On the bi-Hamiltonian Structure of the Trigonometric Spin Ruijsenaars–Sutherland Hierarchy

L. Fehér and I. Marshall

Abstract We report on the trigonometric spin Ruijsenaars–Sutherland hierarchy derived recently by Poisson reduction of a bi-Hamiltonian hierarchy associated with free geodesic motion on the Lie group $\mathrm{U}(n)$. In particular, we give a direct proof of a previously stated result about the form of the second Poisson bracket in terms of convenient variables.

Keywords Integrable system · Spin Ruijsenaars and Sutherland models · bi-Hamiltonian Hierarchy · Hamiltonian reduction

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1 Introduction

The classical integrable many-body models of Calogero–Moser–Sutherland and Ruijsenaars–Schneider as well as their extensions by internal degrees of freedom are in the focus of intense investigations even today, many years after their inception. See [1–4] and references therein. One of the sources of these models is Hamiltonian reduction of obviously integrable 'free motion' on suitable higher dimensional phase spaces, among which cotangent bundles and their Poisson–Lie analogues are the prime examples. In this framework, the emergence of the internal degrees of freedom, colloquially called 'spin', originates from the fact that symplectic

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reductions of cotangent bundles are in general not cotangent bundles, but more complicated phase spaces.

We do not have a single, all encompassing framework for understanding integrable Hamiltonian systems, but there exist several powerful approaches with large intersections of their ranges of applicability. For example, the method of the classical r-matrix incorporates many famous systems, like Toda lattices, that can be derived by Hamiltonian reduction, too, as reviewed in [9, 10]. The r-matrix method and Hamiltonian reduction also have several links to the bi-Hamiltonian approach initiated by Magri [8].

It was pointed out in the recent paper [4] that one of the simplest finite-dimensional integrable systems, the free geodesic motion on the unitary group $\mathrm{U}(n)$, admits a natural bi-Hamiltonian structure, and a suitable reduction of this free system gives rise to the so-called spin Ruijsenaars–Sutherland hierarchy. In this contribution, we overview the results of [4], and give a new, direct proof of a statement formulated in this reference without detailed proof.

2 Bi-Hamiltonian Hierarchy on $T^*\mathrm{U}(n)$ and Its Reduction

In this section we present a terse review of the results of [4].

Our starting point is the manifold $T^*\mathrm{U}(n)$, which we identify with the set

$$\mathfrak{M} := \mathrm{U}(n) \times \mathfrak{H}(n) := \{(g, L) \mid g \in \mathrm{U}(n), L \in \mathfrak{H}(n)\}, \tag{1}$$

using right-trivialization. Here, the vector space of Hermitian matrices, $\mathfrak{H}(n) = \mathrm{i}\mathfrak{u}(n)$, serves as the model of the dual $\mathfrak{u}(n)^*$ of the Lie algebra $\mathfrak{u}(n)$.

Consider the *real* Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ endowed with the non-degenerate bilinear form

$$\langle X, Y\rangle := \Im\, \mathrm{tr}(XY), \quad \forall X, Y \in \mathfrak{gl}(n, \mathbb{C}). \tag{2}$$

Then $\mathfrak{gl}(n, \mathbb{C})$ is the vector space direct sum of its isotropic Lie subalgebras $\mathfrak{u}(n)$ and $\mathfrak{b}(n)$, where $\mathfrak{b}(n)$ contains the upper triangular matrices with real entries along the diagonal. Consequently, we can decompose any $X \in \mathfrak{gl}(n, \mathbb{C})$ as

$$X = X_{\mathfrak{u}(n)} + X_{\mathfrak{b}(n)}, \qquad X_{\mathfrak{u}(n)} \in \mathfrak{u}(n),\ X_{\mathfrak{b}(n)} \in \mathfrak{b}(n). \tag{3}$$

We also have another decomposition into isotropic linear subspaces, $\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{u}(n) + \mathfrak{H}(n)$. Thus both $\mathfrak{b}(n)$ and $\mathfrak{H}(n)$ can serve as models of $\mathfrak{u}(n)^*$.

For any real function $F \in C^\infty(\mathfrak{M})$, introduce the derivatives

$$D_1 F, D_1' F \in C^\infty(\mathfrak{M}, \mathfrak{b}(n)) \quad \text{and} \quad d_2 F \in C^\infty(\mathfrak{M}, \mathfrak{u}(n)) \tag{4}$$

by the relation

$$\langle D_1 F(g, L), X\rangle + \langle D_1' F(g, L), X'\rangle + \langle d_2 F(g, L), Y\rangle = \frac{d}{dt}\bigg|_{t=0} F(e^{tX} g e^{tX'}, L + tY), \tag{5}$$

for every $X, X' \in \mathfrak{u}(n)$ and $Y \in \mathfrak{H}(n)$. The 'free Hamiltonians' of our interest are

$$H_k(g, L) := \frac{1}{k}\operatorname{tr}(L^k), \quad \forall k \in \mathbb{N}. \tag{6}$$

These feature in the 'free bi-Hamiltonian hierarchy' on $\mathfrak{M}$, which is given by the next theorem.

Theorem 1 ([4]) *The following formulae define two compatible Poisson brackets on $\mathfrak{M}$:*

$$\{F, H\}_1(g, L) = \langle D_1 F, d_2 H\rangle - \langle D_1 H, d_2 F\rangle + \langle L, [d_2 F, d_2 H]\rangle, \tag{7}$$

and

$$\{F, H\}_2(g, L) = \langle D_1 F, L d_2 H\rangle - \langle D_1 H, L d_2 F\rangle + 2\left\langle L d_2 F, (L d_2 H)_{\mathfrak{u}(n)}\right\rangle - \frac{1}{2}\left\langle D_1' F, g^{-1}(D_1 H) g\right\rangle, \tag{8}$$

where the derivatives are taken at (g, L) and (3) is applied. The Hamiltonians H_k satisfy

$$\{F, H_k\}_2 = \{F, H_{k+1}\}_1, \qquad \forall F \in C^\infty(\mathfrak{M}), \tag{9}$$

and $\{H_k, H_\ell\}_1 = \{H_k, H_\ell\}_2 = 0$ for every $k, \ell \in \mathbb{N}$. The bi-Hamiltonian flow of the systems $(\mathfrak{M}, \{\ ,\ \}_2, H_k)$ and $(\mathfrak{M}, \{\ ,\ \}_1, H_{k+1})$ is given by $(g(t), L(t)) = \big(\exp(\mathrm{i} t L(0)^k) g(0), L(0)\big)$.

The first Poisson bracket is the canonical one carried by the cotangent bundle of $\mathrm{U}(n)$, while the second one arises from the Heisenberg double [12] of the Poisson–Lie group $\mathrm{U}(n)$. The latter point is explained in [4], where it is also noted that the Lie derivative of the Poisson tensor of $\{\ ,\ \}_2$ along the infinitesimal generator of the flow $(g(t), L(t)) = (g(0), L(0) + t\mathbf{1}_n)$ is the Poisson tensor of $\{\ ,\ \}_1$. This implies [13] compatibility, and the rest of the statements is readily checked as well.

The fact that the flow generated by the Hamiltonian H_1 on the Heisenberg double of $\mathrm{U}(n)$ projects to free motion on $\mathrm{U}(n)$ was pointed out long time ago by S. Zakrzewski [14], which served as one of the motivations behind Theorem 1.

The 'conjugation action' of $\mathrm{U}(n)$ on $\mathfrak{M}$ associates with every $\eta \in \mathrm{U}(n)$ the diffeomorphism A_η of $\mathfrak{M}$ that operates according to

$$A_\eta(g, L) := (\eta g \eta^{-1}, \eta L \eta^{-1}). \tag{10}$$

A key property of the Poisson brackets on $\mathfrak{M}$ is that they can be restricted to the set of invariant functions with respect to this action, denoted $C^\infty(\mathfrak{M})^{\mathrm{U}(n)}$. This means that if $F, H \in C^\infty(\mathfrak{M})^{\mathrm{U}(n)}$, then the same holds for their Poisson brackets $\{F, H\}_i$ for $i = 1, 2$. Because the Hamiltonians H_k are also invariant, we can restrict the 'free hierarchy' to $\mathrm{U}(n)$-invariant observables. This procedure, called Poisson reduction [10], is an algebraic formulation of projection onto the quotient space $\mathfrak{M}/\mathrm{U}(n)$.

Any smooth function on $\mathfrak{M}$ can be recovered from its restriction to the dense open submanifold $\mathfrak{M}_{\mathrm{reg}} \subset \mathfrak{M}$, which contains the points (g, L) with g having distinct eigenvalues. Moreover, $F \in C^\infty(\mathfrak{M}_{\mathrm{reg}})^{\mathrm{U}(n)}$ is uniquely determined by its restriction f on the manifold $\mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{H}(n)$, where $\mathbb{T}^n_{\mathrm{reg}}$ is the set of regular elements in the standard maximal torus of $\mathrm{U}(n)$. In fact, restriction engenders a one-to-one correspondence

$$C^\infty(\mathfrak{M}_{\mathrm{reg}})^{\mathrm{U}(n)} \longleftrightarrow C^\infty(\mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{H}(n))^{\mathcal{N}(n)}, \tag{11}$$

where $\mathcal{N}(n)$ is the normalizer of $\mathbb{T}^n$ in $\mathrm{U}(n)$, whose action preserves $\mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{H}(n)$. Note that $\mathcal{N}(n)$ is the semi-direct product of the permutation group S_n, naturally embedded into $\mathrm{U}(n)$, with $\mathbb{T}^n$. By taking advantage of the correspondence (11), one can encode the Poisson brackets on $C^\infty(\mathfrak{M}_{\mathrm{reg}})^{\mathrm{U}(n)}$ by two compatible Poisson brackets $\{\ ,\ \}_i^{\mathrm{red}}$ on $C^\infty(\mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{H}(n))^{\mathcal{N}(n)}$. The main result of [4] is the formula of these reduced Poisson brackets.

For $f \in C^\infty(\mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{H}(n))$, the $\mathfrak{b}(n)_0$-valued derivative $D_1 f$ and the $\mathfrak{u}(n)$-valued derivative $d_2 f$ are defined by the equality

$$\langle D_1 f(Q, L), X\rangle + \langle d_2 f(Q, L), Y\rangle = \frac{d}{dt}\bigg|_{t=0} f(e^{tX} Q, L + tY), \tag{12}$$

for every $X \in \mathfrak{u}(n)_0$ and $Y \in \mathfrak{H}(n)$, where $\mathfrak{b}(n)_0$ and $\mathfrak{u}(n)_0$ denote the subalgebras of diagonal matrices in $\mathfrak{b}(n)$ and $\mathfrak{u}(n)$, respectively. Decompose $\mathfrak{gl}(n, \mathbb{C})$ as the vector space direct sum of subalgebras

$$\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{gl}(n, \mathbb{C})_+ + \mathfrak{gl}(n, \mathbb{C})_0 + \mathfrak{gl}(n, \mathbb{C})_-, \tag{13}$$

defined by means of the principal gradation. Accordingly, we can decompose any $X \in \mathfrak{gl}(n, \mathbb{C})$ as $X = X_+ + X_0 + X_-$, where X_0 is diagonal and X_+ is strictly upper-triangular. Then, for $Q \in \mathbb{T}^n_{\mathrm{reg}}$, introduce $\mathcal{R}(Q) \in \mathrm{End}(\mathfrak{gl}(n, \mathbb{C}))$ by setting it equal to zero on $\mathfrak{gl}(n, \mathbb{C})_0$ and defining it otherwise as

$$\mathcal{R}(Q)|_{\mathfrak{gl}(n,\mathbb{C})_+ + \mathfrak{gl}(n,\mathbb{C})_-} := \frac{1}{2}(\mathrm{Ad}_Q + \mathrm{id}) \circ \Big((\mathrm{Ad}_Q - \mathrm{id})|_{\mathfrak{gl}(n,\mathbb{C})_+ + \mathfrak{gl}(n,\mathbb{C})_-}\Big)^{-1}, \tag{14}$$

where $\mathrm{Ad}_Q(X) = QXQ^{-1}$ for all $X \in \mathfrak{gl}(n, \mathbb{C})$. The definition makes sense because of the regularity of Q. Note that $\langle \mathcal{R}(Q)X, Y\rangle = -\langle X, \mathcal{R}(Q)Y\rangle$, and introduce the notation

$$[X, Y]_{\mathcal{R}(Q)} := [\mathcal{R}(Q)X, Y] + [X, \mathcal{R}(Q)Y], \quad \forall X, Y \in \mathfrak{gl}(n, \mathbb{C}). \tag{15}$$

Theorem 2 ([4]) *For $f, h \in C^\infty(\mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{H}(n))^{\mathcal{N}(n)}$, the reduced Poisson brackets have the form*

$$\{f, h\}_1^{\mathrm{red}}(Q, L) = \langle D_1 f, d_2 h\rangle - \langle D_1 h, d_2 f\rangle + \langle L, [d_2 f, d_2 h]_{\mathcal{R}(Q)}\rangle, \tag{16}$$

and

$$\{f, h\}_2^{\mathrm{red}}(Q, L) = \langle D_1 f, L d_2 h\rangle - \langle D_1 h, L d_2 f\rangle + 2\langle L d_2 f, \mathcal{R}(Q)(L d_2 h)\rangle, \tag{17}$$

where the derivatives are evaluated at (Q, L), and the notations (14) and 15) are applied.

The reduced system that descends from the free hierarchy generated the Hamiltonians H_k (6) is called *spin Ruijsenaars–Sutherland hierarchy*. The reason for this terminology will become clear in the next section. For the reduced equations of motion and remarks on their integrability, see [4].

3 Useful Changes of Variables

In the first subsection we introduce new variables that behave as canonically conjugate pairs and 'spin variables' with respect to the second Poisson bracket, and allow us to interpret $\mathrm{tr}(L)$ as a spin Ruijsenaars Hamiltonian. These new variables go back to the papers [3, 4]. In the second subsection we describe another, in this case well-known [5, 7], set of new variables, which convert the first Poisson bracket into that of canonical pairs and (other kind of) spin variables, and lead to the interpretation of $\mathrm{tr}(L^2)$ as a spin Sutherland Hamiltonian.

3.1 Interpretation as Spin Ruijsenaars Model

We now discuss the change of variables that the underlie the interpretation of the reduced free system as a spin Ruijsenaars model. For this purpose, we focus on the second Poisson bracket (17), and restrict ourselves to the open submanifold

$$\mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{P}(n) \subset \mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{H}(n), \tag{18}$$

where $\mathfrak{P}(n)$ denotes the set of positive definite Hermitian matrices. It is a standard fact of linear algebra that any $L \in \mathfrak{P}(n)$ can be uniquely written in the form

$$L = bb^{\dagger} \text{ with } b \in \mathrm{B}(n), \tag{19}$$

and $b \in \mathrm{B}(n)$ can be decomposed as

$$b = e^{p} b_{+} \text{ with } p \in \mathfrak{b}(n)_0,\ \ b_{+} \in \mathrm{B}(n)_{+}, \tag{20}$$

where $\mathrm{B}(n)_{+}$ is the group of upper triangular matrices with unit diagonal. We define

$$\lambda := b_{+}^{-1} Q^{-1} b_{+} Q, \tag{21}$$

and obtain the change of variables

$$\mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{P}(n) \ni (Q, L) \longleftrightarrow (Q, p, \lambda) \in \mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{b}(n)_0 \times \mathrm{B}(n)_{+}. \tag{22}$$

A grade by grade inspection of the defining relation (21) shows that this is a diffeomorphism between the respective spaces. Thus every function $f(Q, L)$ corresponds to a unique function $\mathcal{F}(Q, p, \lambda)$. The diffeomorphism (22) induces an action of $\mathcal{N}(n)$ on $\mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{b}(n)_0 \times \mathrm{B}(n)_{+}$, and we are interested in the invariant functions. The action of the subgroup $\mathbb{T}^n < \mathcal{N}(n)$ is especially simple, it is given by

$$(Q, p, \lambda) \mapsto (Q, p, \tau\lambda\tau^{-1}), \quad \forall \tau \in \mathbb{T}^n, \tag{23}$$

since this corresponds to $(Q, L) \mapsto (Q, \tau L \tau^{-1})$.

For any $\mathcal{F} \in C^{\infty}(\mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{b}(n)_0 \times \mathrm{B}(n)_{+})$, we define the derivatives $D_Q\mathcal{F} \in \mathfrak{b}(n)_0,\ d_p\mathcal{F} = \mathfrak{u}(n)_0$ and $D_\lambda\mathcal{F},\ D'_\lambda\mathcal{F} \in \mathfrak{u}(n)_\perp$ by

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{F}(e^{tX_0}Q, p + tY_0, e^{tX_+}\lambda e^{tY_+}) = \langle D_Q\mathcal{F}, X_0\rangle + \langle d_p\mathcal{F}, Y_0\rangle + \langle D_\lambda\mathcal{F}, X_+\rangle + \langle D'_\lambda\mathcal{F}, Y_+\rangle. \tag{24}$$

Here, $X_0 \in \mathfrak{u}(n)_0,\ Y_0 \in \mathfrak{b}(n)_0$ and $X_+, Y_+ \in \mathfrak{b}(n)_+$ are arbitrary, the argument (Q, p, λ) is suppressed on the right hand side, and $\mathfrak{u}(n)_\perp$ denotes the off-diagonal linear subspace of $\mathfrak{u}(n)$.

The next proposition was stated previously without elaborating its proof.

Proposition 3 ([4]) *Consider the functions* $\mathcal{F}, \mathcal{H} \in C^\infty(\mathbb{T}^n_{\rm reg} \times \mathfrak{b}(n)_0 \times \mathrm{B}(n)_+)^{\mathcal{N}(n)}$ *that are related to* $f, h \in C^\infty(\mathbb{T}^n_{\rm reg} \times \mathfrak{P}(n))^{\mathcal{N}(n)}$ *according to*

$$\mathcal{F}(Q, p, \lambda) = f(Q, L),\ \ \mathcal{H}(Q, p, \lambda) = h(Q, L) \textit{ with}$$
$$L = e^p b_+ b_+^\dagger e^p,\ \ \lambda := b_+^{-1} Q^{-1} b_+ Q. \tag{25}$$

In terms of the variables (Q, p, λ)*, the second Poisson bracket (17) takes the form*

$$2\{\mathcal{F}, \mathcal{H}\}_2^{\rm red}(Q, p, \lambda) = \langle D_Q\mathcal{F}, d_p\mathcal{H}\rangle - \langle D_Q\mathcal{H}, d_p\mathcal{F}\rangle + \langle D'_\lambda\mathcal{F}, \lambda^{-1}(D_\lambda\mathcal{H})\lambda\rangle, \tag{26}$$

where the derivatives are evaluated at (Q, p, λ).

Proof Recall that (Q, L), (Q, b) and (Q, p, λ) are alternative sets of variables. In particular, we have the invertible correspondences:

$$(Q, L) \leftrightarrow (Q, b) \leftrightarrow (Q, p, \lambda) \quad \text{with} \quad L = bb^\dagger,\ e^p := b_{\rm diag},\ \lambda := b^{-1}Q^{-1}bQ. \tag{27}$$

Here, we suppressed that λ does not depend on p. Any tangent vector at a fixed (Q, b) can be represented as the velocity vector at $t = 0$ of a curve of the form

$$(Q(t), b(t)) = (e^{t\xi}Q,\ be^{t\beta}),\ \text{with some } \xi \in \mathfrak{u}(n)_0,\ \beta = (\beta_0 + \beta_+) \in \mathfrak{b}(n). \tag{28}$$

In terms of the alternative variables, the corresponding curves are easily seen to satisfy

$$\begin{aligned} L(t) &= L + tb(\beta + \beta^\dagger)b^\dagger + \mathrm{o}(t),\\ \lambda(t) &= \lambda \exp\big(t\big[\xi - Q^{-1}b^{-1}\xi bQ + Q^{-1}\beta Q - \lambda^{-1}\beta\lambda\big] + \mathrm{o}(t)\big),\\ p(t) &= p + t\beta_0 + \mathrm{o}(t). \end{aligned} \tag{29}$$

Of course, the curve that appears in the exponent after λ lies in $\mathfrak{b}(n)_+$. Let us now consider a function on our space, which is either expressed as $(Q, L) \mapsto f(Q, L)$, or equivalently as $(Q, p, \lambda) \mapsto \mathcal{F}(Q, p, \lambda)$. By the definition of derivatives, we obtain the equality

$$\begin{aligned} &\frac{d}{dt}\bigg|_{t=0} f\big(Qe^{t\xi}, L + tb(\beta + \beta^\dagger)b^\dagger + \mathrm{o}(t)\big)\\ &= \frac{d}{dt}\bigg|_{t=0} \mathcal{F}\big(Qe^{t\xi}, p + t\beta_0, \lambda\exp(t[\xi - Q^{-1}b^{-1}\xi bQ + Q^{-1}\beta Q - \lambda^{-1}\beta\lambda] + \mathrm{o}(t))\big). \end{aligned} \tag{30}$$

This generates the following relations between the derivatives of f and $\mathcal{F}$:

$$\langle 2b^\dagger d_2 f b - d_p\mathcal{F} - QD'_\lambda\mathcal{F}Q^{-1} + (\lambda D'_\lambda\mathcal{F}\lambda^{-1})_{\mathfrak{u}(n)}\,,\ \beta\rangle$$
$$+\,\langle D_1 f - D_Q\mathcal{F} - D'_\lambda\mathcal{F} + bQD'_\lambda\mathcal{F}Q^{-1}b^{-1}\,,\ \xi\rangle = 0,\quad \forall\xi\in\mathfrak{u}(n)_0,\ \forall\beta\in\mathfrak{b}(n). \tag{31}$$

The derivatives of f and $\mathcal{F}$ are taken at (Q, L) and at (Q, p, λ), respectively, according to (12) and (24). We have $\langle D'_\lambda\mathcal{F}, \xi\rangle = 0$, and the conventions $D'_\lambda\mathcal{F},\ D_\lambda\mathcal{F}\in\mathfrak{u}(n)_\perp$ imply

$$(\lambda D'_\lambda\mathcal{F}\lambda^{-1})_{\mathfrak{u}(n)} = D_\lambda\mathcal{F} + (\lambda D'_\lambda\mathcal{F}\lambda^{-1})_{\mathrm{im-diag}}. \tag{32}$$

The matrix $X_{\mathrm{im-diag}}$ is obtained from the matrix X by setting to zero the off-diagonal entries and the real parts of the diagonal entries of X, and (3) is used.

From the first term in (31) (the one involving arbitrary β), we must have

$$A := 2b^\dagger d_2 f b - d_p F - QD'_\lambda\mathcal{F}Q^{-1} + (\lambda D'_\lambda\mathcal{F}\lambda^{-1})_{\mathfrak{u}(n)}\in\mathfrak{b}(n). \tag{33}$$

But the formula of A shows that $A\in\mathfrak{u}(n)$, and thence $A = 0$. It is convenient to rewrite

$$2b^\dagger d_2 f b = QD'_\lambda\mathcal{F}Q^{-1} - \lambda D'_\lambda\mathcal{F}\lambda^{-1} + \big[d_p\mathcal{F} + \lambda D'_\lambda\mathcal{F}\lambda^{-1} - (\lambda D'_\lambda\mathcal{F}\lambda^{-1})_{\mathfrak{u}(n)}\big], \tag{34}$$

and, conjugating by b and using $b\lambda = Q^{-1}bQ$, we get

$$\begin{aligned} 2L\,d_2 f &= bQD'_\lambda\mathcal{F}Q^{-1}b^{-1} - b\lambda D'_\lambda\mathcal{F}\lambda^{-1}b^{-1}\\ &\qquad + \mathrm{Ad}_b[d_p\mathcal{F} + \mathrm{Ad}_\lambda D'_\lambda\mathcal{F} - (\lambda D'_\lambda\mathcal{F}\lambda^{-1})_{\mathfrak{u}(n)}]\\ &= (\mathrm{Ad}_Q - \mathrm{id})\mathrm{Ad}_{Q^{-1}bQ}D'_\lambda\mathcal{F} + \mathrm{Ad}_b[d_p\mathcal{F} + \mathrm{Ad}_\lambda D'_\lambda\mathcal{F} - (\lambda D'_\lambda\mathcal{F}\lambda^{-1})_{\mathfrak{u}(n)}], \end{aligned} \tag{35}$$

from which it is easy to obtain

$$\begin{aligned} 2\mathcal{R}(Q)(Ld_2 f) &= \frac{1}{2}(\mathrm{Ad}_Q + \mathrm{id})\mathrm{Ad}_{Q^{-1}bQ}D'_\lambda\mathcal{F} - (bQD'_\lambda\mathcal{F}Q^{-1}b^{-1})_{\mathrm{diag}}\\ &\qquad + \mathcal{R}(Q)\big(\mathrm{Ad}_b[d_p\mathcal{F} + \mathrm{Ad}_\lambda D'_\lambda\mathcal{F} - (\lambda D'_\lambda\mathcal{F}\lambda^{-1})_{\mathfrak{u}(n})]\big). \end{aligned} \tag{36}$$

Of course, we could have written everywhere $\mathrm{Ad}_\lambda D'_\lambda\mathcal{F} - (\lambda D'_\lambda\mathcal{F}\lambda^{-1})_{\mathfrak{u}(n)} \equiv (\mathrm{Ad}_\lambda D'_\lambda\mathcal{F})_{\mathfrak{b}(n)}$. Note also that Ad_m denotes conjugation by m for any $m\in \mathrm{GL}(n,\mathbb{C})$.

A glance at the last equation (36) shows that the expression in the second line belongs to $\mathfrak{b}(n)_+$, and this is crucial for the computation of $\langle Ld_2 f, \mathcal{R}(Q)(Ld_2 h)\rangle$ (cf. (17)):

$$\begin{aligned}
&4\langle Ld_2 f\ ,\ \mathcal{R}(Q)(Ld_2 h)\rangle = \\
&\quad \langle (Ad_Q - \mathrm{id})\mathrm{Ad}_{Q^{-1}bQ} D'_\lambda \mathcal{F} + \mathrm{Ad}_b[d_p\mathcal{F} + \mathrm{Ad}_\lambda D'_\lambda \mathcal{F} - (\lambda D'_\lambda \mathcal{F}\lambda^{-1})_{\mathfrak{u}(n)}]\,, \\
&\qquad - (\mathrm{Ad}_{bQ} D'_\lambda \mathcal{H})_{\mathrm{diag}} + \tfrac{1}{2}(\mathrm{Ad}_Q + \mathrm{id}) Ad_{Q^{-1}bQ} D'_\lambda \mathcal{H} \\
&\qquad + \mathcal{R}(Q)\big(\mathrm{Ad}_b[d_p\mathcal{H} + \mathrm{Ad}_\lambda D'_\lambda \mathcal{H} - (\lambda D'_\lambda \mathcal{H}\lambda^{-1})_{\mathfrak{u}(n)}]\big)\rangle \\
&= \tfrac{1}{2}\langle \mathrm{Ad}_{bQ} D'_\lambda \mathcal{F}, Ad_{Q^{-1}bQ} D'_\lambda \mathcal{H}\rangle + \tfrac{1}{2}\langle d_p\mathcal{F} + \mathrm{Ad}_\lambda D'_\lambda \mathcal{F} - (\lambda D'_\lambda \mathcal{F}\lambda^{-1})_{\mathfrak{u}(n)}\,, \\
&\qquad \mathrm{Ad}_Q D'_\lambda \mathcal{H} + \mathrm{Ad}_\lambda D'_\lambda \mathcal{H} - 2(\mathrm{Ad}_{bQ} D'_\lambda \mathcal{H})_{\mathrm{diag}}\rangle - (\mathcal{F} \leftrightarrow \mathcal{H}) \\
&= \tfrac{1}{2}\langle \mathrm{Ad}_Q D'_\lambda \mathcal{F}, \mathrm{Ad}_\lambda D'_\lambda \mathcal{H}\rangle + \tfrac{1}{2}\langle d_p\mathcal{F}, \mathrm{Ad}_\lambda D'_\lambda \mathcal{H} - 2bQD'_\lambda \mathcal{H} Q^{-1}b^{-1}\rangle \\
&+ \tfrac{1}{2}\langle \mathrm{Ad}_\lambda D'_\lambda \mathcal{F}, \mathrm{Ad}_Q D'_\lambda \mathcal{H}\rangle + \langle (\lambda D'_\lambda \mathcal{F}\lambda^{-1})_{\mathfrak{u}(n)} - \mathrm{Ad}_\lambda D'_\lambda \mathcal{F}, (\mathrm{Ad}_{bQ} D'_\lambda \mathcal{H})_{\mathrm{diag}}\rangle \\
&\qquad - \tfrac{1}{2}\langle (\lambda D'_\lambda \mathcal{F}\lambda^{-1})_{\mathfrak{u}(n)}, \mathrm{Ad}_\lambda D'_\lambda \mathcal{H}\rangle - (\mathcal{F} \leftrightarrow \mathcal{H}).
\end{aligned} \tag{37}$$

Notice that the terms at the beginning of the first two lines after the last equality sign add up to

$$\tfrac{1}{2}\langle \mathrm{Ad}_Q D'_\lambda \mathcal{F}, \mathrm{Ad}_\lambda D'_\lambda \mathcal{H}\rangle + \tfrac{1}{2}\langle Ad_\lambda D'_\lambda \mathcal{F}, Ad_Q D'_\lambda \mathcal{H}\rangle, \tag{38}$$

and this is symmetric with respect to exchange of $\mathcal{F}$ and $\mathcal{H}$; thereby it cancels. Notice also that the second expression in the second line simplifies as follows:

$$\begin{aligned}
&\langle (\lambda D'_\lambda \mathcal{F}\lambda^{-1})_{\mathfrak{u}(n)} - \mathrm{Ad}_\lambda D'_\lambda \mathcal{F}, (\mathrm{Ad}_{bQ} D'_\lambda \mathcal{H})_{\mathrm{diag}}\rangle \\
&\qquad = \langle (\lambda D'_\lambda \mathcal{F}\lambda^{-1})_{\mathfrak{u}(n)} - \mathrm{Ad}_\lambda D'_\lambda \mathcal{F}, (\mathrm{Ad}_{bQ} D'_\lambda \mathcal{H})_{\mathrm{im-diag}}\rangle \\
&\qquad = -\langle \mathrm{Ad}_\lambda D'_\lambda \mathcal{F}, (\mathrm{Ad}_{bQ} D'_\lambda \mathcal{H})_{\mathrm{im-diag}}\rangle,
\end{aligned} \tag{39}$$

which will be shortly shown to vanish. To summarize, we obtained

$$\begin{aligned}
&4\langle Ld_2 f\ ,\ \mathcal{R}(Q)(Ld_2 h)\rangle = -\tfrac{1}{2}\langle \mathrm{Ad}_\lambda D'_\lambda \mathcal{F}, d_p\mathcal{H} + 2(\mathrm{Ad}_{bQ} D'_\lambda \mathcal{H})_{\mathrm{im-diag}}\rangle \\
&\quad - \langle d_p\mathcal{F}, \mathrm{Ad}_{bQ} D'_\lambda \mathcal{H}\rangle - \tfrac{1}{2}\langle (\lambda D'_\lambda \mathcal{F}\lambda^{-1})_{\mathfrak{u}(n)}, \mathrm{Ad}_\lambda D'_\lambda \mathcal{H}\rangle - (\mathcal{F} \leftrightarrow \mathcal{H}).
\end{aligned} \tag{40}$$

Next, we may look at the other terms, and return to the ξ-term of (31). This gives

$$D_1 f = D_Q F - (\mathrm{Ad}_{bQ} D'_\lambda \mathcal{F})_{\mathrm{real-diag}}, \tag{41}$$

which, together with (35)—discarding the term in the range of $(\mathrm{Ad}_Q - \mathrm{id})$ as this is in the annihilator of $\mathfrak{b}(n)_0$ – gives us

$$\begin{aligned} &2\langle D_1 f\,,\, Ld_2 h\rangle \\ &= \langle D_Q\mathcal{F} - (\mathrm{Ad}_{bQ} D'_\lambda \mathcal{F})_{\mathrm{real-diag}}\,,\, \mathrm{Ad}_b[d_p\mathcal{H} + \mathrm{Ad}_\lambda D'_\lambda\mathcal{H} - (\lambda D'_\lambda \mathcal{H}\lambda^{-1})_{\mathfrak{u}(n)}]\rangle \\ &= \langle D_Q\mathcal{F} - (\mathrm{Ad}_{bQ} D'_\lambda \mathcal{F})_{\mathrm{real-diag}}\,,\, d_p\mathcal{H}\rangle = \langle D_Q\mathcal{F} - \mathrm{Ad}_{bQ} D'_\lambda\mathcal{F}\,,\, d_p\mathcal{H}\rangle. \end{aligned} \tag{42}$$

Putting together now (40) and (42), the second term at the very end of (42) cancels, and we arrive at

$$\begin{aligned} 2\{f,h\}_2^{\mathrm{red}}(Q,L) &= 2\langle D_1 f, Ld_2 h\rangle - 2\langle Ld_2 f, D_1 h\rangle + 4\langle Ld_2 f, \mathcal{R}(Q)(Ld_2 h)\rangle \\ &= \langle D_Q\mathcal{F}\,,\, d_p\mathcal{H}\rangle + \tfrac{1}{2}\langle \mathrm{Ad}_\lambda D'_\lambda \mathcal{F}, (\lambda D'_\lambda\mathcal{H}\lambda^{-1})_{\mathfrak{u}(n)}\rangle \\ &\quad - \tfrac{1}{2}\langle \mathrm{Ad}_\lambda D'_\lambda\mathcal{F}, \eta_{\mathcal{H}}\rangle - (\mathcal{F} \leftrightarrow \mathcal{H}), \end{aligned} \tag{43}$$

where $\mathfrak{u}(n)_0 \ni \eta_{\mathcal{H}} := d_p\mathcal{H} + 2(\mathrm{Ad}_{bQ} D'_\lambda \mathcal{H})_{\mathrm{im-diag}}$ represents the diagonal-imaginary entities from the previous formulae. As explained below, for invariant functions $\mathcal{F}$ and $\mathcal{H}$, the term containing $\eta_{\mathcal{H}}$ vanishes, and we also have

$$\begin{aligned} &\langle \mathrm{Ad}_\lambda D'_\lambda\mathcal{F}, (\lambda D'_\lambda\mathcal{H}\lambda^{-1})_{\mathfrak{u}(n)}\rangle \\ &\quad = \langle \mathrm{Ad}_\lambda D'_\lambda\mathcal{F}, D_\lambda\mathcal{H} + (\lambda D'_\lambda\mathcal{H}\lambda^{-1})_{\mathrm{im-diag}}\rangle = \langle \mathrm{Ad}_\lambda D'_\lambda\mathcal{F}, D_\lambda\mathcal{H}\rangle, \end{aligned} \tag{44}$$

where we used (32) and the property (45).

By the above, the claim of the proposition follows from (43) if we can verify that for any $\mathcal{F} \in C^\infty(\mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{b}(n)_0 \times \mathrm{B}(n)_+)^{\mathbb{T}^n}$ we have

$$\langle X, \lambda D'_\lambda\mathcal{F}\lambda^{-1}\rangle = 0, \quad \forall X \in \mathfrak{u}(n)_0. \tag{45}$$

In order to justify this, we remark that

$$\langle X, \lambda D'_\lambda\mathcal{F}\lambda^{-1}\rangle = \langle \lambda^{-1}X\lambda - X, D'_\lambda F\rangle. \tag{46}$$

Since $\lambda^{-1}X\lambda - X \in \mathfrak{b}(n)_+$, we may rewrite this as

$$\begin{aligned} \langle X, \lambda D'_\lambda\mathcal{F}\lambda^{-1}\rangle &= \frac{d}{dt}\Big|_{t=0} \mathcal{F}\big(Q, p, \lambda\exp(t[\lambda^{-1}X\lambda - X])\big) \\ &= \frac{d}{dt}\Big|_{t=0} \mathcal{F}\big(Q, p, e^{tX}\lambda e^{-tX}\big). \end{aligned} \tag{47}$$

In the last step we used that $\frac{d}{dt}\big|_{t=0}\lambda\exp(t[\lambda^{-1}X\lambda - X]) = [X, \lambda]$. We see from (47) that (45) follows from the $\mathbb{T}^n$-invariance of $\mathcal{F}$, and hence the proof is complete. □

Regarding the interpretation of Proposition 3, it is worth pointing out that one may view the restriction to $\mathcal{N}(n)$-invariant functions on $\mathbb{T}^n_{\rm reg} \times \mathfrak{b}(n)_0 \times \mathrm{B}(n)_+$ as the result of a two step process. The first step consists in Hamiltonian reduction of $\mathbb{T}^n_{\rm reg} \times \mathfrak{b}(n)_0 \times \mathrm{B}(n)$ by the normal subgroup $\mathbb{T}^n$. The formula (26) defines a Poisson bracket already on the $\mathbb{T}^n$-invariant functions. In fact, its last term can be identified as the result of reduction of the multiplicative Poisson bracket on $\mathrm{B}(n)$ by the conjugation action of $\mathbb{T}^n$, at the zero value of the pertinent moment map. In other words, the last term of (26) corresponds to the Poisson space $\mathrm{B}(n)//_0\mathbb{T}^n$. (Cf. Theorem 4.3 in [3].) The second step consists in taking quotient by $S_n = \mathcal{N}(n)/\mathbb{T}^n$.

When expressed in the variables (Q, p, λ), the Hamiltonian $\mathrm{tr}(L) = \mathrm{tr}(bb^\dagger) = \mathrm{tr}(e^{2p} b_+ b_+^\dagger)$ can be written as

$$\mathrm{tr}(L) = \sum_{i=1}^n e^{2p_i} V_i(Q, \lambda) \quad \text{with} \quad V_i(Q, \lambda) = \left(b_+(Q, \lambda) b_+(Q, \lambda)^\dagger\right)_{ii}, \tag{48}$$

where λ is a 'spin' variable, and $b_+(Q, \lambda)$ denotes the solution of the equation (21) for b_+. An explicit formula of $b_+(Q, \lambda)$ can be extracted from Section 5.2 in [3]. Comparison of (48) with the light-cone Hamiltonians of the standard RS model [11] justifies calling this a *spin Ruijsenaars type Hamiltonian*. A further justification is that restriction of the system to a one-point symplectic leaf in $\mathrm{B}(n)//_0\mathbb{T}^n$ yields the spinless trigonometric RS model [6].

3.2 Interpretation as Spin Sutherland Model

Concentrating on the first Poisson bracket (16), we present another set of useful variables

$$(Q, p, \phi) \in \mathbb{T}^n_{\rm reg} \times \mathfrak{H}(n)_0 \times \mathfrak{H}(n)_\perp, \tag{49}$$

where the subscripts 0 and $\perp$ refer to diagonal matrices and off-diagonal matrices, respectively. The relevant change of variables is encoded by the diffeomorphism

$$\gamma : \mathbb{T}^n_{\rm reg} \times \mathfrak{H}(n)_0 \times \mathfrak{H}(n)_\perp \to \mathbb{T}^n_{\rm reg} \times \mathfrak{H}(n) \tag{50}$$

operating according to

$$\gamma : (Q, p, \phi) \mapsto (Q, L(Q, p, \phi)) \quad \text{with} \quad L(Q, p, \phi) = p - (\mathcal{R}(Q) + \frac{1}{2}\mathrm{id})(\phi). \tag{51}$$

We now express the functions $f, h \in C^\infty(\mathbb{T}^n_{\rm reg} \times \mathfrak{H}(n))^{\mathcal{N}(n)}$ in the form

$$f \circ \gamma = \mathcal{F},\ \ h \circ \gamma = \mathcal{H}, \quad \mathcal{F}, \mathcal{H} \in C^\infty(\mathbb{T}^n_{\rm reg} \times \mathfrak{H}(n)_0 \times \mathfrak{H}(n)_\perp)^{\mathcal{N}(n)}, \tag{52}$$

where $\mathcal{N}(n)$ acts in the natural manner inherited from the conjugation action. The Poisson bracket $\{\ ,\ \}^{\rm red}_1$ on $C^\infty(\mathbb{T}^n_{\rm reg} \times \mathfrak{H}(n)_0 \times \mathfrak{H}(n)_\perp)^{\mathcal{N}(n)}$ is defined by the formula

$$\{\mathcal{F}, \mathcal{H}\}^{\rm red}_1 \equiv \{\mathcal{F} \circ \gamma^{-1}, \mathcal{H} \circ \gamma^{-1}\}^{\rm red}_1 \circ \gamma, \tag{53}$$

where (51) is used and the right-hand side refers to the Poisson bracket (16).

For any $\mathcal{F} \in C^\infty(\mathbb{T}^n_{\rm reg} \times \mathfrak{H}(n)_0 \times \mathfrak{H}(n)_\perp)$, we have the derivatives

$$D_Q\mathcal{F}(Q, p, \phi) \in \mathfrak{b}(n)_0, \quad d_p\mathcal{F}(Q, p, \phi) \in \mathfrak{u}(n)_0, \quad d_\phi\mathcal{F}(Q, p, \phi) \in \mathfrak{u}(n)_\perp, \tag{54}$$

defined by

$$\langle D_Q\mathcal{F}(Q, p, \phi), X\rangle + \langle d_p\mathcal{F}(Q, p, \phi), Y_0\rangle + \langle d_\phi\mathcal{F}(Q, p, \phi), Y_\perp\rangle = \frac{d}{dt}\bigg|_{t=0} \mathcal{F}(e^{tX}Q, p + tY_0, \phi + tY_\perp), \tag{55}$$

for every $X \in \mathfrak{u}(n)_0$ and $Y = (Y_0 + Y_\perp) \in \mathfrak{H}(n)$.

Proposition 4 ([5, 7]) *In terms of the variables (Q, p, ϕ) defined by (51), the reduced first Poisson bracket (16) has the following form:*

$$\{\mathcal{F}, \mathcal{H}\}^{\rm red}_1(Q, p, \phi) = \langle D_Q\mathcal{F}, d_p\mathcal{H}\rangle - \langle D_Q\mathcal{H}, d_p\mathcal{F}\rangle + \langle \phi, [d_\phi\mathcal{F}, d_\phi\mathcal{H}]\rangle. \tag{56}$$

Here, $\mathcal{F}, \mathcal{H} \in C^\infty(\mathbb{T}^n_{\rm reg} \times \mathfrak{H}(n)_0 \times \mathfrak{H}(n)_\perp)^{\mathcal{N}(n)}$ and the derivatives are taken at (Q, p, ϕ).

The change of variables $(Q, L) \leftrightarrow (Q, p, \phi)$ appeared in the construction of spin Sutherland models via the method of Li and Xu [7], whose relation to Hamiltonian reduction of free motion on Lie groups was clarified in [5]. The proof of Proposition 4 can be extracted from these references. One can also prove it by direct calculation, which is much simpler than the one required for the proof of Proposition 3.

The reduced Hamiltonians $\mathcal{H}^{\rm red}_k$ arising from those in (6) can be written in terms of the variables (Q, p, ϕ) as

$$\mathcal{H}^{\rm red}_k(Q, p, \phi) = \frac{1}{k}\,{\rm tr}(L(Q, p, \phi)^k). \tag{57}$$

For $k = 2$, with $Q = \exp(\mathrm{diag}(\mathrm{i}q_1, \dots, \mathrm{i}q_n))$, and $p = \mathrm{diag}(p_1, \dots, p_n)$ this gives

$$\mathcal{H}_2^{\mathrm{red}}(Q, p, \phi) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{8}\sum_{j\neq l} \frac{|\phi_{jl}|^2}{\sin^2 \frac{q_j - q_l}{2}}, \tag{58}$$

which is a standard spin Sutherland Hamiltonian. The last term in the Poisson bracket (56) represents the Poisson space $\mathfrak{u}(n)^*//_0\mathbb{T}^n$, and only gauge invariant functions of the spin variable ϕ appear in the model.

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