# Introduction to the Algebraic Bethe Ansatz


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**Abstract** We introduce the reader to the basic concepts of the Quantum Inverse Scattering Method and the algebraic Bethe ansatz. We describe a method for constructing integrable systems in this framework. In particular, we obtain the Hamiltonian of the $XXX$ Heisenberg spin chain by this method. We also describe a procedure for finding eigenvectors and the spectrum of quantum integrable Hamiltonians.

**Keywords** Quantum integrable systems · Transfer matrix · Eigenvector

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## 1 Introduction

This is a very short introduction to the method of the algebraic Bethe ansatz (ABA). It is based on the lectures [2]. Interested readers are referred to this course of lectures in which a much more detailed and extended description of the ABA can be found. There one also can find references to numerous publications devoted to this method. In the text below, we only formulate the main statements of the ABA. All the proofs can be found in [2].

The ABA is a part of the Quantum Inverse Scattering Method (QISM) developed in the works of the Leningrad School under the leadership of L. D. Faddeev [1]. This method allows one to effectively describe the spectrum of quantum integrable models. The main advantage of the ABA is that this method deals with a special operator algebra describing a rather wide class of quantum systems. Then different physical systems are nothing but different representations of this algebra.

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## 2 Construction of Integrable Systems

Here we describe a general procedure for constructing quantum integrable systems, which is used within the framework of the QISM. One of the main objects of the QISM is a *monodromy matrix*:

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{1}$$

It's matrix elements are operators. They depend on the complex variable $u$ and act in a Hilbert space $\mathcal{H}$. The latter is called *quantum space*, while the space in which $T(u)$ acts as a $2 \times 2$ matrix is called an *auxiliary space*.

The commutation relations between the monodromy matrix entries are given by an *RTT-relation*:

$$R(u,v)\big(T(u)\otimes I\big)\big(I\otimes T(v)\big) = \big(I\otimes T(v)\big)\big(T(u)\otimes I\big)R(u,v). \tag{2}$$

Here $R(u,v)$ is called an *R-matrix*. This is a $c$-number matrix, acting in the tensor product of the auxiliary spaces.

Let we have some $T(u)$ which satisfies Eq. (2). Then, multiplying (2) with $R^{-1}(u,v)$ from the right and taking the trace with respect to the auxiliary spaces we obtain

$$\mathcal{T}(u)\mathcal{T}(v) = \mathcal{T}(v)\mathcal{T}(u), \tag{3}$$

where

$$\mathcal{T}(u) = \operatorname{tr} T(u) = A(u) + D(u). \tag{4}$$

Let us expand the operator $\mathcal{T}(u)$ into power series over $u$ centered in some point $u_0$

$$\mathcal{T}(u) = \sum_{k=0}^{\infty} (u-u_0)^k I_k. \tag{5}$$

Here the coefficients $I_k$ are operators acting in the quantum space $\mathcal{H}$. Then it follows from (3) that all these operators commute

$$[I_k, I_n] = 0, \qquad \forall k, n. \tag{6}$$

If we now set one of $I_k$ to be the Hamiltonian of a quantum system, then we get a model with, generally speaking, an infinite set of integrals of motion, that is, an integrable system. This is the general scheme for constructing integrable models in the framework of the QISM.

The operator $\mathcal{T}(u) = \operatorname{tr} T(u)$ is called the *transfer matrix*. The main task of the QISM is to find the eigenvectors of the transfer matrix. These vectors are simultaneously common eigenvectors of the Hamiltonian of the corresponding quantum model and all integrals of motion.

## 2.1 Yang–Baxter Equation

One might have impression that $R(u, v)$ can be any invertible matrix. However, there are restrictions which follow from the compatibility condition. The $RTT$-relation allows us to permute two monodromy matrices. We can reorder three or more monodromy matrices using the $RTT$-relation successively. However, in this case, the result may depend on the order in which the permutations are made. It can be shown that the result is independent of the permutation order if the $R$-matrix satisfies a relation

$$R_{23}(u_2, u_3)R_{13}(u_1, u_3)R_{12}(u_1, u_2) = R_{12}(u_1, u_2)R_{13}(u_1, u_3)R_{23}(u_2, u_3)\,, \tag{7}$$

which is called *Yang–Baxter equation*. This equation takes place in the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. The subscripts show in which two of the free spaces the $R$-matrix acts non trivially.

One of the simplest nontrivial solutions to the Yang–Baxter equation (7) is given by

$$R(u, v) = (u - v)\mathbf{1} + cP, \tag{8}$$

where $\mathbf{1}$ is the identity matrix, $P$ is the permutation matrix, and $c$ is a constant.

## 2.2 Hilbert Space

Recall that the elements of the monodromy matrix act in the quantum space $\mathcal{H}$, which we have not yet discussed. In the framework of the QISM, very small requirements are imposed on this space. Namely, it is necessary that there exists a vacuum vector $|0\rangle \in \mathcal{H}$ such that

$$A(u)|0\rangle = a(u)|0\rangle, \qquad D(u)|0\rangle = d(u)|0\rangle, \qquad C(u)|0\rangle = 0. \tag{9}$$

Here $a(u)$ and $d(u)$ are some functions of $u$. Their explicit form depends on the specific model. The action of the operator $B(u)$ on the vacuum is free. It is assumed that acting with this operator on $|0\rangle$ we generate all the space $\mathcal{H}$.

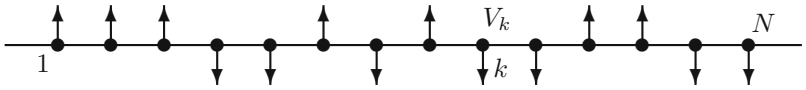


**Fig. 1** $XXX$ Heisenberg chain

## 3 $XXX$ Heisenberg Chain

As we have already mentioned, the main advantage of the ABA is that within this method various physical models can be treated as different representations of the same operator algebra. However, to clarify the main idea of this method, we consider an example of a specific quantum system.

Consider a chain of $N$ spin-$1/2$ particles (see. Fig. 1) and assume that the nearest neighbors in this chain interact with each other. For simplicity, we also assume that the strengths of the interaction along the axis $x$, $y$, and $z$ are the same. This model is called the $XXX$ Heisenberg chain (magnet).[1]

To describe this model at the mathematical language we first introduce a Hilbert space of the corresponding Hamiltonian. For this, we associate a *local quantum space* $V_k$ with $k$-th site of the chain. Each of these spaces is isomorphic to $\mathbb{C}^2$: $V_k \sim \mathbb{C}^2$. The whole quantum space $\mathcal{H}$ of the model is a tensor product of the local spaces $V_k$: $\mathcal{H} = V_1 \otimes \cdots \otimes V_N$.

The spin-$1/2$ operators are given by the standard Pauli matrices

$$\sigma^x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{10}$$

In the case under consideration, the Pauli matrices corresponding to the particle at the $k$-th site of the chain have an additional subscript: $\sigma_k^\alpha$, where $\alpha = x, y, z$. This means that $\sigma_k^\alpha$ acts non trivially in $V_k$ only. In other words,

$$\sigma_k^\alpha = \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{k-1} \otimes \sigma^\alpha \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{N-k}, \qquad \alpha = x, y, z. \tag{11}$$

Thus, these operators are the matrices of the size $2^N \times 2^N$.

Then the Hamiltonian of this quantum system can be written in the form

$$H = \sum_{k=1}^{N} \Big( \sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \sigma_k^z \sigma_{k+1}^z \Big), \tag{12}$$

[1] If the strengths of the interaction along different axis are different, then the model is called $XYZ$ Heisenberg chain.

where we assumed periodic boundary conditions $\sigma_{N+1}^{x,y,z} = \sigma_1^{x,y,z}$.

The standard problem of the quantum mechanics is to find the spectrum of the Hamiltonian. Since in our case the Hamiltonian is a matrix of the size $2^N \times 2^N$, we are dealing with linear algebra. Therefore, we could find the eigenvalues of $H$ by solving the characteristic equation

$$\det(H - \lambda \mathbf{1}) = 0. \tag{13}$$

Let us, however, assume that the chain under consideration is a model of a one-dimensional macroscopic crystal. This means that the number of sites of the chain $N$ is equal to the number of atoms in this macroscopic crystal, say, $N \sim 10^6$. Then the Hamiltonian is a matrix of the size $2^{10^6} \times 2^{10^6}$, and Eq. (13) is an algebraic equation of degree $2^{10^6}$. This number is so huge that there is nothing to compare it with. Clearly, no computer can cope with such a task. We come to the conclusion that the standard methods of the linear algebra in this case are useless, and, therefore, we need to look for some alternative ways. One of these ways is provided by the QISM.

## 4 $XXX$ Chain via QISM

The model of the $XXX$ Heisenberg chain can be formulated within the framework of the QISM. For this, we introduce a set of operator-valued matrices

$$L_k(u) = \begin{pmatrix} u + \frac{c}{2}\sigma_k^z & c\sigma_k^- \\ c\sigma_k^+ & u - \frac{c}{2}\sigma_k^z \end{pmatrix}, \qquad k = 1, \dots, N, \tag{14}$$

where $\sigma_k^{\pm} = \frac{1}{2}(\sigma_k^x \pm i\sigma_k^y)$. Then we define the monodromy matrix as follows:

$$T(u) = L_N(u) \dots L_2(u) L_1(u). \tag{15}$$

This matrix satisfies the $RTT$-relation (2) with the $R$-matrix (8). The corresponding transfer matrix $\mathcal{T}(u) = \operatorname{tr} T(u)$ generates a set of commuting operators. In particular, the Hamiltonian of the $XXX$ chain (12) can be written in the form

$$H = c \frac{d\mathcal{T}(u)}{du} \mathcal{T}^{-1}(u)\Big|_{u=\frac{c}{2}} - N, \tag{16}$$

where we omitted the identity operator at $N$ for brevity.

It is easy to check that a state with all spins up (ferromagnetic state)

$$|0\rangle = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)_1 \otimes \dots \otimes \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)_N \tag{17}$$

satisfies all the properties of the vacuum vector (9). The corresponding functions $a(u)$ and $d(u)$ have the form

$$a(u) = (u + \tfrac{c}{2})^N, \qquad d(u) = (u - \tfrac{c}{2})^N. \tag{18}$$

Thus, we have the complete description of the $XXX$ chain on the language of the QISM.

## 5 Eigenvectors of the Transfer Matrix

The ABA is a procedure for constructing the eigenvectors of the transfer matrix. We formulate this procedure within the abstract scheme described in Sect. 2. We only need an explicit formula for the $R$-matrix (8) and the action formulas (9), however, we do not use formulas (14) and (15) for the monodromy matrix.

The main idea of the ABA is that we look for the eigenvectors of the transfer matrix in the form

$$|\Psi(\bar{u})\rangle = \prod_{k=1}^{n} B(u_k)|0\rangle, \qquad n = 0, 1, \dots . \tag{19}$$

In general, the set $\bar{u} = \{u_1, \dots, u_n\}$ can contain an arbitrary number of elements, and the parameters $u_k$ can take arbitrary complex values.

We should act with the transfer matrix $\mathcal{T}(v) = A(v) + D(v)$ on the vector (19) and check whether $|\Psi(\bar{u})\rangle$ is an eigenvector of this operator. We have all necessary tools for this. First of all, the commutation relations between the monodromy matrix entries follow from the $RTT$-relation. We need only three of them:

$$[B(u), B(v)] = 0, \tag{20}$$

$$\begin{aligned} A(v)B(u) &= f(u, v)B(u)A(v) + g(v, u)B(v)A(u), \\ D(v)B(u) &= f(v, u)B(u)D(v) + g(u, v)B(v)D(u), \end{aligned} \tag{21}$$

where

$$f(u, v) = \frac{u - v + c}{u - v} \qquad \text{and} \qquad g(u, v) = \frac{c}{u - v}. \tag{22}$$

We shall call the first term in the rhs of (21) *the first commutation scheme* (when the operators keep their arguments). The term, in which the operators exchange their arguments, will be called *the second commutation scheme*.

Using these commutation relations we can move the operators $A$ and $D$ through the product of the operators $B$ to the extreme right position. After this, it remains to act with $A$ and $D$ on the vacuum vector via (9).

Let us compute the action of the operator $A(v)$ on the vector $|\Psi(\bar{u})\rangle$. When the operator $A$ is permuted with the operator $B$, it can either keep its argument, or borrow the argument of the operator $B$. As a result, we obtain

$$A(v)|\Psi(\bar{u})\rangle = a(v)\Lambda_0(v|\bar{u})|\Psi(\bar{u})\rangle + \sum_{k=1}^{n} a(u_k)\Lambda_k(v|\bar{u})|\Psi(\{v,\bar{u}_k\})\rangle. \tag{23}$$

Here $\Lambda_k(v|\bar{u})$ are some unknown coefficients to be determined and $\bar{u}_k = \bar{u} \setminus u_k$.

Let us compute the coefficient $\Lambda_0(v|\bar{u})$. Obviously, this coefficient arises in (23) if and only if we use only the first commutation scheme when permuting the operators $A(v)$ and $B(u_j)$. Then, after all permutations we obtain a product of the functions $f(u_j, v)$ over $u_j$, while the operator $A(v)$, approaching the extreme right position and acting on the vacuum, gives the function $a(v)$. As the result we have

$$\Lambda_0(v|\bar{u}) = \prod_{j=1}^{n} f(u_j, v). \tag{24}$$

Let us find now $\Lambda_k(v|\bar{u})$ with $k > 0$. Since, $|\Psi(\bar{u})\rangle$ is symmetric over the set $\bar{u}$ (due to (20)), it is enough to find $\Lambda_1(v|\bar{u})$. Then, in order to obtain a contribution proportional to $a(u_1)$, it is necessary to use the second commutation scheme when permuting $A(v)$ and $B(u_1)$. After this, when moving further to the right, $A(u_1)$ must keep its argument, therefore, we should use the first commutation scheme only. Thus, we arrive at

$$\Lambda_1(v|\bar{u}) = g(v, u_1)\prod_{j=2}^{n} f(u_j, u_1), \tag{25}$$

leading to

$$\Lambda_k(v|\bar{u}) = g(v, u_k)\prod_{\substack{j=1\\ j\neq k}}^{n} f(u_j, u_k). \tag{26}$$

The action of the $D$-operator can be computed exactly in the same way, and we find

$$\begin{aligned}\mathcal{T}(v)|\Psi(\bar{u})\rangle = &\left(a(v)\prod_{j=1}^{n} f(u_j, v) + d(v)\prod_{j=1}^{n} f(v, u_j)\right)|\Psi(\bar{u})\rangle \\ &+ \sum_{k=1}^{n} g(v, u_k)\left(a(u_k)\prod_{\substack{j=1\\ j\neq k}}^{n} f(u_j, u_k) - d(u_k)\prod_{\substack{j=1\\ j\neq k}}^{n} f(u_k, u_j)\right)|\Psi(\{v,\bar{u}_k\})\rangle.\end{aligned} \tag{27}$$

We see that generically $|\Psi(\bar{u})\rangle$ is not an eigenvector of the transfer matrix because of the presence of new vectors $|\Psi(\{v,\bar{u}_k\})\rangle$ in the second line of (27). However, if we impose conditions

$$\frac{a(u_k)}{d(u_k)}=\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{f(u_k,u_j)}{f(u_j,u_k)},\qquad k=1,\dots,n, \tag{28}$$

then the sum in the second line of (27) vanishes. Then the vector $|\Psi(\bar{u})\rangle$ becomes an eigenvector of the operator $\mathcal{T}(v)$ with an eigenvalue

$$\tau(v|\bar{u})=a(v)\prod_{j=1}^{n}f(u_j,v)+d(v)\prod_{j=1}^{n}f(v,u_j). \tag{29}$$

The system (28) is called the *system of Bethe equations*.

Thus, the transfer matrix eigenvectors have the form (19), where the parameters $\bar{u}=\{u_1,\dots,u_n\}$ are the roots of Bethe equations (28).

Applying this scheme to the $XXX$ Heisenberg chain we should only substitute the functions $a(u)$ and $d(u)$ (18) into the Bethe equations

$$\left(\frac{u_k+\frac{c}{2}}{u_k-\frac{c}{2}}\right)^N=\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{f(u_k,u_j)}{f(u_j,u_k)}=-\prod_{j=1}^{n}\frac{u_k-u_j+c}{u_k-u_j-c},\qquad k=1,\dots,n. \tag{30}$$

The Hamiltonian eigenvalues then can be found via (16)

$$E=c\frac{d\log\big(\tau(v|\bar{u})\big)}{dv}\Big|_{v=\frac{c}{2}}-N=\frac{1}{2}\sum_{j=1}^{n}\frac{c^2}{u_j^2-c^2/4}. \tag{31}$$

Thus, the spectrum of the $XXX$ Heisenberg chain is completely determined by the solutions of the Bethe equations (30). Of course, the Bethe equations admit an exact analytic solution only in exceptional cases. This is to be expected, because having an exact solution to the system (30) would mean that we were able to find an exact solution to the algebraic equation (13). It would be naive to hope so. However, the system (30) is easily amenable to numerical analysis even at $n$ and $N$ large enough. Furthermore, in the limit of the chain of large length, the Bethe equations can be replaced by a linear integral equation for the density of roots, which admits an explicit solution.

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