Maximal Surfaces on Two-Step Sub-Lorentzian Structures

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Abstract We describe sufficient maximality conditions for the classes of graph surfaces on two-step Carnot groups with sub-Lorentzian structure. In particular, we introduce a non-holonomic notion of variation of the area functional.

Keywords Two-step Carnot group · Contact mapping · Intrinsic measure · Area formula · Maximal surface · Sufficient maximality condition

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1 Introduction

The aim of this paper is to describe the classes of maximal graph surfaces in sub-Lorentzian geometry, namely, sufficient maximality conditions. The graph mappings are constructed from mappings of two-step nilpotent graded groups. These groups are a particular case of Carnot–Carathéodory spaces well-known in various problems of pure and applied mathematics; see, e. g., [15] and references therein. We also assume that the image and preimage are both subsets of another nilpotent graded group possessing a sub-Lorentzian structure. This structure is a sub-Riemannian generalization of Minkowski geometry. The main characteristic of this geometry is that the distance between points (x_1, t_1) and (x_2, t_2), with $x_1, x_2 \in \mathbb{R}^n$ and $t_1, t_2 \in \mathbb{R}$, equals

$$\sqrt{(x_1 - x_2)^2 - (t_1 - t_2)^2},$$

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i.e., the squared distance along the time-like direction t is negative, while along every space-like direction $x \in \mathbb{R}^n$ it is positive. If all tangent vectors to a surface in $\mathbb{R}^{n+1}_1$ have only positive lengths then this surface is called space-like and it is locally representable as a graph, where the time-like variable depends on the space-like variables: $t = \psi(x)$ with $x \in \mathbb{R}^n$. Under some additional assumptions it is possible to deduce certain equations describing surfaces of maximal area; it follows that their mean curvature vanishes a. e. According to Nielsen's hypothesis, solutions to Einstein's gravity equations are physically meaningful if and only if they are representable as such surfaces in $\mathbb{R}^{n+1}_1$. For the details concerning the properties, applications and interpretations of Minkowski geometry, see [19] and references: e.g., [20, 21] etc.

Sub-Lorentzian geometry is a relatively young branch of analysis; the first results in this area were obtained in the 1990s; see [2]. Later, series of papers studied some fine properties of geodesics together with their connection to relativity theory problems; see, e. g., [5, 6, 16–18]. New cases of Minkowski geometry with multi-dimensional time were studied recently in [1, 3] etc.

In [9], the author deduced necessary maximality conditions for classes of graph surfaces and, moreover, the equations of maximal surfaces. Here the term "maximal surface" means a surface of maximal area (under the assumption that a solution to the corresponding boundary value problem exists). We emphasize that [9], in view of certain fine properties of non-holonomic geometry, the definition of argument increment of the area functional differs substantially from the classical one. Namely, if the horizontal part of the argument changes arbitrarily to order ε then the other part of the formula that corresponds to degree two fields depending on the horizontal ones involves additional summands of order ε^2. Consequently, when we take the second differential of the area functional to obtain sufficient maximality conditions, some new summands appear, which are absent in Riemannian geometry. Recall that generally in non-holonomic structures the notions "maximal area" and "maximal value of the area functional" are not the same. In the latter case, the functional can take some maximal value but it need not correspond to any mapping defining a surface of this area since the PDE problem may lack solutions.

The result of this paper was announced in [12].

2 Graphs on Carnot Groups

Let us recall necessary notions and results.

Definition 1 (See, e. g., [4]) A *two-step Carnot group* is a connected simply connected stratified Lie group $\mathbb{G}$ with a graded Lie algebra V, that is, $V = V_1 \oplus V_2$ with $[V_1, V_1] = V_2$ and $[V_1, V_2] = \{0\}$. If we replace $[V_1, V_1] = V_2$ by $[V_1, V_1] \subset V_2$ and $[V_2, V_2] = \{0\}$ then $\mathbb{G}$ is called a *two-step nilpotent graded* (Lie) *group*. A basis in V is chosen so that each field belongs either to V_1 or V_2. The vector fields in V_1 are called *horizontal* and their *degree* is equal to one. Otherwise the degree is equal to two.

Definition 2 The derivatives along horizontal vector fields are called *horizontal derivatives*.

The group operation is defined by the Baker–Campbell–Hausdorff formula. Now, introduce the distance corresponding to the group structure.

Definition 3 (See, e. g., [12]) Take $w = \exp\Big(\sum\limits_{i=1}^{N} w_i X_i\Big)(v)$ with $w, v \in \mathbb{G}$. Define $d_2(w, v) = \max\Big\{\Big(\sum\limits_{j:\deg X_j=1} w_j^2\Big)^{\frac{1}{2}}, \Big(\sum\limits_{j:\deg X_j=2} w_j^2\Big)^{\frac{1}{4}}\Big\}$. The set $\{w \in \mathbb{G} : d_2(w, v) < r\}$ is called the *radius* $r > 0$ *ball in* d_2 *centered at* v and is denoted by $\mathrm{Box}_2(v, r)$.

Definition 4 ([22]; See Also [23] for the General Case) A mapping $\varphi : U \to \widetilde{\mathbb{K}}$, $U \subset \mathbb{K}$, where $\mathbb{K}$ and $\widetilde{\mathbb{K}}$ are nilpotent graded groups, is *hc-differentiable* at $x \in U$ if there exists a horizontal homomorphism $\mathcal{L}_x : \mathbb{K} \to \widetilde{\mathbb{K}}$ such that $d_2(\varphi(w), \mathcal{L}_x\langle w\rangle) = o(d_2(x, w))$, where $U \ni w \to x$. The *hc-differential* $\mathcal{L}_x$ at x is denoted by $\widehat{D}\varphi(x)$.

Definition 5 (See, e. g., [23]) If the horizontal derivatives of φ exist everywhere and are continuous, while the images of horizontal vector fields are horizontal, then φ is called a *mapping of class* C_H^1, or C_H^1*-mapping*.

Let us now give a precise description of the setup. To this end, we consider a mapping $\varphi : \Omega \to \widetilde{\mathbb{G}}$, where:

1. $\Omega \subset \mathbb{G}$ is an open set and $\varphi : \Omega \to \widetilde{\mathbb{G}}$ is a C_H^1-mapping;
2. $\mathbb{G}$ is a Carnot group of topological dimension N with basis vector fields $X_1, \ldots, X_N$, Lie algebra $V = V_1 \oplus V_2$, where $X_1, \ldots, X_{\dim V_1}$ constitute the basis of V_1, and origin $\mathbf{0}$;
3. each degree two field on $\mathbb{G}$ can be uniquely expressed via the commutators of horizontal fields:

$$X_k = \sum_{i,j=1}^{n} a_{i,j}^{k}[X_i, X_j],\ i < j,\ k = \dim V_1 + 1, \ldots, N \tag{1}$$

   (this enables us to vary the argument arbitrarily; see the details in [10]);
4. $\widetilde{\mathbb{G}}$ is a two-step nilpotent graded group of topological dimension $\widetilde{N}$ with basis fields $\widetilde{X}_1, \ldots, \widetilde{X}_{\widetilde{N}}$, Lie algebra $\widetilde{V} = \widetilde{V}_1 \oplus \widetilde{V}_2$, where $\widetilde{X}_1, \ldots, \widetilde{X}_{\dim \widetilde{V}_1}$ constitute the basis of $\widetilde{V}_1$, structure constants [4] $\{c_{lmq}\}_{l,m,q}$

$$[\widetilde{X}_l, \widetilde{X}_m] = \sum_{q:\deg \widetilde{X}_q=2} c_{lmq}\widetilde{X}_q, \tag{2}$$

   for $l, m = 1, \ldots, \dim \widetilde{V}_1$, and origin $\widetilde{\mathbf{0}}$;

5. $\mathbb{G}, \widetilde{\mathbb{G}} \subset \mathbb{U}$, where $\mathbb{U}$ is a two-step nilpotent graded group of topological dimension $N + \widetilde{N}$, and $\mathbb{G} \cap \widetilde{\mathbb{G}} = \widehat{\mathbf{0}} = (\mathbf{0}, \widetilde{\mathbf{0}})$;
6. the fields $X_1, \dots, X_N$ and $\widetilde{X}_1, \dots, \widetilde{X}_{\widetilde{N}}$ coincide with the restrictions of the basis fields on $\mathbb{U}$ to the groups $\mathbb{G}$ and $\widetilde{\mathbb{G}}$ respectively; moreover, their degrees are equal to those of the corresponding fields on $\mathbb{U}$.

Note that the Cartesian product $\mathbb{G} \times \widetilde{\mathbb{G}}$ is a particular case of $\mathbb{U}$. In general, groups $\mathbb{G}$ and $\widetilde{\mathbb{G}}$ are submanifolds of $\mathbb{U}$ intersecting at their origins. This intersection coincides with the origin $\widehat{\mathbf{0}}$ of $\mathbb{U}$.

The following property is used to obtain the main result.

Theorem 6 ([23]) *Every C^1_H-mapping φ of a Carnot group to a nilpotent graded group is continuously hc-differentiable everywhere, that is, in a neighborhood of each point x it is approximated by a horizontal homomorphism up to $o(d_2(x, \cdot))$. Moreover, the matrix of its hc-differential has a block-diagonal structure with blocks $(\widehat{D}\varphi)_H$ and $(\widehat{D}\varphi)_{H^\perp}$, where the first block corresponds to fields in V_1 and $\widetilde{V}_1$, and the second one, to fields in V_2 and $\widetilde{V}_2$.*

Definition 7 Given φ, the *graph mapping* $\varphi_\Gamma : \Omega \to \mathbb{U}$ assigns to each x the element $\varphi_\Gamma(x) = \exp\Big(\sum\limits_{j=1}^{\widetilde{N}} \varphi_j(x)\widetilde{X}_j\Big)(x)$, where $\exp\Big(\sum\limits_{j=1}^{\widetilde{N}} \varphi_j(x)\widetilde{X}_j\Big)(\widetilde{\mathbf{0}}) = \varphi(x)$.

Straightforward calculations show that the graph mappings of C^1_H-mappings are neither hc-differentiable nor differentiable in the classical sense. Nevertheless, a suitable tool, polynomial hc-differentiability, was created recently in [7, 8]. It enables us to approximate graphs by smooth mappings. The main disadvantage of graph mappings is that the differential of polynomial hc-differential does not have block diagonal structure, which complicates the description of metric properties. The solution is to introduce a new basis [8], called the *intrinsic basis*, close to initial one but ensuring the desired structure of the polynomial hc-differential.

Theorem 8 ([14]) *In a neighborhood of each $\varphi_\Gamma(x)$, where $x \in \Omega$, there exists an intrinsic basis*

$$X_i \mapsto {}^x X_i = X_i + \sum_{k:\deg X_k = 2} a_{ik} X_k + \sum_{l:\deg \widetilde{X}_l = 2} b_{il} \widetilde{X}_l$$

such that the matrix of the differential of polynomial hc-differential has block lower triangle with blocks equal to union of blocks in $\widehat{D}\varphi$ and unit matrices.

3 Sub-Lorentzian Structures

To describe the sub-Lorentzian structure on $\mathbb{U}$, we introduce the following notation. Since we consider non-holonomic generalization of Minkowski geometry with multi-dimensional time, the main idea is to divide basis fields into "positive" and "negative". Here, the squared length of integral curves of "negative" fields is set to be negative.

Definition 9 Put $\{X_1, \dots, X_N, \widetilde{X}_1, \dots, \widetilde{X}_{\widetilde{N}}\} = \{Y_1, \dots, Y_{\widehat{N}}\}$, where $\widehat{N} = N + \widetilde{N}$. Moreover, let the Lie algebra $\widehat{V}$ on $\mathbb{U}$ be equal to $\widehat{V}_1 \oplus \widehat{V}_2$ with

$$[\widehat{V}_1, \widehat{V}_1] \subset \widehat{V}_2,$$

$$(X_1, \dots, X_{\dim V_1}, \widetilde{X}_1, \dots, \widetilde{X}_{\dim \widetilde{V}_1}) = (Y_1, \dots, Y_{\dim \widehat{V}_1^+}, Y_{\dim \widehat{V}_1^+ + 1}, \dots, Y_{\dim \widehat{V}_1}),$$

$$\{Y_{\dim \widehat{V}_1^+ + 1}, \dots, Y_{\dim \widehat{V}_1}\} = \{\widetilde{X}_1, \dots, \widetilde{X}_{\dim \widetilde{V}_1}\},$$

$$(X_{\dim V_1 + 1}, \dots, X_N, \widetilde{X}_{\dim \widetilde{V}_1 + 1}, \dots, X_{\widetilde{N}}) = (Y_{\dim \widehat{V}_1 + 1}, \dots, Y_{\dim \widehat{V}_1 + \dim \widehat{V}_2^+}, Y_{\dim \widehat{V}_1 + \dim \widehat{V}_2^+ + 1}, \dots, Y_{\widehat{N}}),$$

where $(Y_{\dim \widehat{V}_1 + \dim \widehat{V}_2^+ + 1}, \dots, Y_{\widehat{N}}) = (\widetilde{X}_{\dim \widetilde{V}_1 + 1}, \dots, X_{\widetilde{N}})$. Denote $\dim \widehat{V}_2 = \widehat{N} - \dim \widehat{V}_1$, $\dim \widehat{V}_1^- = \dim \widehat{V}_1 - \dim \widehat{V}_1^+ (= \dim \widetilde{V}_1)$ and $\dim \widehat{V}_2^- = \dim \widehat{V}_2 - \dim \widehat{V}_2^+ (= \dim \widetilde{V}_2)$.

Definition 10 For a vector field $T = \sum\limits_{j=1}^{\widehat{N}} y_j Y_j$ with constant coefficients, set the *squared sub-Lorentzian norm* to be

$$\mathbf{d}_2^{SL^2}(T) = \max\Bigg\{ \sum_{j=1}^{\dim \widehat{V}_1^+} y_j^2 - \sum_{k=1}^{\dim \widehat{V}_1^-} y_{\dim \widehat{V}_1^+ + k}^2, \\ \operatorname{sgn}\Bigg(\sum_{j=1}^{\dim \widehat{V}_2^+} y_{\dim \widehat{V}_1 + j}^2 - \sum_{k=1}^{\dim \widehat{V}_2^-} y_{\dim \widehat{V}_1 + \dim \widehat{V}_2^+ + k}^2 \Bigg) \\ \times \Bigg| \sum_{j=1}^{\dim \widehat{V}_2^+} y_{\dim \widehat{V}_1 + j}^2 - \sum_{k=1}^{\dim \widehat{V}_2^-} y_{\dim \widehat{V}_1 + \dim \widehat{V}_2^+ + k}^2 \Bigg|^{1/2} \Bigg\}.$$

If $w = \exp(T)(v)$ then the *squared sub-Lorentzian distance* $\mathfrak{d}_2^2(v, w)$ equals $\mathbf{d}_2^{SL^2}(T)$. The $\mathfrak{d}_2^2$-*ball* of radius r centered at v is $\mathrm{Box}_{\mathfrak{d}_2^2}(v, r) = \{x \in \mathbb{U} : \mathfrak{d}_2^2(v, x) < r^2\}$.

The *intrinsic squared distance* ${}^x\mathfrak{d}_2^2(v, w)$ is defined similarly with Y_j replaced by ${}^xY_j$ for $j = 1, \dots, \widehat{N}$.

Definition 11 For each $x \in \varphi_\Gamma(\Omega)$, consider a neighborhood $\mathcal{U}(\varphi_\Gamma^{-1}(x)) \subset \Omega$ where $o(1)$ from the definition of hc-differentiability is sufficiently small. Consider

$\delta_0 > 0$ such that each ball in Ω of radius $r < T\delta_0$ lies in at least one of these neighborhoods (since we study local property, we may assume without loss of generality that $\Omega \subset \mathbb{G}$ is a compact neighborhood), where T satisfies

$$\frac{1}{T} d_2(v_j, w) \le {}^{v_j}\mathfrak{d}_2^2(\varphi_\Gamma(v_j), \varphi_\Gamma(w))^{1/2} \le T d_2(v_j, w).$$

Define *the intrinsic measure* ${}^{\mathrm{SL}}\mathcal{H}_\Gamma^\nu$ on $S \subset \varphi_\Gamma(\Omega)$ as

$$\omega_{\dim V_1}\omega_{\dim V_2} \lim_{\delta\to 0} \inf \Big\{ \sum_{j\in\mathbb{N}} r_j^\nu : \bigcup_{j\in\mathbb{N}} {}^{\varphi_\Gamma^{-1}(x_j)}\mathrm{Box}_{\mathfrak{d}_2^2}(x_j, r_j) \cap \varphi_\Gamma\big(\mathcal{U}(\varphi_\Gamma^{-1}(x_j))\big) \supset S, \quad x_j \in S,\ r_j < \delta < \delta_0,\ j \in \mathbb{N} \Big\}. \tag{3}$$

To this end, rows $1, \dots, \dim \widetilde{V}_1$ of the matrix of the hc-differential together are denoted by $(\widehat{D}\varphi)_H(x)$. Assume that the squares of its column lengths are at most $\frac{1}{2\dim V_1^2} - c$ with $c > 0$. The block starting from row $\dim \widetilde{V}_1 + 1$ is denoted by $(\widehat{D}\varphi)_{H^\perp}(x)$ and we assume that the squares of its column lengths are at most $\frac{1}{\dim V_2} - c$ with $c > 0$.

Remark 12 The above restrictions guarantee the space-like property of the surface $\varphi_\Gamma(\Omega)$; see the details in [14].

One of the main results of [14] is the following area formula for the graphs of C_H^1-mappings defined on a two-step Carnot group with values in a two-step nilpotent graded group. We formulate it for our case.

Theorem 13 *The surface $\varphi_\Gamma(\Omega)$ is space-like and its ${}^{\mathrm{SL}}\mathcal{H}_\Gamma^\nu$-measure is*

$$\int_\Omega {}^{\mathrm{SL}}\mathcal{J}(\varphi, v)\, d\mathcal{H}^\nu(v) = \int_{\varphi_\Gamma(\Omega)} d\, {}^{\mathrm{SL}}\mathcal{H}_{\mathfrak{b}}^\nu(y), \tag{4}$$

where the sub-Lorentzian Jacobian ${}^{\mathrm{SL}}\mathcal{J}(\varphi, v)$ equals

$$\sqrt{\det\big(E_{\dim V_1} - (\widehat{D}\varphi)_H^*(\widehat{D}\varphi)_H\big)}\sqrt{\det\big(E_{\dim V_2} - (\widehat{D}\varphi)_{H^\perp}^*(\widehat{D}\varphi)_{H^\perp}\big)}$$

and ${}^{\mathrm{SL}}\mathcal{H}_{\mathfrak{b}}^\nu$ is defined the same way as ${}^{\mathrm{SL}}\mathcal{H}_\Gamma^\nu$, where $\omega_{\dim V_1}\omega_{\dim V_2} r_j^\nu$ is replaced by $\mathfrak{b}(x_j, r_j, \nu)$, $j \in \mathbb{N}$; see details in [13]. If the matrix of $D(\widehat{D}_P\varphi_\Gamma)$ has block diagonal structure everywhere then ${}^{\mathrm{SL}}\mathcal{H}_{\mathfrak{b}}^\nu = {}^{\mathrm{SL}}\mathcal{H}_\Gamma^\nu$.

The following notions are important for our description of the main properties of maximal surfaces.

Definition 14 (cf. [11]) The *area functional* $S(\varphi)$ defined on the class of graph mappings constructed from C^1_H-mappings is

$$\int\limits_{\Omega} \sqrt{\det\big(E_{\dim V_1} - (\widehat{D}\varphi)^*_H(\widehat{D}\varphi)_H\big)}\sqrt{\det\big(E_{\dim V_2} - (\widehat{D}\varphi)^*_{H^\perp}(\widehat{D}\varphi)_{H^\perp}\big)}\, d\mathcal{H}^\nu. \tag{5}$$

The *area functional increment* on $\xi : \Omega \to \mathbb{R}^{\dim \widetilde{V}_1}$ with $\xi = (\xi_1, \dots, \xi_{\dim \widetilde{V}_1})$ equals $S(\varphi, \xi, \varepsilon) - S(\varphi)$, where $S(\varphi, \xi, \varepsilon)$ is the integral over Ω of

$$\sqrt{\det\Big(E_{\dim V_1} - \big((\widehat{D}\varphi)_H + \varepsilon D_H\xi\big)^*\big((\widehat{D}\varphi)_H + \varepsilon D_H\xi\big)\Big)}$$
$$\times \sqrt{\det\Big(E_{\dim V_2} - \big((\widehat{D}\varphi)_{H^\perp} + \varepsilon P_1 + \varepsilon^2 P_2\big)^*\big((\widehat{D}\varphi)_{H^\perp} + \varepsilon P_1 + \varepsilon^2 P_2\big)\Big)},$$

D_H denotes differentiation along the horizontal fields only, $P_1(x)\langle X_k\rangle$ and $P_2(x)\langle X_k\rangle$ are equal to

$$2\sum_{i,j=1}^{\dim V_1} a^k_{i,j} \sum_{q>\dim \widetilde{V}_1} \sum_{l,m=1}^{\dim \widetilde{V}_1} \Big((\widehat{D}\varphi(x))_{li} X_j\xi_m(x) - (\widehat{D}\varphi(x))_{lj} X_i\xi_m(x)\Big) c_{lmq} \widetilde{X}_q, \tag{6}$$

and

$$\sum_{i,j=1}^{\dim V_1} a^k_{i,j} \sum_{q>\dim \widetilde{V}_1} \sum_{l,m=1}^{\dim \widetilde{V}_1} \Big(X_i\xi_l(x) X_j\xi_m(x) - X_i\xi_m(x) X_j\xi_l(x)\Big) c_{lmq} \widetilde{X}_q, \tag{7}$$

respectively, $a^k_{i,j}$ are from (1), c_{lmq} are from (2), $i, j = 1, \dots, \dim V_1$ with $i < j$, $k, l, m = 1, \dots, \dim \widetilde{V}_1$, $q = \dim \widetilde{V}_1 + 1, \dots, \widetilde{N}$ (see the details in [10] and [11]).

Definition 15 Take $\Omega \subset \mathbb{G}$, $\xi_1, \dots, \xi_{\dim \widetilde{V}_1} \in C^1_H(\Omega, \mathbb{R})$, and $m \in \mathbb{N}$. Define the norm $\|\xi\|_m$ for ξ as

$$\|\xi\|_m = \bigg(\int\limits_{\Omega} \sum_{k=1}^{\dim \widetilde{V}_1} |\xi_k(x)|^m + \sum_{\beta:\, |\beta|=m} |\widehat{\xi}(x)^\beta|\, d\mathcal{H}^\nu(x)\bigg)^{\frac{1}{m}},$$

and the (semi)norm $\|\xi\|_{H,m}$ for $\xi = (\xi_1, \dots, \xi_{\dim \widetilde{V}_1})$ as

$$\|\xi\|_{H,m} = \Bigl(\int\limits_{\Omega} \sum_{\beta:\,|\beta|=m} |\widehat{\xi}(x)^{\beta}|\, d\mathcal{H}^{\nu}(x)\Bigr)^{\frac{1}{m}},$$

where $\widehat{\xi} = (X_1\xi_1, \dots, X_1\xi_{\dim \widetilde{V}_1}, X_2\xi_1, \dots, X_{\dim V_1}\xi_{\dim \widetilde{V}_1})$.

Definition 16 The domain $\Omega \subset \mathbb{G}$ is called *horizontally attainable* if each interior point of it can be connected to a boundary point by a curve consisting of a finite number of integral lines of horizontal vector fields.

Theorem 17 *The area functional* (5) *is differentiable twice with respect to the norm* $\|\cdot\|_{\max\{6\dim V_1, 12\dim V_2\}}$. *If* Ω *is horizontally attainable then* $\|\cdot\|_{H,\max\{6\dim V_1,12\dim V_2\}}$ *is a norm, and* (5) *is also differentiable twice with respect to it.*

The proof follows the scheme of [10, Theorem 5] almost verbatim with obvious changes. The main idea is to deduce the expression of the third derivative of $\sqrt{f_1(\varepsilon)}\sqrt{f_2(\varepsilon)}$ at ε and then to estimate the maximal degree of $X_1\xi, \dots, X_{\dim V_1}\xi$ in

$$\sqrt{\det\bigl(E_{\dim V_1} - \bigl((\widehat{D}\varphi)_H + \varepsilon D_H\xi\bigr)^*\bigl((\widehat{D}\varphi)_H + \varepsilon D_H\xi\bigr)\bigr)}$$

and

$$f_2(\varepsilon) = \sqrt{\det\bigl(E_{\dim V_2} - \bigl((\widehat{D}\varphi)_{H^\perp} + \varepsilon P_1 + \varepsilon^2 P_2\bigr)^*\bigl((\widehat{D}\varphi)_{H^\perp} + \varepsilon P_1 + \varepsilon^2 P_2\bigr)\bigr)}$$

as well as their derivatives at ε.

Theorem 18 *Assume that* $a^k_{i,j}$ *in* (1), c_{lmq} *in* (2), $i, j = 1, \dots, \dim V_1$ *with* $i < j$, $k, l, m = 1, \dots, \dim \widetilde{V}_1$, $q = \dim \widetilde{V}_1 + 1, \dots, \widetilde{N}$, *are sufficiently small. If there exists* $K > 0$ *such that*

$$\int\limits_{\Omega} \|D_H\xi\|^2\Bigl(\frac{\sqrt{\det\bigl(E_{\dim V_2} - (\widehat{D}\varphi)^*_{H^\perp}(\widehat{D}\varphi)_{H^\perp}\bigr)}}{\sqrt{\det\bigl(E_{\dim V_1} - (\widehat{D}\varphi)^*_{H}(\widehat{D}\varphi)_{H}\bigr)}} + \frac{\sqrt{\det\bigl(E_{\dim V_1} - (\widehat{D}\varphi)^*_{H}(\widehat{D}\varphi)_{H}\bigr)}}{\sqrt{\det\bigl(E_{\dim V_2} - (\widehat{D}\varphi)^*_{H^\perp}(\widehat{D}\varphi)_{H^\perp}\bigr)}}\Bigr)\, d\mathcal{H}^{\nu}(x) \geq K\|\xi\|^2_{\max\{6\dim V_1, 12\dim V_2\}},$$

and the necessary maximality condition

$$\int\limits_{\Omega} \mathcal{D}_1(\varphi,\xi)\frac{\sqrt{\det\bigl(E_{\dim V_2}-(\widehat{D}\varphi)^*_{H^\perp}(\widehat{D}\varphi)_{H^\perp}\bigr)}}{\sqrt{\det\bigl(E_{\dim V_1}-(\widehat{D}\varphi)^*_{H}(\widehat{D}\varphi)_{H}\bigr)}}\,d\mathcal{H}^\nu + \int\limits_{\Omega} \mathcal{D}_2(\varphi,\xi)\frac{\sqrt{\det\bigl(E_{\dim V_1}-(\widehat{D}\varphi)^*_{H}(\widehat{D}\varphi)_{H}\bigr)}}{\sqrt{\det\bigl(E_{\dim V_2}-(\widehat{D}\varphi)^*_{H^\perp}(\widehat{D}\varphi)_{H^\perp}\bigr)}}\,d\mathcal{H}^\nu = 0 \tag{8}$$

holds (*cf.* [11]*), where* $\mathcal{D}_1(\varphi,\xi,x)$ *and* $\mathcal{D}_2(\varphi,\xi,x)$ *are equal to*

$$\sum_{i=1}^{\dim V_1}\sum_{j=1}^{\dim V_1}\bigl\langle D_H\xi_i(x),(\widehat{D}\varphi_j)_H(x)\bigr\rangle\bigl(E_{\dim V_1}-(\widehat{D}\varphi)^*_H(x)(\widehat{D}\varphi)_H(x)\bigr)_{ij} + \sum_{i=1}^{\dim V_1}\sum_{j=1}^{\dim V_1}\bigl\langle (\widehat{D}\varphi_i)_H(x),D_H\xi_j(x)\bigr\rangle\bigl(E_{\dim V_1}-(\widehat{D}\varphi)^*_H(x)(\widehat{D}\varphi)_H(x)\bigr)_{ij},$$

and

$$\sum_{i=1}^{\dim V_2}\sum_{j=1}^{\dim V_2}\bigl\langle (P_1)_i(x),\bigl((\widehat{D}\varphi)_{H^\perp}\bigr)_j(x)\bigr\rangle\bigl(E_{\dim V_2}-(\widehat{D}\varphi)^*_{H^\perp}(x)(\widehat{D}\varphi)_{H^\perp}(x)\bigr)_{ij} + \sum_{i=1}^{\dim V_2}\sum_{j=1}^{\dim V_2}\bigl\langle \bigl((\widehat{D}\varphi)_{H^\perp}\bigr)_i(x),(P_1)_j(x)\bigr\rangle\bigl(E_{\dim V_2}-(\widehat{D}\varphi)^*_{H^\perp}(x)(\widehat{D}\varphi)_{H^\perp}(x)\bigr)_{ij},$$

respectively, then (5) *takes maximal value at* φ *on its neighborhood. For a horizontally attainable domain* Ω*, we may use* $\|\cdot\|_{H,\max\{6\dim V_1,12\dim V_2\}}$ *instead of* $\|\cdot\|_{\max\{6\dim V_1,12\dim V_2\}}$.

Proof This statement is actually a reformulation of the following condition of strong positivity of the area functional: *if the functional* F *is differentiable twice, its first variation at* ζ^* *equals zero, and the second variation is strongly positive in the sense that there exists* $K>0$ *such that* $\delta^2F(\zeta^*,\delta\zeta)\ge K\|\delta\zeta\|^2$ *then* F *has minimum at* ζ^*. The necessary condition (8) is deduced in the same way as in [10, Theorem 6]. To describe sufficiency estimates, put

$$f_1(\varepsilon)=\det\bigl(E_{\dim V_1}-\bigl((\widehat{D}\varphi)_H+\varepsilon D_H\xi\bigr)^*\bigl((\widehat{D}\varphi)_H+\varepsilon D_H\xi\bigr)\bigr),$$
$$f_2(\varepsilon)=\det\bigl(E_{\dim V_2}-\bigl((\widehat{D}\varphi)_{H^\perp}+\varepsilon P_1+\varepsilon^2P_2\bigr)^*\bigl((\widehat{D}\varphi)_{H^\perp}+\varepsilon P_1+\varepsilon^2P_2\bigr)\bigr).$$

Then

$$\left(\sqrt{f_1(\varepsilon)}\sqrt{f_2(\varepsilon)}\right)'' = \frac{f_1''\sqrt{f_2}}{2\sqrt{f_1}} + \frac{f_2''\sqrt{f_1}}{2\sqrt{f_2}} + \frac{f_1'f_2'}{2\sqrt{f_1}\sqrt{f_2}} - \frac{(f_1')^2\sqrt{f_2}}{4f_1^{3/2}} - \frac{(f_2')^2\sqrt{f_1}}{4f_2^{3/2}} \le \frac{f_1''\sqrt{f_2}}{2\sqrt{f_1}} + \frac{f_2''\sqrt{f_1}}{2\sqrt{f_2}}.$$

Consequently, it suffices to estimate the values f_1'' and f_2'' in terms of $D_H\xi$. For f_1'', we see that it coincides with the sum of determinants of the modified matrices $E_{\dim V_1} - (\widehat{D}\varphi)^*_H(\widehat{D}\varphi)_H$ where row k is replaced by $-2X_k\xi \cdot D_H\xi$ or rows i and j with $i \neq j$ are replaced by $-X_i\xi \cdot (\widehat{D}\varphi)_H - X_i\varphi \cdot D_H\xi$ and $-X_j\xi \cdot (\widehat{D}\varphi)_H - X_j\varphi \cdot D_H\xi$ respectively, for $i, j, k = 1, \dots, \dim \widetilde{V}_1$. Here $X_i\varphi$ stands for row i of $(\widehat{D}\varphi)^T_H$.

Applying, if necessary, orthogonal transformations $O_H = O_H(x)$ and $O_{H^\perp} = O_{H^\perp}(x)$, where $x \in \Omega$, we may assume without loss of generality that $(\widehat{D}\varphi)^*_H(\widehat{D}\varphi)_H$ and $(\widehat{D}\varphi)^*_{H^\perp}(\widehat{D}\varphi)_{H^\perp}$ are diagonal matrices. Note that this transformation corresponds to orthogonal transformation of bases within $V_1(x)$ and within $V_2(x)$, thus, all lengths and scalar products are the same at x. Fix $x \in \Omega$. The assumption on the column lengths of these matrices implies that the (diagonal) elements $1 - a_1, \dots, 1 - a_{\dim V_1}$ of $E_{\dim V_1} - (\widehat{D}\varphi)^*_H(\widehat{D}\varphi)_H$ are positive and strictly separated from 0 everywhere in Ω. Thus, if we replace row k of $E_{\dim V_1} - (\widehat{D}\varphi)^*_H(\widehat{D}\varphi)_H$ by $-2X_k\xi \cdot D_H\xi$ then the corresponding determinant equals

$$-2\langle X_k\xi, X_k\xi\rangle \prod_{m:m\neq k}(1 - a_m) = -2\prod_{m:m\neq k}(1 - a_m)\|X_k\xi\|^2 < 0,$$

since $\max\limits_{j=1,\dots,\dim V_1}\{a_j\} \le \frac{1}{3\dim V_1} - c$ with $c > 0$ for $k = 1, \dots, \dim \widetilde{V}_1$. Next, consider the first group of $\dim V_1(\dim V_1 - 1)$ determinants. Each of them equals the sum of four determinants of the modified matrix $E_{\dim V_1} - (\widehat{D}\varphi)^*_H(\widehat{D}\varphi)_H$, where rows i and j with $i \neq j$ are replaced by only one term. Consider the corresponding cases and estimate each value.

Case 1 Rows i and j are replaced by $-X_i\xi \cdot (\widehat{D}\varphi)_H$ and $-X_j\xi \cdot (\widehat{D}\varphi)_H$. Then, the determinant is estimated as

$$\prod_{m:m\neq i,j}(1 - a_m)\langle X_i\xi \cdot (\widehat{D}\varphi)_H, X_j\xi \cdot (\widehat{D}\varphi)_H\rangle \le \frac{1}{2}\prod_{m:m\neq i,j}(1 - a_m)\|(\widehat{D}\varphi)_H\|^2(\|X_i\xi\|^2 + \|X_j\xi\|^2).$$

Case 2 Rows i and j are replaced by $-X_i\varphi \cdot D_H\xi$ and $-X_j\xi \cdot (\widehat{D}\varphi)_H$. The estimate is

$$\prod_{m:m\neq i,j}(1-a_m)\langle X_i\varphi \cdot D_H\xi, X_j\xi \cdot (\widehat{D}\varphi)_H\rangle$$
$$\leq \frac{1}{2}\prod_{m:m\neq i,j}(1-a_m)\Big(a_i\sum_{q=1}^{\dim V_1}\|X_q\xi\|^2+\|(\widehat{D}\varphi)_H\|^2\|X_j\xi\|^2\Big).$$

Case 3 If rows i and j are replaced by $-X_i\xi \cdot (\widehat{D}\varphi)_H$ and $-X_j\varphi \cdot D_H\xi$, then the estimate equals $\frac{1}{2}\prod_{m:m\neq i,j}(1-a_m)\Big(a_j\sum_{q=1}^{\dim V_1}\|X_q\xi\|^2+\|(\widehat{D}\varphi)_H\|^2\|X_i\xi\|^2\Big)$.

Case 4 If rows i and j are replaced by $-X_i\varphi \cdot D_H\xi$ and $-X_j\varphi \cdot D_H\xi$ then we have $\frac{1}{2}\prod_{m:m\neq i,j}(1-a_m)\Big(a_i\sum_{q=1}^{\dim \widetilde{V}_1}\|X_q\xi\|^2+a_j\sum_{q=1}^{\dim \widetilde{V}_1}\|X_q\xi\|^2\Big)$.

Fix i and recall that $\|(\widehat{D}\varphi)_H\|^2=\sum_{q=1}^{\dim V_1}a_q$ as the trace of $\widehat{D}_H\varphi^*\widehat{D}_H\varphi$. We infer that the coefficient at $\|X_i\xi\|^2$ is equal to

$$\sum_{q=1}^{\dim V_1}a_q\sum_{j:j\neq i}\prod_{m:m\neq i,j}(1-a_m)+\sum_{q=1}^{\dim V_1}\sum_{j:j\neq q}\prod_{m:m\neq q,j}(1-a_m)a_q$$
$$-2\prod_{m:m\neq i}(1-a_m)\leq -\prod_{m:m\neq i}(1-a_m)-\widehat{c}<0$$

with $\widehat{c}>0$ since $0<\max_{q=1,\dots,\dim V_1}\{a_q\}\leq \frac{1}{2\dim V_1^2}-c$ with $c>0$ for $i=1,\dots,\dim V_1$.

Consider now f_2'' and its estimates. It coincides with the sum of determinants of the modified matrices $E_{\dim V_1}-(\widehat{D}\varphi)^*_{H^\perp}(\widehat{D}\varphi)_{H^\perp}$ where row k is replaced by $-2(P_2^*)_k\cdot(D\varphi)_{H^\perp}-2(P_1^*)_kP_1-2\big((\widehat{D}\varphi)_{H^\perp}\big)_kP_2$, or rows i and j with $i\neq j$ are replaced by $-(P_1^*)_i\cdot(\widehat{D}\varphi)_{H^\perp}-(\widehat{D}\varphi_i)_H\cdot P_1$ and $-(P_1^*)_j\cdot(\widehat{D}\varphi)_{H^\perp}-(\widehat{D}\varphi_j)_H\cdot P_1$ respectively, for $i,j,k=1,\dots,\dim V_2$. In contrast to the horizontal case, each summand depending on the horizontal derivatives of ξ has coefficients depending on the entries of $\widehat{D}_H\varphi$. Thus, the absolute value of the coefficient at $\|D_H\xi\|^2$ can be considered strictly less than $\prod_{m:m\neq i}(1-a_m)$. Since each summand also contains products of $a^k_{i,j}$ and c_{lmq}, we can easily see that if they are sufficiently small then

$$|f_2''|\leq \|D_H\xi\|^2\cdot\prod_{m:m\neq i}(1-a_m),$$

and finally

$$\left(\sqrt{f_1(\varepsilon)}\sqrt{f_2(\varepsilon)}\right)'' \leq \frac{f_1''\sqrt{f_2}}{2\sqrt{f_1}} + \frac{f_2''\sqrt{f_1}}{2\sqrt{f_2}} \leq -\widehat{c}\|D_H\xi\|^2\left(\frac{\sqrt{f_2}}{\sqrt{f_1}} + \frac{\sqrt{f_1}}{\sqrt{f_2}}\right), \ \widehat{c} > 0.$$

Thus, the functional $-S(\varphi)$ has minimum since its second variation is strongly positive; and therefore the area functional (5) has maximum at φ. The theorem follows. □

Remark 19 We may replace strong restrictions on $|a^k_{i,j}|$ and $|c_{lmq}|$ by adding some restrictions to $\|(\widehat{D}\varphi)_H\|$ since all coefficients at the horizontal derivatives of ξ contain the horizontal derivatives of φ. Moreover, it is possible to deduce restrictions on $(\widehat{D}\varphi)_H$ basing on the given values of $a^k_{i,j}$ and c_{lmq} for $i, j = 1, \ldots, \dim V_1$ with $i < j$ and $k, l, m = 1, \ldots, \dim \widetilde{V}_1$, $q = \dim \widetilde{V}_1 + 1, \ldots, \widetilde{N}$.

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