On Hom-Lie–Rinehart Algebras

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Abstract We describe the notion of hom-Lie–Rinehart algebras as an algebraic analogue of hom-Lie algebroids. We consider modules (left and right) over this hom-structure and describe homology and cohomology complexes by considering coefficient modules. In the sequel, we consider some special classes of hom-Gerstenhaber algebras and their relationship with hom-Lie algebroids by Mandal and Mishra (J Geom Phys 133:287–302, 2018; Commun Algebra 46(9):3722–3744, 2018).

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1 Introduction

The notion of Lie–Rinehart algebras plays an important role in many branches of mathematics. The idea of this notion goes back to the work of N. Jacobson to study certain field extensions. Over the years, this notion appeared with different names in several areas which include differential geometry and differential Galois theory. J. Huebschmann described Lie–Rinehart algebras as an algebraic counterpart of Lie algebroids in [2] and developed systematically through a series of papers. There is also a growing interest in twisted algebraic structures or hom-algebraic structures. The first appearance of a hom-algebra was the notion of hom-Lie algebra, in the context of some particular deformation called q-deformations of Witt and Virasoro algebra of vector fields. Later on, many essential results on hom-Lie algebras and

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hom-associative algebras followed in the works of A. Makhlouf, Y. Sheng, D. Yau and coauthors. In [3], C. Laurent-Gengoux and J. Teles introduced the notion of hom-Lie algebroids through a formulation of hom-Gerstenhaber algebras, following the one-one correspondence between Lie algebroid structures on a vector bundle and Gerstenhaber algebra structures on the exterior algebra of multisections of the vector bundle. In fact, this one-one correspondence is an outcome of a more general categorical result for the algebraic counterpart, namely the existence of adjoint functors between the category of Lie–Rinehart algebras and the category of Gerstenhaber algebras. This adjunction leads us to define the notion of hom-Lie–Rinehart algebras and to construct a pair of adjoint functors between the category of hom-Gerstenhaber algebras and the category of hom-Lie–Rinehart algebras (for details see [4, 5]).

2 Hom-Lie–Rinehart Algebras

Let R be a commutative ring with unity, A be an associative commutative R-algebra, and $\phi : A \to A$ be an algebra endomorphism.

Definition 1 A hom-Lie–Rinehart algebra over (A, ϕ) is given by a tuple $(A, L, [-,-], \phi, \alpha, \rho)$ where L is an A-module, $[-,-] : L \times L \to L$ is a skew symmetric bilinear map, $\alpha : L \to L$ is a ϕ-function linear map satisfying $\alpha([x, y]) = [\alpha(x), \alpha(y)]$, and the map $\rho : L \to \mathrm{Der}_\phi A$ is a ϕ-function linear map such that following conditions hold.

1. The triplet $(L, [-,-], \alpha)$ is a hom-Lie algebra.
2. (ρ, ϕ) is a hom-Lie algebra representation of $(L, [-,-], \alpha)$ on A.
3. $[x, a.y] = \phi(a)[x, y] + \rho(x)(a)\alpha(y)$ for all $a \in A,\ x, y \in L$.

A hom-Lie–Rinehart algebra $(A, L, [-,-], \phi, \alpha, \rho)$ is said to be regular if the map $\phi : A \to A$ is an algebra automorphism and $\alpha : L \to L$ is a bijection.

Example

1. Let L be a Lie–Rinehart algebra over an associative commutative algebra A, and (α, ϕ) be an endomorphism of L, then the tuple $(A, L, [-,-]_\alpha, \phi, \alpha, \rho_\phi)$ is a hom-Lie–Rinehart algebra, where $[-,-]_\alpha := \alpha \circ [-,-]$, and $\rho_\phi := \phi \circ \rho$.
2. Hom-Lie–Rinehart algebra associated to the space of ϕ-derivations: Let $\phi : A \to A$ be an algebra automorphism. Then $(\mathrm{Der}_\phi A, [-,-]_\phi, \alpha_\phi)$ is a hom-Lie algebra, with $\alpha_\phi(D) = \phi \circ D \circ \phi^{-1}$ and the bracket $[D_1, D_2]_\phi = \phi \circ D_1 \circ \phi^{-1} \circ D_2 \circ \phi^{-1} - \phi \circ D_2 \circ \phi^{-1} \circ D_1 \circ \phi^{-1}$, for any $D_1, D_2, D \in \mathrm{Der}_\phi(A)$. In fact, the tuple $(A, \mathrm{Der}_\phi A, [-,-]_\phi, \alpha_\phi, \alpha_\phi)$ is a hom-Lie–Rinehart algebra over (A, ϕ) with the anchor $\rho = \alpha_\phi$.
3. The hom-Lie–Rinehart algebras associated to a Poisson algebra equipped with an automorphism is described in [4].

2.1 Homomorphisms of Hom-Lie–Rinehart Algebras

Definition 2 Let $(A, L, [-,-]_L, \phi, \alpha_L, \rho_L)$ and $(B, L', [-,-]_{L'}, \psi, \alpha_{L'}, \rho_{L'})$ be hom-Lie–Rinehart algebras, then a hom-Lie–Rinehart algebra homomorphism is a pair of maps (g, f), satisfying the following identities:

1. $f \circ \alpha_L = \alpha_{L'} \circ f,\ g \circ \phi = \psi \circ g;$
2. $f([x, y]) = [f(x), f(y)]_{L'}$ for all $x, y \in L$;
3. $g(\rho_L(x)(a)) = \rho_{L'}(f(x))(g(a))$ for all $x \in L,\ a \in A$,

where the map $g : A \to B$ is a R-algebra homomorphism and $f : L \to L'$ is a g-function linear map

Let us denote by hLR the category of hom-Lie-Rinehart algebras and by hGR the category of hom-Gerstenhaber algebras.

Theorem 3 *There are adjoint functors between the categories* hLR *and* hGR.

3 Modules of Hom-Lie–Rinehart Algebras

Let $(\mathcal{L}, \alpha) := (A, L, [-,-], \phi, \alpha, \rho)$ be a hom-Lie–Rinehart algebra over (A, ϕ). Also, let M be an A-module and $\beta : M \to M$ be a ϕ-function linear map.

3.1 Left Modules Over Hom-Lie–Rinehart Algebras

Definition 4 A pair (M, β) is said to be a left module over a hom-Lie Rinehart algebra $(\mathcal{L}, \alpha)$ if the following conditions hold for all $X \in L,\ a \in A,\ m \in M$.

- There is a map $\theta : L \otimes M \to M$, such that the pair (θ, β) is a hom-Lie algebra representation of $(L, [-,-], \alpha)$ on M. Let $\{x, m\} := \theta(x, m)$;
- $\{a.X, m\} = \phi(a)\{X, m\}$;
- $\{X, a.m\} = \phi(a)\{X, m\} + \rho(X)(a).\beta(m)$.

Example

1. If $\alpha = Id_L$ and $\beta = Id_M$ then $(\mathcal{L}, \alpha)$ is a Lie–Rinehart algebra and M is a left Lie–Rinehart algebra module over L.
2. The pair (A, ϕ) is a left module over $(\mathcal{L}, \alpha)$.

Let $(\mathcal{L}, \alpha)$ be a regular hom-Lie–Rinehart algebra. We define a cochain complex $(Alt_A(\mathcal{L}, M), \delta)$, where $Alt_A(\mathcal{L}, M) := \bigoplus_{n\geq 0} Hom_A(\wedge_A^n L, M)$. The coboundary map $\delta : Alt_A^n(\mathcal{L}, M) \to Alt_A^{n+1}(\mathcal{L}, M)$ is defined as follows:

$$\begin{aligned}\delta f(x_1, \cdots, x_{n+1}) \\ &= \sum_{i=1}^{n+1}(-1)^{i+1}\theta(\alpha^{-1}(x_i))(f(\alpha^{-1}(x_1), \cdots, \hat{x_i}, \cdots, \alpha^{-1}(x_{n+1})) \\ &+ \sum_{1\leq i<j\leq n+1}(-1)^{i+j}\beta(f(\alpha^{-2}([x_i, x_j]), \alpha^{-1}(x_1), \cdot, \hat{x_i}, \cdot, \hat{x_j}, \cdot, \alpha^{-1}(x_{n+1}))\end{aligned}$$

for $f \in Alt_A^n(\mathcal{L}, M)$, and $x_i \in L$, for $1 \leq i \leq n+1$. The cohomology of a regular hom-Lie–Rinehart algebra $(\mathcal{L}, \alpha)$ with coefficients in the left module (M, β) is given by the associated cohomology of the cochain complex $(Alt_A(\mathcal{L}, M), \delta)$.

Theorem 5 *Let $(\mathcal{L}, \alpha)$ be a regular hom-Lie–Rinehart algebra over (A, ϕ). If L is a projective A-module of rank n, then there is a bijective correspondence between right $(\mathcal{L}, \alpha)$-module structures on (A, ϕ) and left $(\mathcal{L}, \alpha)$-module structures on $(\wedge_A^n L, \tilde{\alpha})$.*

Corollary 6 *Let $(\mathcal{L}, \alpha)$ be a regular hom-Lie–Rinehart algebra over (A, ϕ). If L is a projective A-module of rank n, then there is a bijective correspondence between left $(\mathcal{L}, \alpha)$-module structures on $(\wedge_A^n L, \tilde{\alpha})$ and exact generators of the hom-Gerstenhaber algebra bracket on $\wedge_A^* L$.*

3.2 Right Modules Over Hom-Lie–Rinehart Algebras

Definition 7 The pair (M, β) is a right module over a hom-Lie Rinehart algebra $(\mathcal{L}, \alpha)$ if the following conditions hold for all $X \in L,\ a \in A,\ m \in M$.

- There is a map $\theta : M \otimes L \to M$ such that the pair (θ, β) is a hom-Lie algebra representation of $(L, [-,-], \alpha)$ on M. Let $\{m, X\} := \theta(m, X)$;
- $\{a.m, X\} = \{m, a.X\} = \phi(a).\{m, X\} - \rho(X)(a).\beta(m)$.

Remark 8 There is no canonical right module structure on (A, ϕ).

For $n \geq 0$, we take $C_n(\mathcal{L}, M) := M \otimes_A \wedge_A^n L$ and define a boundary map $d : C_n(\mathcal{L}, M) \to C_{n-1}(\mathcal{L}, M)$ as

$$\begin{aligned}d(m \otimes (x_1 \otimes \cdots \otimes x_n)) \\ &= \sum_{i=1}^{n}(-1)^{i+1}\{m, x_i\} \otimes (\alpha(x_1) \otimes \cdots \otimes \alpha(\hat{x_i}) \otimes \cdots \otimes \alpha(x_n)) \\ &+ \sum_{i<j}(-1)^{i+j}\beta(m) \otimes ([x_i, x_j], \alpha(x_1) \otimes \cdots \otimes \alpha(\hat{x_i}) \otimes \cdots \otimes \alpha(\hat{x_j}) \otimes \cdots \otimes \alpha(x_n)).\end{aligned}$$

Then it follows that $(C_*(\mathcal{L}, M), d)$ is a chain complex. The homology of $(\mathcal{L}, \alpha)$ with coefficient in the right module (M, β) is given by $H_*^{\mathrm{hLR}}(L; M) := H_*(C_*(\mathcal{L}, M))$.

Example If $\alpha = Id_L$ and $\beta = Id_M$, then $H_*^{\mathrm{hLR}}(L; M)$ is the Lie–Rinehart algebra homology with coefficients in M. For $A = R$, the pair (θ, β) is a representation of hom-Lie algebra $(L, [-,-], \alpha)$ on M and $H_*^{\mathrm{hLR}}(L; M)$ is the homology of a hom-Lie algebra with coefficients in M.

Theorem 9 *Let $(\mathcal{L}, \alpha)$ be a hom-Lie–Rinehart algebra over (A, ϕ). Then there is a bijective correspondence between right $(\mathcal{L}, \alpha)$-module structures on (A, ϕ) and exact generators of the associated hom-Gerstenhaber algebra bracket on $\wedge_A^* L$.*

Corollary 10 *The homology $H_*^{\mathrm{hLR}}(\mathcal{L}, A)$ is isomorphic to the homology of the chain complex associated to the hom-Gerstenhaber algebra structure on $\wedge_A^* L$.*

4 Representation of a Hom-Lie Algebroid

In this section, we consider hom-Lie algebroids as a particular case of hom-Lie–Rinehart algebras. Let $\mathcal{A} := (A, \phi, [-,-], \rho, \alpha)$ be a hom-Lie algebroid and (E, ϕ, β) be a hom-bundle over a smooth manifold M. A bilinear map $\nabla : \Gamma A \otimes \Gamma E \to \Gamma E$, denoted by $\nabla_x(s) := \nabla(x, s)$, is a representation of $\mathcal{A}$ on the hom-bundle (E, ϕ, β) if it satisfies the following properties:

1. $\nabla_{f.x}(s) = \phi^*(f).\nabla_x(s)$ for all $x \in \Gamma A,\ s \in \Gamma E$ and $f \in C^\infty(M)$;
2. $\nabla_x(f.s) = \phi^*(f).\nabla_x(s) + \rho(x)[f].\beta(s)$ for all $x \in \Gamma A,\ s \in \Gamma E$ and $f \in C^\infty(M)$;
3. The pair (∇, β) is a hom-Lie algebra representation of $(\Gamma A, [-,-], \alpha)$ on ΓE.

Example

1. Let $\mathcal{A} = (A, \phi, [-,-], \rho, \alpha)$ be a hom-Lie algebroid over M. Then ∇^{ϕ^*} is a canonical representation of $\mathcal{A}$ on the hom-bundle $(M \times \mathbb{R}, \phi, \phi^*)$, given by $\nabla^{\phi^*}(x, f) = \rho(x)[f]$ for all $x \in \Gamma A$ and $f \in C^\infty(M)$.
2. Let $\mathcal{A} = (A, \phi, [-,-], \rho, \alpha)$ be a hom-Lie algebroid over M and (E, ϕ, β) be a hom-bundle over M, where E is a trivial line bundle over M with $s \in \Gamma E$, a nowhere vanishing section of E over M such that $\beta(s) = c.s$ for some $c \in \mathbb{K}$. Define a map $\nabla : \Gamma A \otimes \Gamma E \to \Gamma E$ by $\nabla(x, f.s) = \rho(x)[f].\beta(s)$ for all $x \in \Gamma A$ and $f \in C^\infty(M)$. Then the map ∇ is a representation of $\mathcal{A}$ on (E, ϕ, β).

Proposition 11 ([5]) *Let $\mathcal{A} = (A, \phi, [-,-], \rho, \alpha)$ be a regular hom-Lie algebroid. Then there is a one-one correspondence between representations of $\mathcal{A}$ on the hom-bundle $(\wedge^n A, \phi, \tilde{\alpha})$ and exact generators of the associated hom-Gerstenhaber algebra $\mathfrak{A} := (\oplus_{k\geq 0}\Gamma \wedge^k A^*, \wedge, [-,-]_{\mathcal{A}}, \tilde{\alpha})$ (here, $\tilde{\alpha}$ is extension of the map α to higher degree elements).*

Example Cohomology of regular hom-Lie algebroids: Let $\mathcal{A} := (A, \phi, [-,-], \rho, \alpha)$ be a regular hom-Lie algebroid over M and the map ∇ be a representation of $\mathcal{A}$ on the hom-bundle (E, ϕ, β). Then we define a cochain complex $(C^*(\mathcal{A}; E), d_{A,E})$ for $\mathcal{A}$ with coefficients in this representation as follows: $C^*(\mathcal{A}; E) := \oplus_{n\geq 0}\Gamma(Hom(\wedge^n A, E))$, and the coboundary map $d_{A,E}$ is defined as follows for $\Xi \in \Gamma(Hom(\wedge^n A, E)), x_i \in \Gamma A$ and $1 \leq i \leq n+1$.

$$
\begin{aligned}
&(d_{A,E}\Xi)(x_1, \cdots, x_{n+1}) \\
&\qquad = \sum_{i=1}^{n+1}(-1)^{i+1}\nabla_{(\alpha^{-1}(x_i))}(\Xi(\alpha^{-1}(x_1), \cdots, \hat{x_i}, \cdots, \alpha^{-1}(x_{n+1}))) \\
&+ \sum_{1\leq i<j\leq n+1}(-1)^{i+j}\beta(\Xi(\alpha^{-2}([x_i, x_j]), \alpha^{-1}(x_1), \cdots, \hat{x_i}, \cdots, \hat{x_j}, \cdots, \alpha^{-1}(x_{n+1}))).
\end{aligned}
$$

- We denote the cohomology of the resulting cochain complex $(C^*(\mathcal{A}; E), d_{A,E})$ by $H^*(\mathcal{A}, E)$.
- If $\alpha = Id_A$ and $\phi = Id_M$, then $\mathcal{A}$ is a Lie algebroid and $H^*(\mathcal{A}, E)$ is the usual de-Rham cohomology of the Lie algebroid $\mathcal{A}$ with coefficients in the representation on the vector bundle E.

Theorem 12 (Dual Description of a Hom-Lie Algebroid, see [1, 5]) *Let (A, ϕ, α) be a regular hom-bundle over M, i.e. the map $\phi : M \to M$ is a diffeomorphism and $\alpha : \Gamma A \to \Gamma A$ is an invertible map. Then a hom-Lie algebroid structure $\mathcal{A} := (A, \phi, [-,-], \rho, \alpha)$ on the hom-bundle (A, ϕ, α) is equivalent to a $(\hat{\alpha}, \hat{\alpha})$-differential graded commutative algebra on $\oplus_{n\geq 0}\Gamma(\wedge^n A^*)$, where the map $\hat{\alpha} : \Gamma(\wedge^n A^*) \to \Gamma(\wedge^n A^*)$ is defined by $\hat{\alpha}(\xi)(x_1, \cdots, x_n) = \phi^*(\xi(\alpha^{-1}(x_1), \cdots, \alpha^{-1}(x_n)))$ for $\xi \in \Gamma(\wedge^n A^*)$, and $x_i \in \Gamma A$, for $1 \leq i \leq n$.*

5 Strong Differential Hom-Gerstenhaber Algebras

A hom-Gerstenhaber algebra $\mathfrak{A} := (\oplus_{i\in\mathbb{Z}}\mathcal{A}_i, \wedge, [-,-], \alpha)$ is called a differential hom-Gerstenhaber algebra if it is equipped with a degree 1 map $d : \mathfrak{A} \to \mathfrak{A}$ such that

1. the map d is a (α, α)-derivation of degree 1 with respect to the graded commutative and associative product $\wedge$, i.e. for $X, Y \in \mathfrak{A}$,

$$d(X \wedge Y) = d(X) \wedge \alpha(Y) + (-1)^{|X|}\alpha(X) \wedge d(Y).$$

2. $d^2 = 0$ and the map d commutes with α, i.e. $d \circ \alpha = \alpha \circ d$.

The hom-Gerstenhaber algebra $\mathfrak{A}$ is said to be a **strong differential hom-Gerstenhaber algebra** if d also satisfies the equation:

$$d[X, Y] = [dX, \alpha(Y)] + [\alpha(X), dY]$$

for $X, Y \in \mathfrak{A}$. A strong differential hom-Gerstenhaber algebra $\mathfrak{A}$ is called regular if the map $\alpha : \mathfrak{A} \to \mathfrak{A}$ is an invertible map.

Example

1. For any hom-Poisson manifold, the hom-Gerstenhaber algebra associated to the tangent hom-Lie algebroid (and to the cotangent hom-Lie algebroid) is a strong differential hom-Gerstenhaber algebra.
2. For any purely hom-Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, the associated hom-Gerstenhaber algebras are equipped with a strong differential.
3. Given a Gerstenhaber algebra $(\mathcal{A}, [-,-], \wedge)$ with a strong differential d and an endomorphism $\alpha : (\mathcal{A}, [-,-], \wedge) \to (\mathcal{A}, [-,-], \wedge)$ satisfying $d \circ \alpha = \alpha \circ d$, the tuple $(\mathcal{A}, \wedge, [-,-]_\alpha = \alpha \circ [-,-], \alpha, d_\alpha = \alpha \circ d)$ is a strong differential hom-Gerstenhaber algebra.

Theorem 13 *The tuple* $(\bigoplus_{i \in \mathbb{Z}_+} \Gamma(\wedge^i A), \wedge, [-,-], \alpha, d)$ *is a strong differential regular hom-Gerstenhaber algebra if and only if* (A, A^*) *is a hom-Lie bialgebroid (see [1, 5]).*

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