\right[.$$

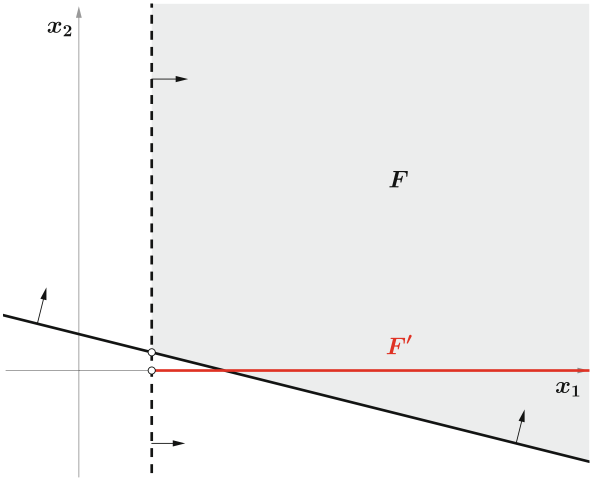


Fig. 2.1 The solution sets of σ and its reduced system σ'

- Case 16. In this case one has $T_0 \neq \emptyset$. It is obvious that $\overline{x} \in F$ implies $\overline{x}' \in F'$. Conversely, given $\overline{x}' \in F'$, a solution $\overline{x} \in F$ is completed by taking any $\overline{x}_n \in \mathbb{R}$. □

We illustrate the Fourier–Motzkin elimination method described in Theorem 2.1 with a simple example.

Example 2.1 Consider the e-polyhedron F in $\mathbb{R}^2$ represented by the system $\sigma = \{-x_1 - 4x_2 \leq -2, -x_1 < -1\}$. We are going to eliminate the variable x_2. For this variable, we have $W_+ = W_0 = S_+ = S_- = \emptyset$, $W_- \neq \emptyset$ and $S_0 \neq \emptyset$, so we are in Case 14, and the reduced system associated to σ is $\sigma' = \{-x_1 < -1\}$. Therefore, the projection of F onto the x_1 axis is $F' =]1, +\infty[$ (see Fig. 2.1).

If we take $\overline{x}_1 \in F'$ and $\overline{x}_2 \in \left[\frac{1}{2} - \frac{1}{4}\overline{x}_1, +\infty\right[$, we have $\overline{x} = (\overline{x}_1, \overline{x}_2) \in F$. This yields the following tomographic description of F as the union of its intersections with vertical lines:

$$F = \bigcup_{x_1 > 1} \left[\frac{1}{2} - \frac{1}{4}x_1, +\infty\right[.$$

According to Theorem 2.1, the orthogonal projection of any e-polyhedron F onto the coordinate hyperplane $x_n = 0$ is an e-polyhedron too. The same applies to the projection of F onto the remaining coordinate hyperplanes after the corresponding adaptation of σ'. By induction, the orthogonal projection of F onto any linear subspace obtained as intersection of coordinate hyperplanes (in particular, the coordinate axis) is an e-polyhedron too.

2.2 Finite Linear Inequality Systems Containing Strict Inequalities

It is well-known that, for an arbitrary ordinary linear system, the characteristic cone introduced by Zhu [193] is a very useful tool to characterize the consistency and the consequent inequalities of the system, among other things. In the particular case of finite ordinary systems, the characteristic cone is finitely generated and, therefore, it is a polyhedral cone, which allows us to simplify the conditions. For non-ordinary linear systems, we have obtained characterizations for consistency (Theorem 1.4) and consequent inequalities (Theorem 1.5) in terms of the characteristic and the moment sets. These conditions can also be simplified when the system is finite, but, in this case, we shall introduce a new dual cone which will allow us to give simpler characterizations for consistency (Theorem 2.2) and consequent inequalities (Theorem 2.3), and will be the fundamental tool to obtain an extension, for e-polyhedra, of the celebrated Motzkin decomposition theorem for polyhedra.

The *representative cone* associated to the system

$$\sigma = \{\langle a_t, x\rangle \leq b_t,\ t \in W;\ \langle a_t, x\rangle < b_t,\ t \in S\}$$

is

$$\mathscr{K}(\sigma) := \operatorname{cone}\left\{ \begin{pmatrix} a_t \\ b_t \\ 0 \end{pmatrix}, t \in W; \begin{pmatrix} a_t \\ b_t \\ -1 \end{pmatrix}, t \in S; \begin{pmatrix} 0_n \\ 1 \\ -1 \end{pmatrix}; \begin{pmatrix} 0_n \\ 0 \\ 1 \end{pmatrix} \right\},$$

whose polar cone is

$$\mathscr{K}(\sigma)^{\circ} = \left\{ (x, y, z) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : \begin{array}{r} \langle a_t, x\rangle + b_t y \leq 0,\ t \in W, \\ \langle a_t, x\rangle + b_t y - z \leq 0,\ t \in S, \\ y - z \leq 0, \\ z \leq 0 \end{array} \right\}.$$

Since $\mathscr{K}(\sigma)$ is a finitely generated cone, by Lemma 2.1(*iii*), it is a polyhedral cone and, therefore, it is closed and convex. The extended Farkas lemma establishes, in this case, that $\mathscr{K}(\sigma)^{\circ\circ} = \mathscr{K}(\sigma)$ (see Sect. 1.6.1).

Here, we also consider the characteristic cone associated to the relaxed system $\overline{\sigma} = \{\langle a_t, x\rangle \leq b_t,\ t \in W \cup S\}$, i.e.,

$$K(\overline{\sigma}) = \operatorname{cone}\left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T; \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\}$$

and its polar cone

$$K(\overline{\sigma})^{\circ} = \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R} : \langle a_t, x\rangle + b_t y \leq 0, t \in W \cup S;\ y \leq 0 \right\}. \tag{2.8}$$

The next result establishes some relations between the representative cone of a finite linear inequality system and the characteristic cone of its associated relaxed system. For that purpose, recall that

$$\operatorname{proj}_{n+2}^{\{1,\ldots,n+1\}}(x,y,z) := (x,y), \quad \forall (x,y,z) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}. \tag{2.9}$$

The reader can find a precise reference for the proof of the next result in Sect. 2.6. The same applies to all the missing proofs in this chapter.

Proposition 2.1 (Comparing Representative and Characteristic Cones) *The following statements hold:*

(*i*) $K(\overline{\sigma}) \times \mathbb{R}_+ \subsetneq \mathscr{K}(\sigma)$.
(*ii*) $\mathscr{K}(\sigma) \subset K(\overline{\sigma}) \times \mathbb{R}$.
(*iii*) $K(\overline{\sigma}) = \operatorname{proj}_{n+2}^{\{1,\ldots,n+1\}}(\mathscr{K}(\sigma))$.
(*iv*) $K(\overline{\sigma})^\circ = \operatorname{proj}_{n+2}^{\{1,\ldots,n+1\}}(\mathscr{K}(\sigma)^\circ)$.

A straightforward consequence of the statements (*i*) and (*ii*) is that, given a scalar $\lambda \in \mathbb{R}_+$, $(a,b,\lambda) \in \mathscr{K}(\sigma)$ if and only if $(a,b) \in K(\overline{\sigma})$, which collapses to

$$\operatorname{proj}_{n+2}^{\{1,\ldots,n+1\}}(\mathscr{K}(\sigma)) = \operatorname{proj}_{n+2}^{\{1,\ldots,n+1\}}(\mathscr{K}(\sigma) \cap (\mathbb{R}^{n+1} \times \{\lambda\})).$$

The inclusion in (*i*) is always strict, as $(0_n, 0, -1) \in \mathscr{K}(\sigma) \setminus (K(\overline{\sigma}) \times \mathbb{R}_+)$, whereas in (*ii*) it can be strict or not, as we can see in the following examples.

Example 2.2 Consider the system $\sigma = \{-x_1 - 4x_2 \leq -2, -x_1 < -1\}$ in $\mathbb{R}^2$ whose solution set F is represented in Fig. 2.1. The unique solution of the linear system

$$\left(-\tfrac{3}{4}, -1, -1, -1\right) = \alpha(-1,-4,-2,0) + \beta(-1,0,-1,-1) + \gamma(0,0,1,-1) + \mu(0,0,0,1),$$

is $\alpha = 1/4$, $\beta = 1/2$, $\gamma = 0$ and $\mu = -1/2 < 0$. This shows that $(-3/4,-1,-1,-1) \notin \mathscr{K}(\sigma)$. However, one has $(-3/4,-1,-1) \in K(\overline{\sigma})$ and so $(-3/4,-1,-1,-1) \in K(\overline{\sigma}) \times \mathbb{R}$.

Example 2.3 Consider the inconsistent system $\sigma = \{x \leq -1, -x < -2\}$ in $\mathbb{R}$ whose representative cone is

$$\mathscr{K}(\sigma) = \operatorname{cone}\left\{\begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}-1\\-2\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}.$$

Since $(0,0,-1) = (1,-1,0) + (-1,-2,-1) + 3(0,1,-1) + 3(0,0,1)$, we have that $(0,0,-1) \in \mathscr{K}(\sigma)$ so, if $(a,b,c) \in K(\overline{\sigma}) \times \mathbb{R}$ with $c < 0$, then

$$(a,b,c) = (a,b,0) - c(0,0,-1) \in \mathscr{K}(\sigma)$$

and, therefore, $\mathscr{K}(\sigma) = K(\overline{\sigma}) \times \mathbb{R}$.

The following result plays a crucial role in both the proof of Theorem 2.2, part $[(i) \Longleftrightarrow (ii)]$, and the decomposition of e-polyhedra (Theorem 2.5).

Proposition 2.2 (Characterizing the Solution Sets via Associated Cones) *Let F and $\overline{F}$ be the solution sets of $\sigma = \{\langle a_t, x\rangle \leq b_t,\ t \in W;\ \langle a_t, x\rangle < b_t,\ t \in S\}$ and its relaxed system $\overline{\sigma} = \{\langle a_t, x\rangle \leq b_t,\ t \in W \cup S\}$, respectively. Then, for any $\overline{x} \in \mathbb{R}^n$, the following statements hold true:*

(i) *$\overline{x} \in \overline{F}$ if and only if $(\overline{x}, -1) \in K(\overline{\sigma})^\circ$.*
(ii) *$\overline{x} \in F$ if and only if $(\overline{x}, -1, \varepsilon) \in \mathscr{K}(\sigma)^\circ$ for some $\varepsilon \in [-1, 0[$.*

Partial Proof (i) By (2.8), $(\overline{x}, -1) \in K(\overline{\sigma})^\circ$ if and only if $\langle a_t, \overline{x}\rangle - b_t \leq 0$, for all $t \in W \cup S$, if and only if $\overline{x} \in \overline{F}$. □

2.2.1 Existence of Solutions

This subsection is devoted to characterize the consistency of finite linear systems possibly containing strict inequalities.

The main result in this subsection provides four characterizations of the consistency of a finite linear system containing strict inequalities in terms of its associated sets.

Theorem 2.2 (Existence for Finite Systems) *For a given system*

$$\sigma = \{\langle a_t, x\rangle \leq b_t,\ t \in W;\ \langle a_t, x\rangle < b_t,\ t \in S\}$$

with $S \neq \emptyset$, the following statements are equivalent:

(i) *σ is consistent.*

(ii) $\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \notin \operatorname{cl}\operatorname{cone}\left(\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in S\right\} + \mathbb{R}_+\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in W\right\}\right)$

and $0_{n+1} \notin \operatorname{conv}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in S\right\} + \operatorname{cone}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in W\right\}$.

(iii) $0_{n+1} \notin \operatorname{conv}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in S;\ \begin{pmatrix} 0_n \\ 1 \end{pmatrix}\right\} + \operatorname{cone}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in W\right\}$.

(iv) $\begin{pmatrix} 0_n \\ 0 \\ -1 \end{pmatrix} \notin \operatorname{cone}\left\{\begin{pmatrix} a_t \\ b_t \\ 0 \end{pmatrix}, t \in W;\ \begin{pmatrix} a_t \\ b_t \\ -1 \end{pmatrix}, t \in S;\ \begin{pmatrix} 0_n \\ 1 \\ -1 \end{pmatrix};\ \begin{pmatrix} 0_n \\ 0 \\ 1 \end{pmatrix}\right\}$.

(v) *$z = 0$ is not a consequence of the system*

$$\left\{\begin{array}{r} \langle a_t, x\rangle + b_t y \leq 0,\ t \in W \\ \langle a_t, x\rangle + b_t y - z \leq 0,\ t \in S \\ y - z \leq 0 \\ z \leq 0 \end{array}\right\}.$$

Sketch of the Proof $[(i) \Longleftrightarrow (ii)]$ It is the finite version of Theorem 1.4. In fact, since σ is a finite system, $N(\overline{\sigma})$ is a closed convex cone (as it is a finitely generated cone) and condition (1.15) reads here

$$\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \notin \operatorname{cl} \operatorname{cone} C(\sigma) = N(\overline{\sigma}). \tag{2.10}$$

Moreover, the set in (1.20) is a polyhedron and, therefore, it is also convex and closed, so that, taking into account (1.21), condition (1.16) can be rewritten as

$$0_{n+1} \notin \operatorname{conv} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, \; t \in S \right\} + \operatorname{cone} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, \; t \in W \right\}. \tag{2.11}$$

$[(ii) \Longleftrightarrow (iii)]$ (iii) is a reformulation of (ii).

$[(iii) \Longleftrightarrow (iv)]$ It is easy to prove the equivalence between the negations of statements (iii) and (iv).

$[(i) \Longleftrightarrow (v)]$ The negation of (v) means that $\mathscr{K}(\sigma)^\circ$ is contained in the hyperplane $x_{n+2} = 0$, and this is equivalent to assert that $F = \emptyset$ by Proposition 2.2(ii). $\square$

The next four results are straightforward consequences of their corresponding extensions in Sect. 1.4.1, called there Motzkin-like and Gordan-like existence theorems, and Rockafellar-like and Carver-like alternative theorems, respectively.

Corollary 2.1 (Motzkin Existence Theorem) *Let*

$$\sigma = \{\langle a_t, x \rangle < 0, \; t \in S; \; \langle a_t, x \rangle \leq 0, \; t \in W; \; \langle a_t, x \rangle = 0, \; t \in E\}$$

be a finite homogeneous system with $S \neq \emptyset$. Then, σ is consistent if and only if

$$0_n \notin \operatorname{conv} \left[\{a_t, \; t \in S\} + \mathbb{R}_+ \{a_t, \; t \in W\} + \mathbb{R} \{a_t, \; t \in E\} \right].$$

Observe that the previous result can also be obtained as a straightforward consequence of Theorem 2.2, part $[(i) \Longleftrightarrow (iii)]$, when $E = \emptyset$.

Corollary 2.2 (Gordan Existence Theorem) *A given finite homogeneous system $\sigma = \{\langle a_t, x \rangle < 0, \; t \in S\}$ is consistent if and only if*

$$0_n \notin \operatorname{conv} \{a_t, \; t \in S\}.$$

Corollary 2.3 (Carver Alternative Theorem) *Let $\sigma = \{\langle a_t, x \rangle < b_t, \; t \in S\}$ be a finite system. Then one and only one of the following alternatives holds:*

(i) *σ is consistent.*
(ii) *There exists $\lambda \in \mathbb{R}_+^{(S)}$ which is legal relative to σ and satisfies*

$$\sum_{t \in S} \lambda_t a_t = 0_n \text{ and } \sum_{t \in S} \lambda_t b_t \leq 0.$$

This result was extended to systems with strict and weak inequalities as follows:

Corollary 2.4 (Rockafellar Alternative Theorem) *Let*

$$\sigma = \{\langle a_t, x\rangle < b_t,\ t \in S;\ \langle a_t, x\rangle \leq b_t,\ t \in W\}$$

be a finite system with $S \neq \emptyset$. Assume that $\{\langle a_t, x\rangle \leq b_t,\ t \in W\}$ is consistent. Then one and only one of the following alternatives holds:

(i) *σ is consistent.*
(ii) *There exists $\lambda \in \mathbb{R}_+^{(T)}$ which is legal relative to σ and satisfies*

$$\sum_{t\in T}\lambda_t a_t = 0_n \ \text{ and } \sum_{t\in T}\lambda_t b_t \leq 0.$$

Corollary 2.5 (Stiemke Existence Theorem I) *The system*

$$\sigma = \left\{\langle a_t, x\rangle = 0,\ t \in W;\ \langle e_j, x\rangle > 0,\ j = 1,\ldots,n\right\},$$

with W finite, is consistent if and only if

$$\operatorname{span}\{a_t, t \in W\} \cap \left(\mathbb{R}_+^n \setminus \{0_n\}\right) = \emptyset. \tag{2.12}$$

Proof According to Theorem 2.2, σ is inconsistent if and only if there exist $\alpha_t \in \mathbb{R}$, $t \in W$, $\beta_j \geq 0,\ j = 1,\ldots,n,\ \gamma \geq 0$ and $\delta \geq 0$ such that

$$\begin{pmatrix} 0_n \\ 0 \\ -1 \end{pmatrix} = \sum_{t\in W}\alpha_t \begin{pmatrix} a_t \\ 0 \\ 0 \end{pmatrix} + \sum \beta_j \begin{pmatrix} e_j \\ 0 \\ -1 \end{pmatrix} + \gamma \begin{pmatrix} 0_n \\ 1 \\ -1 \end{pmatrix} + \delta \begin{pmatrix} 0_n \\ 0 \\ 1 \end{pmatrix},$$

which is equivalent to the negation of (2.12). □

Corollary 2.6 (Stiemke Existence Theorem II) *Given a finite set $\{a_t,\ t \in W\}$, the following statements are equivalent:*

(i) $\{\langle a_t, x\rangle \leq 0,\ t \in W\}$ *has a solution $\overline{x}$ such that $\langle a_t, \overline{x}\rangle \neq 0$ for some $t \in W$.*
(ii) *The system* $\sigma = \left\{\langle a_t, x\rangle \leq 0,\ t \in W;\ \langle \sum_{t\in W} a_t, x\rangle < 0\right\}$ *is consistent.*
(iii) $\operatorname{cone}\{a_t,\ t \in W\}$ *is not a linear subspace of* $\mathbb{R}^n$.
(iv) $0_n \notin \operatorname{rint}\operatorname{conv}\{a_t,\ t \in W\}$.

Proof $[(i) \Longleftrightarrow (ii)]$ It is trivial.

$[(ii) \Longleftrightarrow (iii)]$ We shall do the proof by contraposition. By the equivalence $[(i) \Longleftrightarrow (iv)]$ in Theorem 2.2 applied to system σ, σ is inconsistent if and only if

$$\begin{pmatrix} 0_n \\ 0 \\ -1 \end{pmatrix} \in \operatorname{cone}\left\{ \begin{pmatrix} a_t \\ 0 \\ 0 \end{pmatrix}, t \in W; \begin{pmatrix} \sum_{t\in W} a_t \\ 0 \\ -1 \end{pmatrix}; \begin{pmatrix} 0_n \\ 1 \\ -1 \end{pmatrix}; \begin{pmatrix} 0_n \\ 0 \\ 1 \end{pmatrix} \right\},$$

which is equivalent to the existence of $\lambda \in \mathbb{R}_+^W$, $\eta, \mu, \delta \in \mathbb{R}_+$ such that

$$\begin{pmatrix} 0_n \\ 0 \\ -1 \end{pmatrix} = \sum_{t\in W} \lambda_t \begin{pmatrix} a_t \\ 0 \\ 0 \end{pmatrix} + \eta \begin{pmatrix} \sum_{t\in W} a_t \\ 0 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 0_n \\ 1 \\ -1 \end{pmatrix} + \delta \begin{pmatrix} 0_n \\ 0 \\ 1 \end{pmatrix}. \tag{2.13}$$

From (2.13), we obtain that $\mu = 0$, $\eta = \delta + 1 > 0$ and

$$-\sum_{t\in W} a_t = \sum_{t\in W} \frac{\lambda_t}{\eta} a_t \in \operatorname{cone}\{a_t,\ t \in W\}.$$

Thus, σ is inconsistent if and only if $-\sum_{t\in W} a_t \in \operatorname{cone}\{a_t,\ t \in W\}$.

It remains to prove that $-\sum_{t\in W} a_t \in \operatorname{cone}\{a_t,\ t \in W\}$ if and only if $\operatorname{cone}\{a_t,\ t \in W\}$ is a linear subspace.

If $\operatorname{cone}\{a_t,\ t \in W\}$ is a linear subspace, then $-\sum_{t\in W} a_t \in \operatorname{span}\{a_t,\ t \in W\} = \operatorname{cone}\{a_t,\ t \in W\}$. Conversely, assume that $-\sum_{t\in W} a_t \in \operatorname{cone}\{a_t,\ t \in W\}$, and let $x \in \operatorname{span}\{a_t,\ t \in W\}$. Then, there exist $\alpha_t \in \mathbb{R}$, $t \in W$, such that $x = \sum_{t\in W} \alpha_t a_t$. Let $\delta := \min_{t\in W} \alpha_t$. If $\delta \geq 0$, $x \in \operatorname{cone}\{a_t,\ t \in W\}$ and we are done. So we assume $\delta < 0$. Since $\alpha_t + |\delta| \geq 0$ for all $t \in W$,

$$x + |\delta| \sum_{t\in W} a_t = \sum_{t\in W} (\alpha_t + |\delta|)\, a_t \in \operatorname{cone}\{a_t,\ t \in W\}$$

and, so,

$$x \in -|\delta| \sum_{t\in W} a_t + \operatorname{cone}\{a_t,\ t \in W\} \subset \operatorname{cone}\{a_t,\ t \in W\},$$

showing that $\operatorname{cone}\{a_t,\ t \in W\}$ is a linear subspace.

$[(iii) \Longleftrightarrow (iv)]$ We now prove that $0_n \in \operatorname{rint}\operatorname{conv}\{a_t,\ t \in W\}$ if and only if $\operatorname{cone}\{a_t,\ t \in W\}$ is a linear subspace.

If $0_n \in \operatorname{rint}\operatorname{conv}\{a_t,\ t \in W\}$, by [148, Th. 6.9], there exist $\lambda_t > 0,\ t \in W$, such that $\sum_{t \in W} \lambda_t a_t = 0_n$ and $\sum_{t \in W} \lambda_t = 1$. Given $0 < \varepsilon < \min_{t \in W} \lambda_t$, $-\varepsilon \sum_{t \in W} a_t = \sum_{t \in W} (\lambda_t - \varepsilon)\, a_t \in \operatorname{cone}\{a_t,\ t \in W\}$, so that $-\sum_{t \in W} a_t \in \operatorname{cone}\{a_t,\ t \in W\}$.

Conversely, assume that $\operatorname{cone}\{a_t,\ t \in W\}$ is a linear subspace and let $\{u_1, \ldots, u_d\}$ be a basis of $\operatorname{span}\{a_t,\ t \in W\}$. Since

$$\{\pm u_i, i = 1, \ldots, d\} \subset \operatorname{span}\{a_t,\ t \in W\} = \operatorname{cone}\{a_t,\ t \in W\},$$

all vectors of $\{\pm u_i, i = 1, \ldots, d\}$ are non-negative linear combination of the vectors of $\{a_t,\ t \in W\}$, with not all coefficients equal to zero as $0_n \notin \{\pm u_i, i = 1, \ldots, d\}$. Dividing each vector of $\{\pm u_i, i = 1, \ldots, d\}$ by the sum of the coefficients of $\{a_t,\ t \in W\}$ in the corresponding linear combination, we obtain that 0_n belongs to the convex hull of a set of positive multiples of the vectors $\{\pm u_i, i = 1, \ldots, d\}$, each one belonging to $\operatorname{conv}\{a_t,\ t \in W\}$. So, there exists a polytopal neighborhood of 0_n, for the topology induced by the Euclidean one in $\operatorname{span}\{a_t,\ t \in W\}$, whose extreme points belong to $\operatorname{conv}\{a_t,\ t \in W\}$. Thus, $0_n \in \operatorname{rint}\operatorname{conv}\{a_t,\ t \in W\}$ and we are done. $\square$

2.2.2 *Consequent Inequalities*

In this subsection we provide dual characterizations of linear inequalities which are consequences of a consistent finite linear system

$$\sigma = \{\langle a_t, x\rangle \le b_t,\ t \in W;\ \langle a_t, x\rangle < b_t,\ t \in S$$

in terms of its associated representative cone

$$\mathcal{K}(\sigma) = \operatorname{cone}\left\{\begin{pmatrix} a_t \\ b_t \\ 0 \end{pmatrix}, t \in W; \begin{pmatrix} a_t \\ b_t \\ -1 \end{pmatrix}, t \in S; \begin{pmatrix} 0_n \\ 1 \\ -1 \end{pmatrix}; \begin{pmatrix} 0_n \\ 0 \\ 1 \end{pmatrix}\right\}.$$

Theorem 2.3 (Characterization of Consequent Inequalities of σ) *Assume that $\sigma = \{\langle a_t, x\rangle \le b_t,\ t \in W;\ \langle a_t, x\rangle < b_t,\ t \in S\}$ is consistent and suppose that $(a, b) \in \mathbb{R}^n \times \mathbb{R}$. Then, the following statements hold:*

(*i*) *$\langle a, x\rangle \le b$ is a consequence of σ if and only if $(a, b, 0) \in \mathcal{K}(\sigma)$.*
(*ii*) *$\langle a, x\rangle < b$ is a consequence of σ if and only if $(a, b, c) \in \mathcal{K}(\sigma)$ for some $c < 0$.*
(*iii*) *$\langle a, x\rangle < b$ is a consequence of σ if and only if $\langle a, x\rangle < b$ is a legal linear combination of $\sigma \cup \{\langle 0_n, x\rangle < 1\}$.*

Partial Proof (*iii*) Consider the two conditions in Theorem 1.5(*ii*) adapted to the case when σ is finite. In such a case, condition (1.28) can be reformulated as $\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \in N(\overline{\sigma}) - \mathbb{R}_+\begin{pmatrix} a \\ b \end{pmatrix}$, where the latter cone is polyhedral, whereas the set

$\operatorname{conv} C(\sigma) - \mathbb{R}_+ \begin{pmatrix} a \\ b \end{pmatrix}$ can be written as

$$\operatorname{conv}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in S\right\} + \operatorname{cone}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in W;\ -\begin{pmatrix} a \\ b \end{pmatrix}\right\},$$

which is also a polyhedron by Lemma 2.1(i). So, according to Theorem 1.5, $\langle a, x\rangle < b$ is consequence of a consistent finite system σ if and only if $\langle a, x\rangle < b$ is a legal linear combination of either σ or $\overline{\sigma} \cup \{\langle 0_n, x\rangle < 1\}$. The latter statement is a reformulation of statement (iii). □

Although the negative value c in Theorem 2.3(ii) is not unique (as $(0_n, 0, 1) \in 0^+(\mathscr{K}(\sigma))$), an arbitrary $c < 0$ is not valid as it is shown in the following example.

Example 2.4 Let σ be the system in Example 2.1. The weak inequality $-x_1 - x_2 \leq -1$ is a consequence of σ (we can see in Fig. 2.2 that it is satisfied by any solution of σ). By solving the linear system

$$(-1, -1, -1, 0) = \alpha(-1, -4, -2, 0) + \beta(-1, 0, -1, -1) + \gamma(0, 0, 1, -1) + \mu(0, 0, 0, 1),$$

we obtain $\alpha = \frac{1}{4}$, $\beta = \frac{3}{4}$, $\gamma = \frac{1}{4}$ and $\mu = 1$. Therefore, we have that $(-1, -1, -1, 0) \in \mathscr{K}(\sigma)$, as stated in Theorem 2.3(i).

In the same way, the strict inequality $-\frac{3}{4}x_1 - x_2 < -1$ is a consequence of σ (see the green line in Fig. 2.2), but $(-3/4, -1, -1, -1) \notin \mathscr{K}(\sigma)$ (see Example 2.2). Nevertheless, $(-3/4, -1, -1, -1/2) \in \mathscr{K}(\sigma)$ and so the characterization in Theorem 2.3(ii) holds.

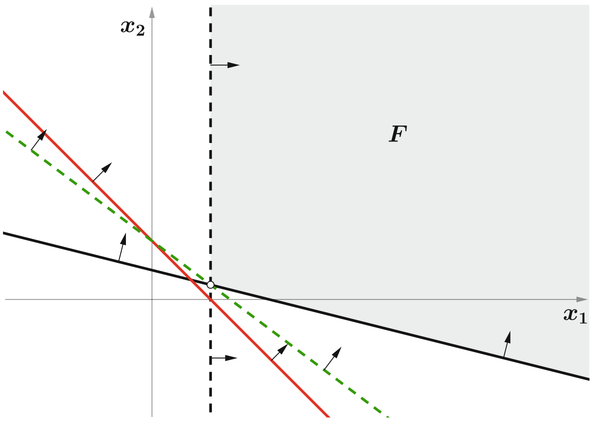


Fig. 2.2 The consequent inequalities $-x_1 - x_2 \leq -1$ (in red) and $-\frac{3}{4}x_1 - x_2 < -1$ (in green)

Observe that an inequality $\langle a, x\rangle < b$ is a consequence of σ if and only if the system $\sigma \cup \{\langle -a, x\rangle \leq -b\}$ is inconsistent. Thus, by applying the equivalence $[(i) \Longleftrightarrow (iii)]$ of Theorem 2.2, one has that $\langle a, x\rangle < b$ is a consequence of σ if and only if

$$0_{n+1} \in \operatorname{conv}\{(a_t, b_t), t \in S; (0_n, 1)\} + \operatorname{cone}\{(a_t, b_t), t \in W; -(a, b)\}.$$

Corollary 2.7 (Characterization of Consequent Inequalities of $\overline{\sigma}$) *Let σ, with relaxed system $\overline{\sigma}$, and (a, b) be as in Theorem 2.3. Then, the following statements hold:*

- (i) *$\langle a, x\rangle \leq b$ is a consequence of $\overline{\sigma}$ if and only if $(a, b) \in K(\overline{\sigma})$.*
- (ii) *$\langle a, x\rangle < b$ is a consequence of $\overline{\sigma}$ if and only if $(a, b) \in K(\overline{\sigma})$ and $(0_n, -1) \in K(\overline{\sigma}) + \operatorname{span}\{(a, b)\}$.*

Sketch of the Proof

- (i) According to Theorem 1.5(i), a weak inequality $\langle a, x\rangle \leq b$ is a consequence of σ if and only if it is a consequence of $\overline{\sigma}$. Moreover, by Proposition 2.1, statements (i) and (ii) , $(a, b, 0) \in \mathscr{K}(\sigma)$ if and only if $(a, b) \in K(\overline{\sigma})$. So, one gets (i) from statement Theorem 2.3(i).
- (ii) Taking into account statement (ii) in Theorem 2.3, one has to prove

$$\exists\, c < 0 : \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathscr{K}(\overline{\sigma}) \Longleftrightarrow \left\{ \begin{array}{l} \begin{pmatrix} a \\ b \end{pmatrix} \in K(\overline{\sigma}), \text{ and} \\ \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \in K(\overline{\sigma}) + \operatorname{span}\left\{\begin{pmatrix} a \\ b \end{pmatrix}\right\}. \end{array}\right.$$

The direct implication is immediate. For the converse, if

$$\begin{pmatrix} a \\ b \end{pmatrix} = \sum_{t\in T} \lambda_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \mu \begin{pmatrix} 0_n \\ 1 \end{pmatrix}, \tag{2.14}$$

with $\lambda_t \geq 0$, for all $t \in T$ and $\mu \geq 0$, and

$$\begin{pmatrix} 0_n \\ -1 \end{pmatrix} = \sum_{t\in T} \gamma_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \delta \begin{pmatrix} 0_n \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} a \\ b \end{pmatrix}, \tag{2.15}$$

with $\gamma_t \geq 0$, for all $t \in T$, $\delta \geq 0$ and $\alpha \in \mathbb{R}$, we have the following cases:

Case 1: If $\mu > 0$, taking $c = -\mu < 0$, we obtain from (2.14) that $(a, b, c) \in \mathscr{K}(\sigma)$.

Case 2: If $\mu = 0$ and $\alpha \geq 0$, replacing (2.14) in (2.15), we obtain that $(0_n, -1) \in K(\overline{\sigma})$, which is a contradiction.

Case 3: If $\mu = 0$ and $\alpha < 0$, taking $c = \frac{\gamma+1}{\alpha} < 0$, we obtain from (2.15) that $(a, b, c) \in \mathscr{K}(\sigma)$.

$\square$

2.2.3 Set Containment of Evenly Convex Polyhedra

As every e-polyhedron is an e-convex set, one may apply the dual characterization stated in Proposition 1.9 to the set containment of e-polyhedra by means of their strict dual cones. Furthermore, since every e-polyhedron F is the solution set of a consistent system σ as in (2.1), by the statement $(i) \iff (ii)$ in Theorem 2.3, we have

$$F \subset \left\{x \in \mathbb{R}^n : \langle a, x\rangle < b\right\} \iff \begin{pmatrix} a \\ b \end{pmatrix} \in \operatorname{proj}_{n+2}^{\{1,\ldots,n+1\}}(\mathscr{K}(\sigma) \cap (\mathbb{R}^{n+1} \times \mathbb{R}_{--})),$$

so we obtain the following expression for the strict dual cone of F in terms of the data of σ and the projection mapping defined in (2.9):

$$K_F^{<} = \operatorname{proj}_{n+2}^{\{1,\ldots,n+1\}}(\mathscr{K}(\sigma) \cap (\mathbb{R}^{n+1} \times \mathbb{R}_{--})). \tag{2.16}$$

The following result provides a characterization for the set containment of e-polyhedra in terms of the representative cones of their external representations.

Proposition 2.3 (Dual Characterization of the Containment of e-Polyhedra) *Let F_1 and F_2 be solution sets of the finite systems σ_1 and σ_2, respectively. Then $F_1 \subset F_2$ if and only if*

$$\operatorname{proj}_{n+2}^{\{1,\ldots,n+1\}}(\mathscr{K}(\sigma_2) \cap (\mathbb{R}^{n+1} \times \mathbb{R}_{--})) \subset \operatorname{proj}_{n+2}^{\{1,\ldots,n+1\}}(\mathscr{K}(\sigma_1) \cap (\mathbb{R}^{n+1} \times \mathbb{R}_{--})).$$

Consequently, $F_1 = F_2$ if and only if

$$\operatorname{proj}_{n+2}^{\{1,\ldots,n+1\}}(\mathscr{K}(\sigma_1) \cap (\mathbb{R}^{n+1} \times \mathbb{R}_{--})) = \operatorname{proj}_{n+2}^{\{1,\ldots,n+1\}}(\mathscr{K}(\sigma_2) \cap (\mathbb{R}^{n+1} \times \mathbb{R}_{--})).$$

Observe that if σ_1 and σ_2 are finite systems defining F_1 and F_2, respectively, by applying Proposition 1.9, $\mathscr{K}(\sigma_2) \subset \mathscr{K}(\sigma_1)$ entails $F_1 \subset F_2$. The converse is not true, as the next example shows.

Example 2.5 Let σ_1 be the consistent system in Example 2.1, and consider $\sigma_2 = \{-\frac{3}{4}x_1 - x_2 < -1\}$, the system whose solution set is the open halfspace (not containing the origin) determined by the green line in Fig. 2.2. Obviously $F_1 \subset F_2$, whereas $\mathscr{K}(\sigma_2) \not\subset \mathscr{K}(\sigma_1)$ as $(-\frac{3}{4}, -1, -1, -1) \in \mathscr{K}(\sigma_2) \setminus \mathscr{K}(\sigma_1)$ (see Example 2.2). However, if we consider $\sigma_2^\star = \{-3x_1 - 4x_2 < -4\}$, which is a different external representation of F_2, we have $\mathscr{K}(\sigma_2^\star) \subset \mathscr{K}(\sigma_1)$, since

$$(-3, -4, -4, -1) = (-1, -4, -2, 0) + 2(-1, 0, -1, -1) + (0, 0, 0, 1) \in \mathscr{K}(\sigma_1).$$

2.3 Double Description of Evenly Convex Polyhedra

H. Weyl proved in 1935 [188] that the (Minkowski) sum of a polytope and a finitely generated cone is a polyhedron and, a year later, Motzkin [131] proved that, conversely, every polyhedron can be expressed as the sum of a polytope and a finitely generated cone. So, any polyhedron admits an internal representation (as the sum of a polytope with a finitely generated convex cone) and an external representation (as solution set of an ordinary finite linear system). It is easy to get an external representation from a given internal one by means of the Fourier–Motzkin elimination method, while it is possible to get an internal representation from an external one from the internal representation of the characteristic cone.

The e-polyhedra also admit a double representation. In this section we first define the internal representations of e-polyhedra, from which it is possible to get an external representation (a finite linear system) by means of the Fourier–Motzkin elimination method described in Theorem 2.1. Then, we show how to obtain an internal representation of an e-polyhedron by using the representative cone associated with an external representation of such an e-polyhedron.

We start by extending the notion of polytope. A subset $Q \subset \mathbb{R}^n$ is said to be an *evenly convex polytope* (*e-polytope*, in brief) if there are two finite sets $U := \{u_1, \ldots, u_m\}$ and $V := \{v_1, \ldots, v_q\}$ in $\mathbb{R}^n$ such that

$$Q = \left\{ \sum_{i=1}^{m} \lambda_i u_i + \sum_{j=1}^{q} \mu_j v_j : \lambda \in \mathbb{R}_+^m, \mu \in \mathbb{R}_+^q, \sum_{i=1}^{m} \lambda_i + \sum_{j=1}^{q} \mu_j = 1, \sum_{i=1}^{m} \lambda_i > 0 \right\}. \tag{2.17}$$

In short, we shall denote $Q = \operatorname{conv}(U : V)$.

Each point of Q can be thought as a kind of legal convex combination, that is, a convex combination of finitely many points in $U \cup V$ with a positive weight associated to at least one point in U. It easily follows from the definition that Q contains the set U, but it does not contain V necessarily. Additionally, one has $\operatorname{conv}(U : V) \neq \operatorname{conv}(V : U)$ In the particular case, in general, although $\operatorname{conv}(U : \emptyset) = \operatorname{conv}(\emptyset : U) = \operatorname{conv} U$. Thus, if either U or V are empty, then Q becomes a polytope in the classical sense.

Proposition 2.4 (Topological Properties of e-Polytopes) *Let* $Q = \operatorname{conv}(U : V)$ *be an e-polytope as in (2.17). Then,*

(i) $\operatorname{cl} Q = \operatorname{conv}(U \cup V)$.
(ii) *Q is closed if and only if $V \subset Q$.*
(iii) *Q is relatively open if and only if $U \subset \operatorname{rint} Q$.*

The two results below establish, conjointly, an evenly polyhedral counterpart for Lemma 2.1(i). The first theorem is a generalization of the well-known Weyl Theorem and its proof provides a method for obtaining an external representation from the internal one for an e-polyhedron. We illustrate this method in Example 2.6.

Theorem 2.4 (Extended Weyl Theorem) *Let $U, V, D \subset \mathbb{R}^n$ be finite sets such that $U \cup V \neq \emptyset$. Then, the Minkowski sum* $\operatorname{conv}(U : V) + \operatorname{cone} D$ *is an e-polyhedron.*

Sketch of the Proof Let $U = \{u_i, i \in I\}$, $V = \{v_j, j \in J\}$, and $D = \{d_k, k \in K\}$, with I, J and K finite. If $x \in \operatorname{conv}(U : V) + \operatorname{cone} D$, it can be written as

$$x = \sum_{i \in I} \lambda_i u_i + \sum_{j \in J} \mu_j v_j + \sum_{k \in K} \alpha_k d_k, \tag{2.18}$$

with

$$\sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j = 1, \tag{2.19}$$

$$\sum_{i \in I} \lambda_i > 0, \tag{2.20}$$

and

$$\lambda_i \geq 0, \mu_j \geq 0, \alpha_k \geq 0, \forall i \in I, \forall j \in J, \forall k \in K. \tag{2.21}$$

Eliminating the parameters $\{\lambda_i, i \in I\}$, $\left\{\mu_j, j \in J\right\}$ and $\{\alpha_k, k \in K\}$ in the finite system {(2.18), (2.19), (2.20), and (2.21)} one gets a linear system with unknowns $x_1, \ldots, x_n$ which is the aimed external representation of $\operatorname{conv}(U : V) + \operatorname{cone} D$. □

In the particular case when $D = \emptyset$, Theorem 2.4 establishes that the e-polytope $\operatorname{conv}(U : V)$ is an e-polyhedron. Moreover, since $\operatorname{cl} \operatorname{conv}(U : V) = \operatorname{conv}(U \cup V)$, Theorem 1.1(*ii*) allows to express the e-polytope $\operatorname{conv}(U : V)$ as the result of eliminating from the polytope $\operatorname{conv}(U \cup V)$ the union of a certain family of its faces.

Example 2.6 Consider $F = \operatorname{conv}(U : V) + \operatorname{cone} D$ in $\mathbb{R}^2$ described by the sets $U = \{(1, 0)\}$, $V = \{(4, 0), (4, 3)\}$ and $D = \{(1, 0), (2, 1)\}$ (see Figs. 2.3 and 2.4).

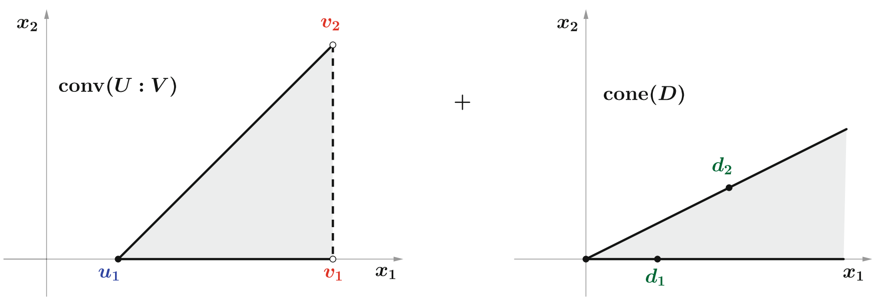


Fig. 2.3 A decomposition of F

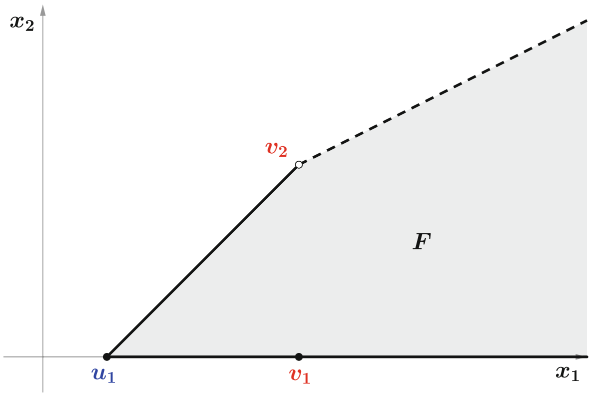


Fig. 2.4 An internal representation of F (drawing the points of U and V in blue and red, respectively)

Then, a given vector $(x_1, x_2) \in \mathbb{R}^2$ belongs to F if and only if

$$(x_1, x_2) = \lambda(1, 0) + \mu_1 (4, 0) + \mu_2(4, 3) + \alpha_1(1, 0) + \alpha_2(2, 1),$$

with $\lambda + \mu_1 + \mu_2 = 1$, $\lambda > 0$, $\mu \in \mathbb{R}^2_+$ and $\alpha \in \mathbb{R}^2_+$, or equivalently, the system

$$\begin{array}{llllll} \lambda & +4\mu_1 & +4\mu_2 & +\alpha_1 & +2\alpha_2 & = x_1, \\ & & +3\mu_2 & & +\alpha_2 & = x_2, \\ \lambda & +\mu_1 & +\mu_2 & & & = 1, \\ \lambda & & & & & > 0, \\ & & & \mu_j & & \geq 0, \quad \forall j = 1, 2, \\ & & & \alpha_l & & \geq 0, \quad \forall l = 1, 2, \end{array}$$

has a solution for $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}^2$. By applying the Gauss method on the three equalities, we obtain $\lambda = -\frac{1}{3}x_1 + \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2 + \frac{4}{3}$, $\mu_1 = \frac{1}{3}x_2 - \frac{1}{3}\alpha_2$ and $\mu_2 = \frac{1}{3}x_1 - \frac{1}{3}x_2 - \frac{1}{3}\alpha_1 - \frac{1}{3}\alpha_2 - \frac{1}{3}$. Now, by replacing these three variables in the inequalities of the system, we have

$$\begin{array}{lllll} -x_1 & & +\alpha_1 & +2\alpha_2 & > -4, \\ & x_2 & & -\alpha_2 & \geq 0, \\ x_1 & -x_2 & -\alpha_1 & -\alpha_2 & \geq 1, \\ & & & \alpha_1 & \geq 0, \\ & & & \alpha_2 & \geq 0. \end{array}$$

Finally, by applying the Fourier–Motzkin method twice in order to eliminate successively the variables α_1 and α_2, one gets the following external representation of F (see Fig. 2.5)

$$\sigma = \{-x_1 + 2x_2 < 2, -x_2 \leq 0, -x_1 + x_2 \leq -1\}. \tag{2.22}$$

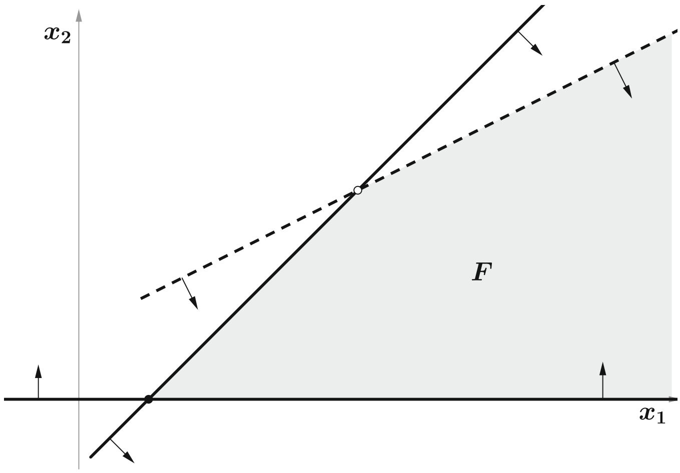


Fig. 2.5 An external representation of F

Although the following result is [152, Th. 4.2], we reproduce a sketch of its proof in order to show how to obtain an internal representation of an e-polyhedron from an external one. The decomposition of an e-polyhedron provided by this theorem coincides with the one given by the classical Motzkin theorem when the e-polyhedron is closed.

Theorem 2.5 (Extended Motzkin Theorem) *Let the e-polyhedron $F \subset \mathbb{R}^n$ be the solution set of $\sigma = \{\langle a_t, x\rangle \leq b_t,\ t \in W;\ \langle a_t, x\rangle < b_t,\ t \in S\}$, with W and S finite. Then, F is the Minkowski sum of an e-polytope and a finitely generated convex cone.*

Sketch of the Proof Let F be the solution set of σ. Since $\mathscr{K}(\sigma)$ is a finitely generated cone, it is also a polyhedral cone (as a consequence of Lemma 2.1), that is,

$$\mathscr{K}(\sigma) = \{(x, y, z) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : \langle(\chi_p, \psi_p, \omega_p), (x, y, z)\rangle \leq 0,\ p = 1, \ldots, s\}.$$

This means that $\mathscr{K}(\sigma) = \left[\text{cone}\{(\chi_p, \psi_p, \omega_p),\ p = 1, \ldots, s\}\right]^\circ$, and so, by taking polars,

$$\mathscr{K}(\sigma)^\circ = \text{cone}\{(\chi_p, \psi_p, \omega_p),\ p = 1, \ldots, s\}.$$

We may assume without loss of generality that $\mathscr{K}(\sigma)^\circ$ has just three kind of generators such that

$$\begin{aligned}\mathscr{K}(\sigma)^\circ = \operatorname{cone}\{&(u_i, -1, \varepsilon_i),\ i = 1, \ldots, m;\\ &(v_j, -1, 0),\ j = 1, \ldots, q;\\ &(d_l, 0, 0),\ l = 1, \ldots, r\}\end{aligned}$$

with $-1 \leq \varepsilon_i < 0$ for all $i = 1, \ldots, m$, $m + q + r = s$ and $m \geq 1$.

If $x \in F$, by Proposition 2.2(*ii*), there exists $\varepsilon \in [-1, 0[$ such that $(x, -1, \varepsilon) \in \mathscr{K}(\sigma)^\circ$ and, so, there exist $\lambda \in \mathbb{R}^m_+$, $\mu \in \mathbb{R}^q_+$ and $\alpha \in \mathbb{R}^r_+$ such that

$$(x, -1, \varepsilon) = \sum_{i=1}^{m} \lambda_i (u_i, -1, \varepsilon_i) + \sum_{j=1}^{q} \mu_j (v_j, -1, 0) + \sum_{l=1}^{r} \alpha_l (d_l, 0, 0).$$

Thus, one has

$$x = \sum_{i=1}^{m} \lambda_i u_i + \sum_{j=1}^{q} \mu_j v_j + \sum_{l=1}^{r} \alpha_l d_l$$

with $\sum_{i=1}^{m} \lambda_i + \sum_{j=1}^{q} \mu_j = 1$ and $\sum_{i=1}^{m} \lambda_i \geq \sum_{i=1}^{m} -\varepsilon_i \lambda_i = -\varepsilon > 0$. Taking $U := \{u_1, \ldots, u_m\}$, $V := \{v_1, \ldots, v_q\}$ and $D := \{d_1, \ldots, d_r\}$, we finally have

$$F = \operatorname{conv}(U : V) + \operatorname{cone} D.$$

□

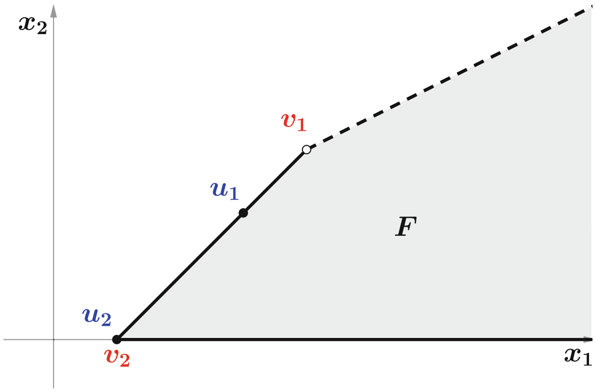
Example 2.7 Consider the e-polyhedron F in $\mathbb{R}^2$ defined by the system σ in (2.22). The representative cone $\mathscr{K}(\sigma)$ is the cone finitely generated by $(0, -1, 0, 0)$, $(-1, 1, -1, 0)$, $(-1, 2, 2, -1)$, $(0, 0, 1, -1)$ and $(0, 0, 0, 1)$. Hence, a given vector $(x_1, x_2, y, z) \in \mathbb{R}^4$ belongs to $\mathscr{K}(\sigma)$ if and only if the linear system

$$\begin{array}{rrrrrl} & -\lambda_2 & -\lambda_3 & & & = x_1,\\ -\lambda_1 & +\lambda_2 & +2\lambda_3 & & & = x_2,\\ & -\lambda_2 & +2\lambda_3 & +\lambda_4 & & = y,\\ & & -\lambda_3 & -\lambda_4 & +\lambda_5 & = z,\\ & & & & \lambda_i & \geq 0, \qquad \forall i = 1, \ldots, 5,\end{array}$$

has a solution for $\lambda \in \mathbb{R}^5$. By applying the Gauss-Fourier method in order to eliminate the variables λ_i, one gets the following external representation of $\mathscr{K}(\sigma)$:

$$\begin{array}{lll} 3x_1 + 2x_2 - y - z \leq 0, & 4x_1 + 3x_2 - y \leq 0, & x_1 \leq 0,\\ x_1 - y - z \leq 0, & x_1 - y \leq 0, & 2x_1 + x_2 \leq 0.\end{array}$$

Fig. 2.6 A different internal representation of F



Thus, according to the above method, an internal representation of F is given by $\operatorname{conv}(U : V) + \operatorname{cone} D$ (see Fig. 2.6) where

$$\begin{aligned} U &= \{(3, 2), (1, 0)\}, \\ V &= \{(4, 3), (1, 0)\}, \\ D &= \{(1, 0), (2, 1)\}. \end{aligned}$$

Corollary 2.8 *An e-polyhedron is bounded if and only if it is an e-polytope.*

Proof Given the e-polytope $\operatorname{conv}(U : V)$, by Proposition 2.4($i$), we know that $\operatorname{cl} \operatorname{conv}(U : V)$ is the polytope $\operatorname{conv}(U \cup V)$. Then, we have

$$0^+ (\operatorname{conv}(U : V)) = 0^+ (\operatorname{conv}(U \cup V)) = \{0_n\},$$

that is, every e-polytope is a bounded e-polyhedron.

Conversely, if F is a bounded polyhedron, then $0^+ F = \{0_n\}$. By applying Theorem 2.5, $F = \operatorname{conv}(U : V) + \operatorname{cone} D$ with $U, V, D \subset \mathbb{R}^n$ finite sets such that $U \cup V \neq \emptyset$. Finally, since $\operatorname{conv}(U : V)$ is a bounded e-polyhedron and $\operatorname{cone} D$ is a polyhedral cone such that $0^+ (\operatorname{conv}(U : V)) \cap 0^+ (\operatorname{cone} D) = \{0_n\}$, then Proposition 1.2($iv$) yields

$$0^+ F = 0^+ (\operatorname{conv}(U : V)) + 0^+ (\operatorname{cone} D) = \operatorname{cone} D, \tag{2.23}$$

and so $F = \operatorname{conv}(U : V)$ is an e-polytope. $\square$

The next result shows that the same operations which are closed in the class of e-convex sets (recall Proposition 1.1) are closed in the class of e-polyhedral, but here the image by linear mappings and the sum also preserve the e-polyhedrality.

Proposition 2.5 (Operations with e-Polyhedra) *The following statements hold:*

(i) *If* $F \subset \mathbb{R}^n$ *is an e-polyhedron, then* αF *(resp.,* $F + v$*) is an e-polyhedron for all* $\alpha \in \mathbb{R}$ *(resp.,* $v \in \mathbb{R}^n$*).*
(ii) *If* $F \subset \mathbb{R}^n$ *is an e-polyhedron and* $A : \mathbb{R}^m \to \mathbb{R}^n$ *is a linear transformation, then* AF *is an e-polyhedron. Moreover, if* $A^{-1}F \neq \emptyset$*, then* $A^{-1}F$ *is an e-polyhedron and* $0^+(A^{-1}F) = A^{-1}\left(0^+F\right)$.
(iii) *If* $F_1 \subset \mathbb{R}^n$ *and* $F_2 \subset \mathbb{R}^m$ *are nonempty sets, then* $F_1 \times F_2$ *is e-polyhedron if and only if* F_1 *and* F_2 *are e-polyhedra.*
(iv) *If* $F_1, F_2 \subset \mathbb{R}^n$ *are e-polyhedra, then* $F_1 + F_2$ *is an e-polyhedron and*

$$0^+ (F_1 + F_2) = 0^+F_1 + 0^+F_2.$$

(v) *If* $\{F_i, i \in I\}$ *is a finite family of e-polyhedra in* $\mathbb{R}^n$ *such that* $\bigcap_{i\in I} F_i \neq \emptyset$*, then* $\bigcap_{i\in I} F_i$ *is an e-polyhedron and*

$$0^+\left(\bigcap_{i\in I} F_i\right) = \bigcap_{i\in I} 0^+F_i.$$

Sketch of the Proof This result contains three types of statements:

- Statements which are immediate application of those of Proposition 1.1 as the e-polyhedra are e-convex sets. This is the case of the assertions related with the recession cones.
- Statements that can be proved turning to the external representations of the e-polyhedra. For instance, regarding (iii), if σ_1 and σ_2 are finite linear systems representing $F_1 \subset \mathbb{R}^n$ *and* $F_2 \subset \mathbb{R}^m$, respectively, the finite system obtained by joining the inequalities of σ_1 to those of σ_2 is an external representation of $F_1 \times F_2$ in $\mathbb{R}^{n+m}$.
- Statements that can be proved by means of internal representations of the involved e-polyhedra. For instance, regarding (ii) and the first part of (iv), if $F = \operatorname{conv}(U : V) + \operatorname{cone}(D)$, where U, V and D are finite sets, then $AF = \operatorname{conv}(AU : AV) + \operatorname{cone}(AD)$, showing that AF is an e-polyhedron. In particular, if $F_1, F_2 \subset \mathbb{R}^n$ are e-polyhedra, then $F_1 + F_2$ is the image of $F_1 \times F_2$ by a linear mapping, showing that $F_1 + F_2$ is an e-polyhedron.

Observe that some statements, like (i), can be proved through either external or internal representations of F. □

In Examples 2.6 and 2.7, we have seen two different decompositions for the same e-polyhedron as a sum of an e-polytope and a finitely generated cone. For an e-polyhedron F, given $U, V, D \subset \mathbb{R}^n$ such that $F = \operatorname{conv}(U : V) + \operatorname{cone} D$, by (2.23), we know that $\operatorname{cone} D = 0^+F$, so that the conical component of any decomposition of F is always 0^+F. Regarding the sets U and V, by employing the notion of minimal face, one can apply the following refinement of Theorem 2.5.

Theorem 2.6 (Minimal Decomposition of e-Polyhedra) *Let $F \subset \mathbb{R}^n$ be a nonempty e-polyhedron. Then,*

$$F = \operatorname{conv}(U : V) + 0^+ F \tag{2.24}$$

where $U := \{u_1, \ldots, u_m\}$, $u_i \in F_i$ with F_i a minimal face of F, for every $i = 1, \ldots, m$, and $V := \{v_1, \ldots, v_q\}$, $v_j \in G_j$ with G_j a minimal face of $\overline{F}$ not intersecting F, for every $j = 1, \ldots, q$.

Example 2.8 Consider again the e-polyhedron F defined in Example 2.7 (see Fig. 2.5). The only minimal face of F is $F_1 = \{(1,0)\}$, whereas the only minimal face of $\overline{F}$ not intersecting F is $G_1 = \{(4,3)\}$. The extreme rays of F are $d_1 = (1,0)$ and $d_2 = (2,1)$, and so $0^+F = \operatorname{cone}(\{d_1, d_2\})$. Then, by choosing any $u_1 \in F_1$ and $v_1 \in G_1$, Theorem 2.6 guarantees that an internal representation of F is

$$F = \operatorname{conv}(\{(1,0)\} : \{(4,3)\}) + \operatorname{cone}(\{(1,0),(2,1)\}).$$

Thus, we observe that the representations given in Examples 2.6 and 2.7 were not minimal in the sense that the sets of generators there can be simplified as shown.

As a straightforward consequence of Theorem 2.6, the following internal representations for e-polyhedra which are either closed or relatively open can be obtained.

Corollary 2.9 *Let $F \subset \mathbb{R}^n$ be a nonempty e-polyhedron.*

(i) *If F is closed, then $F = \operatorname{conv}\{u_1, \ldots, u_m\} + 0^+F$ with $u_i \in F_i$ and F_i a minimal face of F, for every $i = 1, \ldots, m$.*
(ii) *If F is relatively open, then $F = \operatorname{conv}(\{u\} : \{v_1, \ldots, v_q\}) + 0^+F$ with $u \in F$, $v_j \in G_j$ and G_j a minimal face of $\overline{F}$ not intersecting F, for every $j = 1, \ldots, q$.*

Thus, if a polyhedron F contains no line, then $F = \operatorname{extr} F + 0^+F$. Just as an illustration, the e-polyhedron $F = \{x \in \mathbb{R}^2 : x_1 > 1, x_2 > 1\}$ can be represented as stated in Corollary 2.9 by $\operatorname{conv}(\{u\} : \{v\}) + \mathbb{R}^2_+$ where $v = (1,1)$ and u is any point in F.

2.4 Evenly Linear Programming

We now come back to the problems of obtaining geometrical information on the solution set F of a given consistent system

$$\sigma = \{\langle a_t, x\rangle \le b_t,\ t \in W;\ \langle a_t, x\rangle < b_t,\ t \in S\},$$

with $S \neq \emptyset$ and $T = W \cup S$ finite, and deciding whether a given x^* minimizes or not a given linear function on F. To do this, we fall back on the results of Sect. 1.5

on similar questions relative to linear systems containing strict inequalities under constraint qualifications.

Since T is finite, it is a compact space for the discrete topology, so that σ is continuous. Actually, σ satisfies the FMCQ due to the finiteness of T, which entails the closedness of

$$K(\overline{\sigma}) = \operatorname{cone}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T; \begin{pmatrix} 0_n \\ 1 \end{pmatrix}\right\},$$

independently of the satisfaction or not of SCQ. Consequently, σ satisfies the LFMCQ at any $\overline{x} \in \overline{F}$. Similarly, again by the finiteness of T, σ satisfies the LOPCQ at any $\overline{x} \in \overline{F}$.

2.4.1 Feasible Set

The boundedness of F is characterized by Proposition 1.11, with no change. Regarding the formulas for $\dim F$, the following result follows immediately from Proposition 1.12 just taking into account that the involved cones are finitely generated and, so, closed. Let $I := \{1, ..., p\}$ be the set of indices of the linear representation of F, and $I^{=} := \{i \in i : 1, \ldots p : \langle a_i, x\rangle = b_i, \forall x \in F\}$.

Proposition 2.6 (Dimension Formulas) *The following statements hold true:*

(i) $\dim F = n - \dim \operatorname{lin} K(\overline{\sigma})$. *Moreover,*

$$\dim F = n - \dim \operatorname{span}\{a_i : i \in I^{=}\}$$

and

$$\operatorname{aff} F = \left\{x \in \mathbb{R}^n : \langle a_i, x\rangle = b_i,\ i \in I^{=}\right\}.$$

(ii) $\dim 0^+ F = n - \dim \operatorname{lin} \operatorname{cone}\{a_i, i \in I\}$.
(iii) $\dim \operatorname{lin} 0^+ F = n - \dim \operatorname{cone}\{a_i, i \in I\}$.

The next result is straightforward consequence of Proposition 1.13.

Proposition 2.7 (Boundary and Extreme Points) *The following statements hold true:*

(i) *One has*

$$\operatorname{rint} F = \left\{x \in \mathbb{R}^n : \langle a_i, x\rangle = b_i,\ i \in I^{=};\ \langle a_i, x\rangle < b_i,\ i \in I\backslash I^{=}\right\},$$

$$\operatorname{rbd} F = \cup\{F_i : i \in I\backslash I^{=}\}$$

and

$$\operatorname{bd} F = \cup \{F_i : a_i \neq 0_n,\ i \in I\}.$$

(*ii*) *A given* $\overline{x} \in F$ *is an extreme point of* F *if and only if*

$$\dim \operatorname{cone} \{a_i : \langle a_i, \overline{x} \rangle = b_i, i \in I\} = n.$$

2.4.2 Optimality and Duality

We now consider the evenly linear programming (*e-LP* in short) problem

$$\begin{aligned} (P)\ \operatorname*{Min}_{x \in \mathbb{R}^n}\ & \langle c, x \rangle \\ \text{s.t.}\ & \langle a_i, x \rangle \leq b_i, i = 1, \ldots, m, \\ & \langle a_i, x \rangle < b_i, i = m + 1, \ldots, p, \end{aligned}$$

where $c \neq 0_n$. The *optimal value* of (P), say $v(P)$, may be $+\infty$ (when $F = \emptyset$), $-\infty$ (when the set $\{\langle c, x \rangle : x \in F\}$ is not bounded from above), and a real number (when $\{\langle c, x \rangle : x \in F\}$ is a nonempty set bounded from above), in which cases P is said to be *inconsistent*, *unbounded* and *bounded*, respectively. In the last two cases, $F \neq \emptyset$, and P is said to be *consistent*.

The relaxed problem of (P) is

$$\begin{aligned} (\overline{P})\ \operatorname*{Min}_{x \in \mathbb{R}^n}\ & \langle c, x \rangle \\ \text{s.t.}\ & \langle a_i, x \rangle \leq b_i, i = 1, \ldots, p. \end{aligned}$$

Recall that we denote by F and $\overline{F}$ (respectively, F^* and $\overline{F}^*$) the feasible sets (resp., optimal sets) of (P) and $(\overline{P})$. By Proposition 1.14, if the e-LP problem (P) is consistent, then $v(P) = v(\overline{P})$ and $F^* = F \cap \overline{F}^*$.

Regarding duality, let us observe that the Haar dual problem of $(\overline{P})$,

$$\begin{aligned} (D)\ \operatorname*{Max}_{\lambda \in \mathbb{R}_+^p}\ & -\sum_{i=1}^{p} \lambda_i b_i \\ \text{s.t.}\ & -\sum_{i=1}^{p} \lambda_i a_i = c, \end{aligned}$$

is nothing else but its ordinary LP dual.

Theorem 2.7 (Duality in e-LP) *Let* (P) *and* (D) *be consistent. Then* $v(P) = v(D) \in \mathbb{R}$ *and* (D) *is solvable.*

Proof From the LP duality theorem one gets that $v(\overline{P}) = v(D) \in \mathbb{R}$ and, by Theorem 1.6, (D) is solvable. □

The final result of this section is the specialization of Proposition 1.7 to the e-LP setting.

Theorem 2.8 (Optimality and Strong Uniqueness in e-LP) *Given $x^* \in F$, the following statements are equivalent:*

(i) $x^* \in F^*$.
(ii) $-c \in A(x^*)$ (KKT condition).
(iii) *There exists a feasible solution $\overline{\lambda}$ of (D) such that $\overline{\lambda}_i \left(b_i - \langle a_i, x^*\rangle\right) = 0$ for all $i = 1, \ldots, p$* (complementarity condition).

Additionally, x^* is a strongly unique optimal solution of (P) if and only if $-c \in \operatorname{int} A(x^*)$.

We now discuss the use of the Fourier–Motzkin elimination method to solve small size e-LP problems of the form (P) above. In contrast with ordinary LP, in e-LP we may have $v(P) \in \mathbb{R}$ without having optimal solutions. For this reason, we must introduce a more general concept of solution. We say that a sequence $\{x^k\}$ is a *minimizing sequence* for (P) whenever $x^k \in F$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} \langle c, x^k \rangle = v(P)$.

Assume that (P) is consistent and choose $\widehat{x} \in \operatorname{rint} F$ (recall that the relative interior of a convex set is nonempty). By (2.3), such a point can be obtained by applying the Fourier–Motzkin elimination method to the system

$$\{\langle a_i, x\rangle < b_i,\ i \in I \backslash I^=;\ \langle a_i, x\rangle = b_i, i \in I^=\}.$$

By the continuity of linear functions, if (P) is consistent, then $v(\overline{P}) = v(P)$. The LP algorithms provide either an optimal solution $\overline{z}$ of $(\overline{P})$, when $(\overline{P})$ is bounded, or a minimizing sequence $\{z^k\}$ for $(\overline{P})$, when it is unbounded. Then, taking a sequence $\{\lambda_k\}$ of positive scalars such that $\lim_{k\to\infty} \lambda_k = 0$, since $(1 - \lambda_k)\,\overline{z} + \lambda_k \widehat{x} \in F$ for all $k \in \mathbb{N}$ in the first case and $(1 - \lambda_k)\, z^k + \lambda_k \widehat{x} \in F$ for all $k \in \mathbb{N}$ in the second case, we conclude that either $\{(1 - \lambda_k)\,\overline{z} + \lambda_k \widehat{x}\}$ or $\{(1 - \lambda_k)\, z^k + \lambda_k \widehat{x}\}$ is a minimizing sequence for (P). In conclusion, the strategy consists in checking the consistency of σ (by using an existence theorem for linear systems containing strict inequalities) and, if $F \neq \emptyset$, applying some LP solver to the relaxed problem $(\overline{P})$ to get an optimal solution $\overline{z}$ of $(\overline{P})$ (which will be also optimal for (P) if $\overline{z} \in F$, or provides at least a converging minimizing sequence for (P)), or a minimizing sequence $\{z^k\}$ for $(\overline{P})$ with $\lim_{k\to\infty} \langle c, z^k \rangle = -\infty$, which provides a corresponding minimizing sequence for (P). Observe that $\widehat{x}$ can effectively be computed by the ellipsoid method whenever the polyhedron $\overline{F}$ is full dimensional.

As the next example shows, the Fourier–Motzkin elimination method may help to determine whether (P) is solvable or not.

Example 2.9 Consider the consistent e-LP problem consisting in the minimization of the functional $\langle c, x\rangle$ subject to the linear constraints of Example 2.1 when the objective function is either

(i) $c = (1, 0)$, or
(ii) $c = (1, -1)$.

Introducing a new variable $x_3 = \langle c, x\rangle$, (P) is equivalent to

$$\begin{array}{lll}(P_c) & \displaystyle\operatorname*{Min}_{x\in\mathbb{R}^3} & x_3 \\ & \text{s.t.} & c_1x_1 + c_2x_2 - x_3 \leq 0 \\ & & -x_1 - 4x_2 \leq -2, \\ & & -x_1 < -1.\end{array}$$

We must project the feasible set of (P_c), say F_c, onto the x_3 axis. Firstly, by eliminating the variable x_1 in the system

$$\sigma_c = \{c_1x_1 + c_2x_2 - x_3 \leq 0, -x_1 - 4x_2 \leq -2, -x_1 < -1\},$$

we obtain its corresponding reduced system σ_c', whose solution set F_c' is the projection of F_c onto the coordinate hyperplane $x_1 = 0$. Finally, by eliminating the variable x_2 in the system σ_c', we obtain the projection of F_c onto the x_3 axis as the solution set of the reduced system of σ_c'.

(i) If $c = (1, 0)$, for the variable x_1, we are in Case 4 of Table 2.1, and the reduced system associated to $\sigma_c = \{x_1 - x_3 \leq 0, -x_1 - 4x_2 \leq -2, -x_1 < -1\}$ is $\sigma_c' = \{-4x_2 - x_3 \leq -2, -x_3 < -1\}$. Now, for the variable x_2, we are in Case 14 and, therefore, the projection of F_c onto the x_3 axis is $F_c'' =]1, +\infty[$. Consequently, $v(P) = v(P_c) = 1$. Hence, $v(\overline{P}) = v(P) = 1$ but the e-LP problem is not solvable. An optimal solution of $(\overline{P})$ is $\overline{z} = \left(1, \frac{1}{4}\right)$ (the extreme point of the polyhedron provided by the simplex method) and a relative interior point of F (actually a Slater point for σ) is $\widehat{x} = (2, 2)$. Taking $\lambda_k = \frac{1}{k}$ for all $k \in \mathbb{N}$, one gets the bounded minimizing sequence $\{(1 - \lambda_k)\overline{z} + \lambda_k\widehat{x}\} = \left\{\left(\frac{k+1}{k}, \frac{k+7}{4k}\right)\right\}$.

(ii) If $c = (1, -1)$, by applying the Fourier–Motzkin elimination method to the system σ_c, we obtain that the projection of F_c onto the x_3 axis is the real line $\mathbb{R}$ and, therefore, $v(\overline{P}) = v(P) = -\infty$ with the objective function decreasing along the edge $\left\{\left(1, \frac{1}{4}\right) + \lambda\,(0, 1) : \lambda \geq 0\right\}$ of $\overline{F}$. Taking $z^k = (1, k)$ and $\lambda_k = \frac{1}{k}$ for all $k \in \mathbb{N}$, one gets the unbounded minimizing sequence $\{(1 - \lambda_k)z^k + \lambda_k\widehat{x}\} = \left\{\left(\frac{k+1}{k}, \frac{k^2-k+2}{k}\right)\right\}$.

2.5 Selected Applications

Finite linear systems involving strict inequalities and their solutions sets, the evenly convex polyhedra, are frequently used in mathematical programming, ordinary zero-sum two person games, economy, combinatorial problems, formal verification of hardware and software systems, etc.

2.5.1 Mathematical Programming

The existence (alternative or transposition) theorems of Sect. 2.2 have been frequently used to prove important results in mathematical programming. For instance, just to mention classic works, Motzkin [132] and Slater [165] used the Motzkin transposition theorem to derive the duality theorem of linear programming (see [160, Cor. 7.11]) while, according to D. Hilbert [88], Gordan [80] used his transposition theorem to study Hilbert bases of finitely generated convex cones, which became a useful instrument in integer programming some decades later (see [160, pp. 232–233 and 315–316]). From an expository point of view, the alternative theorems, including those involving strict inequalities, are the basic tool of the famous book by Mangasarian on nonlinear programming, whose first edition is dated in 1969 and has been reedited many times, e.g., [116].

In recent times, the systems of strict linear inequalities play a crucial role in single and multiobjective piecewise linear optimization programming. On the one hand, by exploiting the fact that any e-polyhedron is obtained by removing faces from a polyhedron, [52] obtained an internal representation (in terms of linear combinations of generating points and rays) for e-polyhedra which was applied to show that the image of an e-polyhedron under an affine function is always an e-polyhedron (which is not true for arbitrary e-convex sets), and to study sensitivity in a piecewise linear program with possible discontinuity [51]. On the other hand, [190] has shown that the Pareto frontier of any piecewise linear multiobjective optimization can be expressed as finite union of e-polyhedra.

2.5.2 Finite Zero-Sum Two Person Games

A finite two-person zero-sum game is determined by a *payoff matrix* $A\ (m \times n)$, where m and n are the numbers of pure strategies of players I and II. The sets of mixed strategies of both players are discrete probability distributions over the corresponding sets of pure strategies, that we represent by

$$X := \left\{ x \in \mathbb{R}_+^m : \sum_{j=1}^{m} x_j = 1 \right\} \text{ and } Y := \left\{ y \in \mathbb{R}_+^n : \sum_{i=1}^{n} y_i = 1 \right\}.$$

The expected outcome of player II when players I and II choose the mixed strategies x and y is $x^\top Ay$, so that the *maximin* and *minimax values* are

$$v_I := \max_{x\in X}\min_{y\in Y} x^\top Ay \quad \text{and} \quad v_{II} := \min_{y\in Y}\max_{x\in X} x^\top Ay,$$

respectively. Let us mention two classical applications of the alternative theorems for finite systems in this framework:

- The celebrated minimax theorem, proved by John von Neumann in 1937 [134], asserts that, in contrast with semi-infinite games, $v_I = v_{II}$ for finite games. This common value is called *value of the game* and it is usually denoted by $v(A)$. The proof of this crucial result in the popular book of G. Owen [137] (reedited many times since its first edition in 1968) is based on the following matrix version of Gordan alternative theorem (Corollary 2.2): there is a vector $x \in \mathbb{R}^n$ such that $x \geqq 0_n$ and $Ax = 0_m$ if and only if there is no vector $y \in \mathbb{R}^m$ such that $A^\top y \geqq 0_n$ implies $A^\top y = 0_n$, where the symbol $\geqq$ between two vectors of $\mathbb{R}^n$ denotes the componentwise partial order on $\mathbb{R}^n$.
- The famous work [18], published in 1950 by H. F. Bohnenblust, S. Karlin and L. S. Shapley, was devoted to the relationship between the dimensions of the optimal strategy sets, the uniqueness of solution, and the way to construct a game matrix with a given solution. To state the main result in that paper we need some additional notation. Let $A_1, \ldots, A_n$ and $a_1, \ldots, a_m$ be the columns of A and $A^\top$, respectively. Consider the index sets

$$J := \left\{ j \in \{1, \ldots, n\} : x^\top A_j = v(A), \forall x \in X \right\}$$

and

$$I := \left\{ i \in \{1, \ldots, m\} : a_i^\top y = v(A), \forall y \in Y \right\}.$$

The mentioned result of [18] asserts, firstly, that for each maximin strategy x there exists a minimax strategy y such that $y_j > 0$ for all $j \in J$ and, secondly, that for each minimax strategy y there exists a maximin strategy x such that $x_i > 0$ for all $i \in I$. The proof in [18] was based on the following matrix reformulation of one of the Stiemke's alternative theorems of Sect. 2.2.1 (Corollary 2.5): there is a vector $x \in \mathbb{R}^n$ such that $Ax \geqq 0_m$ if and only if there is no vector $y \in \mathbb{R}^m$ such that $A^\top y \geqq 0_n$ and $y > 0_m$ (componentwise).

2.5.3 Economy

The work of von Neumann [134] mentioned in the previous subsection also contains a famous growth model consisting in a finite linear system that we now summarize.

The model is determined by two nonnegative $m \times n$ matrices $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$, the data, to be interpreted as follows:

- Each row-index i stands for a 'good', while each column index j stands for a 'process'.
- Process j can convert (in one time unit, say a year) a_{ij} units of good $i = 1, \ldots, m$ into b_{ij} units of the same good i.

The variables of the model are two vectors, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and two scalars, γ and δ, and have the following meaning:

- x_j denotes the 'intensity' by which we let process j work.
- y_i is the price of one unit of good i.
- The real number γ is the factor by which all intensities are multiplied each year.
- The real number δ denotes the interest factor (i.e., $\delta := 1 + \frac{z}{100}$, where z is the rate of interest).

Regarding the constraints:

- All variables are nonnegative: $x \geqq 0_n$, $y \geqq 0_m$, $\gamma \geq 0$ and $\delta \geq 0$.
- Moreover, it is required that $x \neq 0_n$ and $y \neq 0_m$. Taking into account the sign constraints, the latter constraints can be replaced by the strict inequalities $\sum\limits_{j=1}^{n} x_j > 0$ and $\sum\limits_{i=1}^{m} y_i > 0$.
- The vector inequality $\gamma Ax \leqq Bx$ expresses that for each good i, the amount of good i produced in year k is at least the amount of good i required in year $k+1$.
- If there is strict surplus at the i-th inequality of $\gamma Ax \leqq Bx$, then the equation $y^\top (\gamma Ax - Bx) = 0$ requires the price of good i to be zero.
- The vector inequality $\delta y^\top A \geqq y^\top B$ says that, for each process j, the price of the goods produced at the end of year k is at most the price of the goods necessary for the process at the beginning of year k, added with interest.
- If the j-th inequality at $\delta y^\top A \geqq y^\top B$ is strict, the equation $\left(\delta y^\top A - y^\top B\right) x = 0$ means that the intensity of process j is 0.

Even though the resulting system contains nonlinear equations and inequalities, von Neumann (as well as [160, p. 220]) proved the existence of solution to that nonlinear system turning to the following matrix reformulation of Motzkin alternative theorem (Corollary 2.1): Let A $(m \times n)$ and B $(p \times n)$ be matrices and let $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^p$ (the data); then there exists a vector $x \in \mathbb{R}^n$ with $Ax < b$ and $Bx \leq c$ if and only if for all $y \in \mathbb{R}^m_+$ and $z \in \mathbb{R}^p_+$ it holds

$$A^\top y + B^\top z = 0_n \Longrightarrow y^\top b + z^\top c \geq 0$$

and

$$A^\top y + B^\top z = 0_n \text{ with } y \neq 0_m \Longrightarrow y^\top b + z^\top c > 0.$$

More recently, alternative theorems involving strict inequalities due to Stiemke (Corollary 2.5) and Tucker have been used by H. Nikaidô in his well-known monographs on mathematical economics [135, 136]. There, he basically adopts a Keynesian disequilibrium view of the accumulation process.

2.5.4 Combinatorial Problems

R. Kannan considered in [98] the so-called *Frobenius problem*: given n natural numbers $a_1, .., a_n$ such that their greatest common divisor is 1, find the largest natural number that is not expressible as a nonnegative integer combination of them. The author provides a polynomial time algorithm for a closely related problem: that of finding the covering radius of any polytope $\overline{F}$, i.e., $\inf \left\{ \rho \in \mathbb{R}_+ : \mathbb{Z}^n + \rho\overline{F} = \mathbb{R}^n \right\}$.

Kannan called "copolyhedra" the intersections of finitely many halfspaces not necessarily closed (where the particle "co" before "polyhedra" stands for closed / open), and "copolytopes" those copolyhedra which are bounded. We prefer to call these sets e-polyhedra and e-polytopes, respectively, as other terms like "copolyhedra" and "co-polyhedra" are used with different meanings in functional analysis (e.g., in [185] a co-polyhedron is a subset of some infinite-dimensional space that can be expressed as the sum of the convex hull of finitely many points with a finitely generated convex cone), in algebra, in chromatography, in polymer sciences, etc.

In the main results of [98] one finds the sentence "the algorithm finds an e-polytope F such that ...", meaning that the corresponding algorithm will find a matrix $A \in \mathbb{Q}^{m\times(n+l)}$ and a vector $b \in \mathbb{Q}^m$, where l is at most some polynomial function of n such that

$$F = \left\{ x \in \mathbb{R}^n : A\begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} \leq \\ < \end{pmatrix} b \text{ for some } y \in \mathbb{R}^l \right\},$$

where the symbol $\begin{pmatrix} \leq \\ < \end{pmatrix}$ means that the inequality corresponding to each row of A may be either $\leq$ or $<$.

2.5.5 Formal Verification of Hardware and Software Systems

The term "real-time" derives from its use in the early years of simulation, in which real-world processes were simulated at a rate that matched that of the corresponding real processes. Fortunately, since the 1970s, anybody can use her/his home computer as a real-time system thanks to the existence of coding libraries which offer real time capabilities in a high level language on a variety of operating systems. The purpose of formal verification of the real-time properties of hardware and software is to ensure the correct functionality of complex systems in a complementary way to that of a dynamic verification (empirical testing). Research in this domain of computing science is currently developed by many university departments and institutes, as well as industrial labs.

In particular, there exists an abundant literature on the automatic detection of the linear equations and inequalities which are the consequence of the linear equations and inequalities which are explicit in a given computer program (term meaning, in this setting, collection of instructions or sentences, written in some language programming, that perform specific tasks). Until the mid 1990s, most works dealt with systems of linear equations and weak inequalities, so that the basic mathematical tool then used was the transit between the two possible representations of a given polyhedron: the *external* one, as the solution set of a finite linear system of weak inequalities, and the *internal* one, as the Minkowski sum of a polytope with a finitely generated convex cone (which are called "representation by linear restraints" and "representation by frame", respectively in [32]). The same approach, based in the double representation of polyhedral-like sets, was used in [85] for a particular type of hybrid systems involving both integer and continuous variables as well as both weak and strict linear inequalities. The idea behind was the replacement of the solution set of the linear system with mixed variables and (possibly) strict linear inequalities by a greater set: the solution set of the relaxed system which results of considering all variables continuous and all inequalities weak. The proposed methodology is illustrated by the authors with three examples involving water-level monitors, Fischer mutual exclusion protocols, and scheduling problems.

The systematic handling of linear systems of mixed inequalities and continuous variables in formal verification starts in [86] (paper which subsumes the theory and examples of [85]) under the generic name of "linear relation analysis", and continues with a series of papers of R. Bagnara, P. M. Hill, E. Zaffanella, and co-authors, who coined the term "not necessarily closed (NNC) convex polyhedra" for the objects simply called "e-polyhedra" in this book.

In [11] the authors point out that the existing libraries for the manipulation of polyhedra do not allow to handle linear systems of mixed inequalities, and present their first version of the Parma Polyhedra Library (PPL), which is focussed on the double representation of e-polyhedra. Indeed, the significance of having an internal representation relies on the fact that the ability to switch from one to another representation can be usefully exploited to provide simple implementations for the basic operations on e-polyhedra. Successive improvements of this free software are described in subsequent works [4–10]. Recent works [14–16] have investigated and implemented an improved representation for NNC polyhedra, leading to the software library PPLite, which can achieve impressive time and memory efficiency improvements with respect to the PPL.

To conclude this section containing a selection of applications of e-polyhedra, we point out that finite linear systems containing strict inequalities also arise in the determination of the so-called strict witnesses in computational geometry [97], in the Bayesian approach to probabilistic logic [145, Subsection 1.3], and in other subfields of computer science, e.g., in the treatment of constraint satisfaction problems

[17] and safety verification [163]. Furthermore, evenly linear programming has been used in the framework of natural language processing in order to analyze sentiments [159].

2.6 Bibliographic Notes

The notion of polyhedron, i.e., the solution set of an ordinary finite linear system in $\mathbb{R}^n$, can be extended in several ways, for instance:

- Replacing some weak inequalities by strict ones, giving rise to e-polyhedra.
- Removing the finiteness assumption on the index set, giving rise to the class of closed convex sets.
- If one allows for infinitely many linear inequalities, not necessarily weak, one gets the class of the e-convex sets.
- If one maintains the finite number of inequalities and the weak nature of them, but replaces the space $\mathbb{R}^n$ of variables by a locally convex space one has the so-called *generalized polyhedra.* More precisely, a generalized polyhedron in the sense of Bonnans and Shapiro [19, Def. 2.195] is the intersection of finitely many closed halfspaces with a closed affine subspace.

Three are the main questions regarding all these extensions: guaranteeing the existence of solutions, characterizing the consequent inequalities, and availability of some method for the double description of the elements of the corresponding class.

In this chapter we have analyzed e-polyhedra from this point of view. The first known results on finite linear systems containing strict inequalities are the alternative theorems of Gordan [80], Stiemke [169], Carver [28] and Motzkin [131], published in 1873, 1915, 1921–1922, and 1936, respectively. Table 2.4.1 of the famous Mangasarian's book [116] contains a list of 11 alternative theorems for finite systems, 9 of them involving strict inequalities. These results are also called *transposition theorems* in the literature (e.g., in [132, 160, 165]) as, when they are reformulated as the equivalence of two of statements on matrices, one of the statements involves certain matrix while the other condition involves its transpose (recall the matrix reformulations of Gordan, Stiemke and Motzkin alternative theorems in Sect. 2.5.2). Rockafellar's alternative theorem is relatively recent [148, Th. 22.2] in comparison with the previous ones. All these theorems can be seen as existence theorems for finite linear systems containing strict inequalities (recall Table 1.1 for the comparison with other existence theorems).

All these alternative and transposition theorems can be directly proved from the very general existence Theorem 2.2, whose equivalences $(i) \iff (iii)$, $(i) \iff (iv)$, and $(i) \iff (v)$ are [152, Cor. 3.3], [152, Th. 3.1], and [152, Lem. 3.2(ii)], respectively. These equivalences have been proved falling back on the representative cone of the given system and the characteristic cone of its relaxed one. Proposition 2.1, analyzing the relationships between these two cones, is [152, Lem. 3.1] while statement (*ii*) of Proposition 2.2, on the relationships between these cones and the corresponding solution sets, is [152, Lem. 3.2(i)]. The counterpart of Theorem 2.2 for ordinary finite systems (i.e., with $S = \emptyset$) asserts that $\sigma = \{\langle a_t, x\rangle \leq b_t,\ t \in W\}$ is consistent if and only if

$$(0_n, -1) \notin \operatorname{cone}\{(a_t, b_t)\,,\, t \in W;\, (0_n, 1)\}\,.$$

This is a straightforward consequence of the existence theorem for ordinary linear systems in locally convex spaces due to Y. J. Zhu [193, Th. 1]. The equivalence $(i) \iff (ii)$ in Corollary 2.6 does not have a semi-infinite counterpart as $\sum_{t\in W} a_t$ is not well-defined whenever W is an infinite set. However, the equivalence $(i) \iff$ (iv) for ordinary linear systems appears in [72, Th. 3.3].

Finite linear systems containing finitely many weak and/or strict inequalities were reconsidered in the 1950s by Kuhn [108] in his extension of the Fourier–Motzkin elimination method. Regarding their solution sets, they appear in the literature with different names: *wholefaced polyhedra* [186], *copolyhedra* [98], *NNC polyhedra* [6], *G-polyhedra* [192], *semiclosed polyhedra* [190] and *evenly convex polyhedra* [152] which is, in our modest opinion, the natural name for these sets as it recalls the term introduced by Fenchel for the intersections of halfspaces. In all these papers, evenly convex polyhedra (e-polyhedra, in brief) were defined by means of their external representations, that is, as the solution sets of finite linear systems possibly containing strict inequalities.

The three statements of Theorem 2.3 were proved in [152]: (i) in Prop. 3.1, (*ii*) in Prop. 3.2, and (*iii*) in Cor. 3.6 (see also [108, Th. II]). Corollary 2.7(i) is the non-homogeneous Farkas lemma [128], later extended by Zhu [193, Th. 2] to infinite systems posed in locally convex spaces while different proofs of Corollary 2.7(*ii*) can be found in [78, Prop. 1.2] and [152, Cor. 3.7].

Regarding the characterization of the containment of e-polyhedra in Proposition 2.3, [$\Leftarrow$] is [152, Prop. 3.4] while [$\Rightarrow$] is [152, Prop 3.5].

Concerning Sect. 2.3, on the double description of the evenly convex polyhedra, statements (i) , and (*ii*) and (*iii*) of Proposition 2.4 are proved in [152, Lem. 4.1 (i)] and [152, Prop. 4.1], respectively, while Theorem 2.4 is [152, Th. 4.1]. Motzkin proved in his 1936 Inaugural Dissertation [131] that any polyhedral convex set in $\mathbb{R}^n$ can be written as the sum of a polytope and a polyhedral convex cone. Inspired by this result, different authors have investigated those sets which can be decomposed as the (Minkowski) sum of a convex compact set and a closed convex cone [66, 67, 77], those which can be decomposed as the sum of a convex compact set and a convex cone [92], and those that can be expressed as the sum of a bounded convex set and a convex cone [125]. Theorem 2.5 (proved in [152, Th. 4.2]), also inspired by the mentioned Motzkin's decomposition theorem, deals with those sets which can be decomposed as the sum of an e-polytope and a polyhedral convex cone. In fact, this class of decomposition constitutes an internal representation for e-polyhedra. The double representation of e-polyhedra is also studied by Fang and co-authors in [51–53] and by Bagnara and co-authors in [6, 11]. In [52, Th. 2.1], the authors give an internal representation for a e-polyhedron based on the Motzkin decomposition of its closure (which is a polyhedron) and on the fact that any e-polyhedron is the result of eliminating from a polyhedron (its closure) the union of a family of its faces. The obtained result is different of the one in Theorem 2.5 and, as a consequence, the minimal decomposition given in [53, Th. 4.1] cannot be recovered in Theorem 2.6, proved in [152, Th. 4.3]. Likewise, the internal representation for relatively open e-polyhedral sets in Corollary 2.9(*ii*) differs from the one given in [52, Lem. 2.3]. On the other hand, although the internal representation for e-polyhedra obtained in [6, Th. 4.4] coincides with the one given in Theorem 2.5, the proof of the latter does not use the standard version of Motzkin theorem but a generalization of the technique employed by him and provides an accurate method to obtain the internal representation from an external one. Statements (*ii*) and (*iv*) in Proposition 2.5 are Theorem 2.2 and Proposition 2.1(i) in [52], respectively.

The feasibility problem regarding finite linear systems containing finitely many weak and/or strict inequalities can be tackled in three different ways: Fourier–Motzkin elimination, the ellipsoid method and reduction to a nonlinear programming problem (e.g., minimizing a barrier function). By the end of the 1980s, only finite linear strict inequality systems with small number of variables could be solved with the Fourier–Motzkin elimination method. The proposal by Khachiyan and co-authors [177], in 1988, of the so-called Khachiyan's ellipsoid method, a polynomial-time algorithm allowing to solve large infinite linear strict inequality systems, provoked an increasing use of this type of systems in different fields. Inspired in the logarithmic barrier method of Fiacco and McCormick [56], one can compute a solution of $\sigma = \left\{ \langle a_i, x \rangle \begin{pmatrix} \leq \\ < \end{pmatrix} b_i, i = 1, \ldots, m \right\}$, provided its solution set S is nonempty, by solving the unconstrained optimization problem

$$\underset{x \in \mathbb{R}^n}{\text{Min}} - \sum_{t \in T} \ln \left[b_t - \langle a_t, x \rangle \right],$$

whose optimal solution (which is unique thanks to the strict convexity of the objective function, usually named the barrier of S) is the so-called analytic center of S.

Chapter 3
Evenly Quasiconvex Functions

It is well-known that a real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous if and only if its *graph*, $\operatorname{gph} f := \{(x, f(x)) : x \in \mathbb{R}^n\}$, is a closed subset of $\mathbb{R}^{n+1}$. Since gph f can be written as the intersection of the sets $\operatorname{epi} f := \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq \lambda\}$ and $\operatorname{hypo} f := \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R} : f(x) \geq \lambda\}$, called *epigraph* and *hypograph* of f, respectively, one can split the continuity in two weaker properties: f is said to be *lower* (*upper*) *semicontinuous* whenever epi f (hypo f, resp.) is closed, so that f is continuous if and only if it is lower and upper semicontinuous.

In the same vein, the modern treatment of extended real-valued functions emphasizes the role played by the set-based approach. So, functions of the form $f : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ such that

$$f((1-\mu)x + \mu y) \leq (1-\mu) f(x) + \mu f(y), \forall x, y \in \mathbb{R}^n, \forall \mu \in [0, 1],$$

are called *convex* and are characterized by the convexity of their epigraph epi f while the *concave* functions f (for which $-f$ is convex) are those functions whose hypograph hypo f is convex. Similarly, functions $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ such that

$$f((1-\mu)x + \mu y) \leq \max\{f(x), f(y)\}, \forall x, y \in \mathbb{R}^n, \forall \mu \in [0, 1],$$

which are called *quasiconvex*, are those functions whose *lower level sets* $[f \leq r] := \{x \in \mathbb{R}^n : f(x) \leq r\}$ (or equivalently, their *strict lower level sets*, $[f < r] := \{x \in \mathbb{R}^n : f(x) < r\}$) are convex for all $r \in \mathbb{R}$. The opposite of the quasiconvex functions are called *quasiconcave*, and they are characterized by the convexity of their *upper level sets* $[f \geq r] := \{x \in \mathbb{R}^n : f(x) \geq r\}$, $r \in \mathbb{R}$.

One of the ways of facing the unconstrained minimization of a given function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ whose bad behavior does not allow the application of known numerical methods consists in the replacement of f by a function $\widehat{f} \leq f$ (that is, a *minorant* of f) enjoying desirable properties and such that the *approximation gap* $\inf_{x \in \mathbb{R}^n} f(x) - \inf_{x \in \mathbb{R}^n} \widehat{f}(x) \geq 0$ is sufficiently small.

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In this sense, if $\mathscr{F} = \{f_i, i \in I\}$ is a family of minorants of f, then the *supremum* of $\mathscr{F}$, $\sup \mathscr{F}$, defined as $(\sup \mathscr{F})(x) := \sup\{f_i(x), i \in I\}$, is also a minorant of f and its epigraph and lower level sets are the intersection of the epigraphs and the lower level sets, respectively, of the family members. Since the intersection of closed sets is closed, there exists a greatest lower semicontinuous minorant of f, which is called *lower semicontinuous hull* of f, denoted by $\operatorname{cl} f$ and defined as the supremum of the family of all lower semicontinuous minorants of f. In the same way, since the intersection of convex sets is convex, one can consider the *convex* (*quasiconvex*) *hull* of f, denoted by $\operatorname{co} f$ ($\operatorname{qco} f$, resp.) and defined as the largest convex (quasiconvex, resp.) minorant of f. Similarly, $\operatorname{cu} f$ denotes the *upper semicontinuous hull* of f (defined as its smallest upper semicontinuous majorant, whose hypograph is the closure of $\operatorname{hypo} f$). The reader is referred to [59, 91, 148, 191], among many other textbooks on convex analysis, for a comprehensive introduction on these notions.

The approximation gap decreases when one replaces the given minorant $\widehat{f}$ by a greater one. In Sect. 3.1 we define a class of functions, the evenly quasiconvex ones, which provide greater minorants than the smaller class of the lower semicontinuous quasiconvex functions (analogously, in the next chapter, we analyze the class of evenly convex functions, which extends the class of lower semicontinuous convex functions). Section 3.2 studies the evenly quasiconvex hull providing the largest evenly quasiconvex minorant of a given function. Section 3.3 analyzes conjugates and subdifferentials for evenly quasiconvex functions, while Sect. 3.4 provides a sketch of quasiconvex duality. Finally, Sect. 3.5 describes an application in mathematical economy.

3.1 Evenly Quasiconvex Functions

A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be *lower semicontinuous*, lsc in brief, (*upper semicontinuous*, usc in brief) at a point $\overline{x} \in \mathbb{R}^n$ if for any $\lambda \in \mathbb{R}$, $\lambda < f(\overline{x})$ ($\lambda > f(\overline{x})$, resp.), there exists a neighbourhood of $\overline{x}$, $V_{\overline{x}}$, such that $\lambda < f(x)$ ($\lambda > f(x)$, resp.) for all $x \in V_{\overline{x}}$. It is well-known that a function f is lsc at any point of $\mathbb{R}^n$ if and only if $\operatorname{epi} f$ is closed, or equivalently, if $[f \leq r]$ is closed for every $r \in \mathbb{R}$. As a function f is usc if and only if $-f$ is lsc, usc functions turn out to be those functions whose strict lower level sets are open.

An extended real-valued function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be *evenly quasiconvex*, e-quasiconvex in brief, (*strictly evenly quasiconvex*, resp.) if the lower level set $[f \leq r]$ (the strict lower level set $[f < r]$, resp.) is e-convex for every $r \in \mathbb{R}$. In particular, if $f : \mathbb{R} \to \overline{\mathbb{R}}$ is an univariate quasiconvex function, then all its lower level sets are e-convex (since all of them are intervals) and, therefore, f is e-quasiconvex. Since every convex set being closed or open is also e-convex, it is obvious that every lsc quasiconvex function is e-quasiconvex and every usc quasiconvex function is strictly e-quasiconvex. Moreover, since $[f \leq r] = \bigcap_{r<q} [f < q]$ and the intersection of e-convex sets is e-convex, every strictly

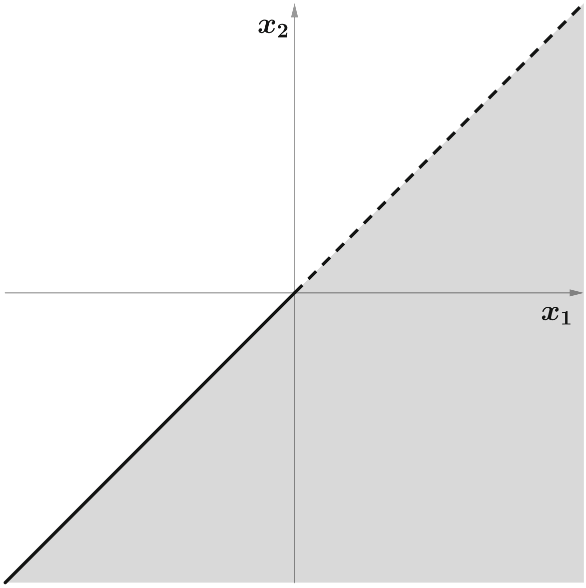


Fig. 3.1 The strict lower level set $[f < 1]$ is not e-convex

e-quasiconvex function is e-quasiconvex. The converse is not true, as the following example shows.

Example 3.1 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined as follows:

$$f(x_1, x_2) = \begin{cases} 0, & \text{if } x_1 \geq x_2 \text{ and } x_2 \leq 0, \\ x_2/x_1, & \text{if } x_1 > x_2 > 0, \\ 1, & \text{elsewhere.} \end{cases}$$

All the lower level sets of f are closed and convex, and so e-convex, showing that f is e-quasiconvex. However,

$$[f < 1] = \{x \in \mathbb{R}^2 : x_1 > x_2 > 0\} \cup \{x \in \mathbb{R}^2 : x_1 \geq x_2,\ x_2 \leq 0\}$$

is not e-convex (see Fig. 3.1).

The relationships between the different families of quasiconvex functions are summarized in Diagram 3.1. In Example 3.2, we show a strictly e-quasiconvex (and so, e-quasiconvex) function which is neither lsc nor usc.

Example 3.2 Consider the function $f : \mathbb{R} \to \overline{\mathbb{R}}$ defined by

$$f(x) = \begin{cases} x^2, & \text{if } -1 < x < 1, \\ 3, & \text{if } x = 1, \\ +\infty, & \text{otherwise,} \end{cases}$$

Diagram 3.1 Evenly quasiconvex and related functions

usc quasiconvex

$\Downarrow$

strictly e-quasiconvex

$\Downarrow$

e-quasiconvex

$\Uparrow$

lsc quasiconvex

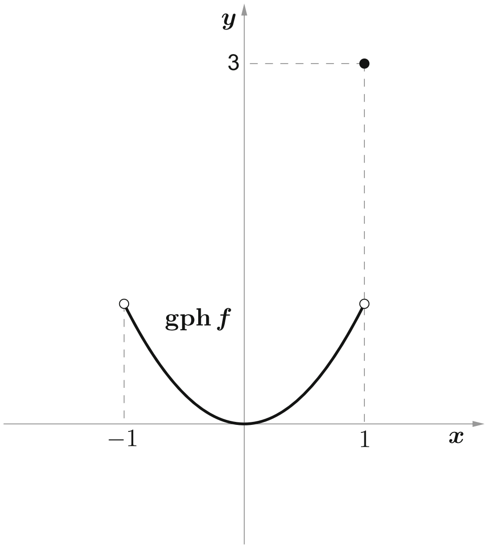


Fig. 3.2 Strictly e-quasiconvex function which is neither usc nor lsc

whose graph is represented in Fig. 3.2. It is easy to see that all the strict lower level sets are e-convex. However, $[f < 4] = [f \leq 4] =]-1, 1]$ is neither open nor closed, so that f is a strictly e-quasiconvex function which is not usc and it is also an e-quasiconvex function which is not lsc.

As a straightforward consequence of the definitions given in this section, we observe that the solution set of a system of the form $\{g_t(x) \leq 0, t \in W; g_t(x) < 0, t \in S\}$ with $W \cap S = \emptyset$, g_t e-quasiconvex (in particular, lsc quasiconvex) for all $t \in W$ and g_t strictly e-quasiconvex for all $t \in S$, is e-convex.

Proposition 3.1 (Operations with e-Quasiconvex Functions) *The following statements hold:*

(*i*) *If $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is an e-quasiconvex function and $\alpha > 0$, then αf is e-quasiconvex.*
(*ii*) *If $\{f_i : \mathbb{R}^n \to \overline{\mathbb{R}},\ i \in I\}$ is a family of e-quasiconvex functions, then $\sup_{i\in I} f_i$ is an e-quasiconvex function.*

Proof

(i) It is a direct consequence of the equalities

$$[\alpha f \leq r] = \left\{x \in \mathbb{R}^n : (\alpha f)(x) \leq r\right\} = \left\{x \in \mathbb{R}^n : f(x) \leq \frac{r}{\alpha}\right\} = \left[f \leq \frac{r}{\alpha}\right].$$

(ii) As $\left[\left(\sup_{i \in I} f_i\right) \leq r\right] = \cap_{i \in I} [f_i \leq r]$ and even convexity is preserved under intersections, we have that the family of e-quasiconvex functions is closed under pointwise suprema.

□

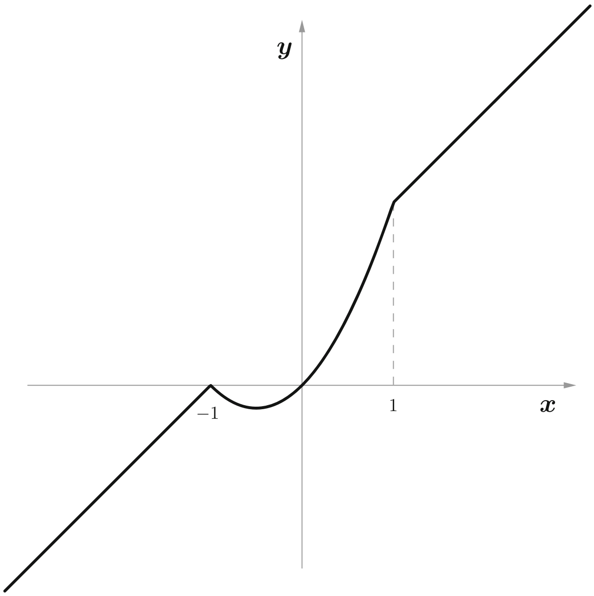
The sum of e-quasiconvex functions is not, in general, e-quasiconvex as we can see in the following example.

Example 3.3 Let $f, g : \mathbb{R} \to \mathbb{R}$ be the functions defined by $f(x) = x$, for all $x \in \mathbb{R}$ and

$$g(x) = \begin{cases} x^2, & \text{if } -1 \leq x \leq 1, \\ 1, & \text{otherwise.} \end{cases}$$

It is easy to see that f and g are e-quasiconvex functions, whereas the function $f + g$, whose graph is represented in Fig. 3.3, is not e-quasiconvex.

Fig. 3.3 Graphical representation of $f + g$



3.2 Evenly Quasiconvex Hull

As a consequence of Proposition 3.1(*ii*), every function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ has a largest e-quasiconvex minorant, which is called its *evenly quasiconvex hull* (*e-quasiconvex hull*, in short) and denoted by eqco f. Obviously,

$$\operatorname{cl}\operatorname{qco} f \leq \operatorname{eqco} f \leq \operatorname{qco} f \leq f \tag{3.1}$$

and, therefore, $\operatorname{cl}\operatorname{eqco} f = \operatorname{cl}\operatorname{qco} f$. We shall say that a function f is *e-quasiconvex* at $\overline{x} \in \mathbb{R}^n$ if $f(\overline{x}) = (\operatorname{eqco} f)(\overline{x})$. Clearly, f is e-quasiconvex if and only if it is e-quasiconvex at every $\overline{x} \in \mathbb{R}^n$.

The following examples show that the inequalities in (3.1) can be strict.

Example 3.4 Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x+1, & \text{if } x \leq -1, \\ x^2 + x, & \text{if } -1 < x < 1, \\ x+2, & \text{otherwise,} \end{cases}$$

whose graph is represented in Fig. 3.4. Obviously, f is not a quasiconvex function. Figure 3.5 represents the graphs of qco f and cl qco f and, as we can see, one has:

$$\operatorname{cl}\operatorname{qco} f \lneqq \operatorname{qco} f \lneqq f.$$

In this case, as qco f is an univariate quasiconvex function, it is also e-quasiconvex and we have that $\operatorname{eqco} f = \operatorname{qco} f$.

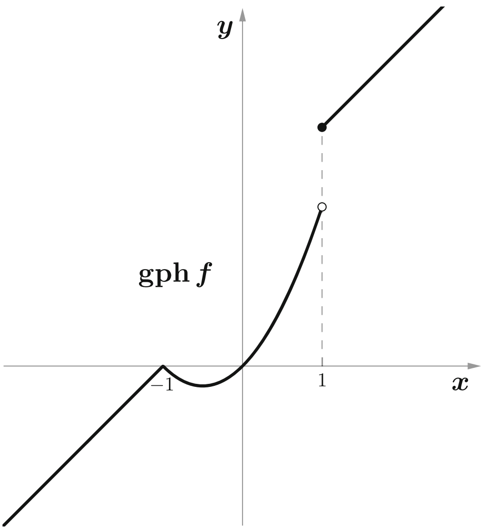


Fig. 3.4 The function f is not quasiconvex

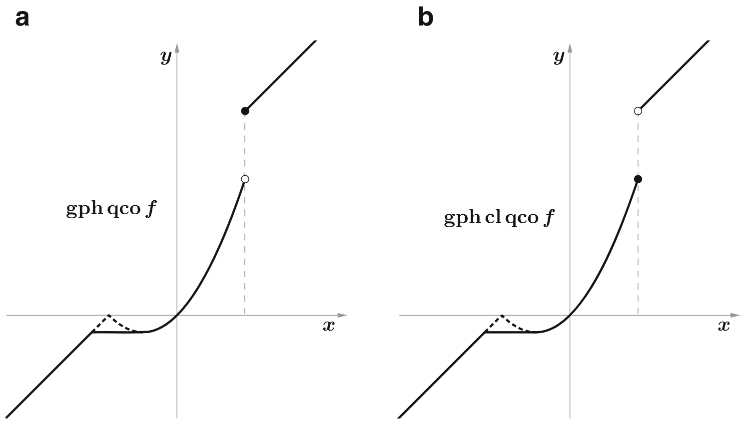


Fig. 3.5 (**a**) The graph of qco f; (**b**) The graph of cl qco f

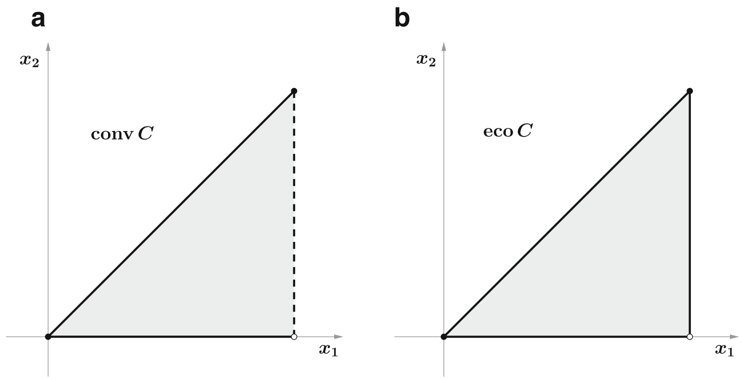


Fig. 3.6 (**a**) convex hull of C; (**b**) e-convex hull of C

Example 3.5 Consider the set $C = \{x \in \mathbb{R}^2 : 0 \leq x_1 < 1, x_2 = 0\} \cup \{(1, 1)\}$ and its *indicator function* $\delta_C : \mathbb{R}^2 \to \overline{\mathbb{R}}$ defined by $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$ if $x \in \mathbb{R}^2 \backslash C$. Since $C \subsetneq \operatorname{conv} C \subsetneq \operatorname{eco} C$ (see Fig. 3.6), one gets

$$\operatorname{eqco} \delta_C = \delta_{\operatorname{eco} C} \lneqq \operatorname{qco} \delta_C = \delta_{\operatorname{conv} C} \lneqq \delta_C.$$

It is easy to see that, for every function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and for all $r, r' \in \mathbb{R}$ with $r \leq r'$, the following inclusion holds,

$$[f \leq r] \subset \left[f \leq r'\right].$$

In general, we say that the family $\mathscr{A} := \{A_t\}_{t \in \mathbb{R}}$ of (possibly empty) subsets of $\mathbb{R}^n$ is an *ascending family* if $A_t \subset A_{t'}$ for all $t, t' \in \mathbb{R}$ with $t \leq t'$. We also associate to the ascending family $\mathscr{A}$ the function $\psi_{\mathscr{A}} : \mathbb{R}^n \to \overline{\mathbb{R}}$, which we call the *extended* *gauge* of $\mathscr{A}$, defined by

$$\psi_{\mathscr{A}}(x) := \inf\{t \in \mathbb{R} : x \in A_t\}, \ \forall x \in \mathbb{R}^n, \tag{3.2}$$

with the convention $\inf \emptyset = +\infty$. It is easy to prove that, for every $r \in \mathbb{R}$,

$$A_r \subset [\psi_{\mathscr{A}} \leq r] = \bigcap_{t>r} A_t. \tag{3.3}$$

The motivation of the name given to the function $\psi_{\mathscr{A}}$ relies on the following fact. For a convex set $A \subset \mathbb{R}^n$ such that $0_n \in \operatorname{int} A$, the family $\mathscr{A} = \{A_t\}_{t \in \mathbb{R}}$, defined by $A_t = \emptyset$ if $t < 0$ and $A_t = tA$ if $t \geq 0$, is ascending and the function

$$\psi_{\mathscr{A}}(x) = \inf\{t \geq 0 : x \in A_t\}, \ \forall x \in \mathbb{R}^n,$$

is nothing else but the *gauge* (or Minkowski functional) of A.

Example 3.6 Consider the ascending family $\mathscr{A} = \{A_t\}_{t \in \mathbb{R}}$ given by $A_t =]-\infty, t[$ if $t < 0$, $A_t =]-\infty, 1[$ if $t = 0$ and $A_t =]-\infty, e^t[$ if $t > 0$. The gauge function of $\mathscr{A}$ is

$$\psi_{\mathscr{A}}(x) = \begin{cases} x, & \text{if } x < 0, \\ 0, & \text{if } 0 \leq x < 1, \\ \ln(x), & \text{if } x \geq 1. \end{cases}$$

Obviously, the family of all lower level sets of a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, $\mathscr{A}_f := \{[f \leq t]\}_{t \in \mathbb{R}}$, is an ascending family whose extended gauge is the function f. In fact, by (3.3), one has

$$[f \leq r] \subset \left[\psi_{\mathscr{A}_f} \leq r\right] = \bigcap_{t>r} [f \leq t] = [f \leq r], \ \forall r \in \mathbb{R},$$

so $\left[\psi_{\mathscr{A}_f} \leq r\right] = [f \leq r]$ for all $r \in \mathbb{R}$, which implies that $\psi_{\mathscr{A}_f} = f$.

Proposition 3.2 (Even Quasiconvexity of the Extended Gauge) *Let $\mathscr{A} = \{A_t\}_{t \in \mathbb{R}}$ be an ascending family of e-convex sets of $\mathbb{R}^n$. Then, the extended gauge $\psi_{\mathscr{A}}$ is e-quasiconvex.*

Proof Taking into account (3.3) and that the intersection of a family of e-convex sets is an e-convex set, we have that $[\psi_{\mathscr{A}} \leq r]$ is e-convex for all $r \in \mathbb{R}$. □

Next, for a family of sets $\mathscr{A} = \{A_t\}_{t \in \mathbb{R}}$, we shall denote $\operatorname{eco} \mathscr{A} := \{\operatorname{eco} A_t\}_{t \in \mathbb{R}}$. Observe that $\operatorname{eco} \mathscr{A}$ is an ascending family of e-convex sets whenever $\mathscr{A}$ is an ascending family.

Proposition 3.3 (Evenly Quasiconvex Hull of the Extended Gauge) *Let $\mathscr{A} = \{A_t\}_{t\in\mathbb{R}}$ be an ascending family of sets in $\mathbb{R}^n$. Then,*

$$\operatorname{eqco}\psi_{\mathscr{A}} = \psi_{\operatorname{eco}\mathscr{A}}.$$

Proof Since $A_t \subset \operatorname{eco} A_t$ for every $t \in \mathbb{R}$, one has $\psi_{\operatorname{eco}\mathscr{A}} \leq \psi_{\mathscr{A}}$, with $\psi_{\operatorname{eco}\mathscr{A}}$ being an e-quasiconvex function due to Proposition 3.2. Now, by the definition of the e-quasiconvex hull of $\psi_{\mathscr{A}}$, we obtain

$$\psi_{\operatorname{eco}\mathscr{A}} \leq \operatorname{eqco}\psi_{\mathscr{A}} \leq \psi_{\mathscr{A}}.$$

On the other hand, since $A_t \subset [\psi_{\mathscr{A}} \leq t] \subset \left[\operatorname{eqco}\psi_{\mathscr{A}} \leq t\right]$ for every $t \in \mathbb{R}$, then

$$\operatorname{eco} A_t \subset \operatorname{eco}[\psi_{\mathscr{A}} \leq t] \subset \operatorname{eco}\left[\operatorname{eqco}\psi_{\mathscr{A}} \leq t\right] = \left[\operatorname{eqco}\psi_{\mathscr{A}} \leq t\right], \ \forall t \in \mathbb{R}.$$

Therefore,

$$[\psi_{\operatorname{eco}\mathscr{A}} \leq r] = \bigcap_{t>r} \operatorname{eco} A_t \subset \bigcap_{t>r} \left[\operatorname{eqco}\psi_{\mathscr{A}} \leq t\right] = \left[\operatorname{eqco}\psi_{\mathscr{A}} \leq r\right], \ \forall r \in \mathbb{R},$$

which implies $\psi_{\operatorname{eco}\mathscr{A}} \geq \operatorname{eqco}\psi_{\mathscr{A}}$. □

Corollary 3.1 (Characterization of the e-Quasiconvex Hull) *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $\mathscr{A} = \{A_t\}_{t\in\mathbb{R}}$ be an ascending family such that $[f < t] \subset A_t \subset [f \leq t]$ for every $t \in \mathbb{R}$. Then,*

$$\operatorname{eqco} f = \psi_{\operatorname{eco}\mathscr{A}}.$$

Proof From the hypothesis, one obtains

$$[f \leq r] = \bigcap_{t>r} [f < t] \subset \bigcap_{t>r} A_t \subset \bigcap_{t>r} [f \leq t] = [f \leq r].$$

Hence, $[\psi_{\mathscr{A}} \leq r] = [f \leq r]$ for every $r \in \mathbb{R}$, and so $\psi_{\mathscr{A}} = f$. Now, by Proposition 3.3, we have that $\operatorname{eqco} f = \operatorname{eqco}\psi_{\mathscr{A}} = \psi_{\operatorname{eco}\mathscr{A}}$. □

In the previous result, we can consider the ascending family $\mathscr{A}_f = \{[f \leq t]\}_{t\in\mathbb{R}}$, (equivalently, $\mathscr{A} = \{[f < t]\}_{t\in\mathbb{R}}$). Then, one obtains the following representation for the e-quasiconvex hull of f:

$$(\operatorname{eqco} f)(x) = \inf\{r \in \mathbb{R} : x \in \operatorname{eco}[f \leq r]\}, \ x \in \mathbb{R}^n \tag{3.4}$$

(equivalently, $(\operatorname{eqco} f)(x) = \inf\{r \in \mathbb{R} : x \in \operatorname{eco}[f < r]\}$, for every $x \in \mathbb{R}^n$).

Proposition 3.4 (A Sufficient Condition for e-Quasiconvexity) *For a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, if there is an ascending family $\mathscr{A} = \{A_t\}_{t\in\mathbb{R}}$ of e-convex sets in $\mathbb{R}^n$ such that $[f < t] \subset A_t \subset [f \leq t]$ for all $t \in \mathbb{R}$, then f is e-quasiconvex.*

Proof As $f = \psi_{\mathscr{A}}$ and $\mathscr{A}$ is an ascending family of e-convex sets, due to Proposition 3.2, one has that f is e-quasiconvex. □

A direct consequence of Proposition 3.4 is that any function whose strict lower level sets are e-convex (that is, any strictly e-quasiconvex function), is e-quasiconvex as well (as pointed out in Diagram 3.1).

Next result gathers together four characterizations of e-quasiconvexity at a point given in the literature. For illustrative purposes, we give a proof of $(i) \iff (iii)$ based on the notion of ascending family, whereas precise references for the complete proof can be found in Sect. 3.6. The latter also applies to all the missing proofs in this chapter.

Theorem 3.1 (Characterizations of e-Quasiconvexity at a Point) *Consider $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $\overline{x} \in \mathbb{R}^n$. The following statements are equivalent:*

(i) *f is e-quasiconvex at $\overline{x}$.*
(ii) $f(\overline{x}) = \inf\{t \in \mathbb{R} : \overline{x} \in \operatorname{eco}[f \le t]\}$.
(iii) $\overline{x} \notin \operatorname{eco}[f \le r]$ *for all* $r < f(\overline{x})$.
(iv) *For every $r < f(\overline{x})$, there exists $q \in \mathbb{R}^n$ such that $\langle q, x - \overline{x}\rangle < 0$ for all $x \in [f \le r]$.*
(v) *f is quasiconvex and for every $\overline{y} \in \mathbb{R}^n$ such that $f(\overline{y}) < f(\overline{x})$, every sequence $y^k \subset \mathbb{R}^n$ such that $\left\{y^k\right\} \to \overline{y}$, and every $\{\mu_k\} \subset (0, +\infty)$, one has*

$$f(\overline{x}) \le \liminf_{k\to+\infty} f(\overline{x} + \mu_k(\overline{x} - y^k)).$$

Partial Proof Consider the ascending family $\operatorname{eco}\mathscr{A}_f := \{\operatorname{eco}[f \le t]\}_{t\in\mathbb{R}}$.

$[(i) \Rightarrow (iii)]$ Let $r \in \mathbb{R}$ be such that $\overline{x} \in \operatorname{eco}[f \le r] \subset \left[\psi_{\operatorname{eco}\mathscr{A}_f} \le r\right]$. By applying (i) and Corollary 3.1, one has

$$f(\overline{x}) = (\operatorname{eqco} f)(\overline{x}) = \psi_{\operatorname{eco}\mathscr{A}_f}(\overline{x}) \le r.$$

$[(iii) \Rightarrow (i)]$ Suppose that (iii) holds. Then, $f(\overline{x}) \le r$, for every $r \in \mathbb{R}$ such that $\overline{x} \in \operatorname{eco}[f \le r]$, so that, by applying Corollary 3.1, we have

$$\begin{aligned}(\operatorname{eqco} f)(\overline{x}) = \psi_{\operatorname{eco}\mathscr{A}_f}(\overline{x}) &= \inf\{t \in \mathbb{R} : \overline{x} \in \operatorname{eco}[f \le t]\} \\ &\ge \inf\{t \in \mathbb{R} : \overline{x} \in [f \le t]\} = f(\overline{x}).\end{aligned}$$ □

3.3 Conjugacy and Subdifferentiability

Duality theory plays an essential role in convex optimization. Its construction is based on Fenchel conjugation, an important tool in convex analysis. Recall that the *Fenchel conjugate* of a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by $f^*(\cdot) := \sup\{\langle\cdot, x\rangle - f(x) : x \in \operatorname{dom} f\}$, where $\operatorname{dom} f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$

is the *effective domain* of f. If f is a *proper* function (that is, $\operatorname{dom} f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in \mathbb{R}^n$), then f^* is a proper lsc convex function (since it is the pointwise supremum of a collection of affine functions). In particular, the second conjugate of f, f^{**}, is the largest proper lsc convex minorant of f. Therefore, a proper function f is lsc and convex if and only if $f = f^{**}$, result that turns out to be crucial for convex duality (see, e.g., [148, Th. 12.2]).

Another essential tool in duality theory is the notion of subdifferential due to Moreau and Rockafellar [129, 148]. Given $\varepsilon \geq 0$, a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be *ε-subdifferentiable* at a point $\overline{x} \in f^{-1}(\mathbb{R})$ if there exists $u \in \mathbb{R}^n$ such that

$$f(x) \geq f(\overline{x}) + \langle u, x - \overline{x}\rangle - \varepsilon, \quad \forall x \in \mathbb{R}^n. \tag{3.5}$$

The set of those points $u \in \mathbb{R}^n$ satisfying (3.5) is the *ε-subdifferential* of f at $\overline{x}$, denoted by $\partial_\varepsilon f(\overline{x})$. If $f(\overline{x}) \notin \mathbb{R}$, it is assumed to be empty. When $\varepsilon = 0$, we just write $\partial f(\overline{x})$ and it is called the *subdifferential* of f at $\overline{x}$. The function f is said to be ε-subdifferentiable on a subset A of $\mathbb{R}^n$ if it is ε-subdifferentiable at each point of A.

Subdifferentiability is used to obtain optimality conditions in both convex and nonconvex optimization (see, e.g., [89]). Thus, given a proper convex function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $\overline{x} \in \mathbb{R}^n$, [148, Th. 23.5] establishes that $\overline{x} \in \operatorname{argmin} f := \{x \in \mathbb{R}^n : f(x) \leq f(y), \forall y \in \mathbb{R}^n\}$ if and only if $0_n \in \partial f(\overline{x})$.

However, convex duality is not adequate for nonconvex problems. With the aim of providing a basis for duality theory in the e-quasiconvex case, some conjugation and subdifferentiability notions were developed in the literature. Next we show some of them.

3.3.1 $\mathscr{H}$-Conjugation

Let $\mathscr{H}$ be a family of univariate extended real-valued functions, closed under pointwise supremum. A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be *$\mathscr{H}$-convex* if it can be expressed as the supremum of those minorants obtained by composing some $h \in \mathscr{H}$ with some linear function, that is, if for each $\overline{x} \in \mathbb{R}^n$ one has

$$f(\overline{x}) = \sup\{h(\langle y, \overline{x}\rangle) : h \in \mathscr{H}, y \in \mathbb{R}^n, h(\langle y, x\rangle) \leq f(x), \ \forall x \in \mathbb{R}^n\}.$$

The supremum of $\mathscr{H}$-convex functions is again an $\mathscr{H}$-convex function, and the greatest $\mathscr{H}$-convex function majorized by a given function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is called the *$\mathscr{H}$-convex hull* of f.

For $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, its *$\mathscr{H}$-conjugate* is $f^{\mathscr{H}} : \mathbb{R}^n \to \mathscr{H}$ given by

$$f^{\mathscr{H}}(y) := \sup\{h : h \in \mathscr{H}, h(\langle y, x\rangle) \leq f(x) \ \forall x \in \mathbb{R}^n\}.$$

Conversely, given $g : \mathbb{R}^n \to \mathscr{H}$, its *$\mathscr{H}'$-conjugate* function $g^{\mathscr{H}'} : \mathbb{R}^n \to \overline{\mathbb{R}}$ is defined by

$$g^{\mathscr{H}'}(x) := \sup_{y \in \mathbb{R}^n} g(y)(\langle y, x \rangle). \tag{3.6}$$

As a consequence of these definitions the following statements hold.

Proposition 3.5 (Properties of $\mathscr{H}$-Conjugation) *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$. Then, the following statements hold:*

(i) $f^{\mathscr{H}}(y)(\langle y, x \rangle) \leq f(x)$*, for all* $y, x \in \mathbb{R}^n$.
(ii) $f^{\mathscr{H}\mathscr{H}'}$ *is* $\mathscr{H}$*-convex.*
(iii) *The* $\mathscr{H}$*-convex hull of* f *is equal to the second conjugate* $f^{\mathscr{H}\mathscr{H}'}$.
(iv) f *is* $\mathscr{H}$*-convex if and only if* $f = f^{\mathscr{H}\mathscr{H}'}$.

This generalized concept of conjugation encompasses the classical Fenchel conjugation method by considering $\mathscr{H}$ the family of functions $h_b(t) = t - b$, with $b \in \overline{\mathbb{R}}$, in which case the $\mathscr{H}$-convex functions are the lsc convex functions. It also includes conjugation schemes which are suitable both for lsc quasiconvex functions and e-quasiconvex functions.

Now we give the details of the conjugation for the family of e-quasiconvex functions. For that purpose, recall that an *e-quasiaffine* function is a function which is both e-quasiconvex and quasiconcave. If $\mathscr{H}$ is assumed to be the family of all nondecreasing univariate extended real-valued functions, then the $\mathscr{H}$-conjugation method introduced above provides a characterization of the family of e-quasiconvex functions. More precisely, one gets the following results.

Proposition 3.6 (Characterization of e-Quasiaffine Functions) *Let* $q : \mathbb{R}^n \to \overline{\mathbb{R}}$. *Then, the following statements are equivalent:*

(i) q *is e-quasiaffine.*
(ii) $q(x) = h(\langle y, x \rangle)$ *for all* $x \in \mathbb{R}^n$*, for some* $y \in \mathbb{R}^n$ *and* $h : \mathbb{R} \to \overline{\mathbb{R}}$ *nondecreasing.*
(iii) *For each* $r \in \mathbb{R}$, $[q \leq r]$ *is either a closed or open halfspace or* $\emptyset$ *or* $\mathbb{R}^n$.

Proposition 3.7 (Conjugation for e-Quasiconvex Functions) *For a function* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$*, if* $\mathscr{H}$ *is the family of all nondecreasing univariate extended real-valued functions, then the following statements hold:*

(i) $f^{\mathscr{H}}(y)(t) = \inf\{f(x) : \langle y, x \rangle \geq t\}$ *for all* $y \in \mathbb{R}^n$ *and* $t \in \mathbb{R}$.
(ii) $f^{\mathscr{H}\mathscr{H}'} = \operatorname{eqco} f$.
(iii) f *is e-quasiconvex if and only if* $f = f^{\mathscr{H}\mathscr{H}'}$.

The conjugation theory described above (called $\mathscr{H}$-conjugation) is not symmetric, since the conjugate of an extended real-valued function is a function whose values are taken in a family of functions $\mathscr{H}$ instead of $\overline{\mathbb{R}}$. However, it provides a good geometric interpretation of e-quasiconvexity, since any e-quasiconvex function

f is the supremum of e-quasiaffine minorants. Recall from (3.6) that

$$f^{\mathscr{H}\mathscr{H}'}(\cdot) := \sup_{y \in \mathbb{R}^n} f^{\mathscr{H}}(y)(\langle y, \cdot \rangle),$$

where $f^{\mathscr{H}}(y)(\langle y, \cdot \rangle)$, for each $y \in \mathbb{R}^n$, is an e-quasiaffine minorant of f in virtue of Propositions 3.5 and 3.6.

3.3.2 Moreau's Generalized Conjugation Theory

Next we show how the conjugacy for e-quasiconvex functions can be derived from the generalized conjugation theory that was developed by Moreau [130] in an abstract framework. We begin by recalling the essentials of Moreau's conjugation theory (see, for instance, [124]). Let X and Y be two arbitrary sets and let $c : X \times Y \to \overline{\mathbb{R}}$ be a function, called *the coupling function*. For any function $f : X \to \overline{\mathbb{R}}$, its c-conjugate $f^c : Y \to \overline{\mathbb{R}}$ is defined by

$$f^c(y) = \sup_{x \in X}\{c(x, y) - f(x)\}, \quad \forall y \in Y,$$

where the conventions $+\infty + (-\infty) = -\infty + (+\infty) = +\infty - (+\infty) = -\infty - (-\infty) = -\infty$ are assumed. In the same way, for every $g : Y \to \overline{\mathbb{R}}$, its c'-conjugate is the function $g^{c'} : X \to \overline{\mathbb{R}}$ defined by

$$g^{c'}(x) = \sup_{y \in Y}\{c(x, y) - g(y)\}, \quad \forall x \in X.$$

This notation is consistent with considering the coupling function $c' : Y \times X \to \overline{\mathbb{R}}$ given by $c'(y, x) = c(x, y)$. Functions of the form $x \in X \mapsto c(x, y) - \beta \in \overline{\mathbb{R}}$, with $y \in Y$ and $\beta \in \overline{\mathbb{R}}$, are called *c-elementary*. Similarly, *c'-elementary* functions are those of the form $y \in Y \mapsto c'(y, x) - \beta \in \overline{\mathbb{R}}$, with $x \in X$ and $\beta \in \overline{\mathbb{R}}$. We denote by Γ_c and $\Gamma_{c'}$ the sets of c-elementary and c'-elementary functions, respectively. A function $f : X \to \overline{\mathbb{R}}$ is said to be *Γ_c-convex* if it is the pointwise supremum of a subset of Γ_c. Hence, every function $f : X \to \overline{\mathbb{R}}$ has a largest Γ_c-convex minorant, which is called its *Γ_c-convex hull*. Similarly, a function $g : Y \to \overline{\mathbb{R}}$ is said to be *$\Gamma_{c'}$-convex* if it is the pointwise supremum of a subset of $\Gamma_{c'}$, being the largest $\Gamma_{c'}$-convex minorant of g its *$\Gamma_{c'}$-convex hull*. We summarize in the following proposition the main properties of this conjugation theory.

Proposition 3.8 *Let $f : X \to \overline{\mathbb{R}}$ and $g : Y \to \overline{\mathbb{R}}$. Then,*

(i) $f^c(y) \le c(x, y) - f(x)$ *and* $g^{c'}(x) \le c(x, y) - g(y)$, *for all* $x \in X, y \in Y$.
(ii) f^c *and* $g^{c'}$ *are* $\Gamma_{c'}$*-convex and* Γ_c*-convex, respectively.*
(iii) $f = f^{cc'}$ *if and only if* f *is* Γ_c*-convex, and* $g = g^{c'c}$ *if and only if* g *is* $\Gamma_{c'}$*-convex.*

Following this generalized conjugation theory in order to get an appropriate conjugation scheme for the family of e-quasiconvex functions, we shall consider the coupling function $c : \mathbb{R}^n \times (\mathbb{R}^n \times \mathbb{R}) \to \overline{\mathbb{R}}$ defined by

$$c(x, (y, \alpha)) = \begin{cases} 0, & \text{if } \langle y, x \rangle \geq \alpha, \\ -\infty, & \text{otherwise.} \end{cases} \tag{3.7}$$

The conjugation formulas are then

$$f^c(y, \alpha) = -\inf\{f(x) : \langle y, x \rangle \geq \alpha\} \tag{3.8}$$

for $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $(y, \alpha) \in \mathbb{R}^n \times \mathbb{R}$, and

$$g^{c'}(x) = -\inf\{g(y, \alpha) : \langle y, x \rangle \geq \alpha\}$$

for $g : \mathbb{R}^n \times \mathbb{R} \to \overline{\mathbb{R}}$ and $x \in \mathbb{R}^n$. The second c-conjugate of $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is, for every $\overline{x} \in \mathbb{R}^n$,

$$f^{cc'}(\overline{x}) = \sup_{y \in \mathbb{R}^n} \inf\{f(x) : \langle y, x \rangle \geq \langle y, \overline{x} \rangle\}. \tag{3.9}$$

We illustrate these formulas with an example.

Example 3.7 Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$. By (3.8), its c-conjugate (with c the coupling function in (3.7)) is, for every $(y, \alpha) \in \mathbb{R}^2$,

$$f^c(y, \alpha) = \begin{cases} -\infty, & \text{if } y = 0, \alpha > 0, \\ -(\alpha/y)^3, & \text{if } y > 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Now, given $\overline{x} \in \mathbb{R}$, we observe that if $y > 0$ then $\inf\{f(x) : \langle y, x \rangle \geq \langle y, \overline{x} \rangle\} = \overline{x}^3$, and if $y \leq 0$ then $\inf\{f(x) : \langle y, x \rangle \geq \langle y, \overline{x} \rangle\} = -\infty$. Hence, in virtue of (3.9) we get

$$f^{cc'}(\overline{x}) = \sup_{y \in \mathbb{R}^n} \inf\{f(x) : \langle y, x \rangle \geq \langle y, \overline{x} \rangle\} = \overline{x}^3 = f(\overline{x}).$$

Therefore, as a straightforward consequence of the general theory of conjugation, one has that eqco $f = f^{cc'}$ for any $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, getting a further characterization of the even quasiconvexity at a point besides the ones given in Theorem 3.1.

Theorem 3.2 (Characterization of e-Quasiconvexity at a Point) *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $\overline{x} \in \mathbb{R}^n$. The following statements are equivalent:*

(i) *f is e-quasiconvex at $\overline{x}$.*
(ii) $f(\overline{x}) = \sup_{y \in \mathbb{R}^n} \inf\{f(x) : \langle y, x\rangle \geq \langle y, \overline{x}\rangle\}$.

The above result shows that every e-quasiconvex function can be expressed as a supremum of a family of e-quasiaffine functions. More precisely, if f is e-quasiconvex, then

$$f = \sup_{y \in \mathbb{R}^n} f_y,$$

with $f_y = \inf\{f(x) : \langle y, x\rangle \geq \langle y, \cdot\rangle\}$.

3.3.3 Subdifferentials

Quasiconvex functions are not (Moreau-Rockafellar) subdifferentiable in general. Because of that, several subdifferential notions have been proposed for quasiconvex functions in the literature, being the Greenberg–Pierskalla subdifferential [82] the first to be proposed and the key for subgradient methods in quasiconvex optimization problems. Next we aim to link e-quasiconvexity of a function and subdifferentiability (in the sense of Greenberg–Pierskalla) via its strict lower level sets.

Given $\varepsilon \geq 0$, a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be *ε-GP-subdifferentiable* at a point $\overline{x} \in f^{-1}(\mathbb{R})$ if there exists $u \in \mathbb{R}^n$ such that

$$\langle u, x - \overline{x}\rangle \geq 0 \Rightarrow f(x) \geq f(\overline{x}) - \varepsilon, \quad \forall x \in \mathbb{R}^n. \tag{3.10}$$

We define the *ε -GP-subdifferential* of f at $\overline{x}$, denoted by $(\partial_{\varepsilon}^{GP} f)(\overline{x})$ as the set of those points $u \in \mathbb{R}^n$ satisfying (3.10). If $f(\overline{x}) \notin \mathbb{R}$, it is assumed to be empty. When $\varepsilon = 0$, we just write $(\partial^{GP} f)(\overline{x})$ and it is called the *GP-subdifferential* of f at $\overline{x}$. The function f is said to be ε-GP-subdifferentiable on a subset A of $\mathbb{R}^n$ if it is ε-GP-subdifferentiable at each point of A.

Next result characterizes the ε-GP-subdifferentiability of a function at a given point in terms of the even convexity of a given strict lower level set.

Proposition 3.9 (Non-emptiness of the ε-GP-Subdifferential) *Consider $\varepsilon \geq 0$, $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $\overline{x} \in f^{-1}(\mathbb{R})$. Then, the following statements are equivalent:*

(i) $(\partial_{\varepsilon}^{GP} f)(\overline{x}) \neq \emptyset$.
(ii) $\overline{x} \notin \operatorname{eco}[f < f(\overline{x}) - \varepsilon]$.

Proof It follows from the definition of ε-GP-subdifferential since (3.10) is equivalent to

$$\langle u, x\rangle < \langle u, \overline{x}\rangle, \quad \forall x : f(x) < f(\overline{x}) - \varepsilon,$$

and this is equivalent to (ii) according to (1.8). □

Corollary 3.2 *Let* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *and* $\overline{x} \in f^{-1}(\mathbb{R})$. *Then:*

(i) $(\partial^{GP} f)(\overline{x}) \neq \emptyset$ *if and only if* $\overline{x} \notin \operatorname{eco}[f < f(\overline{x})]$.
(ii) *If* $[f < f(\overline{x})]$ *is e-convex, then* f *is GP-subdifferentiable at* $\overline{x}$.

Corollary 3.2(ii) provides a sufficient condition for GP-subdifferentiability based on the even convexity of an strict lower level set. The next result shows that, under certain even convexity assumptions on either the function or its domain, the even convexity of all strict lower level sets is a necessary condition for the subdifferentiability. The results in this section are in line with the fact that a given e-quasiconvex function at a point is not necessarily GP-subdifferentiable at that point (cf. [183]).

Proposition 3.10 (Necessary Condition for GP-Subdifferentiability) *Assume that a function* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *is GP-subdifferentiable on* $f^{-1}(\mathbb{R})$ *and either* f *is e-quasiconvex or* dom f *is e-convex. Then,* $[f < r]$ *is e-convex for every* $r \in \mathbb{R}$ *(i.e.,* f *is strictly e-quasiconvex).*

Proof Let $r \in \mathbb{R}$ and $\overline{x} \notin [f < r]$, that is, $f(\overline{x}) \geq r$.

Firstly, assume that $f(\overline{x}) < +\infty$ and so, $f(\overline{x}) \in \mathbb{R}$. As f is GP-subdifferentiable on $f^{-1}(\mathbb{R})$, there exists $u \in (\partial^{GP} f)(\overline{x})$ such that if $\langle u, x - \overline{x}\rangle \geq 0$, then $f(x) \geq f(\overline{x}) \geq r$. Hence, $\overline{x} \notin \operatorname{eco}[f < r]$ and so, $[f < r]$ is e-convex.

Now, if $f(\overline{x}) = +\infty$ and dom f is e-convex, there exists $u \in \mathbb{R}^n$ such that $\langle u, \overline{x}\rangle > \langle u, x\rangle$ for all $x \in \operatorname{dom} f$. Since $[f < r] \subset \operatorname{dom} f$, then $\overline{x} \notin \operatorname{eco}[f < r]$ and, so, $[f < r]$ is e-convex.

Finally, assume that $f(\overline{x}) = +\infty$ and f is e-quasiconvex. If $\overline{x} \in \operatorname{eco}[f < r]$, then $\overline{x} \in [f \leq r] = \operatorname{eco}[f \leq r]$, but this is impossible as $f(\overline{x}) = +\infty$. Consequently, $\overline{x} \notin \operatorname{eco}[f < r]$ and the conclusion follows. □

Following Moreau's generalized conjugation theory described in Sect. 3.3.2, it is possible to define alternative notions of subdifferentials based on the coupling function employed for the conjugacy. Thus, $f : X \to \overline{\mathbb{R}}$ is said to be *c-subdifferentiable* at $\overline{x} \in X$ if $f(\overline{x}) \in \mathbb{R}$ and there exists $\overline{y} \in Y$ such that

$$c(\overline{x}, \overline{y}) \in \mathbb{R} \text{ and } f(x) - f(\overline{x}) \geq c(x, \overline{y}) - c(\overline{x}, \overline{y}), \quad \forall x \in X. \tag{3.11}$$

The set of those points $\overline{y} \in Y$ satisfying (3.11) is called the *c-subdifferential* of f at $\overline{x}$, denoted by $\partial_c f(\overline{x})$. If $f(\overline{x}) \notin \mathbb{R}$, $\partial_c f(\overline{x}) = \emptyset$ by definition. The following properties hold.

Proposition 3.11 (Properties of the c-Subdifferential) *Let* $f : X \to \overline{\mathbb{R}}$, $\overline{x} \in X$ *and* $\overline{y} \in Y$. *If* $c(\overline{x}, \overline{y}) \in \mathbb{R}$, *then:*

(*i*) $\overline{y} \in \partial_c f(\overline{x})$ *if and only if* $f(\overline{x}) + f^c(\overline{y}) = c(\overline{x}, \overline{y})$.
(*ii*) $\overline{y} \in \partial_c f^{cc'}(\overline{x})$ *if and only if* $\overline{x} \in \partial_{c'} f^c(\overline{y})$.
(*iii*) *If* $\partial_c f(\overline{x}) \neq \emptyset$, *then* f *is* Γ_c*-convex at* $\overline{x}$.
(*iv*) *If* f *is* Γ_c*-convex at* $\overline{x}$, *then* $\partial_c f^{cc'}(\overline{x}) = \partial_c f(\overline{x})$.

In particular, if we consider again the coupling function $c : \mathbb{R}^n \times (\mathbb{R}^n \times \mathbb{R}) \to \overline{\mathbb{R}}$ introduced in (3.7) for the appropriate conjugation scheme of the e-quasiconvex functions, one gets a corresponding notion of c-subdifferential, say ∂_c. The relationship between this coupling-based subdifferential and the GP-subdifferential is as follows: for $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $\overline{x} \in f^{-1}(\mathbb{R})$, one has

$$\overline{y} \in \partial^{GP} f(\overline{x}) \iff (\overline{y}, \langle \overline{y}, \overline{x} \rangle) \in \partial_c f(\overline{x}).$$

Finally, taking into account the $\mathscr{H}$-conjugation method described in Sect. 3.3.2, we can define further notions of subdifferentials based on the family $\mathscr{H}$ of univariate extended real-valued functions. Thus, $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be *$\mathscr{H}$-subdifferentiable* at $\overline{x} \in \mathbb{R}^n$ if $f(\overline{x}) \in \mathbb{R}$ and there exist $h \in \mathscr{H}$ and $y \in \mathbb{R}^n$ such that

$$h(\langle y, \overline{x} \rangle) = f(\overline{x}) \text{ and } h(\langle y, x \rangle) \leq f(x), \quad \forall x \in \mathbb{R}^n. \tag{3.12}$$

The set of those points $y \in \mathbb{R}^n$ satisfying (3.12) is called the *$\mathscr{H}$-subdifferential* of f at $\overline{x}$, denoted by $\partial_{\mathscr{H}} f(\overline{x})$. If $f(\overline{x}) \notin \mathbb{R}$, $\partial_{\mathscr{H}} f(\overline{x}) = \emptyset$ by definition. When $\mathscr{H}$ is the family of all nondecreasing functions, then $\partial_{\mathscr{H}} f(\cdot) = \partial^{GP} f(\cdot)$.

3.4 Duality in Quasiconvex Optimization

Consider an arbitrary unconstrained optimization problem

$$(GP) \qquad \operatorname*{Min}_{x \in \mathbb{R}^n} F(x),$$

where $F : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a proper function. We now recall (see, e.g., [41, 149]) the well-known *perturbational approach to duality*, whose key is the use of a perturbation function $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$, such that $\Phi(x, 0_m) = F(x)$ for all $x \in \mathbb{R}^n$, $\mathbb{R}^m$ being the space of perturbation variables. Thus, for each $y \in \mathbb{R}^m$, we have the perturbed optimization problem

$$(GP_y) \qquad \operatorname*{Min}_{x \in \mathbb{R}^n} \Phi(x, y),$$

and we associate to the family of perturbed problems, the *infimum value function* $p : \mathbb{R}^m \to \overline{\mathbb{R}}$ defined by $p(y) := \inf_{x \in \mathbb{R}^n} \Phi(x, y)$. It is obvious that $p(0_m) = \inf_{x \in \mathbb{R}^n} F(x)$. In order to apply the conjugation scheme for e-quasiconvex functions, we consider the coupling function $c : \mathbb{R}^m \times (\mathbb{R}^m \times \mathbb{R}) \to \overline{\mathbb{R}}$ introduced in (3.7) as

$$c(y, (y^*, \alpha)) = \begin{cases} 0, & \text{if } \langle y^*, y\rangle \geq \alpha, \\ -\infty, & \text{otherwise,} \end{cases}$$

for all $y \in \mathbb{R}^m$ and $(y^*, \alpha) \in \mathbb{R}^m \times \mathbb{R}$, and the c-conjugate of p, $p^c : \mathbb{R}^m \times \mathbb{R} \to \overline{\mathbb{R}}$, defined by

$$p^c(y^*, \alpha) = \sup_{y \in \mathbb{R}^m} \{c(y, (y^*, \alpha)) - p(y)\}, \quad \forall (y^*, \alpha) \in \mathbb{R}^m \times \mathbb{R}. \tag{3.13}$$

From (3.1), it is easy to obtain

$$p(0_m) \geq c(0_m, (y^*, \alpha)) - p^c(y^*, \alpha), \ \forall (y^*, \alpha) \in \mathbb{R}^m \times \mathbb{R},$$

so that we define the dual problem of (GP) as

$$(GD) \qquad \underset{y^* \in \mathbb{R}^m, \alpha \in \mathbb{R}}{\text{Max}} \; c(0_m, (y^*, \alpha)) - p^c(y^*, \alpha),$$

which can be equivalently written as

$$\begin{array}{ll} (GD) & \underset{y^* \in \mathbb{R}^m, \alpha \in \mathbb{R}}{\text{Max}} \quad \inf\{\Phi(x, y) : x \in \mathbb{R}^n, \langle y^*, y\rangle \geq \alpha\} \\ & \qquad \text{s.t.} \qquad \alpha \leq 0. \end{array}$$

Proposition 3.12 (Duality Theorem) *The following statements hold:*

- (i) $v(GD) \leq v(GP)$.
- (ii) $v(GD) = (\operatorname{eqco} p)(0_m)$. *If it is finite, the optimal solution set is* $\partial_c(\operatorname{eqco} p)(0_m)$.
- (iii) $v(GD) = v(GP)$ *if and only if* p *is e-quasiconvex at* 0_m. *In this case, if the optimal value is finite, then the optimal solution set to* (GD) *is* $\partial_c p(0_m)$.

3.5 An Application to Consumer Theory

In economics it is rather common the following duality framework (see, e.g., [141, 142]). Given a real-valued, non-decreasing function u on the non-negative orthant of $\mathbb{R}^n$, a dual function v is defined by the relation $v(y) = \sup_{x \in \mathbb{R}^n_+} \{u(x) : \langle x, y\rangle \leq 1\}$. It is easy to see that v is non-increasing and quasiconvex. If u is quasiconcave, some additional assumptions allow to obtain a complete duality between the primal function u and the dual function v in the sense that u can be obtained from v through the relation $u(x) = \inf_{y\in\mathbb{R}^n_+} \{v(y) : \langle x, y\rangle \leq 1\}$. A case in which this scheme is applied is the one of the duality between direct and indirect utility functions in consumer theory.

In an economy in which n different type of commodities are available, each vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n_+$, with x_i denoting the quantity of i-commodity, represents a consumption option. The preferences of a consumer in the set of commodity bundles, $\mathbb{R}^n_+$, are usually represented by a so-called *utility function* $u : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$, that is, for any $x, y \in \mathbb{R}^n_+$, x is preferred to y if and only if $u(x) > u(y)$. If $M > 0$ is the maximal amount of money that the consumer can spend and $p \in \mathbb{R}^n_+$ is the vector of commodity prices, the consumer chooses a commodity bundle x by maximizing $u(x)$ subject to the budget constraint $\langle p, x\rangle \leq M$. As $M > 0$, one can consider the vector of normalized prices $y = p/M$ and the consumer's utility problem may be written as

$$(P(y)) \qquad \sup\left\{u(x) : \langle y, x\rangle \leq 1, x \in \mathbb{R}^n_+\right\}.$$

The function v that associates to y the optimal value of the parameterized problem $P(y)$, $v(y) = \sup\left\{u(x) : \langle y, x\rangle \leq 1, x \in \mathbb{R}^n_+\right\}$, is called the *indirect utility function* associated with u and it gives the maximum utility level that the consumer can attain when he or she faces the vector y of normalized prices. A complete duality between $u(x)$ and $v(y)$ allows to obtain u from v through the relation $u(x) = \inf\left\{v(y) : \langle x, y\rangle \leq 1, y \in \mathbb{R}^n_+\right\}$, which implies that the behavior of the consumer can be equivalently described through the indirect utility function, whose variables are the prices.

The duality between the utility function of a consumer and the corresponding indirect utility function has been studied extensively. In 1977, Crouzeix established quite symmetric conditions for the utility functions when they are continuous [34] and, later, in 1983, for the differentiable case [36]. In 1991, Martínez-Legaz [122] obtained a symmetric duality under the weakest possible assumptions.

Proposition 3.13 *Let $v : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$. There exists a utility function $u : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ having v as its associated indirect utility function if and only if v is non-increasing, evenly quasiconvex and satisfies*

$$v(y) \leq \lim_{\alpha\to 1^-} (\operatorname{cl} v)(\alpha y), \ \forall y \in \operatorname{bd}\mathbb{R}^n_+. \tag{3.14}$$

In this case, one can take u non-decreasing, evenly quasiconcave and satisfying

$$u(x) \geq \lim_{\alpha\to 1^-} \operatorname{cu} u(\alpha x), \ \forall x \in \operatorname{bd}\mathbb{R}^n_+. \tag{3.15}$$

Under these conditions, u is unique, namely,

$$u(x) = \inf\left\{v(y) : \langle x, y\rangle \leq 1, y \in \mathbb{R}^n_+\right\}, \ \forall x \in \mathbb{R}^n_+.$$

According to this theorem, any non-increasing e-quasiconvex function $v : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ satisfying (3.14) is the indirect utility function associated with a unique non-decreasing e-quasiconcave function $u : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ satisfying (3.15).

3.6 Bibliographic Notes

The concept of e-quasiconvex function first appeared in the PhD thesis of J.E. Martínez-Legaz [119] (see also [120]) on generalized conjugation under the name of "normal quasiconvex function". The term "evenly quasiconvex function" was introduced by Passy and Prisman in [138], a work on conjugacy in quasiconvex programming which was followed by a sequel on the same subject [139]. After that, Martínez-Legaz [121] presented a survey on quasiconvex duality theory based on generalized conjugation methods and showed that e-quasiconvex functions constitute the class of regular functions in most of the conjugation schemes.

This chapter is based on [37, 79, 121, 123, 138]. More precisely, the concept of strictly e-quasiconvex function and the relationships reflected in Diagram 3.1 appeared in [138]. Although Passy and Prisman provide an example of an e-quasiconvex function which is not strictly e-quasiconvex, our Example 3.1 has been taken from [37, p. 64]. The e-quasiconvex hull of a function is introduced in [138, Def. 2.5] and its representation as in (3.4) is [138, Th. 2.1]. The notions regarding ascending families were previously considered in [35, Sec. 2]. The equivalences in Theorem 3.1 appear in [37] in the more general context of separable Banach spaces. In particular, $(i) \iff (ii)$ is [37, Prop. 8], $(i) \iff (iii)$ is [37, Prop. 10], $(i) \iff (iv)$ is [37, Prop. 11] and $(i) \iff (v)$ is [37, Prop. 12].

Several generalizations and particularizations of e-quasiconvex functions have been studied in the literature. Next we review two of them.

- According to Thach [179], a convex set $C \subset \mathbb{R}^n$ is R-convex if $\lambda x \in C$ for all $x \in C$ and $\lambda \geq 1$. Thus, in the same way that quasiconvex functions are defined by the convexity of its lower level sets, a function f is said to be R-quasiconvex when its lower level sets are R-convex, and it is R-evenly quasiconvex when these lower level sets are R-evenly convex, i.e., intersection of a family of open halfspaces whose closures do not contain 0_n. Martínez-Legaz [123] characterized the R-evenly quasiconvex functions as those evenly quasiconvex functions f that satisfy a certain simple relation with their lower semicontinuous hull $\operatorname{cl} f$.
- Rubinov and Glover [153] defined, for a given pair of sets (X, V) with a coupling function $[,] : V \times X \to \overline{\mathbb{R}}$, the so-called evenly-$(X, V)$-convex sets as those sets $Z \subset X$ such that, for each $x \in X \backslash Z$ there exists $v \in V$ such that $[v, x] > [v, z]$ for all $z \in Z$. Then, they define the evenly-(X, V)-quasiconvex functions as those whose lower level sets are evenly-(X, V)-convex.

Regarding the conjugation schemes, the $\mathscr{H}$-conjugation method described in Sect. 3.3.1 was introduced by Martínez-Legaz in [119, 120], whereas the alternative conjugation method in Sect. 3.3.2 was pointed out in [124] and obtained as a particular case from the generalized conjugation theory developed by Moreau [130]. More precisely, Proposition 3.5 is [120, Prop. 1], [120, Prop. 2] and [120, Cor. 3] for statements (i), (iii) and (iv), respectively, Proposition 3.6 is [120, Prop. 23'], Proposition 3.7 is [120, Props. 24'–26'], Proposition 3.8 is [124, Props. 6.1 and 6.2] and the result in Theorem 3.2 is given in [124, p. 258]. Finally, the proofs of Propositions 3.11 and 3.12 can be found in [124, Secs. 6.3 and 6.5].

Independently to the above methods, a related symmetric conjugation scheme for quasiconvex functions was introduced by Passy and Prisman in [138]. In this case, however, the biconjugate function coincides with the proper homogeneous e-quasiconvex hull instead of just the e-quasiconvex hull. The motivation of this result is as follows. For $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, its *perspective function of order* 0 (the perspective function in [90] can be understood as the one of order 1) $g_f : \mathbb{R}^n \times \mathbb{R} \to \overline{\mathbb{R}}$ is defined by

$$g_f(x,\lambda) := \begin{cases} f(x/\lambda), & \text{if } \lambda > 0, \\ \sup f, & \text{otherwise.} \end{cases}$$

By construction, g_f is a *positively homogeneous* function of degree zero (that is, $g_f(tx, t\lambda) = g_f(x,\lambda)$ for all $t > 0$), and f is recoverable from g_f in the sense that $f(\cdot) = g_f(\cdot, 1)$. It holds that f is e-quasiconvex if and only if g_f is e-quasiconvex. Only quasiconvex positively homogeneous of degree zero functions (those functions whose lower level sets are convex cones) are considered in [138]. In this case, if f is quasiconvex, then g_f is *proper* in the sense that if $\alpha < \sup g_f$, then one has $0_{n+1} \notin [g_f \leq \alpha]$ (which implies $g_f(0_{n+1}) = \sup g_f(x,\lambda)$). Hence, for any function $g : \mathbb{R}^n \to \overline{\mathbb{R}}$, its *quasi-conjugate* function $g^{\odot} : \mathbb{R}^n \to \overline{\mathbb{R}}$ is given by

$$g^{\odot}(y) = -\inf\{g(x) : \langle y, x\rangle \geq 0\}, \quad \forall y \in \mathbb{R}^n,$$

having that $g^{\odot}$ is a proper homogeneous e-quasiconvex function, and $g = g^{\odot\odot}$ if and only if g is a proper homogeneous e-quasiconvex function (cf. [138, Th. 3.1]). Consequently, the quasi-conjugate $^{\odot}$ induces a one-to-one mapping on the family of proper homogeneous e-quasiconvex functions. The study of those functions whose lower level sets are e-convex cones allows us to recover, with some improvements, some results of Passy and Prisman, and Martínez-Legaz. For instance, a function that attains its maximum at the origin is e-quasiconvex and homogeneous if and only if all its lower level sets are evenly convex cones (cf. [182, Th. III.1.2]).

The notion of λ-quasiconjugate ($\lambda \in \mathbb{R}$) of a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, defined by $f^*_\lambda(y) = \lambda - \inf\{f(x) : \langle y, x\rangle \geq \lambda\}$ for all $y \in \mathbb{R}$, was introduced by Greenberg and Pierskalla [82] and plays an important role in quasiconvex optimization and in the theory of surrogate duality (as well as the Fenchel conjugate does in convex conjugation and Lagrangian duality). Thach [178, 179] established two dualities for a general quasiconvex optimization problem, restricting himself to particular classes of quasiconvex functions. For that purpose, he introduced the notions of H-quasiconjugate and R-quasiconjugate. On the one hand, the *H-quasiconjugate* of f is defined by $f^H(y) = -\inf\{f(x) : \langle y, x\rangle \geq 1\}$ if $y \neq 0_n$, and $f^H(0_n) = -\sup\{f(x) : x \in \mathbb{R}^n\}$. The fundamental theorem says that f is H-evenly quasiconvex (i.e., all its lower level sets are evenly convex containing 0_n) and $f(0_n) = \inf\{f(x) : x \in \mathbb{R}^n \backslash \{0_n\}\}$ if and only if $f = f^{HH}$. On the other hand, the *R-quasiconjugate* of f is defined by $f^R(y) = -\inf\{f(x) : \langle y, x\rangle \geq -1\}$ for all $y \in \mathbb{R}^n$. In this case, the fundamental theorem says that a function f is R-evenly quasiconvex if and only if $f = f^{RR}$. These concepts have been applied by Suzuki and Kuroiwa [171, 172, 174, 175] to provide duality theorems and set containment characterizations for quasiconvex programming. Furthermore, the paper [179] provides an application to a decentralization by prices for the von Neumann equilibrium problem.

Rubinov and Dutta [154] obtained a Hadamard type inequality for non-negative evenly quasiconvex functions that attain their minimum. The asymptotically sharp constant associated with the inequality over the unit square in the two-dimensional plane is explicitly calculated. An extension of this Hadamard type inequality to non-negative quasiconvex functions was obtained a year later by Hadjisavvas [84].

Quasiconvex analysis has always been deeply related with economic theory. Thus, the seminal work of de Finetti [57] was motivated by problems in Paretian ordinal utility and, later, research in quasiconvex duality was motivated by the dual description of preferences and technologies in microeconomics (see, e.g., [38]). The application of duality theory to problems arising in economics, provides dual problems which usually have nice interpretations that give new perspectives for analyzing the associated primal problems. That is the case of the application to consumer theory described in Sect. 3.5. Proposition 3.13, which establishes a symmetric duality between direct and indirect utility functions under the weakest possible assumptions, is [124, Th. 6.16], although the original result appeared in [122, Th. 2.2] in the more general framework of locally convex topological vector spaces.

Other similar duality schemes, involving suitable extensions of the concept of e-convex set, enjoy a number of applications in finance and economics, particularly, in the context of decision theory [29] and risk measures [30, 60]. For instance, Frittelli and Maggis [60, 115] introduced a generalization of the concept of e-convex set in the conditional framework, providing also the corresponding generalized version of the bipolar theorem. Then, they applied this notion to obtain the dual representation of conditionally evenly quasiconvex maps, which turns out to be a key tool in the study of quasiconvex dynamic risk measures.

Chapter 4
Evenly Convex Functions

In this chapter we introduce *evenly convex functions* as those whose epigraphs are evenly convex sets, and develop a duality theory for nonlinear programming problems involving evenly convex functions, that is, evenly convex optimization problems. In Sect. 4.1 we present the main properties of this class of convex functions that contains the important subclass of lower semicontinuous convex functions, whose relevance in convex analysis comes from the fact that the Fenchel conjugacy is an involution on most of them. More precisely, any proper lower semicontinuous convex function coincides with its biconjugate. In Sects. 4.2 and 4.3 we introduce the evenly convex hull of a function and appropriate conjugation schemes for evenly convex functions, respectively. Finally, in Sect. 4.4, we use the perturbational approach for developing the so-called *c-conjugate duality theory*, providing closedness-type regularity conditions. These conditions will be expressed in terms of the even convexity of the involved functions, for both strong and stable strong duality for convex optimization problems.

4.1 Evenly Convex Functions

In the same way that convex functions are defined by the convexity of their epigraphs, we will say that an extended real-valued function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is *evenly convex* (in brief, *e-convex*) provided that its epigraph $\operatorname{epi} f$ is an e-convex set in $\mathbb{R}^{n+1}$. Obviously, any e-convex function is convex and, since any closed convex set is e-convex, lsc convex functions constitute a subclass of e-convex functions. In particular, any finite-valued convex function is e-convex.

The next two examples show that not every convex function is e-convex and not every e-convex function is a lower semicontinuous convex function, respectively.

Example 4.1 (Example 3.2 Revisited) The epigraph of the function f defined in Example 3.2, which is represented in Fig. 4.1, is obviously a convex set, but it is

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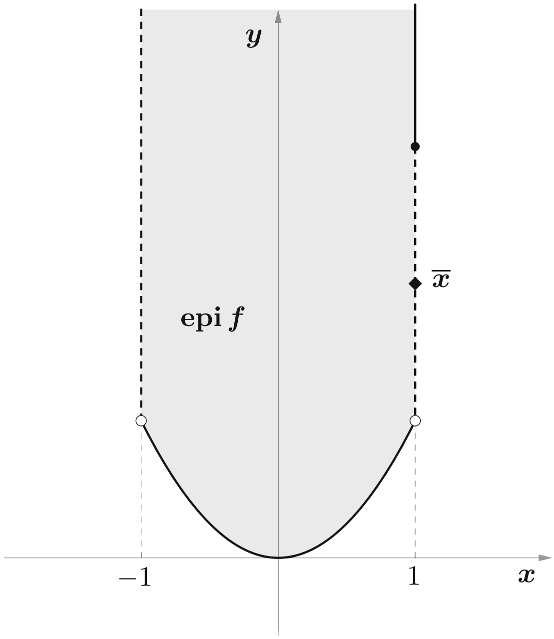

Fig. 4.1 epi f is not e-convex

not e-convex. In fact, considering $\overline{x} = (1, 2) \in \mathbb{R}^2 \backslash (\text{epi } f)$, we have that, for any hyperplane H containing $\overline{x}$, $H \cap (\text{epi } f) \neq \emptyset$.

Example 4.2 One can easily check that the epigraph, represented in Fig. 4.2, of the function $f : \mathbb{R} \to \overline{\mathbb{R}}$ defined by

$$f(x) = \begin{cases} x^2, & \text{if } x > -1, \\ +\infty, & \text{if } x \leq -1, \end{cases}$$

is e-convex, but it is not closed. Then, f is an e-convex function which is not lower semicontinuous.

Recall that closedness of lower level sets characterizes the class of lower semicontinuous functions. While lower level sets are convex for convex functions, a function whose lower level sets are all convex needs not be convex (see, e.g., the function f in Example 3.7). Analogously, lower level sets of e-convex functions are all e-convex, that is, every e-convex function is e-quasiconvex as well. This fact follows from the identity

$$[f \leq r] \times \{r\} = (\text{epi } f) \cap (\mathbb{R}^n \times \{r\}), \quad \forall r \in \mathbb{R},$$

and the properties of e-convex sets studied in Chap. 1 (see Proposition 1.2(*iii*) and (v)).

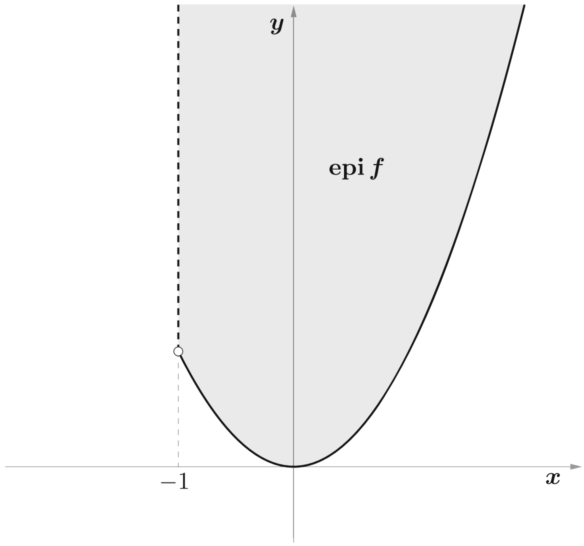

Fig. 4.2 f is an e-convex function but it is not lower semicontinuous

Having these facts in mind, the following diagram shows the relations between different types of convex and quasiconvex functions.

Convex convex epigraph	$\Rightarrow$	**Quasiconvex** convex lower level sets
$\Uparrow$		$\Uparrow$
Evenly convex evenly convex epigraph	$\Rightarrow$	**Evenly quasiconvex** evenly convex lower level sets
$\Uparrow$		$\Uparrow$
Lsc convex closed and convex epigraph	$\Rightarrow$	**Lsc quasiconvex** closed and convex lower level sets

Diagram 4.1 Evenly convex and related functions

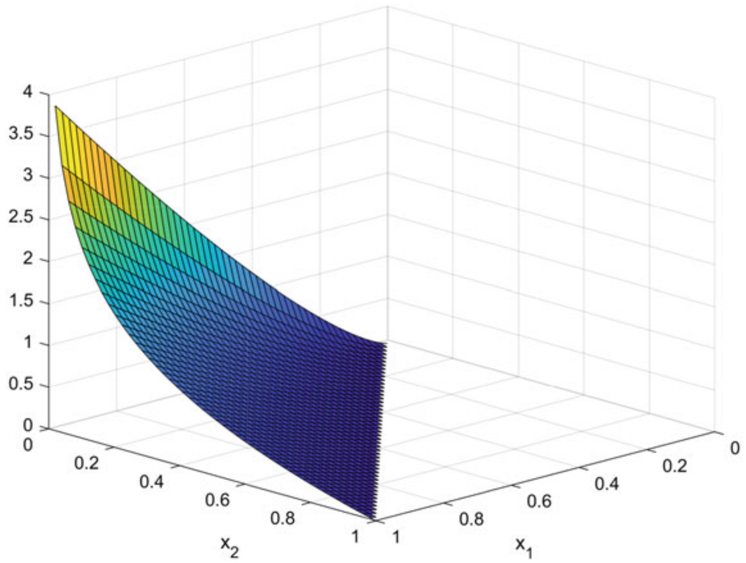

Fig. 4.3 The graph of f

It is well-known that the effective domain of a convex function is a convex set, since it is the projection on $\mathbb{R}^n$ of the epigraph. However, the projection of an e-convex set is not, in general, e-convex. Then, the effective domain of an e-convex function is convex but not necessarily e-convex, as the following example shows Example 4.3.

Example 4.3 Let $f : \mathbb{R}^2 \to \overline{\mathbb{R}}$ be the function defined by

$$f(x) = \begin{cases} x_1 \ln \frac{x_1}{x_2}, & \text{if } x \in E, \\ 0, & \text{if } x = 0_2, \\ +\infty, & \text{otherwise,} \end{cases}$$

where $E = \left\{x \in \mathbb{R}^2 : 0 < x_1 \leq 1,\ 0 < x_2 \leq x_1\right\}$, whose graph is represented in Fig. 4.3. Observe that $\operatorname{dom} f = E \cup \{0_2\}$ (represented in Fig. 4.4) is not e-convex, although, as we shall see, f is a lower semicontinuous convex function and, therefore, it is e-convex.

Consider the function $g : \mathbb{R}^2_{++} \to \mathbb{R}$ defined by $g(x) = x_1 \ln \frac{x_1}{x_2}$, where $\mathbb{R}^2_{++} = (]0, +\infty[)^2$. Since g is a twice continuously differentiable real-valued function on the open convex set $\mathbb{R}^2_{++}$ and its Hessian matrix is positive semi-definite for every $x \in \mathbb{R}^2_{++}$, according to [148, Th. 4.5], we have that g is a convex function on $\mathbb{R}^2_{++}$. On the other hand, both functions f and g coincide on the convex subset E of $\mathbb{R}^2_{++}$ and, hence, f is convex on E. The convexity of f on $\operatorname{dom} f = E \cup \{0_2\}$ can be

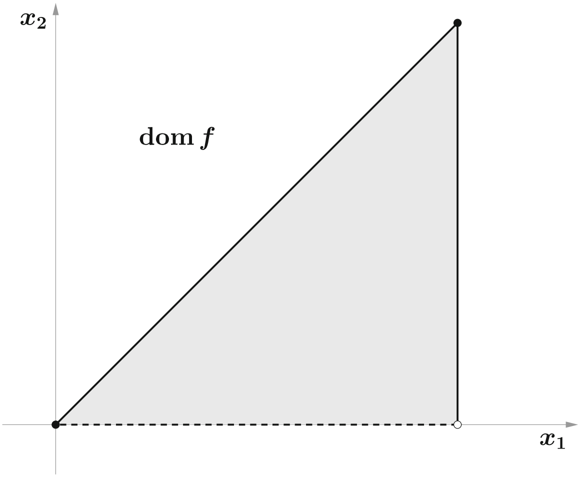

Fig. 4.4 The domain of f is not e-convex

easily proved showing that $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$ for $x \in E$, $y = 0_2$ and $0 < \lambda < 1$. So, f is a convex function.

Since every convex function is always lsc except perhaps at relative boundary points of its domain, we only have to prove that f is lsc on $\operatorname{rbd}(\operatorname{dom} f)$.

Since g is a finite-valued convex function on $\mathbb{R}^2_{++}$, g is lsc at any $\overline{x} \in \mathbb{R}^2_{++}$, i.e., for all $\lambda < g(\overline{x})$, there exists a neighborhood of $\overline{x}$, $V_{\overline{x}}$, in $\mathbb{R}^2_{++}$ such that $\lambda < g(x)$ for all $x \in V_{\overline{x}}$. Then, given $\overline{x} \in \operatorname{rbd}(\operatorname{dom} f) \cap E \subset \mathbb{R}^2_{++}$, we have that, for all $\lambda < f(\overline{x}) = g(\overline{x})$, there exists a neighborhood of $\overline{x}$, $V_{\overline{x}}$, in $\mathbb{R}^2_{++} \subset \mathbb{R}^2$ such that $\lambda < g(x) = f(x)$ for all $x \in V_{\overline{x}} \cap E$. Obviously, if $x \in V_{\overline{x}} \cap (\mathbb{R}^2 \backslash E) \subset \mathbb{R}^2 \backslash (\operatorname{dom} f)$, then $f(x) = +\infty > \lambda$, so that f is lsc at $\overline{x}$.

On the other hand, lower semicontinuity of f at 0_2 is a direct consequence of $f(x) \geq 0 = f(0_2)$ for all $x \in \mathbb{R}^2$.

Finally, for $\overline{x} \in]0, 1] \times \{0\}$, we have $f(\overline{x}) = +\infty > \lambda$ for all $\lambda \in \mathbb{R}$. Moreover, since

$$\lim_{\substack{x \to \overline{x} \\ x \in E}} f(x) = \lim_{\substack{x \to \overline{x} \\ x \in E}} g(x) = +\infty,$$

we have that, given any $\lambda \in \mathbb{R}$, there exists a neighborhood of $\overline{x}$, $V_{\overline{x}}$, in $\mathbb{R}^2 \backslash \{0_2\}$ such that $f(x) > \lambda$, for all $x \in V_{\overline{x}} \cap E$, and $f(x) = +\infty > \lambda$, for all $x \in V_{\overline{x}} \cap (\mathbb{R}^2 \backslash E)$. Then, f is also lsc on $]0, 1] \times \{0\}$.

The next result states conditions ensuring the even convexity of the effective domain of an e-convex function. Recall that a function is said to be *improper* if it is not proper (i.e., if it is identically $+\infty$ or takes the value $-\infty$ at some point). Clearly, the improper e-convex functions identically either $+\infty$ or $-\infty$ have e-convex effective domains. Precise references to all missing proofs in this chapter are appropriately given in Sect. 4.5.

Theorem 4.1 (On the Effective Domain of an e-Convex Function) *Let* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$.

(i) *If* f *is an improper function such that* $f(\overline{x}) = -\infty$ *for some* $\overline{x} \in \mathbb{R}^n$, *then* f *is e-convex if and only if* $\operatorname{dom} f$ *is e-convex and* $f(x) = -\infty$ *for all* $x \in \operatorname{dom} f$.
(ii) *If* f *is a proper e-convex function bounded from above on* $\operatorname{dom} f$, *then* $\operatorname{dom} f$ *is e-convex.*

So, the even convexity of the effective domain is a necessary condition for the even convexity of proper functions which are bounded from above. However, this is not a sufficient condition, as illustrates the function considered in Example 4.1, which is bounded on $\operatorname{dom} f =]-1, 1]$ and $\operatorname{dom} f$ is e-convex, but f is not e-convex.

As stated in Diagram 4.1, the class of e-convex functions is intermediate between the class of lower semicontinuous convex functions and the class of convex functions (that are always lower semicontinuous except perhaps at relative boundary points of their domains). Next we provide a characterization of proper e-convex functions in terms of lower semicontinuity.

Theorem 4.2 (Characterization of e-Convex Functions I) *Let* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *be a proper function. Then,* f *is e-convex if and only if* f *is convex and lower semicontinuous on* $\operatorname{eco}(\operatorname{dom} f)$.

Corollary 4.1 *Let* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *be an e-convex function whose effective domain is closed. Then,* f *is a lower semicontinuous convex function.*

It is well-known that any convex function f is lsc/usc/continuous relative to $\operatorname{rint}\operatorname{dom} f$ (see [148, Th. 10.1]). Moreover, if f is a proper e-convex function, then f is lower semicontinuous on the greater set $\operatorname{eco}\operatorname{dom} f$. When we ask whether any proper convex function f can be assumed upper semicontinuous on $\operatorname{rbd}(\operatorname{dom} f) \cap \operatorname{dom} f$ relative to $\operatorname{dom} f$, the answer is negative in general (see [148, p. 83]). However, it is easy to prove that this property holds for univariate functions. Consequently, we consider the concept of upper semicontinuity along lines (as in [114]).

Given a nonempty convex set $A \subset \mathbb{R}^n$, a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be *upper (resp. lower) semicontinuous along lines on* $A \subset \mathbb{R}^n$ if, for every $x, y \in A$, the function $f_{x,y} : [0, 1] \to \overline{\mathbb{R}}$, given by

$$f_{x,y}(t) := f(x + t(y - x)),$$

is upper semicontinuous (resp. lower semicontinuous) at t relative to $[0, 1]$, for any $t \in [0, 1]$. Moreover, f is said to be *continuous along lines* on A if f is upper and lower semicontinuous along lines on A.

For any proper convex function f, $\operatorname{dom} f$ is a nonempty convex set and, for every $x, y \in \operatorname{dom} f$, $f_{x,y}$ is a univariate convex function and, therefore, it is upper semicontinuous relative to $[0, 1]$. As a consequence, any proper convex function is upper semicontinuous along lines on its domain. Furthermore, it is easy to prove that any proper convex function f that is lower semicontinuous on a nonempty convex set $A \subset \operatorname{dom} f$, is also lower semicontinuous along lines on A.

Proposition 4.1 (Necessary Conditions for Even Convexity) *If f is a proper e-convex function, then it is continuous along lines on its domain, and its image set* $\operatorname{Im} f := \{f(x) : x \in \operatorname{dom} f\}$ *is convex.*

It is well-known that convexity and lower semicontinuity are preserved by the most important functional operations. The following theorem shows that the same happens with even convexity.

Theorem 4.3 (Operations with e-Convex Functions) *Let $f, g, f_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i \in I$.*

(i) *If f is an e-convex function and $\alpha > 0$, then αf is e-convex.*
(ii) *If $\{f_i\}_{i \in I}$ is a family of e-convex functions, then $\sup_{i \in I} f_i$ is an e-convex function.*
(iii) *If f and g are two proper e-convex functions, then $f + g$ is e-convex.*
(iv) *If f and g are two e-convex functions and f is improper, then $f+g$ is e-convex if and only if* $\operatorname{dom}(f + g)$ *is an e-convex set.*
(v) *If f and g are two e-convex functions, where f is improper and g is proper with e-convex domain, then $f + g$ is e-convex.*
(vi) *If f and g are two e-convex functions, where f is improper and g is proper and bounded on its domain, then $f + g$ is e-convex.*
(vii) *If f and g are two improper e-convex functions, then $f + g$ is e-convex.*

In general, (iii) is not true whenever one of the functions is not proper.

Example 4.4 Let $f, g : \mathbb{R}^2 \to \overline{\mathbb{R}}$, where f is the proper e-convex function defined in Example 4.3 and g is the improper e-convex function defined by

$$g(x) = \begin{cases} -\infty, & \text{if } x \in [0, 1]^2, \\ +\infty, & \text{otherwise.} \end{cases}$$

Obviously, $\operatorname{dom}(f + g) = (\operatorname{dom} f) \cap (\operatorname{dom} g) = \operatorname{dom} f$, that is not an e-convex set, and, in particular,

$$(f + g)(x) = \begin{cases} -\infty, & \text{if } x \in \operatorname{dom} f, \\ +\infty, & \text{otherwise,} \end{cases}$$

so, by Theorem 4.1, $f + g$ is not an e-convex function.

4.2 Evenly Convex Hull

Given a function f, its *evenly convex hull*, abbreviated as *e-convex hull* and denoted by $\operatorname{eco} f$, is defined to be the largest e-convex function minorizing f. Obviously, thanks to Theorem 4.3(*ii*), $\operatorname{eco} f$ coincides with the pointwise supremum of all the e-convex functions minorizing f. Moreover, a function f is said to be *evenly convex at a point* $\overline{x}$ provided that $f(\overline{x}) = (\operatorname{eco} f)(\overline{x})$.

With each subset $A \subset \mathbb{R}^{n+1}$, we associate the so-call *lower-bound function* $\varphi_A : \mathbb{R}^n \to \overline{\mathbb{R}}$ (cf. [90]) defined by

$$\varphi_A(x) := \inf\{t \in \mathbb{R} : (x, t) \in A\}.$$

If $A = \emptyset$ then $\varphi_A(x) = +\infty$ for all $x \in \mathbb{R}^n$. A set $A \subset \mathbb{R}^{n+1}$ is said to be *ascending* if either $A = \emptyset$ or there exists $(\overline{x}, \overline{t}) \in A$ such that $(\overline{x}, t) \in A$ for all $t \geq \overline{t}$. Thanks to Proposition 1.1(v), which can be equivalently written as follows: "if there exist $x \in C \subset \mathbb{R}^n$ and $y \neq 0_n$ such that $\{x + \lambda y : \lambda \geq 0\} \subset C$, then $d \in 0^+(\operatorname{eco} C)$", one easily gets that for a nonempty e-convex set $A \subset \mathbb{R}^{n+1}$, A is ascending if and only if $(0_n, 1) \in 0^+ A$. This fact allows to get the following results, where the strict epigraph of a function f is denoted by

$$\operatorname{epi}_s f := \left\{(x, \lambda) \in \mathbb{R}^{n+1} : x \in \operatorname{dom} f,\ f(x) < \lambda\right\}.$$

Theorem 4.4 (Sufficient Conditions for Even Convexity) *Let $A \subset \mathbb{R}^{n+1}$ be an e-convex set and $f : \mathbb{R}^n \to \overline{\mathbb{R}}$. Then*

(i) *If A is ascending, then φ_A is an e-convex function.*
(ii) *If A is such that* $\operatorname{epi}_s f \subset A \subset \operatorname{epi} f$, *then f is an e-convex function. In particular, any function whose strict epigraph is e-convex, is e-convex as well.*

Concerning statement (i) in the above theorem, it is worth saying that the assumption that A is ascending is not superfluous in order to guarantee the even convexity of φ_A (see the e-convex set in [101, Ex. 3.1]). This makes a difference with a well-known result ensuring that if $A \subset \mathbb{R}^n \times \mathbb{R}$ is a closed convex set, then φ_A is a lower semicontinuous convex function (see [148, Th. 5.3]), no matter A is ascending or not.

Regarding statement (ii), whose proof derives from (i), not every e-convex function has an e-convex strict epigraph, as we can see in the following example.

Example 4.5 Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be the function defined by

$$f(x) = \begin{cases} -\sqrt{1 - x^2}, & \text{if } -1 \leq x \leq 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

In Fig. 4.5a, we can see that the epigraph of f is e-convex (so f is an e-convex function), whereas Fig. 4.5b shows that its strict epigraph is not.

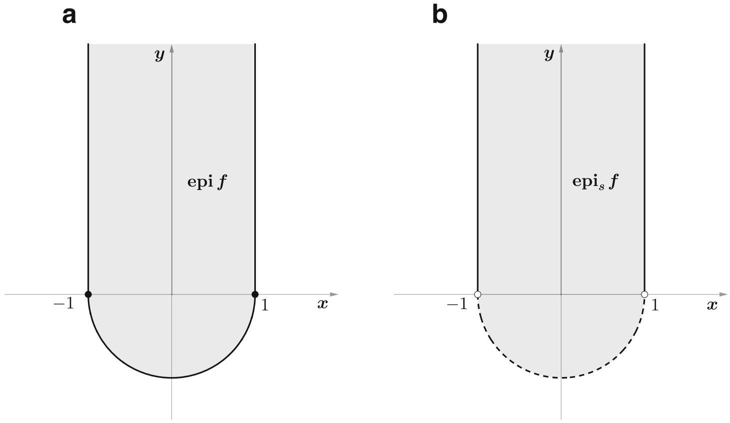

Fig. 4.5 (**a**) The epigraph of f; (**b**) The strict epigraph of f

Theorem 4.5 *Let* $A \subset \mathbb{R}^{n+1}$ *and* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$.

(i) *If* eco A *is ascending, then*

$$\operatorname{eco} \varphi_A = \varphi_{\operatorname{eco} A}.$$

(ii) *If* A *is such that* $\operatorname{epi}_s f \subset A \subset \operatorname{epi} f$, *then*

$$\operatorname{eco} f = \varphi_{\operatorname{eco} A}.$$

Consequently,

$$(\operatorname{eco} f)(x) = \inf \{a \in \mathbb{R} : (x, a) \in \operatorname{eco} \operatorname{epi} f\}, \ \forall x \in \mathbb{R}^n.$$

Next we summarize some basic properties regarding the domain and the epigraph of the e-convex hull.

Theorem 4.6 (Properties of the e-Convex Hull) *Let* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *be a convex function. Then:*

(i) $\operatorname{cl} f \leq \operatorname{eco} f \leq f$.
(ii) $\operatorname{epi}_s(\operatorname{eco} f) \subset \operatorname{eco} \operatorname{epi} f \subset \operatorname{epi}(\operatorname{eco} f)$.
(iii) $\operatorname{dom} f \subset \operatorname{dom}(\operatorname{eco} f) \subset \operatorname{dom}(\operatorname{cl} f) \subset \operatorname{cl} \operatorname{dom} f$.
(iv) $\operatorname{dom}(\operatorname{eco} f) \subset \operatorname{eco} \operatorname{dom} f \subset \operatorname{cl} \operatorname{dom} f$.
(v) $\operatorname{rint} \operatorname{dom}(\operatorname{eco} f) = \operatorname{rint} \operatorname{dom} f$.

The inequalities in statement (i) and the inclusions in statements (ii), (iii) and (iv) can be strict, as the following example shows.

Example 4.6 (Example 3.2 Revisited) Consider the function $f : \mathbb{R} \to \overline{\mathbb{R}}$ defined in Example 3.2. Taking into account the epigraph of f, represented in Fig. 4.1, it is easy to conclude that

$$(\operatorname{eco} f)(x) = \begin{cases} x^2, & \text{if } -1 < x \leq 1, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$(\operatorname{cl} f)(x) = \begin{cases} x^2, & \text{if } -1 \leq x \leq 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Thus, $\operatorname{cl} f \lneqq \operatorname{eco} f \lneqq f$ and

$$\operatorname{dom} f = \operatorname{dom}(\operatorname{eco} f) = \operatorname{eco}(\operatorname{dom} f) =]-1, 1] \subsetneq [-1, 1] = \operatorname{dom}(\operatorname{cl} f) = \operatorname{cl} \operatorname{dom} f.$$

On the other hand, in Fig. 4.6, we can see that

$$\operatorname{epi}_s(\operatorname{eco} f) \subsetneq \operatorname{eco} \operatorname{epi} f \subsetneq \operatorname{epi}(\operatorname{eco} f).$$

The following result characterizes the even convexity of a function at a point. Corresponding characterizations have been given in Chap. 3 for even quasiconvexity of a function at a point by using lower level sets instead of epigraphs.

Theorem 4.7 (Characterization of the Even Convexity at a Point) *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $\overline{x} \in \mathbb{R}^n$. Then:*

(i) *f is e-convex at $\overline{x}$ if and only if $(\overline{x}, a) \notin \operatorname{eco} \operatorname{epi} f$ for every $a < f(\overline{x})$.*
(ii) *f is e-convex at $\overline{x}$ if and only if, for any $a < f(\overline{x})$, there exists $q \in \mathbb{R}^{n+1}$ such that $\langle q, (x, \lambda) - (\overline{x}, a)\rangle > 0$, for all $(x, \lambda) \in \operatorname{epi} f$.*
(iii) *f is e-convex if and only if it is e-convex at every $\overline{x} \in \mathbb{R}^n$.*

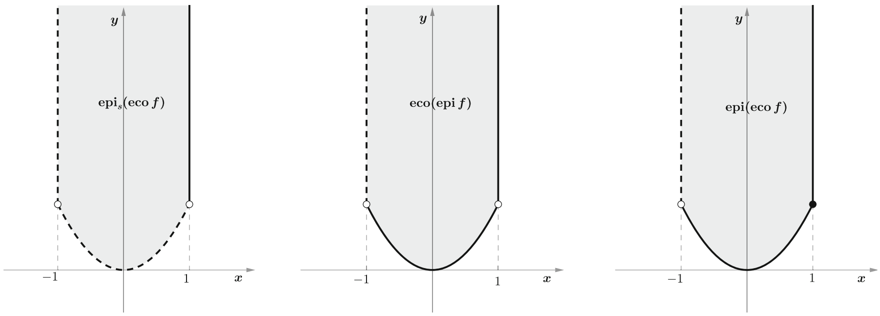

Fig. 4.6 The inclusions in Theorem 4.6(*ii*) are strict

The even convexity of a function can also be characterized in most cases through the set $\mathscr{E}_f$ of all the *e-affine minorants* of f, that is,

$$\mathscr{E}_f := \left\{ a : \mathbb{R}^n \to \overline{\mathbb{R}} : a \text{ is e-affine and } a \leq f \right\}.$$

Here, a function $a : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be *e-affine* if there exist $c, z \in \mathbb{R}^n$ and $\alpha, t \in \mathbb{R}$ such that, for all $x \in \mathbb{R}^n$,

$$a(x) = \begin{cases} \langle x, c \rangle - \alpha, & \text{if } \langle x, z \rangle < t, \\ +\infty, & \text{if } \langle x, z \rangle \geq t, \end{cases}$$

i.e., a is the restriction of an ordinary affine function to some open half-space, and it is identically $+\infty$ on its complement. For instance, the function $a : \mathbb{R} \to \overline{\mathbb{R}}$ defined by

$$a(x) = \begin{cases} \frac{3x-6}{5}, & \text{if } x > \frac{-6}{5}, \\ +\infty, & \text{if } x \leq \frac{-6}{5}, \end{cases}$$

is an e-affine minorant of the function $f : \mathbb{R} \to \overline{\mathbb{R}}$ in Example 4.5, that is, $a \in \mathscr{E}_f$ (see Fig. 4.7).

Theorem 4.8 (Characterization of e-Convex Functions II) *Let f be a proper function. Then, f is e-convex if and only if*

$$f = \sup \mathscr{E}_f. \tag{4.1}$$

The representation of e-convex functions as suprema of their e-affine minorants in (4.1) also applies to improper e-convex functions identically either $-\infty$ or $+\infty$, by considering $\mathscr{E}_f$ the empty set and the set of all e-affine functions, respectively.

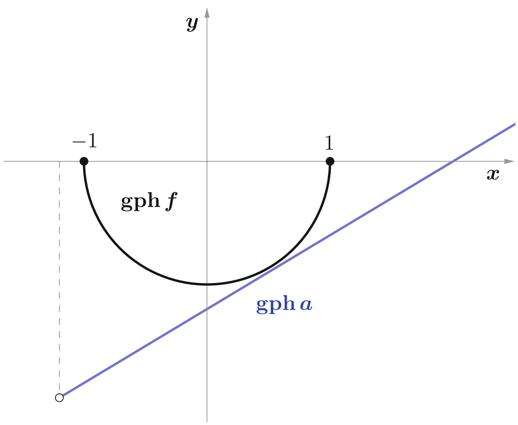

Fig. 4.7 An e-affine minorant of f

However, such a representation does not apply to those improper e-convex functions f such that $f(\overline{x}) = -\infty$ for some and $\overline{x} \in \mathbb{R}^n$ and $\operatorname{dom} f \neq \mathbb{R}^n$, as in this case $\mathscr{E}_f = \emptyset$.

Corollary 4.2 *Let* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$.

(i) *If* f *has a proper e-convex minorant, then*

$$\operatorname{eco} f = \sup \mathscr{E}_f.$$

(ii) *If* f *is a proper e-convex function, then*

$$\operatorname{eco} \operatorname{dom} f = \bigcap_{a \in \mathscr{E}_f} \operatorname{dom} a.$$

4.3 Conjugacy and Subdifferentiability

In this section we shall adopt the generalized conjugation theory of Moreau described in Sect. 3.3 in order to provide up to three different conjugation schemes which are appropriate for e-convex functions, in the sense that a (proper) function is e-convex if and only if it coincides with its biconjugate. For this purpose, we shall recall that every lsc convex function f admitting a continuous affine minorant coincides with its Fenchel biconjugate f^{**} (see, e.g., [41, Prop. 3.1]).

4.3.1 First Conjugacy Scheme

Let us consider the space $W := \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and the coupling function $c : \mathbb{R}^n \times W \to \mathbb{R} \cup \{+\infty\}$ defined by

$$c(x, (x^*, u^*, \alpha)) := \begin{cases} \langle x^*, x \rangle, & \text{if } \langle u^*, x \rangle < \alpha, \\ +\infty, & \text{if } \langle u^*, x \rangle \geq \alpha. \end{cases} \tag{4.2}$$

Then, the *c-conjugate* of a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is the function $f^c : W \to \overline{\mathbb{R}}$ given, for every $(x^*, u^*, \alpha) \in W$, by

$$\begin{aligned} f^c(x^*, u^*, \alpha) &= \sup_{x \in \mathbb{R}^n} \left\{ c(x, (x^*, u^*, \alpha)) - f(x) \right\} \\ &= \begin{cases} f^*(x^*), & \text{if } \operatorname{dom} f \subset [u^* < \alpha], \\ +\infty, & \text{otherwise.} \end{cases} \end{aligned}$$

From this expression, one easily gets that

$$\operatorname{dom} f^c = \operatorname{dom} f^* \times \left\{(u^*, \alpha) \in \mathbb{R}^n \times \mathbb{R} : \operatorname{dom} f \subset [u^* < \alpha]\right\}.$$

Similarly, the *c'-conjugate* of a function $g : W \to \overline{\mathbb{R}}$ is the function $g^{c'} : \mathbb{R}^n \to \overline{\mathbb{R}}$ given, for every $x \in \mathbb{R}^n$, by

$$\begin{aligned} g^{c'}(x) &= \sup_{(x^*,u^*,\alpha)\in W} \left\{c(x, (x^*, u^*, \alpha)) - g(x^*, u^*, \alpha)\right\} \\ &= \begin{cases} g^*(x, 0_n, 0), & \text{if } \operatorname{dom} g \subset [(0_n, x, -1) < 0], \\ +\infty, & \text{otherwise.} \end{cases} \end{aligned}$$

For this particular coupling function in (4.2), that is, for this particular conjugation scheme, we have that the class of c-elementary functions is precisely the family of e-affine functions, and then, by Theorem 4.8, the class of Γ_c-convex functions coincides with the class of e-convex functions from $\mathbb{R}^n$ to $\mathbb{R} \cup \{+\infty\}$ along with the function identically $-\infty$. We will say that a function $g : W \to \overline{\mathbb{R}}$ is *e$'$-convex* if it is $\Gamma_{c'}$-convex, and the e$'$-convex hull of $g : W \to \overline{\mathbb{R}}$ will be denoted by $\mathrm{e}' \operatorname{co} g$.

The most remarkable properties of this c-conjugation scheme are summarized in the following theorem.

Theorem 4.9 (Properties of c-Conjugation) *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, $g : W \to \overline{\mathbb{R}}$ and $c : \mathbb{R}^n \times W \to \mathbb{R} \cup \{+\infty\}$ be as in (4.2). Then,*

(i) *f^c is e$'$-convex, and $g^{c'}$ is e-convex.*

(ii) $f^{cc'} = \begin{cases} f^{**} + \delta_{\operatorname{eco} \operatorname{dom} f}, & \text{if } \operatorname{dom} f^* \neq \emptyset, \\ -\infty, & \text{otherwise.} \end{cases}$

(iii) *If f is minorized by a proper e-convex function, then* $\operatorname{eco} f = f^{cc'}$.

(iv) *If f does not take the value $-\infty$, then f is e-convex if and only if $f = f^{cc'}$.*

(v) $\mathrm{e}' \operatorname{co} g = g^{c'c}$.

(vi) *g is e$'$-convex if and only if $g = g^{c'c}$.*

Consequently, $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is e-convex at $x \in \mathbb{R}^n$ if and only if $f(x) = f^{cc'}(x)$, and $g : W \to \overline{\mathbb{R}}$ is e$'$-convex at $(x^*, u^*, \alpha) \in W$ if and only if $g(x^*, u^*, \alpha) = g^{c'c}(x^*, u^*, \alpha)$.

Statement (iv) also holds whenever f is the function identically $-\infty$, but it fails if f is an arbitrary function such that $\emptyset \neq \operatorname{dom} f \neq \mathbb{R}^n$ and $f(\overline{x}) = -\infty$ for some $\overline{x} \in \mathbb{R}^n$, as in that case $f^{cc'}$ is identically $-\infty$. Furthermore, if $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ does not admit any proper e-convex minorant, then the relation $\operatorname{eco} f = f^{cc'}$ may be false, as it is shown by the following example, which involves the so-called *valley function* υ_C of a set $C \subset \mathbb{R}^n$, defined as $\upsilon_C(x) := -\infty$ if $x \in C$ and $\upsilon_C(x) := +\infty$ if $x \notin C$.

Example 4.7 Consider the function f defined on the real line given by

$$f(x) = \begin{cases} 0, & \text{if } x = 0, \\ -\frac{1}{|x|}, & \text{if } 0 < |x| < 1, \\ +\infty, & \text{if } |x| \geq 1. \end{cases}$$

Its effective domain is $]-1, 1[$, which is an e-convex set. We have $f^* \equiv +\infty$ and, by Theorem 4.9(*ii*), $f^{cc'} \equiv -\infty$. However, it is easy to see that $\operatorname{eco} f = v_{]-1,1[}$. Hence, in this case the identity $\operatorname{eco} f = f^{cc'}$ fails because of the fact that f does not have a proper e-convex minorant.

Next example illustrates how this conjugation scheme works with a well-known function.

Example 4.8 By applying the conjugation scheme developed in this subsection to the indicator function δ_C of $C \subset \mathbb{R}^n$, for every $(x^*, u^*, \alpha) \in W$ one has

$$\delta_C^c(x^*, u^*, \alpha) = \begin{cases} \sigma_C(x^*), & \text{if } C \subset [u^* < \alpha], \\ +\infty, & \text{otherwise.} \end{cases}$$

The function $\delta_C^c : W \to \overline{\mathbb{R}}$ can be regarded as a kind of support function of C. The second conjugate is

$$\delta_C^{cc'}(x) = \begin{cases} \delta_{\operatorname{cl}\operatorname{co} C}(x), & \text{if } x \in \operatorname{eco} C, \\ +\infty, & \text{if } x \notin \operatorname{eco} C. \end{cases}$$

Consequently, $\operatorname{eco} \delta_C = \delta_C^{cc'} = \delta_{\operatorname{eco} C}$.

4.3.2 Second Conjugacy Scheme

Next we apply the e-quasiconvex conjugation scheme developed in Sect. 3.3 in order to get another conjugation scheme for e-convex functions. More precisely, we will consider a slightly modification of the coupling function in (3.7), in order to get a conjugation scheme for e-convex functions in $\mathbb{R}^n$ by means of the conjugation scheme for e-quasiconvex functions in $\mathbb{R}^{n+1}$. Thus, consider the coupling function $c : \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \times \mathbb{R}) \to \overline{\mathbb{R}}$ defined by

$$c\left((x, t), \left(x^*, t^*, \alpha\right)\right) := \begin{cases} 0, & \text{if } \langle x^*, x\rangle + tt^* \geq \alpha, \\ -\infty, & \text{if } \langle x^*, x\rangle + tt^* < \alpha. \end{cases} \tag{4.3}$$

With each function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, we associate the function $\widetilde{f} : \mathbb{R}^{n+1} \to \overline{\mathbb{R}}$ defined, for every $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, by

$$\widetilde{f}(x, t) := f(x) + t.$$

Observe that $\operatorname{dom} \widetilde{f} = \operatorname{dom} f \times \mathbb{R}$. If we assume that $\operatorname{dom} f \neq \emptyset$ and consider the coupling function c in (4.3), then the *c-conjugate* of $\widetilde{f}$ is

$$\widetilde{f}^c\left(x^*, t^*, r\right) = \begin{cases} f^*\left(\frac{x^*}{t^*}\right) - \frac{r}{t^*}, & \text{if } t^* > 0, \\ +\infty, & \text{if } t^* < 0, \\ +\infty, & \text{if } t^* = 0 \text{ and } \operatorname{dom} f \not\subset \left[x^* < r\right], \\ -\infty, & \text{if } t^* = 0 \text{ and } \operatorname{dom} f \subset \left[x^* < r\right], \end{cases}$$

for every $(x^*, t^*, r) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$. From Sect. 3.3, $\widetilde{f}^{cc'}$, the biconjugate of $\widetilde{f}$, coincides with the e-quasiconvex hull of $\widetilde{f}$. This fact, together with the following equivalences,

$$f \text{ is e-convex} \iff \widetilde{f} \text{ is e-convex} \iff \widetilde{f} \text{ is e-quasiconvex}, \tag{4.4}$$

allow to obtain expressions for the e-convex hulls of $\widetilde{f}$ and f.

Theorem 4.10 (Properties of c-Conjugation) *Let $c : \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \times \mathbb{R}) \to \overline{\mathbb{R}}$ be as in* (4.3)*. For any $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and any $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, the following statements hold:*

(*i*) $(\operatorname{eco} \widetilde{f})(x, t) = (\operatorname{eco} f)(x) + t$.
(*ii*) $\widetilde{f}^{cc'}(x, t) = f^{**}(x) + \delta_{\operatorname{eco}\operatorname{dom} f}(x) + t$.
(*iii*) $(\operatorname{eco} \widetilde{f})(x, t) = f^{**}(x) + \delta_{\operatorname{eco}\operatorname{dom} f}(x) + t$.
(*iv*) $\operatorname{eco} f = f^{**} + \delta_{\operatorname{eco}\operatorname{dom} f}$.

The conjugation method described in this subsection does not apply directly on a given function f on $\mathbb{R}^n$, but on a certain extension $\widetilde{f}$ defined on $\mathbb{R}^{n+1}$. The results obtained in Theorem 4.10 follow from the relationship between f and $\widetilde{f}$ in (4.4), and the conjugation method for e-quasiconvex functions. Among the given results, we observe that the identity $\operatorname{eco} f = f^{**} + \delta_{\operatorname{eco}\operatorname{dom} f}$ is guaranteed for any extended real-valued function f. However, with the method described in Sect. 4.3.1, such equality can be just asserted for functions having an e-convex minorant (besides the function identically $-\infty$).

4.3.3 Third Conjugacy Scheme

Now, consider the coupling function $c : \mathbb{R}^n \times (\mathbb{R}^n \times \mathbb{R} \times \{0, 1\}) \to \overline{\mathbb{R}}$, defined by

$$c\left(x, \left(x^*, r, i\right)\right) := \begin{cases} \langle x, x^* \rangle, & \text{if } i = 0, \\ \upsilon_{[x^* < r]}(x), & \text{if } i = 1. \end{cases} \tag{4.5}$$

In this case, the c-conjugate of $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is the function $f^c : \mathbb{R}^n \times \mathbb{R} \times \{0, 1\} \to \overline{\mathbb{R}}$ given, for every $(x^*, r) \in \mathbb{R}^n \times \mathbb{R}$ by

$$f^c\left(x^*, r, i\right) = \begin{cases} f^*(x^*), & \text{if } i = 0, \\ \upsilon_{\{(x^*, r) \in \mathbb{R}^n \times \mathbb{R} \,:\, \operatorname{dom} f \subset [x^* < r]\}}, & \text{if } i = 1. \end{cases}$$

Observe that the c-elementary functions are, on the one hand, the continuous affine functions on $\mathbb{R}^n$, and, on the other hand, the valley functions of the open halfspaces of $\mathbb{R}^n$, plus the constant functions $+\infty$ and $-\infty$.

The main properties of this conjugation scheme are as follows.

Theorem 4.11 (Properties of c-Conjugation) *Let $c : \mathbb{R}^n \times (\mathbb{R}^n \times \mathbb{R} \times \{0, 1\}) \to \overline{\mathbb{R}}$ be as in (4.5). For any function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, the following statements hold:*

(*i*) $f^{cc'} = \operatorname{eco} f$.
(*ii*) *f is e-convex if and only if $f^{cc'} = f$.*
(*iii*) $f^{cc'} = \max\{f^{**}, \upsilon_{\operatorname{eco}\operatorname{dom} f}\} = f^{**} + \delta_{\operatorname{eco}\operatorname{dom} f}$.
(*iv*) $\operatorname{eco} f$ *coincides with the supremum of all the continuous affine functions and all the open halfspaces valley functions that are minorants of f.*

From the above theorem we get that $\operatorname{eco} f = f^{**} + \delta_{\operatorname{eco}\operatorname{dom} f}$ for any extended real-valued function f. This identity was also obtained in Sect. 4.3.2 by using a different approach. It required a transformation of the function f, which is not needed here. Furthermore, we observe that the identity $\operatorname{eco} f = f^{cc'}$ holds for any function f, while in Sect. 4.3.1 such equality was just asserted for functions having an e-convex minorant (besides the function identically $-\infty$). Finally, Theorem 4.11 points out that $\operatorname{eco} f = \max\{f^{**}, \upsilon_{\operatorname{eco}\operatorname{dom} f}\}$, which is another representation of the e-convex hull of f (and of its biconjugate $f^{cc'}$) that was not obtained with the two approaches in Sects. 4.3.1 and 4.3.2. Such representation is related with the following geometric interpretation: the e-convex hull of a function f is the supremum of all the continuous affine minorants and all the open halfspaces valley functions that are minorants of f.

4.3.4 Subdifferentials

As pointed out in Sect. 3.3, each coupling functional c defining a conjugacy has an associated c-subdifferential ∂_c. Thus, if we consider the first coupling function c in this section, the one in (4.2), the following result provides the link between the c-subdifferential ∂_c and the Moreau–Rockafellar subdifferential ∂. We would like to clarify that this subsection is not related to Clark subdifferentials at all.

Proposition 4.2 *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $\overline{x} \in f^{-1}(\mathbb{R})$. Then,*

$$\partial_c f(\overline{x}) = \partial f(\overline{x}) \times \{(u^*, \alpha) \in \mathbb{R}^n \times \mathbb{R} : \operatorname{dom} f \subset [u^* < \alpha]\}.$$

Example 4.9 Consider the function f in Example 4.5 and $\overline{x} = 0$. Then,

$$\partial_c f(0) = \{0\} \times \{(a, b) \in \mathbb{R}^2 : ax < b \text{ is a consequence of } \{-x \le 1, x \le 1\}\}.$$

More precisely, the latter set is characterized in Theorem 2.3 as

$$\{0\} \times \operatorname{proj}_3^{\{1,2\}} \left(\operatorname{cone} \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \cap \left(\mathbb{R}^2 \times (-\mathbb{R}_{++}) \right) \right)$$
$$= \{0\} \times \operatorname{int} \operatorname{epi} |\cdot| ,$$

where $|\cdot| : \mathbb{R} \to \mathbb{R}$ is the absolute value function (see Fig. 4.8).

As a consequence of Proposition 4.2, f is c-subdifferentiable at $\overline{x}$ if and only if it is subdifferentiable at this point. This motivates our focus on the (Moreau-Rockafellar) subdifferential in the next results. Firstly, we provide a characterization of the ε-subdifferentiability of a function at a given point in terms of the even convexity of its strict epigraph.

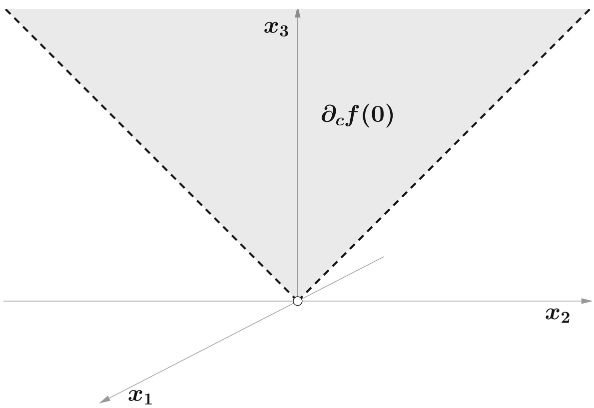

Fig. 4.8 The c-subdifferential of f at 0

Proposition 4.3 (Non-emptiness of the ε-Subdifferential) *Let* $\varepsilon \geq 0$*,* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *and* $\overline{x} \in f^{-1}(\mathbb{R})$*. Then, the following statements are equivalent:*

(i) $\partial_\varepsilon f(\overline{x}) \neq \emptyset$.
(ii) $(\{\overline{x}\} \times \mathbb{R}) \cap \operatorname{eco}(\operatorname{epi}_s f) \subset (\{\overline{x}\} \times \mathbb{R}) \cap \operatorname{epi}_s(f - \varepsilon)$.
(iii) $(\overline{x}, f(\overline{x}) - \varepsilon) \notin \operatorname{eco}(\operatorname{epi}_s f)$.

The following notion is inspired by the concept of *closedness regarding to a set* (see [20, p. 56]): given two sets A and B, one says that A is *e-convex regarding to* B provided that $B \cap \operatorname{eco} A = B \cap A$.

Corollary 4.3 *Let* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *and* $\overline{x} \in f^{-1}(\mathbb{R})$*. Then, the following statements are equivalent:*

(i) $\partial f(\overline{x}) \neq \emptyset$.
(ii) $\operatorname{epi}_s f$ *is e-convex regarding to* $\{\overline{x}\} \times \mathbb{R}$.
(iii) $(\overline{x}, f(\overline{x})) \notin \operatorname{eco}\left(\operatorname{epi}_s f\right)$.

Regarding the function in Example 4.5 and $\overline{x} = -1$, we have

$$(\{-1\} \times \mathbb{R}) \cap \operatorname{epi}_s f = \{-1\} \times \mathbb{R}_{++},$$

while $(\{-1\} \times \mathbb{R}) \cap \operatorname{eco} \operatorname{epi}_s f = \{-1\} \times \mathbb{R}_+$ (see, Fig. 4.9). Since (ii) fails, (i) and (iii) also fail. In the case $\overline{x} = 0$, (i), (ii) and (iii) hold.

In particular, if the strict epigraph of a function f is e-convex, then f is subdifferentiable on $f^{-1}(\mathbb{R})$.

Proposition 4.4 (Necessary Condition for Subdifferentiability) *Assume that* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *is subdifferentiable on* $f^{-1}(\mathbb{R})$ *and either* f *is e-convex or* $\operatorname{dom} f$ *is e-convex. Then,* $\operatorname{epi}_s f$ *is e-convex.*

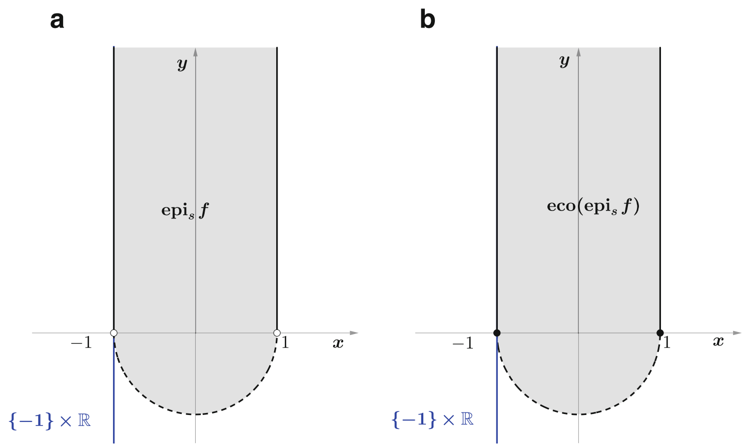

Fig. 4.9 (**a**) $(\{-1\} \times \mathbb{R}) \cap \operatorname{epi}_s f = \{-1\} \times \mathbb{R}_{++}$; (**b**) $(\{-1\} \times \mathbb{R}) \cap \operatorname{eco}\left(\operatorname{epi}_s f\right) = \{-1\} \times \mathbb{R}_+$

We conclude this section by providing two additional characterizations of the even convexity at a given point (cf. Ths. 4.5 and 4.7).

Theorem 4.12 (Characterization of the Even Convexity at a Point) *Consider $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, $\overline{x} \in f^{-1}(\mathbb{R})$ and $A \subset \mathbb{R}^n \times \mathbb{R}$ such that* $\operatorname{epi}_s f \subset A \subset \operatorname{epi} f$. *Then, the following statements are equivalent:*

(i) $f(\overline{x}) = (\operatorname{eco} f)(\overline{x})$.
(ii) *For all* $\varepsilon > 0$, $\partial_\varepsilon f(\overline{x}) \neq \emptyset$.
(iii) *For all* $\varepsilon > 0$, $(\overline{x}, f(\overline{x}) - \varepsilon) \notin \operatorname{eco} A$.

There exits also an ε-subdifferentiability notion associated with the c-conjugation pattern described in Sect. 4.3.1. For $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $\varepsilon \geq 0$, it is said that $(x^*, u^*, \alpha) \in W$ is an $\varepsilon - c$*-subgradient* of f at $x_0 \in f^{-1}(\mathbb{R})$, if $\langle u^*, x_0\rangle < \alpha$ and

$$f(x) - f(x_0) \geq c\left(x, \left(x^*, u^*, \alpha\right)\right) - c\left(x_0, \left(x^*, u^*, \alpha\right)\right) - \varepsilon,$$

for all $x \in \mathbb{R}^n$. The set of all the $\varepsilon - c$-subgradients of f at x_0 is denoted by $\partial_{c,\varepsilon} f(x_0)$, and it is called the $\varepsilon - c$*-subdifferential* of f at x_0. Clearly, for $\varepsilon = 0$ we obtain the c-subdifferential of f at x_0.

The most important properties of $\varepsilon - c$-subdifferentiability, whose counterparts for ε-subdifferentiability and Fenchel conjugation are very well known, are summarized in the following theorem. Previously, we introduce a set of e-affine functions associated to a pair of functions $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$, due to the lack of additivity property in the set of e-affine functions. This new set, denoted by $\widetilde{\mathscr{E}}_{f+g}$ is defined in the following way: $a \in \widetilde{\mathscr{E}}_{f+g}$ if and only if there exist $a_1 \in \mathscr{E}_f$, $a_2 \in \mathscr{E}_g$ such that, if

$$a_1(x) = \begin{cases} \langle x, y_1\rangle - \beta_1, & \text{if } \langle x, z_1\rangle < \alpha_1, \\ +\infty, & \text{otherwise,} \end{cases} \quad \text{and} \quad a_2(x) = \begin{cases} \langle x, y_2\rangle - \beta_2, & \text{if } \langle x, z_2\rangle < \alpha_2, \\ +\infty, & \text{otherwise,} \end{cases}$$

then

$$a(x) = \begin{cases} \langle x, y_1 + y_2\rangle - (\beta_1 + \beta_2), & \text{if } \langle x, z_1 + z_2\rangle < \alpha_1 + \alpha_2, \\ +\infty, & \text{otherwise.} \end{cases} \tag{4.6}$$

Clearly $\widetilde{\mathscr{E}}_{f+g} \subset \mathscr{E}_{f+g}$.

Theorem 4.13 (Properties of c-Subdifferentiability) *Let $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper functions, such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. *Then,*

(i) *For* $x_0 \in \operatorname{dom} f$,

$$\operatorname{epi} f^c = \bigcup_{\varepsilon \geq 0} \left\{\left(x^*, u^*, \alpha, \langle x^*, x_0\rangle + \varepsilon - f(x_0)\right) : \left(x^*, u^*, \alpha\right) \in \partial_{c,\varepsilon} f(x_0)\right\}.$$

(*ii*) *If* $f+g=\sup\{a : a \in \widetilde{\mathscr{E}}_{f+g}\}$, $\operatorname{epi} f^c + \operatorname{epi} g^c$ *is* e'*-convex if and only if*

$$\partial_{c,\varepsilon}(f+g)(x) = \bigcup_{\varepsilon_1+\varepsilon_2=\varepsilon} \partial_{c,\varepsilon_1} f(x) + \partial_{c,\varepsilon_2} g(x).$$

for all $\varepsilon \geq 0$ *and* $x \in \operatorname{dom} f \cap \operatorname{dom} g$.

(*iii*) *If* $f+g=\sup\{a : a \in \widetilde{\mathscr{E}}_{f+g}\}$ *and* $\operatorname{epi} f^c + \operatorname{epi} g^c$ *is* e'*-convex, then*

$$\partial_c(f+g)(x) = \partial_c f(x) + \partial_c g(x).$$

for all $x \in \operatorname{dom} f \cap \operatorname{dom} g$.

4.4 Duality in Evenly Convex Optimization

4.4.1 General Regularity Conditions

An important part of mathematical programming from both theoretical and computational points of view is duality theory. We consider an arbitrary unconstrained optimization problem

$$(GP) \qquad \operatorname*{Min}_{x\in\mathbb{R}^n} F(x), \tag{4.7}$$

where $F : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a proper function, and we apply the *perturbational approach to duality* as in Sect. 3.4. By taking a perturbation function $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ such that $\Phi(x, 0_m) = F(x)$ for all $x \in \mathbb{R}^n$, $\mathbb{R}^m$ being the space of perturbation variables, the so-called *conjugate dual problem* of (GP) can be formulated as follows:

$$(GD) \qquad \operatorname*{Max}_{y^*\in\mathbb{R}^m} -\Phi^*(0_n, y^*),$$

where $\Phi^* : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ is the Fenchel conjugate of Φ, that is,

$$\Phi^*\left(x^*, y^*\right) = \sup_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^m} \left\{\left\langle x, x^*\right\rangle + \left\langle y, y^*\right\rangle - \Phi(x, y)\right\}.$$

Problem (GD) can also be expressed by means of the *infimum value function* $p : \mathbb{R}^m \to \overline{\mathbb{R}}$,

$$p(y) := \inf_{x\in\mathbb{R}^n} \Phi(x, y). \tag{4.8}$$

In fact, since $p(0_m) = \inf_{x\in\mathbb{R}^n} F(x)$, and $p^*(y^*) = \Phi^*(0_n, y^*)$, one has

$$(GD) \qquad \underset{y^*\in\mathbb{R}^m}{\operatorname{Max}}\ p^*\left(y^*\right).$$

From the so-called *Fenchel–Young inequality*

$$p^*\left(y^*\right) + p(y) \geq \left\langle y, y^*\right\rangle, \forall y, y^* \in \mathbb{R}^m,$$

it follows that $-p^*(y^*) \leq p(0_m)$ and, denoting by $v(GP)$ and $v(GD)$ the optimal values of the primal and the dual problems, respectively, we have $v(GD) \leq v(GP)$, situation known as *weak duality*. The difference between the optimal values of the primal and the dual problems is called *duality gap*, and it is said that there exists *strong duality* when there is no duality gap and the dual problem is solvable. Sufficient conditions for strong duality are called *regularity conditions*, and they are classified, mainly, in two different groups: interiority-type and closedness-type conditions, being the last ones used recently as a viable alternative to their interiority-type counterparts. This well-known framework will be named *the classical setting* throughout this section. In this classical framework, strong duality is characterized through the subdifferential of the infimum value function at 0_m; [191, Th. 2.6.1] states that, if the perturbation function is convex, $\partial p(0_m) \neq \emptyset$ if and only if strong duality holds.

However, from the point of view of applicability, it is also necessary to find out conditions guaranteeing strong duality even when F is perturbed with linear functions, situation called *stable strong duality.*

Since e-convex functions can be viewed as a generalization of convex lower semicontinuous functions and, moreover, c-conjugation (with c the coupling function in (4.2)) is suitable for this kind of functions, it is natural trying the extension of well-known results for convex duality in the classical setting to that more general framework. In this moment we introduce the notion of e$'$-convexity, which appeared firstly in [46], for the calculus of the counterpart of the classical Moreau–Rockafellar formula (see, e.g., [24]). The idea consisted in creating a kind of hull for the convex set $\operatorname{epi} f^c + \operatorname{epi} g^c$, for $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$, trying to link it with $\operatorname{epi}(f+g)^c$. For this problem, lower semicontinuous and evenly convex hulls do not properly work. Recall that $\operatorname{epi} f^c + \operatorname{epi} g^c$ and $\operatorname{epi}(f+g)^c$ are both contained in $W \times \mathbb{R}$, where $W = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$.

A subset $D \subset W \times \mathbb{R}$ is said to be *e$'$-convex* if there exists an e$'$-convex function $h : W \to \overline{\mathbb{R}}$ such that $D = \operatorname{epi} h$. Clearly, the intersection of an arbitrary family of e$'$-convex sets is an e$'$-convex set, and the *e$'$-convex hull* of a set $D \subset W \times \mathbb{R}$ is defined as the smallest e$'$-convex set containing D, which will be denoted by $\mathrm{e}' \operatorname{co} D$. Actually, it is the epigraph of the e$'$-convex hull of the function $f_D : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by $f_D(x^*, u^*, \alpha) := \inf\{a \in \mathbb{R} : (x^*, u^*, \alpha, a) \in D\}$.

Considering now the general dual problem (4.7) with $F : \mathbb{R}^n \to \overline{\mathbb{R}}$ a proper function, let us take a perturbation function $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$. Denoting by $Z = \mathbb{R}^n \times \mathbb{R}^m$, the c-conjugate of the perturbation function Φ, $\Phi^c : Z \times Z \times \mathbb{R} \to \overline{\mathbb{R}}$, is

$$\Phi^c\left(\left(x^*, y^*\right), \left(u^*, v^*\right), \alpha\right) = \sup_{(x,y)\in Z}\left\{\overline{c}\left((x,y), \left(\left(x^*, y^*\right), \left(u^*, v^*\right), \alpha\right)\right) - \Phi(x,y)\right\},$$

where $\overline{c} : Z \times Z \times Z \times \mathbb{R} \to \overline{\mathbb{R}}$ is the coupling function

$$\overline{c}\left((x,y), \left(\left(x^*, y^*\right), \left(u^*, v^*\right), \alpha\right)\right) = \begin{cases} \langle x, x^*\rangle + \langle y, y^*\rangle, & \text{if } \langle x, u^*\rangle + \langle y, v^*\rangle < \alpha, \\ +\infty, & \text{otherwise.} \end{cases}$$

In [46], the general problem (GP) is associated with the dual problem

$$\begin{array}{lll} (GD_c) & \underset{y^*, v^* \in \mathbb{R}^m, \alpha \in \mathbb{R}}{\text{Max}} & -\Phi^c\left(\left(0_n, y^*\right), \left(0_n, v^*\right), \alpha\right) \\ & \text{s.t.} & \alpha > 0, \end{array} \tag{4.9}$$

verifying weak duality, $v(GD_c) \leq v(GP)$. Since

$$\Phi^c\left(\left(0_n, y^*\right), \left(0_n, v^*\right), \alpha\right) = p^c\left(y^*, v^*, \alpha\right), \forall y^*, v^* \in \mathbb{R}^m, \forall \alpha > 0,$$

(GD_c) can also be expressed by means of the infimum value function p in (4.8), as follows:

$$\begin{array}{lll} (GD_c) & \underset{y^*, v^* \in \mathbb{R}^m, \alpha \in \mathbb{R}}{\text{Max}} & -p^c\left(y^*, v^*, \alpha\right) \\ & \text{s.t.} & \alpha > 0. \end{array}$$

We focus firstly in obtaining interior point regularity conditions for strong duality between (GP) and (GD_c). It is evident that strong duality holds if $v(GP) = -\infty$, hence we deal with the case $v(GP) \in \mathbb{R}$. In first place, we will characterize strong duality in terms of the c-subdifferential associated to the coupling c in (4.2) as considered in Sect. 4.3.

Proposition 4.5 (General Characterization of Strong Duality) *Let us assume that $v(GP) \in \mathbb{R}$. Then the duality pair $(GP) - (GD_c)$ verifies strong duality if and only if $\partial_c p\left(0_m\right) \neq \emptyset$. In this case, $\partial_c p\left(0_m\right)$ is the solution set of (GD_c).*

We introduce an interior point condition expressed in terms of the relative interior of a set, and a closedness-type one, expressed in terms of closures of epigraphs, as usual in this kind of conditions.

Denoting by $\operatorname{proj}(\operatorname{dom}\Phi)$ the projection of $\operatorname{dom}\Phi$ onto $\mathbb{R}^m$, let us also observe that $\operatorname{epi}\Phi^c \subset Z \times Z \times \mathbb{R} \times \mathbb{R}$, $Z = \mathbb{R}^n \times \mathbb{R}^m$, so denoting by $W = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, we refer to the projection of $\operatorname{epi}\Phi^c$ onto $W \times \mathbb{R}$ by $\operatorname{proj}\left(\operatorname{epi}\Phi^c\right)$. These two projections could be written more precisely as $\operatorname{proj}_{n+m}^{\{n+1,\ldots,n+m\}}$ and $\operatorname{proj}_{2n+2m+1}^{\{1,\ldots,n,m+1,\ldots,m+n,2n+2m+1\}}$, respectively, but this notation would provide cumbersome formulae.

Theorem 4.14 (General Regularity Conditions) *Let us consider the general primal problem* (GP) *and its dual* (GD_c)*, and assume that* Φ *is e-convex. The following conditions ensure* $\partial_c p(0_m) \neq \emptyset$ *and, therefore, strong duality between* (GP) *and* (GD_c)*:*

$(C1)$ $0_m \in \operatorname{rint} \operatorname{proj} (\operatorname{dom} \Phi)$.
$(C2)$ $\operatorname{proj} (\operatorname{epi} \Phi^c)$ *is* e'*-convex or, equivalently,*

$$\operatorname{proj}\left(\operatorname{epi} \Phi^c\right) = \operatorname{epi} \Phi^c\left(\cdot, 0_m\right).$$

Moreover, this equality can be reformulated as the fulfilment of the following condition

$$\Phi^c\left(\cdot, 0_m\right) = \min_{y^*, v^* \in \mathbb{R}^m} \Phi^c\left(\left(\cdot, y^*\right), \left(\cdot, v^*\right), \cdot\right),$$

where $\Phi\left(\cdot, 0_m\right) : \mathbb{R}^n \to \overline{\mathbb{R}}$, $\Phi^c\left(\cdot, 0_m\right) : W \to \overline{\mathbb{R}}$, $\Phi^c\left(\left(\cdot, y^*\right), \left(\cdot, v^*\right), \cdot\right) : W \to \overline{\mathbb{R}}$ *and* $W = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$.

Remark 4.1 As it is shown in [47], conditions $(C1)$ and $(C2)$ are also sufficient for stable strong duality. The perturbed problems with linear functionals and the corresponding duals are

$$\begin{array}{lll} (GP_{x^*}) & \underset{x \in \mathbb{R}^n}{\operatorname{Min}} & \Phi(x, 0_m) + \left\langle x, x^*\right\rangle, \\ (GD_{c,x^*}) & \underset{y^*, v^* \in \mathbb{R}^m, \alpha \in \mathbb{R}}{\operatorname{Max}} & -\Phi^c\left(\left(-x^*, y^*\right), \left(0_n, v^*\right), \alpha\right) \\ & \text{s.t.} & \alpha > 0, \end{array}$$

for an arbitrary $x^* \in \mathbb{R}^n$.

4.4.2 *Regularity Conditions for Fenchel Duality*

In this subsection, we consider a particular primal problem together with a particular perturbation function, whose c-conjugate allows us to obtain a Fenchel-type dual problem. We analyze regularity conditions for this pair of problems and do a comparison among them. Let us consider the following optimization problem

$$(P_1) \ \underset{x \in \mathbb{R}^n}{\operatorname{Min}} \ f(x) + g(x),$$

where $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$ are proper functions, with $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. The problem (P_1) is a particular case of (GP) with $F = f + g$.

We will consider the perturbation function $\Phi_F : \mathbb{R}^n \times \mathbb{R}^n \to \overline{\mathbb{R}}$ given by

$$\Phi_F(x, u) := f(x + u) + g(x). \tag{4.10}$$

Calculating the c-conjugate of Φ_F as in (4.9) we obtain the *Fenchel dual problem* of (P_1):

$$\begin{array}{lll} (D_F) & \underset{x^*, u^* \in \mathbb{R}^n, \alpha_1, \alpha_2 \in \mathbb{R}}{\text{Max}} & \{-f^c(x^*, u^*, \alpha_1) - g^c(-x^*, -u^*, \alpha_2)\} \\ & \text{s.t.} & \alpha_1 + \alpha_2 > 0. \end{array} \tag{4.11}$$

We state the next theorem which gathers all the studied regularity conditions for Fenchel duality, being $(C1_F)$ and $(C2_F)$ the particularized versions of the general conditions $(C1)$ and $(C2)$ in Theorem 4.14, respectively. Let us recall that for proper convex functions $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$, the *infimal convolution* of f with g, denoted by $f \square g : \mathbb{R}^n \to \overline{\mathbb{R}}$, is defined by

$$(f \square g)(x) := \inf_{x_1 + x_2 = x} \{f(x_1) + g(x_2)\},$$

and it is said to be *exact at* $x \in \mathbb{R}^n$ if $(f \square g)(x) = f(a) + g(x - a)$ for some $a \in \mathbb{R}^n$. Moreover, the infimal convolution is called *exact* when it is exact at any $x \in \mathbb{R}^n$.

Observe that, in this case, $Z = \mathbb{R}^n \times \mathbb{R}^n$, and $W = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, and when we refer to the projection of $\operatorname{epi} \Phi_F^c$ onto $W \times \mathbb{R}$, $\operatorname{proj}\left(\operatorname{epi} \Phi_F^c\right)$, we mean

$$\operatorname{proj}\left(\operatorname{epi} \Phi_F^c\right) = \left\{\left(x^*, u^*, \alpha, \beta\right) \in W \times \mathbb{R} : \left(x^*, y^*, u^*, v^*, \alpha, \beta\right) \in \operatorname{epi} \Phi_F^c,\ y^*, v^* \in \mathbb{R}^n\right\}.$$

Theorem 4.15 (Fenchel Regularity Conditions) *Let us consider the primal problem (P_1), where $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$ are proper e-convex functions, and its Fenchel dual (D_F). The following conditions ensure strong duality between both problems:*

$(C1_F)$ $0_n \in \operatorname{rint}(\operatorname{dom} f - \operatorname{dom} g)$.
$(C2_F)$ $\operatorname{proj}\left(\operatorname{epi} \Phi_F^c\right)$ *is e′-convex, or, equivalently,*

$$\Phi_F^c(\cdot, 0_m) = \min_{y^*, v^* \in \mathbb{R}^n} \Phi_F^c\left(\left(\cdot, y^*\right), \left(\cdot, v^*\right), \cdot\right).$$

$(C3_F)$ $f + g = \sup \widetilde{\mathscr{E}}_{f,g}$ *and* $\operatorname{epi} f^c + \operatorname{epi} g^c$ *is e′-convex.*

Moreover, strong duality is characterized by the following condition:

$(C4_F)$ *There exists $\alpha > 0$ such that $(f + g)^c(0_n, 0_n, \alpha) \geq (f^c \square g^c)(0_n, 0_n, \alpha)$ and the infimal convolution is exact at $(0_n, 0_n, \alpha)$, which is equivalent to saying that*

$$\operatorname{epi}(f + g)^c \cap \{(0_n, 0_n, \alpha) \times \mathbb{R}\} \subseteq \left(\operatorname{epi} f^c + \operatorname{epi} g^c\right) \cap \{(0_n, 0_n, \alpha) \times \mathbb{R}\}.$$

Comparing regularity conditions $(C1_F)$, $(C2_F)$ and $(C3_F)$, the unique relationship among them is that $(C3_F)$ implies $(C2_F)$, as it is pointed out ahead in Proposition 4.6. The following example shows that $(C3_F)$ does not imply $(C1_F)$.

Example 4.10 Let us take $n = 1$, $f = \delta_{[0,+\infty[}$ and $g = \delta_{]-\infty,0]}$. Since

$$\operatorname{dom} f - \operatorname{dom} g = [0, +\infty[,$$

$0 \in \operatorname{int}(\operatorname{dom} f - \operatorname{dom} g)$ and $(C1_F)$ does not hold. We now check condition $(C3_F)$.

We have $f + g = \delta_{\{0\}}$ and let $h = \sup \widetilde{\mathscr{E}}_{f,g}$. Now, an e-affine function $a_1 \in \mathscr{E}_f$ if and only if

$$a_1(x) = \begin{cases} \alpha_1 x - \beta_1, & \text{if } \gamma_1 x < \delta_1, \\ +\infty, & \text{otherwise,} \end{cases}$$

with $\alpha_1 \leq 0$, $\beta_1 \geq 0$, $\gamma_1 \leq 0$ and $\delta_1 > 0$. On the other hand, $a_2 \in \mathscr{E}_g$ if and only if

$$a_2(x) = \begin{cases} \alpha_2 x - \beta_2, & \text{if } \gamma_2 x < \delta_2, \\ +\infty, & \text{otherwise,} \end{cases}$$

with $\alpha_2 \geq 0$, $\beta_2 \geq 0$, $\gamma_2 \geq 0$ and $\delta_2 > 0$. Then, $a \in \widetilde{\mathscr{E}}_{f,g}$ if and only if

$$a(x) = \begin{cases} \alpha x - \beta, & \text{if } \gamma x < \delta, \\ +\infty, & \text{otherwise,} \end{cases}$$

with $\alpha, \gamma \in \mathbb{R}$, $\beta \geq 0$ and $\delta > 0$.

We obtain $h = \sup \widetilde{\mathscr{E}}_{f,g} = \delta_{\{0\}} = f + g$. The following step is to calculate $\operatorname{epi} f^c + \operatorname{epi} g^c$. We have $(\alpha, \beta, \gamma, \delta) \in \operatorname{epi} f^c$ if and only if, for all $x \geq 0$, $\alpha x \leq \delta$ and $\beta x < \gamma$, hence $\operatorname{epi} f^c = \mathbb{R}_- \times \mathbb{R}_- \times \mathbb{R}_{++} \times \mathbb{R}_+$. Similarly, $(\alpha, \beta, \gamma, \delta) \in \operatorname{epi} g^c$ if and only if, for all $x \leq 0$, $\alpha x \leq \delta$ and $\beta x < \gamma$, hence $\operatorname{epi} \delta_A^c = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_{++} \times \mathbb{R}_+$. We obtain

$$\operatorname{epi} f^c + \operatorname{epi} g^c = \mathbb{R} \times \mathbb{R} \times \mathbb{R}_{++} \times \mathbb{R}_+.$$

This set is e$'$-convex, because it is the epigraph of the c$'$-elementary function $c'(\cdot, 0)$, which allows us to conclude that conditions in $(C3_F)$ fulfills.

Proposition 4.6 (Relations Between Fenchel Regularity Conditions) $(C3_F)$ *implies* $(C2_F)$. *Moreover, if* $f + g = \sup \widetilde{\mathscr{E}}_{f,g}$, $(C2_F)$ *implies* $(\mathrm{C}3_F)$ *if and only if*

$$f^c \Box g^c = \min_{y^*, v^* \in \mathbb{R}^n} \Phi^c \left((\cdot, y^*), (\cdot, v^*), \cdot \right).$$

The following example shows that in general $(C2_F)$ does not imply $(C3_F)$.

Example 4.11 Let us take $n = 1$, $g = \delta_{[0,+\infty[}$ and $f = \delta_{]0,+\infty[}$.

It is easy to see that $f+g=\sup \widetilde{\mathscr{E}}_{f,g}$. The most important fact to prove is that

$$\left(f^c \Box g^c\right)\left(x^*, u^*, \alpha\right) > \inf_{y^*, v^* \in \mathbb{R}} \Phi^c\left(\left(x^*, y^*\right), \left(u^*, v^*\right), \alpha\right)$$

at some point $(x^*, u^*, \alpha) \in \mathbb{R}^3$, implying, by Proposition 4.6, that $(C3_F)$ does not hold. So, we must give $(u^*, \alpha) \in \mathbb{R}^2$ such that, if we have

$$\left(v^* - u^*\right) x + u^* u < \alpha, \forall x \in \operatorname{dom} g, \forall u \in \operatorname{dom} f, \tag{4.12}$$

for some $v^* \in \mathbb{R}$, then

$$\left(u^* - w^*\right) x < \alpha_1 \text{ and } w^* u < \alpha_2 \tag{4.13}$$

whatever $w^* \in \mathbb{R}$ implies $\alpha_1 + \alpha_2 > \alpha$, meaning that $(f^c \Box g^c)(x^*, u^*, \alpha) = +\infty$. Take $(u^*, \alpha) = (-1, 0)$. For any $-1 \leq v^* < 0$, (4.12) holds. However for any $w^* \in \mathbb{R}$ verifying (4.13) it must be $\alpha_1 + \alpha_2 > 0$, since $\alpha_1 > 0$ and $\alpha_2 \geq 0$ necessarily.

On the other hand, if $x^* \leq 0$,

$$\inf_{y^*, v^* \in \mathbb{R}} \Phi^c\left(\left(x^*, y^*\right), \left(-1, v^*\right), 0\right) = \inf_{y^* \in \mathbb{R}} \left\{ \sup_{x \geq 0} \left(x^* - y^*\right) x + \sup_{z > 0} y^* z \right\} = 0.$$

Now we are going to check that

$$\sup_{x \in \mathbb{R}} \left\{ c\left(x, \left(x^*, u^*, \alpha\right)\right) - \Phi(x, 0) \right\} = \min_{y^*, v^* \in \mathbb{R}} \Phi^c\left(\left(x^*, y^*\right), \left(u^*, v^*\right), \alpha\right), \tag{4.14}$$

for all $(x^*, u^*, \alpha) \in \mathbb{R}^3$, which means that $(C2_F)$ holds.

In the case $\sup_{x \in \mathbb{R}} \{c(x, (x^*, u^*, \alpha)) - \Phi(x, 0)\} = +\infty$, (4.14) holds trivially. Hence, let us assume that $\sup_{x \in \mathbb{R}} \{c(x, (x^*, u^*, \alpha)) - \Phi(x, 0)\} < +\infty$. It is equivalent to saying that $(x^*, u^*, \alpha) \in \mathbb{R}_- \times (\mathbb{R}_- \times \mathbb{R}_+ \backslash \{0_2\})$ and then

$$\sup_{x \in \mathbb{R}} \left\{ c\left(x, \left(x^*, u^*, \alpha\right)\right) - \Phi(x, 0) \right\} = 0.$$

We now compute $\Phi^c((x^*, y^*), (u^*, v^*), \alpha)$, for any points $(x^*, u^*, \alpha) \in \mathbb{R}_- \times (\mathbb{R}_- \times \mathbb{R}_+ \backslash \{0_2\})$ and $y^*, v^* \in \mathbb{R}$:

$$\begin{aligned}
&\Phi^c\left(\left(x^*, y^*\right), \left(u^*, v^*\right), \alpha\right) \\
&= \sup_{x, z \in \mathbb{R}} \left\{ \overline{c}\left((x, y), \left(x^* - y^*, y^*\right), \left(u^* - v^*, v^*\right), \alpha\right) - f(y) - g(x) \right\} \\
&= \sup_{x \geq 0, y > 0} \left\{ \overline{c}\left((x, y), \left(x^* - y^*, y^*\right), \left(u^* - v^*, v^*\right), \alpha\right) \right\}.
\end{aligned}$$

Since we are interested in those suprema which are finite, if $u^* < 0$ and $\alpha \geq 0$, take any $v^* \in \left[u^*, 0\right[$ and, if $u^* = 0$ and $\alpha > 0$, take $v^* = 0$. Then

$$\begin{aligned} &\inf_{y^*, v^* \in \mathbb{R}} \sup_{x \geq 0, y > 0} \left\{ \overline{c}\left((x, y), \left(x^* - y^*, y^*\right), \left(u^* - v^*, v^*\right), \alpha \right) \right\} \\ &= \inf_{y^* \leq x^*} \sup_{x \geq 0, y > 0} \left\{ \left(x^* - y^*\right) x + y^* z \right\} = 0, \end{aligned}$$

and this infimum is a minimum.

We finish the comparison with an example showing that $(C1_F)$ does not imply $(C2_F)$.

Example 4.12 Let us take $n = 1$, $g = \delta_{[0,+\infty[}$ and $f = \delta_{]1,+\infty[}$. At the point $(x^*, -1, -1) \in \mathbb{R}^3$, $x^* \leq 0$, we obtain

$$\sup_{x \in \mathbb{R}} \left\{ c\left(x, (-1, -1, -1)\right) - \Phi\left(x, 0\right) \right\} = \sup_{x > 1} \left\{ c\left(x, \left(x^*, -1, -1\right)\right) \right\} = x^*.$$

On the other hand,

$$\begin{aligned} &\inf_{y^*, v^* \in \mathbb{R}} \Phi^c\left(\left(x^*, y^*\right), \left(-1, v^*\right), -1 \right) \\ &= \inf_{y^*, v^* \in \mathbb{R}} \sup_{z > 1, x \geq 0} \left\{ \overline{c}\left((x, y), \left(x^* - y^*, y^*\right), \left(-1 - v^*, v^*\right), -1 \right) \right\}. \end{aligned}$$

Let us observe that a necessary condition *(*depending on $v^* \in \mathbb{R}$*)* for these suprema to be finite is that, for all $x \geq 0$ and $y > 1$,

$$\left(-1 - v^*\right) x + v^* y < -1.$$

In particular for $x = 0$ and $y > 1$, it must be $v^* < -1$, but this implies that $(-1 - v^*) > 0$ and hence $(-1 - v^*)\, x$ cannot be bounded from above, since $x \geq 0$. We conclude that

$$\overline{c}\left((x, y), \left(x^* - y^*, y^*\right), \left(-1 - v^*, v^*\right), -1 \right) = +\infty,$$

for all $y^*, v^* \in \mathbb{R}$ and

$$\min_{y^*, v^* \in \mathbb{R}} \Phi^c\left(\left(x^*, y^*\right), \left(-1, v^*\right), -1 \right) = +\infty.$$

Hence $(C2_F)$ does not hold despite $(C1_F)$ being fulfilled, because of the fact that $\operatorname{dom} f - \operatorname{dom} g = \mathbb{R}$.

Remark 4.2 As it is shown in [47], $(C3_F)$ is a sufficient condition for stable strong Fenchel duality, i.e., for strong duality between the pair of problems

$$\begin{array}{lll} (P_{1,x^*}) & \displaystyle\operatorname*{Min}_{x\in\mathbb{R}^n} & f(x)+g(x)+\left\langle x,x^*\right\rangle \\ (D_{F,x^*}) & \displaystyle\operatorname*{Max}_{y^*,u^*\in\mathbb{R}^n,\alpha_1,\alpha_2\in\mathbb{R}} & \left\{-f^c\left(y^*-x^*,u^*,\alpha_1\right)-g^c\left(-y^*,-u^*,\alpha_2\right)\right\} \\ & \text{s.t.} & \alpha_1+\alpha_2>0, \end{array}$$

for all $x^*\in\mathbb{R}^n$.

4.4.3 Regularity Conditions for Lagrange Duality

In this subsection, we consider the following primal problem

$$\begin{array}{ll} (P_2) \displaystyle\operatorname*{Min}_{x\in\mathbb{R}^n} & f(x) \\ \quad\text{s.t.} & g_i(x)\le 0, i=1,\ldots,m, \end{array} \tag{4.15}$$

where $f,g_i:\mathbb{R}^n\to\overline{\mathbb{R}}$, for all $i=1,\ldots,m$, are proper functions. Let us suppose that the feasible set $A=\{x\in\mathbb{R}^n:g_i(x)\le 0, i=1,\ldots,m\}$ is nonempty. The problem (P_2) is a particular case of (GP) with $F=f+\delta_A$. We also consider the following perturbation function $\Phi_L:\mathbb{R}^n\times\mathbb{R}^m\to\overline{\mathbb{R}}$,

$$\Phi_L(x,b)=\begin{cases} f(x), \text{ if } g_i(x)\le b_i, i=1,\ldots,m, \\ +\infty, \quad \text{otherwise.}\end{cases} \tag{4.16}$$

In this case, the perturbation variable is $b\in\mathbb{R}^m$. We can describe A as the set $\{x\in\mathbb{R}^n:g(x)\in-\mathbb{R}^m_+\}$, where $g(x)=(g_1(x),\ldots,g_m(x))$, for all $x\in\mathbb{R}^n$, and the perturbation function Φ_L reads

$$\Phi_L(x,b)=\begin{cases} f(x), \text{ if } g(x)-b\in-\mathbb{R}^m_+, \\ +\infty, \quad \text{otherwise.}\end{cases}$$

Calculating the c-conjugate of Φ_L makes it possible to associate to (P_2) a dual problem verifying weak duality. In [45] it was obtained the following *Lagrange dual problem*

$$(D_L) \quad \operatorname*{Max}_{\lambda\in\mathbb{R}^m_+}\ \inf_{x\in\mathbb{R}^n}\{f(x)+\langle\lambda,g(x)\rangle\} \tag{4.17}$$

which is the classical Lagrange dual problem actually.

This subsection is devoted to study strong duality between (P_2) and (D_L). Trivially, if $v(P_2) = -\infty$ strong duality holds. The function $\langle \lambda, g(\cdot) \rangle : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined as $\langle \lambda, g(\cdot) \rangle (x) := \langle \lambda, g(x) \rangle$, for any $\lambda \in \mathbb{R}^m_+$, will be denoted by λg.

We will say that (P_2) verifies the *e-convex cone constraint qualification* $(ECCQ)$ if the cone $\bigcup_{\lambda \in \mathbb{R}^m_+} \operatorname{epi} (\lambda g)^c$ is an e$'$-convex set. It can be viewed as the counterpart of the Farkas-Minkowski CQ in [69], and can be reformulated in the following way, if the e$'$-convex hull of $\bigcup_{\lambda \in \mathbb{R}^m_+} \operatorname{epi} (\lambda g)^c$ is computed:

$$(ECCQ) \quad \bigcup_{\lambda \in \mathbb{R}^m_+} \operatorname{epi} (\lambda g)^c = \operatorname{epi} \delta_A^c. \tag{4.18}$$

The next theorem shows the main regularity conditions for Lagrange duality. Again, as in Fenchel case, conditions $(C1_L)$ and $(C2_L)$ are the particularized versions of the general conditions $(C1)$ and $(C2)$ in Theorem 4.14, respectively.

In this context, $\operatorname{epi} \Phi_L^c \subset Z \times Z \times \mathbb{R} \times \mathbb{R}$, $Z = \mathbb{R}^n \times \mathbb{R}^m$, $W = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, and when we refer to the projection of $\operatorname{epi} \Phi_L^c$ onto $W \times \mathbb{R}$, $\operatorname{proj}\left(\operatorname{epi} \Phi_L^c\right)$, we mean

$$\operatorname{proj}\left(\operatorname{epi} \Phi_L^c\right) = \left\{\left(x^*, u^*, \alpha, \beta\right) \in W \times \mathbb{R} : \left(x^*, \lambda, u^*, \delta, \alpha, \beta\right) \in \operatorname{epi} \Phi_L^c, \lambda, \delta \in \mathbb{R}^m\right\}.$$

Theorem 4.16 (Lagrange Regularity Conditions) *Let us consider the primal problem (P_2), where $f, g_t : \mathbb{R}^n \to \overline{\mathbb{R}}$ are proper e-convex functions, and its Lagrange dual (D_L). If $f + \delta_A = \sup \widetilde{\mathscr{E}}_{f,\delta_A}$ and* $\operatorname{epi} f^c + \operatorname{epi} \delta_A^c$ *is e$'$-convex, any of the following conditions ensures strong duality between $(P_2) - (D_L)$.*

$(C1_L)$ $0_m \in \operatorname{rint}\left(g(\operatorname{dom} f) + \mathbb{R}^m_+\right)$.
$(C2_L)$ $\operatorname{proj}\left(\operatorname{epi} \Phi_L^c\right)$ *is e$'$-convex, or, equivalently,*

$$\Phi_L(\cdot, 0_m)^c = \min_{\lambda, \beta \in \mathbb{R}^m} \Phi_L^c(\cdot, (\lambda, \beta)).$$

$(C3_L)$ *(P_2) verifies $(ECCQ)$.*
$(C4_L)$ *For all $(x^*, y^*, \alpha) \in W$ such that $A \subset \{x \in \mathbb{R}^n : \langle x, y^* \rangle < \alpha\}$ it holds*

$$\inf_{x \in A} c\left(x, \left(x^*, y^*, \alpha\right)\right) = \max_{\mathbb{R}^m_+} \left\{ \inf_{x \in \mathbb{R}^n} \left\{c\left(x, \left(x^*, y^*, \alpha\right)\right) + \lambda g(x)\right\}\right\}. \tag{4.19}$$

and there exists a solution $\overline{\lambda}$ of (4.19) which, in addition, verifies

$$\inf_{x \in A} c\left(x, \left(x^*, y^*, \alpha\right)\right) = \inf_{x \in \operatorname{dom} \overline{\lambda} g} \left\{-c\left(x, \left(-x^*, y^*, \alpha\right)\right) + \overline{\lambda} g(x)\right\}.$$

Actually, $(C3_L)$ and $(C4_L)$ are equivalent.

Next we compare the regularity conditions $(C1_L)$, $(C2_L)$ and $(C3_L)$. As we shall see, the unique relationship between them is that $(C3_L)$ implies $(C2_L)$.

Proposition 4.7 (Relation Between Lagrange Regularity Conditions) *Regularity condition* $(C3_L)$ *implies* $(C2_L)$.

The following example shows that $(C2_L)$ does not imply $(C3_L)$.

Example 4.13 Let us take $n = 1$, $f = \delta_{[0,+\infty[}$, $m = 1$ and $g_1(x) = x$. We have $A =]-\infty, 0]$.

It was shown, in Example 4.10, that $f + \delta_A = \sup \widetilde{\mathscr{E}}_{f,\delta_A}$ and

$$\operatorname{epi} f^c + \operatorname{epi} \delta_A^c = \mathbb{R} \times \mathbb{R} \times \mathbb{R}_{++} \times \mathbb{R}_+,$$

which is e′-convex. We shall see that (ECCQ) does not hold, i.e.,

$$\bigcup_{\lambda \geq 0} \operatorname{epi} (\lambda g)^c \subsetneq \operatorname{epi} \delta_A^c.$$

Since $\operatorname{epi} \delta_A^c = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_{++} \times \mathbb{R}_+$ (see again Example 4.10), a point $(\alpha, \beta, \gamma, \delta) \in \operatorname{epi} \delta_A^c$ verifies $\alpha \geq 0$, $\beta \geq 0$, $\gamma > 0$ and $\delta \geq 0$. This point will be in $\operatorname{epi} (\lambda g)^c$ for some $\lambda \geq 0$ if $c(x, (\alpha, \beta, \gamma)) - \lambda x \leq \delta$, for all $x \in \operatorname{dom} (\lambda g) = \mathbb{R}$, which implies that $\beta x < \gamma$, for all $x \in \mathbb{R}$, and this is impossible if $\beta \neq 0$. Hence $(C3_L)$ does not fulfill.

We now prove that $(C2_L)$ holds. The set $\operatorname{proj}\left(\operatorname{epi} \Phi_L^c\right)$ is e′-convex if and only if

$$\operatorname{epi} \Phi_L(\cdot, 0)^c \subset \operatorname{proj}\left(\operatorname{epi} \Phi_L^c\right),$$

according to the equivalent formulation of $(C2_L)$. Since $(\operatorname{dom} f) \cap A \neq \emptyset$, $f + \delta_A = h_{f,\delta_A}$ and $\operatorname{epi} f^c + \operatorname{epi} \delta_A^c$ is e′-convex, applying Theorem 4.16, we have

$$\operatorname{epi} \Phi_L(\cdot, 0)^c = \operatorname{epi} (f + \delta_A)^c = \operatorname{epi} f^c + \operatorname{epi} \delta_A^c,$$

hence we will see that

$$\mathbb{R} \times \mathbb{R} \times \mathbb{R}_{++} \times \mathbb{R}_+ = \operatorname{epi} f^c + \operatorname{epi} \delta_A^c \subset \operatorname{proj}\left(\operatorname{epi} \Phi_L^c\right).$$

Take a point $(\alpha, \beta, \gamma, \delta) \in \operatorname{epi} f^c + \operatorname{epi} \delta_A^c$. Hence $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $\gamma > 0$ and $\delta \geq 0$. This point will be in $\operatorname{proj}\left(\operatorname{epi} \Phi_L^c\right)$ if and only if there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\Phi_L^c((\alpha, \lambda_1), (\beta, \lambda_2), \gamma) \leq \delta$, meaning that, for all $(x, b) \in \operatorname{dom} \Phi_L$,

$$c_1((x, b), (\alpha, \lambda_1), (\beta, \lambda_2), \gamma) \leq \delta;$$

equivalently,

$$\beta x + \lambda_2 b < \gamma \text{ and } \alpha x + \lambda_1 b \leq \delta. \tag{4.20}$$

Since $\operatorname{dom} \Phi_L = \{(x, b) \in \mathbb{R} \times \mathbb{R} : 0 \leq x \leq b\}$, taking in particular $x = 0$ and $b \geq 0$, from (4.20) we deduce that $\lambda_1, \lambda_2 \leq 0$. Now, for $x > 0$ and $b \geq x$,

$$\beta x + \lambda_2 b \leq x (\beta + \lambda_2).$$

Taking $\lambda_2 \leq 0$ satisfying $(\beta + \lambda_2) \leq 0$, we have $\beta x + \lambda_2 b < \gamma$, If we now take the second inequality in (4.20), we also deduce that the choice of λ_1 only depends of the chosen α, which is fixed, and again clearly $\lambda_1 \leq 0$ can be found. We conclude that $(\alpha, \beta, \gamma, \delta) \in \operatorname{proj}\left(\operatorname{epi} \Phi_L^c\right)$.

We continue with an example showing that ($C3_L$) does not imply $(C1_L)$.

Example 4.14 Consider $n = 1$, $f = \delta_{[0,+\infty[}$, $m = 2$ and $g_i(x) = (i-1)x + \delta_{]-\infty,i-1]}(x)$, $i = 1, 2$. We have $A =]-\infty, 0]$. Then, as in the previous example, $f + \delta_A = h_{f,\delta_A}$ and $\operatorname{epi} f^c + \operatorname{epi} \delta_A^c$ is e$'$-convex. For the fulfilment of $(C3_L)$ we only need to show that $(ECCQ)$ holds, i.e.,

$$\operatorname{epi} \delta_A^c = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_{++} \times \mathbb{R}_+ \subset \bigcup_{\lambda \in \mathbb{R}_+^2} \operatorname{epi} (\lambda g)^c.$$

Take any point $(\alpha, \beta, \gamma, \delta) \in \operatorname{epi} \delta_A^c$, with $\alpha \geq 0$, $\beta \geq 0$, $\gamma > 0$ and $\delta \geq 0$. Then $(\alpha, \beta, \gamma, \delta) \in \operatorname{epi} (\lambda g)^c$ for some $\lambda \in \mathbb{R}_+^2$ if $c(x, (\alpha, \beta, \gamma)) - \lambda g(x) \leq \delta$, for all $x \in \mathbb{R}$, i.e.,

$$\operatorname{dom} (\lambda g) \subset \{x \in \mathbb{R} : \beta x < \gamma\} \text{ and } \alpha x - \lambda g(x) \leq \delta, \ \forall x \in \operatorname{dom} (\lambda g).$$

We distinguish two cases.

Case 1: If $\beta = 0$, it is enough to take $\lambda = (\lambda_1, \lambda_2) = (0, \alpha)$. Then $\operatorname{dom} (\lambda g) = \operatorname{dom} g_2 =]-\infty, 1] \subset \{x \in \mathbb{R} : \beta x < \gamma\} = \mathbb{R}$. Moreover, $\alpha x - \lambda g(x) = (\alpha - \alpha) x = 0 \leq \delta$, for all $x \leq 1$, since $\delta \geq 0$.

Case 2: If $\beta > 0$, take $\lambda = (\lambda_1, \lambda_2) = (1, 0)$. Then $\operatorname{dom} (\lambda g) = \operatorname{dom} g_1 =]-\infty, 0] \subset \{x \in \mathbb{R} \mid \beta x < \gamma\}$. Moreover, $\alpha x - \lambda g(x) = \alpha x \leq \delta$, for all $x \in \operatorname{dom} (\lambda g)$, since $\alpha \geq 0$ and $\delta \geq 0$.

Then, we conclude that $\operatorname{epi} \delta_A^c \subset \bigcup_{\lambda \in \mathbb{R}_+^2} \operatorname{epi} (\lambda g)^c$.

It remains to prove that $0 \notin \operatorname{rint}\left(g(\operatorname{dom} f) + +\mathbb{R}_+^2\right)$, meaning that $(C1_L)$ does not hold. Since $\operatorname{dom} f = [0, +\infty[$, $x = 0$ is the only point verifying $g(0) \in \mathbb{R}^2$. Hence

$$g(\operatorname{dom} f) + \mathbb{R}_+^2 = \mathbb{R}_+^2,$$

and $(C1_L)$ does not fulfill.

We finish with an example showing that $(C1_L)$ does not imply $(C2_L)$.

Example 4.15 Let us take $n = 2$, $m = 2$, $g_1(x) = x_1 + x_2$ and $g_2(x) = x_1 - x_2$, and consider the function $f : \mathbb{R}^2 \to \overline{\mathbb{R}}$ such that

$$f(x_1, x_2) = \begin{cases} \frac{x_2^2}{x_1}, & \text{if } x_1 > 0, \\ 0, & \text{if } x_1 = x_2 = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

It was shown, in Example 4.3, that f is a proper e-convex function. We have

$$A = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 \le 0, x_1 \le x_2 \right\}.$$

It is clear that $(C1_L)$ holds, since $\operatorname{dom} f = (]0, +\infty[\times \mathbb{R}) \cup \{0_2\}]$, and we obtain in this case that $g(\operatorname{dom} f) + \mathbb{R}^2_+ = \mathbb{R}^2$.

Now, we use the equivalent condition to $(C2_L)$, and we will see that there exists at least a point $(x^*, y^*, \alpha) \in W$ such that

$$\Phi_L(\cdot, 0)^c \left(x^*, y^*, \alpha\right) < \min_{\lambda, \beta \in \mathbb{R}^2} \Phi_L^c \left(\left(x^*, \lambda\right), \left(y^*, \beta\right), \alpha \right).$$

Let $y^* = (0, 1)$, any $x^* \in \mathbb{R}^2$ and $\alpha = 1$. Then,

$$\begin{aligned} \Phi_L(\cdot, 0)^c \left(x^*, y^*, \alpha\right) &= \sup_{x \in A \cap \operatorname{dom} f} \left\{ c\left(x, \left(x^*, y^*, \alpha\right)\right) - \Phi_L(x, 0) \right\} \\ &= \sup_{x = 0_2} \left\{ \left\langle x, x^* \right\rangle \right\} = 0. \end{aligned}$$

Now, take any $\lambda, \beta \in \mathbb{R}^2$. Then,

$$\Phi_L^c \left(\left(x^*, \lambda\right), \left(y^*, \beta\right), \alpha \right) = \sup_{\substack{x \in \operatorname{dom} f \\ g(x) - b \in -\mathbb{R}^2_+}} \left\{ c_1 \left((x, b), \left(x^*, \lambda\right), \left(y^*, \beta\right), \alpha \right) - f(x) \right\}.$$

It is clear that $c_1((x, b), (x^*, \lambda), (y^*, \beta), \alpha) < +\infty$ only if

$$\left\langle x, y^* \right\rangle + \langle \beta, b \rangle < \alpha, \tag{4.21}$$

for all (x, b) such that $x \in \operatorname{dom} f$ and $g(x) - b \in -\mathbb{R}^2_+$. Since the sequence $\left\{ \left(x_k^0, b_k^0\right) \right\}$, where $x_k^0 = (1, k)$ and $b_k^0 = (1 + k, 1 - k)$ for $k \in \mathbb{N}$ is contained in the set $D := \left\{ (x, b) : x \in \operatorname{dom} f,\ g(x) - b \in -\mathbb{R}^2_+ \right\}$, the fulfilment of (4.21) implies that, denoting $\beta = (\beta_1, \beta_2)$,

$$1 + \beta_1 - \beta_2 \le 0.$$

On the other hand, taking the sequence $\left\{\left(x_k^1, b_k^1\right)\right\}$, where $x_k^1 = (1, -k)$ and $b_k^1 = (1-k, 1+k)$ for $k \in \mathbb{N}$, also contained in D, (4.21) forces

$$1 + \beta_1 - \beta_2 \geq 0,$$

and we conclude that $1+\beta_1-\beta_2 = 0$. Finally, considering the sequence $\left\{\left(x_k^2, b_k^2\right)\right\}$, where $x_k^2 = \left(\frac{1}{k}, k\right)$ and $b_k^2 = \left(\frac{1}{k} + k, \frac{1}{k} - k\right)$ for $k \in \mathbb{N}$, again contained in D, we obtain, if (4.21) holds and taking into account that $1 + \beta_1 = \beta_2$,

$$k + \frac{2}{k}\beta_1 < 1,$$

for $k \in \mathbb{N}$, which is impossible. Hence $c_1 \left((x, b), (x^*, \lambda), (y^*, \beta), \alpha\right) = +\infty$, for all $\lambda, \beta \in \mathbb{R}^2$, and

$$\min_{\lambda, \beta \in \mathbb{R}^2} \Phi_L^c \left(\left(x^*, \lambda\right), \left(y^*, \beta\right), \alpha\right) = +\infty.$$

Remark 4.3 $(C3_L)$ is a sufficient condition for stable strong duality for $(P_2)-(D_L)$. Here the extended primal and dual problems are

$$\begin{aligned}
(P_{2,x^*}) \qquad & \underset{x \in \mathbb{R}^n}{\text{Min}} \ f(x) + \left\langle x, x^* \right\rangle \\
& \text{s.t.} \quad g_i(x) \leq 0, i = 1, \ldots, m, \\
(D_{L,x^*}) \qquad & \underset{\lambda \in \mathbb{R}^m_+}{\text{Max}} \left\{ \inf_{x \in \mathbb{R}^n} \left\{ f(x) + \left\langle x, x^* \right\rangle + \lambda g(x) \right\} \right\},
\end{aligned}$$

for all $x^* \in \mathbb{R}^n$, as it is shown in [47, Prop. 5.1]. Moreover, in that work, it was introduced another sufficient condition which has its counterpart in the classical setting, where f and g_i, for all $i = 1, \ldots, m$, are proper convex and lsc functions. As it is proved in [22], stable strong Lagrange duality in the classical setting is equivalent to the closedness of the set

$$\bigcup_{\lambda \in \mathbb{R}^m_+} \text{epi} \left(f + \lambda g\right)^*.$$

Then, Proposition 5.3 and Corollary 5.4 in [47] show that condition

$$(C'_L) \qquad \bigcup_{\lambda \in \mathbb{R}^m_+} \text{epi} \left(f + \lambda g\right)^c \ \text{ is an e}'\text{-convex set}$$

is sufficient for stable strong Lagrange duality, however not necessary, as we can see in the following example, since $(C3_L)$ does not imply $\left(C'_L\right)$.

Example 4.16 Let us take $n = 1$, $f = \delta_{[0,+\infty[}$, $m = 2$ and $g_i(x) = (i-1)x + \delta_{]-\infty,i-1]}(x)$, $i = 1, 2$. We have $A =]-\infty, 0]$. From Example 4.14, it is shown that $(C3_L)$ holds, so $f + \delta_A = \sup \widetilde{\mathscr{E}}_{f,\delta_A}$ and the sets $\operatorname{epi} f^c + \operatorname{epi} \delta_A^c$ and $\bigcup_{\lambda\in\mathbb{R}^2_+} \operatorname{epi}(\lambda g)^c$ are e′-convex in $\mathbb{R}^4$. Furthermore,

$$\operatorname{epi} f^c + \operatorname{epi} \delta_A^c = \mathbb{R} \times \mathbb{R} \times \mathbb{R}_{++} \times \mathbb{R}_+.$$

We are going to see that

$$\bigcup_{\lambda\in\mathbb{R}^2_+} \operatorname{epi}(f + \lambda g)^c \subsetneqq \mathrm{e}' \operatorname{co}\left(\bigcup_{\lambda\in\mathbb{R}^2_+} \operatorname{epi}(f + \lambda g)^c\right)$$

being, in this case,

$$\mathrm{e}' \operatorname{co}\left(\bigcup_{\lambda\in\mathbb{R}^2_+} \operatorname{epi}(f + \lambda g)^c\right) = \operatorname{epi}(f + \delta_A)^c = \operatorname{epi} f^c + \operatorname{epi} \delta_A^c.$$

Let us take any $\lambda \in \mathbb{R}^2_+$. Then $(y^*, z^*, \alpha, \beta) \in \operatorname{epi}(f + \lambda g)^c$ if and only if

$$c\left(x, \left(y^*, z^*, \alpha\right)\right) - f(x) - \lambda g(x) \leq \beta, \forall x \in \mathbb{R}.$$

This is equivalent to the fulfilment of

$$\left\langle x, y^*\right\rangle - \lambda_1 \left(\delta_{]-\infty,0]}(x)\right) - \lambda_2 \left(x + \delta_{-\infty,1}(x)\right) \leq \beta \text{ and } \left\langle x, z^*\right\rangle < \alpha, \forall x \geq 0.$$

It implies, in particular, that $z^* \leq 0$, and it happens for any $\lambda \in \mathbb{R}^2_+$. Then

$$\bigcup_{\lambda\in\mathbb{R}^2_+} \operatorname{epi}(f + \lambda g)^c \subsetneqq \mathbb{R} \times \mathbb{R} \times \mathbb{R}_{++} \times \mathbb{R}_+.$$

4.4.4 Regularity Conditions for Fenchel–Lagrange Duality

The primal optimization problem treated in this subsection will be again (4.15)

$$\begin{aligned} (P_2)\ &\operatorname*{Min}_{x\in\mathbb{R}^n}\ f(x) \\ &\text{s.t.}\ \ g_i(x) \leq 0, i = 1, \ldots, m, \end{aligned}$$

where $f, g_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, \ldots, m$, are proper functions and the feasible set $A = \{x \in \mathbb{R}^n : g_i(x) \leq 0, i = 1, \ldots, m\}$ is nonempty. In [48], using the c-conjugation scheme and the perturbation function $\Phi_{FL} : \mathbb{R}^n \times (\mathbb{R}^n \times \mathbb{R}^m) \to \overline{\mathbb{R}}$ defined as

$$\Phi_{FL}(x, (u, b)) := \begin{cases} f(x+u), & g_i(x) \leq 0, i = 1, \ldots, m, \\ +\infty, & \text{otherwise}, \end{cases} \tag{4.22}$$

the *Fenchel–Lagrange dual problem* for (P_2) was obtained:

$$\begin{array}{lll} (D_{FL}) & \underset{\lambda \in \mathbb{R}^m_+, y^*, v^* \in \mathbb{R}^n, \alpha_1, \alpha_2 \in \mathbb{R}}{\text{Max}} & \{-f^c(x^*, u^*, \alpha_1) - (\lambda g)^c(-x^*, -u^*, \alpha_2)\} \\ & \text{s.t.} & \alpha_1 + \alpha_2 > 0, \end{array} \tag{4.23}$$

where $\lambda g = \langle \lambda, g(\cdot) \rangle : \mathbb{R}^n \to \overline{\mathbb{R}}$ is defined as $\langle \lambda, g(\cdot) \rangle (x) := \langle \lambda, g(x) \rangle$, for any $\lambda \in \mathbb{R}^m_+$. Observe that the perturbation function (4.22) is a combination of the perturbation functions (4.10) and (4.16) to build the Fenchel and the Lagrange dual problems, respectively. It is natural, then, to try to connect the regularity conditions for both dualities presented in the previous subsections in order to obtain regularity conditions for Fenchel–Lagrange duality.

We state, in the next theorem, all the studied regularity conditions for Fenchel–Lagrange duality, where $(C1_{FL})$ and $(C2_{FL})$ are the particularized versions of the general regularity conditions $(C1)$ and $(C2)$ in Theorem 4.14, respectively. Moreover, two of them are characterizations. In this result, we use the set $\text{gph}(-g) = \{(x, -g(x))\} \subset \mathbb{R}^n \times \mathbb{R}^m$. For the definition of $\widetilde{\mathscr{E}}_{f,g}$ in $(C3_F)$, we recall (4.6).

In this setting, $\text{epi}\,\Phi^c_{FL} \subset Z \times Z \times \mathbb{R} \times \mathbb{R}$ and the spaces $Z = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$, $W = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, and when we refer to the projection of $\text{epi}\,\Phi^c_{FL}$ onto $W \times \mathbb{R}$, $\text{proj}\left(\text{epi}\,\Phi^c_{FL}\right)$, we mean

$$\text{proj}\left(\text{epi}\,\Phi^c_{FL}\right) = \left\{ \begin{array}{r} (x^*, u^*, \alpha, \beta) \in W \times \mathbb{R} : (x^*, y^*, \lambda, u^*, v^*, \delta, \alpha, \beta) \in \text{epi}\,\Phi^c_L, \\ \lambda, \delta \in \mathbb{R}^m, y^*, v^* \in \mathbb{R}^n \end{array} \right\}.$$

Theorem 4.17 (Fenchel–Lagrange Regularity Conditions) *Let us consider the primal problem (P_2), where f, g_i $(i = 1, \ldots, m) : \mathbb{R}^n \to \overline{\mathbb{R}}$ are proper e-convex functions, and its Fenchel–Lagrange dual (D_{FL}). Any of the following conditions ensure strong duality between both problems:*

$(C1_{FL})$ $0_{n+m} \in \text{rint}\left(\left(\text{dom}\, f \times \mathbb{R}^m_+\right) - \text{gph}(-g)\right)$.
$(C2_{FL})$ $\text{proj}\left(\text{epi}\,\Phi^c_{FL}\right) = \text{epi}(f + \delta_A)^c$*, or, equivalently,*

$$\Phi^c_{FL}(\cdot, 0_n, 0_m) = \min_{\substack{y^*, v^* \in \mathbb{R}^n \\ \lambda, \beta \in \mathbb{R}^m}} \Phi_{FL}((\cdot, y^*, \lambda), (\cdot, v^*, \beta), \cdot).$$

$(C3_{FL})$ *$f+\lambda g = \sup \widetilde{\mathscr{E}}_{f,\lambda g}$ for all $\lambda \in \mathbb{R}_+^m$ and the set*

$$\operatorname{epi} f^c + \bigcup_{\lambda \in \mathbb{R}_+^m} \operatorname{epi}(\lambda g)^c$$

is e'-convex, which is equivalent to saying that $\operatorname{epi} f^c + \bigcup_{\lambda \in \mathbb{R}_+^m} \operatorname{epi}(\lambda g)^c = \operatorname{epi}(f+\delta_A)^c$.

Moreover, strong duality is characterized by the followings conditions:

$(C4_{FL})$ *For some $\overline{\alpha} > 0$, it holds*

$$\operatorname{epi}(f+\delta_A)^c \bigcap \{(0_n, 0_n, \overline{\alpha}) \times \mathbb{R}\} \subseteq \left(\operatorname{epi} f^c + \bigcup_{\lambda \in \mathbb{R}_+^m} \operatorname{epi}(\lambda g)^c\right) \bigcap \{(0_n, 0_n, \overline{\alpha}) \times \mathbb{R}\},$$

where $\{(0_n, 0_n, \overline{\alpha}) \times \mathbb{R}\} := \{(0_n, 0_n, \overline{\alpha}, \beta) : \beta \in \mathbb{R}\}$.

$(C5_{FL})$ *For some $\overline{\lambda} \in \mathbb{R}_+^m$ and $\overline{\alpha} > 0$, it holds*

$$(f^c \square (\overline{\lambda} g)^c)(0_n, 0_n, \overline{\alpha}) \le (f+\delta_A)^c(0_n, 0_n, \overline{\alpha})$$

and $f^c \square (\overline{\lambda} g)^c$ is exact at $(0_n, 0_n, \overline{\alpha})$.

We study the relationships between the regularity conditions $(C2_{FL})$ and $(C3_{FL})$. The only relation between them is stated in the next result.

Proposition 4.8 (Relation Between Fenchel–Lagrange Regularity Conditions) *If the functions f, g_i, $i = 1, \ldots, m$, are e-convex and $f+\lambda g = \sup \widetilde{\mathscr{E}}_{f,\lambda g}$ for all $\lambda \in \mathbb{R}_+^m$, then $(C3_{FL})$ implies $(C2_{FL})$.*

The following example shows that the converse of the above proposition does not hold in general. From this example and Theorem 4.17 we deduce that $(C3_{FL})$ is not necessary for strong Fenchel–Lagrange duality because otherwise, $(C2_{FL})$ would imply $(C3_{FL})$.

Example 4.17 Let $n = 1$, $f = \delta_{]0,+\infty[}$, $m = 1$ and $g(x) = -x + \delta_{]-1,+\infty[}(x)$. Since f and g are e-convex functions, first of all we will check that $f+\lambda g = \sup \widetilde{\mathscr{E}}_{f,\lambda g}$ for any $\lambda \ge 0$. In that case, we have $(f+\lambda g)(x) = -\lambda x + \delta_{]0,+\infty[}(x)$. Identifying any e-affine function

$$a(x) = \begin{cases} \alpha x - \beta, & \text{if } \gamma x < \delta, \\ +\infty, & \text{otherwise,} \end{cases}$$

with $a = (\alpha, \beta, \gamma, \delta) \in \mathbb{R}^4$, we see that $\mathscr{E}_f = \mathbb{R}_- \times \mathbb{R}_+ \times (\mathbb{R}_- \times \mathbb{R}_+ \setminus \{0_2\})$ and, since $\lambda g(x) = -\lambda x + \delta_{]-1,+\infty[}(x)$, we have $\mathscr{E}_{\lambda g} = \{-\lambda\} \times \mathbb{R}_+ \times \{0\} \times \mathbb{R}_{++}$. Then,

$$\widetilde{\mathscr{E}}_{f,\lambda g} =]-\infty, -\lambda] \times \mathbb{R}_+ \times (\mathbb{R}_- \times \mathbb{R}_+ \setminus \{0_2\}),$$

and $\sup \widetilde{\mathscr{E}}_{f,\lambda g} = f + \lambda g$.

In order to show that $(C2_{FL})$ holds, and taking into account that $\operatorname{epi}(f + \delta_A)^c = \operatorname{epi} f^c = \mathbb{R}_- \times (\mathbb{R}_- \times \mathbb{R}_+ \setminus \{0_2\}) \times \mathbb{R}_+$, it will be enough to show that

$$\mathbb{R}_- \times (\mathbb{R}_- \times \mathbb{R}_+ \setminus \{0_2\}) \times \mathbb{R}_+ \subseteq \operatorname{proj}\left(\operatorname{epi} \Phi_{FL}^c\right).$$

Let us fix $(\alpha, \beta, \gamma, \delta) \in \mathbb{R}_- \times (\mathbb{R}_- \times \mathbb{R}_+ \setminus \{0_2\}) \times \mathbb{R}_+$ arbitrarily. The key is to find $\kappa, \nu \in \mathbb{R}$ and $\lambda, \mu \in \mathbb{R}$ such that, for all $(x, u, b) \in \operatorname{dom} \Phi_{FL}$

$$\beta x + \nu u + \mu b < \gamma \text{ and } \alpha x + \kappa u + \lambda b \leq \delta. \tag{4.24}$$

Now, $\operatorname{dom} \Phi_{FL} = \{(x, u, b) : x + u > 0,\ x > -1,\ b \geq -x\}$, and if we consider any sequence $\{u_k\} \subseteq \mathbb{R}_{++}$ converging to zero, and any $b \geq 0$, $\{(0, u_k, b)\} \subseteq \operatorname{dom} \Phi_{FL}$. Then, from (4.24), taking limits when k tends to $+\infty$, it follows that, necessarily, $\lambda, \mu \geq 0$. For every $(x, u, b) \in \operatorname{dom} \Phi_{FL}$, if $\nu \in \mathbb{R}$ and $\mu \geq 0$, $\beta x + \nu u + \mu b \leq (\beta - \mu)x + \nu u$. In the case $\beta = 0$ (let us observe that $\gamma > 0$), take $\mu = 0$ ($\nu = 0$), and if $\beta < 0$, take $\mu \geq 0$ such that $\beta - \mu < 0$, and name $\nu = \beta - \mu$. We would have, in both cases, $\beta x + \nu u + \mu b < \gamma$. Proceeding in the same way with the second inequality in (4.24), we can also find κ and λ verifying it. Hence, $(\alpha, \beta, \gamma, \delta) \in \operatorname{proj}\left(\operatorname{epi} \Phi_{FL}^c\right)$ and $(C2_{FL})$ is fulfilled. Finally, let us check that $(C3_{FL})$ does not hold, i.e., $\operatorname{epi} f^c + \bigcup_{\lambda \geq 0} \operatorname{epi}(\lambda g)^c \subsetneq \operatorname{epi}(f + \delta_A)^c$. Let $\lambda \geq 0$ be arbitrary. Then, $\operatorname{epi}(\lambda g)^c = \{(\alpha, \beta, \gamma, \delta) : \alpha \leq \lambda, \beta \leq 0, \gamma > -\beta, \delta \geq \lambda - \alpha\}$ and $\cup_{\lambda \geq 0} \operatorname{epi}(\lambda g)^c = \mathbb{R}_- \times \mathbb{R}_- \times \mathbb{R}_{++} \times \mathbb{R}_+$, so that $(C3_{FL})$ is not fulfilled.

To close this subsection, let us show that $(C2_{FL})$ is not necessary for strong Fenchel–Lagrange duality either.

Example 4.18 (Example 4.15 Revisited) Let $n = 2$, $m = 2$, $g_1(x) = x_1 + x_2$, $g_2(x) = x_1 - x_2$, and $f : \mathbb{R}^2 \to \overline{\mathbb{R}}$ such that

$$f(x_1, x_2) = \begin{cases} \frac{x_2^2}{x_1}, & \text{if } x_1 > 0, \\ 0, & \text{if } x_1 = x_2 = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

The feasible set is $A = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq 0,\ x_1 \leq x_2\right\}$ and we obtain $v(P_2) = 0$. On the other hand, taking $\lambda = 0$ and $\overline{\alpha}_1, \overline{\alpha}_2 > 0$, one has

$$v(D_{FL}) \geq -f^c(0, 0, \overline{\alpha}_1) - (0g)^c(0, 0, \overline{\alpha}_2) = \inf_{x \in \mathbb{R}^2} \{f(x)\} = 0.$$

Hence, we have shown that $v(D_{FL}) \geq v(P_2)$. Due to the weak duality, it follows that strong Fenchel–Lagrange duality holds. Now, let us see that $(C2_{FL})$ is not fulfilled. We will use its equivalent formulation which is stated in Theorem 4.17 and we will see that there exists, at least a point $(\overline{x}^*, \overline{u}^*, \overline{\alpha})$, such that

$$\Phi_{FL}(\cdot, 0, 0)^c(\overline{x}^*, \overline{u}^*, \overline{\alpha}) < \min_{\substack{y^*, v^* \in \mathbb{R} \\ \lambda, \beta \in \mathbb{R}^2}} \Phi_{FL}((\overline{x}^*, y^*, \lambda), (\overline{u}^*, v^*, \beta), \overline{\alpha}).$$

Let any $\overline{x}^* \in \mathbb{R}^2$, $\overline{u}^* = (0, 1)$ and $\overline{\alpha} = 1$. Then, it is not difficult to see that

$$\Phi_{FL}(\cdot, 0, 0)^c(\overline{x}^*, \overline{u}^*, \overline{\alpha}) = \sup_{x = 0_2} \left\{ \left\langle x, \overline{x}^* \right\rangle \right\} = 0,$$

and following an analogous argument to the one of Example 4.15, it follows

$$\min_{\substack{y^*, v^* \in \mathbb{R} \\ \lambda, \beta \in \mathbb{R}^2}} \Phi_{FL}^c((\overline{x}^*, y^*, \lambda), (\overline{u}^*, v^*, \beta), \overline{\alpha}) = +\infty.$$

Remark 4.4 It is easy to check that if $f + \lambda g = \sup \widetilde{\mathscr{E}}_{f,\lambda g}$ for all $\lambda \in \mathbb{R}_+^m$, $(C3_{FL})$ assures stable strong duality between $(P_2) - (D_{FL})$. Here the extended primal and dual problems are

$$\begin{array}{lll} (P_{2,x^*}) & \displaystyle\operatorname*{Min}_{x \in \mathbb{R}^n} & f(x) + \left\langle x, x^* \right\rangle \\ & \text{s.t.} & g_i(x) \leq 0, i = 1, \ldots, m, \\ (D_{FL}) & \displaystyle\operatorname*{Max}_{\lambda \in \mathbb{R}_+^m,\ y^*, v^* \in \mathbb{R}^n, \alpha_1, \alpha_2 \in \mathbb{R}} & \left\{ -\left(f + \left\langle \cdot, x^* \right\rangle\right)^c (y^*, u^*, \alpha_1) - (\lambda g)^c(-y^*, -u^*, \alpha_2) \right\} \\ & \text{s.t.} & \alpha_1 + \alpha_2 > 0, \end{array}$$

for all $x^* \in \mathbb{R}^n$.

4.4.5 A Comparison Between Optimal Values and Optimal Solutions

We devote this subsection to make a comparison of the optimal values of Fenchel, Lagrange and Fenchel–Lagrange dual problems for the primal problem (P_2) in (4.15). We point out that in the case of the Fenchel dual, since the objective function in (P_2) is $F = f + \delta_A$, being the feasible set $A = \{x \in \mathbb{R}^n : g_i(x) \leq 0, i = 1, \ldots, m\}$, we obtain

$$\begin{array}{lll} (D_F) & \displaystyle\operatorname*{Min}_{x^*, u^* \in \mathbb{R}^n, \alpha_1, \alpha_2 \in \mathbb{R}} & \left\{ -f^c(x^*, u^*, \alpha_1) - \delta_A^c(-x^*, -u^*, \alpha_2) \right\} \\ & \text{s.t.} & \alpha_1 + \alpha_2 > 0. \end{array} \tag{4.25}$$

Recall that $f, g_i : \mathbb{R}^n \to \overline{\mathbb{R}}$ for all $i = 1, \ldots, m$ are proper functions. We assume also that the feasible set is nonempty. We will provide sufficient conditions under which the optimal value of Fenchel–Lagrange dual problem is equal, on the one hand, to the one of Fenchel dual problem and, on the other hand, to the optimal value of Lagrange dual problem. Finally, we study the relations between the optimal solutions of these three dual problems and their solvability.

First of all, we will establish the main inequalities that their optimal values satisfy as well as some examples where the inequalities are strictly fulfilled. Recall that the space $W = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$.

Theorem 4.18 (Optimal Values Relationships) *Let (D_F), (D_L) and (D_{FL}) be the dual problems defined in (4.25), (4.17) and (4.23), respectively. The following statements hold:*

(i) $v(D_L) \geq v(D_{FL})$.
(ii) $v(D_F) \geq v(D_{FL})$.
(iii) *If $f, g_i : \mathbb{R}^n \to \overline{\mathbb{R}}$ are convex, for all $i = 1, \ldots, m$, then $v(D_L) = v(D_{FL})$.*
(iv) *If there exist $\overline{\alpha} > 0$ and $((\overline{y}^*, \overline{v}^*, \overline{\alpha}_1), \overline{\alpha}_2, \overline{\lambda}) \in W \times \mathbb{R} \times \mathbb{R}^m_+$ such that $\overline{\alpha}_1 + \overline{\alpha}_2 = \overline{\alpha}$ and*

$$f^c(\overline{y}^*, \overline{v}^*, \overline{\alpha}_1) + (\overline{\lambda} g)^c(-\overline{y}^*, -\overline{v}^*, \overline{\alpha}_2) \leq \inf_{\lambda \in \mathbb{R}^m_+} \left\{ (f + \lambda g)^c(0_n, 0_n, \overline{\alpha}) \right\},$$

then $v(D_L) = v(D_{FL})$.
(v) *If $(ECCQ)$ (4.18) holds, and g_i, for all $i = 1, \ldots, m$, are e-convex functions, then $v(D_F) = v(D_{FL})$.*
(vi) *If there exist $\overline{\lambda} \in \mathbb{R}^{(T)}_+$ and $\overline{\alpha} > 0$ such that $(f + \delta_A)^c(0_n, 0_n, \overline{\alpha}) = (f^c \square (\overline{\lambda} g)^c)(0_n, 0_n, \overline{\alpha})$ and the infimal convolution is exact at $(0_n, 0_n, \overline{\alpha})$, then*

$$v(P_2) = v(D_L) = v(D_F) = v(D_{FL}).$$

Remark 4.5 In statement (*iii*), the convexity assumption on the involved functions in the primal problem cannot be removed.

Example 4.19 Let us take $n = 1$, $f(x) = -x^2$, $m = 1$ and $g_1(x) = x^2$. Clearly $A = \{0\}$ and

$$v(D_L) = \sup_{\lambda \geq 0} \left\{ \inf_{x \in \mathbb{R}} \left\{ -x^2 + \lambda x^2 \right\} \right\} = 0.$$

On the other hand,

$$v(D_{FL}) = \sup_{\substack{y^*, v^* \in \mathbb{R}, \\ \alpha_1 + \alpha_2 > 0, \\ \lambda \geq 0}} \left\{ - \sup_{x \in \mathbb{R}} \left\{ c\left(x, \left(y^*, v^*, \alpha_1\right)\right) + x^2 \right\} \right.$$

$$\left. - \sup_{x \in \mathbb{R}} \left\{ c\left(x, \left(-y^*, -v^*, \alpha_2\right)\right) - \lambda x^2 \right\} \right\}.$$

It is clear that we can restrict ourselves to $v^* = 0$ and $\alpha_1, \alpha_2 > 0$, and we get

$$(D_{FL}) = \sup_{\substack{y^*, \in \mathbb{R}, \\ \lambda \geq 0}} \left\{ \inf_{x \in \mathbb{R}} \left\{ -xy^* - x^2 \right\} + \inf_{x \in \mathbb{R}} \left\{ xy^* + \lambda x^2 \right\} \right\} = -\infty.$$

In the following example it is shown that condition (ECCQ) is necessary in Theorem 4.18(v), even when the involved functions in (P_2) are e-convex.

Example 4.20 Let us take $n = 2$, $m = 2$, $g_1(x) = -x_2$, $g_2(x) = x_1 - x_2$ and

$$f(x) = \begin{cases} x_2 & \text{if } x_1 \leq 0,\ x_2 \in \mathbb{R}, \\ +\infty & \text{otherwise.} \end{cases}$$

Firstly, let us see that f is e-convex. Naming $H = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 \geq x_2\right\}$, it is easy to calculate $\operatorname{epi} f = H \cap (\operatorname{dom} f \times \mathbb{R})$. This set is clearly convex and closed, so f is e-convex. The feasible set A is

$$A = \left\{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_2 \geq 0,\ x_2 \geq x_1\right\}.$$

Hence, since $A \cap \operatorname{dom} f = \left\{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 \leq 0, x_2 \geq 0\right\}$, a simple calculation shows that $v(P_2) = 0$. Now, we calculate the optimal value of the Lagrange dual problem

$$v(D_L) = \sup_{\lambda_1, \lambda_2 \geq 0} \inf_{x \in \operatorname{dom} f} \{\lambda_1 x_1 + (1 - \lambda_1 - \lambda_2) x_2\} = -\infty.$$

Since the involved functions are convex, from Theorem 4.18, we get that $v(D_{FL}) = v(D_L) = -\infty$. If we compute the optimal value of the Fenchel dual problem, we have

$$v(D_F) = \sup_{\substack{y^*, v^* \in \mathbb{R}^2, \\ \alpha_1 + \alpha_2 > 0}} \left\{ -f^c(y^*, v^*, \alpha_1) - (\delta_A)^c(-y^*, -v^*, \alpha_2) \right\}.$$

It is not difficult to see that, at least, one of these c-conjugate functions equals $+\infty$ whenever $v^* \neq 0_2$. Analyzing the trivial case where $v^* = 0_2$ and $\alpha_1, \alpha_2 > 0$, we get that $f^c(y^*, v^*, \alpha_1)$ and $(\delta_A)^c(-y^*, -v^*, \alpha_2)$ are finite, and

$$\begin{aligned} v(D_F) &= \sup_{y^* \in \mathbb{R}^2} \left\{ -\sup_{x \in \operatorname{dom} f} \left\{ x_1 y_1^* + x_2 y_2^* - x_2 \right\} - \sup_{x \in A} \left\{ -x_1 y_1^* - x_2 y_2^* \right\} \right\} \\ &\geq -\sup_{x \in \operatorname{dom} f} \{x_1 \cdot 0 + x_2 \cdot 1 - x_2\} + \inf_{x \in A} \{x_1 \cdot 0 + x_2 \cdot 1\} = \inf_{x \in A} x_2 = 0. \end{aligned}$$

We have just shown that $v(D_F) \geq 0$ and, by the weak duality, $v(D_F) \leq 0$, so $v(D_F) = 0$. To conclude this example, it remains to see that

$$\operatorname{epi} \delta_A^c \nsubseteq \bigcup_{\lambda \in \mathbb{R}_+^2} \operatorname{epi}(\lambda g)^c. \tag{4.26}$$

Clearly, $((0, -1), (0, -1), 1, 0) \in \operatorname{epi} \delta_A^c$. However, this element does not belong to any $\operatorname{epi}(\overline{\lambda} g)^c$ with $\overline{\lambda} \in \mathbb{R}_+^2$ since this fact would imply the fulfilment of

$$c((x_1, x_2), ((0, -1), (0, -1), 1)) - (\overline{\lambda} g)(x_1, x_2) \leq 0,$$

for all $(x_1, x_2) \in \operatorname{dom} \overline{\lambda} g = \mathbb{R}^2$, or, equivalently, $\langle (x_1, x_2), (0, -1) \rangle < 1$, for all $(x_1, x_2) \in \operatorname{dom}(\lambda g) = \mathbb{R}^2$, which is not true. Therefore, (4.26) holds.

Remark 4.6 If the involved functions are proper and convex, applying Theorem 4.18, it always holds

$$v(D_L) = v(D_{FL}) \leq v(D_F),$$

but without any assumption over the primal problem, $v(D_L)$ and $v(D_F)$ cannot be related.

It is worth studying conditions under which the solvability of one of these dual problems implies the solvability of the others.

Theorem 4.19 (Optimal Solutions Relationships) *Let (D_F), (D_L) and (D_{FL}) be the dual problems defined in (4.25), (4.17) and (4.23), respectively. The following statements hold:*

(i) *If $v(P) = v(D_{FL})$ and $(y_0^*, v_0^*, \overline{\alpha}_1, \overline{\alpha}_2, \overline{\lambda}) \in W \times \mathbb{R} \times \mathbb{R}_+^m$ is an optimal solution of (D_{FL}) with $\alpha_1 + \alpha_2 > 0$, then $\overline{\lambda}$ is optimal to (D_L), $(y_0^*, v_0^*, \overline{\alpha}_1, \overline{\alpha}_2)$ is optimal to (D_F), and*

$$v(P) = v(D_L) = v(D_F) = v(D_{FL}).$$

Then if there exists strong Fenchel–Lagrange duality, there also exist Fenchel and Lagrange strong dualities.

(*ii*) *If* $v(D_L) = v(D_F) = v(D_{FL})$ *and either* (D_L) *or* (D_F) *is not solvable, then* (D_{FL}) *is not solvable.*

We finish by showing that the converse statement of Theorem 4.19(i) does not hold in general.

Example 4.21 Let us take $n = 1$, $f = \delta_{[0,+\infty[}$, $m = 1$ and $g_1(x) = x$. Then, $A =]-\infty, 0]$ and trivially $v(P) = 0$. On the other hand,

$$v(D_L) = \sup_{\lambda \geq 0} \left\{ \inf_{x \geq 0} \{\lambda g(x)\} \right\} = 0,$$

so every $\lambda \geq 0$ is an optimal solution of (D_L). Since $f^c(y^*, v^*, \alpha_1) < +\infty$ if and only if $(y^*, v^*, \alpha_1) \in \mathbb{R}_- \times \mathbb{R}_- \times \mathbb{R}_{++}$ and its value is 0, and $\delta_A^c(-y^*, -v^*, \alpha_2) < +\infty$ if and only if $(y^*, v^*, \alpha_2) \in \mathbb{R}_- \times \mathbb{R}_- \times \mathbb{R}_{++}$ being its value, again, 0, it follows that $v(D_F) = 0$ and the solution set of (D_F) is $\mathbb{R}_- \times \mathbb{R}_- \times \mathbb{R}_{++} \times \mathbb{R}_{++}$.

Now, taking in particular $\overline{y}^* = \overline{v}^* = 0, \overline{\alpha}_1, \overline{\alpha}_2 > 0$ and $\overline{\lambda} = 1$, which are optimal solutions of (D_F) and (D_L), respectively, we get that $f^c(0, 0, \overline{\alpha}_1) = 0$, but

$$(\overline{\lambda} g)^c(0, 0, \overline{\alpha}_2) = \sup_{x \in \mathbb{R}} \{-x\} = +\infty.$$

Then, $(0, 0, \overline{\alpha}_1, \overline{\alpha}_2, \overline{\lambda})$ is not optimal to (D_{FL}) since, according to Theorem 4.18, $v(D_{FL}) = v(D_L) = 0$.

4.5 Bibliographic Notes

Convex and lower semicontinuous functions represent a crucial ingredient in variational analysis (see, e.g., [41]), subdifferential calculus [90, 147], conjugate duality theory and optimization [20, 149]. The Fenchel-Moreau Theorem, or the biconjugation Theorem, gives necessary and sufficient conditions for a real extended valued function to be equal to its biconjugate. This happens, in particular, if the function is proper, convex and lower semicontinuous (see [191, Th. 2.3.3]). This theorem represents a fundamental tool when we deal with duality in convex optimization, where a convex optimization problem is embedded in a family of perturbed problems, and by using Fenchel conjugation, a dual problem is associated with the primal one.

In the literature we can find good references concerning the perturbational approach to conjugate duality theory. It has been well-described in the monographs from Rockafellar [149] in the finite-dimensional context, from Ekeland and Temam [41] in Banach spaces, and from Zălinescu [191] in locally convex spaces. There exist two main classes of regularity conditions, named generalized interior-point and closedness-type conditions. In [21] it is provided an overview on some classical interior-point regularity conditions as well as several new ones, which are indeed generalized interior-point ones. In [24–27, 94, 95] the reader can find closedness-type regularity conditions for particular cases of primal and dual problems, see for instance [20] as a presentation of the state of art in this field. The mechanisms behind the closedness-type regularity conditions can be seen in [81]. Also in the general case, sufficient interior-point-type conditions for stable strong duality can be found in [191], while it is characterized by a much more general closedness-type condition in [27].

Evenly convex functions, introduced in [151], extend in a natural way the concept of evenly convex set to functions, and also allow a generalization of the convex and lower semicontinuous functions class. Although Fenchel conjugation theory is not suitable for evenly convex functions, in the sense that different evenly convex functions may have the same lower semicontinuous hull and hence identical second conjugates, we have described in this chapter different conjugation schemes fulfilling that, in the case of an evenly convex function, the second conjugate is identical to the original function. Thus, invoking to the c-conjugation scheme in particular, a perturbational duality theory for evenly convex optimization problems has been developed by Fajardo and her co-authors [45–49], while regularity conditions for strong duality based on even convexity are given in [43, 45, 46, 48, 49, 181].

This chapter is based on [43, 45, 48, 49, 126, 151, 181, 184]. Evenly convex functions definition, properties and examples in Sect. 4.1 can be found almost entirely in [151]: statements (i) and (ii) of Theorem 4.1 are [151, Th. 2.6] and [151, Prop. 2.7], respectively, Theorem 4.2 and Corollary 4.1 correspond to [151, Th. 2.9] and [151, Cor. 2.10], respectively. For Proposition 4.1 see [151, Prop. 2.12], while operations with evenly convex functions in Theorem 4.3 are stated in [151, Sec. 3].

Results in Sect. 4.2 on the e-convex hull of a given function can be found in [126, 151, 181]. The representation for the e-convex hull of a function f in (4.5) was proved for the first time in [151, Prop. 3.10] and it still remains true if the set $\operatorname{epi} f$ is replaced by $\operatorname{epi}_s f$, or even by any set A such that $\operatorname{epi}_s f \subset A \subset \operatorname{epi} f$. More precisely, Theorem 4.4 is [181, Prop. 2.3 and Cor. 2.4], Theorem 4.5 is [181, Prop. 2.6 and Cor. 2.7], Theorems 4.6 and 4.7 are taken from [151], Theorem 4.8 is [126, Th. 16], and finally, Corollary 4.2 encompasses [126, Cors. 18 and 19].

Section 4.3 is devoted to three possible conjugation schemes for evenly convex functions, depending on the chosen coupling functions, and the notion of subdifferentibility associated to one of these schemes. It is based on [46, 126, 184]. On the one hand, regarding the conjugation patterns, the first approach is taken from [46, 126], whereas the details to the second and the third approaches can be found in [184]. On the other hand, c-subdifferentiability is described in [126]. Regarding the precise references for the given results in this section, Theorem 4.9 is [46, Prop. 2], Theorem 4.10 encompasses [184, Props. 4.2, 5.2 and Cor. 5.1] while the key tool (4.4) is proved in [184, Prop. 4.1], and Theorem 4.11 is [184, Th. 6.1 and Cor. 6.1]. In addition to that, Proposition 4.2 is [126, Prop. 45], Proposition 4.3 is [181, Th. 3.1], Corollary 4.3 is [181, Cor. 3.2], Proposition 4.4 is [181, Prop. 3.4], and finally, Theorem 4.12 is [181, Th. 4.1]. Readers who are familiar with the classical concept of subdifferentiability and its relationship with Fenchel conjugation will find that the three statements given in Theorem 4.13 are generalizations of well-known formulas (see, e.g., [20]). Statement (i) in that theorem is [46, Lem. 9], while *(ii)* is [46, Th. 11] and *(iii)* is [46, Cor. 12].

Section 4.4 is divided into five subsections, corresponding to the following objectives. Section 4.4.1 is devoted to regularity conditions for general optimization problems and duals obtained via perturbational approach and by means of c-conjugation, and is based mostly on [43], where the general optimization problem framework does not have any restrictions about dimensionality, and the involved spaces are Hausdorff locally convex. Actually the Fenchel dual problem and the regularity conditions were obtained in the particular case of g being an indicator function of any subset, but all the results can be generalized easily to any function g. It is necessary to point out that the regularity condition $(C1)$ in Theorem 4.5 is expressed in terms of the relative algebraic interior of certain projection of $\operatorname{dom} \Phi$. Theorem 4.5 is proved in [43, Prop. 4.3]. This result can be derived as a particular case in [124, Prop. 6.4, Th. 6.7 and Cor. 6.2], because the dual problem with $u_0 = 0$ suggested by Martínez-Legaz is equivalent to (GD_c), as it is shown in [46]. The general regularity conditions in Theorem 4.14 are stated in [43, Prop. 4.5] and [43, Props. 5.4 and 5.5], respectively.

The topic in Sect. 4.4.2 is Fenchel duality, obtaining regularity conditions and comparing them. Most results there can be found in [43, Section 6]. Condition $(C3_F)$ in Theorem 4.15 is studied in detail in [46, Section 5], and strong duality characterization $(C4_F)$ is discussed in [47, Lem. 4.3]. This characterization was motivated by [109], where, in the classical setting, strong Fenchel duality is equivalent to the following inequality, which is called $(FRC)_A$,

$$(f + g \circ A)^* (0) \geq \left(f^* \Box A^* g^*\right) (0),$$

together with the exactness of the infimal convolution at the point 0. In that paper, X and Y are assumed to be Hausdorff locally convex spaces, $A : X \to Y$ is a linear operator and $f : X \to \overline{\mathbb{R}}$ and $g : Y \to \overline{\mathbb{R}}$ are proper convex functions such that $A(\operatorname{dom} f) \cap \operatorname{dom} g \neq \emptyset$. Finally, the relationships between regularity conditions in Proposition 4.6 are established in [43, Props. 6.3 and 6.5].

Section 4.4.3 develops an analogous study for Lagrange duality, with a similar structure than in the previous subsection. It is based mostly on [45], where the primal problem is stated in an infinite-dimensional framework and the number of constraints is arbitrary. The calculus of the e′-convex hull of $\bigcup_{\lambda \in \mathbb{R}_+^m} (\lambda g)^c$ is computed in [45, Prop. 4.1], which allows formula (4.18). The general regularity condition $(C1_L)$ in Lagrange duality Theorem 4.16 is provided in [45, Prop. 5.1], while $(C3_L)$, $(C4_L)$ and their equivalence can be found in [45, Prop. 4.2 and Th. 4.1]. The comparison between regularity conditions in Proposition 4.7 is stated in [45, Prop. 5.2].

Regarding Fenchel–Lagrange duality, Sect. 4.4.4 follows the same scheme as Sects. 4.4.2 and 4.4.3. All the results can be found in [49], in a more general framework with infinite dimensional spaces and an arbitrary number of constraints in the primal problem. The regularity condition $(C3_{FL})$ from Theorem 4.17 is proved in [49, Prop. 3.4], and it was motivated by the regularity condition $(ECCQ)$ in (4.18). Actually, it is a direct generalization of a regularity condition from the closed convex case; see $\left(CQ^{FL}\right)$ in [22]. On the other hand, strong duality characterizations $(C4_{FL})$ and $(C5_{FL})$ can be found in [49, Prop. 3.1], and the comparison between regularity conditions in Proposition 4.8 is provided in [49, Prop. 4.1]. It is worth to remark that in the infinite dimensional case, (i) in Proposition 4.8 is not enough for the equality between optimal values $v(D_L)$ and $v(D_{FL})$. We also need that int(epi $f) \neq \emptyset$, as it is pointed out in [49, Rem. 4.2 and Ex. 4.3].

The last subsection within Sect. 4.4 extends what Boţ and Wanka obtained in [23], where they compared the three dual problems on finite-dimensional spaces having a finite number of constraints and dealing with the classical Fenchel conjugation scheme. In our setting we also deal with a finite number of inequalities, but work with the c-conjugation pattern. The relations between optimal values in Theorem 4.18 can be found, in a more general setting, with an arbitrary number of constraints and within Hausdorff locally convex spaces, in the following references from [48]: (i), *(iii)* and *(iv)* in Proposition 4.1, *(ii)* in Proposition 4.5, (v) in Proposition 4.6 and *(vi)* in Proposition 5.1. Finally, Theorem 4.19 is stated in Theorems 5.3 and 5.5.

For readers who could be interested in evenly convex optimization, a very recent work [44] deals with converse and total duality, situations where, in the first case, there is no duality gap and the primal problem is solvable, and, in the second case, strong and converse duality hold together. Total duality is characterized by means of the saddle-point theory approach. Furthermore, one can find there formulae for the c-subdifferential and biconjugate of the objective function in a general optimization problem.

Appendix A
Extensions to Infinite-Dimensional Spaces

Some of the results collected in this book have infinite-dimensional versions, where the finite-dimensional Euclidean space $\mathbb{R}^n$ (decision space for the optimization problems considered in this book) is replaced by some Banach spaces or even a locally convex separated (i.e., Hausdorff) topological vector space (lcHtvs in short), as Table A.1 shows:

Table A.1 Infinite-dimensional extensions

Item	Banach	lcHtvs
$[(i) \iff (iii) \iff (vi)]$ in Theorem 1.1	[37]	–
Section 1.4.3	[39]	–
Section 2.3	[192]	[112, 113]
Theorem 3.1	[37]	–
Section 3.3	–	[121]
Sections 4.1 and 4.2	–	[126, 181]
Section 4.3	–	[126, 184]
Section 4.4.1	–	[43]
Section 4.4.2	–	[43, 46]
Section 4.4.3	–	[47]
Section 4.4.4	–	[49]
Section 4.4.5	–	[48]

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