2-Hom-Associative Bialgebras and Hom-Left Symmetric Dialgebras

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Abstract From the definition and properties of unital hom-associative algebras, and the use of the Kaplansky's construction, we develop algebraic structures called *2-hom-associative bialgebras, 2-hom-bialgebras, and 2-2-hom-bialgebras*. Besides, we define and characterize the hom-associative dialgebras, hom-Leibniz algebra and hom-left symmetric dialgebras, and discuss their main relevant properties. Explicit examples are given to illustrate the developed formalism.

Keywords 2-hom-associative bialgebra · 2-hom-bialgebra · 2-2-hom-bialgebra · Hom-left symmetric dialgebra · Hom-Leibniz algebra

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1 Introduction

A theory of 2-associative algebras was developed by J-L. Loday and M. Ronco in [8], where the operad of 2-associative algebras was also introduced as a Koszul operad. The notion of infinitesimal bialgebra was given for the first time by S. Joni and G.-C. Rota in [6]. The basic theory was developed by M. Aguiar in [1] and in [2]. J-L. Loday in [7] also performed a non-antisymmetric version of Lie algebras, called Leibniz algebras, whose the bracket satisfies the Leibniz relation. The Leibniz rule, combined with the antisymmetry property, leads to the Jacobi identity. Therefore, the Lie algebras are anti-symmetric Leibniz algebras. In the same work, Loday formulated an associative version of Leibniz algebras, called diassociative algebras, equipped with two bilinear and associative operations, which satisfy three axioms, all of them being various forms of the associative law. Recently [5], R. Felipe built the left-symmetric dialgebras which include, as a particular case,

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the notion of dialgebras.This gave a new impulse to the construction of Leibniz algebras.

The hom-algebra structures first arose in quasi-deformations of Lie algebras of vector fields. Discrete modifications of vector fields, via twisted derivations, provide hom-Lie and quasi-hom-Lie structures, in which the Jacobi condition is twisted. Other interesting hom-type algebras of classical structures were studied. They include hom-associative algebras, hom-Lie admissible algebras[11], and, more generally, G-hom-associative algebras [10], enveloping algebras of hom-Lie algebras[12], hom-Lie admissible hom-coalgebras and hom-Hopf algebras [9], hom-alternative algebras, hom-Malcev algebras and hom-Jordan algebras [14], L-modules, L-comodules and hom-Lie quasi-bialgebras [3], and Laplacian of hom-Lie quasi-bialgebras [4].

In this paper, we devise a hom-type generalization of 2-associative algebras, 2-bialgebras, 2-associative bialgebras, 2-2-bialgebras and left symmetric dialgebras, leading to the concepts of hom-bialgebras, 2-hom-associative algebras, 2-hom-bialgebras, 2-hom- associative bialgebras, 2-2-hom-bialgebras and hom-left symmetric dialgebras, respectively. The hom-type algebras are usually defined by twisting the defining axioms of a type of algebras by a certain twisting map. When the twisting map happens to be the identity map, we get an ordinary algebraic structure. A hom-counital condition can be given as follows:

$$(\varepsilon \otimes \alpha)\Delta(x) = (\alpha \otimes \varepsilon)\Delta(x) = \alpha^2(x), \ \forall x \in V.$$

This leads to new definitions of counital hom-coassociative coalgebra and unital hom-bialgebra structures. A unital infinitesimal hom-bialgebra condition can be formulated by the relation:

$$\Delta \circ \mu = (\mu \otimes \alpha) \circ (\alpha \otimes \Delta) + (\alpha \otimes \mu) \circ (\Delta \otimes \alpha) - \alpha^2 \otimes \alpha.$$

This unital infinitesimal twisted condition permits to define the unital infinitesimal hom-bialgebra structure. Then, we deal with the concepts of 2-hom-associative bialgebras, 2-hom-bialgebras and 2-2-hom bialgebras. Besides, we provide a hom-algebra version of Kaplansky's construction of hom-bialgebras in order to build unital analogs of 2-hom-associative bialgebras, 2-hom-bialgebras and 2-2-hom-bialgebras. Finally, we define the notion of hom-left symmetric dialgebras generalizing the classical left symmetric dialgebras, and discuss their relevant properties.

The paper is organized as follows. In Sect. 2, we give the definitions of hom-bialgebra, 2-hom-associative algebra, 2-hom-associative bialgebra, 2-hom-bialgebra, 2-2-hom-bialgebra, and derive their main properties. In Sect. 3, we provide a hom-algebra version of Kaplansky's constructions of hom-bialgebras from a unital hom-associative algebra. We show that these constructions induce a large class of 2-hom-bialgebras, 2-hom-associative bialgebras, and 2-2-hom-bialgebras. In Sect. 4, we define and characterize the hom-associative dialgebras, hom-Leibniz algebra and hom-left symmetric dialgebras, and discussed their main relevant properties. Section 5 is devoted to concluding remarks.

2 Definitions of Unital 2-Hom-Associative Bialgebras

2.1 Unital Hom-Bialgebra and Unital Infinitesimal Hom-Bialgebra

Definition 1 ([10]) A hom-associative algebra is a triple (V, μ, α) consisting of a linear space V, a bilinear map $\mu : V \times V \to V$ and a homomorphism $\alpha : V \to V$ satisfying the multiplicativity and hom-associativity properties, i.e.

$$\alpha \circ \mu = \mu \circ \alpha^{\otimes^2} := \mu \circ (\alpha \otimes \alpha), \tag{1}$$

$$\mu \circ (\alpha \otimes \mu) = \mu \circ (\mu \otimes \alpha), \tag{2}$$

respectively.

Definition 2 ([15]) A unital hom-associative algebra is given by a quadruple $(\mathcal{A}, \mu, \alpha, e)$, where $e \in \mathcal{A}$, such that:

- $(\mathcal{A}, \mu, \alpha)$ is a hom-associative algebra,
- $\mu(x, e) = \mu(e, x) = \alpha(x), \forall x \in \mathcal{A}$,
- $\alpha(e) = e$.

Example Let $\mathcal{A}$ be an n-dimensional vector space, ($n = 2, 3$), over a field $\mathcal{K}$ with a basis $\{e_i\}_{i=1,\dots,n}$. The following product μ and linear map α on $\mathcal{A}$ define a unital hom-associative algebra in each of the following cases:

- $\mu(e_1 \otimes e_1) = e_1, \mu(e_2 \otimes e_2) = e_2, \mu(e_1 \otimes e_2) = \mu(e_2 \otimes e_1) = 0, \alpha(e_1) = e_1$ and $\alpha(e_2) = 0$.
- $\mu(e_1 \otimes e_1) = e_1,\ \mu(e_2 \otimes e_2) = e_1,\ \mu(e_1 \otimes e_2) = \mu(e_2 \otimes e_1) = -e_2, \alpha(e_1) = e_1$ and $\alpha(e_2) = -e_2$.
- $\mu(e_1 \otimes e_1) = e_1,\ \mu(e_2 \otimes e_2) = e_2,\ \mu(e_1 \otimes e_3) = -e_3,\ \mu(e_3 \otimes e_1) = -e_3, \mu(e_3 \otimes e_3) = e_1,\ \alpha(e_1) = e_1$ and $\alpha(e_3) = -e_3$.

Definition 3 ([13]) A hom-coassociative coalgebra is a triple (V, Δ, α) consisting of a linear space V, a linear map $\Delta : V \to V \otimes V$, and a homomorphism $\alpha : V \to V$ satisfying

$$\alpha^{\otimes^2} \circ \Delta = \Delta \circ \alpha \text{ (comultiplicativity)} \tag{3}$$

$$(\alpha \otimes \Delta) \circ \Delta = (\Delta \otimes \alpha) \circ \Delta \text{ (hom-coassociativity).} \tag{4}$$

Example Let $\mathcal{A}$ be a 3-dimensional vector space over $\mathcal{K}$ with a basis $\{e_1, e_2, e_3\}$. The following coproduct Δ and linear map α on $\mathcal{A}$ define a hom-coassociative coalgebra:

$$\Delta(e_1) = e_1 \otimes e_1, \quad \Delta(e_2) = e_2 \otimes e_2,$$

$$\Delta(e_3) = \frac{\sqrt{5}-1}{2\sqrt{5}}(e_1 \otimes e_3 + e_3 \otimes e_1) + \frac{1}{\sqrt{5}}(-e_1 \otimes e_1 + e_3 \otimes e_3),$$

$$\alpha(e_1) = e_1, \alpha(e_2) = 0 \text{ and } \alpha(e_3) = e_3.$$

Definition 4 A counital hom-coassociative coalgebra is defined as a quadruple $(V, \Delta, \varepsilon, \alpha)$ such that the triple (V, Δ, α) is a hom-coassociative coalgebra satisfying the hom-counital condition

$$(\varepsilon \otimes \alpha)\Delta(x) = (\alpha \otimes \varepsilon)\Delta(x) = \alpha^2(x), \forall x \in V. \tag{5}$$

Example Let $\mathcal{A}$ be a 3-dimensional vector space over $\mathcal{K}$ with a basis $\{e_1, e_2, e_3\}$. The following coproduct Δ and linear map α on $\mathcal{A}$ define a counital hom- coassociative coalgebra:

$$\Delta(e_1) = e_1 \otimes e_1, \quad \Delta(e_2) = e_2 \otimes e_2,$$

$$\Delta(e_3) = \frac{\sqrt{5}-1}{2\sqrt{5}}(e_1 \otimes e_3 + e_3 \otimes e_1) + \frac{1}{\sqrt{5}}(-e_1 \otimes e_1 + e_3 \otimes e_3),$$

$$\alpha(e_1) = e_1, \alpha(e_2) = 0, \alpha(e_3) = e_3,$$

$$\epsilon(e_1) = 1, \epsilon(e_2) = 1 \text{ and } \epsilon(e_3) = \frac{1+\sqrt{5}}{2}.$$

Definition 5 A unital hom-bialgebra is a system $(V, \mu, \eta, \Delta, \varepsilon, \alpha)$, where $\mu : V \otimes V \to V$(multiplication), $\eta : \mathcal{K} \to V$(unit), $\Delta : V \to V \otimes V$ (comultiplication), $\varepsilon : V \to \mathcal{K}$ (counit), and $\alpha : V \to V$ (endomorphism) are linear maps satisfying the following properties:

(1) the quadruple (V, μ, η, α) is a unital hom-associative algebra;
(2) the quadruple $(V, \Delta, \varepsilon, \alpha)$ is a counital hom-coassociative coalgebra;
(3) the compatibility condition is expressed by the following three identities:

 (a) $\Delta(\mu(x \otimes y)) = \Delta(x) \bullet \Delta(y), \forall x, y \in V,$
 (b) $\alpha^{\otimes^2} \circ \Delta = \Delta \circ \alpha,$
 (c) $\varepsilon(\mu(x \otimes y)) = \varepsilon(x)\varepsilon(y),$
 (d) $\varepsilon \circ \alpha(x) = \varepsilon(x).$

Example Let $\mathcal{A}$ be a n-dimensional vector space, ($n = 2, 3$), over a field $\mathcal{K}$ with a basis $\{e_i\}_{i=1,\dots,n}$. The following product μ, coproduct Δ and linear maps α, ε on $\mathcal{A}$ define a unital hom-bialgebra:

- $\mu(e_1 \otimes e_1) = e_1,\ \mu(e_2 \otimes e_2) = 0,\ \mu(e_1 \otimes e_2) = \mu(e_2 \otimes e_1) = e_2,$
  $\alpha(e_1) = e_1,\ \alpha(e_2) = e_2,$
  $\Delta(e_1) = e_1 \otimes e_1,\ \Delta(e_2) = \theta e_2 \otimes e_2,$
  $\varepsilon(e_1) = 1$ and $\varepsilon(e_2) = \dfrac{1}{\theta}(\theta \neq 0).$
- $\mu(e_1 \otimes e_1) = e_1,\ \mu(e_1 \otimes e_3) = \mu(e_3 \otimes e_1) = e_3,\ \mu(e_2 \otimes e_2) = e_2,$
  $\mu(e_3 \otimes e_3) = e_1 + e_3,$
  $\alpha(e_1) = e_1,\ \alpha(e_2) = 0, \alpha(e_3) = e_3,$
  $\Delta(e_1) = e_1 \otimes e_1,\ \Delta(e_2) = e_2 \otimes e_2,$
  $\Delta(e_3) = \dfrac{\sqrt{5}-1}{2\sqrt{5}} e_1 \otimes e_1 + \dfrac{1}{\sqrt{5}}(e_1 \otimes e_3 + e_3 \otimes e_1 - 2e_3 \otimes e_3),$
  $\varepsilon(e_1) = 1,\ \varepsilon(e_2) = 1$ and $\varepsilon(e_3) = \dfrac{1-\sqrt{5}}{2}.$

It is worth noticing that the conditions (3.a) and (3.b) of Definition 5 do not lead to define a unital hom-bialgebra structure in $(\mathcal{A}, \mu)$ given by $\mu(e_1 \otimes e_1) = e_1,\ \mu(e_2 \otimes e_2) = e_1,\ \mu(e_1 \otimes e_2) = \mu(e_2 \otimes e_1) = -e_2, \alpha(e_1) = e_1,\ \alpha(e_2) = -e_2.$

Definition 6 A unital infinitesimal hom-bialgebra $(V, \mu, \eta, \Delta, \varepsilon, \alpha)$ is a $\mathcal{K}$-vector space V equipped with a unital hom-associative multiplication μ and a counital hom-coassociative comultiplication Δ, which are related by the unital hom-infinitesimal relation

$$\Delta \circ \mu = (\mu \otimes \alpha) \circ (\alpha \otimes \Delta) + (\alpha \otimes \mu) \circ (\Delta \otimes \alpha) - \alpha^2 \otimes \alpha. \tag{6}$$

Note that the unital hom-bialgebras, given in the previous example, are not unital infinitesimal hom-bialgebras. The presence of the term $\alpha^2 \otimes \alpha$ in Eq. (6) complicates the construction of non trivial examples of unital infinitesimal hom-bialgebras, unital 2-hom-associative bialgebras and unital 2-2-hom-bialgebras. Finding such more relevant examples is a task in the core of our current concerns. It deserves further works, which will complete and enrich the present study.

Example Let $\mathcal{A}$ be a 2-dimensional vector space over $\mathcal{K}$ with a basis $\{e_1, e_2\}$. The next product μ, coproduct Δ and linear maps α, ε on $\mathcal{A}$ define a unital hom-bialgebra:

$$\mu(e_1 \otimes e_1) = e_1,\ \mu(e_2 \otimes e_2) = e_2,\ \mu(e_1 \otimes e_2) = \mu(e_2 \otimes e_1) = 0,$$

$$\alpha(e_1) = e_1,\ \alpha(e_2) = 0,$$

$$\Delta(e_1) = e_1 \otimes e_1,\ \Delta(e_2) = 0, \varepsilon(e_1) = 1 \text{ and } \varepsilon(e_2) = 0.$$

2.2 2-Hom-Associative Algebra

The 2-hom-associative algebras generalize the 2-associative algebras in the sense where the associativity laws are twisted.

Definition 7 A 2-hom-associative algebra over $\mathcal{K}$ is a vector space equipped with two hom-associative structures. A 2-hom-associative algebra is said to be unital if there is a unit e, which is a unit for both operations.

Example Let $\mathcal{A}$ be a 3-dimensional vector space over $\mathcal{K}$ with a basis $\{e_1, e_2, e_2\}$. The following multiplications μ_1, μ_2 and linear map α on $\mathcal{A}$ define a unital 2- hom-associative algebra:

$$\mu_1(e_1 \otimes e_1) = e_1,\ \ \mu_1(e_3 \otimes e_3) = e_3, \mu_1(e_1 \otimes e_3) = \mu_1(e_3 \otimes e_1) = e_3,$$

$$\mu_2(e_1 \otimes e_1) = e_1,\ \ \mu_2(e_3 \otimes e_3) = e_1 + e_3, \mu_2(e_1 \otimes e_3) = \mu_2(e_3 \otimes e_1) = e_3,$$

$$\mu_2(e_2 \otimes e_2) = e_2 \text{ and } \alpha(e_1) = e_1,\ \ \alpha(e_2) = 0, \alpha(e_3) = e_3.$$

Definition 8 Let $(V, \mu_1, \mu_2, \alpha)$ and $(V', \mu'_1, \mu'_2, \alpha')$ be two 2-hom-associative algebras. A liner map $f : V \to V'$ is a morphism of 2-hom-associative algebras if

$$\mu'_1 \circ (f \otimes f) = f \circ \mu_1,\ \ \mu'_2 \circ (f \otimes f) = f \circ \mu_2 \text{ and } f \circ \alpha = \alpha' \circ f.$$

In particular, the 2-hom-associative algebras $(V, \mu_1, \mu_2, \alpha)$ and $(V', \mu'_1, \mu'_2, \alpha')$ are isomorphic if f is a bijective linear map such that

$$\mu_1 = f^{-1} \circ \mu'_1 \circ (f \otimes f),\ \ \mu_2 = f^{-1} \circ \mu'_2 \circ (f \otimes f) \text{ and } \alpha = f^{-1} \circ \alpha' \circ f.$$

Theorem 9 *Let (V, μ_1, μ_2) be a 2-associative algebra, and $\alpha : V \to V$ be an associative algebra endomorphism. Then, $V_\alpha = (V, \alpha \circ \mu_1, \alpha \circ \mu_2, \alpha)$ is a 2-hom-associative algebra. Moreover, suppose that (V', μ'_1, μ'_2) is another 2-associative algebra and $\alpha' : V' \to V'$ an associative algebra endomorphism. If $f : V \to V'$ is an associative algebra morphism that satisfies $f \circ \alpha = \alpha' \circ f$, then*

$$f : (V, \alpha \circ \mu_1, \alpha \circ \mu_2, \alpha) \to (V', \alpha \circ \mu'_1, \alpha \circ \mu'_2, \alpha')$$

is a morphism of 2-hom-associative algebra.

2.3 Unital 2-Hom-Associative Bialgebra

We give the notion of unital 2-hom-associative bialgebras generalizing unital 2-associative bialgebras.

Definition 10 A unital 2-hom-associative bialgebra $(V, \mu_1, \mu_2, \eta, \Delta, \varepsilon, \alpha)$ is a vector space V equipped with two multiplications μ_1 and μ_2, a unit η, a comultiplication Δ, a counit ε, and a linear map $\alpha : V \to V$ such that

- $(V, \mu_1, \eta, \Delta, \varepsilon, \alpha)$ is a unital hom-bialgebra, and
- $(V, \mu_2, \eta, \Delta, \varepsilon, \alpha)$ is a unital infinitesimal hom-bialgebra.

Example Let $\mathcal{A}$ be a 2-dimensional vector space over $\mathcal{K}$ with a basis $\{e_1, e_2\}$. The products μ_1, μ_2, the coproduct Δ and the linear maps α and ε given by

$$\mu_1(e_1 \otimes e_1) = e_1,\ \mu_1(e_2 \otimes e_2) = e_2, \mu_1(e_1 \otimes e_2) = \mu_1(e_2 \otimes e_1) = 0,$$
$$\mu_2(e_1 \otimes e_1) = e_1,\ \mu_2(e_2 \otimes e_2) = 0, \mu_2(e_1 \otimes e_2) = \mu_2(e_2 \otimes e_1) = 0,$$
$$\Delta(e_1) = e_1 \otimes e_1,\ \Delta(e_2) = 0,$$
$$\alpha(e_1) = e_1, \alpha(e_2) = 0, \varepsilon(e_1) = 1, \varepsilon(e_2) = 0$$

define a unital 2-hom-associative bialgebra structure on $\mathcal{A}$.

Definition 11 Let $(V, \mu_1, \mu_2, \eta, \Delta, \varepsilon, \alpha)$ and $(V', \mu_1', \mu_2', \eta', \Delta', \varepsilon', \alpha')$ be two unital 2-associative hom-bialgebras. A linear map $f : V \to V'$ is a morphism of unital 2-associative hom-bialgebras if:

- $\mu_1' \circ (f \otimes f) = f \circ \mu_1$,
- $\mu_2' \circ (f \otimes f) = f \circ \mu_2$,
- $f \circ \eta = \eta'$,
- $(f \otimes f) \circ \Delta = \Delta' \circ f$,
- $\varepsilon' = \varepsilon \circ f$, $f \circ \alpha = \alpha' \circ f$.

2.4 Unital 2-Hom-Bialgebra

Definition 12 A unital 2-hom-bialgebra $(V, \mu_1, \mu_2, \eta, \Delta_1, \Delta_2, \varepsilon_1, \varepsilon_2, \alpha)$ is a vector space V equipped with two multiplications μ_1, μ_2, the unit η, two comultiplications Δ_1, Δ_2, two counits $\varepsilon_1, \varepsilon_2$, and a linear map $\alpha : V \to V$ such that: $(V, \mu_1, \eta, \Delta_1, \varepsilon_1, \alpha)$, $(V, \mu_2, \eta, \Delta_2, \varepsilon_2, \alpha)$, $(V, \mu_1, \eta, \Delta_2, \varepsilon_2, \alpha)$, and $(V, \mu_2, \eta, \Delta_1, \varepsilon_1, \alpha)$ are unital hom-bialgebras.

Example Let $\mathcal{A}$ be a 2-dimensional vector space over $\mathcal{K}$ with a basis $\{e_1, e_2\}$. The products μ_1, μ_2, the coproducts $\Delta = \Delta_1 = \Delta_2$ and the linear maps α and $\varepsilon = \varepsilon_1 = \varepsilon_2$ given by

$$\mu_1(e_1 \otimes e_1) = e_1,\ \mu_1(e_2 \otimes e_2) = e_2, \mu_1(e_1 \otimes e_2) = \mu_1(e_2 \otimes e_1) = 0,$$
$$\mu_2(e_1 \otimes e_1) = e_1,\ \mu_2(e_2 \otimes e_2) = 0, \mu_2(e_1 \otimes e_2) = \mu_2(e_2 \otimes e_1) = 0,$$
$$\Delta(e_1) = e_1 \otimes e_1,\ \Delta(e_2) = 0,$$
$$\alpha(e_1) = e_1, \alpha(e_2) = 0, \varepsilon(e_1) = 1, \varepsilon(e_2) = 0$$

define a unital 2-hom-bialgebra structure on $\mathcal{A}$.

Example Let $\mathcal{A}$ be a 3-dimensional vector space over $\mathcal{K}$ with a basis $\{e_1, e_2, e_3\}$. The products $\mu = \mu_1 = \mu_2$, the coproducts Δ_1, Δ_2 and the linear maps α and $\varepsilon_1, \varepsilon_2$ given by

$$\mu(e_1 \otimes e_1) = e_1,\ \mu(e_2 \otimes e_2) = e_1 + e_2,$$

$$\mu_1(e_1 \otimes e_2) = \mu(e_2 \otimes e_1) = e_2, \mu(e_3 \otimes e_3) = e_3,$$

$$\Delta_1(e_1) = \Delta_2(e_1) = e_1 \otimes e_1,\ \Delta_1(e_3) = \Delta_2(e_3) = e_3 \otimes e_3,$$

$$\Delta_1(e_2) = \frac{\sqrt{5}-1}{2\sqrt{5}}(e_1 \otimes e_2 + e_2 \otimes e_1) + \frac{1}{\sqrt{5}}(-e_1 \otimes e_1 + e_2 \otimes e_2),$$

$$\Delta_2(e_2) = \frac{\sqrt{5}+1}{2\sqrt{5}}(e_1 \otimes e_2 + e_2 \otimes e_1) + \frac{1}{\sqrt{5}}(e_1 \otimes e_1 - e_2 \otimes e_2),$$

$$\alpha(e_1) = e_1,\ \alpha(e_2) = e_2,\ \alpha(e_3) = 0,$$

$$\varepsilon_1(e_1) = \varepsilon_2(e_1) = 1,\ \varepsilon_1(e_3) = \varepsilon_2(e_3) = 1,$$

$$\varepsilon_1(e_2) = \frac{1+\sqrt{5}}{2}, \varepsilon_2(e_2) = \frac{1-\sqrt{5}}{2}$$

define a unital 2-hom-bialgebra structure on $\mathcal{A}$.

The unital 2-hom-bialgebra is called of type (1-1), (resp. of type (2-2)), if the two multiplications and the two comultiplications are identical, (resp. distinct). The unital 2-hom-bialgebra is called of type (1-2), (resp. of type (2-1)), if the two multiplications are identical, (resp. distinct), and the two comultiplications are distinct, (resp. identical).

Proposition 13 *Let $(V, \mu, \eta, \Delta, \varepsilon, \alpha)$ be a unital hom-bialgebra. Then, we have that $(V, \mu, \mu, \eta, \Delta, \Delta, \varepsilon, \alpha)$ and $(V, \mu, \mu^{op}, \eta, \Delta, \Delta^{cop}, \varepsilon, \alpha)$ are unital 2-hom-bialgebras, where $\mu^{op}(x \otimes y) = \mu(y \otimes x)$ and $\Delta^{cop}(x) = \tau \circ \Delta(x)$, with $\tau(x \otimes y) = y \otimes x$. The first unital 2-hom-bialgebra is of type (1-1), and the second one is of type (2-2).*

Proof It comes from a direct computation. □

2.5 Unital 2-2-Hom-Bialgebra

Definition 14 A unital 2-2-hom-bialgebra $(V, \mu_1, \mu_2, \eta, \Delta_1, \Delta_2, \varepsilon_1, \varepsilon_2, \alpha)$ is a vector space V equipped with two multiplications μ_1, μ_2, two comultiplications Δ_1, Δ_2, two counits $\varepsilon_1, \varepsilon_2$, one unit η, and a linear map $\alpha : V \to V$ such that

(1) $(V, \mu_1, \eta, \Delta_1, \varepsilon_1, \alpha)$ and $(V, \mu_2, \eta, \Delta_2, \varepsilon_2, \alpha)$ are unital hom-bialgebras,

(2) $(V, \mu_1, \eta, \Delta_2, \varepsilon_2, \alpha)$ and $(V, \mu_2, \eta, \Delta_1, \varepsilon_1, \alpha)$ are unital infinitesimal hom-bialgebras.

Example Let $\mathcal{A}$ be a 2-dimensional vector space over $\mathcal{K}$ with a basis $\{e_1, e_2\}$. The products μ_1, μ_2, the coproducts $\Delta = \Delta_1 = \Delta_2$ and the linear maps α and $\varepsilon = \varepsilon_1 = \varepsilon_2$ given by

$$\mu_1(e_1, e_1) = e_1,\ \mu_1(e_2, e_2) = e_2, \mu_1(e_1, e_2) = \mu_1(e_2, e_1) = 0,$$

$$\mu_2(e_1, e_1) = e_1,\ \mu_2(e_2, e_2) = 0, \mu_2(e_1, e_2) = \mu_2(e_2, e_1) = 0,$$

$$\Delta(e_1) = e_1 \otimes e_1,\ \Delta(e_2) = 0,$$

$$\alpha(e_1) = e_1, \alpha(e_2) = 0, \varepsilon(e_1) = 1, \varepsilon(e_2) = 0$$

define a unital 2-2-hom-bialgebra structure on $\mathcal{A}$.

A unital 2-2-hom-bialgebra is called of type (1-1), (resp. of type (2-2)), if the two multiplications and the two comultiplications are identical, (resp. distinct). A unital 2-2-hom-bialgebra is called of type (1-2), (resp. of type (2-1)), if the two multiplications are identical, (resp. distinct), and the two comultiplications are distinct, (resp. identical).

The definition of unital 2-2-hom-bialgebra morphism is similar to that of unital 2-hom-bialgebra morphism.

3 Kaplansky's Construction of Hom-Bialgebras

In this section, we give a hom-algebra version of Kaplansky's construction of hom-bialgebras in order to build unital 2-associative hom-bialgebras, unital 2-hom-bialgebras, and unital 2-2-hom-bialgebras. The following statement is in order.

Proposition 15 *Let $\mathcal{A} = (V, \mu, \eta, \alpha)$ be a unital hom-associative algebra, where $e_2 := \eta(1)$ is the unit. Let $\tilde{V}$ be the vector space spanned by V and e_1, $\tilde{V} = \mathrm{span}(V, e_1)$. Then, $\mathcal{K}_1(\mathcal{A}) := (\tilde{V}, \mu_1, \eta_1, \Delta_1, \varepsilon_1, \alpha_1)$ is a unital hom-bialgebra where the multiplication μ_1 is defined by:*

$$\mu_1(e_1 \otimes x) = \mu_1(x \otimes e_1) = \alpha_1(x)\ \forall x \in \tilde{V},$$

$$\mu_1(x \otimes y) = \mu(x \otimes y)\ \ \forall x, y \in V,$$

the unit η_1 *is given by* $\eta_1(1) = e_1$, *while the comultiplication* Δ_1, *the counit* ε_1, *and the linear map* α_1 *are defined by,* $\forall x \in V$*:*

$$\Delta_1(e_1) = e_1 \otimes e_1$$
$$\Delta_1(x) = \alpha(x) \otimes e_1 + e_1 \otimes \alpha(x) - e_2 \otimes \alpha(x)$$
$$\varepsilon_1(e_1) = 1, \varepsilon_1(x) = 0$$
$$\alpha_1(e_1) = e_1, \alpha_1(x) = \alpha(x),$$

respectively.

Proof

◇ (V, μ, η, α) is a unital hom-associative algebra. We have: $\forall x \in V$,

$$\mu_1(\alpha_1(x), \mu_1(e_2, e_1)) = \mu_1(\alpha(x), e_2) = \mu(\alpha(x), e_2) = \alpha^2(x), \text{ and}$$
$$\mu_1(\mu_1(x, e_2), \alpha_1(e_1)) = \mu_1(\mu(x, e_2), e_1) = \mu_1(\alpha(x), e_1) = \alpha^2(x).$$

Then, $\mu_1(\alpha_1(x), \mu_1(e_2, e_1)) = \mu_1(\mu_1(x, e_2), \alpha_1(e_1)) = \alpha^2(x)$.

By permuting x, e_1, e_2, we find the same result. Therefore, $\forall x, y, z \in \tilde{V}$, $\mu_1(\alpha_1(x), \mu_1(y, z)) = \mu_1(\mu_1(x, y), \alpha_1(z))$. Hence, $(\tilde{V}, \mu_1, \eta_1, \alpha_1)$ is a unital hom-associative algebra.

◇ We have

$$(\alpha_1 \otimes \Delta_1) \circ \Delta_1(x) = (\alpha_1 \otimes \Delta_1)(\alpha_1(x) \otimes e_1 + e_1 \otimes \alpha_1(x) - e_2 \otimes \alpha_1(x))$$
$$= \Delta_1(\alpha_1(x)) \otimes e_1 + \Delta_1(e_1) \otimes \alpha_1^2(x) + \Delta_1(e_2) \otimes \alpha_1^2(x)$$
$$= (\Delta_1 \otimes \alpha_1) \circ \Delta_1(x),$$

and

$$(\varepsilon_1 \otimes \alpha_1) \circ \Delta_1(x) = 1 \otimes \alpha_1^2(x) = \alpha_1^2(x)$$
$$(\alpha_1 \otimes \varepsilon_1) \circ \Delta_1(x) = \alpha_1^2(x) \otimes 1 = \alpha_1^2(x).$$

Then, $(\varepsilon_1 \otimes \alpha_1) \circ \Delta_1(x) = (\alpha_1 \otimes \varepsilon_1) \circ \Delta_1(x) = \alpha_1^2(x)$. Hence, we can conclude that $(\tilde{V}, \Delta_1, \varepsilon_1, \alpha_1)$ is a counital hom-coassociative coalgebra.

◇

$$\Delta_1(x) \bullet \Delta_1(y) = \mu_1(\alpha(x), \alpha(y)) \otimes e_1 + e_1 \otimes \mu_1(\alpha(x), \alpha(y))$$
$$- e_2 \otimes \mu_1(\alpha(x), \alpha(y)) = \mu_1 \circ \alpha_1^{\otimes^2}(x \otimes y) \otimes e_1 + e_1 \otimes \mu_1 \circ \alpha_1^{\otimes^2}(x \otimes y)$$
$$- e_2 \otimes \mu_1 \circ \alpha_1^{\otimes^2}(x \otimes y) = \alpha_1(\mu_1(x \otimes y)) \otimes e_1 + e_1 \otimes \alpha_1(\mu_1(x \otimes y))$$
$$- e_2 \otimes \alpha_1(\mu_1(x \otimes y)) = \Delta_1(\mu_1(x \otimes y));$$
$$\Delta_1(\alpha_1(x)) = \alpha_1^2(x) \otimes e_1 + e_1 \otimes \alpha_1^2(x) - e_2 \otimes \alpha_1^2(x)$$
$$= \alpha_1^{\otimes^2}(\alpha_1(x) \otimes e_1 + e_1 \otimes \alpha_1(x) - e_2 \otimes \alpha_1(x)) = \alpha_1^{\otimes^2} \circ \Delta_1(x).$$

Then, Δ is a homomorphism of the hom-associative algebras (V, μ, α) and $(V \otimes V, \bullet, \alpha \otimes \alpha)$.

Therefore, we can conclude that $\mathcal{K}_1(\mathcal{A}) := (\tilde{V}, \mu_1, \eta_1, \Delta_1, \varepsilon_1, \alpha_1)$ is a unital hom-bialgebra. □

Proposition 16 *Let $\mathcal{A} = (V, \mu, \eta, \alpha)$ be a unital hom-associative algebra, where $e_2 := \eta(1)$ is the unit. Let $\tilde{V}$ be the vector space spanned by V and e_1, $\tilde{V} =$* span(V, e_1). *$\mathcal{K}_2(\mathcal{A}) := (\tilde{V}, \mu_2, \eta_2, \Delta_2, \varepsilon_2, \alpha_2)$ is a unital hom-bialgebra, where the multiplication μ_2 is defined by:*

$$\mu_2(e_1 \otimes x) = \mu_2(x \otimes e_1) = \alpha_2(x), \ \forall x \in \tilde{V},$$
$$\mu_2(x \otimes y) = \mu(x \otimes y), \ \ \forall x, y \in V,$$

the unit η_2 is given by $\eta_2(1) = e_1$, while the comultiplication Δ_2, the counit ε_2, and the linear map α_2 are defined as follows:

$$\Delta_2(e_1) = e_1 \otimes e_1,$$
$$\Delta_2(e_2) = e_2 \otimes e_1 + e_1 \otimes e_2 - e_2 \otimes e_2,$$
$$\Delta_2(x) = (e_1 - e_2) \otimes \alpha(x) + \alpha(x) \otimes (e_1 - e_2) \ \ \forall x \in V \backslash \{e_2\},$$
$$\varepsilon_2(e_1) = 1, \varepsilon_2(x) = 0 \ \ \forall x \in V,$$
$$\alpha_2(e_1) = e_1, \alpha_2(x) = \alpha(x), \ \forall x \in V,$$

respectively.

Proof We have:

◇

$$\Delta_2(\alpha_2(x)) = (e_1 - e_2) \otimes \alpha_2^2(x) + \alpha_2^2(x) \otimes (e_1 - e_2)$$
$$= \alpha_2^{\otimes^2}((e_1 - e_2) \otimes \alpha_2(x) + \alpha_2(x) \otimes (e_1 - e_2)) = \alpha_2^{\otimes^2}(\Delta_2(x)).$$

◇

$$
\begin{aligned}
\Delta_2(x)\bullet\Delta_2(y) = {} & \mu_2[(e_1-e_2);(e_1-e_2)]\otimes\mu_2[\alpha_2(x);\alpha_2(y)]\\
& +\mu_2[(e_1-e_2);\alpha_2(y)]\otimes\mu_2[\alpha_2(x);(e_1-e_2)]\\
& +\mu_2[\alpha_2(x);(e_1-e_2)]\otimes\mu_2[(e_1-e_2);\alpha_2(y)]\\
& +\mu_2(\alpha_2(x);\alpha_2(y))\otimes\mu_2[(e_1-e_2);(e_1-e_2)]\\
= {} & (e_1-e_2)\otimes\alpha_2(\mu_2(x,y))+\alpha_2(\mu_2(x,y))\otimes(e_1-e_2)\\
= {} & \Delta_2(\mu_2(x,y)).
\end{aligned}
$$

◇

$$
\begin{aligned}
(\alpha_2\otimes\Delta_2)\circ\Delta_2(x) = {} & e_1\otimes e_1\otimes\alpha_2^2(x)-e_1\otimes e_2\otimes\alpha_2^2(x)-e_2\otimes e_1\otimes\alpha_2^2(x)\\
& +e_2\otimes e_2\otimes\alpha_2^2(x)+e_1\otimes\alpha_2^2(x)\otimes e_1-e_1\otimes\alpha_2^2(x)\otimes e_2-e_2\otimes\alpha_2^2(x)\otimes e_2\\
& +\alpha_2^2(x)\otimes e_1\otimes e_1-\alpha_2^2(x)\otimes e_2\otimes e_1-\alpha_2^2(x)\otimes e_1\otimes e_2+\alpha_2^2(x)\otimes e_2\otimes e_2\\
= {} & (\Delta_2\otimes\alpha_2)\circ\Delta_2(x).
\end{aligned}
$$

◇ The condition (5) is easily established.

Hence, $\mathcal{K}_2(\mathcal{A}) := (\tilde{V},\mu_2,\eta_2,\Delta_2,\varepsilon_2,\alpha_2)$ is a unital hom-bialgebra. □

3.1 Construction of Unital 2-Hom-Associative Bialgebras

Here, we construct $(n+1)$-dimensional unital 2-hom-associative bialgebras from n-dimensional unital hom-associative algebras.

Lemma 17 *Let $\mathcal{A}=(V,\mu,\eta,\alpha)$ be a unital hom-associative algebra. The unital hom-bialgebra $\mathcal{K}_1(\mathcal{A}) = (\tilde{V},\mu_1,\eta_1,\Delta_1,\varepsilon_1,\alpha_1)$ is a unital infinitesimal hom-bialgebra.*

Proof We know that $\mathcal{K}_1(\mathcal{A})$ is a unital hom-bialgebra. Then, we only have to show the unital hom-infinitesimal condition. For all $x, y\in V$, we have:

$$
\begin{aligned}
&(\mu\otimes\alpha)\circ(\alpha\otimes\Delta)(x\otimes y)+(\alpha\otimes\mu)\circ(\Delta\otimes\alpha)(x\otimes y)-\alpha^2(x)\otimes\alpha(y)\\
&\quad=\mu(\alpha(x),\alpha(y))\otimes e_1+e_1\otimes\mu(\alpha(x),\alpha(y))-e_2\otimes\mu(\alpha(x),\alpha(y))\\
&\quad=\alpha(\mu(x,y))\otimes e_1+e_1\otimes\alpha(\mu(x,y))-e_2\otimes\alpha(\mu(x,y))=\Delta(\mu(x,y)).
\end{aligned}
$$

Hence, $\mathcal{K}_1(\mathcal{A})$ is a unital infinitesimal hom-bialgebra. □

Let us point out the following:

(i) Let $\mathcal{A} = (V, \mu, \eta, \alpha)$ be a unital hom-associative algebra. The unital hom-bialgebra $\mathcal{K}_2(\mathcal{A})$ is not a unital infinitesimal hom-bialgebra since the unital hom-infinitesimal condition is not satisfied.

(ii) Let $\mathcal{A}_2 = (V, \mu_1, \mu_2, \eta, \alpha)$ be a unital 2-hom-associative algebra. Then, we have the same hom-coalgebra structure in the associated hom-bialgebra, (or unital infinitesimal hom-bialgebra), related to unital hom-associative algebras (V, μ_1, η, α) and (V, μ_2, η, α).

Proposition 18 *Let $\mathcal{A} = (V, \mu, \eta, \alpha)$ and $\mathcal{A}' = (V, \mu', \eta, \alpha)$ be two unital hom-associative algebras over an n-dimensional vector space V. Let $\mathcal{K}_1(\mathcal{A}) = (\tilde{V}, \mu_1, \eta_1, \Delta_1, \varepsilon_1, \alpha_1)$ and $\mathcal{K}_1(\mathcal{A}') = (\tilde{V}, \mu'_1, \eta_1, \Delta_1, \varepsilon_1, \alpha_1)$ be the above defined associated hom-bialgebras. Then, we have that $\mathcal{B}_1 = (\tilde{V}, \mu_1, \mu'_1, \eta_1, \Delta_1, \varepsilon_1, \alpha_1)$ is an $(n+1)$-dimensional unital 2-hom-associative bialgebra over the vector space $\tilde{V} = \text{span}(V, e_1)$, where $\eta_1(1) = e_1$.*

Proof From Lemma 17, $\mathcal{K}_1(\mathcal{A}')$ is a unital infinitesimal hom-bialgebra, and $\mathcal{K}_1(\mathcal{A})$ is a unital hom-bialgebra, and hence $\mathcal{B}_1 = (\tilde{V}, \mu_1, \mu'_1, \eta_1, \Delta_1, \varepsilon_1, \alpha_1)$ is a unital 2-hom-associative bialgebra. □

Remark 19 Let $(V, \mu, \eta, \Delta, \varepsilon, \alpha)$ be a unital hom-bialgebra. If the comultiplication satisfies the unital hom-infinitesimal condition, then $(V, \mu, \mu, \eta, \Delta, \varepsilon, \alpha)$ is a unital 2-associative-hom-bialgebra.

3.2 Construction of Unital 2-Hom Bialgebras

Proposition 20 *Let V be an n-dimensional vector space over $\mathcal{K}$. Let $\mathcal{A}_1 = (V, \mu_1, \eta_1, \alpha)$ and $\mathcal{A}_2 = (V, \mu_2, \eta_2, \alpha)$ be two unital hom-associative algebras, and $\mathcal{K}_j(\mathcal{A}_i) = (\tilde{V}, \tilde{\mu}_i, \eta, \Delta_i, \varepsilon, \tilde{\alpha})$, $i, j = 1, 2$, the above defined associated hom-bialgebras. Then,*

$$\mathcal{B}_1 = (\tilde{V}, \tilde{\mu}_1, \tilde{\mu}_2, \eta, \Delta_1, \Delta_2, \varepsilon, \tilde{\alpha}) \text{ and } \mathcal{B}_2 = (\tilde{V}, \tilde{\mu}_1, \tilde{\mu}_2, \eta, \Delta_1^{cop}, \Delta_2, \varepsilon, \tilde{\alpha})$$

are two $(n+1)$-dimensional unital 2-hom bialgebras on $\tilde{V} = \text{span}(V, e_1)$, where $\eta(1) = e_1$

Proof From Proposition 13, we establish, by a straightforward computation, that $\mathcal{B}_1$ and $\mathcal{B}_2$ are unital 2-hom-bialgebras. □

The next corollary gives a unital 2-2-hom-bialgebra from two unital hom-associative algebras.

Corollary 21 *Under the above conditions, $\mathcal{B}_1 = (\tilde{V}, \tilde{\mu}_1, \tilde{\mu}_2, \eta, \Delta_1, \Delta_2, \varepsilon, \tilde{\alpha})$ is an $(n+1)$-dimensional unital 2-2-hom bialgebra on $\tilde{V} = \text{span}(V, e_1)$, where $\eta(1) = e_1$.*

4 Hom-Left Symmetric Dialgebras

4.1 Hom-Associative Dialgebra

Definition 22 We call differential hom-associative algebra the quadruple $(\mathcal{A}, \cdot, \alpha, d)$ such that $(\mathcal{A}, \cdot, \alpha)$ is a hom-associative algebra, $d(a \cdot b) = da \cdot b + a \cdot db, \forall a, b \in \mathcal{A},\ d^2 = 0$, and $d \circ \alpha = \alpha \circ d$.

Let us immediately emphasize that this definition has a quite strong condition. Indeed, using the Leibniz rule twice and noting the nilpotency of d leads to $2d(a)d(b) = 0$, for all $a,\ b \in A$. If the ground field has characteristic different from 2, this puts a restriction on the image of d.

Proposition 23 *Let $(\mathcal{A}, \cdot, \alpha, d)$ be a differential hom-associative algebra. Consider the products $\vdash$ and $\dashv$ on $\mathcal{A}$ given by $x \dashv y = \alpha(x)d\alpha(y)$ and $x \vdash y = \alpha(x)d\alpha(y)$. Then, $(\mathcal{A}, \dashv, \vdash, \alpha)$ is a hom-associative dialgebra.*

Proof By hypothesis, $(\mathcal{A}, \cdot, \alpha, d)$ is a differential hom-associative algebra. Hence we have:

- $\alpha(x) \dashv (y \dashv z) = \alpha(x) \dashv (\alpha(y)d\alpha(z)) = \alpha^2(x)d\alpha(\alpha(y)d\alpha(z))$
  $= \alpha^2(x)d\alpha^2(y)d\alpha^2(z) = \alpha^2(x)d[\alpha(d\alpha(y))\alpha^2(z)] = \alpha^2(x) \dashv \alpha[d\alpha(y)\alpha(z)]$
  $= \alpha(x) \dashv (d\alpha(y)\alpha(z)) = \alpha(x) \dashv (y \vdash z)$;
- $(x \vdash y) \dashv \alpha(z) = (d\alpha(x)\alpha(y)) \dashv \alpha(z) = d\alpha^2(x)\alpha^2(y)d\alpha^2(z)$
  $= \alpha(x) \vdash (\alpha(y)d\alpha(z)) = \alpha(x) \vdash (y \dashv z)$;
- $(x \vdash y) \vdash \alpha(z) = (d\alpha(x)\alpha(y)) \vdash \alpha(z) = d[d\alpha^2(x)\alpha^2(y)]\alpha^2(z)$
  $= d(\alpha^2(x)d\alpha^2(y))\alpha^2(z) = (\alpha(x)d\alpha(y)) \vdash \alpha(z) = (x \dashv y) \vdash \alpha(z)$.

Therefore, $(A, \dashv, \vdash, \alpha)$ is a hom-associative dialgebra. □

Theorem 24 *Let $(D, \dashv, \vdash)$ be an associative dialgebra, and $\alpha : D \to D$ be an associative dialgebra endomorphism. Then $D_\alpha = (D, \dashv_\alpha, \vdash_\alpha, \alpha)$, where $\dashv_\alpha = \alpha \circ \dashv$ and $\vdash_\alpha = \alpha \circ \vdash$, is a hom-associative dialgebra. Moreover, suppose that $(D', \dashv', \vdash')$ is another associative dialgebra, and $\alpha' : D' \to D'$ is an associative dialgebra endomorphism. If $f : D \to D'$ is an associative dialgebra morphism that satisfies $f \circ \alpha = \alpha' \circ f$, then $f : D_\alpha \to D'_{\alpha'}$ is a morphism of hom-associative dialgebras.*

Proof We have:

-

$$\begin{aligned}
\alpha(x) \dashv_\alpha (y \dashv_\alpha z) &= \alpha(\alpha(x) \dashv (y \dashv_\alpha z)) = \alpha(\alpha(x) \dashv \alpha(y \dashv z)) \\
&= \alpha^2(x \dashv (y \dashv z)) = \alpha^2(x \dashv (y \vdash z)) = \alpha(\alpha(x) \dashv (\alpha(y \vdash z))) \\
&= \alpha(\alpha(x) \dashv (y \vdash_\alpha z)) = \alpha(x) \dashv_\alpha (y \vdash_\alpha z);
\end{aligned}$$

- $$(x \vdash_\alpha y) \dashv_\alpha \alpha(z) = \alpha^2((x \vdash y) \dashv z) = \alpha^2(x \vdash (y \dashv z))$$
  $$= \alpha(x) \vdash_\alpha (y \dashv_\alpha z);$$

- $$(x \vdash_\alpha y) \vdash_\alpha \alpha(z) = \alpha^2((x \vdash y) \vdash z) = \alpha^2((x \dashv y) \vdash z)$$
  $$= (x \dashv_\alpha y) \vdash_\alpha \alpha(z);$$

- $$f \circ \dashv_\alpha = f \circ (\alpha \circ \dashv) = (f \circ \alpha) \circ \dashv = (\alpha' \circ f) \circ \dashv = \alpha' \circ (f \circ \dashv)$$
  $$= \alpha' \circ (\dashv' \circ (f \otimes f)) = (\alpha' \circ \dashv') \circ (f \otimes f) = \dashv'_{\alpha'} \circ (f \otimes f),$$
  and we also obtain that $f \circ \vdash_\alpha = \vdash'_{\alpha'} \circ (f \otimes f)$.

Hence, we can conclude that D_α is a hom-associative dialgebra, and f a morphism of hom-associative dialgebras. □

4.2 *Hom-Leibniz Algebra*

Definition 25 A hom-Leibniz algebra is a triple $(L, [.,.]\,\alpha)$ consisting of a linear space L, a bilinear product $[.,.] : L \times L \to L$, and a homomorphism $\alpha : L \to L$ satisfying

$$[[x, y], \alpha(z)] = [[x, z], \alpha(y)] + [\alpha(x), [y, z]]. \tag{7}$$

Proposition 26 *Let $(\mathcal{A}, \cdot, \alpha, d)$ be a differential hom-associative algebra. Define the bracket on $\mathcal{A}$ by*

$$[x, y] := \alpha(x) \cdot d\alpha(y) - d\alpha(y) \cdot \alpha(x).$$

Then, the vector space $\mathcal{A}$ equipped with this bracket is a hom-Leibniz algebra.

Proof By direct computation, we obtain:

- $[\alpha(x), [y, z]] = \alpha(x)d\alpha(y)d\alpha(z) - d\alpha(y)d\alpha(z)\alpha(x) - \alpha(x)d\alpha(z)d\alpha(y)$ $+d\alpha(z)d\alpha(y)\alpha(x);$
- $[[x, y], \alpha(z)] = \alpha(x)d\alpha(y)d\alpha(z) - d\alpha(z)\alpha(x)d\alpha(y) - d\alpha(y)\alpha(x)d\alpha(z)$ $+d\alpha(z)d\alpha(y)\alpha(x);$
- $[[x, z], \alpha(y)] = \alpha(x)d\alpha(z)d\alpha(y) - d\alpha(y)\alpha(x)d\alpha(z) - d\alpha(z)\alpha(x)d\alpha(y)$ $+d\alpha(y)d\alpha(z)\alpha(x).$

Then, $[\alpha(x), [y, z]] = [[x, y], \alpha(z)] - [[x, z], \alpha(y)]$. Hence, the pair $(A, [., .])$ is a hom-Leibniz algebra. □

Theorem 27 *Let $(L, [., .])$ be a Leibniz algebra, and $\alpha : L \to L$ be a Leibniz algebra endomorphism. Then, $L_\alpha = (L, [., .]_\alpha, \alpha)$ is a hom-Leibniz algebra. Moreover, suppose that $(L', [., .]')$ is another Leibniz algebra, and $\alpha' : L' \to L'$ a Leibniz algebra endomorphism. If $f : L \to L'$ is a Leibniz algebra morphism satisfying $f \circ \alpha = \alpha' \circ f$, then $L_\alpha \to L'_{\alpha'}$ is a morphism of Leibniz algebras.*

Proof Since

- $[[x, y]_\alpha, \alpha(z)]_\alpha = \alpha([\alpha([x, y]), \alpha(z)]) = \alpha^2([[x, y], z])$
  $= \alpha^2([[x, z], y] + [x, [y, z]]) = [[x, z]_\alpha, \alpha(y)]_\alpha + [\alpha(x), [y, z]_\alpha]_\alpha,$
  and
- $f \circ [., .]_\alpha = f \circ (\alpha \circ [., .]) = (f \circ \alpha) \circ [., .] = (\alpha' \circ f) \circ [., .] = \alpha' \circ (f \circ [., .])$
  $= \alpha' \circ ([., .]' \circ (f \otimes f)) = (\alpha' \circ [., .]') \circ (f \otimes f) = [., .]'_{\alpha'} \circ (f \otimes f).$

Therefore, we have the results. □

Theorem 28 *Let $(D, \dashv, \vdash, \alpha)$ be a hom-associative dialgebra. Consider a linear map $[., .] : D \otimes D \to D$ defined, for $x, y \in D$, by*

$$[x, y] = x \dashv y - x \vdash y.$$

Then, $(D, [., .], \alpha)$ is a hom-Leibniz algebra.

Proof $(D, \dashv, \vdash, \alpha)$ is a hom-associative dialgebra, then, we have:

$$\alpha(y) \vdash (z \vdash x) = (y \vdash z) \vdash \alpha(x) = (y \dashv z) \vdash \alpha(x);$$

$$(x \dashv z) \dashv \alpha(y) = \alpha(x) \dashv (z \dashv y) = \alpha(x) \dashv (z \vdash y).$$

Therefore, by direct computation, we obtain the identity (7). □

4.3 Hom-Left Symmetric Dialgebras

Now, we generalize the notion of left symmetric dialgebra introduced by R. Felipe, twisting the identities by a linear map, as well as some theorems established in [5].

Definition 29 Let S be a vector space over a field K. Let us assume that S is equipped with two bilinear products $\dashv, \vdash$: $S \otimes S \to S$, and a homomorphism $\alpha : S \to S$ satisfying the identities:

$$\alpha(x) \dashv (y \dashv z) = \alpha(x) \dashv (y \vdash z), \tag{8}$$

$$(x \vdash y) \vdash \alpha(z) = (x \dashv y) \vdash \alpha(z), \tag{9}$$

$$\alpha(x) \dashv (y \dashv z) - (x \dashv y) \dashv \alpha(z) = \alpha(y) \vdash (x \dashv z) - (y \vdash x) \dashv \alpha(z), \qquad (10)$$

$$\alpha(x) \vdash (y \vdash z) - (x \vdash y) \vdash \alpha(z) = \alpha(y) \vdash (x \vdash z) - (y \vdash x) \vdash \alpha(z). \qquad (11)$$

Then, we say that S is a hom-left symmetric dialgebra (HLSDA), or left disymmetric hom-algebra.

Example Any hom-associative algebra $(\mathcal{A}, \cdot, \alpha)$ is a hom-left symmetric dialgebra with $\vdash = \dashv = \cdot$.

The definition of a hom-left symmetric dialgebra morphism is similar to that of a hom-associative dialgebra morphism. Note also that we can construct a hom-left symmetric dialgebra by the composition method from a classical left-symmetric dialgebra $(D, \dashv, \vdash)$ and an algebra endomorphism α, by considering $(D, \dashv_\alpha, \vdash_\alpha, \alpha)$, where $x \dashv_\alpha y = \alpha(x \dashv y)$ and $x \vdash_\alpha y = \alpha(x \vdash y)$. Let us denote by $\mathcal{HS}$ the set of all hom-left symmetric dialgebras, and $\mathcal{HD}$ the set of all hom-associative dialgebras.

Proposition 30 *Any hom-associative dialgebra is a hom-left symmetric dialgebra. Then,* $\mathcal{HD} \subseteq \mathcal{HS}$.

Proof Let $(D, \dashv, \vdash, \alpha)$ be a hom-associative dialgebra. Then, Eqs. (8) and (9) are satisfied. Since the products $\dashv$ and $\vdash$ are associative, then Eqs. (10) and (11) are established. □

Remark 31 Any hom-left symmetric algebra is a hom-left symmetric dialgebra in which $\vdash = \dashv$. A non associative hom-left symmetric algebra is not a hom-left symmetric dialgebra. Hence, we have $\mathcal{HD} \neq \mathcal{HS}$.

Proposition 32 *A hom-left symmetric dialgebra S is a hom-associative dialgebra if and only if both products of S are hom-associative.*

Proof Let $(S, \dashv, \vdash, \alpha)$ be a hom-left symmetric dialgebra. If S is a hom-associative dialgebra, then the products $\dashv$ and $\vdash$ defined on S are hom-associative. Conversely, suppose that the products $\dashv$ and $\vdash$ are hom-associative. Since S has a hom-left symmetric dialgebra structure, then, from Eq. (10), S is a hom-associative dialgebra. □

Theorem 33 *Let* $(S, \vdash, \dashv, \alpha)$ *be a hom-left symmetric dialgebra. Then, the commutator given by* $[x, y] = x \dashv y - y \vdash x$ *defines a structure of hom-Leibniz algebra on* S. *In other words,* $(S, [.,.], \alpha)$ *is a hom-Leibniz algebra.*

Proof We have:

- $[[x, y], \alpha(z)] = (x \dashv y) \dashv \alpha(z) - \alpha(z) \vdash (x \dashv y) - (y \vdash x) \dashv \alpha(z)$
  $+\alpha(z) \vdash (y \vdash x)$;
- $[[x, z], \alpha(y)] = (x \dashv z) \dashv \alpha(y) - \alpha(y) \vdash (x \dashv z) + (z \vdash x) \dashv \alpha(y)$
  $-\alpha(y) \vdash (z \vdash x)$;
- $[\alpha(x), [y, z]] = \alpha(x) \dashv (y \dashv z) - (y \dashv z) \vdash \alpha(x) - \alpha(x) \dashv (z \vdash y)$
  $+(z \vdash y) \vdash \alpha(x)$.

From Eqs. (10) and (11), we obtain the condition (7). □

Definition 34 Let $(L, [., .], \alpha)$ be a hom-Leibniz algebra. The pair of bilinear mappings $\triangledown_1, \triangledown_2 : L \times L \to L$ is called an affine hom-Leibniz structure obeying the relations:

$$\triangledown_2(x, y) - \triangledown_1(y, x) = [x, y], \tag{12}$$

$$\begin{aligned} \triangledown_1(\triangledown_1(x, y), \alpha(z)) &= \triangledown_1(\triangledown_2(x, y), \alpha(z)); \\ \triangledown_2(\alpha(x), \triangledown_2(y, z)) &= \triangledown_2(\alpha(x), \triangledown_1(y, z)) \end{aligned} \tag{13}$$

$$\triangledown_2(\alpha(x), \triangledown_2(y; z)) - \triangledown_1(\alpha(y), \triangledown_2(x, z)) = \triangledown_2([x, y], \alpha(z)) \tag{14}$$

and

$$\triangledown_1(\alpha(x), \triangledown_1(y, z)) - \triangledown_1(\alpha(y), \triangledown_1(x, z)) = \triangledown_1([x, y], \alpha(z)) \tag{15}$$

for all $x, y, z \in L$.

Theorem 35 *Let $(L, [., .], \alpha)$ be a hom-Leibniz algebra, and let $\triangledown_1, \triangledown_2$ define an affine hom-Leibniz structure. Then, L is a hom-left symmetric dialgebra with $\vdash$ and $\dashv$ defined as*

$$x \vdash y = \triangledown_1(x, y); \; x \dashv y = \triangledown_2(x, y). \tag{16}$$

Proof (13) implies (8) and (9). Then, (10) and (11) follow from (14) and (15), respectively. □

5 Concluding Remarks

In this work, from the hom-counital and unital infinitesimal hom-bialgebra conditions, and following Kaplansky's construction based on unital hom-associative algebras, we have built unital 2-hom-associative bialgebras, unital 2-hom-bialgebras, and unital 2-2-hom-bialgebras, and derived their main relevant properties. Finally, we have defined and characterized the hom-associative dialgebras, hom-Leibniz algebra and hom-left symmetric dialgebras generalizing the ordinary left symmetric dialgebras. The study of relevant properties of unital 2-hom-associative bialgebras, unital 2-hom-bialgebras, and unital 2-2-hom-bialgebras will be in the core of our forthcoming works.

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