One Step Degeneration of Trigonal Curves and Mixing of Solitons and Quasi-Periodic Solutions of the KP Equation

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To the memory of Victor Enolski

Abstract We consider certain degenerations of trigonal curves and hyperelliptic curves, which we call one step degeneration. We compute the limits of corresponding quasi-periodic solutions using the Sato Grassmannian. The mixing of solitons and quasi-periodic solutions is clearly visible in the obtained solutions.

Keywords KP equation · Soliton · Quasi-periodic solution · Sato Grassmannian · Trigonal curve

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1 Introduction

The aim of this paper is to compute explicitly the limits of quasi-periodic solutions of the KP (Kadomtsev-Petviashvili) equation according to certain degenerations of trigonal and hyperelliptic curves, which we call one step degeneration.

The KP equation is the 2 + 1 dimensional equation given by

$$3u_{t_2t_2} + (-4u_{t_3} + 6uu_{t_1} + u_{t_1t_1t_1})_{t_1} = 0, \tag{1}$$

where (t_1, t_2) and t_3 are space and time variables respectively. It can be rewritten in the Hirota bilinear form:

$$(D_{t_1}^4 - 4D_{t_2}D_{t_3} + 3D_{t_2}^2)\tau \cdot \tau = 0, \tag{2}$$

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where D_{t_i}'s are Hirota derivatives defined by

$$f(t+s)g(t-s)=\sum_{n=0}^{\infty} D_t^n f\cdot g\frac{s^n}{n!}.$$

For a solution τ of (2) $u=2(\log\tau)_{t_1t_1}$ gives a solution of (1). The KP hierarchy is the infinite system of differential equations which contains the KP equation (2) as its first member [6]. It is given by

$$\int \tau(t-s-[z^{-1}])\tau(t+s+[z^{-1}])\mathrm{e}^{-2\sum_{j=1}^{\infty} s_j z^j}\frac{dz}{2\pi i}=0, \tag{3}$$

where $t=(t_1,t_2,\ldots)$, $s=(s_1,s_2,\ldots)$, $[z^{-1}]=[z^{-1},z^{-2}/2,z^{-3}/3,\ldots]$ and the integral signifies taking the coefficient of z^{-1} in the series expansion of the integrand. Expanding (3) by s we get differential equations for $\tau(t)$ in the Hirota bilinear form. A solution $\tau(t)$ is sometimes called a tau function. The introduction of the infinitely many variables is indispensable to the Sato theory which we use in this paper.

The KP hierarchy has a variety of solutions. Among them soliton solutions and algebro-geometric solutions are relevant to us. Soliton solutions are the solutions expressed by exponential functions given as follows (see [11] for example). Take positive integers $N<M$, non-zero parameters κ_j, $1\leq j\leq M$ and an $N\times M$-matrix $A=(a_{i,j})$. Then soliton solution is given by

$$\tau(t)=\sum_{I=(i_1<\cdots<i_N)}\Delta_I A_I \mathrm{e}^{\eta(\kappa_{i_1})+\cdots+\eta(\kappa_{i_M})}, \tag{4}$$

$$\Delta_I=\prod_{p<q}(\kappa_{i_q}-\kappa_{i_p}),\quad A_I=\det(a_{p,i_q})_{1\leq p,q\leq N},\quad \eta(\kappa)=\sum_{i=1}^{\infty} t_i\kappa^i.$$

Recently it was discovered that the shapes of soliton solutions form various web patterns and that they are related with the geometry of Grassmann manifolds, cluster algebras (see [11] and references therein).

Quasi-periodic solutions, which is also called algebro-geometric solutions, constitute a class of solutions expressed by theta functions of algebraic curves with positive genus. Periodic solutions are contained in this class. Soliton solutions can be considered as the limits of quasi-periodic solutions when periods go to infinity. In terms of curves soliton solutions are the genus zero limits of quasi-periodic solutions. Our original motivation of the research was to take these limits and compare the structure of the quasi-periodic solutions and that of solitons described in [11]. However in the course of study [2] we come to the recognition that the limits to positive genus solutions are more fundamental. Anyhow the difficulty here is that to take a limit of a theta function or, in other words, to take a limit of the period matrix of an algebraic curve, is not very easy.

In [2, 18, 19] we have demonstrated that the Sato Grassmannian (UGM) approach to this kind of problem is very effective. The reason, roughly speaking, is explained as follows. There is a one to one correspondence between points of UGM and solutions of the KP-hierarchy up to constants. Using UGM an algebro-geometric solution can be described as a series whose coefficients are constructed from some rational functions on an algebraic curve. In this way the difficult problem on taking limits of period matrices reduces to much easier problem on taking limits of rational functions. In this paper we develop the UGM approach further.

We consider the following degeneration of algebraic curves, which we call one step degeneration, given by

$$y^m = \prod_{j=1}^{mn+1} (x-\alpha_j) \longrightarrow y^m = (x-\alpha)^m \prod_{j=1}^{m(n-1)+1} (x-\alpha_j), \tag{5}$$

for $m = 2, 3$. Fix m and denote by C_n the non-singular curve before taking the limit. We define some canonical tau function $\tau_{n,0}(t)$ (see (35)) corresponding to the curve C_n. Then we express the limit of $\tau_{n,0}(t)$ in terms of $\tau_{n-1,0}(t)$ with the variable t being appropriately shifted. Then a solitonic structure can be seen clearly in the degeneration of the algebro-geometric solution $\tau_{n,0}(t)$. This is another crucial idea in this paper.

The results are as follows. For $m = 2$, that is, the case of a hyperelliptic curve, we have (Theorem 24),

$$\lim \tau_{n,0}(t) = C\mathrm{e}^{-2\sum_{l=1}^{\infty} \alpha^l t_{2l}}$$
$$\times \left(\mathrm{e}^{\eta(\alpha^{1/2})}\tau_{n-1,0}(t-[\alpha^{-1/2}]) + (-1)^n \mathrm{e}^{\eta(-\alpha^{1/2})}\tau_{n-1,0}(t-[-\alpha^{-1/2}])\right), \tag{6}$$

for some constant C. It is observed that the soliton factors $\mathrm{e}^{\eta(\pm\alpha^{1/2})}$ pop out from $\tau_{n,0}(t)$. Then the solution (6) looks like a mixture of solitons and quasi-periodic solutions. Using the formula repeatedly and noting that $\tau_{0,0}(t) = 1$ if $\alpha_1 = 0$ we get well known soliton solutions of the KdV equation.

For $m = 3$ we have (Theorem 18)

$$\lim \tau_{n,0}(t) = \mathrm{e}^{-6\sum_{l=1}^{\infty} \alpha^l t_{3l}}$$
$$\times \sum_{0\le i<j\le 2, 0\le k\le 2} \frac{\partial}{\partial\beta}\Big(\tilde{C}_{i,j,k}(\alpha,\beta)\mathrm{e}^{\eta(z_i(\alpha)^{-1})+\eta(z_j(\alpha)^{-1})+\eta(z_k(\beta)^{-1})}$$
$$\times\, \tau_{n-1,0}(t-[z_i(\alpha)]-[z_j(\alpha)]-[z_k(\beta)])\Big)\Big|_{\beta=\alpha},$$
$$z_i(\alpha) = \omega^{-i}\alpha^{-1/3}, \quad \omega = \mathrm{e}^{2\pi i/3}, \tag{7}$$

for some constants $\tilde{C}_{i,j,k}(\alpha, \beta)$. A new feature in this case is the appearance of the derivative with respect to the parameter β. This corresponds to the fact that the limit of $\tau_{n,0}$ to genus zero curve in this case is not a soliton but a generalized soliton [18]. The constants $\tilde{C}_{i,j,k}(\alpha, \beta)$ should be expressed by some derivatives of the sigma function. The explicit formulas for them are important for the further analysis of the solutions.

We remark that the formula of the forms (6), (7) can be generalized for $m \geq 4$ in (5). They should be treated in a subsequent papers. A generalization of the results in this paper to other class of curves such as that treated in [1] is also interesting.

The paper is organized as follows. In Sect. 2 we first review the theory of the Sato Grassmannian (UGM). Then we explain how to embed the space of functions on an algebraic curve to UGM. Next we apply the general theory to our concrete examples and define the frame $\tilde{\xi}_n$ of a point of UGM corresponding to the space of regular rational functions on $C_n \backslash \{\infty\}$. Then we study the degeneration of $\tilde{\xi}_n$ and define the frame ξ_n as a gauge transformation of $\tilde{\xi}_n$. In order to express ξ_n by an object associated with the curve C_{n-1} we study the frame associated with the space of rational functions on $C_{n-1} \backslash \{\infty\}$ which are singular at three points. Decomposing some rational functions we derive the degeneration formula of the tau function $\tau(t; \tilde{\xi}_n)$ corresponding to $\tilde{\xi}_n$ in terms of some tau functions associated with the curve C_{n-1} in the final subsection of Sect. 2. In Sect. 3 we first review the sigma function of a so called (N, M) curve. Then we recall the sigma function expression of $\tau(t; \tilde{\xi}_n)$. Next we express the tau function corresponding to the space of functions with additional singularities as a shift of $\tau(t; \tilde{\xi}_n)$. By substituting these formulas to the degeneration formula derived in Sect. 2 we express the limit of $\tau(t; \tilde{\xi}_n)$ in terms of the shift of $\tau(t; \tilde{\xi}_{n-1})$. In Sect. 4 we derive a similar degeneration formula for hyperelliptic curves based on the results of [2].

2 Sato Grassmannian and τ-Function

In this section we briefly recall the definition and basic properties of the Sato Grassmannian.

2.1 Sato Grassmannian

Let $V = \mathbb{C}((z))$ be the vector space of Laurent series in the variable z and $V_\phi = \mathbb{C}[z^{-1}]$, $V_0 = z\mathbb{C}[[z]]$ two subspaces of V. Then V is isomorphic to $V_\phi \oplus V_0$. Let $\pi : V \longrightarrow V_\phi$ be the projection map. Then the Sato Grassmannian UGM is defined as the set of subspaces U of V such that the restriction $\pi|_U$ has the finite dimensional kernel and cokernel whose dimensions coincide.

To an element $\sum a_n z^n \in V$ we associate the infinite column vector $(a_n)_{n\in\mathbb{Z}}$. Then a frame of a point U of UGM is expressed by a $\mathbb{Z}\times\mathbb{N}_{\le 0}$ matrix $\xi = (\xi_{i,j})_{i\in\mathbb{Z}, j\in\mathbb{N}_{\le 0}}$, where columns, and therefore a basis of U, are labeled by the set of non-positive integers $\mathbb{N}_{\le 0}$. A frame ξ is written in the form

$$\xi = \begin{pmatrix} & \vdots & \vdots \\ \cdots & \xi_{-1,-1} & \xi_{-1,0} \\ \cdots & \xi_{0,-1} & \xi_{0,0} \\ - - - & - - - - & - - - \\ \cdots & \xi_{1,-1} & \xi_{1,0} \\ \cdots & \xi_{2,-1} & \xi_{2,0} \\ & \vdots & \vdots \end{pmatrix} \tag{8}$$

It is always possible to take a frame satisfying the following condition, there exists a negative integer l such that

$$\xi_{i,j} = \begin{cases} 1 & \text{if } j < l \text{ and } i = j \\ 0 & \text{if } (j < l \text{ and } i < j) \text{ or } (j \ge l \text{ and } i < l). \end{cases} \tag{9}$$

In the sequel we always take a frame which satisfies this condition, although it is not unique.

A Maya diagram $M = (m_j)_{j=0}^{\infty}$ is a sequence of decreasing integers such that $m_j = -j$ for all sufficiently large j. For a Maya diagram $M = (m_j)_{j=0}^{\infty}$ the corresponding partition is defined by $\lambda(M) = (j + m_j)_{j=0}^{\infty}$. By this correspondence the set of Maya diagrams and the set of partitions bijectively correspond to each other.

For a frame ξ and a Maya diagram $M = (m_j)_{j=0}^{\infty}$ define the Plücker coordinate by

$$\xi_M = \det(\xi_{m_i,j})_{-i,j\le 0}$$

Due to the condition (9) and the condition of the Maya diagram M this infinite determinant can be computed as the finite determinant $\det(\xi_{m_i,j})_{k\le -i,j\le 0}$ for sufficiently small k.

Define the elementary Schur function $p_n(t)$ by

$$\mathrm{e}^{\sum_{n=1}^{\infty} t_n \kappa^n} = \sum_{n=0}^{\infty} p_n(t)\kappa^n.$$

The Schur function [13] corresponding to a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ is defined by

$$s_\lambda(t) = \det(p_{\lambda_i - i + j}(t))_{1\le i,j\le l}.$$

Assign the weight j to the variable t_j. Then it is known that $s_\lambda(t)$ is homogeneous of weight $|\lambda| = \lambda_1 + \cdots + \lambda_l$. To a point U of UGM take a frame ξ and define the tau function by

$$\tau(t;\xi) = \sum_M \xi_M s_{\lambda(M)}(t). \tag{10}$$

If we change the frame ξ $\tau(t;\xi)$ is multiplied by a constant. We call $\tau(t;\xi)$, for any frame ξ of U, a tau function corresponding to U. So tau functions of a point of UGM differ by constant multiples to each other.

Then

Theorem 1 ([24]) *The tau function $\tau(t;\xi)$ is a solution of the KP-hierarchy. Conversely for a formal power series solution $\tau(t)$ of the KP-hierarchy there exists a point U of UGM such that $\tau(t)$ coincides with a tau function of U.*

The point U of UGM corresponding to a solution $\tau(t)$ in Theorem 1 is given as follows [10, 16, 23, 24].

Let $\Psi^*(t;z)$ be the adjoint wave function [6] corresponding to $\tau(t)$ which is defined by

$$\Psi^*(t;z) = \frac{\tau(t+[z])}{\tau(t)} e^{-\sum_{i=1}^\infty t_i z^{-i}}. \tag{11}$$

Define $\Psi_i^*(z)$ by the following expansion

$$\left(\tau(t)\Psi^*(t;z)\right)|_{t=(x,0,0,0,\ldots)} = \tau((x,0,0,0,\ldots)+[z])e^{-xz^{-1}} = \sum_{i=0}^\infty \Psi_i^*(z)x^i. \tag{12}$$

Then

$$U = \sum_{i=0}^\infty \mathbb{C}\Psi_i^*(z). \tag{13}$$

By this correspondence between points of UGM and tau functions the following property follows. Let U be a point of UGM, $\tau(t)$ be a tau function corresponding to U and $f(z) = e^{\sum_{i=1}^\infty a_i \frac{z^i}{i}}$ be an invertible formal power series. Then $f(z)U$ belongs to UGM and the corresponding tau function is given by

$$e^{\sum_{i=1}^\infty a_i t_i}\tau(t). \tag{14}$$

It is sometimes called the gauge transformation of $\tau(t)$.

2.2 Embedding of Algebro-Geometric Data to UGM

In this section we recall the construction of points of UGM from algebraic curves (see [14, 19] for more details).

Let C be a compact Riemann surface of genus g, p_∞ a point on it, z a local coordinate around p_∞. For $m \geq 0$ and points p_i, $1 \leq i \leq m$, on C, such that $p_j \neq \infty$ for any j, we denote by

$$H^0(C, \mathcal{O}(\sum_{j=1}^{m} p_j + *p_\infty)) \tag{15}$$

the vector space of meromorphic functions on C which have a pole at each p_j of order at most 1 and have a pole at p_∞ of any order. By expanding functions in the local coordinate z we can consider $H^0(C, \mathcal{O}(\sum_{j=1}^{m} p_j + *p_\infty))$ as a subspace of $V = \mathbb{C}((z))$. Then

Proposition 2 ([14, 19]) *The subspace* $z^{g-m}H^0(C, \mathcal{O}(\sum_{j=1}^{m} p_j + *p_\infty))$ *belongs to UGM.*

Remark 3 This Proposition was proved in [19] from the general results [14], for $m \leq g$. But the case $m > g$ can be proved in the same way.

2.3 Tau Function Corresponding to Zero Point Space

For $n \geq 1$ and mutually distinct complex numbers $\{\alpha_i\}_{i=1}^{3n}$ consider the compact Riemann surface C_n corresponding to the algebraic curve defined by the equation

$$y^3 = \prod_{j=1}^{3n+1} (x - \alpha_j). \tag{16}$$

The genus of C_n is $g = 3n$ and there is a unique point on C_n over $x = \infty$ which we denote by ∞.

Consider the space $H^0(C_n, \mathcal{O}(*\infty))$ which corresponds to $m = 0$ in (15). It is the space of meromorphic functions on C which are regular on $C_n \backslash \{\infty\}$. It can be easily proved that it coincides with the vector space $\mathbb{C}[x, y]$ of polynomials in x, y. A basis of this vector space is given by

$$x^i, \quad x^i y, \quad x^i y^2 \quad i \geq 0. \tag{17}$$

We take the local coordinate z around ∞ such that

$$x = z^{-3}, \qquad y = z^{-(3n+1)} F_n(z), \qquad F_n(z) = \left(\prod_{j=1}^{3n+1} (1 - \alpha_j z^3) \right)^{1/3}. \tag{18}$$

In the following we denote by z this local coordinate unless otherwise stated. The function $F_n(z)$ is considered as a power series in z by the Taylor expansion at $z = 0$.

By Proposition 2 $z^g H^0(C_n, \mathcal{O}(*\infty))$ determines a point of UGM. Writing (17) in terms of z and multiplying them by z^g we get a basis of it,

$$z^{3n-3i}, \qquad z^{-1-3i} F_n(z), \qquad z^{-3n-2-3i} F_n(z)^2 \qquad i \geq 0. \tag{19}$$

We define the frame $\tilde{\xi}_n$ from this basis as follows.

For an element $v(z) = \sum_{n \leq i} a_i z^i$, $a_n \neq 0$, define the order of $v(z)$ to be $-n$ and write $\mathrm{ord}\, v(z) = -n$.

Definition 4 Label the elements of (19) by $\tilde{v}_i$, $i \leq 0$, in such a way that $\mathrm{ord}\, \tilde{v}_0 < \mathrm{ord}\, \tilde{v}_{-1} < \mathrm{ord}\, \tilde{v}_{-2} < \cdots$ and define the frame $\tilde{\xi}_n$ of $z^g H^0(C_n, \mathcal{O}(*\infty))$ by

$$\tilde{\xi}_n = (\ldots, \tilde{v}_{-2}, \tilde{v}_{-1}, \tilde{v}_0). \tag{20}$$

By the construction of $\tilde{\xi}_n$ the tau function $\tau(t; \tilde{\xi}_n)$ has the following expansion (see [16])

$$\tau(t; \tilde{\xi}_n) = s_{\lambda^{(n)}}(t) + \text{h.w.t}, \tag{21}$$

where h.w.t means the higher weight terms, $\lambda^{(n)}$ is the partition determined from the gap sequence $w_1 < \cdots < w_g$ at ∞ of C_n and is given by

$$\lambda^{(n)} = (w_g - (g-1), \ldots, w_2 - 1, w_1).$$

Example 5 $\lambda^{(1)} = (3, 1, 1)$, $\lambda^{(2)} = (6, 4, 2, 2, 1, 1)$, $\lambda^{(3)} = (9, 7, 5, 3, 3, 2, 2, 1, 1)$.

2.4 Degeneration

Let us take a complex number α which is different from α_i, $1 \leq i \leq 3n-2$ and consider the limit

$$\alpha_{3n+1}, \alpha_{3n}, \alpha_{3n-1} \to \alpha, \tag{22}$$

which means that the curve C_n degenerates to

$$y^3 = (x-\alpha)^3 \prod_{j=1}^{3n-2} (x-\alpha_j). \tag{23}$$

which we call one step degeneration of C_n.

In the limit

$$F_n(z) \quad \longrightarrow \quad (1-\alpha z^3) F_{n-1}(z),$$

and the basis (19) tends to

$$z^{3n-3i},\ z^{-1-3i}(1-\alpha z^3)F_{n-1}(z),\ z^{-3n-2-3i}(1-\alpha z^3)^2 F_{n-1}(z)^2,\ i \geq 0. \tag{24}$$

Let W_n be the point of UGM generated by this basis. Multiply (24) by $(1-\alpha z^3)^{-2}$ we have

$$\frac{z^{3n-3i}}{(1-\alpha z^3)^2}, \quad \frac{z^{-1-3i}}{(1-\alpha z^3)} F_{n-1}(z), \quad z^{-3n-2-3i} F_{n-1}(z)^2 \qquad i \geq 0. \tag{25}$$

By taking linear combinations we have

Lemma 6 *The following set of elements gives a basis of* $(1-\alpha z^3)^{-2} W_n$.

$$\begin{aligned} & z^{3n-6-3i}, \quad z^{-4-3i} F_{n-1}(z), \quad z^{-3n-2-3i} F_{n-1}(z)^2, \quad i \geq 0, \\ & \frac{z^{3n}}{(1-\alpha z^3)^2}, \quad \frac{z^{3n-3}}{1-\alpha z^3}, \quad \frac{z^{-1}}{1-\alpha z^3} F_{n-1}(z). \end{aligned} \tag{26}$$

We arrange the basis elements of this lemma according as their orders and define the frame ξ_n as follows.

Definition 7 Define the frame ξ_n of W_n by

$$\xi_n = (\ldots, v_{-2}, v_{-1}, v_0),$$

with

$$\begin{aligned} v_0 &= \frac{z^{3n}}{(1-\alpha z^3)^2}, \\ v_{-1} &= \frac{z^{3n-3}}{1-\alpha z^3}, \\ v_{-(2+i)} &= z^{3n-6-3i}, \qquad\qquad 0 \leq i \leq n-2, \end{aligned}$$

$$
\begin{aligned}
v_{-(n+1)} &= \frac{z^{-1}}{1-\alpha z^3} F_{n-1}(z), && \\
v_{-(n+2+2i)} &= z^{-3-3i}, && 0 \le i \le n-1, \\
v_{-(n+3+2i)} &= z^{-4-3i} F_{n-1}(z), && 0 \le i \le n-1, \\
v_{-(3n+2+3i)} &= z^{-3n-2-3i} F_{n-1}(z)^2, && i \ge 0, \\
v_{-(3n+3+3i)} &= z^{-3n-3-3i}, && i \ge 0, \\
v_{-(3n+4+3i)} &= z^{-3n-4-3i} F_{n-1}(z), && i \ge 0.
\end{aligned}
$$

Since we have the expansion

$$\log(1-\alpha z^3)^{-2} = 6 \sum_{l=1}^{\infty} \alpha^l \frac{z^{3l}}{3l},$$

the following relation holds by (14),

$$\tau(t; \xi_n) = \mathrm{e}^{6 \sum_{l=1}^{\infty} \alpha^l t_{3l}} \lim \tau(t; \tilde{\xi}_n), \tag{27}$$

where the lim signifies taking the limit (22).

2.5 Three Point Insertion

Consider the curve C_{n-1} defined by (16) where n is replaced by $n-1$. The genus of C_{n-1} is $g' = 3n-3 = g-3$. Let

$$Q_j = (c_j, Y_j), \quad j = 0, 1, 2, \tag{28}$$

be points on C_{n-1}. We assume $c_j \neq \alpha_i$ for any i, j. Define φ_j by

$$\varphi_j = \frac{y^2 + Y_j y + Y_j^2}{x - c_j}.$$

The pole divisor of this function is $Q_j + (2g'-1)\infty$. Consider the space $H^0(C_{n-1}, \mathcal{O}(Q_0 + Q_1 + Q_2 + *\infty))$. A basis of it is given by

$$x^i, \quad x^i y, \quad x^i y^2, \quad \varphi_j, \quad i \ge 0, \quad j = 0, 1, 2.$$

Write this basis in terms of the local coordinate z and multiply it by $z^{g'-3}$ we have

$$z^{3n-6-3i},\ z^{-4-3i}F_{n-1}(z),\ z^{-3n-2-3i}F_{n-1}(z)^2,\ z^{3n-6}\varphi_j,\ i\geq 0,\ j=0,1,2. \tag{29}$$

By Proposition 2 $z^{g'-3}H^0(C_{n-1},\mathcal{O}(Q_0+Q_1+Q_2+*\infty))$ is a point of UGM and the set of functions (29) is a basis of it. Using this basis define the frame of $z^{g'-3}H^0(C_{n-1},\mathcal{O}(Q_0+Q_1+Q_2+*\infty))$ by

$$\begin{aligned}&\xi_{n-1}(Q_0,Q_1,Q_2)\\&\quad=(\ldots,v_{-(n+3)},v_{-(n+2)},v_{-n},\ldots,v_{-2},z^{3n-6}\varphi_0,z^{3n-6}\varphi_1,z^{3n-6}\varphi_2),\end{aligned}$$

where v_j is the same as that in ξ_n.

2.6 Degeneration Formula in Algebraic Form

Corresponding to the parameter α in (22) let $P_i(\alpha)=(\alpha,\omega^i y_0(\alpha))$, $i=0,1,2$ be points on C_{n-1}, where $\omega=e^{2\pi i/3}$. Take $Q_j=P_j(\alpha)$ in (28) and denote the function φ_j by $\varphi_j(\alpha)$. Then

$$\varphi_j(\alpha)=\frac{y^2+(\omega^j y_0(\alpha))y+(\omega^j y_0(\alpha))^2}{x-\alpha}.$$

Lemma 8 *For* $0\leq i\leq 2$ *we have*

$$\frac{y^i}{x-\alpha}=\frac{1}{3y_0(\alpha)^{2-i}}\sum_{j=0}^{2}\omega^{(i+1)j}\varphi_j(\alpha)$$

The lemma can be verified by direct computation. From these relations we have

$$v_{-1}=\frac{z^{3n-3}}{1-\alpha z^3}=\frac{1}{3y_0(\alpha)^2}\sum_{i=0}^{2}\omega^i z^{3n-6}\varphi_i(\alpha) \tag{30}$$

$$v_{-(n+1)}=\frac{z^{-1}F_{n-1}(z)}{1-\alpha z^3}=\frac{1}{3y_0(\alpha)}\sum_{i=0}^{2}\omega^{2i} z^{3n-6}\varphi_i(\alpha) \tag{31}$$

$$v_0=\frac{z^{3n}}{(1-\alpha z^3)^2}=\frac{\partial}{\partial\beta}\left(\frac{1}{3y_0(\beta)^2}\sum_{i=0}^{2}\omega^i z^{3n-6}\varphi_i(\beta)\right)\Bigg|_{\beta=\alpha}. \tag{32}$$

The third equation is obtained by differentiating the first equation in α.

Let λ be a partition and consider the Plücker coordinate of $(\xi_n)_\lambda$. Substitute the above expression to the definition of $(\xi_n)_\lambda$ of ξ_n. Then Eqs. (30)–(32) mean that each of the column vectors of ξ_n corresponding to v_0, v_{-1}, $v_{-(n+1)}$ is a sum of vectors. So we have

$$(\xi_n)_\lambda = \frac{(-1)^n}{27 y_{n-1,0}(\alpha)^5} \sum_{0\le i<j\le 2, 0\le k\le 2} \omega^{i+k+2j}(1-\omega^{i-j}) \times \frac{\partial}{\partial \beta}\left(\xi_{n-1}(P_i(\alpha), P_j(\alpha), P_k(\beta))_\lambda\right)|_{\beta=\alpha}.$$

Multiplying this equation by $s_\lambda(x)$ and summing up in λ we get

$$\tau(t;\xi_n) = \frac{(-1)^n}{27 y_{n-1,0}(\alpha)^5} \sum_{0\le i<j\le 2, 0\le k\le 2} \omega^{i+k+2j}(1-\omega^{i-j}) \times \frac{\partial}{\partial \beta}\left(\tau(t;\xi_{n-1}(P_i(\alpha), P_j(\alpha), P_k(\beta)))\right)|_{\beta=\alpha}.$$

Finally using (27) we obtain

Theorem 9 *Consider the limit (22). Then the limit of the tau function of the frame $\tilde{\xi}_n$ defined by (20) is given by the following formula:*

$$\lim \tau(t;\tilde{\xi}_n) = \frac{(-1)^n}{27 y_{n-1,0}(\alpha)^5} e^{-6\sum_{l=1}^{\infty}\alpha^l t_{3l}} \sum_{0\le i<j\le 2, 0\le k\le 2} \omega^{i+k+2j}(1-\omega^{i-j}) \times \frac{\partial}{\partial \beta}\left(\tau\left(t;\xi_{n-1}(P_i(\alpha), P_j(\alpha), P_k(\beta))\right)\right)\Big|_{\beta=\alpha}.$$

Remark 10 The new feature of the trigonal case compared with the hyperelliptic case studied in [2] (see Theorem 20) is the existence of a derivative in the parameter β. In [18] the degeneration to genus zero curve in the trigonal case was directly studied. The obtained solutions are not solitons but generalized solitons. The appearance of the derivative corresponds to this phenomenon.

3 Analytic Expression of Tau Functions

In this section we derive the analytic expression of tau functions appeared in Theorem 9 in terms of the multivariate sigma function [3–5, 15, 16]. The fundamental idea behind constructing the expression is due to Krichever [12].

3.1 The Sigma Function of an (N, M) Curve

We consider the general (N, M)-curve [5] defined by $f(x, y) = 0$ with

$$f(x, y) = y^N - x^M - \sum_{Ni+Mj<NM} \lambda_{ij} x^i y^j, \tag{33}$$

where N, M are relatively prime integers such that $1 < N < M$. We assume that the curve is non singular. We denote the corresponding compact Riemann surface by C. Then the genus of C is given by $g = 1/2(N-1)(M-1)$. There is one point on C over $x = \infty$ which is also denoted by ∞. Here we recall several necessary facts related with the curve C. See [15, 16] for details.

We assign the order $Ni + Mj$ to the monomial $x^i y^j$, $i, j \geq 0$, and define f_i, $i \geq 1$, to be the i-th monomial in this order. For example $f_1 = 1$, $f_2 = x$. Then the set of differentials

$$du_i = -\frac{f_{g+1-i}dx}{f_y}, \quad 1 \leq i \leq g$$

constitutes a basis of holomorphic one forms. We choose an algebraic fundamental form $\widehat{\omega}(p_1, p_2)$ on $C \times C$ as in [15]. It has the decomposition of the form

$$\widehat{\omega}(p_1, p_2) = d_{p_2}\Omega(p_1, p_2) + \sum_{i=1}^{g} du_i(p_1)dr_i(p_2),$$

where $\Omega(p_1, p_2)$ is a certain meromorphic one form on $C \times C$ and $dr_i(p)$ is a certain differential of the second kind on C with a pole only at ∞ (see [15] for more precise form of $\widehat{\omega}$, Ω, dr_i). Taking a symplectic basis $\{\alpha_i, \beta_i\}_{i=1}^{g}$ of the homology group of C we define the period matrices ω_k, η_k, $k = 1, 2$, Π by

$$2\omega_1 = \left(\int_{\alpha_j} du_i\right), \qquad 2\omega_2 = \left(\int_{\beta_j} du_i\right),$$

$$-2\eta_1 = \left(\int_{\alpha_j} dr_i\right), \qquad -2\eta_2 = \left(\int_{\beta_j} dr_i\right),$$

and $\Pi = \omega_1^{-1}\omega_2$. Define Riemann's theta function by

$$\theta[\epsilon](z, \Pi) = \sum_{m\in\mathbb{Z}^g} e^{\pi i\, {}^t(m+\epsilon')\Pi(m+\epsilon') + 2\pi i\, {}^t(m+\epsilon')(z+\epsilon'')},$$

where $\epsilon = {}^t(\epsilon', \epsilon'') \in \mathbb{R}^{2g}$, $\epsilon', \epsilon'' \in \mathbb{R}^g$. Let $\Pi\delta' + \delta''$, $\delta', \delta'' \in (1/2)\mathbb{Z}^g$, be a representative of Riemann's constant with respect to the choice of the base point ∞ and $\{\alpha_i, \beta_i\}_{i=1}^g$, and $\delta = {}^t(\delta', \delta'') \in (1/2)\mathbb{Z}^{2g}$.

Let $(w_1, \dots, w_g)$, $w_1 < \cdots < w_g$, be the gap sequence of the curve C at ∞ (see [7, 15] for example). Define the partition $\lambda^{(N,M)}$ by

$$\lambda^{(N,M)} = (w_g - (g-1), \dots, w_2 - 1, w_1).$$

By the definition $\lambda^{(n)} = \lambda^{(3,3n+1)}$.

Definition 11 The sigma function is defined by

$$\sigma(u) = C e^{\frac{1}{2}{}^t u \eta_1 \omega_1^{-1} u} \theta[-\delta]((2\omega_1)^{-1} u, \Pi),$$
$$u = {}^t(u_1, \dots, u_g)$$

for some constant C.

Assign the weight w_i to u_i. Then the constant C is specified by the condition that $\sigma(u)$ has the expansion of the form

$$\sigma(u) = s_{\lambda^{(N,M)}}(t)|_{t_{w_i} = u_i} + \text{h.w.t.}$$

It is known that C is explicitly expressed by some derivatives of the Riemann's theta function [17, 20]. The sigma function satisfies the following quasi-periodicity property:

$$\sigma(u + \sum_{i=1}^{2} 2\omega_i m_i)$$
$$= (-1)^{{}^t m_1 m_2 + 2({}^t\delta' m_1 - {}^t\delta'' m_2)} e^{{}^t(\sum_{i=1}^{2} 2\eta_i m_i)(u + \sum_{i=1}^{2} \omega_i m_i)} \sigma(u). \qquad (34)$$

3.2 Sigma Function Expression of Tau Functions

Here we derive sigma function expressions for the tau functions corresponding to the spaces in Proposition 2 in the case of (N, M) curves.

We take the local coordinate z around ∞ such that

$$x = z^{-N}, \qquad y = z^{-M}(1 + O(z)).$$

Expand du_i, $\widehat{\omega}$ in z as

$$du_i = \sum_{j=1}^{\infty} b_{i,j} z^{j-1},$$

$$\widehat{\omega}(p_1, p_2) = \left(\frac{1}{(z_1 - z_2)^2} + \sum_{i,j \geq 1} \widehat{q}_{i,j} z_1^{i-1} z_2^{j-1} \right) dz_1 dz_2,$$

where $z_i = z(p_i)$. The differential du_g has a zero of order $2g - 2$ at ∞ and has the expansion of the form

$$du_g = z^{2g-2}(1 + \sum_{j=2g}^{\infty} b_{g,j} z^{j-2g+1}) dz.$$

Define c_i by the expansion

$$\log \left(\sqrt{z^{-2g+2} \frac{du_g}{dz}} \right) = \sum_{i=1}^{\infty} c_i \frac{z^i}{i}.$$

In [16] there is a misprint, $c_i z^i$ should be $c_i z^i / i$ as above. Define $g \times \mathbb{N}$ matrix B and the quadratic form $\widehat{q}$ by

$$B = (b_{i,j})_{1 \leq i \leq g, j \geq 1}, \qquad \widehat{q}(t) = \sum_{i,j=1}^{\infty} \widehat{q}_{i,j} t_i t_j.$$

The following theorem is proved in [16].

Theorem 12 ([16]) *A tau function corresponding to $z^g H^0(C, \mathcal{O}(*\infty))$ is given by*

$$\tau_0(t) := \mathrm{e}^{-\sum_{i=1}^{\infty} c_i t_i + \frac{1}{2} \widehat{q}(t)} \sigma(Bt). \tag{35}$$

It has the expansion of the form

$$\tau_0(t) = s_{\lambda(N,M)}(t) + \text{h.w.t.} \tag{36}$$

Remark 13 In [16] it is proved that $\tau_0(t)$ defined by (35) is a solution of the N-reduced KP-hierarchy [6].

More generally the tau function corresponding to the m-point space with $m \geq 1$ given by Proposition 2 is described in terms of the shift of $\tau_0(t)$.

Theorem 14 *Let p_i, $1 \le i \le m$, be points on $C \backslash \{\infty\}$ and $z_i = z(p_i)$. A tau function corresponding to $z^{g-m} H^0(C, \mathcal{O}(\sum_{i=1}^m p_i + *\infty))$ is given by*

$$\tau(t|p_1, \ldots, p_m) := \mathrm{e}^{\sum_{i=1}^\infty \eta(z_i^{-1})} \tau_0(t - \sum_{i=1}^m [z_i]), \tag{37}$$

where $\eta(\kappa) = \sum_{i=1}^\infty t_i \kappa^i$, $[w] = [w, w^2/2, w^3/3, \ldots]$.

By (14) and by that the KP-hierarchy is the system of autonomous equations, if $\tau(t)$ is a solution of the KP-hierarchy, so is $\mathrm{e}^{\sum_{i=1}^\infty \gamma_i t_i} \tau(t + \zeta)$ for any set of constants $\{\gamma_i\}$ and a constant vector ζ. Therefore $\tau(t|p_1, \ldots, p_m)$ is a solution of the KP-hierarchy.

Then the theorem is proved by calculating the adjoint wave function using (13). To this end we need some notation.

Let $E(p_1, p_2)$ be the prime form [8] (see also [10]). Define $E(z_1, z_2)$, $E(q, p)$ with $z_i = z(p_i)$ and q being a fixed point on C by

$$E(p_1, p_2) = \frac{E(z_1, z_2)}{\sqrt{dz_1}\sqrt{dz_2}}, \qquad E(q, p) = \frac{E(z(q), z(p))}{\sqrt{dz(p)}}.$$

Define $\tilde{E}(q, p)$ for q fixed by

$$\tilde{E}(q, p) = E(q, p)\sqrt{du_g(p)}\mathrm{e}^{\frac{1}{2}\int_q^p {}^t du (\eta_1 \omega_1^{-1}) \int_q^p du},$$
$$du = {}^t(du_1, \ldots, du_g).$$

In [15] two variables $\tilde{E}(p_1, p_2)$ and one variable $\tilde{E}(\infty, p)$ were introduced and studied. It should be noticed that $\tilde{E}(q, p)$ is a multiplicative function of p while $E(q, p)$ is a $-1/2$ form. Similarly to the case of $\tilde{E}(\infty, p)$ in [15] the following lemma can be proved.

Lemma 15

(i) *The function $\tilde{E}(q, p)$ has the expansion in $z = z(p)$ near ∞ of the form*

$$\tilde{E}(q, p) = (z - z(q)) z^{g-1}(1 + O(z)).$$

(ii) *Let γ be an element of $\pi_1(C, \infty)$ and its Abelian image be $\sum_{i=1}^g (m_{1,i}\alpha_i + m_{2,i}\beta_i)$. Then*

$$\tilde{E}(q, \gamma(p))/\tilde{E}(q, p)$$
$$= (-1)^{{}^t m_1 m_2 + 2({}^t\delta' m_1 - {}^t\delta'' m_2)} \mathrm{e}^{{}^t(\sum_{i=1}^2 2\eta_i m_i)(\int_q^p du + \sum_{i=1}^2 \omega_i m_i)}, \tag{38}$$

where $m_i = {}^t(m_{i,1}, \ldots, m_{i,g})$.

By (i) of this lemma $\tilde{E}(\infty, p)$ has a zero of order g at ∞.

Let $d\tilde{r}_i$ be the normalized differential of the second kind with a pole only at ∞, that is, it satisfies

$$\int_{\alpha_j} d\tilde{r}_i = 0, \quad 1 \le j \le g, \qquad d\tilde{r}_i = d(z^{-i} + O(1)).$$

Define

$$d\hat{r}_i = d\tilde{r}_i + \sum_{j,k=1}^{g} b_{j,i} (\eta_1 \omega_1^{-1})_{j,k} du_k.$$

By the construction their periods can be computed as (Lemma 5 in [16])

$$\int_{\alpha_j} d\widehat{r_i} = \left({}^t(2\eta_1)B\right)_{j,i}, \qquad \int_{\beta_j} d\widehat{r_i} = \left({}^t(2\eta_2)B\right)_{j,i}. \tag{39}$$

In Lemma 5 of [16] there is a misprint: the right hand side is not the (i, j) component but the (j, i) component.

Proof of Theorem 14 The adjoint wave function (11) corresponding to the tau function (37) is computed as

$$\Psi^*(t, z) = C(z_1, \ldots, z_m) z^{g-m} \frac{\tilde{E}(\infty, p)^{m-1} \sigma\left(\int_\infty^p du - \sum_{i=1}^m \int_\infty^{p_i} du + Bt\right)}{\prod_{i=1}^m \tilde{E}(p_i, p)\ \sigma\left(-\sum_{i=1}^m \int_\infty^{p_i} du + Bt\right)} \times e^{-\sum_{i=1}^\infty t_i \int^p d\widehat{r_i}},$$

$$C(z_1, \ldots, z_m) = (-1)^m \left(\prod_{i=1}^m z_i\right) e^{\frac{1}{2} \sum_{i=1}^m \int_\infty^{p_i} {}^t du (\eta_1 \omega_1^{-1}) \int_\infty^{p_i} du}.$$

By Lemma 15 and (39) we can check that $z^{-g+m}\Psi^*(t, z)$ is, as a function of $p \in C$, $\pi_1(C, \infty)$ invariant. Then the same is true for any expansion coefficient of $\Psi^*(t, z)$ in t. Expansion coefficients in t are regular except p_i, $1 \le i \le m$, ∞ and have at most a simple pole at p_i. Therefore the point U of UGM corresponding to $\tau(t|p_1, \ldots, p_m)$ is contained in $z^{g-m} H^0(C, \mathcal{O}(\sum_{i=1}^m p_i + *\infty))$. Since a strict inclusion relation is impossible for two points of UGM [2, Lemma 4.17], these two points of UGM coincide. □

3.3 Degeneration Formula in Analytic Form

In this section we apply the results in the previous section to the curves C_n, C_{n-1} and associated tau functions in Theorem 9. So, in this section $\tau_{n,0}(t)$ denotes the function defined by (35) for the curve C_n.

Lemma 16 *We have*

$$\tau(t;\tilde{\xi}_n)=\tau_{n,0}(t). \tag{40}$$

Proof Since $\tilde{\xi}_n$ is a tau function corresponding to $z^g H^0(C_n, \mathcal{O}(*\infty))$, we have, by Theorem 12,

$$\tau(t;\tilde{\xi}_n)=C\tau_{n,0}(t),$$

for some constant C. Comparing the expansions (21) and (36) we have $C=1$. □

Next we consider tau functions appearing in the right hand side of the equation in Theorem 9. We need a point $(\alpha, y_0(\alpha))$ of C_{n-1}. To specify $y_0(\alpha)$ is equivalent to specify one value of z such that $z^{-3}=\alpha$, that is, $\alpha^{-1/3}$. In fact, if $z=\alpha^{-1/3}$ is given the value of $y_0(\alpha)$ is determined by (18) as

$$y_0(\alpha)=\alpha^{n-1}\alpha^{1/3}F_{n-1}(\alpha^{-1/3}). \tag{41}$$

Since $P_i(\alpha)=(\alpha,\omega^i y_0(\alpha))$, we have

$$z(P_i(\alpha))=\omega^{-i}\alpha^{-\frac{1}{3}}. \tag{42}$$

For simplicity we set

$$z_i(\alpha)=\omega^{-i}\alpha^{-\frac{1}{3}}. \tag{43}$$

Since, in general $\xi_{n-1}(Q_0, Q_1, Q_2)$ is a frame of the point

$$z^{g'-3}H^0(C_{n-1},\mathcal{O}(\sum_{i=0}^{2}Q_i+*\infty))\in \mathrm{UGM}$$

we have, by Theorem 14,

$$\begin{aligned}\tau(t;\xi_{n-1}(P_i(\alpha),P_j(\alpha),P_k(\beta)))\\ &=C_{i,j,k}(\alpha,\beta)\mathrm{e}^{\eta(z_i(\alpha)^{-1})+\eta(z_j(\alpha)^{-1})+\eta(z_k(\beta)^{-1})}\\ &\qquad\times\tau_{n-1,0}(t-[z_i(\alpha)]-[z_j(\alpha)]-[z_k(\beta)]),\end{aligned} \tag{44}$$

for some constant $C_{i,j,k}(\alpha, \beta)$.

Remark 17 The explicit forms of the constants $C_{i,j,k}(\alpha, \beta)$ are not yet determined. They should be calculated by comparing the Schur function expansions and are expected to be expressed by some derivatives of the sigma function.

Substituting (40), (44) into the relation in Theorem 9 we get

Theorem 18 *Let $\tau_{n,0}(t)$ be defined by the right hand side of (35) for the curve C_n and $z_i(\alpha)$ defined by (43). Then, in the limit $\alpha_j \to \alpha$ for $j = 3n, 3n \pm 1$, we have*

$$\begin{aligned}
&\lim \tau_{n,0}(t) \\
&= \frac{(-1)^n}{27 y_0(\alpha)^5} e^{-6\sum_{l=1}^{\infty} \alpha^l t_{3l}} \sum_{0 \le i < j \le 2, 0 \le k \le 2} \omega^{i+k+2j}(1-\omega^{i-j}) \\
&\quad \times \frac{\partial}{\partial \beta}\Big(C_{i,j,k}(\alpha, \beta) e^{\eta(z_i(\alpha)^{-1}) + \eta(z_j(\alpha)^{-1}) + \eta(z_k(\beta)^{-1})} \\
&\quad \times \tau_{n-1,0}(t - [z_i(\alpha)] - [z_j(\alpha)] - [z_k(\beta)]) \Big)\Big|_{\beta=\alpha}. \qquad (45)
\end{aligned}$$

for the constants $C_{i,j,k}(\alpha, \beta)$ in (44), where $y_0(\alpha)$ is given by (41).

Remark 19 In the right hand side of (45) the exponential factor, which is characteristic to soliton solutions, is clearly visible. Since it can be shown that $\tau_{0,0} = 1$ for the genus zero curve $y^3 = x$ which corresponds to the case $\alpha_1 = 0$, using repeatedly the formula (18) we obtain the formula which contains only exponential functions and their derivatives with respect to parameters. The formulas for them were computed in [18] independently of Theorem 18, where all constants are explicitly given as functions of $\{\alpha_j\}$. These solutions are called generalized solitons in [18].

4 The Case of Hyperelliptic Curves

In this section, based on the results of [2], we derive the corresponding formula to (45) in the case of hyperelliptic curve X_g defined by

$$y^2 = \prod_{j=1}^{2g+1}(x - \alpha_j) \qquad (46)$$

and its degeneration

$$\alpha_{2g+1}, \alpha_{2g-1} \to \alpha, \qquad (47)$$

where $\alpha \neq \alpha_j$ for $1 \leq j \leq 2g-2$. The curve X_g has the unique point over $x = \infty$ which we also denote by ∞. We take the local coordinate z around ∞ such that

$$x = z^{-2}, \qquad y = z^{-2g-1} F_g(z), \quad F_g(z) = \left(\prod_{j=1}^{2g+1} (1 - \alpha_i z^2) \right)^{1/2}. \tag{48}$$

Let

$$\mu^{(g)} = (g, g-1, \ldots, 1)$$

be the partition and $\tilde{\xi}_g$ a frame of $z^g H^0(X_g, \mathcal{O}(*\infty))$ such that the corresponding tau function has the expansion of the form

$$\tau(t; \tilde{\xi}_g) = s_{\mu^{(g)}}(t) + \text{h.w.t.} \tag{49}$$

Fix one of the square root $\alpha^{-1/2}$ and define y_0 by

$$y_0 = \alpha^{g-1/2} F_{g-1}(\alpha^{-1/2}). \tag{50}$$

Then (α, y_0) is a point of X_{g-1}. Set

$$p_\pm = (\alpha, \pm y_0). \tag{51}$$

Then the values of the local coordinates of $p_\pm$ are

$$z(p_\pm) = \pm\alpha^{-1/2}.$$

Let $\xi_{g-1}(p_\pm)$ be a frame of $z^{g-2} H^0\left(X_{g-1}, \mathcal{O}(p_\pm + *\infty)\right)$ such that their tau functions have the following expansions

$$\tau(t; \xi_{g-1}(p_\pm)) = s_{\mu^{(g-2)}}(t) + \text{h.w.t.} \tag{52}$$

The following theorem is proved in [2] in a similar way to Theorem 9.

Theorem 20 ([2]) *The following relation holds.*

$$\begin{aligned} &\lim \tau(t; \tilde{\xi}) \\ &\quad = (-1)^{g-1} (2y_0)^{-1} \mathrm{e}^{-2\sum_{l=1}^{\infty} \alpha^l t_{2l}} \left(\tau(t; \xi_{g-1}(p_+)) - \tau(t; \xi_{g-1}(p_-)) \right), \end{aligned} \tag{53}$$

where lim *in the left hand side means the limit taking* α_{2g+1}, α_{2g} *to* α.

Let $\tau_{g,0}(t)$ denote the function defined by the right hand side of (35) for X_g.

Lemma 21

(i) $\tau(t;\tilde{\xi}_g) = \tau_{g,0}(t)$.
(ii) *For some constant* $C_\epsilon(\alpha)$

$$\tau(t;\xi_{g-1}(p_\epsilon)) = C_\epsilon(\alpha)\mathrm{e}^{\sum_{l=1}^{\infty}(\epsilon\alpha^{-1/2})^{-l}t_l}\tau_{g-1,0}(t-[\epsilon\alpha^{-1/2}]), \quad \epsilon = \pm.$$

Proof

(i) Both $\tau(t;\tilde{\xi}_g)$ and $\tau_{g,0}(t)$ are tau functions corresponding to $z^g H^0(X_g, \mathcal{O}(*\infty))$. By comparing the expansions (36) and (49) we get the result.
(ii) Since the right hand side and the left hand side without $C_\epsilon(\alpha)$ of the equation in the assertion are the tau functions corresponding to $z^{g-2}H^0(X_{g-1}, \mathcal{O}(p_\epsilon + *\infty))$ by the definition of $\xi_{g-1}(p_\epsilon)$ and Theorem 14, the assertion follows.

□

This lemma is proved in [2] in a different form. The explicit form of the constant $C_\epsilon(\alpha)$ can be extracted from there. Let us give the formula.

Let $m^{(g)} = \left[\frac{g+1}{2}\right]$. Define the sequence $A^{(g)}$ and $s^{(g)} \in \{\pm 1\}$ by

$$A^{(g)} = (a_1^{(g)}, \ldots, a_{m^{(g)}}^{(g)}) = (2g-1, 2g-5, 2g-9, \ldots),$$
$$s^{(g)} = (-1)^{(g-1)m^{(g)}}.$$

Example 22 $A^{(1)} = (1)$, $A^{(2)} = (3)$, $A^{(3)} = (5,1)$, $A^{(4)} = (7,3)$.
$s^{(1)} = 1$, $s^{(2)} = -1$, $s^{(3)} = 1$, $s^{(4)} = 1$.

The following property of $A^{(g)}$ is known [20, 22],

$$|A^{(g)}| := \sum_{j=1}^{m^{(g)}} a_j^{(g)} = \frac{1}{2}g(g+1). \tag{54}$$

Denote the sigma function of X_{g-1} by $\sigma^{(g-1)}(u)$. Set

$$b_i = (a_i^{(g-2)}+1)/2 \in \{1,2,\ldots,g-2\}, \quad 1 \le i \le m^{(g-2)},$$

and define

$$\sigma_{A^{(g-2)}}^{(g-1)}(u) = \frac{\partial^{m^{(g-2)}}}{\partial u_{b_1}\cdots\partial u_{b_{m^{(g-2)}}}}\sigma^{(g-1)}(u).$$

Then, by Theorem 4.14 of [2], we can deduce that

$$C_\epsilon(\alpha)=s^{(g-2)}\sigma^{(g-1)}_{A^{(g-2)}}(-\int_\infty^{p_\epsilon}du)^{-1},\quad du={}^t(du_1,\dots,du_g). \tag{55}$$

Lemma 23 *The following relation is valid.*

$$C_-(\alpha)=(-1)^{g-1}C_+(\alpha). \tag{56}$$

Proof It is known that the sigma function satisfies the following relation [15, 22]

$$\sigma^{(g-1)}(-u)=(-1)^{\frac{1}{2}g(g-1)}\sigma^{(g-1)}(u).$$

By differentiating it we get

$$\sigma^{(g-1)}_{A^{(g-2)}}(-u)=(-1)^{\frac{1}{2}g(g-1)+m^{(g-2)}}\sigma^{(g-1)}(u). \tag{57}$$

We can easily verify that

$$\frac{1}{2}g(g-1)+m^{(g-2)}=g-1 \quad \text{mod.2}. \tag{58}$$

For the hyperelliptic curve X_{g-1} the following relation holds,

$$\int_\infty^{p_-}du=-\int_\infty^{p_+}du. \tag{59}$$

The assertion of the lemma follows from (55), (57), (58), (59). □

Substituting the equations of (i), (ii) in Lemma 21 into (53) and using (56) we get

Theorem 24 *Let $\tau_{g,0}(t)$ be given by the right hand side of (35) for the hyperelliptic curve X_g defined by (46). Then in the limit $\alpha_{2g+1},\alpha_{2g}\to\alpha$ we have the following formula,*

$$\lim\tau_{g,0}(t)=(-1)^g(2y_0)^{-1}C_+(\alpha)\mathrm{e}^{-2\sum_{l=1}^\infty\alpha^l t_{2l}}$$
$$\times\left(\mathrm{e}^{\eta(\alpha^{1/2})}\tau_{g-1,0}(t-[\alpha^{-1/2}])+(-1)^g\mathrm{e}^{\eta(-\alpha^{1/2})}\tau_{g-1,0}(t-[-\alpha^{-1/2}])\right),$$

where y_0, $p_\pm$, $C_+(\alpha)$ are given by (50), (51), (55) respectively.

Remark 25 The tau function $\tau_{g,0}(t)$ gives a solution of the KdV hierarchy (see Remark 13). Again it can be shown that $\tau_{0,0}=1$ for the genus zero curve $y^2=x$ which corresponds to $\alpha_1=0$. Using the formula repeatedly we get the well known soliton solution [9, 21]. For $\alpha_1 \neq 0$ we can show that $\tau_{0,0}(t) = \mathrm{e}^{L(t)+Q(t)}$, where $L(t)$ and $Q(t)$ are certain linear and quadratic functions of t.

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