Fock Quantization of Canonical Transformations and Semiclassical Asymptotics for Degenerate Problems

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Abstract The aim of this work is to explain the role played by the Fock quantization of canonical transformations in the construction of the global semiclassical (high-frequency) asymptotic approximation. This role may well pass unnoticed as long as one deals with nondegenerate differential equations. However, the situation is different for some classes of equations with degeneration, where the Fock quantization of canonical transformations becomes instrumental in the construction of asymptotic solutions.

Keywords Semiclassical asymptotics · Canonical transformation · Quantization · Degenerate equation · Maslov's canonical operator

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1 Introduction

Maslov's canonical operator [14, 15] is a powerful tool for constructing global semiclassical asymptotics of solutions of differential equations with a small parameter multiplying the derivatives. The asymptotic solutions produced by this operator have the form of sums of *WKB elements*[1] in coordinate and momentum representations, with the $1/h$-Fourier transform $\mathcal{F}^{1/h}_{p\to x}$ applied to the latter to make them functions

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[1] See Sect. 4 for more details.

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of the coordinate rather than the momentum.[2] The operator $\mathcal{F}^{1/h}_{p\to x}$ is actually the Fock quantization of the rotation by an angle of $\pi/2$ in the phase plane; however, this, in a sense, works behind the scenes, and one may not know that but still apply the canonical operator successfully to problems with nondegenerate characteristics. Things become more difficult when one deals with degenerate operators to which the standard scheme of the canonical operator does not apply. In that case, to desingularize the problem, one may need more complicated canonical transformations than mere rotations by $\pi/2$, and then the Fock quantization rule gives the right recipe of what to do with the WKB elements arising in the new variables and how to construct a modified canonical operator suitable for the degeneration in question.

This is exactly what happens for the class of operators with boundary degeneration arising in the linear theory of run-up of long waves on a shallow beach [19, 21]. The theory of global semiclassical asymptotics for this class of problems has been developed in the recent years by the authors and their colleagues [1, 2, 5–10, 16, 17]. The aim of the present note is to explain how the Fock quantization of canonical transformations enters the construction of semiclassical asymptotics. As an example, we use the simplest problem of this class in dimension 1, that is, a problem for an ordinary differential equation (ODE).

2 Degenerate Boundary Value Problem

Let $D(x) \in C^\infty([-1,1])$ be a function such that $D(x) > 0$ for $x \in (-1,1)$, $D(-1) = D(1) = 0$, $D'(-1) > 0$, and $D'(1) < 0$. Further, consider the operator

$$L_0 = -\frac{d}{dx}D(x)\frac{d}{dx} \quad \text{with domain} \quad \mathcal{D}(L_0) = C_0^\infty((-1,1))$$

in the space $L^2([-1,1])$. The operator L_0 degenerates at the endpoints of the interval $(-1,1)$, and hence one cannot define any self-adjoint extensions of L_0 with the use of classical boundary conditions such as the Dirichlet or Neumann conditions [18]. Thus, one has to use "generalized boundary conditions." Define the operator L in $L^2([-1,1])$ as the Friedrichs extension [3, Sec. 10.3] of L_0, which is equivalent to the finiteness of the energy integral [22, Sec. 33.1]. Consider the *eigenvalue problem*

$$L\eta = \lambda\eta, \tag{1}$$

[2]For simplicity, we only deal here with the case of one spatial variable x (i.e., $x \in \mathbb{R}^1$); if $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, then the construction also involves partial Fourier transforms (Fourier transforms with respect to part of the variables).

which naturally arises in the approximation given by the linearized shallow water equations as the one-dimensional model of harmonic water waves (such as seiches) in a basin of variable depth $D(x)$. Here $\eta(x)e^{i\omega t}$, $\omega = \sqrt{\lambda}$, has the meaning of the free surface elevation at the point x at time t. The motion of water is assumed to be potential, and we use a system of units in which the acceleration due to gravity has the value $g = 1$.

We will be interested in the behavior of solutions of this eigenvalue problem with large λ. One defines an *asymptotic series* of solutions as a sequence $\lambda_n \to \infty$ of numbers (called *asymptotic eigenvalues*) and a sequence of functions $\eta_n \in \mathcal{D}(L)$ such that $\|\eta_n\| \ge C > 0$ (where the norm is taken in $L^2([-1, 1])$) and these functions are *almost eigenfunctions* in the sense that $\|L\eta_n - \lambda_n\eta_n\| = O(1)$ as $n \to \infty$. By the well-known estimates for the resolvent of a self-adjoint operator, an asymptotic series satisfies the relation $\operatorname{dist}(\lambda_n, \sigma(L)) = O(1)$, where $\sigma(L)$ is the spectrum of L, and has other useful properties.

Equation (1) is an ODE with singular points, and there is a vast literature concerning the theory of such equations (e.g., see the books by Fedoryuk [11] and Slavyanov [20] and references therein). Needless to say, problem (1) can be solved by methods of that theory; for example, one can use the method of standard equations with the Bessel equation serving as a standard equation (see [6, Sec. 2]). However, these methods have a drawback in that they cannot be transferred to the multidimensional case automatically; for us, Eq. (1) only serves as a simple example, and we will use an approach is free from this drawback. This approach is based on the geometry of the characteristics of the problem and extends Maslov's canonical operator.

3 Quantization of Canonical Transformations

The idea of quantization of canonical transformations is apparently due to Dirac, who wrote [4, Sec. 26]:

> ... for a quantum dynamic system that has a classical analogue, unitary transformation in the quantum theory is the analogue of contact transformation in the classical theory.

The definition of quantization of canonical transformations was given by Fock [12]. Since then, there have been an extensive literature on the topic. In particular, a comprehensive theory including global aspects and featuring far-reaching generalizations was developed by Karasev and Maslov [13]. We will need the simplest local version essentially defined by Fock himself. In this paper, we restrict ourselves to the one-dimensional case. Consider a canonical transformation $g\colon \mathbb{R}^2_{(x,p)} \to \mathbb{R}^2_{(y,q)}$. The quantized canonical transformation is given by

$$T(g)\colon L^2(\mathbb{R}_y) \longrightarrow L^2(\mathbb{R}_x), \qquad [T(g)u](x) = \int K_g(x, y)u(y)\,dy,$$

where the kernel K_g depends on the small parameter $h > 0$ and is defined via the *generating function* of g as follows.

1. If g is defined by a generating function $\Phi(x, y)$ by the formulas $q = -\Phi_y(x, y)$, $p = \Phi_x(x, y)$, where, by definition, $\Phi''_{xy}(x, y) \neq 0$, then the kernel is given by

$$K_g(x, y) = \left(\frac{-i}{2\pi h}\right)^{1/2} e^{\frac{i}{h}\Phi(x,y)} \sqrt{\Phi''_{xy}(x, y)}, \qquad \arg i = \frac{\pi}{2}.$$

2. If g is defined by a generating function $\Phi(x, q)$ by the formulas $y = \Phi_q(x, q)$, $p = \Phi\ x(x, q)$, where, by definition, $\Phi''_{xq}(x, q) \neq 0$, then

$$K_g(x, y) = \frac{1}{2\pi h} \int e^{\frac{i}{h}(\Phi(x,q)-yq} \sqrt{\Phi''_{xq}(x, q)}\, dq.$$

The choice of the argument of the radicand is irrelevant to our discussion.

Let us present two examples.

1. Let $\Phi(x, y) = -xy$, so that $p = \Phi_x = -y$, $q = -\Phi_y = x$, and the transformation is the counterclockwise rotation by $\pi/2$. Then the quantized transformation has the kernel

$$K_g(x, y) = \left(\frac{-i}{2\pi h}\right)^{1/2} e^{-\frac{i}{h}xy}, \quad [T(g)u](x) = \left(\frac{-i}{2\pi h}\right)^{1/2} \int e^{-\frac{i}{h}xy} u(y)\, dy;$$

   thus, $T(g) = \mathcal{F}^{1/h}_{y\to x}$ is the $1/h$-Fourier transform.
2. Now let $\Phi(x, q) = qf(x)$ (where $f'(x) \neq 0$); then $y = f(x)$, $p = (f'(x))^{-1}q$ is the classical canonical transformation associated with a change of variables. The kernel has the form

$$K_g(x, y) = \frac{\sqrt{f'(x)}}{2\pi h} \int e^{\frac{i}{h}q(f(x)-y)}\, dq = \sqrt{f'(x)}\delta(y - f(x)),$$

   and the transformation $T(g)$ itself is the same change of variables in a function followed by the multiplication by a factor ensuring the unitarity of $T(g)$ in L^2.

4 Semiclassical Asymptotics

The semiclassical theory deals with equations of the form $\widehat{H}u = 0$, where $\widehat{H} = H(x, \widehat{p})$, $\widehat{p} = -ih\frac{\partial}{\partial x}$, is a differential operator with a small parameter $h > 0$ multiplying the derivatives. Semiclassical asymptotic theory provides rapidly oscillating asymptotic solutions of the equation $\widehat{H}u = 0$ as $h \to 0$. Let us recall the standard construction of the canonical operator [14, 15], again sticking to the case of $n = 1$. To define the canonical operator, we need a Lagrangian manifold $\Lambda \subset \mathbb{R}^2_{(x,p)}$

with a smooth measure $d\mu$ (volume form) on it. The canonical operator K_Λ^h takes smooth functions on Λ to rapidly oscillating functions on $\mathbb{R}_x$. The manifold Λ must be compact (or at least the projection $\Lambda \to \mathbb{R}_x$ must be proper).

The function $K_\Lambda^h \phi$ is pasted together from local elements corresponding to parts of Λ with "good" projection onto one of the coordinate axes. There can be two possible cases:

(i) Assume that the projection of $\operatorname{supp}\phi \subset \Lambda$ onto the x-axis is good. Then $[K_\Lambda^h \phi](x)$ is the WKB element

$$[K_\Lambda^h \phi](x) = \exp\left(\frac{iS(x)}{h}\right)\phi(x)\left(\frac{d\mu}{dx}\right)^{1/2}, \quad \text{where } \Lambda = \left\{p = \frac{\partial S}{\partial x}(x)\right\}.$$

(ii) Assume that the projection of $\operatorname{supp}\phi \subset \Lambda$ onto the p-axis is good. Then we can in a similar way define the WKB element

$$\exp\left(\frac{i\widetilde{S}(p)}{h}\right)\phi(p)\left(\frac{d\mu}{dp}\right)^{1/2}, \quad \text{where } \Lambda = \left\{x = -\frac{\partial \widetilde{S}}{\partial p}(p)\right\},$$

but we cannot make it the value of the canonical operator, because it depends on the wrong variable! To obtain a function of x, we transpose the axes by rotating the picture by an angle of $\pi/2$. The Fock quantization of this rotation gives the Fourier transform, and we obtain

$$[K_\Lambda^h \phi](x) = \left(\frac{i}{2\pi h}\right)^{1/2} \int \exp\left(\frac{i(\widetilde{S}(p) + px)}{h}\right)\phi(p)\left(\frac{d\mu}{dp}\right)^{1/2} dp.$$

Now, to define $K_\Lambda^h \phi$ for an arbitrary compactly supported smooth function ϕ on Λ, one uses a partition of unity to split ϕ into a sum of terms each of which can be treated with the use of (i) or (ii). The consistency of (i) and (ii) in case they both apply is ensured by additional unimodular factors; in turn, these can be chosen consistently if Λ satisfies the quantization conditions (see [14, 15]).

5 Solution of the Degenerate Problem

5.1 Geometric Construction

We rewrite problem (1) in the semiclassical form

$$\widehat{H}\eta = \eta, \qquad \widehat{H} = \widehat{p}D(x)\widehat{p},$$

with Hamiltonian $H(x, p) = D(x)p^2$. The semiclassical asymptotics is associated with a Lagrangian manifold Λ_0 contained in the set $\{(x, p)\colon H(x, p) = 1\}$. In the one-dimensional case, this set is a curve, and the Lagrangian manifold necessarily coincides with it. The difficulty is that the Lagrangian manifold is singular (namely, the projection onto the base is improper). The solution is to extend the phase space.

The geometric construction was suggested in [16] based on the idea in [23] that one should proceed from the momentum variable p to its reciprocal, $1/p$. The natural next step (which however was not made in [23]) is to accompany this transformation with a transformation of the variable x so as to obtain a canonical transformation. This was done in [16]. The desired change of variables in the phase space $T^*((-1, 1))$ over a neighborhood of the left end $x = -1$ of the interval $(-1, 1)$ has the form

$$\theta = p^2(x+1), \quad q = -\frac{1}{p} \qquad \Leftrightarrow \qquad x = q^2\theta - 1, \qquad p = -\frac{1}{q}. \tag{2}$$

This transformation is canonical, $dp \wedge dx = dq \wedge d\theta$. We add the open half-line $\{q = 0, \theta > 0\}$ to this chart of the phase space in the new coordinates and carry out a similar construction near the right end $x = 1$. The resulting new phase space Φ is diffeomorphic to a plane with two deleted points, $\Phi \simeq R^2 \setminus \{(-1, 0), (1, 0)\}$. The closure Λ of the manifold Λ_0 in the phase space Φ is obtained by the addition of two points; it is a smooth Lagrangian manifold diffeomorphic to a circle. To construct asymptotic eigenfunctions, we must define the canonical operator on Λ in the vicinity of the newly added points.

5.2 Modified Canonical Operator

Consider a neighborhood of a point in $\Lambda \setminus \Lambda_0$. This point is projected into one of the endpoints of $[-1, 1]$ and is defined by the equation $q = 0$ in the corresponding new coordinates. Thus, the endpoints are a special kind of caustic. To define the canonical operator near these points, we use the same idea as earlier for the "standard" canonical operator. Namely, we write a WKB element that is a function of q and then define a function of the variable x by applying the Fock quantized canonical transformation corresponding to the classical canonical transformation (2). To be definite, consider a neighborhood of the left endpoint $x = -1$. Then the canonical transformation (2) can be defined by the generating function $\Phi(x, q) = -x/q$, and accordingly the quantized canonical transformation is

$$[T(g)u](x) = \int K(x, \theta)u(\theta)\, d\theta,$$

where

$$K(x,\theta)=\frac{1}{2\pi h}\int_{-\infty}^{\infty}e^{-\frac{i}{h}\left(\frac{x}{q}+\theta q\right)}\frac{dq}{q}=-\frac{i}{h}J_0\Big(\frac{2\sqrt{x\theta}}{h}\Big)$$

and $J_0(z)$ is the Bessel function of the first kind and zero order. Thus, we have the Hankel transform instead of the usual Fourier transform in the definition of the canonical operator. In other words, the canonical operator in a neighborhood of the boundary point acts as an application of the Hankel transform (composed with the Fourier transform) to a WKB element. The corresponding integral formulas can be found in [17]; the kernels of these integrals are products of $K(x,\theta)$ by certain rapidly oscillating exponentials. Computing these Bessel type integrals according to [5], we arrive at the form of the modified canonical operator given in [1]. In the one-dimensional case, these formulas do not contain any integrals and hence express the asymptotic solution in closed form. We refer the reader for the general formulas to [1, 6] and restrict ourselves in the present paper to the solution formulas for our specific problem.

5.3 Formulas for the Asymptotic Eigenfunctions

The final answer in problem (1) reads [6, Eq. (1.6)]

$$\eta_n(x)\asymp\begin{cases}\sqrt{2\pi\omega_n}\,J_0(\omega_n S(-1,x))\Big(\dfrac{S(-1,x)}{c(x)}\Big)^{1/2}, & x\in[-1,1-\varepsilon],\\ (-1)^n\sqrt{2\pi\omega_n}\,J_0(\omega_n S(x,1))\Big(\dfrac{S(x,1)}{c(x)}\Big)^{1/2}, & x\in[-1+\varepsilon,1],\end{cases}$$

where $\varepsilon>0$ is fixed,

$$c(x)=\sqrt{D(x)},\qquad S(x_0,x)=\int_{x_0}^{x}\frac{d\xi}{c(\xi)},\qquad -1\le x_0,x\le 1,$$

and

$$\omega_n=\frac{\pi}{S(-1,1)}\Big(n+\frac{1}{2}\Big),\qquad n=1,2,\ldots,$$

are the asymptotic eigenvalues of the problem.

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