Trends in Mathematics

Piotr Kielanowski
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Editors

Geometric Methods in Physics XXXVIII

Workshop, Białowieża, Poland, 2019

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Trends in Mathematics

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Preface

The Workshops on Geometric Methods in Physics (WGMP) were established in 1982 by Anatol Odzijewicz and they have been running yearly without interruption ever since under the sponsorship of the University of Białystok.

This volume contains original scholarship based on the talks presented at the Thirty-Eighth Workshop (June 30–July 6, 2019); the posters presented; and the Eighth School on Geometry and Physics, which immediately preceded the workshop. The schools were created in 2012, to precede or follow the workshops, so as to offer an introductory series of lectures on cutting-edge areas of research related to the workshop themes. The materials of recent WGMPs, such as program, abstracts of the talks and lectures, and participants' lists, are posted on the website: http://wgmp.uwb.edu.pl. The website includes bibliographical information on the proceedings volumes for each year.

Each workshop comes with main themes; in 2019, the themes were "Integrable Systems, Classical and Quantum Field Theories, Quantum Information, Lie Groupoids and Lie Algebroids, and Poisson Geometry." The school's lecturers have each contributed an extended abstract.

The WGMP has played an exceptional role in bringing together the two communities of mathematicians and physicists. The workshop has grown from its original contingent of mainly eastern-European scholars, to widely international, with participants from several continents in 2019.

The venue plays no small role in fostering a close-knit, intense experience: situated in the Białowieża Forest, a UNESCO World Heritage Place, the village hosts the participants in two small hotels and a variety of other local accommodations with hospitality and customs providing the background for communal outings, bear and bison sightings, evenings around campfires, and afternoons of discussion where new collaborations are established and old ones come to fruition. The plenary lectures, talks, and poster sessions take place in the Nature and Forest Museum, the oldest museum in the Polish national parks; the school's lectures take place in an Open-air Museum of Wooden Architecture of the Russian People of Podlasie (in Polish: Skansen Architektury Drewnianej Ludności Ruskiej Podlasia).

We hope that this collection of articles may provide the readers with an overview of the latest knowledge in a wide variety of areas, as well as stimulate interest in the threads represented by the school's lecture series.

Ciudad de México, Mexico — Piotr Kielanowski
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April 2020

Participants of the XXXVIII Workshop on Geometric Methods in Physics, Białowieża, 2018 (Photo by T. Goliński)

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Part I
Contributions to the XXXVIII Workshop

Toeplitz Extensions in Noncommutative Topology and Mathematical Physics

Francesca Arici and Bram Mesland

Abstract We review the theory of Toeplitz extensions and their role in operator K-theory, including Kasparov's bivariant K-theory. We then discuss the recent applications of Toeplitz algebras in the study of solid-state systems, focusing in particular on the bulk-edge correspondence for topological insulators.

Keywords Toeplitz algebras · C^*-algebras · Extensions · KK-theory · Bulk-edge correspondence

Mathematics Subject Classification (2010) Primary 46L85; Secondary 19K35, 46L80, 47B35, 81T75, 81V70

1 Introduction

Noncommutative topology is rooted in the equivalence of categories between locally compact topological spaces and commutative C^*-algebras. This duality allows for a transfer of ideas, constructions, and results between topology and operator algebras. This interplay has been fruitful for the advancement of both fields. Notable examples are the Connes–Skandalis foliation index theorem [17], the K-theory proof of the Atiyah–Singer index theorem [4, 5], and Cuntz's proof of Bott periodicity in K-theory [22]. Each of these demonstrates how techniques from operator algebras lead to new results in topology, or simplify their proofs. In the other direction, Connes' development of *noncommutative geometry* [19] by using techniques from Riemannian geometry to study C^*-algebras, led to the discovery of cyclic homology

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[18], a homology theory for noncommutative algebras that generalises de Rham cohomology.

Noncommutative geometry and topology techniques have found ample applications in mathematical physics, ranging from Connes' reformulation of the standard model of particle physics [20], to quantum field theory [21], and to solid-state physics. The noncommutative approach to the study of complex solid-state systems was initiated and developed in [6, 8], focusing on the quantum Hall effect and resulting in the computation of topological invariants via pairings between K-theory and cyclic homology. Noncommutative geometry techniques have proven to be a key tool in this field, and applications include the study of disordered systems, quasi-crystals and aperiodic solids [44, 45]. The correct framework to describe such systems, as has been shown recently, is via KK-theory elements for certain observable C^*-algebras.

This review is dedicated to a discussion of Toeplitz algebras and more generally C^*-extensions, and their role in noncommutative index theory. It is aimed at readers interested in the more recent applications of Toeplitz extensions and should serve as a brief overview and introduction to the subject. We shall provide an exposition of operator algebra techniques recently used in mathematical physics, in particular in the study of solid-state systems.

The paper is structured as follows. In Sect. 2 we review the construction of the classical one-dimensional Toeplitz algebra as the universal C^*-algebra generated by a single isometry, and we recall its role in the Noether–Gohberg–Krein index theorem, which relates the index of Toeplitz operators to the winding number of their symbol. We conclude the section by discussing how the construction can be extended to higher dimensions. In Sect. 3 we take a deep dive into the world of noncommutative topology and discuss the role of Toeplitz extensions in operator K-theory, namely in Cuntz's proof of Bott periodicity and in the development of Kasparov's bivariant K-theory. This rather technical section allows us to introduce the tools that are needed in the noncommutative approach to solid-state physics. In Sect. 4, we describe two constructions of universal C^*-algebras that will later play a crucial role in the study of solid-state systems, namely crossed products by the integers, Cuntz–Pimsner algebras, and their Toeplitz algebras. Finally, Sect. 5 is devoted to describing how Toeplitz extensions and the associated maps in K-theory provide the natural framework for implementing the bulk-edge correspondence from solid-state physics.

2 Toeplitz Algebras of Operators

2.1 Shifts, Winding Numbers, and the Noether–Gohberg–Krein Index Theorem

In view of the Gelfand–Naimark theorem [25], every abstract C^*-algebra, commutative or not, admits a faithful representation as a subalgebra of the algebra $B(H)$ of bounded operators on some Hilbert space H. In this section, we will start by constructing two concrete examples of C^*-algebras of operators. As mentioned in the Introduction, we are interested in how the commutative algebra of functions on the circle and the noncommutative algebra generated by a single isometry fit together in a short exact sequence. This extension will later serve as our prototypical example illustrating the use of C^*-algebraic techniques in solid-state physics.

Let $S^1 := \{z \in \mathbb{C} \mid \overline{z}z = 1\}$ denote the unit circle in the complex plane. The corresponding C^*-algebra, $C(S^1)$, is the closure in the supremum norm of the algebra of *Laurent polynomials*

$$\mathcal{O}(S^1) = \frac{\mathbb{C}[z, \overline{z}]}{\langle \overline{z}z = 1\rangle}.$$

The algebra $C(S^1)$ admits a convenient representation on the Hilbert space $L^2(S^1)$ of square-integrable functions on S^1. This Hilbert space is isomorphic to the Hilbert space of sequences $\ell^2(\mathbb{Z})$, and the isomorphism is implemented by the discrete Fourier transform

$$\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2(S^1), \qquad (\mathcal{F}\phi)(z) = (2\pi)^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}} \phi_n e^{-in\cdot z}. \tag{1}$$

Under this isomorphism, the operator of multiplication by z is mapped to the bilateral shift operator U, defined on the standard basis $\{e_n\}_{n\in\mathbb{Z}}$ of $\ell^2(\mathbb{Z})$ via

$$U(e_n) = (e_{n+1}), \quad U^*(e_n) = e_{n-1}. \tag{2}$$

It is easy to see that U is a *unitary* operator, i.e. $U^*U = 1 = UU^*$. The algebra $C(S^1)$ is then isomorphic to the smallest C^*-subalgebra of $B(\ell^2(\mathbb{Z}))$ that contains U.

In order to define the second C^*-algebra we are interested in, which is genuinely non-commutative, we shall consider the Hardy space $H^2(S^1)$. This is defined as the subset of $L^2(S^1)$ consisting of continuous functions that extend holomorphically to the unit disk. The projection $P : L^2(S^1) \to H(S^1)$ is called the *Hardy projection*. Under the discrete Fourier transform, it corresponds to the projection $p : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{N})$.

Multiplication by z on the Hardy space corresponds to a shift operator on $\ell^2(\mathbb{N})$, called the *unilateral shift*, expressed on the standard basis $\{f_n\}_{n\in\mathbb{N}}$ of $\ell^2(\mathbb{N})$ via:

$$T(f_n) = (f_{n+1}).$$

Its adjoint is not invertible, as

$$T^*(f_n) = \begin{cases} f_{n-1} & n \geq 1 \\ 0 & n = 0 \end{cases}.$$

This motivates the following:

Definition 1 The *Toeplitz algebra* $\mathcal{T}$ is the smallest C^*-subalgebra of $B(\ell^2(\mathbb{N}))$ that contains T.

It is easy to see that the Toeplitz algebra $\mathcal{T}$ is not commutative, as

$$T^*T = 1, \qquad TT^* = 1 - \mathrm{p}_{\ker(T^*)}. \tag{3}$$

In particular, it follows from (3) that elements of $\mathcal{T}$ commute up to compact operators, and in particular the generator T is unitary modulo compact operators. In other words, the Toeplitz algebra can be viewed as the C^*-algebra extension of continuous functions on the circle by the compact operators:

$$0 \longrightarrow \mathcal{K}(\ell^2(\mathbb{N})) \longrightarrow \mathcal{T} \xrightarrow{\pi} C(S^1) \longrightarrow 0. \tag{4}$$

The extension (4) admits a completely positive and completely contractive splitting given by the Hardy projection P. Indeed, for every $f \in C(S^1)$, the assignment

$$T_f(g) = \mathrm{P}(fg), \quad g \in H^2(S^1) \tag{5}$$

defines a bounded operator on the Hardy space $H^2(S^1)$, where, under Fourier transform, T_z corresponds to the unilateral shift. As the function z generates $C(S^1)$ as a C^*-algebra, every such T_f is an element of $\mathcal{T}$.

The following result implies that the Toeplitz algebra is the universal C^*-algebra generated by an element T satisfying $T^*T = 1$:

Theorem 2 (Coburn [16]) *Suppose v is an isometry in a unital C^*-algebra A. Let $T = T_z \in \mathcal{T}$. Then there exists a unique unital $*$-homomorphism $\phi : \mathcal{T} \to A$ such that $\phi(T) = v$. Moreover, if $vv^* \neq 1$, then the map ϕ is isometric.*

2.1.1 The Noether–Gohberg–Krein Index Theorem

Recall that an operator $F \in B(H)$ is a *Fredholm operator* if F has closed range and both $\ker F$ and $\ker F^*$ are finite-dimensional. The *Fredholm index* of such an operator is the integer

$$\mathrm{Ind}(F) = \dim\ker F - \dim\ker F^* \in \mathbb{Z}.$$

One of the key properties of the Fredholm index is that it is constant along continuous paths of Fredholm operators. As such it is a homotopy invariant.

The completely positive linear splitting $f \mapsto T_f$ allows one to give a precise characterisation of which Toeplitz operators T_f are Fredholm. Moreover, the index of a Fredholm Toeplitz operator T_f can be described entirely in terms of a familiar homotopy invariant of the complex function f. This is the content of the Toeplitz index theorem, due to F. Noether and later reproved independently by Gohberg and Krein. It was one of the first results linking index theory to topology and should be viewed as an ancestor to the celebrated Atiyah–Singer index theorem.

Theorem 3 (Noether [41], Gohberg–Krein [27]) *For $f : S^1 \to \mathbb{C}^\times$ the operator $T_f : H^2(S^1) \to H^2(S^1)$ is Fredholm and*

$$\operatorname{Ind}\left(T_f\right) = -w(f),$$

with $w(f)$ the winding number of f. If f is a C^1-function, then the winding number can be computed as

$$w(f) = \int_{S^1} \frac{f'(z)}{f(z)} \mathrm{d}z.$$

The latter, explicit expression for the winding number shows that the Toeplitz index should be viewed as a result of *differential* topology: By choosing a nice representative in the homotopy class of the function f, the differential calculus can be employed to compute a topological invariant. We will see an application of this computation in Sect. 5.

2.2 Generalisation: Higher Toeplitz Algebras

2.2.1 Toeplitz Operators on Strongly Pseudo-Convex Domains

The definition of Toeplitz operators on the circle in terms of the Hardy space lends itself to generalisations to higher dimensions. The crucial observation here is that the Hardy space $H^2(S^1)$ can be defined as the closure of the space of boundary values of holomorphic functions on the unit disk that admit a continuous extension to the closed unit disk.

Definition 4 ([48, Definition 1.2.18]) Let Ω be a smooth domain in $\mathbb{C}^n$ with defining function $\rho \in C^\infty(\mathbb{C}^n)$:

$$\Omega = \{z \in \mathbb{C}^n \ : \rho(z) < 0\}$$

and boundary $\partial\Omega = \{z \in \mathbb{C}^n \ : \rho(z) = 0\}$. For every $z \in \partial\Omega$, the Levi form $\langle\, ,\, \rangle_z$ is defined as

$$\langle u, v\rangle_z := \sum_{1\le i,j\le n} \frac{\partial^2\rho}{\partial z_i \partial \overline{z}_j}(z) u_j \overline{v}_j, \qquad u, v \in \mathbb{C}^n.$$

Then Ω is called a *strongly pseudo-convex domain* if the Levi form is positive semi-definite on the complex tangent space at every point $z \in \partial\Omega$. That, for every nonzero $u \in T_z(\partial\Omega)$ it holds that $\langle u, u\rangle_z > 0$.

Open balls in $\mathbb{C}^n$ are examples of strongly pseudo-convex domains. However, the product of two open balls is not strongly pseudo-convex, showing the notion is somewhat subtle.

Given a strongly pseudo-convex domain $\Omega \subseteq \mathbb{C}^n$ with smooth boundary, we denote by $L^2(\partial\Omega)$ the Hilbert space of square integrable functions on the boundary $\partial\Omega$. The Hardy space $H^2(\partial\Omega)$ is defined as the Hilbert space closure in $L^2(\partial\Omega)$ of boundary values of holomorphic functions on Ω that admit a continuous extensions to the boundary $\partial\Omega$ (cf. [48, Definition 2.3]). The orthogonal projection

$$\mathrm{P}_{CS} : L^2(\partial\Omega) \to H^2(\partial\Omega),$$

called the Cauchy–Szegö projection, is used to define Toeplitz operators, in analogy with (5). Indeed, let f be a continuous function on $\partial\Omega$, the Toeplitz operator with symbol f is defined as

$$T_f(g) = \mathrm{P}_{CS}(fg),$$

for all $g \in H^2(\partial\Omega)$.

For any two $f, f' \in C(\partial\Omega)$, the product of Toeplitz operators $T_f \circ T_{f'}$ is equal to $T_{ff'}$ modulo compact operators. Moreover, for any $f \in C(\partial\Omega)$, the operator T_f is compact if and only if f is identically zero. These two facts combined lead to the following:

Theorem 5 *Let Ω be a strongly pseudo-convex domain. Let $\mathcal{T}(\partial\Omega)$ be the closed subalgebra of $B(H^2(\partial\Omega))$ that contains all the Toeplitz operators. There is an extension of C^*-algebras*

$$0 \longrightarrow \mathcal{K}(H^2(\partial\Omega)) \longrightarrow \mathcal{T}(\partial\Omega) \longrightarrow C(\partial\Omega) \longrightarrow 0.$$

The extension admits a completely positive and completely contractive linear splitting given by the Cauchy–Szegö projection.

Applied to the unit ball in $\mathbb{C}^n$ this construction yields the Toeplitz extensions for odd-dimensional spheres as a special case:

$$0 \longrightarrow \mathcal{K}(H^2(S^{2d-1})) \longrightarrow \mathcal{T}(S^{2d-1}) \longrightarrow C(S^{2d-1}) \longrightarrow 0,$$

which clearly recover (4) for $d = 1$.

The Toeplitz algebra $\mathcal{T}(S^{2d-1})$ admits an equivalent description in terms of so-called d-shifts, as described in [3, Theorem 5.7]. For an overview of the interplay of Toeplitz C^*-algebras and index theory, as well as their role in the computation of noncommutative invariants, we refer the reader to the excellent survey [38].

3 Toeplitz Algebras in Operator K-Theory and Bivariant K-Theory

An indispensable tool in Fredholm index theory is operator K-theory, a functor associating to a C^*-algebra A two Abelian groups $K_*(A),\ * = 0, 1$. Functoriality means that for a $*$-homomorphism $\varphi : A \to B$ between C^*-algebras A and B, there are induced homomorphism of Abelian groups

$$\varphi_* : K_*(A) \to K_*(B).$$

The key properties of the operator K-theory functor are that it is *homotopy invariant*, *half-exact* and *Morita invariant*. We now define each of these properties more precisely.

Homotopy invariance is the property that if φ and ψ are connected by a continuous path of $*$-homomorphisms, then the induced maps on K-theory coincide, that is $\varphi_* = \psi_*$.

Half-exactness is the property that for any extension of C^*-algebras

$$0 \longrightarrow I \xrightarrow{\ i\ } E \xrightarrow{\ p\ } A \longrightarrow 0, \tag{6}$$

the corresponding sequence of groups

$$K_*(I) \xrightarrow{\ i_*\ } K_*(E) \xrightarrow{\ p_*\ } K_*(A),$$

is exact at $K_*(E)$.

Lastly, Morita invariance entails that for any rank-one projection $p \in \mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$, the $*$-homomorphism

$$A \to \mathcal{K} \otimes A, \quad a \mapsto p \otimes a,$$

induces an isomorphism in K-theory.

Recall that the *suspension* SA of a C^*-algebra A is defined to be

$$SA := C_0(0, 1) \otimes A \simeq C_0((0, 1), A),$$

which is a C^*-algebra in the sup-norm, and pointwise product and involution inherited from A.

The operation $A \to SA$ is functorial for $*$-homomorphisms, and it is customary to define the *higher* K-groups as $K_n(A) := K_0(S^n A)$. Via a general construction in topology, it follows that the extension (6) induces a long exact sequence

$$\cdots \to K_{n+1}(A) \to K_n(I) \to K_n(E) \to K_n(A) \to K_{n-1}(I) \to \cdots, \tag{7}$$

of Abelian groups.

The boundary maps in such exact sequences are often related to index theory. For instance, for the Toeplitz extension (4), the boundary map

$$\partial : K_1(C(S^1)) \to K_0(\mathcal{K}(\ell^2(\mathbb{N})) \simeq \mathbb{Z}, \tag{8}$$

maps the class of a nonzero function $f \in C(S^1)$ to the index of the corresponding Toeplitz operator T_f.

One of the key features of operator K-theory is *Bott periodicity*. It states that for any C^*-algebra A there are natural isomorphisms between its K-theory and the K-theory of its double suspension $S^2 A$. It turns out that the three properties of homotopy invariance, half-exactness and Morita invariance suffice to deduce the existence of natural *Bott periodicity* isomorphisms $K_*(A) \simeq K_*(S^2 A)$. As a consequence, there are only two K-functors, K_0 and K_1, and the exact sequence (7) reduces the cyclic six-term exact sequence

$$\begin{array}{ccccc} K_0(I) & \xrightarrow{i_*} & K_0(E) & \xrightarrow{p_*} & K_0(A) \\ \uparrow & & & & \downarrow \\ K_1(A) & \xleftarrow[p_*]{} & K_1(E) & \xleftarrow[i_*]{} & K_1(I). \end{array}$$

3.1 Cuntz's Proof of Bott Periodicity

Apart from the invariance properties of the K-functor, Cuntz's proof of Bott periodicity (cf. [22]) exploits essential properties of the Toeplitz extension (4). By composing the projection homomorphism $\pi : \mathcal{T} \to C(S^1)$ with the evaluation map $\mathrm{ev}_1 : C(S^1) \to \mathbb{C}$, given by $\mathrm{ev}_1(f) = f(1)$, we obtain a character of $\mathcal{T}$:

$$\chi := \mathrm{ev}_1 \circ \pi : \mathcal{T} \to \mathbb{C}. \tag{9}$$

The unital embedding $\iota : \mathbb{C} \to \mathcal{T}$ splits the homomorphism χ in the sense that $\chi \circ \iota = \mathrm{id}_{\mathbb{C}}$. It is a non-trivial fact that these $*$-homomorphisms are mutually inverse in K-theory, in a strong sense made precise below.

To state the result, which lies at the heart of the proof of the Bott periodicity theorem, we shall recall the construction of the *spatial* or *minimal* tensor product $A_1 \overline{\otimes} A_2$ of C^*-algebras $A_i, i = 1, 2$. Choose faithful representations $\pi_i : A_i \to B(\mathcal{H}_i)$ and let $\mathcal{H}_1 \otimes \mathcal{H}_2$ be the completed tensor product of Hilbert spaces. One defines $A\overline{\otimes}B$ to be the completion of the algebraic tensor product $A \otimes B$ in the norm inherited from the representation

$$\pi_1 \otimes \pi_2 : A_1 \otimes A_2 \to B(\mathcal{H}_1 \otimes \mathcal{H}_2).$$

Proposition 6 ([22, Proposition 4.3]) *Let A be a C^*-algebra. The map $\chi \otimes 1 : \mathcal{T}\overline{\otimes}A \to A$ induces an isomorphism $\chi_* \otimes 1 : K_0(\mathcal{T}\overline{\otimes}A) \xrightarrow{\sim} K_0(A)$.*

Tensor products of C^*-algebras are not unique, and the spatial tensor product is the completion in the minimal C^*-norm on the algebraic tensor product $A \otimes B$. There is also a maximal C^*-norm on $A \otimes B$, which involves taking the supremum over all representations. A C^*-algebra N is *nuclear*, if for any other C^*-algebra A, the minimal and maximal C^*-tensor norms on $N \otimes A$ coincide. For our purposes it suffices to know that all commutative C^*-algebras are nuclear. Given an extension of C^*-algebras

$$0 \longrightarrow I \longrightarrow E \longrightarrow B \longrightarrow 0\ , \tag{10}$$

the sequence of tensor products

$$0 \longrightarrow I\overline{\otimes}A \longrightarrow E\overline{\otimes}A \longrightarrow B\overline{\otimes}A \longrightarrow 0\ , \tag{11}$$

may fail to be exact in the middle. However, nuclearity of the C^*-algebra B guarantees exactness.

Lemma 7 (cf. [15, Corollary 3.7.4]) *Let A be a C^*-algebra and consider an extension (10). If the C^*-algebra B is nuclear, then the sequence (11) is exact.*

We can now exploit Proposition 6, Lemma 7, and the exactness properties of the K-functor to deduce Bott periodicity.

Theorem 8 *For any C^*-algebra A there are natural isomorphisms $K_n(A) \simeq K_{n+2}(A)$.*

Proof Consider the character χ defined in (9) and let $\mathcal{T}_0 := \ker \chi$, so that we have an extension

$$0 \longrightarrow \mathcal{T}_0 \longrightarrow \mathcal{T} \longrightarrow \mathbb{C} \longrightarrow 0\ .$$

As $\mathbb{C}$ is nuclear, this extension has the property that the induced sequence

$$0 \longrightarrow \mathcal{T}_0\overline{\otimes}A \longrightarrow \mathcal{T}\overline{\otimes}A \longrightarrow A \longrightarrow 0\ ,$$

is exact for any C^*-algebra A as well, by Lemma 7.

The long exact sequence (7), together with the fact that $S(A\overline{\otimes}B) \simeq A\overline{\otimes}SB$ and Proposition 6, imply that $\chi_* : K_n(\mathcal{T}\overline{\otimes}A) \to K_n(A)$ is an isomorphism for all n.

Consequently $K_n(\mathcal{T}_0\overline{\otimes}A) = 0$ for all n. Now observe that, after identifying $\ker \mathrm{ev}_1$ with $C_0(0, 1)$, we can construct a second extension

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T}_0 \longrightarrow C_0(0,1) \longrightarrow 0\,.$$

As $C_0(0, 1)$ is nuclear, this extension, too, has the property that

$$0 \longrightarrow \mathcal{K}\overline{\otimes}A \longrightarrow \mathcal{T}_0\overline{\otimes}A \longrightarrow C_0(0,1)\overline{\otimes}A \longrightarrow 0$$

is exact for any C^*-algebra A, by Lemma 7. Since $C_0(0, 1)\overline{\otimes}A \simeq SA$, the long exact sequence (7) gives an isomorphism

$$K_{n+1}(C(0,1)\overline{\otimes}A) \xrightarrow{\sim} K_n(\mathcal{K}\overline{\otimes}A).$$

Now we use the Morita invariance isomorphism $K_n(\mathcal{K}\overline{\otimes}A) \simeq K_n(A)$ and the fact that $C(0, 1)\overline{\otimes}A \simeq SA$ to deduce that

$$K_{n+2}(A) \simeq K_{n+1}(C(0,1)\overline{\otimes}A) \xrightarrow{\sim} K_n(\mathcal{K}\overline{\otimes}A) \simeq K_0(A),$$

which yields the Bott periodicity isomorphism. □

We remark that, in fact, the theorem holds if we replace K by any functor that is homotopy invariant, half-exact and Morita invariant. We also note that earlier work of Karoubi [31] provides another short and conceptual proof of Bott periodicity. Although Bott periodicity does not hold in algebraic K-theory, Karoubi's proof puts algebraic and topological K-theory of Banach algebras on the same footing.

3.2 Toeplitz Extensions and Bivariant K-Theory

As we have seen so far in the Toeplitz index and Bott periodicity theorems, extensions of C^*-algebras play a crucial role in K-theory and henceforth in index theory. An extension of a C^*-algebra A by B should be viewed as a new C^*-algebra, built by "gluing together" A and B in a possibly topologically nontrivial way.

In [14], Brown, Douglas, and Fillmore initiated the study of extensions by considering exact sequences of the form

$$0 \longrightarrow \mathcal{K}(H) \longrightarrow E \longrightarrow C(M) \longrightarrow 0,$$

for some Hilbert space H and some compact Hausdorff topological space M. They proved that such extensions form an Abelian group by defining addition via an appropriate version of the Baer sum. They also showed that their Abelian group is dual to K-theory in a precise sense governed by Fredholm index theory.

Kasparov generalised this construction to extensions

$$0 \longrightarrow \mathcal{K}(X) \longrightarrow E \longrightarrow A \longrightarrow 0,$$

where A is a separable C^*-algebra and X a countably generated Hilbert C^*-module over a second, σ-unital C^*-algebra B. A technical assumption on such extensions is that they admit a completely positive and completely contractive linear splitting $\ell : A \to E$ such that $\ell \circ \pi = \mathrm{id}_A$. This assumption is automatically satisfied when the quotient algebra in the extension is nuclear. Commutative C^*-algebras are nuclear, and thus the Toeplitz extensions discussed previously satisfy this assumption. The isomorphism classes of such extensions form an Abelian group $\mathrm{Ext}^1(A, B)$ which is isomorphic to the Kasparov group $KK_1(A, B)$. This section is devoted to making this statement more precise. An excellent reference for this discussion is [28, Chapter 3].

3.2.1 Hilbert Modules and C^*-Correspondences

Before we proceed, we need to recall some results from the theory of Hilbert C^*-modules. For more details on the latter, we refer the interested reader to the monograph [37] and to the recent article [36].

Definition 9 A *pre-Hilbert module* over a C^*-algebra B is a right B-module X with a B-valued Hermitian product, i.e. a map $\langle \cdot, \cdot \rangle_B : X \times X \to B$ satisfying

$$\langle \xi, \eta + \zeta \rangle_B = \langle \xi, \eta \rangle_B + \langle \xi, \zeta \rangle_B,$$
$$\langle \xi, \eta \rangle_B = \langle \eta, \xi \rangle_B^*, \qquad \langle \xi, \eta b \rangle_B = \langle \xi, \eta \rangle_B b,$$
$$\langle \xi, \xi \rangle_B \geq 0, \qquad \langle \xi, \xi \rangle_B = 0 \Leftrightarrow \xi = 0,$$

for all $\xi, \eta, \zeta \in X$ and for all $b \in B$.

Note that using the existence of approximate units in C^*-algebras, one can prove that the inner product automatically satisfies $\langle \xi, \lambda\eta \rangle_B = \lambda \langle \xi, \eta \rangle_B$ for all $\xi, \eta \in X$ and $\lambda \in \mathbb{C}$ (cf. [36, Section 2]).

For a pre-Hilbert module X, one can define a scalar valued norm $\| \cdot \|$ using the C^*-norm on B:

$$\|\xi\|^2 = \| \langle \xi, \xi \rangle_B \|_B. \tag{12}$$

Definition 10 A *Hilbert C^*-module* is a pre-Hilbert module that is complete in the norm (12).

If one defines $\langle X, X \rangle$ to be the linear span of elements of the form $\langle \xi, \eta \rangle$ for $\xi, \eta \in X$, then its closure is a two-sided ideal in B. We say that the Hilbert module X is *full* whenever $\langle X, X \rangle$ is dense in B.

Let now X, Y be two Hilbert C^*-modules over the same C^*-algebra B.

Definition 11 A map $T : X \to Y$ is said to be an *adjointable operator* if there exists another map $T^* : Y \to X$ with the property that

$$\langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle \qquad \text{for all} \quad \xi \in X, \eta \in Y\ .$$

Every adjointable operator is automatically right B-linear and bounded. However, the converse is in general not true: a bounded linear map between Hilbert modules need not be adjointable. We denote the collection of adjointable operators from X to Y by $\mathrm{Hom}^*_B(X, Y)$. When $X = Y$, the adjointable operators form a C^*-algebra in the operator norm, that is denoted by $\mathrm{End}^*_B(X)$.

Inside the adjointable operators one can single out a particular subspace, which is analogous to that of finite-rank operators on a Hilbert space. More precisely, for every $\xi \in Y, \eta \in X$ one defines the operator $\theta_{\xi,\eta} : X \to Y$ as

$$\theta_{\xi,\eta}(\zeta) = \xi\langle\eta, \zeta\rangle, \qquad \forall\zeta \in X. \tag{13}$$

This is an adjointable operator, with adjoint $\theta^*_{\xi,\eta} : Y \to X$ given by $\theta_{\eta,\xi}$.

We denote by $\mathcal{K}_B(X, Y)$ the closure of the linear span of

$$\{\theta_{\xi,\eta} \mid \xi, \eta \in X\} \subseteq \mathrm{Hom}^*_B(X, Y), \tag{14}$$

and we refer to it as the space of *compact adjointable operators*. In particular $\mathcal{K}_B(X) := \mathcal{K}_B(X, X) \subseteq \mathrm{End}^*_B(X)$ is a closed two-sided ideal in the C^*-algebra $\mathrm{End}^*_B(X)$, hence a C^*-subalgebra, whose elements are referred to as *compact endomorphisms*. Elements of $\mathcal{K}_B(X)$ and of $\mathrm{End}^*_B(X)$ act on X from the left, motivating the following:

Definition 12 A *C^*-correspondence* (X, ϕ) *from A to B*, is a right Hilbert B-module X endowed with a $*$-homomorphism $\phi : A \to \mathrm{End}^*_B(X)$. If $\phi : A \to \mathcal{K}_B(X)$ we refer to (X, ϕ) as a *compact* C^*-correspondence and in the case $A = B$ we refer to (X, ϕ) as a C^*-correspondence *over B*.

When no confusion arises, we will omit the map ϕ and simply write X.

Two C^*-correspondences (X, ϕ) and (Y, ψ) over the same algebra B are called *isomorphic* if and only if there exists a unitary $U \in \mathrm{End}^*_B(X, Y)$ intertwining ϕ and ψ.

Given an (A, B)-correspondence (X, ϕ) and a (B, C)-correspondence (Y, ψ), one can construct an (A, C)-correspondence, named the *interior tensor product* of (X, ϕ) and (Y, ψ). As a first step, one constructs the *balanced tensor product* $X \otimes_B Y$ which is a quotient of the algebraic tensor product $X \otimes_{\mathrm{alg}} Y$ by the subspace generated by elements of the form

$$\xi b \otimes \eta - \xi \otimes \psi(b)\eta, \tag{15}$$

for all $\xi \in X,\ \eta \in Y,\ b \in B$.

This has a natural structure of right module over C given by

$$(\xi \otimes \eta)c = \xi \otimes (\eta c),$$

and a C-valued inner product defined on simple tensors as

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle_C := \langle \eta_1, \psi(\langle \xi_1, \xi_2 \rangle_B)\eta_2 \rangle_C, \tag{16}$$

and extended by linearity.

The inner product is well-defined (cf. [37, Proposition 4.5]); in particular, the null space $N = \{\zeta \in X \otimes_{\mathrm{alg}} Y\, ;\ \langle \zeta, \eta \rangle = 0\}$ can be shown to coincide with the subspace generated by elements of the form in (15).

One then defines $X\widehat{\otimes}_\psi Y$ to be the right Hilbert module obtained by completing $X \otimes_B Y$ in the norm induced by (16). Moreover for every $T \in \mathrm{End}^*_B(X)$, the operator defined on simple tensors by

$$\xi \otimes \eta \mapsto T(\xi) \otimes \eta$$

extends to a well-defined operator $\phi_*(T) := T \otimes 1$. It is adjointable with adjoint given by $T^* \otimes 1 = \phi_*(T^*)$. In particular, this means that there is a left action of A defined on simple tensors by

$$(\phi \otimes_\psi 1)(a)(\xi \otimes \eta) = \phi(a)\xi \otimes \eta,$$

and extended by linearity to a map

$$\phi \otimes_\psi 1 : A \to \mathrm{End}^*_C(X\widehat{\otimes}_\psi Y),$$

thus turning $X\widehat{\otimes}_\psi Y$ into an (A, C)-correspondence. For all the details, we refer the reader once more to [37, Chapter 4].

We remark that the interior tensor product induces an associative operation on isomorphism classes of C^*-correspondences.

3.2.2 Kasparov Modules and the Theory of Extensions

We now come to defining the key objects in Kasparov's bivariant K-theory [34], which are inspired by the geometry of elliptic operators on manifolds. For technical reasons, Kasparov theory is developed under some mild countability assumptions. Recall that a C^*-algebra B is *σ-unital* if it admits a countable approximate unit, and *separable* if it admits a countable dense subset. Separable C^*-algebras are σ-unital. A Hilbert C^*-module X over B is *countably generated* if there is a countable subset $\{x_i\} \subset X$ such that the right B submodule generated by $\{x_i\}$ is dense in X.

Definition 13 An *odd* Kasparov (A, B)-bimodule is a pair (Y, F) where $Y = (Y, \phi)$ is a countably generated Hilbert C^*-correspondence from A to B, and $F \in \mathrm{End}^*_B(Y)$ is a self-adjoint operator such that $F^2 = 1$ and $[F, \phi(a)] \in \mathcal{K}(Y)$. An *even* Kasparov module is a triple (Y, F, γ) such that (Y, F) is an odd Kasparov module and $\gamma \in \mathrm{End}^*_B(Y)$ is a self-adjoint unitary that commutes with A and anticommutes with F.

The natural equivalence relation of homotopy of Kasparov modules is conveniently defined via Kasparov modules for $(A, C([0, 1], B))$. The homotopy classes of odd Kasparov (A, B)-modules form an Abelian group denoted $KK_1(A, B)$. Similarly, the homotopy classes of even Kasparov modules form an Abelian group $KK_0(A, B)$. If we choose $A = \mathbb{C}$ then there are natural isomorphisms $KK_*(\mathbb{C}, B) \simeq K_*(B)$, and as such KK-theory generalises K-theory. The main feature of the theory is the existence of an associative, bilinear product structure

$$KK_i(A, B) \times KK_j(B, C) \to KK_{i+j}(A, C), \tag{17}$$

the *Kasparov product*, defined whenever A is separable and B is σ-unital. Again, if we set $A = \mathbb{C}$, we see that elements in $KK_j(B, C)$ induce maps $K_*(B) \to K_{*+j}(C)$ by taking products from the right.

There is a close relationship between the Abelian groups $KK_1(A, B)$ and $\mathrm{Ext}^1(A, B)$ which can be understood via the following *Kasparov–Stinespring theorem*, first proved in [33].

Theorem 14 (See the Proof of Theorem 3.2.7 in [28]) *Let A, B be C^*-algebras, with A separable and B σ-unital. Let X be a countably generated Hilbert C^*-module over B and $\rho : A \to \mathrm{End}^*_B(X)$ be a completely positive contraction. There exists a countably generated Hilbert C^*-module Y over B, a $*$-homomorphism $\pi : A \to \mathrm{End}^*_B(Y)$ and an isometry $v : X \to Y$ such that $\rho(a) = v^*\pi(a)v$.*

A proof of the above theorem is obtained by combining the proof of Theorem 3.2.7 in [28] with Kasparov's stabilisation theorem for countably generated C^*-modules [33, Theorem 3.2]. For our KK-theoretic purposes, remaining in the countably generated category is of vital importance, but the reader is invited to consult the more general versions of this result that are available, see for instance [37, Theorem 5.6].

It is worth noting that such an isometry $v : X \to Y$ immediately gives rise to a Toeplitz type algebra

$$\mathcal{T}_v := vv^*\mathrm{End}^*_B(Y)vv^* \simeq \mathrm{End}^*_B(X).$$

To an extension

$$0 \longrightarrow \mathcal{K}(X) \longrightarrow E \longrightarrow A \longrightarrow 0,$$

with a completely positive linear splitting $\ell : A \to E$, we can associate an odd Kasparov module by observing that, as $\mathcal{K}(X)$ is an ideal in E, there is a

$*$-homomorphism $\varphi : E \to \mathrm{End}^*_B(X)$. We consider the completely positive contraction $\rho := \varphi \circ \ell : A \to \mathrm{End}^*_B(X)$ and obtain an (A, B)-bimodule Y and an isometry $v : X \to Y$ via Theorem 14.

Theorem 15 *Let X be a countably generated Hilbert C^*-module over the σ-unital C^*-algebra B and A a separable C^*-algebra. If*

$$0 \longrightarrow \mathcal{K}(X) \longrightarrow E \longrightarrow A \longrightarrow 0,$$

is a semisplit extension with completely contractive and completely positive linear splitting $\ell : A \to E$, then the Stinespring dilation $v : X \to Y$ of $\rho := \varphi \circ \ell : A \to \mathrm{End}^*_B(X)$ makes $(Y, 2vv^* - 1)$ into an odd Kasparov module for (A, B).

Proof As Y is an (A, B)-correspondence and $F = 2vv^* - 1$ it holds that $F^2 = 1$ and $F^* = F$. Hence all we need to check is that $[F, \pi(a)] = 2[vv^*, \pi(a)]$ is an element of $\mathcal{K}(Y)$. Write $p = vv^*$, so $p^2 = p^* = p$ and

$$[p, \pi(a)] = p\pi(a)(1 - p) - (1 - p)\pi(a)p.$$

It thus suffices to show that $p\pi(a)(1 - p)\pi(a)^*p \in \mathcal{K}(Y)$, for $\mathcal{K}(Y)$ is an ideal in $\mathrm{End}^*_B(Y)$ and thus for $T \in \mathrm{End}^*_B(Y)$ it holds that $T \in \mathcal{K}(Y)$ if and only if $TT^* \in \mathcal{K}(Y)$ (see for instance [10, Proposition II.5.1.1.ii]). Now $v\mathcal{K}(X)v^* \subset \mathcal{K}(Y)$, since for $x_1, x_2 \in X$ it holds that $v\theta_{x_1,x_2}v^* = \theta_{v(x_1),v(x_2)}$, and we compute

$$\begin{aligned} p\pi(a)(1 - p)\pi(a^*)p &= vv^*\pi(a)(1 - vv^*)\pi(a^*)vv^* \\ &= v(v^*\pi(a)vv^*\pi(a^*)v - v^*\pi(aa^*)v)v^* \\ &= v(\ell(a)\ell(a^*) - \ell(aa^*))v^* \in v\mathcal{K}(X)v^*. \end{aligned}$$

This proves that (Y, F) is a Kasparov module. □

By the previous theorem, we see that an extension of C^*-algebras induces an element in $KK_1(A, B)$. Using the product structure (17), this leads to the elegant viewpoint that an extension induces maps

$$\otimes_A[(Y, F)] : K_*(A) \to K_{*+1}(B),$$

via the Kasparov product. These maps coincide with the boundary maps in the long exact sequence associated to the extension. For instance, the product with the extension

$$0 \longrightarrow \mathcal{K}\overline{\otimes}A \longrightarrow \mathcal{T}_0\overline{\otimes}A \longrightarrow C_0(0,1)\overline{\otimes}A \longrightarrow 0\ ,$$

of the previous section induces the Bott periodicity isomorphisms $K_n(S^2A) \simeq K_n(A)$. In fact, the extension above, in combination with the Kasparov product, can be used to prove the general bivariant Bott periodicity isomorphisms

$$KK_*(S^2A, B) \simeq KK_*(A, B) \simeq KK_*(A, S^2B),$$

for any pair of separable C^*-algebras (A, B).

The Kasparov–Stinespring construction can be inverted up to homotopy, yielding the statement that $KK_1(A, B)$ is isomorphic to $\mathrm{Ext}^1(A, B)$. Effectively, this amounts to the observation that KK-theory is nothing but the study of extensions of C^*-algebras.

To conclude, let us sketch the inverse construction. An odd Kasparov module (X, F) for (A, B) defines an adjointable projection $P := \frac{1}{2}(F + 1)$ and hence a complemented submodule $X := PY \subset Y$. The C^*-subalgebra

$$E := \left\{(PTP, a) \in \mathrm{End}^*_B(X) \oplus A : T \in \mathrm{End}^*_B(Y), \quad P(T - a)P \in \mathcal{K}(Y)\right\},$$

of $\mathrm{End}^*_B(Y) \oplus A$ is an extension of A by $\mathcal{K}(X)$. To see that E is closed under products, we use that

$$\begin{aligned} PSPTP - PabP &= P(S - a)PTP + PaP(T - b)P - Pa(1 - P)bP \\ &= P(S - a)PTP + PaP(T - b)P - [P, a](1 - P)bP, \end{aligned}$$

which is an element of $\mathcal{K}(X)$. The quotient map $E \to A$, given by $(PTP, a) \mapsto a$ has kernel $\mathcal{K}(X) = \mathcal{K}(PY)$. Moreover, it admits the completely contractive linear splitting

$$\ell : A \to E, \quad \ell : a \mapsto (PaP, a).$$

The C^*-algebra E can be viewed as an *abstract Toeplitz algebra* associated to the Kasparov module (Y, F). This inverts the Kasparov–Stinespring construction, as is easily checked.

4 Toeplitz Algebras, Crossed Products by the Integers, and Cuntz–Pimsner Algebras

We will now describe two constructions of Toeplitz C^*-algebras and quotients thereof that appear in the study of solid-state systems, as they provide the natural framework for implementing the bulk-edge correspondence.

4.1 *Crossed Products by the Integers and the Pimsner–Voiculescu Toeplitz Algebra*

Our first object of study are crossed products by the integers. They constitute one of the simplest and most well-understood examples of C^*-algebras associated to C^*-dynamical systems, a class of objects which were introduced to study group actions on C^*-algebras.

Let α be an automorphism of a unital C^*-algebra B. This defines an action of the additive group $\mathbb{Z}$ of integers on B given by

$$\mathbb{Z} \to \mathrm{Aut}(B), \quad n \mapsto \alpha^n.$$

The crossed product C^*-algebra $B \rtimes_\alpha \mathbb{Z}$ is realised as the universal C^*-algebra generated by B and a unitary u satisfying the covariance condition

$$\alpha^n(b) = u^n b u^{*n}, \quad \forall b \in B, n \in \mathbb{Z}.$$

As described in [42], crossed products by a single automorphism can be realised as quotients in a Toeplitz exact sequence of C^*-algebras, constructed starting from the Toeplitz extension (4).

Definition 16 Let B a unital C^*-algebra and α an automorphism of B. Let $\mathcal{T} = C^*(T)$ be the Toeplitz algebra of the unilateral shift. The *Pimsner–Voiculescu Toeplitz algebra* $\mathcal{T}(B, \alpha)$ is defined as the C^*-subalgebra of $(B \rtimes_\alpha \mathbb{Z})\overline{\otimes}\mathcal{T}$ generated by $B \otimes 1$ and $u \otimes T$.

The Pimsner–Voiculescu Toeplitz algebra $\mathcal{T}(B, \alpha)$ and the crossed product C^*-algebra $B \rtimes_\alpha \mathbb{Z}$ fit into a short exact sequence involving the stabilisation of B:

$$0 \longrightarrow \mathcal{K}\overline{\otimes}B \longrightarrow \mathcal{T}(B, \alpha) \longrightarrow B \rtimes_\alpha \mathbb{Z} \longrightarrow 0. \tag{18}$$

Proof of exactness of the above sequence follows after tensoring the Toeplitz exact sequence (4) with the algebra B, using nuclearity of $C(S^1)$ together with Lemma 7, and by realising $B \rtimes_\alpha \mathbb{Z}$ as a subalgebra of $B\overline{\otimes}C(S^1)$ (see [42, Section 2]).

The Pimsner–Voiculescu Toeplitz algebra $\mathcal{T}(B, \alpha)$ is KK-equivalent to the algebra B itself. The exact sequence (18) then induces six-term exact sequences that allow for an elegant computation of the K-theory and K-homology groups of the crossed product algebra $B \rtimes_\alpha \mathbb{Z}$ in terms of those of the algebra B. These exact sequences are a special case of those described in Sect. 4.2.2.

4.2 Pimsner's Construction: Universal C*-Algebras from C*-Correspondences

The construction which we shall describe now generalises that of crossed products by the integers. In [43], starting from a C^*-correspondence (X, ϕ) such that ϕ is injective, Pimsner constructed two C^*-algebras $\mathcal{T}_X$ and $\mathcal{O}_X$, which are now referred to as the *Toeplitz algebra* and the *Cuntz–Pimsner algebra* of the pair (X, ϕ), respectively. Both algebras are characterised by universal properties and depend only on the isomorphism class of the pair (X, ϕ). We will describe the construction for compact correspondences, i.e. such that $\mathrm{Im}(\phi) \subseteq \mathcal{K}_B(X)$.

4.2.1 The Toeplitz Algebra

As one can take balanced tensor products of C^*-correspondences, as described in Sect. 3.2.1, we consider the modules

$$X^{(k)} := X^{\widehat{\otimes}_\phi^k} \quad k > 0, \tag{19}$$

and we take the infinite direct sum

$$F_X = B \oplus \bigoplus_{k=1}^{\infty} X^{(k)}, \tag{20}$$

which is referred to as the *(positive) Fock correspondence* associated to the correspondence (X, ϕ).

One can naturally associate to any element $\xi \in X$ a shift map:

$$T_\xi(\xi_1 \otimes \cdots \otimes \xi_k) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_k, \qquad T_\xi(b) = \xi b. \tag{21}$$

This is an adjointable operator on F_X, with adjoint

$$T_\xi^*(\xi_1 \otimes \cdots \otimes \xi_k) = \phi(\langle \xi, \xi_1 \rangle)\xi_2 \otimes \cdots \otimes \xi_k, \qquad T_\xi^*(b) = 0. \tag{22}$$

Definition 17 The Toeplitz algebra of the C^*-correspondence X_ϕ is the smallest C^*-subalgebra of $\mathrm{End}_B^*(F_X)$ that contains all the T_ξ for $\xi \in X$.

When (X, ϕ) is a compact C^*-correspondence, the compact operators on the Fock module sit inside $\mathcal{T}_E$ as a two-sided ideal, motivating the following:

Definition 18 The Cuntz–Pimsner algebra $\mathcal{O}_X$ of a compact C^*-correspondence (X, ϕ) is the quotient algebra appearing in the exact sequence

$$0 \longrightarrow \mathcal{K}_B(F_X) \longrightarrow \mathcal{T}_X \xrightarrow{\pi} \mathcal{O}_X \longrightarrow 0. \tag{23}$$

The image of an element $T_\xi \in \mathcal{T}_X$ under the quotient map π will be denoted by S_ξ.

Changing the ideal in the exact sequence (23), one can define the Cuntz–Pimsner algebra of a general (i.e. non-compact, and possibly non-injective) C^*-correspondence. We will not be concerned with this more elaborate construction here. For details see the original papers of Pimsner [43] and Katsura [35], as well as [15, Section 4.6].

Many well-known examples of C^*-algebras admit a description as Toeplitz–Pimsner or Cuntz–Pimsner algebras. The theory provides a unifying framework for a variety of examples, ranging from the study of discrete dynamics to more geometric situations.

Example Let $B = \mathbb{C}$ and $X = \mathbb{C}^n$ and ϕ the left action by multiplication. If one chooses a basis for $\mathbb{C}^n$, then the Toeplitz algebra of (X, ϕ) is the universal C^*-algebra generated by n isometries $V_1, \ldots, V_n$ satisfying $\sum_i V_i V_i^* \leq 1$.

This yields the well known Toeplitz extension for the Cuntz algebras $\mathcal{O}_n$:

$$0 \longrightarrow \mathcal{K}(\mathcal{F}) \longrightarrow C^*(V_1, \ldots, V_n) \longrightarrow \mathcal{O}_n \longrightarrow 0,$$

where $\mathcal{F}$ is the full Fock space on $\mathbb{C}^n$. In particular, for $n = 1$ one gets back the classical Toeplitz extension of (4).

Example (cf. [29, Section 2]) If the correspondence X is a finitely generated and projective module over a unital C^*-algebra, the Pimsner algebra of (X, ϕ) can be realised explicitly in terms of generators and relations. Indeed, since X is finitely generated and projective, there exists a finite set $\{\eta_j\}_{j=1}^n$ of elements of X such that

$$\xi = \sum\nolimits_{j=1}^{n} \eta_j \langle \eta_j, \xi \rangle_B, \qquad \forall \xi \in X.$$

Then, using the above formula, one can spell out the left B-action on X as

$$\phi(b)\eta_j = \sum_{j=1}^{n} \eta_i \langle \eta_i, \phi(b)\eta_j \rangle_B, \quad \forall b \in B.$$

The C^*-algebra $\mathcal{O}_X$ is then the universal C^*-algebra generated by B together with n operators $S_1, \ldots, S_n$, satisfying

$$S_i^* S_j = \langle \eta_i, \eta_j \rangle_B, \quad \sum\nolimits_j S_j S_j^* = 1, \quad \text{and} \quad b S_j = \sum\nolimits_i S_i \langle \eta_i, \phi(b)\eta_j \rangle_B, \tag{24}$$

for $b \in B$, and $j = 1, \ldots, n$. The generators S_i are partial isometries if and only if $\langle \eta_i, \eta_j \rangle = 0$ for $i \neq j$. For $B = \mathbb{C}$ and E a Hilbert space of dimension n, one recovers the Cuntz algebra $\mathcal{O}_n$ of Example 4.2.1.

Example Let B be a C^*-algebra and $\alpha : B \to B$ an automorphism of B. Then $X = B$, seen as a module over itself, can be naturally made into a compact C^*-correspondence.

The right Hilbert B-module structure is the standard one, with right B-valued inner product $\langle a, b\rangle_B = a^*b$. The automorphism α is used to define the left action via $a \cdot b = \alpha(a)b$.

Each module $X^{(k)}$ is isomorphic to B as a right-module, with left action

$$a \cdot (x_1 \otimes \cdots \otimes x_k) = \alpha^k(a)\alpha^{k-1}(x_1)\cdots\alpha(x_{k-1})x_k. \tag{25}$$

The corresponding Pimsner algebra $\mathcal{O}_X$ coincides with the crossed product algebra $B \rtimes_\alpha \mathbb{Z}$, while the Toeplitz algebra $\mathcal{T}_X$ agrees with the Toeplitz algebra $\mathcal{T}(B, \alpha)$. The extension (23) then reduces to (18).

4.2.2 Six-Term Exact Sequences

The Toeplitz extension (23) induces a six-term exact sequence in K-theory. In case the extension is semi-split, it induces six-term exact sequences in KK-theory as well. Split-exactness is automatic, for instance, when the coefficient algebra B is nuclear. These exact sequences can be simplified to a great extent after making the following observations:

- For a compact C^*-correspondence (X, ϕ), the triple $(X, \phi, 0)$ gives a well-defined even Kasparov module (with trivial grading), whose class we denote by $[X]$.
- The ideal $\mathcal{K}(F_X)$ is naturally Morita equivalent to the algebra B itself.
- By [43, Theorem 4.4.], the Toeplitz algebra $\mathcal{T}_X$ is KK-equivalent to the coefficient algebra B.

In K-theory, the induced six-term exact sequence reads

$$\begin{array}{ccccc} K_0(B) & \xrightarrow{\otimes(1-[X])} & K_0(B) & \xrightarrow{i_*} & K_0(\mathcal{O}_X) \\ \uparrow \partial & & & & \downarrow \partial \\ K_1(\mathcal{O}_X) & \xleftarrow[i_*]{} & K_1(B) & \xleftarrow[\otimes(1-[X])]{} & K_1(B) \end{array} \tag{26}$$

where i_* is the map induced by the inclusion $B \hookrightarrow \mathcal{O}_X$ and the maps ∂ are connecting homomorphisms. Up to Morita equivalence, the latter can be computed as Kasparov products with the class of the extension (23). An unbounded representative for the extension class was constructed [26] in the setting bi-Hilbertian bimodules of finite Jones–Watatani index (cf. [30]), subject to some additional assumptions.

We conclude this section by remarking that, in the case of a self-Morita equivalence bimodule—i.e., whenever X is full and ϕ implements an isomorphism between B and $\mathcal{K}_B(X)$—the exact sequence (26) can be interpreted as a generalisation of the classical *Gysin sequence* in K-theory (see [32, IV.1.13]) for the module of sections E of a noncommutative line bundle. The Kasparov product with the map $1 - [X]$ can be interpreted as a *noncommutative Euler class*. This analogy was exploited in [2] to compute K-theory groups of algebras presenting a circle bundle structure.

5 Applications to Topological Insulators

We conclude by discussing the *bulk-edge correspondence*, a principle in solid-state physics, according to which one should be able to *read* the topology of the bulk physical system from the effects it induces on boundary states. This principle underlies, for example, the quantization of the Hall current on the boundary of a sample of a quantum Hall system.

In this section, we illustrate how Toeplitz extensions and the maps they induce in (bivariant) K-theory are essential for a mathematical understanding of these phenomena.

5.1 The Bulk-Boundary Correspondence for the One-Dimensional Su–Schrieffer–Heeger Model and the Noether–Gohberg–Krein Index Theorem

We will now give an exposition of the key ideas behind the bulk-edge correspondence for the one-dimensional Su–Schrieffer–Heeger model [47], a lattice model with chiral symmetry. Our main reference for this Subsection is [45, Chapter 1]. On the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^n \otimes \ell^2(\mathbb{Z})$ we consider the one dimensional Hamiltonian

$$H := \frac{1}{2}(\sigma_1 + i\sigma_2) \otimes 1_n \otimes U + \frac{1}{2}(\sigma_1 - i\sigma_2) \otimes 1_n \otimes U^* + m\sigma_2 \otimes 1_n \otimes 1, \qquad (27)$$

where 1_n and 1 are identity operators on $\mathbb{C}^n$ and $\mathbb{C}^2$, respectively, m is a mass term, U is the right shift on $\ell^2(\mathbb{Z})$ defined in (2), and the σ_i are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This Hamiltonian goes back to work of [47] and models a conducting polymer, namely polyacetilene. It possess a chiral symmetry, implemented by the unitary operator

$$J = \sigma_3 \otimes 1_n \otimes 1,$$

i.e., $J^* H J = -H$.

The model has a spectral gap at $m = 0$ so there exists $\varepsilon > 0$ and a continuous function

$$\chi : \mathbb{R} \to \mathbb{R}, \quad \chi(x) = \begin{cases} 0 \text{ for } x \in (-\infty, -\varepsilon] \\ 1 \text{ for } x \in \quad [0, \infty), \end{cases}$$

so that we can form the *Fermi projection* $P_F := \chi(H)$ through functional calculus with χ. The projection P_F satisfies the identity $J P_F J = 1 - P_F$, so that the flat band Hamiltonian

$$Q := 1 - 2P_F = \operatorname{sgn}(H)$$

satisfies again $J^* Q J = -Q$. Moreover, $Q^2 = 1$, hence its spectrum consists of the two isolated points $+1$ and -1, allowing us to write

$$Q = \begin{pmatrix} 0 & U_F^* \\ U_F & 0 \end{pmatrix}$$

for U_F a unitary on $\mathbb{C}^n \otimes \ell^2(\mathbb{Z})$. This unitary operator, called the *Fermi unitary*, provides us with a natural topological invariant for the boundary system, the first odd Chern number, which can be computed as follows.

We use the discrete Fourier transform mentioned in (1) to write $\mathcal{F} Q \mathcal{F}^*$ as a direct integral $\int_{S^1}^{\oplus} Q_z \mathrm{d}_z$ where each of the Q_z's has the form

$$Q_z = \begin{pmatrix} 0 & U_z^* \\ U_z & 0 \end{pmatrix}.$$

The family of unitary operators is differentiable and the first Chern class can be computed as the integral

$$\mathrm{Ch}_1(U_F) := \frac{i}{2\pi} \int_{S^1}^{\oplus} \mathrm{tr}(U_z \partial_z U_z) \mathrm{d}z \tag{28}$$

This quantity is an invariant under *small perturbations*.

5.1.1 The Bulk Boundary Correspondence

We now introduce an edge for the Hamiltonian (27) by restricting it to the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^n \otimes \ell^2(\mathbb{N})$ and imposing Dirichlet boundary conditions. The resulting Hamiltonian is

$$\widehat{H} := \frac{1}{2}(\sigma_1 + i\sigma_2) \otimes 1_n \otimes T + \frac{1}{2}(\sigma_1 - i\sigma_2) \otimes 1_n \otimes T^* + m\sigma_2 \otimes 1_n \otimes 1, \qquad (29)$$

with conventions as above, and with S the unilateral shift on $\ell^2(\mathbb{N})$ described in Sect. 2.1. Similarly to the bulk Hamiltonian, the edge Hamiltonian has a chiral symmetry implemented by the half-space chiral operator $\widehat{J} = \sigma_3 \otimes 1_n \otimes 1$. Moreover, it has a spectral gap at 0 that we denote by Δ.

Let us now consider the Hilbert space obtained as the span of all the eigenvectors with eigenvalues in $[-\delta, \delta] \subset \Delta$, which we denote by $\mathcal{E}^\delta$. The chirality operator $\widehat{J}$ can be diagonalised on $\mathcal{E}^\delta$, and we have a splitting $\mathcal{E}^\delta = \mathcal{E}^\delta_+ \oplus \mathcal{E}^\delta_-$.

The difference of the dimensions of the spaces $\mathcal{E}^\delta_\pm$ is the *boundary invariant* of the system and it can be computed as a trace:

$$\mathrm{tr}(\widehat{J}\widehat{P_\delta}) = N_+ - N_-, \qquad N_\pm = \dim \mathcal{E}^\delta_\pm,$$

where $\widehat{P_\delta} := \chi(|\widehat{H}| \leq \delta)$ is the spectral projection. This invariant is independent of the choice of δ, as long as it lies in the central gap.

The bulk-edge correspondence is contained in the following identity, that relates the bulk invariant (winding number of the Fermi unitary) to the boundary invariant we just introduced.

Theorem 19 ([45, Theorem 1.2.2]) *Consider the Hamiltonian* (27) *and its half-space restriction* (29)*. If U_F is the Fermi unitary and* $\mathrm{Ch}_1(U_F)$ *its winding number defined in* (28)*, then*

$$\mathrm{Ch}_1(U_F) = \mathrm{Tr}(\tilde{J}\tilde{P}(\delta)).$$

We remark that the Toeplitz extension (4) offers an index theoretic interpretation of this identity. The above equality of classes follows from the six-term exact sequence coming from the Toeplitz extension (4). Indeed, the boundary map described in (8) maps classes of unitaries in the bulk algebra $C(S^1)$ to classes of projections in the boundary algebra $\mathcal{K}(\ell^2(\mathbb{N}))$, whose integer K-theory classes are given by the winding number of the relevant unitary.

5.2 The Role of Toeplitz Extensions in the Bulk-Edge Correspondence

The example of the Su–Schrieffer–Heeger model is in some sense paradigmatic, as other solid-state systems can be modelled using related C^*-algebraic extensions, where Toeplitz algebras serve as models for the half-space system, while quotients of Toeplitz algebras are used to model the edge system. Likewise, the K-theory boundary map coming from the extension can be used to implement the bulk-edge correspondence, relating bulk invariants to edge invariants.

The idea to model the algebra of observables of a solid-state system via crossed product C^*-algebras of some disorder space goes back to Bellissard [7]. His approach culminated in outlining a full-fledged mathematical programme for solid-state physics based on Delone sets [6, 9]. These are uniformly discrete and relatively dense subsets of Euclidean space, but are not required to possess any translational symmetry. In order to work with them, one needs to replace crossed products by groupoid C^*-algebras. The recent developments around the bulk-edge correspondence gave new impetus to this program [44]. We will now present a selection of contemporary results that make use of Toeplitz extensions and KK-theory.

In [12], the authors use the techniques from unbounded KK-theory to prove the bulk-edge correspondence in K-theory for the quantum Hall effect. In their approach, they are able to represent bulk topological invariants as a Kasparov product of boundary invariants with the class of a Toeplitz extension that links the bulk and boundary algebras.

A topological boundary map associated to an extension of a bulk algebra of observables by a boundary algebra is also used in [40]. The bulk algebra is constructed as a crossed product of the codimension-one boundary algebra by the integers, and the K-theoretic invariants are obtained from the associated Toeplitz extension. In their approach the authors use methods from noncommutative T-duality [39].

In [13], the observable algebra of the physical system is a *twisted* crossed product C^*-algebra. The Toeplitz extensions for twisted crossed products by $\mathbb{Z}^n$ offers the natural framework for the investigation of the bulk-edge correspondence, as it elegantly links the algebras of the bulk and the edge systems.

Crossed product C^*-algebras are also used to describe disordered systems. The recent paper [1] describes the bulk-boundary correspondence for disordered free-fermion topological phases in terms of Van Daele K-theory for graded C^*-algebras [49, 50]. The relevant observable algebra is the crossed product of the algebra of continuous functions on a compact disorder space by the action of a lattice.

In [11], the authors replaced crossed products C^*-algebras by groupoid C^*-algebras. While crossed products of commutative C^*-algebras are naturally an example of groupoid C^*-algebras, the advantage of this more general setting lies in the possibility of studying systems without translational symmetries, like those resulting from non-periodic $\mathbb{R}^d$-actions and the above mentioned Delone sets. The systems are still linked by a short exact sequence of the form

$$0 \longrightarrow C_r^*(\mathcal{Y}, \sigma) \otimes \mathcal{K} \longrightarrow \mathcal{T} \longrightarrow C_r^*(\mathcal{G}, \sigma) \longrightarrow 0\,,$$

where σ is a 2-cocycle encoding the magnetic field, $\mathcal{Y}$ is a closed subgroupoid of the groupoid $\mathcal{G}$, and the algebra $\mathcal{T}$ models the half-space system.

Quite remarkably, in the one-dimensional case, the groupoid C^*-algebra admits an alternative description as Cuntz–Pimsner algebra of a self-Morita equivalence bimodule (cf. [11, Subsection 2.3]). The map implementing the bulk-edge correspondence is realised as a Kasparov product with the unbounded representative for the class of the extension (23), as constructed in [26] (see also [2]). It remains an interesting open question whether groupoid C^*-algebras of higher dimensional systems admit a description in terms of C^*-algebras associated to families of C^*-correspondences, for instance in terms of product and subproduct systems [23, 24, 46, 51].

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Standard Groupoids of von Neumann Algebras

Daniel Beltiţă and Anatol Odzijewicz

Abstract We point out the deep relation between the Poisson geometry and the standard form representation of any von Neumann algebra. This is done via a canonical presymplectic groupoid structure of the representation Hilbert space $\mathcal{H}$ endowed with a suitable Banach manifold structure $\widetilde{\mathcal{H}}$ for which the identity mapping is a bijective weak immersion $\widetilde{\mathcal{H}} \to \mathcal{H}$.

Keywords Poisson bracket · Banach-Lie groupoid · von Neumann algebra

Mathematics Subject Classification (2010) Primary 53D17; Secondary 22A22, 22E65, 46L10, 46L60, 58B25

1 Introduction

Motivated by the similarities between finite-dimensional Poisson geometry and the theory of von Neumann algebras that were pointed out by A. Weinstein in [10], we discuss the Poisson geometric background of the standard form of an arbitrary von Neumann algebra $\mathfrak{M}$. The Banach-Lie groupoid $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$, where $\mathcal{U}(\mathfrak{M})$ is the set of partial isometries and $\mathcal{L}(\mathfrak{M})$ the projections lattice of $\mathfrak{M}$ was already investigated in [6] and [9] with a view to Poisson geometry, while the Poisson bracket on the predual $\mathfrak{M}_*$ of $\mathfrak{M}$ had been studied in [8].

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In the present note, for any standard form realization $(\mathfrak{M}, \mathcal{H}, J, \mathcal{P})$ in the sense of [5], we find a canonical foliation of the Hilbert space $\mathcal{H}$, whose leaves are Banach manifolds that are weakly immersed into $\mathcal{H}$. The manifold structure that underlies the Hilbert space $\mathcal{H}$ is thus enriched to a Banach manifold structure to be denoted by $\widetilde{\mathcal{H}}$. It turns out that $\widetilde{\mathcal{H}}$ has the structure of a Banach-Lie groupoid $\widetilde{\mathcal{H}} \rightrightarrows \mathfrak{M}_*^+$ which is isomorphic to the action groupoid $\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+ \rightrightarrows \mathfrak{M}_*^+$ defined by the natural action of the groupoid $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ on the positive cone $\mathfrak{M}_*^+$ in the predual. There is also a presymplectic form $\widetilde{\boldsymbol{\omega}} \in \Omega^2(\widetilde{\mathcal{H}})$ that comes from the scalar product of $\mathcal{H}$ and is multiplicative in the usual sense of finite-dimensional Lie groupoid theory. We further show that the groupoid $(\widetilde{\mathcal{H}}, \widetilde{\boldsymbol{\omega}}) \rightrightarrows \mathfrak{M}_*^+$ shares several other properties of finite-dimensional presymplectic groupoids from [4] and we investigate the Banach manifold structures of its orbits as well as the leaf spaces of the foliation defined by the degeneracy kernel of the presymplectic form $\widetilde{\boldsymbol{\omega}}$.

Details on the facts presented in this note can be found in [2].

2 Preliminaries on W^*-Algebras and Their Standard Forms

We recall here a few facts concerning W^*-algebras indispensable of this presentation.

A C^**-algebra* is an associative Banach $*$-algebra $\mathfrak{M}$ whose topology can be defined by a norm satisfying $\|x^*x\| = \|x\|^2$ for every $x \in \mathfrak{M}$. One can equivalently define a C^*-algebra as a $*$-algebra that is $*$-isomorphic to a norm-closed $*$-subalgebra of $L^\infty(\mathcal{H})$, where $L^\infty(\mathcal{H})$ is the algebra of all bounded operators on a complex Hilbert space $\mathcal{H}$.

A W^**-algebra* is a C^*-algebra $\mathfrak{M}$ that admits a predual Banach space $\mathfrak{M}_*$, hence $\mathfrak{M} = (\mathfrak{M}_*)^*$. Then $\mathfrak{M}_*$ is uniquely determined and we define its positive cone

$$\mathfrak{M}_*^+ := \{\varphi \in \mathfrak{M}_* \mid (\forall x \in \mathfrak{M}) \quad \langle \varphi, x^*x \rangle \geq 0\}.$$

One can equivalently define a W^*-algebra as a $*$-algebra that is $*$-isomorphic to a *von Neumann algebra*, that is, a weakly closed $*$-subalgebra of $L^\infty(\mathcal{H})$.

Since for every smooth functions $f, g \in C^\infty(\mathfrak{M}_*)$ their derivatives at any $\varphi \in \mathfrak{M}_*$ satisfy $Df(\varphi), Dg(\varphi) \in (\mathfrak{M}_*)^* = \mathfrak{M}$, one defines the Lie-Poisson bracket as follows

$$\{f, g\}(\varphi) := \langle \varphi, [Df(\varphi), Dg(\varphi)] \rangle.$$

See [8] for details. In particular, since $L^1(\mathcal{H})^* = L^\infty(\mathcal{H})$, one has a Lie-Poisson structure on the ideal of the trace-class operators

$$L^1(\mathcal{H}) := \{\rho \in L^\infty(\mathcal{H}) \mid \operatorname{Tr}|\rho| < +\infty\}.$$

One also defines the lattice of projections

$$\mathcal{L}(\mathfrak{M}) = \{p \in \mathfrak{M} \mid p = p^* = p^2\}$$

and the set of partial isometries

$$\mathcal{U}(\mathfrak{M}) = \{v \in \mathfrak{M} \mid v^*v \in \mathcal{L}(\mathfrak{M})\}.$$

If $\mathfrak{M} \subseteq L^\infty(\mathcal{H})$ is a von Neumann algebra, then one has

$$\mathcal{L}(\mathfrak{M}) = \mathfrak{M} \cap \mathcal{L}(L^\infty(\mathcal{H})) \text{ and } \mathcal{U}(\mathfrak{M}) = \mathfrak{M} \cap \mathcal{U}(L^\infty(\mathcal{H})).$$

Every W^*-algebra can be realized as a *standard von Neumann algebra* (see [5]). That is, a von Neumann algebra $\mathfrak{M} \subseteq L^\infty(\mathcal{H})$, along with an antilinear unitary $J = J^{-1}: \mathcal{H} \to \mathcal{H}$ and a self-dual cone $\mathcal{P} \subset \mathcal{H}$ satisfying

(i) $J\mathfrak{M}J = \mathfrak{M}'$ $(:= \{y \in L^\infty(\mathcal{H}) \mid xy = yx \text{ for all } x \in \mathfrak{M}\})$;
(ii) $JxJ = x^*$ for $x \in \mathfrak{M} \cap \mathfrak{M}'$;
(iii) $J\gamma = \gamma$ for $\gamma \in \mathcal{P}$;
(iv) $xJxJ\mathcal{P} \subset \mathcal{P}$ for $x \in \mathfrak{M}$.

The *standard form* $(\mathfrak{M}, \mathcal{H}, J, \mathcal{P})$ of any W^*-algebra is unique up to unitary equivalence.

In the above setting, the *expectation map* $E: \mathcal{H} \to \mathfrak{M}_*^+$ defined by

$$\langle E(\gamma), x\rangle := \langle \gamma \mid x\gamma\rangle$$

gives the homeomorphism $E|_{\mathcal{P}}: \mathcal{P} \xrightarrow{\sim} \mathfrak{M}_*^+$. Using that homeomorphism one obtains the *polar decomposition of vectors*

$$\gamma = v_\gamma |\gamma| \in \mathcal{H}, \tag{1}$$

where $v_\gamma \in \mathcal{U}(\mathfrak{M})$ and $|\gamma| \in \mathcal{P}$, are uniquely determined if $v_\gamma^* v_\gamma = [\mathfrak{M}'|\gamma|] \in \mathcal{L}(\mathfrak{M})$, where $[\mathfrak{M}'|\gamma|]$ denotes the orthogonal projection of $\mathcal{H}$ onto the closure of $\mathfrak{M}'|\gamma|$. Also, $|\gamma| \in \mathcal{P}$ is uniquely determined by the condition $E(\gamma) = E(|\gamma|)$.

3 The Groupoids $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ and $\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+ \rightrightarrows \mathfrak{M}_*^+$

Loosely speaking, a groupoid is defined by an associative *partial* multiplication on a set ("of arrows") $\mathcal{G}$. In more detail, a groupoid is a set $\mathcal{G}$ with the following structures:

- a set of unit elements ("objects") $\mathcal{G}^{(0)}$
- an object inclusion map $\boldsymbol{\epsilon}: \mathcal{G}^{(0)} \to \mathcal{G}$

- source/target maps $\boldsymbol{s}, \boldsymbol{t} \colon \mathcal{G} \to \mathcal{G}^{(0)}$
- a set of composable pairs $\mathcal{G} * \mathcal{G} := \{(\gamma_1, \gamma_2) \in \mathcal{G} \times \mathcal{G} \mid \boldsymbol{s}(\gamma_1) = \boldsymbol{t}(\gamma_2)\}$
- the multiplication map $\mathcal{G} * \mathcal{G} \to \mathcal{G}$, $(\gamma_1, \gamma_2) \mapsto \gamma_1 \cdot \gamma_2$
- inversion map $\boldsymbol{\iota} \colon \mathcal{G} \to \mathcal{G}$

satisfying associativity, unit element, inversion axioms whenever they make sense. This groupoid should be denoted $\mathcal{G} \underset{\boldsymbol{s}}{\overset{\boldsymbol{t}}{\rightrightarrows}} \mathcal{G}^{(0)}$ but for the sake of simplicity it is denoted as $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$. The groups are precisely the groupoids $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ for which the set $\mathcal{G}^{(0)}$ is a singleton.

More briefly, a *groupoid* is a small category whose morphisms are invertible.

A *Banach-Lie groupoid* is a groupoid $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ as above for which the sets $\mathcal{G}$ and $\mathcal{G}_0$ are Banach manifolds, the source/target maps are submersions and the multiplication and the inversion maps are smooth (see [3] and [1] for more details).

To any W^*-algebra one can directly associate a few Banach-Lie groupoids. The groupoid of partial isometries $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ has $\mathcal{U}(\mathfrak{M})$ as the set of arrows and $\mathcal{L}(\mathfrak{M})$ as the set of objects. Its structural maps as defined as follows:

- the object inclusion map $\varepsilon : \mathcal{L}(\mathfrak{M}) \to \mathcal{U}(\mathfrak{M})$ is the inclusion $\mathcal{L}(\mathfrak{M}) \hookrightarrow \mathcal{U}(\mathfrak{M})$;
- the source $\boldsymbol{s} : \mathcal{U}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ and target $\boldsymbol{t} : \mathcal{U}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ maps are

$$\boldsymbol{s}(v) := v^* v \qquad \text{and} \qquad \boldsymbol{t}(v) := v v^*;$$

- the inverse groupoid map $\iota : \mathcal{U}(\mathfrak{M}) \to \mathcal{U}(\mathfrak{M})$ is $\iota(v) = v^*$;
- the product of $(v_1, v_2) \in \mathcal{U}(\mathfrak{M}) * \mathcal{U}(\mathfrak{M}) := \{(v_1, v_2) \in \mathcal{U}(\mathfrak{M}) \times \mathcal{U}(\mathfrak{M}) : \boldsymbol{s}(v_1) = \boldsymbol{t}(v_2)$ is their algebraic product $v_1 v_2 \in \mathcal{U}(\mathfrak{M})$.

The groupoid $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ is a real Banach-Lie groupoid which was studied in [6] and [9].

One has the action of $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ on $\mathfrak{M}_*^+$ (called as the co-adjoint action in this note)

$$\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+ \ni (u, \rho) \mapsto u \rho u^* \in \mathfrak{M}_*^+ \tag{2}$$

where

$$\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+ := \{(u, \rho) \in \mathcal{U}(\mathfrak{M}) \times \mathfrak{M}_*^+; \quad u^* u = \sigma_*(\rho)\}$$

for which $\sigma_* : \mathfrak{M}_*^+ \to \mathcal{L}(\mathfrak{M})$, defined as the support $\sigma_*(\rho)$ of $\rho \in \mathfrak{M}_*^+$, is the momentum map.

Using the action (2), one defines $\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+ \rightrightarrows \mathfrak{M}_*^+$, the co-adjoint action groupoid whose structural maps are defined as follows:

- the object inclusion map $\varepsilon_* : \mathfrak{M}_*^+ \to \mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+$

$$\varepsilon_*(\rho) := (\sigma_*(\rho), \rho)$$

- the source $\boldsymbol{s}_* : \mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+ \to \mathfrak{M}_*^+$ and target $\boldsymbol{t}_* : \mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+ \to \mathfrak{M}_*^+$ maps are

$$\boldsymbol{s}_*(u, \rho) := \rho \qquad \text{and} \qquad \boldsymbol{t}_*(u, \rho) := u\rho u^*;$$

- the inverse groupoid map $\iota_* : \mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+ \to \mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+$ is $\iota_*(u, \rho) := (u^*, u\rho u^*)$;
- the product of elements $(u, \rho), (w, \delta) \in \mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+$ such that $\boldsymbol{s}_*(u, \rho) = \boldsymbol{t}_*(w, \delta)$ is defined by

$$(u, \rho)(w, \delta) := (uw, \delta)$$

For the co-adjoint action groupoid the following assertions hold.

Theorem 1

(i) *The groupoid $\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+ \rightrightarrows \mathfrak{M}_*^+$ is a Banach-Lie groupoid whose various connected components are modeled on different Banach spaces.*

(ii) *Every orbit*

$$\mathcal{O}_\rho := \{v^* \rho v : \quad (v, \rho) \in \mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+\}$$

of $\rho \in \mathfrak{M}_*^+$ is a weakly symplectic Banach manifold.

See [2, Sections 3 and 5] for details.

4 Poisson Geometry on the Standard Groupoid $\widetilde{\mathcal{H}} \rightrightarrows \mathfrak{M}_*^+$

Throughout this section, unless otherwise mentioned, $(\mathfrak{M}, \mathcal{H}, J, \mathcal{P})$ is a standard form of an arbitrary W^*-algebra.

Theorem 2 *There exists a groupoid $\mathcal{H} \rightrightarrows \mathfrak{M}_*^+$ having as structural maps*

- *the source map $\boldsymbol{s} = E$,*
- *the inversion map J,*
- *the target map $\boldsymbol{t} = E \circ J$,*
- *the object inclusion map $\boldsymbol{\epsilon} = (E|_{\mathcal{P}})^{-1} : \mathfrak{M}_*^+ \to \mathcal{P} \hookrightarrow \mathcal{H}$,*
- *the multiplication $\gamma_1 \cdot \gamma_2 = v_{\gamma_1} v_{\gamma_2} |\gamma_2|$ if $\boldsymbol{s}(\gamma_1) = \boldsymbol{t}(\gamma_2)$.*

Example 3 Assume that $\mathfrak{M}$ is commutative. Then the standard form of $\mathfrak{M}$ is the 4-tuple $(\mathfrak{M}, \mathcal{H}, J, \mathcal{P})$ and its corresponding standard groupoid $\mathcal{H} \rightrightarrows \mathcal{P}$ can be described as follows. For a suitable measure space (T, μ), one has $\mathfrak{M}_* \simeq L^1(T, \mu)$, $\mathfrak{M}_*^+ \simeq L^1(T, \mu)^+ := \{\varphi \in L^1(T, \mu) \mid \varphi \geq 0 \text{ a.e.}\}$ and moreover

- $\mathcal{H} = L^2(T, \mu)$ and $\mathcal{P} = L^2(T, \mu)^+ := \{\gamma \in L^2(T, \mu) \mid \gamma \geq 0 \text{ a.e.}\}$
- $\boldsymbol{s} = \boldsymbol{t} \colon L^2(T, \mu) \to L^1(T, \mu)^+, \gamma \mapsto |\gamma|^2$

Also $\mathfrak{M} = \{M_f \mid f \in L^\infty(T,\mu)\}$, where $M_f : \mathcal{H} \to \mathcal{H}$, $\gamma \mapsto f\gamma$ is the *multiplication-by-f operator* for any $f \in L^\infty(T,\mu)$. Finally, $J: \mathcal{H} \to \mathcal{H}$, $\gamma \mapsto \overline{\gamma}$, and we note that $J\mathfrak{M}J = \mathfrak{M} = \mathfrak{M}'$ since $\mathfrak{M}$ is commutative.

Since $\boldsymbol{s} = \boldsymbol{t}$, the standard groupoid $\mathcal{H} \rightrightarrows \mathfrak{M}_*^+$ is a *group bundle*.

The expectation map $E : \mathcal{H} \to \mathfrak{M}_*^+$ allows us to define the bijection $\phi : \mathcal{H} \to \mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+$ by

$$\phi(\gamma) := (v_\gamma, E|\gamma|), \tag{3}$$

where the partial isometry v_γ is related to $\gamma \in \mathcal{H}$ by the polar decomposition (1).

Theorem 4

(i) *The map* (3) *defines the groupoid isomorphism between* $\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+ \rightrightarrows \mathfrak{M}_*^+$ *and* $\mathcal{H} \rightrightarrows \mathfrak{M}_*^+$.

(ii) *This bijection is a weak immersion of Banach manifolds* $\mathcal{H}$ *and* $\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+$.

Corollary 5 (smooth structure of the standard groupoid) *We thus obtain a (foliation-like) Banach manifold structure* $\widetilde{\mathcal{H}}$ *on the Hilbert space* $\mathcal{H}$ *for which the identity map is a weak immersion* $\widetilde{\mathcal{H}} \to \mathcal{H}$. *Thus the standard groupoid* $\widetilde{\mathcal{H}} \rightrightarrows \mathfrak{M}_*^+$ *is a Banach-Lie groupoid.*

For details see [2].

We assume again that $(\mathfrak{M}, \mathcal{H}, J, \mathcal{P})$ is a standard form of an arbitrary W^*-algebra, unless otherwise specified.

The mapping $\mathfrak{M} \hookrightarrow \mathcal{C}^\infty(\mathfrak{M}_*)$, $x \mapsto \langle x, \cdot\rangle$, is a Lie algebra morphism for a unique Poisson bracket $\{\cdot,\cdot\}$ on $\mathcal{C}^\infty(\mathfrak{M}_*)$ and for the Lie bracket $[x,y] := xy - yx$ on $\mathfrak{M}$. (See [8] and [7].) Also the Hilbert space $\mathcal{H}$ has the symplectic structure defined by $\operatorname{Im}\langle\cdot \mid \cdot\rangle$.

Theorem 6 *The following assertions hold.*

(i) *The source/target maps* $\boldsymbol{s}, \boldsymbol{t} : \mathcal{H} \to \mathfrak{M}_*$ *are Poisson/anti-Poisson maps.*

(ii) $\operatorname{Im}\langle\cdot \mid \cdot\rangle$ *gives a* multiplicative *presymplectic form* $\widetilde{\boldsymbol{\omega}} \in \Omega^2(\widetilde{\mathcal{H}})$*:*
$\boldsymbol{m}^*\widetilde{\boldsymbol{\omega}} = \mathrm{pr}_1^*\widetilde{\boldsymbol{\omega}} + \mathrm{pr}_2^*\widetilde{\boldsymbol{\omega}}$, *where* $\boldsymbol{m}, \mathrm{pr}_1, \mathrm{pr}_2 : \widetilde{\mathcal{H}} * \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ *are the groupoid multiplication and the Cartesian projections, respectively.*

(iii) *The leaves of the null-foliation of* $\widetilde{\boldsymbol{\omega}}$ *are the fibers of the submersion* $(\boldsymbol{t},\boldsymbol{s}) : \widetilde{\mathcal{H}} \to \mathfrak{M}_*^+ \times \mathfrak{M}_*^+$.

(iv) *Every orbit* $\mathcal{O} \subseteq \mathfrak{M}_*^+$ *has a weakly symplectic structure* $\omega_{\mathcal{O}} \in \Omega^2(\mathcal{O})$ *and a Poisson bracket for which* $\mathcal{O} \hookrightarrow \mathfrak{M}_*$ *is a Poisson map. Moreover, on the transitive subgroupoid* $\boldsymbol{t}^{-1}(\mathcal{O}) \subseteq \widetilde{\mathcal{H}}$, *one has*

$$\widetilde{\boldsymbol{\omega}} = \boldsymbol{t}^*\omega_{\mathcal{O}} - \boldsymbol{s}^*\omega_{\mathcal{O}}.$$

Example 7 Here we study the special case of the type I factor $L^\infty(\mathcal{H}_0) = (L^1(\mathcal{H}_0))^*$, where $\mathcal{H}_0$ is any complex Hilbert space. To this end we introduce the notation

- $\mathcal{H} = L^2(\mathcal{H}_0) := \{x \in \mathfrak{M}_0 \mid \mathrm{Tr}(x^*x) < \infty\}$,
- $J\colon L^2(\mathcal{H}_0) \to L^2(\mathcal{H}_0),\ J(x) := x^*$,
- $\mathcal{P} := \{x \in L^2(\mathcal{H}_0) \mid x \geq 0\}$,
- for all $a \in L^\infty(\mathcal{H}_0)$ we define $\lambda(a)\colon L^2(\mathcal{H}_0) \to L^2(\mathcal{H}_0)$, $\lambda(a)\gamma := a\gamma$, and then $L^\infty(\mathcal{H}_0) \simeq \lambda(L^\infty(\mathcal{H}_0)) =: \mathfrak{M} \subseteq L^\infty(\mathcal{H})$.

Then $(\mathfrak{M}, \mathcal{H}, J, \mathcal{P})$ is a standard form of the W^*-algebra $L^\infty(\mathcal{H}_0)$, whose corresponding standard groupoid is

$$L^2(\mathcal{H}_0) \rightrightarrows L^1(\mathcal{H}_0)^+ := L^1(\mathcal{H}_0) \cap \mathcal{P}$$

having its source/target maps

$$\boldsymbol{s}, \boldsymbol{t}\colon L^2(\mathcal{H}_0) \to L^1(\mathcal{H}_0)^+, \quad \boldsymbol{s}(\gamma) = \gamma^*\gamma,\ \boldsymbol{t}(\gamma) = \gamma\gamma^*$$

and its multiplication

$$\gamma_1 \cdot \gamma_2 := v_1 v_2 |\gamma_2|$$

if $\gamma_j = v_j|\gamma_j| \in L^2(\mathcal{H}_0)$ for $j = 1, 2$ with $\gamma_1^*\gamma_1 = \gamma_2\gamma_2^*$. Moreover, the orbit of any $\rho \in L^1(\mathcal{H}_0)^+$ is

$$\mathcal{O}_\rho = \{v\rho v^* \mid v \in L^\infty(\mathcal{H}_0),\ v^*v = [\rho\mathcal{H}_0]\}.$$

5 Further Facts

5.1 Standard Groupoids of Finite W*-Algebras

A W^*-algebra $\mathfrak{M}$ is *finite* if every isometry in $\mathfrak{M}$ is unitary, i.e.,

$$u \in \mathfrak{M} \quad \& \quad u^*u = 1 \implies uu^* = 1$$

Examples of finite W^*-algebras include

1. Type I: $M_n(\mathbb{C}),\ n = 1, 2, \ldots$
2. Type II: the von Neumann of any countable discrete group.

Proposition 8 *For any W^*-algebra $\mathfrak{M}$ the following conditions are equivalent:*

(i) *$\mathfrak{M}$ is finite.*
(ii) *The orbits of the standard groupoid of $\mathfrak{M}$ are connected.*

5.2 Standard Groupoids of Purely Infinite W*-Algebras

A W^*-algebra $\mathfrak{M}$ is *purely infinite* (or type III) if every nonzero projection is infinite (i.e., is equivalent to a strictly smaller projection):

$$p \in \mathcal{L}(\mathfrak{M}) \setminus \{0\} \implies (\exists u \in \mathfrak{M}) \quad p = u^*u > uu^*$$

Recall that $\mathfrak{M}$ is a *factor* if $\{x \in \mathfrak{M} \mid (\forall y \in \mathfrak{M})\ xy = yx\} = \mathbb{C}\mathbf{1}$.

Proposition 9 *If $\mathfrak{M}$ is a factor with separable predual, the following are equivalent:*

(i) $\mathfrak{M}$ *is purely infinite*
(ii) *Each nonzero orbit of the standard groupoid of $\mathfrak{M}$ has exactly two connected components.*

5.3 Standard Groupoids of Type III_1

Special (type III_1) purely infinite W^*-algebras occur in some models of the relativistic quantum field theory due to Haag-Kastler-Araki-Borchers-... We recall that a W^*-algebra $\mathfrak{M}$ is *type III_1* if

1. $\mathfrak{M}$ is a purely infinite factor;
2. $\mathfrak{M}_*$ is separable;
3. for every faithful state $\rho \in \mathfrak{M}_*^+$ its modular operator $\Delta_\rho \geq 0$ has its spectrum equal to $[0, \infty)$.

Proposition 10 *If $\mathfrak{M}$ is a factor with separable predual, the following are equivalent:*

(i) $\mathfrak{M}$ *is type III_1*
(ii) *The norm-closure of every orbit of the standard groupoid of $\mathfrak{M}$ is a sphere in $\mathfrak{M}_*^+$.*

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Quantum Differential Equations and Helices

Giordano Cotti

Abstract We give an overview of recent results obtained in joint works with Dubrovin and Guzzetti (Helix structures in quantum cohomology of Fano varieties, 2018, arXiv:1811.09235), and Cotti and Varchenko (Equivariant quantum differential equation and *qKZ* equations for a projective space: Stokes bases as exceptional collections, Stokes matrices as Gram matrices, and B-Theorem. In: Krichever, I., Novikov, S., Ogievetsky, O., Shlosman, S. (Eds.) Integrability, quantization and geometry–Dubrovin's memorial volume. Proceedings of Symposia in Pure Mathematics (PSPUM) book series, AMS).

Keywords Quantum cohomology · Frobenius manifolds · Monodromy data · Exceptional collections · Dubrovin's conjecture

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1 Quantum Cohomology

1.1 Notations and Conventions

Let X be a smooth projective variety over $\mathbb{C}$ with vanishing odd-cohomology, i.e. $H^{2k+1}(X, \mathbb{C}) = 0$, for $k \geq 0$. Fix a homogeneous basis $(T_1, \dots, T_n)$ of the complex vector space $H^\bullet(X) := \bigoplus_k H^{2k}(X, \mathbb{C})$, and denote by $\boldsymbol{t} := (t^1, \dots, t^n)$ the corresponding dual coordinates. Without loss of generality, we assume that $T_1 = 1$. The *Poincaré pairing* on $H^\bullet(X)$ will be denoted by

$$\eta(u, v) := \int_X u \cup v, \quad u, v \in H^\bullet(X), \tag{1}$$

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and we put $\eta_{\alpha\beta} := \eta(T_\alpha, T_\beta)$, for $\alpha, \beta = 1, \dots, n$, to be the Gram matrix w.r.t. the fixed basis. The entries of the inverse matrix will be denoted by $\eta^{\alpha\beta}$, for $\alpha, \beta = 1, \dots, n$. In all the paper, the Einstein rule of summation over repeated indices is used. General references for this section are [7, 8, 10–13, 27, 28, 31].

1.2 Gromov–Witten Invariants in Genus 0

For a fixed $\beta \in H_2(X, \mathbb{Z})/\text{torsion}$, denote by $\overline{\mathcal{M}}_{0,k}(X, \beta)$ the Deligne–Mumford moduli stack of k-pointed stable rational maps with target X of degree β:

$$\overline{\mathcal{M}}_{0,k}(X, \beta) := \{f\colon (C, \boldsymbol{x}) \to X,\ f_*[C] = \beta\}/\text{equivalencies}, \tag{2}$$

where C is an algebraic curve of genus 0 with at most nodal singularities, $\boldsymbol{x} := (x_1, \dots, x_k)$ is a k-tuple of pairwise distinct marked points of C, and equivalencies are automorphisms of $C \to X$ identical on X and the markings.

Gromov–Witten invariants (GW-invariants for short) of X, and their *descendants*, are defined as intersection numbers of cycles on $\overline{\mathcal{M}}_{0,k}(X, \beta)$, by the integrals

$$\langle \tau_{d_1}\gamma_1, \dots, \tau_{d_k}\gamma_k \rangle_{k,\beta}^X := \int_{[\overline{\mathcal{M}}_{0,k}(X,\beta)]^{\text{virt}}} \prod_{i=1}^{k} \text{ev}_i^*\gamma_i \wedge \psi_i^{d_i}, \tag{3}$$

for $\gamma_1, \dots, \gamma_k \in H^\bullet(X)$, $d_i \in \mathbb{N}$. In formula (3),

$$\text{ev}_i\colon \overline{\mathcal{M}}_{0,k}(X, \beta) \to X, \quad f \mapsto f(x_i), \quad i = 1, \dots, k, \tag{4}$$

are evaluation maps, and $\psi_i := c_1(\mathcal{L}_i)$ are the first Chern classes of the universal cotangent line bundles

$$\mathcal{L}_i \to \overline{\mathcal{M}}_{0,k}(X, \beta), \quad \mathcal{L}_i|_f = T_{x_i}^*C, \quad i = 1, \dots, k. \tag{5}$$

The *virtual fundamental cycle* $[\overline{\mathcal{M}}_{0,k}(X, \beta)]^{\text{virt}}$ is an element of the Chow ring $A_\bullet\left(\overline{\mathcal{M}}_{0,k}(X, \beta)\right)$, namely

$$[\overline{\mathcal{M}}_{0,k}(X, \beta)]^{\text{virt}} \in A_D\left(\overline{\mathcal{M}}_{0,k}(X, \beta)\right), \quad D := \dim_{\mathbb{C}} X - 3 + k + \int_\beta c_1(X).$$

See [1] for its construction.

1.3 Quantum Cohomology as a Frobenius Manifold

Introduce infinitely many variables $\boldsymbol{t}_\bullet := (t_p^\alpha)_{\alpha,p}$ with $\alpha = 1, \dots, n$ and $p \in \mathbb{N}$.

Definition 1 The *genus 0 total descendant potential* of X is the generating function $\mathcal{F}_0^X \in \mathbb{C}[\![\boldsymbol{t}_\bullet]\!]$ of descendant GW-invariants of X defined by

$$\mathcal{F}_0^X(\boldsymbol{t}_\bullet) := \sum_{k=0}^{\infty} \sum_{\beta} \sum_{\alpha_1,\dots,\alpha_k=1}^{n} \sum_{p_1,\dots,p_k=0}^{\infty} \frac{t_{p_1}^{\alpha_1} \dots t_{p_k}^{\alpha_k}}{k!} \langle \tau_{p_1} T_{\alpha_1}, \dots, \tau_{p_k} T_{\alpha_k} \rangle_{k,\beta}^X.$$

Setting $t_0^\alpha = t^\alpha$ and $t_p^\alpha = 0$ for $p > 0$, we obtain the *Gromov–Witten potential* of X

$$F_0^X(\boldsymbol{t}) := \sum_{k=0}^{\infty} \sum_{\beta} \sum_{\alpha_1,\dots,\alpha_k=1}^{n} \frac{t^{\alpha_1} \dots t^{\alpha_k}}{k!} \langle T_{\alpha_1}, \dots, T_{\alpha_k} \rangle_{k,\beta}^X. \tag{6}$$

Let $\Omega \subseteq H^\bullet(X)$ be the domain of convergence of $F_0^X(\boldsymbol{t})$, assumed to be non-empty. We denote by $T\Omega$ and $T^*\Omega$ its holomorphic tangent and cotangent bundles, respectively. Each tangent space $T_p\Omega$, with $p \in \Omega$, is canonically identified with the space $H^\bullet(X)$, via the identification $\frac{\partial}{\partial t^\alpha} \mapsto T_\alpha$. The Poincaré metric η defines a flat non-degenerate $\mathcal{O}_\Omega$-bilinear pseudo-Riemannian metric on Ω. The coordinates $\boldsymbol{t}$ are manifestly flat. Denote by ∇ the Levi-Civita connection of η.

Definition 2 Define the tensor $c \in \Gamma(T\Omega \otimes \bigodot^2 T^*\Omega)$ by

$$c_{\beta\gamma}^\alpha := \eta^{\alpha\lambda} \nabla^3_{\lambda\beta\gamma} F_0^X, \quad \alpha, \beta, \gamma = 1, \dots, n, \tag{7}$$

and let us introduce a product $*$ on vector fields on Ω by

$$\frac{\partial}{\partial t^\beta} * \frac{\partial}{\partial t^\gamma} := c_{\beta\gamma}^\alpha \frac{\partial}{\partial t^\alpha}, \quad \beta, \gamma = 1, \dots, n. \tag{8}$$

Theorem 3 ([27, 31]) *The Gromov–Witten potential $F_0^X(\boldsymbol{t})$ is a solution of $WDVV$ equations*

$$\frac{\partial^3 F_0^X(\boldsymbol{t})}{\partial t^\alpha \partial t^\beta \partial t^\gamma} \eta^{\gamma\delta} \frac{\partial^3 F_0^X(\boldsymbol{t})}{\partial t^\delta \partial t^\epsilon \partial t^\phi} = \frac{\partial^3 F_0^X(\boldsymbol{t})}{\partial t^\phi \partial t^\beta \partial t^\gamma} \eta^{\gamma\delta} \frac{\partial^3 F_0^X(\boldsymbol{t})}{\partial t^\delta \partial t^\epsilon \partial t^\alpha}, \tag{9}$$

for $\alpha, \beta, \epsilon, \phi = 1, \dots, n$.

On each tangent space $T_p\Omega$, the product $*_p$ defines a structure of associative, commutative algebra with unit $\frac{\partial}{\partial t^1} \equiv 1$. Furthermore, the product $*$ is compatible with the Poincaré metric, namely

$$\eta(u * v, w) = \eta(u, v * w), \quad u, v, w \in \Gamma(T\Omega). \tag{10}$$

This endows $(T_p\Omega, *_p, \eta_p, \left.\frac{\partial}{\partial t^1}\right|_p)$ with a complex *Frobenius algebra* structure.

Definition 4 The vector field

$$E = c_1(X) + \sum_{\alpha=1}^{n} \left(1 - \frac{1}{2} \deg T_\alpha\right) t^\alpha \frac{\partial}{\partial t^\alpha}, \tag{11}$$

is called the *Euler vector field*. Here, $\deg T_\alpha$ denotes the cohomological degree of T_α, i.e. $\deg T_\alpha := r_\alpha$ if and only if $T_\alpha \in H^{r_\alpha}(X, \mathbb{C})$. We denote by $\mathcal{U}$ the (1, 1)-tensor defined by the multiplication with the Euler vector field, i.e.

$$\mathcal{U}\colon \Gamma(T\Omega) \to \Gamma(T\Omega), \quad v \mapsto E * v. \tag{12}$$

Proposition 5 ([11, 13]) *The Euler vector field E is a Killing conformal vector field, whose flow preserves the structure constants of the Frobenius algebras:*

$$\mathfrak{L}_E \eta = (2 - \dim_{\mathbb{C}} X)\eta, \quad \mathfrak{L}_E c = c. \tag{13}$$

The structure $(\Omega, c, \eta, \frac{\partial}{\partial t^1}, E)$ gives an example of analytic *Frobenius manifold*, called *quantum cohomology of* X and denoted by $QH^\bullet(X)$, see [11–13, 28].

1.4 Extended Deformed Connection

Definition 6 The *grading operator* $\mu \in \mathrm{End}(T\Omega)$ is the tensor defined by

$$\mu(v) := \frac{2 - \dim_{\mathbb{C}} X}{2} v - \nabla_v E, \quad v \in \Gamma(T\Omega). \tag{14}$$

Consider the canonical projection $\pi : \mathbb{C}^* \times \Omega \to \Omega$, and the pull-back bundle $\pi^* T\Omega$. Denote by

1. $\mathscr{T}_\Omega$ the sheaf of sections of $T\Omega$,
2. $\pi^* \mathscr{T}_\Omega$ the pull-back sheaf, i.e. the sheaf of sections of $\pi^* T\Omega$
3. $\pi^{-1} \mathscr{T}_\Omega$ the sheaf of sections of $\pi^* T\Omega$ constant on the fibers of π.

All the tensors $\eta, c, E, \mathcal{U}, \mu$ can be lifted to $\pi^* T\Omega$, and their lifts will be denoted by the same symbols. The Levi-Civita connection ∇ is lifted on $\pi^* T\Omega$, and it acts so that

$$\nabla_{\frac{\partial}{\partial z}} v = 0 \quad \text{for } v \in (\pi^{-1} \mathscr{T}_\Omega)(\Omega), \tag{15}$$

where z is the coordinate on $\mathbb{C}^*$.

Definition 7 The *extended deformed connection* is the connection $\widehat{\nabla}$ on the bundle $\pi^*T\Omega$ defined by

$$\widehat{\nabla}_w v = \nabla_w v + z \cdot w * v, \tag{16}$$

$$\widehat{\nabla}_{\frac{\partial}{\partial z}} v = \nabla_{\partial_z} v + \mathcal{U}(v) - \frac{1}{z}\mu(v), \tag{17}$$

for $v, w \in \Gamma(\pi^*T\Omega)$.

Theorem 8 ([11, 13]) *The connection $\widehat{\nabla}$ is flat.*

1.5 Semisimple Points and Orthonormalized Idempotent Frame

Definition 9 A point $p \in \Omega$ is *semisimple* if and only if the corresponding Frobenius algebra $(T_p\Omega, *_p, \eta_p, \frac{\partial}{\partial t^1}|_p)$ is without nilpotents. Denote by Ω_{ss} the open dense subset of Ω of semisimple points.

Theorem 10 ([24]) *The set Ω_{ss} is non-empty only if X is of Hodge–Tate*[1] *type, i.e. $h^{p,q}(X) = 0$ for $p \neq q$.*

On Ω_{ss} there are n well-defined idempotent vector fields $\pi_1, \ldots, \pi_n \in \Gamma(T\Omega_{ss})$, satisfying

$$\pi_i * \pi_j = \delta_{ij}\pi_i, \quad \eta(\pi_i, \pi_j) = \delta_{ij}\eta(\pi_i, \pi_i), \quad i, j = 1, \ldots, n. \tag{18}$$

Theorem 11 ([10, 11, 13]) *The idempotent vector fields pairwise commute: $[\pi_i, \pi_j] = 0$ for $i, j = 1, \ldots, n$. Hence, there exist holomorphic local coordinates $(u_1, \ldots, u_n)$ on Ω_{ss} such that $\frac{\partial}{\partial u_i} = \pi_i$ for $i = 1, \ldots, n$.*

Definition 12 The coordinates $(u_1, \ldots, u_n)$ of Theorem 11 are called *canonical coordinates*.

Proposition 13 ([11, 13]) *Canonical coordinates are uniquely defined up to ordering and shifts by constants. The eigenvalues of the tensor $\mathcal{U}$ define a system of canonical coordinates in a neighborhood of any semisimple point of Ω_{ss}.*

Definition 14 We call *orthonormalized idempotent frame* a frame $(f_i)_{i=1}^n$ of $T\Omega_{ss}$ defined by

$$f_i := \eta(\pi_i, \pi_i)^{-\frac{1}{2}}\pi_i, \quad i = 1, \ldots, n, \tag{19}$$

[1]Here $h^{p,q}(X) := \dim_{\mathbb{C}} H^q(X, \Omega_X^p)$, with Ω_X^p the sheaf of holomorphic p-forms on X, denotes the (p,q)-Hodge number of X.

for arbitrary choices of signs of the square roots. The Ψ-matrix is the matrix $(\Psi_{i\alpha})_{i,\alpha=1}^{n}$ of change of tangent frames, defined by

$$\frac{\partial}{\partial t^\alpha} = \sum_{i=1}^{n} \Psi_{i\alpha} f_i, \quad \alpha = 1, \dots, n. \tag{20}$$

Remark 15 In the orthonormalized idempotent frame, the operator $\mathcal{U}$ is represented by a diagonal matrix, and the operator μ by an antisymmetric matrix:

$$U := \text{diag}(u_1, \dots, u_n), \quad \Psi \mathcal{U} \Psi^{-1} = U, \tag{21}$$

$$V := \Psi \mu \Psi^{-1}, \quad V^T + V = 0. \tag{22}$$

2 Quantum Differential Equation

The connection $\widehat{\nabla}$ induces a flat connection on $\pi^*(T^*\Omega)$. Let $\xi \in \Gamma(\pi^*(T^*\Omega))$ be a flat section. Consider the corresponding vector field $\zeta \in \Gamma(\pi^*(T\Omega))$ via musical isomorphism, i.e. such that $\xi(v) = \eta(\zeta, v)$ for all $v \in \Gamma(\pi^*(T\Omega))$.

The vector field ζ satisfies the following system[2] of equations

$$\frac{\partial}{\partial t^\alpha}\zeta = z\mathcal{C}_\alpha \zeta, \quad \alpha = 1, \dots, n, \tag{23}$$

$$\frac{\partial}{\partial z}\zeta = \left(\mathcal{U} + \frac{1}{z}\mu\right)\zeta. \tag{24}$$

Here $\mathcal{C}_\alpha$ is the $(1, 1)$-tensor defined by $(\mathcal{C}_\alpha)_\gamma^\beta := c_{\alpha\gamma}^\beta$.

Definition 16 The *quantum differential equation* (qDE) of X is the differential equation (24).

The qDE is an ordinary differential equation with rational coefficients. It has two singularities on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$:

1. a Fuchsian singularity at $z = 0$,
2. an irregular singularity (of Poincaré rank 1) at $z = \infty$.

[2]We consider the joint system (23) and (24) in matrix notations (ζ a column vector whose entries are the components $\zeta^\alpha(\boldsymbol{t}, z)$ w.r.t. $\frac{\partial}{\partial t^\alpha}$). Bases of solutions are arranged in invertible $n \times n$-matrices, called *fundamental systems of solutions*.

Points of Ω are parameters of deformation of the coefficients of the qDE. Solutions $\zeta(\boldsymbol{t}, z)$ of the joint system of Eqs. (23) and (24) are "multivalued" functions w.r.t. z, i.e. they are well-defined functions on $\Omega \times \widehat{\mathbb{C}^*}$, where $\widehat{\mathbb{C}^*}$ is the universal cover of $\mathbb{C}^*$.

2.1 Solutions in Levelt form at $z = 0$ and Topological-Enumerative Solution

Theorem 17 ([7, 11, 13]) *There exist fundamental systems of solutions $Z_0(\boldsymbol{t}, z)$ of the joint system (23) and (24) with expansions at $z = 0$ of the form*

$$Z_0(\boldsymbol{t}, z) = F(\boldsymbol{t}, z) z^{\mu} z^{R}, \quad R = \sum_{k \geq 1} R_k, \quad F(\boldsymbol{t}, z) = I + \sum_{j=1}^{\infty} F_j(\boldsymbol{t}) z^j \tag{25}$$

where $(R_k)_{\alpha\beta} \neq 0$ only if $\mu_\alpha - \mu_\beta = k$. The series $F(\boldsymbol{t}, z)$ is convergent and satisfies the orthogonality condition

$$F(\boldsymbol{t}, -z)^T \eta F(\boldsymbol{t}, z) = \eta. \tag{26}$$

Definition 18 A fundamental system of solutions $Z_0(\boldsymbol{t}, z)$ of the form described in Theorem 17 are said to be in *Levelt form* at $z = 0$.

Remark 19 Fundamental systems of solutions in Levelt form are not unique. The exponent R is not uniquely determined. Moreover, even for a fixed exponent R, the series $F(\boldsymbol{t}, z)$ is not uniquely determined, see [7]. It can be proved that the matrix R can be chosen as the matrix of the operator $c_1(X) \cup (-) \colon H^\bullet(X) \to H^\bullet(X)$ w.r.t. the basis $(T_\alpha)_{\alpha=1}^n$ [13, Corollary 2.1].

Remark 20 Let $Z_0(\boldsymbol{t}, z)$ be a fundamental system of solutions in Levelt form (25). The monodromy matrix $M_0(\boldsymbol{t})$, defined by

$$Z_0(\boldsymbol{t}, e^{2\pi\sqrt{-1}} z) = Z_0(\boldsymbol{t}, z) M_0(\boldsymbol{t}), \quad z \in \widehat{\mathbb{C}^*}, \tag{27}$$

is given by

$$M_0(\boldsymbol{t}) = \exp(2\pi\sqrt{-1}\mu) \exp(2\pi\sqrt{-1}R). \tag{28}$$

In particular, M_0 does not depend on $\boldsymbol{t}$.

Definition 21 Define the functions $\theta_{\beta,p}(\boldsymbol{t},z)$, $\theta_\beta(\boldsymbol{t},z)$, with $\beta = 1,\dots,n$ and $p \in \mathbb{N}$, by

$$\theta_{\beta,p}(\boldsymbol{t}) := \left.\frac{\partial^2 \mathcal{F}_0^X(\boldsymbol{t}_\bullet)}{\partial t_0^1 \partial t_p^\beta}\right|_{t_p^\alpha=0 \text{ for } p>1,\quad t_0^\alpha=t^\alpha \text{ for } \alpha=1,\dots,n}, \tag{29}$$

$$\theta_\beta(\boldsymbol{t},z) := \sum_{p=0}^\infty \theta_{\beta,p}(\boldsymbol{t}) z^p. \tag{30}$$

Define the matrix $\Theta(\boldsymbol{t},z)$ by

$$\Theta(\boldsymbol{t},z)^\alpha_\beta := \eta^{\alpha\lambda}\frac{\partial \theta_\beta(\boldsymbol{t},z)}{\partial t^\lambda}, \quad \alpha,\beta = 1,\dots,n. \tag{31}$$

Theorem 22 ([7, 13]) *The matrix* $Z_{\text{top}}(\boldsymbol{t},z) := \Theta(\boldsymbol{t},z) z^\mu z^{c_1(X)\cup}$ *is a fundamental system of solutions of the joint system* (23)–(24) *in Levelt form at* $z = 0$.

Definition 23 The solution $Z_{\text{top}}(\boldsymbol{t},z)$ is called *topological-enumerative solution* of the joint system (23) and (24).

2.2 Stokes Rays and ℓ-Chamber Decomposition

Definition 24 We call *Stokes rays* at a point $p \in \Omega$ the oriented rays $R_{ij}(p)$ in $\mathbb{C}$ defined by

$$R_{ij}(p) := \left\{-\sqrt{-1}\,(\overline{u_i(p)} - \overline{u_j(p)})\rho\colon \rho \in \mathbb{R}_+\right\}, \tag{32}$$

where $(u_1(p),\dots,u_n(p))$ is the spectrum of the operator $\mathcal{U}(p)$ (with a fixed arbitrary order).

Fix an oriented ray ℓ in the universal cover $\widehat{\mathbb{C}^*}$.

Definition 25 We say that ℓ is *admissible* at $p \in \Omega$ if the projection of the ray ℓ on $\mathbb{C}^*$ does not coincide with any Stokes ray $R_{ij}(p)$.

Definition 26 Define the open subset O_ℓ of points $p \in \Omega$ by the following conditions:

1. the eigenvalues $u_i(p)$ are pairwise distinct,
2. ℓ is admissible at p.

We call *ℓ-chamber* of Ω any connected component of O_ℓ.

2.3 Stokes Fundamental Solutions at $z = \infty$

Fix an oriented ray $\ell \equiv \{\arg z = \phi\}$ in $\widehat{\mathbb{C}^*}$. For $m \in \mathbb{Z}$, define the sectors in $\widehat{\mathbb{C}^*}$

$$\Pi_{L,m}(\phi) := \left\{z \in \widehat{\mathbb{C}^*} : \phi + 2\pi m < \arg z < \phi + \pi + 2\pi m\right\}, \tag{33}$$

$$\Pi_{R,m}(\phi) := \left\{z \in \widehat{\mathbb{C}^*} : \phi - \pi + 2\pi m < \arg z < \phi + 2\pi m\right\}. \tag{34}$$

Definition 27 The *coalescence locus* of Ω is the set

$$\Delta_\Omega := \left\{p \in \Omega : u_i(p) = u_j(p), \quad \text{for some } i \neq j\right\}. \tag{35}$$

Theorem 28 ([11, 13]) *There exists a unique formal solution $Z_{\text{form}}(\boldsymbol{t}, z)$ of the joint system (23) and (24) of the form*

$$Z_{\text{form}}(\boldsymbol{t}, z) = \Psi(\boldsymbol{t})^{-1} G(\boldsymbol{t}, z) \exp(zU(\boldsymbol{t})), \tag{36}$$

$$G(\boldsymbol{t}, z) = I + \sum_{k=1}^{\infty} \frac{1}{z^k} G_k(\boldsymbol{t}), \tag{37}$$

where the matrices $G_k(\boldsymbol{t})$ are holomorphic on $\Omega \setminus \Delta_\Omega$.

Theorem 29 ([11, 13]) *Let $m \in \mathbb{Z}$. There exist unique fundamental systems of solutions $Z_{L,m}(\boldsymbol{t}, z)$, $Z_{R,m}(\boldsymbol{t}, z)$ of the joint system (23) and (24) with asymptotic expansion*

$$Z_{L,m}(\boldsymbol{t}, z) \sim Z_{\text{form}}(\boldsymbol{t}, z), \quad |z| \to \infty, \quad z \in \Pi_{L,m}(\phi), \tag{38}$$

$$Z_{R,m}(\boldsymbol{t}, z) \sim Z_{\text{form}}(\boldsymbol{t}, z), \quad |z| \to \infty, \quad z \in \Pi_{R,m}(\phi), \tag{39}$$

respectively.

Definition 30 The solutions $Z_{L,m}(\boldsymbol{t}, z)$ and $Z_{R,m}(\boldsymbol{t}, z)$ are called *Stokes fundamental solutions* of the joint system (23) and (24) on the sectors $\Pi_{L,m}(\phi)$ and $\Pi_{R,m}(\phi)$ respectively.

2.4 Monodromy Data

Let $\ell \equiv \{\arg z = \phi\}$ be an oriented ray in $\widehat{\mathbb{C}^*}$ and consider the corresponding Stokes fundamental systems of solutions $Z_{L,m}(\boldsymbol{t}, z)$, $Z_{R,m}(\boldsymbol{t}, z)$, for $m \in \mathbb{Z}$.

Definition 31 We define the *Stokes* and *central connection* matrices $S^{(m)}(p)$, $C^{(m)}(p)$, with $m \in \mathbb{Z}$, at the point $p \in O_\ell$ by the identities

$$Z_{L,m}(\boldsymbol{t}(p), z) = Z_{R,m}(\boldsymbol{t}(p), z) S^{(m)}(p), \tag{40}$$

$$Z_{R,m}(\boldsymbol{t}(p), z) = Z_{\text{top}}(\boldsymbol{t}(p), z) C^{(m)}(p). \tag{41}$$

Set $S(p) := S^{(0)}(p)$ and $C(p) := C^{(0)}(p)$.

Definition 32 The *monodromy data* at the point $p \in O_\ell$ are defined as the 4-tuple $(\mu, R, S(p), C(p))$, where

- μ is the (matrix associated to) the grading operator,
- R is the (matrix associated to) the operator $c_1(X)\cup\colon H^\bullet(X) \to H^\bullet(X)$,
- $S(p), C(p)$ are the Stokes and central connection matrices at p, respectively.

Remark 33 The definition of the Stokes and central connection matrices is subordinate to several non-canonical choices:

1. the choice of an oriented ray ℓ in $\widehat{\mathbb{C}^*}$,
2. the choice of an ordering of canonical coordinates $u_1, \dots, u_n$ on each ℓ-chamber,
3. the choice of signs in (19), and hence of the branch of the Ψ-matrix on each ℓ-chamber.

Different choices affect the numerical values of the data (S, C), see [7]. In particular, for different choices of ordering of canonical coordinates, the Stokes and central connection matrices transform as follows:

$$S \mapsto \Pi S \Pi^{-1}, \quad C \mapsto C\Pi^{-1}, \quad \Pi \text{ permutation matrix.} \tag{42}$$

Definition 34 Fix a point $p \in O_\ell$ with canonical coordinates $(u_i(p))_{i=1}^n$. Define the oriented rays $L_j(p, \phi)$, $j = 1, \dots, n$, in the complex plane by the equations

$$L_j(p, \phi) := \left\{ u_j(p) + \rho e^{\sqrt{-1}(\frac{\pi}{2} - \phi)} \colon \rho \in \mathbb{R}_+ \right\}. \tag{43}$$

The ray $L_j(p, \phi)$ is oriented from $u_j(p)$ to ∞. We say that $(u_i(p))_{i=1}^n$ are in ℓ-*lexicographical order* if $L_j(p, \phi)$ is on the left of $L_k(p, \phi)$ for $1 \le j < k \le n$.

In what follows, it is assumed that the ℓ-lexicographical order of canonical coordinates is fixed at all points of ℓ-chambers.

Lemma 35 ([7, 13]) *If the canonical coordinates $(u_i(p))_{i=1}^n$ are in ℓ-lexicographical order at $p \in O_\ell$, then the Stokes matrices $S^{(m)}(p)$, $m \in \mathbb{Z}$, are upper triangular with 1's along the diagonal.*

By Remarks 19 and 20, the matrices μ and R determine the monodromy of solutions of the qDE,

$$M_0 := \exp(2\pi\sqrt{-1}\mu)\exp(2\pi\sqrt{-1}R). \tag{44}$$

Moreover, μ and R do not depend on the point p. The following theorem furnishes a refinement of this property.

Theorem 36 ([7, 11, 13]) *The monodromy data* (μ, R, S, C) *are constant in each* ℓ*-chamber. Moreover, they satisfy the following identities:*

$$CS^TS^{-1}C^{-1} = M_0, \tag{45}$$

$$S = C^{-1}\exp(-\pi\sqrt{-1}R)\exp(-\pi\sqrt{-1}\mu)\eta^{-1}(C^T)^{-1}, \tag{46}$$

$$S^T = C^{-1}\exp(\pi\sqrt{-1}R)\exp(\pi\sqrt{-1}\mu)\eta^{-1}(C^T)^{-1}. \tag{47}$$

Theorem 37 ([7]) *The Stokes and central connection matrices* S_m, C_m*, with* $m \in \mathbb{Z}$*, can be reconstructed from the monodromy data* (μ, R, S, C)*:*

$$S^{(m)} = S, \quad C^{(m)} = M_0^{-m}C, \quad m \in \mathbb{Z}. \tag{48}$$

Remark 38 Points of O_ℓ are semisimple. The results of [4, 5, 7, 9] imply that the monodromy data (μ, R, S, C) are well defined also at points $p \in \Omega_{ss} \cap \Delta_\Omega$, and that Theorem 36 still holds true.

Remark 39 From the knowledge of the monodromy data (μ, R, S, C) the Gromov–Witten potential $F_0^X(\boldsymbol{t})$ can be reconstructed via a Riemann–Hilbert boundary value problem, see [7, 8, 13, 23]. Hence, the monodromy data may be interpreted as a *system of coordinates* in the space of solutions of $WDVV$ equations.

2.5 Action of the Braid Group $\mathcal{B}_n$

Consider the braid group $\mathcal{B}_n$ with generators $\beta_1, \dots, \beta_{n-1}$ satisfying the relations

$$\beta_i\beta_j = \beta_j\beta_i, \quad |i-j| > 1, \tag{49}$$

$$\beta_i\beta_{i+1}\beta_i = \beta_{i+1}\beta_i\beta_{i+1}. \tag{50}$$

Let $\mathcal{U}_n$ be the set of upper triangular $(n\times n)$-matrices with 1's along the diagonal.

Definition 40 Given $U \in \mathcal{U}_n$ define the matrices $A^{\beta_i}(U)$, with $i = 1, \dots, n-1$, as follows

$$\left(A^{\beta_i}(U)\right)_{hh} := 1, \quad h = 1, \dots, n, \quad h \neq i, i+1, \tag{51}$$

$$\left(A^{\beta_i}(U)\right)_{i+1,i+1} = -U_{i,i+1}, \tag{52}$$

$$\left(A^{\beta_i}(U)\right)_{i,i+1} = \left(A^{\beta_i}(U)\right)_{i+1,i} = 1, \tag{53}$$

and all other entries of $A^{\beta_i}(U)$ are equal to zero.

Lemma 41 ([7, 11, 13]) *The braid group $\mathcal{B}_n$ acts on $\mathcal{U}_n \times GL(n, \mathbb{C})$ as follows:*

$$\mathcal{B}_n \times \mathcal{U}_n \times GL(n, \mathbb{C}) \longrightarrow \mathcal{U}_n \times GL(n, \mathbb{C})$$

$$(\beta_i, U, C) \longmapsto (A^{\beta_i}(U) \cdot U \cdot A^{\beta_i}(U),\ C \cdot A^{\beta_i}(U)^{-1})$$

We denote by $(U, C)^{\beta_i}$ the action of β_i on (U, C).

Fix an oriented ray $\ell \equiv \{\arg z = \phi\}$ in $\widehat{\mathbb{C}^*}$, and denote by $\overline{\ell}$ its projection on $\mathbb{C}^*$. Let $\Omega_{\ell,1}, \Omega_{\ell,2}$ be two ℓ-chambers and let $p_i \in \Omega_{\ell,i}$ for $i = 1, 2$. The difference of values of the Stokes and central connection matrices (S_1, C_1) and (S_2, C_2), at p_1 and p_2 respectively, can be described by the action of the braid group $\mathcal{B}_n$ of Lemma 41.

Theorem 42 ([7, 11, 13]) *Consider a continuous path $\gamma\colon [0, 1] \to \Omega$ such that*

- *$\gamma(0) = p_1$ and $\gamma(1) = p_2$,*
- *there exists a unique $t_o \in [0, 1]$ such that ℓ is not admissible at $\gamma(t_o)$,*
- *there exist $i_1, \dots, i_k \in \{1, \dots, n\}$, with $|i_a - i_b| > 1$ for $a \neq b$, such that the rays[3] $\left(R_{i_j,i_j+1}(t)\right)_{j=1}^{r}$ (resp. $\left(R_{i_j,i_j+1}(t)\right)_{j=r+1}^{k}$) cross the ray $\overline{\ell}$ in the clockwise (resp. counterclockwise) direction, as $t \to t_o^-$.*

Then, we have

$$(S_2, C_2) = (S_1, C_1)^{\beta}, \quad \beta := \left(\prod_{j=1}^{r} \beta_{i_j}\right) \cdot \left(\prod_{h=r+1}^{k} \beta_{i_h}\right)^{-1}. \tag{54}$$

Remark 43 In the general case, the points p_1 and p_2 can be connected by concatenations of paths γ satisfying the assumptions of Theorem 42.

Remark 44 The action of $\mathcal{B}_n$ on (S, C) also describes the analytic continuation of the Frobenius manifold structure on Ω, see [13, Lecture 4].

[3] Here the labeling of Stokes rays is the one prolonged from the initial point $t = 0$.

3 Derived Category, Exceptional Collections, and Helices

3.1 Notations and Basic Notions

Denote by $\mathrm{Coh}(X)$ the abelian category of coherent sheaves on X, and by $\mathcal{D}^b(X)$ its bounded derived category. Objects of $\mathcal{D}^b(X)$ are bounded complexes $A^\bullet$ of coherent sheaves on X. Morphisms are given by *roofs*: if $A^\bullet$, $B^\bullet$ are two bounded complexes, a morphism $f\colon A^\bullet \to B^\bullet$ in $\mathcal{D}^b(X)$ is the datum of

- a third object $C^\bullet$ in $\mathcal{D}^b(X)$,
- two homotopy classes of morphisms of complexes $q\colon C^\bullet \to A^\bullet$ and $g\colon C^\bullet \to B^\bullet$,
- the morphism q is required to be a *quasi-isomorphism*, i.e. it induces isomorphism in cohomology.

$$\begin{array}{ccc} & C^\bullet & \\ {}^{q}\swarrow & & \searrow^{g} \\ A^\bullet & \underset{f}{\dashrightarrow} & B^\bullet \end{array} \tag{55}$$

The derived category $\mathcal{D}^b(X)$ admits a triangulated structure, the *shift functor* $[1]\colon \mathcal{D}^b(X) \to \mathcal{D}^b(X)$ being defined by

$$A^\bullet[1] := A^{\bullet+1}, \quad A^\bullet \in \mathcal{D}^b(X). \tag{56}$$

Denote by $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet) := \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}(A^\bullet, B^\bullet[k])$. General references for this section are [17, 20, 21, 32].

3.2 Exceptional Collections

Definition 45 An object $E \in \mathcal{D}^b(X)$ is called *exceptional* iff

$$\mathrm{Hom}^\bullet(E, E) \cong \mathbb{C}. \tag{57}$$

Definition 46 An *exceptional collection* is an ordered family $(E_1, \dots, E_n)$ of exceptional objects of $\mathcal{D}^b(X)$ such that

$$\mathrm{Hom}^\bullet(E_j, E_i) \cong 0 \quad \text{for } j > i. \tag{58}$$

An exceptional collection is *full* if it generates $\mathcal{D}^b(X)$ as a triangulated category, i.e. if any full triangulated subcategory of $\mathcal{D}^b(X)$ containing all the objects E_i's is equivalent to $\mathcal{D}^b(X)$ via the inclusion functor.

Example In [2] A. Beilinson showed that the collection of line bundles

$$\mathfrak{B} := (\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n)) \tag{59}$$

on $\mathbb{P}^n$ is a full exceptional collection. M. Kapranov generalized this result in [25], where full exceptional collections on Grassmannians, flag varieties of group SL_n, and smooth quadrics are constructed.

Denote by $\mathbb{G}(k, n)$ the Grassmannian of k-dimensional subspaces in $\mathbb{C}^n$, by $\mathcal{S}^\vee$ the dual of its tautological bundle. Let $\mathbb{S}^\lambda$ be the Schur functor (see [15]) labelled by a Young diagram λ inside a rectangle $k \times (n-k)$. The collection $\mathfrak{K} := \left(\mathbb{S}^\lambda \mathcal{S}^\vee\right)_\lambda$ is full and exceptional in $\mathcal{D}^b(\mathbb{G}(k, n))$. The order of the objects of the collection is the partial order defined by inclusion of Young diagrams.

3.3 Mutations and Helices

Let E be an exceptional object in $\mathcal{D}^b(X)$. For any $X \in \mathcal{D}^b(X)$, we have natural evaluation and co-evaluation morphisms

$$j^*\colon \operatorname{Hom}^\bullet(E, X) \otimes E \to X, \quad j_*\colon X \to \operatorname{Hom}^\bullet(X, E)^* \otimes E. \tag{60}$$

Definition 47 The *left* and *right mutations* of X with respect to E are the objects $\mathbb{L}_E X$ and $\mathbb{R}_E X$ uniquely defined by the distinguished triangles

$$\mathbb{L}_E X[-1] \longrightarrow \operatorname{Hom}^\bullet(E, X) \otimes E \xrightarrow{j^*} X \longrightarrow \mathbb{L}_E X, \tag{61}$$

$$\mathbb{R}_E X \longrightarrow X \xrightarrow{j_*} \operatorname{Hom}^\bullet(X, E)^* \otimes E \longrightarrow \mathbb{R}_E X[1], \tag{62}$$

respectively.

Remark 48 In general, the third object of a distinguished triangle is not canonically defined by the other two terms. Nevertheless, the objects $\mathbb{L}_X E$ and $\mathbb{R}_E X$ are uniquely defined *up to unique isomorphism*, because of the exceptionality of E, see [8, Section 3.3].

Definition 49 Let $\mathfrak{E} = (E_1, \dots, E_n)$ be an exceptional collection. For any $i = 1, \dots, n-1$ define the *left* and *right mutations*

$$\mathbb{L}_i\mathfrak{E} := (E_1, \dots, \mathbb{L}_{E_i}E_{i+1}, E_i, \dots, E_n), \tag{63}$$

$$\mathbb{R}_i\mathfrak{E} := (E_1, \dots, E_{i+1}, \mathbb{R}_{E_{i+1}}E_i, \dots, E_n). \tag{64}$$

Theorem 50 ([21, 32]) *For all $i = 1, \dots, n-1$ the collections $\mathbb{L}_i\mathfrak{E}$ and $\mathbb{R}_i\mathfrak{E}$ are exceptional. Moreover, we have that*

$$\mathbb{L}_i\mathbb{R}_i = \mathbb{R}_i\mathbb{L}_i = \mathrm{Id}, \quad \mathbb{L}_{i+1}\mathbb{L}_i\mathbb{L}_{i+1} = \mathbb{L}_i\mathbb{L}_{i+1}\mathbb{L}_i, \quad i = 1, \dots, n, \tag{65}$$

$$\mathbb{L}_i\mathbb{L}_j = \mathbb{L}_j\mathbb{L}_i, \quad |i-j| > 1. \tag{66}$$

According to Theorem 50, we have a well-defined action of $\mathcal{B}_n$ on the set of exceptional collections of length n in $\mathcal{D}^b(X)$: the action of the generator β_i is identified with the action of the mutation $\mathbb{L}_i$ for $i = 1, \dots, n-1$.

Definition 51 Let $\mathfrak{E} = (E_1, \dots, E_n)$ be a full exceptional collection. We define the *helix* generated by $\mathfrak{E}$ to be the infinite family $(E_i)_{i\in\mathbb{Z}}$ of exceptional objects obtained by iterated mutations

$$E_{n+i} := \mathbb{R}_{E_{n+i-1}} \dots \mathbb{R}_{E_{i+1}} E_i, \quad E_{i-n} := \mathbb{L}_{E_{i-n+1}} \dots \mathbb{L}_{E_{i-1}} E_i, \quad i \in \mathbb{Z}.$$

Any family of n consecutive exceptional objects $(E_{i+k})_{k=1}^n$ is called a *foundation* of the helix.

Lemma 52 ([21]) *For $i, j \in \mathbb{Z}$, we have* $\mathrm{Hom}^\bullet(E_i, E_j) \cong \mathrm{Hom}^\bullet(E_{i-n}, E_{j-n})$.

3.4 Exceptional Bases in K-Theory

Consider the Grothendieck group $K_0(X) \equiv K_0(\mathcal{D}^b(X))$, equipped with the Grothendieck–Euler–Poincaré bilinear form

$$\chi([V],[F]) := \sum_k (-1)^k \dim_{\mathbb{C}} \mathrm{Hom}(V, F[i]), \quad V, F \in \mathcal{D}^b(X). \tag{67}$$

Definition 53 A basis $(e_i)_{i=1}^n$ of $K_0(X)_{\mathbb{C}}$ is called *exceptional* if $\chi(e_i, e_i) = 1$ for $i = 1, \dots, n$, and $\chi(e_j, e_i) = 0$ for $1 \le i < j \le n$.

Lemma 54 *Let $(E_i)_{i=1}^n$ be a full exceptional collection in $\mathcal{D}^b(X)$. The K-classes $([E_i])_{i=1}^n$ form an exceptional basis of $K_0(X)_{\mathbb{C}}$.*

The action of the braid group on the set of exceptional collections in $\mathcal{D}^b(X)$ admits a K-theoretical analogue on the set of exceptional bases of $K_0(X)_{\mathbb{C}}$, see [8, 21].

4 Dubrovin's Conjecture

4.1 Γ-Classes and Graded Chern Character

Let V be a complex vector bundle on X of rank r, and let $\delta_1, \dots, \delta_r$ be its Chern roots, so that $c_j(V) = s_j(\delta_1, \dots, \delta_r)$, where s_j is the j-th elementary symmetric polynomial.

Definition 55 Let Q be an indeterminate, and $F \in \mathbb{C}[\![Q]\!]$ be of the form $F(Q) = 1 + \sum_{n\geq 1} \alpha_n Q^n$. The *$F$-class* of V is the characteristic class $\widehat{F}_V \in H^\bullet(X)$ defined by $\widehat{F}_V := \prod_{j=1}^r F(\delta_j)$.

Definition 56 The *$\Gamma^\pm$-classes* of V are the characteristic classes associated with the Taylor expansions

$$\Gamma(1 \pm Q) = \exp\left(\mp\gamma Q + \sum_{m=2}^{\infty} (\mp 1)^m \frac{\zeta(m)}{m} Q^n\right) \in \mathbb{C}[\![Q]\!], \tag{68}$$

where γ is the Euler–Mascheroni constant and ζ is the Riemann zeta function.

If $V = TX$, then we denote $\widehat{\Gamma}_X^\pm$ its Γ-classes.

Definition 57 The *graded Chern character* of V is the characteristic class $\mathrm{Ch}(V) \in H^\bullet(X)$ defined by $\mathrm{Ch}(V) := \sum_{j=1}^r \exp(2\pi\sqrt{-1}\delta_j)$.

4.2 Statement of the Conjecture

Let X be a Fano variety. In [12] Dubrovin conjectured that many properties of the qDE of X, in particular its monodromy, Stokes and central connection matrices, are encoded in the geometry of exceptional collections in $\mathcal{D}^b(X)$. The following conjecture is a refinement of the original version in [12].

Conjecture 58 ([8]) Let X be a smooth Fano variety of Hodge–Tate type.

1. The quantum cohomology $QH^\bullet(X)$ has semisimple points if and only if there exists a full exceptional collection in $\mathcal{D}^b(X)$.
2. If $QH^\bullet(X)$ is generically semisimple, for any oriented ray ℓ of slope $\phi \in [0, 2\pi[$ there is a correspondence between ℓ-chambers and helices with a marked foundation.
3. Let Ω_ℓ be an ℓ-chamber and $\mathfrak{E}_\ell = (E_1, \dots, E_n)$ the corresponding exceptional collection (the marked foundation). Denote by S and C Stokes and central connection matrices computed in Ω_ℓ.

(a) The matrix S is the inverse of the Gram matrix of the χ-pairing in $K_0(X)_{\mathbb{C}}$ w.r.t. the exceptional basis $[\mathfrak{E}_\ell]$,

$$(S^{-1})_{ij} = \chi(E_i, E_j); \tag{69}$$

(b) The matrix C coincides with the matrix associated with the $\mathbb{C}$-linear morphism

$$\mathcal{I}^-_X\colon K_0(X)_{\mathbb{C}} \longrightarrow H^\bullet(X) \tag{70}$$

$$F \longmapsto \frac{(\sqrt{-1})^{\overline{d}}}{(2\pi)^{\frac{d}{2}}}\widehat{\Gamma}^-_X \exp(-\pi\sqrt{-1}c_1(X))\mathrm{Ch}(F), \tag{71}$$

where $d := \dim_{\mathbb{C}} X$, and $\overline{d}$ is the residue class $d \pmod 2$. The matrix is computed w.r.t. the exceptional basis $[\mathfrak{E}_\ell]$ and the pre-fixed basis $(T_\alpha)_{\alpha=1}^n$ of $H^\bullet(X)$.

Remark 59 Conjecture 58 relates two different aspects of the geometry of X, namely its *symptectic structure* (GW-theory) and its *complex structure* (the derived category $\mathcal{D}^b(X)$). Heuristically, Conjecture 58 follows from Homological Mirror Symmetry Conjecture of M. Kontsevich, see [8, Section 5.5].

Remark 60 In the paper [26] it was underlined the role of Γ-classes for refining the original version of Dubrovin's conjecture [12]. Subsequently, in [14] and [16, Γ-conjecture II] two equivalent versions of point (3.b) above were given. However, in both these versions, different choices of solutions in Levelt form of the qDE at $z = 0$ are chosen w.r.t. the natural ones in the theory of Frobenius manifolds, see Remark 19, and [8, Section 5.6].

Remark 61 If point (3.b) holds true, then automatically also point (3.a) holds true. This follows from the identity (46) and Hirzebruch–Riemann–Roch Theorem, see [8, Corollary 5.8].

Remark 62 Assume the validity of points (3.a) and (3.b) of Conjecture 58. The action of the braid group $\mathcal{B}_n$ on the Stokes and central connection matrices (Lemma 41) is compatible with the action of $\mathcal{B}_n$ on the marked foundations attached at each ℓ-chambers. Different choices of the branch of the Ψ-matrix correspond to shifts of objects of the marked foundation. The matrix M_0^{-1} is identified with the canonical operator $\kappa\colon K_0(X)_{\mathbb{C}} \to K_0(X)_{\mathbb{C}}$, $[F] \mapsto (-1)^d[F \otimes \omega_X]$. Equations (48) imply that the connection matrices $C^{(m)}$, with $m \in \mathbb{Z}$, correspond to the matrices of the morphism $\mathcal{I}^-_X$ w.r.t. the foundations $(\mathfrak{E}_\ell \otimes \omega_X^{\otimes m})[md]$. The statement $S^{(m)} = S$ coincides with the periodicity described in Lemma 52, see [8, Theorem 5.9].

Remark 63 Point (3.b) of Conjecture 58 allows to identify K-classes with solutions of the joint system of Eqs. (23) and (24). Under this identification, Stokes fundamental solutions correspond to exceptional bases of K-theory. In the approach of [6, 33], where the equivariant case is addressed, such an identification is more fundamental and a priori, see Sect. 6.

5 Results for Grassmannians

Conjecture 58 has been proved for complex Grassmannians $\mathbb{G}(k, n)$ in [8, 16]. See also [22, 34]. The proof is based on direct computation of the monodromy data of the qDE at points of the *small quantum cohomology*, namely the subset $H^2(\mathbb{G}(k, n), \mathbb{C})$ of Ω. Here we summarize the main results obtained.

Remark 64 If[4] $\pi_1(n) \leq k \leq n - \pi_1(n)$, the small quantum locus of $\mathbb{G}(k, n)$ is contained in the coalescence locus Δ_Ω, see [3]. In these cases, the computations of the monodromy data is justified by the results of [4, 5, 7, 9]. See also Remark 38.

5.1 The Case of Projective Spaces

Denote by $\sigma \in H^2(\mathbb{P}^{n-1}, \mathbb{C})$ the hyperplane class and fix the basis $(\sigma^k)_{k=0}^{n-1}$ of $H^\bullet(\mathbb{P}^{n-1})$. The joint system (23) and (24) for $\mathbb{P}^{n-1}$, restricted at the point $t\sigma \in H^2(\mathbb{P}^{n-1}, \mathbb{C})$, with $t \in \mathbb{C}$, is

$$\frac{\partial Z}{\partial t} = z\mathcal{C}(t)Z, \tag{72}$$

$$\frac{\partial Z}{\partial z} = \left(\mathcal{U}(t) + \frac{1}{z}\mu\right) Z, \tag{73}$$

with

$$\mathcal{U}(t) = \begin{pmatrix} 0 & & & & nq \\ n & 0 & & & \\ & n & 0 & & \\ & & \ddots & \ddots & \\ & & & n & 0 \end{pmatrix}, \quad q := e^t, \quad \mathcal{C}(t) = \frac{1}{n}\mathcal{U}(t), \tag{74}$$

$$\mu = \text{diag}\left(-\frac{n-1}{2}, -\frac{n-3}{2}, \dots, \frac{n-3}{2}, \frac{n-1}{2}\right). \tag{75}$$

The canonical coordinates are given by the eigenvalues of the matrix $\mathcal{U}(t)$,

$$u_h(t) = ne^{\frac{2\pi i(h-1)}{n}} q^{\frac{1}{n}} \quad h = 1, \dots, n. \tag{76}$$

[4]Here $\pi_1(n)$ denotes the smallest prime number which divides n.

Fix the orthonormalized idempotent vector fields, $f_1(t), \dots, f_n(t)$, given by

$$f_h(t) := \sum_{\ell=1}^{n} f_h^\ell(t)\sigma^{\ell-1}, \quad f_h^\ell(t) := n^{-\frac{1}{2}} q^{\frac{n+1-2\ell}{2n}} e^{(1-2\ell)i\pi\frac{(h-1)}{n}} \quad h, \ell = 1, \dots, n,$$

and consider the following branch of the Ψ-matrix,

$$\Psi(t) := \left(\begin{array}{c|c|c} f_1^1(t) & \dots & f_n^1(t) \\ \vdots & & \vdots \\ f_1^n(t) & \dots & f_n^n(t) \end{array} \right)^{-1}. \tag{77}$$

Theorem 65 ([8]) *Fix the oriented ray ℓ in $\widehat{\mathbb{C}^*}$ of slope $\phi \in [0, \frac{\pi}{n}[$. For suitable choices of the signs of the columns of the Ψ-matrix (77), the central connection matrix computed at $0 \in H^\bullet(\mathbb{P}^{n-1})$ coincides with the matrix attached to the morphism*

$$Д^-_{\mathbb{P}^{n-1}} : K_0(\mathbb{P}^{n-1})_{\mathbb{C}} \to H^\bullet(\mathbb{P}^{n-1})$$

computed w.r.t. the exceptional bases

$$\mathcal{O}\left(\frac{n}{2}\right), \bigwedge^1 \mathcal{T}\left(\frac{n}{2}-1\right), \mathcal{O}\left(\frac{n}{2}+1\right), \bigwedge^3 \mathcal{T}\left(\frac{n}{2}-2\right), \dots, \mathcal{O}(n-1), \bigwedge^{n-1} \mathcal{T} \tag{78}$$

for n even, and

$$\mathcal{O}\left(\frac{n-1}{2}\right), \mathcal{O}\left(\frac{n+1}{2}\right), \bigwedge^2 \mathcal{T}\left(\frac{n-3}{2}\right),$$
$$\mathcal{O}\left(\frac{n+3}{2}\right), \bigwedge^4 \mathcal{T}\left(\frac{n-5}{2}\right), \dots, \mathcal{O}(n-1), \bigwedge^{n-1} \mathcal{T} \tag{79}$$

for n odd. In particular, Conjecture 58 holds true for $\mathbb{P}^{n-1}$.

Remark 66 Exceptional collections (78) and (79) are related to Beilinson's exceptional collection (59) by mutations and shifts. For different choices of the ray ℓ, the exceptional collections attached to the monodromy data computed at $0 \in H^\bullet(\mathbb{P}^{n-1})$ are given (up to shifts) by the following list, see [6, 8].

1. **Case n odd:** an exceptional collection either of the form

$$\mathcal{O}\left(-k-\frac{n-1}{2}\right), \mathcal{T}\left(-k-\frac{n-1}{2}-1\right), \mathcal{O}\left(-k-\frac{n-1}{2}+1\right),$$
$$\bigwedge^3 \mathcal{T}\left(-k-\frac{n-1}{2}-2\right), \mathcal{O}\left(-k-\frac{n-1}{2}+2\right), \dots, \bigwedge^{n-4} \mathcal{T}(-k-n+2),$$
$$\mathcal{O}(-k-1), \bigwedge^{n-2} \mathcal{T}(-k-n+1), \mathcal{O}(-k),$$

or of the form

$$\mathcal{O}\left(-k-\frac{n-1}{2}\right),\ \mathcal{O}\left(-k-\frac{n-1}{2}+1\right),\ \bigwedge^2\mathcal{T}\left(-k-\frac{n-1}{2}-1\right),$$
$$\mathcal{O}\left(-k-\frac{n-1}{2}+2\right),\ \bigwedge^3\mathcal{T}\left(-k-\frac{n-1}{2}-2\right)\dots,\ \mathcal{O}(-k-1),$$
$$\bigwedge^{n-3}\mathcal{T}(-k-n+2),\ \mathcal{O}(-k),\ \bigwedge^{n-1}\mathcal{T}(-k-n+1),$$

for some $k \in \mathbb{Z}$

2. **Case n even:** an exceptional collection either of the form

$$\mathcal{O}\left(-k-\frac{n}{2}\right),\ \mathcal{O}\left(-k-\frac{n}{2}+1\right),\ \bigwedge^2\mathcal{T}\left(-k-\frac{n}{2}-1\right),\ \mathcal{O}\left(-k-\frac{n}{2}+2\right),\dots,$$
$$\dots,\ \bigwedge^{n-4}\mathcal{T}(-k-n+2),\ \mathcal{O}(-k-1),\ \bigwedge^{n-2}\mathcal{T}(-k-n+1),\ \mathcal{O}(-k),$$

or of the form

$$\mathcal{O}\left(-k-\frac{n}{2}+1\right),\ \mathcal{T}\left(-k-\frac{n}{2}\right),\ \mathcal{O}\left(-k-\frac{n}{2}+2\right),\ \bigwedge^3\mathcal{T}\left(-k-\frac{n}{2}-1\right),\dots,$$
$$\dots,\ \mathcal{O}(-k-1),\ \bigwedge^{n-3}\mathcal{T}(-k-n+2),\ \mathcal{O}(-k),\ \bigwedge^{n-1}\mathcal{T}(-k-n+1),$$

for some $k \in \mathbb{Z}$.

5.2 *The Case of Grassmannians*

Denote by $\mathbb{G}$ the Grassmannian $\mathbb{G}(k, n)$ parametrizing k-dimensional subspaces in $\mathbb{C}^n$, and by $\mathbb{P}$ the projective space $\mathbb{P}^{n-1}$. Let $\xi_1, \dots, \xi_k$ be the Chern roots of the dual of the tautological bundle $\mathcal{S}$ on $\mathbb{G}$, and denote by $h_j(\boldsymbol{\xi})$ the j-th complete symmetric polynomial in $\xi_1, \dots, \xi_k$. An additive basis of the cohomology ring

$$H^\bullet(\mathbb{G}) \cong \mathbb{C}[\xi_1, \dots, \xi_k]^{\mathfrak{S}_k} \Big/ \langle h_{n-k+1}, \dots, h_n \rangle, \tag{80}$$

is given by the Schubert classes $(\sigma_\lambda)_{\lambda \subseteq k\times(n-k)}$, labelled by partitions λ with Young diagram inside a $k \times (n-k)$ rectangle. Under the presentation (80), the Schubert classes are given by Schur polynomials in $\boldsymbol{\xi}$,

$$\sigma_\lambda := \frac{\det\left(\xi_i^{\lambda_j+k-j}\right)_{1\le i,j\le k}}{\prod_{i<j}(\xi_i-\xi_j)}. \tag{81}$$

Denote by $\eta_{\mathbb{P}}$ and $\eta_{\mathbb{G}}$ the Poincaré metrics on $H^\bullet(\mathbb{P})$ and $H^\bullet(\mathbb{G})$ respectively. The metric $\eta_{\mathbb{P}}$ induces a metric $\eta_{\mathbb{P}}^{\wedge k}$ on the exterior power $\bigwedge^k H^\bullet(\mathbb{P})$:

$$\eta_{\mathbb{P}}^{\wedge k}(\alpha_1 \wedge \dots, \wedge\alpha_k, \beta_1 \wedge \dots, \wedge\beta_k) := \det\left(\eta_{\mathbb{P}}(\alpha_i, \beta_j)\right)_{1\leq i,j\leq k}. \tag{82}$$

Theorem 67 ([8, 16]) *We have a $\mathbb{C}$-linear isometry*

$$\mathcal{I}\colon \left(\bigwedge^k H^\bullet(\mathbb{P}),\ (-1)^{\binom{k}{2}}\eta_{\mathbb{P}}^{\wedge k}\right) \to (H^\bullet(\mathbb{G}), \eta_{\mathbb{G}}), \quad \sigma^{\nu_1}\wedge\cdots\wedge\sigma^{\nu_k} \mapsto \sigma_{\tilde{\nu}},$$

where $n-1 \geq \nu_1 > \nu_2 > \cdots > \nu_k \geq 0$ and $\tilde{\nu} := (\nu_1 - k + 1, \nu_2 - k + 2, \dots, \nu_k)$.

Consider the domain $\Omega_{\mathbb{G}} \subset H^\bullet(\mathbb{G})$ (resp. $\Omega_{\mathbb{P}} \subset H^\bullet(\mathbb{P})$) where the GW-potential $F_0^{\mathbb{G}}$ (resp. $F_0^{\mathbb{P}}$) converges. Let $t \in \mathbb{C}$ and consider the points

$$p := t\sigma_1 \in H^2(\mathbb{G}, \mathbb{C}), \quad \hat{p} := \left(t + \pi\sqrt{-1}(k-1)\right)\sigma \in H^2(\mathbb{P}, \mathbb{C}), \tag{83}$$

in the small quantum cohomology of $\mathbb{G}$ and $\mathbb{P}$ respectively. Theorem 67 allow us to identify[5] the tangent spaces $T_p\Omega_{\mathbb{G}}$ and $\bigwedge^k T_{\hat{p}}\Omega_{\mathbb{P}}$.

Lemma 68 ([8, 16]) *Let $\Psi^{\mathbb{P}}(t)$ be the Ψ-matrix defined by (77). Then the matrix $\Psi^{\mathbb{G}}(t) := (\sqrt{-1})^{\binom{k}{2}} \bigwedge^k \Psi^{\mathbb{P}}(t + \pi\sqrt{-1}(k-1))$ defines a branch of the Ψ-matrix for $\mathbb{G}$.*

The following results show that under the identification of Theorem 67, solutions and monodromy data of the joint system (23) and (24) for $\mathbb{G}$ can be reconstructed from solutions for the joint system for $\mathbb{P}$.

Theorem 69 ([8]) *Let $Z^{\mathbb{P}}(t,z)$ be a solution of the joint system (72) and (73). The function*

$$Z^{\mathbb{G}}(t,z) := \bigwedge^k \left(Z^{\mathbb{P}}(t + \pi\sqrt{-1}(k-1), z)\right) \tag{84}$$

is a solution for the joint system for $\mathbb{G}$, namely

$$\frac{\partial Z^{\mathbb{G}}}{\partial t} = z\mathcal{C}_{\mathbb{G}}(t)Z^{\mathbb{G}}, \tag{85}$$

$$\frac{\partial Z^{\mathbb{G}}}{\partial z} = \left(\mathcal{U}_{\mathbb{G}}(t) + \frac{1}{z}\mu_{\mathbb{G}}\right)Z^{\mathbb{G}}. \tag{86}$$

[5] In what follows, if A is a $n \times n$-matrix, we denote by $\bigwedge^k A$ the matrix of $k \times k$-minors of A, ordered in lexicographical order.

Corollary 70 ([8]) *Fix an oriented ray ℓ in $\widehat{\mathbb{C}^*}$ admissible at both points $p, \hat{p}$ in (83). Denote by $S^{\mathbb{P}}(\hat{p}), S^{\mathbb{G}}(p)$ and $C^{\mathbb{P}}(\hat{p}), C^{\mathbb{G}}(p)$ the Stokes and central connection matrices at $\hat{p}$ and p, respectively. We have*

$$S^{\mathbb{G}}(p) = \bigwedge^k S^{\mathbb{P}}(\hat{p}), \tag{87}$$

$$C^{\mathbb{G}}(p) = (\sqrt{-1})^{-\binom{k}{2}} \left(\bigwedge^k C^{\mathbb{P}}(\hat{p})\right) \exp(\pi\sqrt{-1}(k-1)\sigma_1 \cup). \tag{88}$$

Proof Denote by

- $Z^{\mathbb{P}}_{\mathrm{top}}(t, z)$ and $Z^{\mathbb{G}}_{\mathrm{top}}(t, z)$ the topological-enumerative solutions for $\mathbb{P}$ and $\mathbb{G}$ respectively, restricted at their small quantum cohomologies;
- $Z^{\mathbb{P}/\mathbb{G}}_{L/R,m}(t, z)$, with $m \in \mathbb{Z}$, the Stokes fundamental solutions of the joint systems (23) and (24) for $\mathbb{P}$ and $\mathbb{G}$ respectively.

We have

$$Z^{\mathbb{G}}_{\mathrm{top}}(t, z) = \left(\bigwedge^k Z^{\mathbb{P}}_{\mathrm{top}}(t + \pi\sqrt{-1}(k-1), z)\right) \cdot \exp(-\pi\sqrt{-1}(k-1)\sigma_1 \cup),$$

$$Z^{\mathbb{G}}_{L/R,m}(t, z) = (\sqrt{-1})^{-\binom{k}{2}} \bigwedge^k Z^{\mathbb{P}}_{L/R,m}(t + \pi\sqrt{-1}(k-1), z).$$

See [8] for proofs of these identities. □

Corollary 71 ([8]) *The central connection matrix computed at $0 \in H^\bullet(\mathbb{G})$ coincides with the matrix attached to the morphism*

$$\mathit{\Pi}^-_{\mathbb{G}} \colon K_0(\mathbb{G})_{\mathbb{C}} \to H^\bullet(\mathbb{G})$$

computed w.r.t. an exceptional basis of $K_0(\mathbb{G})_{\mathbb{C}}$. Such a basis is the projection in K-theory of an exceptional collection of $\mathcal{D}^b(\mathbb{G})$ related by mutations and shifts to the twisted Kapranov exceptional collection

$$(\mathbb{S}^\lambda \mathcal{S}^\vee \otimes \mathcal{L}), \quad \mathcal{L} := \det\left(\bigwedge^2 \mathcal{S}^\vee\right). \tag{89}$$

In particular, Conjecture 58 holds true for $\mathbb{G}$.

6 Results on the Equivariant qDE of $\mathbb{P}^{n-1}$

Gromov–Witten theory, as described in Sect. 1.2, can be suitably adapted to the equivariant case [18]. Given a variety X equipped with the action of a group G, a quantum deformation of the equivariant cohomology algebra $H^\bullet_G(X, \mathbb{C})$ can be defined.

Consider the projective space $\mathbb{P}^{n-1}$ equipped with the diagonal action of the torus $\mathbb{T} := (\mathbb{C}^*)^n$. Although the isomonodromic system (72) and (73) does not admit an equivariant analog, the differential equation (72) only can be easily modified. By change of coordinates $q := \exp(t)$, setting $z = 1$, and replacing the quantum multiplication $*_q$ by the corresponding equivariant one $*_{q,z}$, Eq. (72) takes the form

$$q\frac{d}{dq}Z = \sigma *_{q,z} Z. \tag{90}$$

Here the equivariant parameters $\boldsymbol{z} = (z_1, \dots, z_n)$ correspond to the factors of $\mathbb{T}$, and $Z(q, \boldsymbol{z})$ takes values in $H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1}, \mathbb{C})$. Equation (90) admits a compatible system of difference equations, called qKZ difference equations

$$Z(q, z_1, \dots, z_i - 1, \dots, z_n) = K_i(q, \boldsymbol{z})Z(q, \boldsymbol{z}), \quad i = 1, \dots, n, \tag{91}$$

for suitable linear operators K_i's, introduced in [33]. The joint system (90) and (91) is a suitable limit of an analogue one for the cotangent bundle $T^*\mathbb{P}^{n-1}$, see [19, 30]. The existence and compatibility of such a joint system for more general Nakajima quiver varieties is justified by the general theory of D. Maulik and A. Okounkov [29].

In [33], the study of the monodromy and Stokes phenomenon at $q = \infty$ of solutions of the joint system (90) and (91) is addressed. Furthermore, elements of $K_0^{\mathbb{T}}(\mathbb{P}^{n-1})_{\mathbb{C}}$ are identified with solutions of the joint system (90) and (91): Stokes bases of solutions correspond to exceptional bases.

In [6], the authors describe relations between the monodromy data of the joint system of the equivariant qDE (90) and qKZ Eqs. (91) and characteristic classes of objects of the derived category $\mathcal{D}^b_{\mathbb{T}}(\mathbb{P}^{n-1})$ of equivariant coherent sheaves on $\mathbb{P}^{n-1}$. Equivariant analogs of results of [8, Section 6] are obtained.

The Б-Theorem of [6] is the equivariant analog of Theorem 65. Moreover, in [6] the Stokes bases of solutions of the joint system (90) and (91) are identified with explicit $\mathbb{T}$-full exceptional collections in $\mathcal{D}^b_{\mathbb{T}}(\mathbb{P}^{n-1})$, which project to those listed in Remark 66 via the forgetful functor $\mathcal{D}^b_{\mathbb{T}}(\mathbb{P}^{n-1}) \to \mathcal{D}^b(\mathbb{P}^{n-1})$. This refines results of [33]. Finally, in [6] it is proved that the Stokes matrices of the joint system (90) and (91) equal the Gram matrices of the equivariant Grothendieck–Euler–Poincaré pairing on $K_0^{\mathbb{T}}(\mathbb{P}^{n-1})_{\mathbb{C}}$ w.r.t. the same exceptional bases.

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Periodic One-Point Rank One Commuting Difference Operators

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Abstract In this paper we study one-point rank one commutative rings of difference operators. We find conditions on spectral data which specify such operators with periodic coefficients.

Keywords Commuting difference operators

Mathematics Subject Classification (2010) 39A13, 39A23, 17B80

1 Introduction and Main Results

In this paper we study one-point rank one commuting difference operators with periodic coefficients.

Let us consider a (maximal) commutative ring $\mathcal{A}$ of difference operators consisting of operators of the form

$$L_m = \sum_{i=-N_-}^{i=N_+} u_i(n)T^i, \quad u_{N_+} = 1, N_+, N_- \geq 0,$$

where T is a shift operator, $T\psi(n) = \psi(n+1)$, $m = N_+ + N_-$ is the order of L_m (assuming $u_{N_-} \neq 0$). The operator L_m acts on the space of formal functions $\{\psi : \mathbb{Z} \longrightarrow \mathbb{C}\}$. The ring $\mathcal{A}$ is isomorphic to a ring of rational functions on spectral curve Γ with poles in points $q_1, \dots, q_s \in \Gamma$ (see [1]). Common eigenfunctions of

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operators from $\mathcal{A}$ form a vector bundle of rank l over $\Gamma\backslash\{\cup_{j=1}^{s} q_j\}$. More precisely, there is a vector-function $\psi(n, P) = (\psi_1(n, P), \ldots, \psi_l(n, P))$, $P \in \Gamma\backslash\{\cup_{j=1}^{s} q_j\}$ which is called *Baker–Akhiezer function*, such that every operator $L_m \in \mathcal{A}$ corresponds to a meromorphic function $f(P)$ on Γ with poles in $q_1, \ldots, q_s$

$$L_m\psi = f\psi.$$

Moreover, $m = lm'$, where m' is the degree of the pole divisor of f. The operators from $\mathcal{A}$ are called *s-point rank l operators*.

Two-point rank one operators were classified in [1, 7]. Baker–Akhiezer function of such operators can be reconstructed from Krichever's spectral data [1]. One-point rank $l > 1$ operators were discovered by Krichever and Novikov in [3]. Spectral data for one-point rank one operators were found in [4]. Those operators contain the shift operator T only in positive power. Recall that the spectral data for such operators has the form (see [4])

$$S = \{\Gamma, \gamma, P_n, q, k^{-1}\}.$$

Here Γ is a Riemannian surface of genus g (we do not consider singular spectral curves), $\gamma = \gamma_1 + \ldots + \gamma_g$ is a non-special divisor on Γ, $P_n \in \Gamma$, $n \in \mathbb{Z}$ is a set of general points, $q \in \Gamma$ is a fixed point, k^{-1} is a local parameter near q. There is a unique Baker–Akhezer function $\psi(n, P)$, $n \in \mathbb{Z}$, $P \in \Gamma$ which is rational function on Γ and satisfies the following properties

- if $n > 0$, then the zero and pole divisor of ψ has the form

$$(\psi(n, P)) = \gamma(n) + P_0 + \ldots + P_{n-1} - \gamma - nq,$$

- if $n < 0$, then the zero and pole divisor of ψ has the form

$$(\psi(n, P)) = \gamma(n) - P_{-1} - \ldots - P_n - \gamma - nq,$$

- if $n = 0$ then $\psi(n, P) = 1$,
- in a neighborhood of q the function ψ has the following expansion

$$\psi(n, P) = k^n + O(k^{n-1}).$$

Here $\gamma(n) = \gamma_1(n) + \ldots + \gamma_g(n)$, $n \neq 0$ is some divisor on Γ. Further we will use the following notation $\gamma(0) = \gamma$. For arbitrary meromorphic function $f(P)$ on Γ with the unique pole in q of order m there is a unique operator

$$L_m = T^m + u_{m-1}(n)T^{m-1} + \ldots + u_0(n)$$

such that $L_m\psi = f\psi$, see [4].

If in the spectral data S all points P_n coincide, $P_n = q^+$, then we get the two-point Krichever's construction [1].

One-point rank one operators were studied in [4–6], in particular some explicit examples of such operators were given. That class of operators is very interesting because, for example with the help of those operators one can construct a discretization of the Lamé operator preserving the spectral curve. More precisely, let $\wp(x)$, $\zeta(x)$ be the Weierstrass functions. We define the function $A_g(x,\varepsilon)$ by the following formulas

$$A_1 = -2\zeta(\varepsilon) - \zeta(x-\varepsilon) + \zeta(x+\varepsilon),$$

$$A_2 = -\frac{3}{2}\left(\zeta(\epsilon) + \zeta(3\varepsilon) + \zeta(x-2\varepsilon) - \zeta(x+2\varepsilon)\right),$$

$$A_g = A_1 \prod_{i=1}^{g_1} \left(1 + \frac{\zeta(x-(2i+1)\varepsilon) - \zeta(x+(2i+1)\varepsilon)}{\zeta(\varepsilon) + \zeta((4i+1)\varepsilon)}\right), \text{ for odd } g = 2g_1 + 1,$$

$$A_g = A_2 \prod_{i=2}^{g_1} \left(1 + \frac{\zeta(x-2i\varepsilon) - \zeta(x+2i\varepsilon)}{\zeta(\varepsilon) + \zeta((4i-1)\varepsilon)}\right), \text{ for even } g = 2g_1.$$

The operator

$$L_2 = \frac{1}{\varepsilon^2} T_\varepsilon^2 + A_g(x,\varepsilon)\frac{1}{\varepsilon} T_\varepsilon + \wp(\varepsilon) \tag{1}$$

commutes with the operator L_{2g+1}, operators L_2, L_{2g+1} are rank one one-point operators. In the above formulas it is assumed that $T_\varepsilon \psi(x) = \psi(x+\varepsilon)$. The operator L_2 has the following expansion

$$L_2 = \partial_x^2 - g(g+1)\wp(x) + O(\varepsilon).$$

For small g it is checked that the spectral curve of the pair L_2, L_{2g+1} coincides with the spectral curve of the Lamé operator $\partial_x^2 - g(g+1)\wp(x)$, see [5]. Probably this class of difference operators can be used for the construction of a discretization of arbitrary finite-gap one dimensional Schrödinger operators. Note that the operator (1) is periodic. So, for the discretization of the finite-gap operators it is useful to find the condition when rank one one-point operators are periodic with real coefficients. This is the main motivation of this paper.

In the next theorem we formulate periodicity and reality conditions of the coefficients of the operators.

Theorem 1 *Coefficients of one-point rank one operators corresponding to the spectral data*

$$S = \{\Gamma, \gamma, P_n, q, k^{-1}\}$$

are N-periodic, $N \in \mathbb{N}$, if and only if

$$P_{n+N} = P_n,$$

and there is a meromorphic function $\lambda(P)$ on Γ with a divisor of zeros and poles of the form

$$(\lambda(P)) = P_0 + \ldots + P_{N-1} - Nq.$$

Let us assume that the spectral curve Γ admits an antiholomorphic involution

$$\tau : \Gamma \longrightarrow \Gamma, \quad \tau^2 = id.$$

If

$$\tau(P_n) = P_n, \quad \tau(\gamma) = \gamma, \quad \tau(q) = q, \quad \tau(k) = \bar{k}, \tag{2}$$

then the Baker–Akhiezer function satisfies the identity

$$\psi(n, P) = \overline{\psi(n, \tau(P))}, \tag{3}$$

and if additionally

$$\tau(f(P)) = \overline{f(P)},$$

then the coefficients of the operator L_m corresponding to the function $f(P)$, $L_m\psi = f\psi$, are real.

In the case of two-point rank one operators the analogue of Theorem 1 was proved in [2]. In the two-point case we have $(\lambda) = Nq^+ - Nq$.

2 Proof of Theorem 1

At the beginning we prove the second part of the theorem. The proof of this part is usual. The identity (3) follows from the uniqueness of the Baker–Akhiezer function with the fixed spectral data. Indeed, from (2) it follows that the function $\overline{\psi(n, \tau(P))}$ satisfies the same conditions as $\psi(n, P)$, hence we get (3).

We have

$$L_m\psi(n, \tau(P)) = f(\tau(P))\psi(n, \tau(P)).$$

Consequently,

$$\overline{L_m \psi(n, \tau(P))} = \bar{L}_m \overline{\psi(n, \tau(P))} = \bar{L}_m \psi(n, P) = \overline{f(\tau(P))\psi(n, \tau(P))}.$$

Hence

$$\bar{L}_m \psi(n, P) = f(P)\psi(n, P).$$

From the uniqueness of the operator corresponding to the meromorphic function $f(P)$ we get

$$\bar{L}_m = L_m,$$

hence, the coefficients of L_m are real.

To prove the first part of the theorem we introduce the following function

$$\chi(n, P) = \frac{\psi(n+1, P)}{\psi(n, P)}.$$

From the definition of the Baker–Akhiezer function we obtain that the zero and pole divisor of χ has the form

$$(\chi(n, P)) = \gamma(n+1) + P_n - \gamma(n) - q, \quad n \in \mathbb{Z}. \tag{4}$$

Lemma 1 *Operators from $\mathcal{A}$ have N-periodic coefficients if and only if*

$$\chi(n+N, P) = \chi(n, P).$$

Proof Let us prove the inverse part of the lemma. We assume that the coefficients of all operators from $\mathcal{A}$ are periodic. This means that the operator T^N commutes with all operators from $\mathcal{A}$, i.e., $T^N \in \mathcal{A}$. This also means that there is a meromorphic function $\lambda(P)$ on Γ with the unique pole in q of order N such that

$$T^N \psi(n, P) = \psi(n+N, P) = \lambda(P)\psi(n, P). \tag{5}$$

We have

$$\chi(n+N, P) = \frac{\psi(n+1+N, P)}{\psi(n+N, P)} = \frac{\lambda(P)\psi(n+1, P)}{\lambda(P)\psi(n, P)} = \chi(n, P). \tag{6}$$

Let us prove the direct part of the lemma. We assume that $\chi(n+N, P) = \chi(n, P)$. We introduce a rational function on Γ

$$\lambda(P) = \chi(0, P) \dots \chi(N-1, P) = \psi(N, P).$$

Then we obtain

$$T^N\psi(n,P)=\psi(n+N,P)=\chi(n+N-1,P)\psi(n+N-1,P)=$$
$$\chi(n+N-1,P)\chi(n+N-2,P)\psi(n+N-2,P)=\ldots=$$
$$\chi(n+N-1,P)\ldots\chi(n,P)\psi(n,P)=$$
$$\chi(0,P)\ldots\chi(N-1,P)\psi(n,P).$$

Hence,

$$T^N\psi(n,P)=\lambda(P)\psi(n,P). \tag{7}$$

From (7) it follows that $T^N \in \mathcal{A}$ since T^N and operators from $\mathcal{A}$ have common Baker–Akhiezer eigenfunction. Moreover $\lambda(P)$ has the unique pole of order N in q. Lemma 1 is proved. □

Now we can finish the proof of Theorem 1. Let us assume that coefficients of the operators are periodic. Then by Lemma 1 the function χ is periodic and from (4) we have

$$(\chi(n+N,P))=\gamma(n+N+1)+P_{n+N}-\gamma(n+N)-q. \tag{8}$$

Hence, comparing the pole divisors of (4) and (8) we get $\gamma(n)=\gamma(n+N)$, and after comparing the zero divisors of (4) and (8) we get $P_{n+N}=P_n$.

From the proof of Lemma 1 it follows that the function $\lambda(P)=\psi(N,P)$ has an unique pole q of order N, moreover

$$(\lambda(P))=\gamma(N)+P_0+\ldots+P_{N-1}-\gamma(0)-Nq=P_0+\ldots+P_{N-1}-Nq.$$

Hence the direct part of Theorem 1 is proven.

Let us assume that there is a meromorphic function $\lambda(P)$ such that

$$(\lambda(P))=P_0+\ldots+P_{N-1}-Nq,$$

and $P_{n+N}=P_n$. We can suppose that in the neighborhood of q we have the expansion

$$\lambda=k^N+O(k^{n-1}).$$

Then from (4) we have

$$(\chi(n+N))=\gamma(n+N+1)+P_{n+N}-\gamma(n+N)-q$$
$$=\gamma(n+1)+P_n-\gamma(n)-q=\chi(n).$$

Since $\chi(n) = k + O(1)$ in the neighborhood of q, we get $\chi(n+N) = \chi(n)$. Hence by Lemma 1 the coefficient of the operators are periodic. Theorem 1 is proven.

2.1 Example

Let us consider the case of elliptic spectral curve Γ given by the equation

$$w^2 = F(z) = z^3 + c_2 z^2 + c_1 z + c_0.$$

The degree of the divisor $\gamma(n)$ is 1. Let

$$\gamma(n) = (\alpha_n, \beta_n) \in \Gamma, \quad \beta_n^2 = F(\alpha_n).$$

Commuting operators of orders 2 and 3 have the forms (see [4])

$$L_2 = (T + U_n)^2 + W_n, \quad U_n = \frac{\beta_n + \beta_{n+1}}{\alpha_{n+1} - \alpha_n}, \quad W_n = -c_2 - \alpha_n - \alpha_{n+1},$$

$$\begin{aligned} L_3 = {} & T^3 + (U_n + U_{n+1} + U_{n+2})T^2 \\ & + (U_n^2 + U_{n+1}^2 + U_n U_{n+1} + W_n - \alpha_{n+2})T + (U_n(U_n^2 + W_n - \alpha_n) + \beta_n). \end{aligned}$$

The function $\chi(n, P)$ has the form

$$\chi = \frac{w + \beta_n}{z - \alpha_n} + \frac{\beta_n + \beta_{n+1}}{\alpha_n - \alpha_{n+1}}.$$

The point $P_n = (z_n, w_n) \in \Gamma$ has the coordinates

$$z_n = \frac{c_1(\alpha_n + \alpha_{n+1}) + \alpha_n\alpha_{n+1}(\alpha_n + \alpha_{n+1}) + 2c_2\alpha_n\alpha_{n+1} + 2(c_0 + \beta_n\beta_{n+1})}{(\alpha_n - \alpha_{n+1})^2},$$

$$w_n = \frac{\beta_{n+1}(\alpha_n - z_n) + \beta_n(\alpha_{n+1} - z_n)}{\alpha_n - \alpha_{n+1}}.$$

If $\alpha_{n+N} = \alpha_n,\ \beta_{n+N} = \beta_n$, then

$$\gamma(n+N) = (\alpha_{n+N}, \beta_{n+N}) = (\alpha_n, \beta_n) = \gamma(n), \quad P_{n+N} = P_n,$$

and the meromorphic function

$$\lambda(P) = \chi(0, P) \dots \chi(N-1, P)$$

satisfies the conditions of Theorem 1.

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On the bi-Hamiltonian Structure of the Trigonometric Spin Ruijsenaars–Sutherland Hierarchy

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Abstract We report on the trigonometric spin Ruijsenaars–Sutherland hierarchy derived recently by Poisson reduction of a bi-Hamiltonian hierarchy associated with free geodesic motion on the Lie group $\mathrm{U}(n)$. In particular, we give a direct proof of a previously stated result about the form of the second Poisson bracket in terms of convenient variables.

Keywords Integrable system · Spin Ruijsenaars and Sutherland models · bi-Hamiltonian Hierarchy · Hamiltonian reduction

Mathematics Subject Classification (2010) 70H06, 37J15, 37K10

1 Introduction

The classical integrable many-body models of Calogero–Moser–Sutherland and Ruijsenaars–Schneider as well as their extensions by internal degrees of freedom are in the focus of intense investigations even today, many years after their inception. See [1–4] and references therein. One of the sources of these models is Hamiltonian reduction of obviously integrable 'free motion' on suitable higher dimensional phase spaces, among which cotangent bundles and their Poisson–Lie analogues are the prime examples. In this framework, the emergence of the internal degrees of freedom, colloquially called 'spin', originates from the fact that symplectic

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reductions of cotangent bundles are in general not cotangent bundles, but more complicated phase spaces.

We do not have a single, all encompassing framework for understanding integrable Hamiltonian systems, but there exist several powerful approaches with large intersections of their ranges of applicability. For example, the method of the classical r-matrix incorporates many famous systems, like Toda lattices, that can be derived by Hamiltonian reduction, too, as reviewed in [9, 10]. The r-matrix method and Hamiltonian reduction also have several links to the bi-Hamiltonian approach initiated by Magri [8].

It was pointed out in the recent paper [4] that one of the simplest finite-dimensional integrable systems, the free geodesic motion on the unitary group $\mathrm{U}(n)$, admits a natural bi-Hamiltonian structure, and a suitable reduction of this free system gives rise to the so-called spin Ruijsenaars–Sutherland hierarchy. In this contribution, we overview the results of [4], and give a new, direct proof of a statement formulated in this reference without detailed proof.

2 Bi-Hamiltonian Hierarchy on $T^*\mathrm{U}(n)$ and Its Reduction

In this section we present a terse review of the results of [4].

Our starting point is the manifold $T^*\mathrm{U}(n)$, which we identify with the set

$$\mathfrak{M} := \mathrm{U}(n) \times \mathfrak{H}(n) := \{(g, L) \mid g \in \mathrm{U}(n), L \in \mathfrak{H}(n)\}, \tag{1}$$

using right-trivialization. Here, the vector space of Hermitian matrices, $\mathfrak{H}(n) = \mathrm{i}\mathfrak{u}(n)$, serves as the model of the dual $\mathfrak{u}(n)^*$ of the Lie algebra $\mathfrak{u}(n)$.

Consider the *real* Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ endowed with the non-degenerate bilinear form

$$\langle X, Y\rangle := \Im\,\mathrm{tr}(XY), \quad \forall X, Y \in \mathfrak{gl}(n, \mathbb{C}). \tag{2}$$

Then $\mathfrak{gl}(n, \mathbb{C})$ is the vector space direct sum of its isotropic Lie subalgebras $\mathfrak{u}(n)$ and $\mathfrak{b}(n)$, where $\mathfrak{b}(n)$ contains the upper triangular matrices with real entries along the diagonal. Consequently, we can decompose any $X \in \mathfrak{gl}(n, \mathbb{C})$ as

$$X = X_{\mathfrak{u}(n)} + X_{\mathfrak{b}(n)}, \qquad X_{\mathfrak{u}(n)} \in \mathfrak{u}(n),\ X_{\mathfrak{b}(n)} \in \mathfrak{b}(n). \tag{3}$$

We also have another decomposition into isotropic linear subspaces, $\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{u}(n) + \mathfrak{H}(n)$. Thus both $\mathfrak{b}(n)$ and $\mathfrak{H}(n)$ can serve as models of $\mathfrak{u}(n)^*$.

For any real function $F \in C^\infty(\mathfrak{M})$, introduce the derivatives

$$D_1 F, D_1' F \in C^\infty(\mathfrak{M}, \mathfrak{b}(n)) \quad \text{and} \quad d_2 F \in C^\infty(\mathfrak{M}, \mathfrak{u}(n)) \tag{4}$$

by the relation

$$\langle D_1 F(g, L), X\rangle + \langle D_1' F(g, L), X'\rangle + \langle d_2 F(g, L), Y\rangle = \frac{d}{dt}\bigg|_{t=0} F(e^{tX} g e^{tX'}, L + tY), \qquad (5)$$

for every $X, X' \in \mathfrak{u}(n)$ and $Y \in \mathfrak{H}(n)$. The 'free Hamiltonians' of our interest are

$$H_k(g, L) := \frac{1}{k}\operatorname{tr}(L^k), \quad \forall k \in \mathbb{N}. \qquad (6)$$

These feature in the 'free bi-Hamiltonian hierarchy' on $\mathfrak{M}$, which is given by the next theorem.

Theorem 1 ([4]) *The following formulae define two compatible Poisson brackets on* $\mathfrak{M}$*:*

$$\{F, H\}_1(g, L) = \langle D_1 F, d_2 H\rangle - \langle D_1 H, d_2 F\rangle + \langle L, [d_2 F, d_2 H]\rangle, \qquad (7)$$

and

$$\{F, H\}_2(g, L) = \langle D_1 F, L d_2 H\rangle - \langle D_1 H, L d_2 F\rangle + 2\left\langle L d_2 F, (L d_2 H)_{\mathfrak{u}(n)}\right\rangle - \frac{1}{2}\left\langle D_1' F, g^{-1}(D_1 H) g\right\rangle, \qquad (8)$$

where the derivatives are taken at (g, L) *and (3) is applied. The Hamiltonians* H_k *satisfy*

$$\{F, H_k\}_2 = \{F, H_{k+1}\}_1, \qquad \forall F \in C^\infty(\mathfrak{M}), \qquad (9)$$

and $\{H_k, H_\ell\}_1 = \{H_k, H_\ell\}_2 = 0$ *for every* $k, \ell \in \mathbb{N}$*. The bi-Hamiltonian flow of the systems* $(\mathfrak{M}, \{\ ,\ \}_2, H_k)$ *and* $(\mathfrak{M}, \{\ ,\ \}_1, H_{k+1})$ *is given by* $(g(t), L(t)) = \big(\exp(\mathrm{i} t L(0)^k) g(0), L(0)\big)$.

The first Poisson bracket is the canonical one carried by the cotangent bundle of $\mathrm{U}(n)$, while the second one arises from the Heisenberg double [12] of the Poisson–Lie group $\mathrm{U}(n)$. The latter point is explained in [4], where it is also noted that the Lie derivative of the Poisson tensor of $\{\ ,\ \}_2$ along the infinitesimal generator of the flow $(g(t), L(t)) = (g(0), L(0) + t\mathbf{1}_n)$ is the Poisson tensor of $\{\ ,\ \}_1$. This implies [13] compatibility, and the rest of the statements is readily checked as well.

The fact that the flow generated by the Hamiltonian H_1 on the Heisenberg double of $\mathrm{U}(n)$ projects to free motion on $\mathrm{U}(n)$ was pointed out long time ago by S. Zakrzewski [14], which served as one of the motivations behind Theorem 1.

The 'conjugation action' of $\mathrm{U}(n)$ on $\mathfrak{M}$ associates with every $\eta \in \mathrm{U}(n)$ the diffeomorphism A_η of $\mathfrak{M}$ that operates according to

$$A_\eta(g, L) := (\eta g \eta^{-1}, \eta L \eta^{-1}). \tag{10}$$

A key property of the Poisson brackets on $\mathfrak{M}$ is that they can be restricted to the set of invariant functions with respect to this action, denoted $C^\infty(\mathfrak{M})^{\mathrm{U}(n)}$. This means that if $F, H \in C^\infty(\mathfrak{M})^{\mathrm{U}(n)}$, then the same holds for their Poisson brackets $\{F, H\}_i$ for $i = 1, 2$. Because the Hamiltonians H_k are also invariant, we can restrict the 'free hierarchy' to $\mathrm{U}(n)$-invariant observables. This procedure, called Poisson reduction [10], is an algebraic formulation of projection onto the quotient space $\mathfrak{M}/\mathrm{U}(n)$.

Any smooth function on $\mathfrak{M}$ can be recovered from its restriction to the dense open submanifold $\mathfrak{M}_{\mathrm{reg}} \subset \mathfrak{M}$, which contains the points (g, L) with g having distinct eigenvalues. Moreover, $F \in C^\infty(\mathfrak{M}_{\mathrm{reg}})^{\mathrm{U}(n)}$ is uniquely determined by its restriction f on the manifold $\mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{H}(n)$, where $\mathbb{T}^n_{\mathrm{reg}}$ is the set of regular elements in the standard maximal torus of $\mathrm{U}(n)$. In fact, restriction engenders a one-to-one correspondence

$$C^\infty(\mathfrak{M}_{\mathrm{reg}})^{\mathrm{U}(n)} \longleftrightarrow C^\infty(\mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{H}(n))^{\mathcal{N}(n)}, \tag{11}$$

where $\mathcal{N}(n)$ is the normalizer of $\mathbb{T}^n$ in $\mathrm{U}(n)$, whose action preserves $\mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{H}(n)$. Note that $\mathcal{N}(n)$ is the semi-direct product of the permutation group S_n, naturally embedded into $\mathrm{U}(n)$, with $\mathbb{T}^n$. By taking advantage of the correspondence (11), one can encode the Poisson brackets on $C^\infty(\mathfrak{M}_{\mathrm{reg}})^{\mathrm{U}(n)}$ by two compatible Poisson brackets $\{\ ,\ \}^{\mathrm{red}}_i$ on $C^\infty(\mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{H}(n))^{\mathcal{N}(n)}$. The main result of [4] is the formula of these reduced Poisson brackets.

For $f \in C^\infty(\mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{H}(n))$, the $\mathfrak{b}(n)_0$-valued derivative $D_1 f$ and the $\mathfrak{u}(n)$-valued derivative $d_2 f$ are defined by the equality

$$\langle D_1 f(Q, L), X \rangle + \langle d_2 f(Q, L), Y \rangle = \frac{d}{dt}\bigg|_{t=0} f(e^{tX} Q, L + tY), \tag{12}$$

for every $X \in \mathfrak{u}(n)_0$ and $Y \in \mathfrak{H}(n)$, where $\mathfrak{b}(n)_0$ and $\mathfrak{u}(n)_0$ denote the subalgebras of diagonal matrices in $\mathfrak{b}(n)$ and $\mathfrak{u}(n)$, respectively. Decompose $\mathfrak{gl}(n, \mathbb{C})$ as the vector space direct sum of subalgebras

$$\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{gl}(n, \mathbb{C})_+ + \mathfrak{gl}(n, \mathbb{C})_0 + \mathfrak{gl}(n, \mathbb{C})_-, \tag{13}$$

defined by means of the principal gradation. Accordingly, we can decompose any $X \in \mathfrak{gl}(n, \mathbb{C})$ as $X = X_+ + X_0 + X_-$, where X_0 is diagonal and X_+ is strictly upper-triangular. Then, for $Q \in \mathbb{T}^n_{\mathrm{reg}}$, introduce $\mathcal{R}(Q) \in \mathrm{End}(\mathfrak{gl}(n, \mathbb{C}))$ by setting it equal to zero on $\mathfrak{gl}(n, \mathbb{C})_0$ and defining it otherwise as

$$\mathcal{R}(Q)|_{\mathfrak{gl}(n,\mathbb{C})_+ + \mathfrak{gl}(n,\mathbb{C})_-} := \frac{1}{2}(\mathrm{Ad}_Q + \mathrm{id}) \circ \Big((\mathrm{Ad}_Q - \mathrm{id})|_{\mathfrak{gl}(n,\mathbb{C})_+ + \mathfrak{gl}(n,\mathbb{C})_-} \Big)^{-1}, \tag{14}$$

where $\mathrm{Ad}_Q(X) = QXQ^{-1}$ for all $X \in \mathfrak{gl}(n, \mathbb{C})$. The definition makes sense because of the regularity of Q. Note that $\langle \mathcal{R}(Q)X, Y\rangle = -\langle X, \mathcal{R}(Q)Y\rangle$, and introduce the notation

$$[X, Y]_{\mathcal{R}(Q)} := [\mathcal{R}(Q)X, Y] + [X, \mathcal{R}(Q)Y], \quad \forall X, Y \in \mathfrak{gl}(n, \mathbb{C}). \tag{15}$$

Theorem 2 ([4]) *For $f, h \in C^\infty(\mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{H}(n))^{\mathcal{N}(n)}$, the reduced Poisson brackets have the form*

$$\{f, h\}_1^{\mathrm{red}}(Q, L) = \langle D_1 f, d_2 h\rangle - \langle D_1 h, d_2 f\rangle + \langle L, [d_2 f, d_2 h]_{\mathcal{R}(Q)}\rangle, \tag{16}$$

and

$$\{f, h\}_2^{\mathrm{red}}(Q, L) = \langle D_1 f, L d_2 h\rangle - \langle D_1 h, L d_2 f\rangle + 2\langle L d_2 f, \mathcal{R}(Q)(L d_2 h)\rangle, \tag{17}$$

where the derivatives are evaluated at (Q, L), and the notations (14) and 15) are applied.

The reduced system that descends from the free hierarchy generated the Hamiltonians H_k (6) is called *spin Ruijsenaars–Sutherland hierarchy*. The reason for this terminology will become clear in the next section. For the reduced equations of motion and remarks on their integrability, see [4].

3 Useful Changes of Variables

In the first subsection we introduce new variables that behave as canonically conjugate pairs and 'spin variables' with respect to the second Poisson bracket, and allow us to interpret $\mathrm{tr}(L)$ as a spin Ruijsenaars Hamiltonian. These new variables go back to the papers [3, 4]. In the second subsection we describe another, in this case well-known [5, 7], set of new variables, which convert the first Poisson bracket into that of canonical pairs and (other kind of) spin variables, and lead to the interpretation of $\mathrm{tr}(L^2)$ as a spin Sutherland Hamiltonian.

3.1 Interpretation as Spin Ruijsenaars Model

We now discuss the change of variables that the underlie the interpretation of the reduced free system as a spin Ruijsenaars model. For this purpose, we focus on the second Poisson bracket (17), and restrict ourselves to the open submanifold

$$\mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{P}(n) \subset \mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{H}(n), \tag{18}$$

where $\mathfrak{P}(n)$ denotes the set of positive definite Hermitian matrices. It is a standard fact of linear algebra that any $L \in \mathfrak{P}(n)$ can be uniquely written in the form

$$L = bb^\dagger \text{ with } b \in \mathrm{B}(n), \tag{19}$$

and $b \in \mathrm{B}(n)$ can be decomposed as

$$b = e^p b_+ \text{ with } p \in \mathfrak{b}(n)_0,\ \ b_+ \in \mathrm{B}(n)_+, \tag{20}$$

where $\mathrm{B}(n)_+$ is the group of upper triangular matrices with unit diagonal. We define

$$\lambda := b_+^{-1} Q^{-1} b_+ Q, \tag{21}$$

and obtain the change of variables

$$\mathbb{T}^n_{\text{reg}} \times \mathfrak{P}(n) \ni (Q, L) \longleftrightarrow (Q, p, \lambda) \in \mathbb{T}^n_{\text{reg}} \times \mathfrak{b}(n)_0 \times \mathrm{B}(n)_+. \tag{22}$$

A grade by grade inspection of the defining relation (21) shows that this is a diffeomorphism between the respective spaces. Thus every function $f(Q, L)$ corresponds to a unique function $\mathcal{F}(Q, p, \lambda)$. The diffeomorphism (22) induces an action of $\mathcal{N}(n)$ on $\mathbb{T}^n_{\text{reg}} \times \mathfrak{b}(n)_0 \times \mathrm{B}(n)_+$, and we are interested in the invariant functions. The action of the subgroup $\mathbb{T}^n < \mathcal{N}(n)$ is especially simple, it is given by

$$(Q, p, \lambda) \mapsto (Q, p, \tau\lambda\tau^{-1}), \quad \forall \tau \in \mathbb{T}^n, \tag{23}$$

since this corresponds to $(Q, L) \mapsto (Q, \tau L \tau^{-1})$.

For any $\mathcal{F} \in C^\infty(\mathbb{T}^n_{\text{reg}} \times \mathfrak{b}(n)_0 \times \mathrm{B}(n)_+)$, we define the derivatives $D_Q\mathcal{F} \in \mathfrak{b}(n)_0$, $d_p\mathcal{F} = \mathfrak{u}(n)_0$ and $D_\lambda\mathcal{F}$, $D'_\lambda\mathcal{F} \in \mathfrak{u}(n)_\perp$ by

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{F}(e^{tX_0}Q, p + tY_0, e^{tX_+}\lambda e^{tY_+})$$
$$= \langle D_Q\mathcal{F}, X_0\rangle + \langle d_p\mathcal{F}, Y_0\rangle + \langle D_\lambda\mathcal{F}, X_+\rangle + \langle D'_\lambda\mathcal{F}, Y_+\rangle. \tag{24}$$

Here, $X_0 \in \mathfrak{u}(n)_0$, $Y_0 \in \mathfrak{b}(n)_0$ and $X_+, Y_+ \in \mathfrak{b}(n)_+$ are arbitrary, the argument (Q, p, λ) is suppressed on the right hand side, and $\mathfrak{u}(n)_\perp$ denotes the off-diagonal linear subspace of $\mathfrak{u}(n)$.

The next proposition was stated previously without elaborating its proof.

Proposition 3 ([4]) *Consider the functions* $\mathcal{F}, \mathcal{H} \in C^\infty(\mathbb{T}^n_{\rm reg} \times \mathfrak{b}(n)_0 \times \mathrm{B}(n)_+)^{\mathcal{N}(n)}$ *that are related to* $f, h \in C^\infty(\mathbb{T}^n_{\rm reg} \times \mathfrak{P}(n))^{\mathcal{N}(n)}$ *according to*

$$\begin{aligned}
&\mathcal{F}(Q, p, \lambda) = f(Q, L),\ \ \mathcal{H}(Q, p, \lambda) = h(Q, L) \textit{ with}\\
&L = e^p b_+ b_+^\dagger e^p,\ \ \lambda := b_+^{-1} Q^{-1} b_+ Q.
\end{aligned} \tag{25}$$

In terms of the variables (Q, p, λ)*, the second Poisson bracket (17) takes the form*

$$2\{\mathcal{F}, \mathcal{H}\}_2^{\rm red}(Q, p, \lambda) = \langle D_Q \mathcal{F}, d_p \mathcal{H}\rangle - \langle D_Q \mathcal{H}, d_p \mathcal{F}\rangle + \langle D'_\lambda \mathcal{F}, \lambda^{-1}(D_\lambda \mathcal{H})\lambda\rangle, \tag{26}$$

where the derivatives are evaluated at (Q, p, λ).

Proof Recall that (Q, L), (Q, b) and (Q, p, λ) are alternative sets of variables. In particular, we have the invertible correspondences:

$$(Q, L) \leftrightarrow (Q, b) \leftrightarrow (Q, p, \lambda) \quad \text{with} \quad L = bb^\dagger,\ e^p := b_{\rm diag},\ \lambda := b^{-1} Q^{-1} b Q. \tag{27}$$

Here, we suppressed that λ does not depend on p. Any tangent vector at a fixed (Q, b) can be represented as the velocity vector at $t = 0$ of a curve of the form

$$(Q(t), b(t)) = (e^{t\xi} Q,\ b e^{t\beta}),\ \text{with some } \xi \in \mathfrak{u}(n)_0,\ \beta = (\beta_0 + \beta_+) \in \mathfrak{b}(n). \tag{28}$$

In terms of the alternative variables, the corresponding curves are easily seen to satisfy

$$\begin{aligned}
&L(t) = L + tb(\beta + \beta^\dagger)b^\dagger + \mathrm{o}(t),\\
&\lambda(t) = \lambda \exp\big(t\big[\xi - Q^{-1} b^{-1} \xi b Q + Q^{-1}\beta Q - \lambda^{-1}\beta\lambda\big] + \mathrm{o}(t)\big),\\
&p(t) = p + t\beta_0 + \mathrm{o}(t).
\end{aligned} \tag{29}$$

Of course, the curve that appears in the exponent after λ lies in $\mathfrak{b}(n)_+$. Let us now consider a function on our space, which is either expressed as $(Q, L) \mapsto f(Q, L)$, or equivalently as $(Q, p, \lambda) \mapsto \mathcal{F}(Q, p, \lambda)$. By the definition of derivatives, we obtain the equality

$$\begin{aligned}
&\frac{d}{dt}\bigg|_{t=0} f\big(Qe^{t\xi}, L + tb(\beta + \beta^\dagger)b^\dagger + \mathrm{o}(t)\big)\\
&= \frac{d}{dt}\bigg|_{t=0} \mathcal{F}\big(Qe^{t\xi}, p + t\beta_0, \lambda \exp(t[\xi - Q^{-1} b^{-1} \xi b Q + Q^{-1}\beta Q - \lambda^{-1}\beta\lambda] + \mathrm{o}(t))\big).
\end{aligned} \tag{30}$$

This generates the following relations between the derivatives of f and $\mathcal{F}$:

$$\langle 2b^\dagger d_2 f b - d_p\mathcal{F} - QD'_\lambda\mathcal{F}Q^{-1} + (\lambda D'_\lambda\mathcal{F}\lambda^{-1})_{\mathfrak{u}(n)}\,,\ \beta\rangle$$
$$+\langle D_1 f - D_Q\mathcal{F} - D'_\lambda\mathcal{F} + bQD'_\lambda\mathcal{F}Q^{-1}b^{-1}\,,\ \xi\rangle = 0,\quad \forall\xi\in\mathfrak{u}(n)_0,\ \forall\beta\in\mathfrak{b}(n). \tag{31}$$

The derivatives of f and $\mathcal{F}$ are taken at (Q, L) and at (Q, p, λ), respectively, according to (12) and (24). We have $\langle D'_\lambda\mathcal{F}, \xi\rangle = 0$, and the conventions $D'_\lambda\mathcal{F},\ D_\lambda\mathcal{F}\in\mathfrak{u}(n)_\perp$ imply

$$(\lambda D'_\lambda\mathcal{F}\lambda^{-1})_{\mathfrak{u}(n)} = D_\lambda\mathcal{F} + (\lambda D'_\lambda\mathcal{F}\lambda^{-1})_{\mathrm{im-diag}}. \tag{32}$$

The matrix $X_{\mathrm{im-diag}}$ is obtained from the matrix X by setting to zero the off-diagonal entries and the real parts of the diagonal entries of X, and (3) is used.

From the first term in (31) (the one involving arbitrary β), we must have

$$A := 2b^\dagger d_2 f b - d_p F - QD'_\lambda\mathcal{F}Q^{-1} + (\lambda D'_\lambda\mathcal{F}\lambda^{-1})_{\mathfrak{u}(n)}\in\mathfrak{b}(n). \tag{33}$$

But the formula of A shows that $A\in\mathfrak{u}(n)$, and thence $A = 0$. It is convenient to rewrite

$$2b^\dagger d_2 f b = QD'_\lambda\mathcal{F}Q^{-1} - \lambda D'_\lambda\mathcal{F}\lambda^{-1} + \big[d_p\mathcal{F} + \lambda D'_\lambda\mathcal{F}\lambda^{-1} - (\lambda D'_\lambda\mathcal{F}\lambda^{-1})_{\mathfrak{u}(n)}\big], \tag{34}$$

and, conjugating by b and using $b\lambda = Q^{-1}bQ$, we get

$$\begin{aligned} 2L\,d_2 f &= bQD'_\lambda\mathcal{F}Q^{-1}b^{-1} - b\lambda D'_\lambda\mathcal{F}\lambda^{-1}b^{-1}\\ &\qquad + \mathrm{Ad}_b[d_p\mathcal{F} + \mathrm{Ad}_\lambda D'_\lambda\mathcal{F} - (\lambda D'_\lambda\mathcal{F}\lambda^{-1})_{\mathfrak{u}(n)}]\\ &= (\mathrm{Ad}_Q - \mathrm{id})\mathrm{Ad}_{Q^{-1}bQ}D'_\lambda\mathcal{F} + \mathrm{Ad}_b[d_p\mathcal{F} + \mathrm{Ad}_\lambda D'_\lambda\mathcal{F} - (\lambda D'_\lambda\mathcal{F}\lambda^{-1})_{\mathfrak{u}(n)}], \end{aligned} \tag{35}$$

from which it is easy to obtain

$$\begin{aligned} 2\mathcal{R}(Q)(Ld_2 f) &= \frac{1}{2}(\mathrm{Ad}_Q + \mathrm{id})\mathrm{Ad}_{Q^{-1}bQ}D'_\lambda\mathcal{F} - (bQD'_\lambda\mathcal{F}Q^{-1}b^{-1})_{\mathrm{diag}}\\ &\qquad + \mathcal{R}(Q)\big(\mathrm{Ad}_b[d_p\mathcal{F} + \mathrm{Ad}_\lambda D'_\lambda\mathcal{F} - (\lambda D'_\lambda\mathcal{F}\lambda^{-1})_{\mathfrak{u}(n}]\big). \end{aligned} \tag{36}$$

Of course, we could have written everywhere $\mathrm{Ad}_\lambda D'_\lambda\mathcal{F} - (\lambda D'_\lambda\mathcal{F}\lambda^{-1})_{\mathfrak{u}(n)} \equiv (\mathrm{Ad}_\lambda D'_\lambda\mathcal{F})_{\mathfrak{b}(n)}$. Note also that Ad_m denotes conjugation by m for any $m\in\mathrm{GL}(n,\mathbb{C})$.

A glance at the last equation (36) shows that the expression in the second line belongs to $\mathfrak{b}(n)_+$, and this is crucial for the computation of $\langle Ld_2 f, \mathcal{R}(Q)(Ld_2 h)\rangle$ (cf. (17)):

$$\begin{aligned}
4\langle Ld_2 f\,,\ \mathcal{R}(Q)(Ld_2 h)\rangle &= \\
\langle (Ad_Q - \mathrm{id})\mathrm{Ad}_{Q^{-1}bQ} D'_\lambda \mathcal{F} &+ \mathrm{Ad}_b[d_p\mathcal{F} + \mathrm{Ad}_\lambda D'_\lambda\mathcal{F} - (\lambda D'_\lambda \mathcal{F}\lambda^{-1})_{\mathfrak{u}(n)}]\,, \\
&- (\mathrm{Ad}_{bQ} D'_\lambda \mathcal{H})_{\mathrm{diag}} + \tfrac{1}{2}(\mathrm{Ad}_Q + \mathrm{id}) Ad_{Q^{-1}bQ} D'_\lambda \mathcal{H} \\
&+ \mathcal{R}(Q)\big(\mathrm{Ad}_b[d_p\mathcal{H} + \mathrm{Ad}_\lambda D'_\lambda \mathcal{H} - (\lambda D'_\lambda \mathcal{H}\lambda^{-1})_{\mathfrak{u}(n)}]\big)\rangle \\
= \tfrac{1}{2}\langle \mathrm{Ad}_{bQ} D'_\lambda \mathcal{F}, Ad_{Q^{-1}bQ} D'_\lambda \mathcal{H}\rangle &+ \tfrac{1}{2}\langle d_p\mathcal{F} + \mathrm{Ad}_\lambda D'_\lambda \mathcal{F} - (\lambda D'_\lambda \mathcal{F}\lambda^{-1})_{\mathfrak{u}(n)}\,, \\
\mathrm{Ad}_Q D'_\lambda \mathcal{H} &+ \mathrm{Ad}_\lambda D'_\lambda \mathcal{H} - 2(\mathrm{Ad}_{bQ} D'_\lambda \mathcal{H})_{\mathrm{diag}}\rangle - (\mathcal{F} \leftrightarrow \mathcal{H}) \\
= \tfrac{1}{2}\langle \mathrm{Ad}_Q D'_\lambda \mathcal{F}, \mathrm{Ad}_\lambda D'_\lambda \mathcal{H}\rangle &+ \tfrac{1}{2}\langle d_p\mathcal{F}, \mathrm{Ad}_\lambda D'_\lambda \mathcal{H} - 2bQD'_\lambda \mathcal{H} Q^{-1}b^{-1}\rangle \\
+ \tfrac{1}{2}\langle \mathrm{Ad}_\lambda D'_\lambda \mathcal{F}, \mathrm{Ad}_Q D'_\lambda \mathcal{H}\rangle &+ \langle (\lambda D'_\lambda \mathcal{F}\lambda^{-1})_{\mathfrak{u}(n)} - \mathrm{Ad}_\lambda D'_\lambda \mathcal{F}, (\mathrm{Ad}_{bQ} D'_\lambda \mathcal{H})_{\mathrm{diag}}\rangle \\
&- \tfrac{1}{2}\langle (\lambda D'_\lambda \mathcal{F}\lambda^{-1})_{\mathfrak{u}(n)}, \mathrm{Ad}_\lambda D'_\lambda \mathcal{H}\rangle - (\mathcal{F} \leftrightarrow \mathcal{H}).
\end{aligned} \tag{37}$$

Notice that the terms at the beginning of the first two lines after the last equality sign add up to

$$\tfrac{1}{2}\langle \mathrm{Ad}_Q D'_\lambda \mathcal{F}, \mathrm{Ad}_\lambda D'_\lambda \mathcal{H}\rangle + \tfrac{1}{2}\langle Ad_\lambda D'_\lambda \mathcal{F}, Ad_Q D'_\lambda \mathcal{H}\rangle, \tag{38}$$

and this is symmetric with respect to exchange of $\mathcal{F}$ and $\mathcal{H}$; thereby it cancels. Notice also that the second expression in the second line simplifies as follows:

$$\begin{aligned}
&\langle (\lambda D'_\lambda \mathcal{F}\lambda^{-1})_{\mathfrak{u}(n)} - \mathrm{Ad}_\lambda D'_\lambda \mathcal{F}, (\mathrm{Ad}_{bQ} D'_\lambda \mathcal{H})_{\mathrm{diag}}\rangle \\
&\qquad = \langle (\lambda D'_\lambda \mathcal{F}\lambda^{-1})_{\mathfrak{u}(n)} - \mathrm{Ad}_\lambda D'_\lambda \mathcal{F}, (\mathrm{Ad}_{bQ} D'_\lambda \mathcal{H})_{\mathrm{im-diag}}\rangle \\
&\qquad = -\langle \mathrm{Ad}_\lambda D'_\lambda \mathcal{F}, (\mathrm{Ad}_{bQ} D'_\lambda \mathcal{H})_{\mathrm{im-diag}}\rangle,
\end{aligned} \tag{39}$$

which will be shortly shown to vanish. To summarize, we obtained

$$\begin{aligned}
4\langle Ld_2 f\,,\ \mathcal{R}(Q)(Ld_2 h)\rangle &= -\tfrac{1}{2}\langle \mathrm{Ad}_\lambda D'_\lambda \mathcal{F}, d_p\mathcal{H} + 2(\mathrm{Ad}_{bQ} D'_\lambda \mathcal{H})_{\mathrm{im-diag}}\rangle \\
- \langle d_p\mathcal{F}, \mathrm{Ad}_{bQ} D'_\lambda \mathcal{H}\rangle &- \tfrac{1}{2}\langle (\lambda D'_\lambda \mathcal{F}\lambda^{-1})_{\mathfrak{u}(n)}, \mathrm{Ad}_\lambda D'_\lambda \mathcal{H}\rangle - (\mathcal{F} \leftrightarrow \mathcal{H}).
\end{aligned} \tag{40}$$

Next, we may look at the other terms, and return to the ξ-term of (31). This gives

$$D_1 f = D_Q F - (\mathrm{Ad}_{bQ} D'_\lambda \mathcal{F})_{\mathrm{real-diag}}, \tag{41}$$

which, together with (35)—discarding the term in the range of $(\mathrm{Ad}_Q - \mathrm{id})$ as this is in the annihilator of $\mathfrak{b}(n)_0$ – gives us

$$\begin{aligned}
&2\langle D_1 f\,,\ Ld_2 h\rangle\\
&= \langle D_Q\mathcal{F} - (\mathrm{Ad}_{bQ} D'_\lambda \mathcal{F})_{\mathrm{real-diag}}\,,\ \mathrm{Ad}_b[d_p\mathcal{H} + \mathrm{Ad}_\lambda D'_\lambda\mathcal{H} - (\lambda D'_\lambda \mathcal{H}\lambda^{-1})_{\mathfrak{u}(n)}]\rangle\\
&= \langle D_Q\mathcal{F} - (\mathrm{Ad}_{bQ} D'_\lambda \mathcal{F})_{\mathrm{real-diag}}\,,\ d_p\mathcal{H}\rangle = \langle D_Q\mathcal{F} - \mathrm{Ad}_{bQ} D'_\lambda \mathcal{F}\,,\ d_p\mathcal{H}\rangle. \qquad (42)
\end{aligned}$$

Putting together now (40) and (42), the second term at the very end of (42) cancels, and we arrive at

$$\begin{aligned}
2\{f,h\}_2^{\mathrm{red}}(Q,L) &= 2\langle D_1 f, Ld_2 h\rangle - 2\langle Ld_2 f, D_1 h\rangle + 4\langle Ld_2 f, \mathcal{R}(Q)(Ld_2 h)\rangle\\
&= \langle D_Q\mathcal{F}\,,\ d_p\mathcal{H}\rangle + \tfrac{1}{2}\langle \mathrm{Ad}_\lambda D'_\lambda\mathcal{F}, (\lambda D'_\lambda\mathcal{H}\lambda^{-1})_{\mathfrak{u}(n)}\rangle\\
&\quad - \tfrac{1}{2}\langle \mathrm{Ad}_\lambda D'_\lambda \mathcal{F}, \eta_{\mathcal{H}}\rangle - (\mathcal{F} \leftrightarrow \mathcal{H}), \qquad (43)
\end{aligned}$$

where $\mathfrak{u}(n)_0 \ni \eta_{\mathcal{H}} := d_p\mathcal{H} + 2(\mathrm{Ad}_{bQ} D'_\lambda\mathcal{H})_{\mathrm{im-diag}}$ represents the diagonal-imaginary entities from the previous formulae. As explained below, for invariant functions $\mathcal{F}$ and $\mathcal{H}$, the term containing $\eta_{\mathcal{H}}$ vanishes, and we also have

$$\begin{aligned}
&\langle \mathrm{Ad}_\lambda D'_\lambda\mathcal{F}, (\lambda D'_\lambda\mathcal{H}\lambda^{-1})_{\mathfrak{u}(n)}\rangle\\
&\quad = \langle \mathrm{Ad}_\lambda D'_\lambda\mathcal{F}, D_\lambda\mathcal{H} + (\lambda D'_\lambda\mathcal{H}\lambda^{-1})_{\mathrm{im-diag}}\rangle = \langle \mathrm{Ad}_\lambda D'_\lambda\mathcal{F}, D_\lambda\mathcal{H}\rangle, \qquad (44)
\end{aligned}$$

where we used (32) and the property (45).

By the above, the claim of the proposition follows from (43) if we can verify that for any $\mathcal{F} \in C^\infty(\mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{b}(n)_0 \times \mathrm{B}(n)_+)^{\mathbb{T}^n}$ we have

$$\langle X, \lambda D'_\lambda\mathcal{F}\lambda^{-1}\rangle = 0, \quad \forall X \in \mathfrak{u}(n)_0. \qquad (45)$$

In order to justify this, we remark that

$$\langle X, \lambda D'_\lambda\mathcal{F}\lambda^{-1}\rangle = \langle \lambda^{-1}X\lambda - X, D'_\lambda F\rangle. \qquad (46)$$

Since $\lambda^{-1}X\lambda - X \in \mathfrak{b}(n)_+$, we may rewrite this as

$$\begin{aligned}
\langle X, \lambda D'_\lambda\mathcal{F}\lambda^{-1}\rangle &= \frac{d}{dt}\Big|_{t=0} \mathcal{F}\big(Q, p, \lambda\exp(t[\lambda^{-1}X\lambda - X])\big)\\
&= \frac{d}{dt}\Big|_{t=0} \mathcal{F}\big(Q, p, e^{tX}\lambda e^{-tX}\big). \qquad (47)
\end{aligned}$$

In the last step we used that $\frac{d}{dt}\big|_{t=0} \lambda\exp(t[\lambda^{-1}X\lambda - X]) = [X,\lambda]$. We see from (47) that (45) follows from the $\mathbb{T}^n$-invariance of $\mathcal{F}$, and hence the proof is complete. □

Regarding the interpretation of Proposition 3, it is worth pointing out that one may view the restriction to $\mathcal{N}(n)$-invariant functions on $\mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{b}(n)_0 \times \mathrm{B}(n)_+$ as the result of a two step process. The first step consists in Hamiltonian reduction of $\mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{b}(n)_0 \times \mathrm{B}(n)$ by the normal subgroup $\mathbb{T}^n$. The formula (26) defines a Poisson bracket already on the $\mathbb{T}^n$-invariant functions. In fact, its last term can be identified as the result of reduction of the multiplicative Poisson bracket on $\mathrm{B}(n)$ by the conjugation action of $\mathbb{T}^n$, at the zero value of the pertinent moment map. In other words, the last term of (26) corresponds to the Poisson space $\mathrm{B}(n)//_0\mathbb{T}^n$. (Cf. Theorem 4.3 in [3].) The second step consists in taking quotient by $S_n = \mathcal{N}(n)/\mathbb{T}^n$.

When expressed in the variables (Q, p, λ), the Hamiltonian $\mathrm{tr}(L) = \mathrm{tr}(bb^\dagger) = \mathrm{tr}(e^{2p}b_+b_+^\dagger)$ can be written as

$$\mathrm{tr}(L) = \sum_{i=1}^{n} e^{2p_i} V_i(Q, \lambda) \quad \text{with} \quad V_i(Q, \lambda) = \Big(b_+(Q, \lambda)b_+(Q, \lambda)^\dagger\Big)_{ii}, \tag{48}$$

where λ is a 'spin' variable, and $b_+(Q, \lambda)$ denotes the solution of the equation (21) for b_+. An explicit formula of $b_+(Q, \lambda)$ can be extracted from Section 5.2 in [3]. Comparison of (48) with the light-cone Hamiltonians of the standard RS model [11] justifies calling this a *spin Ruijsenaars type Hamiltonian*. A further justification is that restriction of the system to a one-point symplectic leaf in $\mathrm{B}(n)//_0\mathbb{T}^n$ yields the spinless trigonometric RS model [6].

3.2 Interpretation as Spin Sutherland Model

Concentrating on the first Poisson bracket (16), we present another set of useful variables

$$(Q, p, \phi) \in \mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{H}(n)_0 \times \mathfrak{H}(n)_\perp, \tag{49}$$

where the subscripts 0 and $\perp$ refer to diagonal matrices and off-diagonal matrices, respectively. The relevant change of variables is encoded by the diffeomorphism

$$\gamma : \mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{H}(n)_0 \times \mathfrak{H}(n)_\perp \to \mathbb{T}^n_{\mathrm{reg}} \times \mathfrak{H}(n) \tag{50}$$

operating according to

$$\gamma : (Q, p, \phi) \mapsto (Q, L(Q, p, \phi)) \quad \text{with} \quad L(Q, p, \phi) = p - (\mathcal{R}(Q) + \frac{1}{2}\mathrm{id})(\phi). \tag{51}$$

We now express the functions $f, h \in C^\infty(\mathbb{T}^n_{\rm reg} \times \mathfrak{H}(n))^{\mathcal{N}(n)}$ in the form

$$f \circ \gamma = \mathcal{F}, \; h \circ \gamma = \mathcal{H}, \quad \mathcal{F}, \mathcal{H} \in C^\infty(\mathbb{T}^n_{\rm reg} \times \mathfrak{H}(n)_0 \times \mathfrak{H}(n)_\perp)^{\mathcal{N}(n)}, \tag{52}$$

where $\mathcal{N}(n)$ acts in the natural manner inherited from the conjugation action. The Poisson bracket $\{\, ,\, \}_1^{\rm red}$ on $C^\infty(\mathbb{T}^n_{\rm reg} \times \mathfrak{H}(n)_0 \times \mathfrak{H}(n)_\perp)^{\mathcal{N}(n)}$ is defined by the formula

$$\{\mathcal{F}, \mathcal{H}\}_1^{\rm red} \equiv \{\mathcal{F} \circ \gamma^{-1}, \mathcal{H} \circ \gamma^{-1}\}_1^{\rm red} \circ \gamma, \tag{53}$$

where (51) is used and the right-hand side refers to the Poisson bracket (16).

For any $\mathcal{F} \in C^\infty(\mathbb{T}^n_{\rm reg} \times \mathfrak{H}(n)_0 \times \mathfrak{H}(n)_\perp)$, we have the derivatives

$$D_Q\mathcal{F}(Q, p, \phi) \in \mathfrak{b}(n)_0, \quad d_p\mathcal{F}(Q, p, \phi) \in \mathfrak{u}(n)_0, \quad d_\phi\mathcal{F}(Q, p, \phi) \in \mathfrak{u}(n)_\perp, \tag{54}$$

defined by

$$\langle D_Q\mathcal{F}(Q, p, \phi), X\rangle + \langle d_p\mathcal{F}(Q, p, \phi), Y_0\rangle + \langle d_\phi\mathcal{F}(Q, p, \phi), Y_\perp\rangle = \frac{d}{dt}\Big|_{t=0} \mathcal{F}(e^{tX}Q, p + tY_0, \phi + tY_\perp), \tag{55}$$

for every $X \in \mathfrak{u}(n)_0$ and $Y = (Y_0 + Y_\perp) \in \mathfrak{H}(n)$.

Proposition 4 ([5, 7]) *In terms of the variables (Q, p, ϕ) defined by (51), the reduced first Poisson bracket (16) has the following form:*

$$\{\mathcal{F}, \mathcal{H}\}_1^{\rm red}(Q, p, \phi) = \langle D_Q\mathcal{F}, d_p\mathcal{H}\rangle - \langle D_Q\mathcal{H}, d_p\mathcal{F}\rangle + \langle \phi, [d_\phi\mathcal{F}, d_\phi\mathcal{H}]\rangle. \tag{56}$$

Here, $\mathcal{F}, \mathcal{H} \in C^\infty(\mathbb{T}^n_{\rm reg} \times \mathfrak{H}(n)_0 \times \mathfrak{H}(n)_\perp)^{\mathcal{N}(n)}$ and the derivatives are taken at (Q, p, ϕ).

The change of variables $(Q, L) \leftrightarrow (Q, p, \phi)$ appeared in the construction of spin Sutherland models via the method of Li and Xu [7], whose relation to Hamiltonian reduction of free motion on Lie groups was clarified in [5]. The proof of Proposition 4 can be extracted from these references. One can also prove it by direct calculation, which is much simpler than the one required for the proof of Proposition 3.

The reduced Hamiltonians $\mathcal{H}_k^{\rm red}$ arising from those in (6) can be written in terms of the variables (Q, p, ϕ) as

$$\mathcal{H}_k^{\rm red}(Q, p, \phi) = \frac{1}{k}\,{\rm tr}(L(Q, p, \phi)^k). \tag{57}$$

For $k = 2$, with $Q = \exp(\text{diag}(\mathrm{i}q_1, \ldots, \mathrm{i}q_n))$, and $p = \text{diag}(p_1, \ldots, p_n)$ this gives

$$\mathcal{H}_2^{\text{red}}(Q, p, \phi) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{8}\sum_{j\neq l} \frac{|\phi_{jl}|^2}{\sin^2 \frac{q_j - q_l}{2}}, \tag{58}$$

which is a standard spin Sutherland Hamiltonian. The last term in the Poisson bracket (56) represents the Poisson space $\mathfrak{u}(n)^*//_0\mathbb{T}^n$, and only gauge invariant functions of the spin variable ϕ appear in the model.

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Hermitian–Einstein Metrics from Non-commutative $U(1)$ Solutions

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Abstract We show that Hermitian–Einstein metrics can be locally constructed by a map from (anti-)self-dual two-forms on Euclidean $\mathbb{R}^4$ to symmetric two-tensors introduced in "Gravitational instantons from gauge theory" Yang and Salizzoni (Phys Rev Lett 201602, 2006 [hep-th/0512215]). This correspondence is valid not only for a commutative space but also for a noncommutative space. We choose $U(1)$ instantons on a noncommutative $\mathbb{C}^2$ as the self-dual two-form, from which we derive a family of Hermitian–Einstein metrics. We also discuss the condition when the two-forms are not instantons but they are solutions to the Yang–Mills equations.

Keywords Noncommutative geometry · Gauge field theory

Mathematics Subject Classification (2010) Primary 53Z05; Secondary 83C05

1 Background: Gravity and Gauge Theory

The coordinate transformation on the coordinate neighborhood is

$$z_1 := x^2 + \mathrm{i}x^1, z_2 := x^4 + \mathrm{i}x^3.$$

1.1 Gravity/Einstein Manifold

Definition 1 Assume that (M, g) is a (semi)Riemannian manifold, $R_{\mu\nu}$ is Ricci curvature and R is Scalar curvature.If

$$R_{\mu\nu}(x) + \frac{1}{2}R(x)\, g_{\mu\nu}(x) = 0$$

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then (M, g) is called Einstein manifold.

Fact 2 *If $M = \mathbb{C}^2$ and g is a Hermitian metric then Ricci curvature tensor with a Hermitian connection is calculated as follows:*

$$R_{\bar{j}k}(x) = \partial_{\bar{j}} \partial_k \log\left(\det\left[g(x)\right]\right). \tag{1}$$

1.2 Electromagnetism (Gauge Theory)

$Alt_n(\mathbb{R})$ is defined as the set of 4×4 alternative matrices.

$$Alt_n(\mathbb{R}) := \left\{F \in M_n(\mathbb{R}) \mid F^T = -F\right\}$$

.

Definition 3 An automorphism $* : Alt_4(\mathbb{R}) \longrightarrow Alt_4(\mathbb{R})$ is defined as

$$* \left[\begin{pmatrix} 0 & \hat{F}_{12} & \hat{F}_{13} & \hat{F}_{14} \\ -\hat{F}_{12} & 0 & \hat{F}_{23} & \hat{F}_{24} \\ -\hat{F}_{13} & -\hat{F}_{23} & 0 & \hat{F}_{34} \\ -\hat{F}_{14} & -\hat{F}_{24} & -\hat{F}_{34} & 0 \end{pmatrix}\right] := \begin{pmatrix} 0 & \hat{F}_{34} & -\hat{F}_{24} & \hat{F}_{23} \\ -\hat{F}_{34} & 0 & \hat{F}_{14} & -\hat{F}_{13} \\ \hat{F}_{24} & -\hat{F}_{14} & 0 & \hat{F}_{12} \\ -\hat{F}_{23} & \hat{F}_{13} & -\hat{F}_{12} & 0 \end{pmatrix}.$$

Assume that $\hat{F}(x)$ is a Alt_4-valued function on $\mathbb{R}^4$ and define the 2-form on $\mathbb{R}^4$ as

$$\hat{F}(x) := \begin{pmatrix} 0 & \hat{F}_{12}(x) & \hat{F}_{13}(x) & \hat{F}_{14}(x) \\ -\hat{F}_{12}(x) & 0 & \hat{F}_{23}(x) & \hat{F}_{24}(x) \\ -\hat{F}_{13}(x) & -\hat{F}_{23}(x) & 0 & \hat{F}_{34}(x) \\ -\hat{F}_{14}(x) & -\hat{F}_{24}(x) & -\hat{F}_{34}(x) & 0 \end{pmatrix}$$

$$\hat{F}(x)_{\mu\nu}\, \mathrm{d}x_\mu \wedge \mathrm{d}x_\nu, \quad \left(*\hat{F}(x)\right)_{\mu\nu} \mathrm{d}x_\mu \wedge \mathrm{d}x_\nu$$

(using the Einstein summation convention).This $\hat{F}(x)$ is assumed to be an electromagnetic tensor in electromagnetism. If

$$*\hat{F}(x) = -\hat{F}(x)$$

then $\hat{F}(x)$ is called instanton. This name means that it is the solution of a differential equation. An instanton is a solution of a differential equation named Yang–Mills equation as follows.

Fact 4 *If $\hat{F}(x)$ is a instanton and $\hat{F}_{\mu\nu}(x)\,\mathrm{d}x_\mu \wedge \mathrm{d}x_\nu$ is closed then $\hat{F}(x)$ is a solution of Yang–Mills equation which means*

$$*\hat{F}(x) = -\hat{F}(x),\ \mathrm{d}\left(\hat{F}_{\mu\nu}(x)\,\mathrm{d}x_\mu \wedge \mathrm{d}x_\nu\right) = 0 \Longrightarrow \mathrm{d}\left(\left(*\hat{F}(x)\right)_{\mu\nu}\mathrm{d}x_\mu \wedge \mathrm{d}x_\nu\right) = 0.$$

It is well known that the Bianchi identity leads to Gauss's law for magnetic fields and Faraday's law of induction, and the Yang–Mills equation leads to Gauss's law and Ampere's law for electric fields.

2 Previous Research

With the result of [4] $F(x)$ and $g\left(\hat{F}(x)\right)$ are defined as

$$g\left(\hat{F}(x)\right) := 2\left(E_4 - \hat{F}(x)\,\theta\right)^{-1} - E_4. \tag{2}$$

and

$$\hat{F}_{\mu\nu}(x) = \left(\frac{1}{1 + F(x)\,\theta}F(x)\right)_{\mu\nu}$$

where E_4 is the 4×4 unit matrix and

$$\theta := \begin{pmatrix} 0 & -\eta & 0 & 0 \\ \eta & 0 & 0 & 0 \\ 0 & 0 & 0 & -\eta \\ 0 & 0 & \eta & 0 \end{pmatrix} \tag{3}$$

in [3]. It is shown that $g\left(\hat{F}(x)\right)$ is the Eguchi Hanson metric if F is the solution which satisfies the Bianchi identity and $\hat{F}$ is a self-dual solution. This metric is an example of the well-known Kähler–Einstein metric. It has been known that $g\left(\hat{F}(x)\right)$ is a Kähler metric when F satisfies the Bianchi identity, and we generalize this result in [1]. We have shown that $g\left(\hat{F}(x)\right)$ is Hermitian–Einstein metric when $\hat{F}$ is self-dual in [1].

Lemma 5 *Assume that $\hat{F}^- \in Alt_4(\mathbb{R})$ is an alternative matrix, θ is defined as (3) and the symmetric matrix $g\left(\hat{F}(x)\right) \in M_4\mathbb{R}$ is defined as (2). If $*\hat{F}^-(x) = -\hat{F}^-(x)$ then*

$$\det\left[g\left(\hat{F}^-(x)\right)\right] = 1.$$

Theorem 6 *As a result if* $*\hat{F}^{-}(x) = -\hat{F}^{-}(x)$ *then*

$$R_{\bar{j}k}\left(\hat{F}^{-}(x)\right) = 0$$

In other words, it means that $g\left(\hat{F}^{-}(x)\right)$ *is a Hermitian–Einstein metric.*

As an example of self-dual $\hat{F}$, instanton solutions are well known, and instanton solutions on noncommutative manifolds are calculated in [2]. The Hermitian–Einstein metric corresponding to this instanton solution is calculated in [1].

3 Instantons from Ricci-Flat Metrics (Main Result)

Next, we will think of a converse of the last Theorem. That means "Is the $\hat{F}$ instanton if the metric is Ricci flat?".

First $g\left(\hat{F}(x)\right)$ needs to be a metric. Since the metric matrix is a symmetric matrix, therefore $g\left(\hat{F}(x)\right)$ should be a symmetric matrix, in which case a condition that is "fairly close" to an anti-self-dual condition is derived.

Then we consider "asymptotic to zero". In conclusion, imposing a Ricci flat condition on the metric when $\hat{F}(x)$ satisfies this condition leads to $\hat{F}(x)$ being anti-self-dual.

Remark 7 In the last section $\hat{F}$ was an instanton but this is not assumed in this section. Since $\hat{F}$ is an alternative matrix but not assumed to be an instanton now, $\hat{F}$ is as follows.

$$\hat{F}(x) = \begin{pmatrix} 0 & \hat{F}_{12}(x) & \hat{F}_{13}(x) & \hat{F}_{14}(x) \\ -\hat{F}_{12}(x) & 0 & \hat{F}_{23}(x) & \hat{F}_{24}(x) \\ -\hat{F}_{13}(x) & -\hat{F}_{23}(x) & 0 & \hat{F}_{34}(x) \\ -\hat{F}_{14}(x) & -\hat{F}_{24}(x) & -\hat{F}_{34}(x) & 0 \end{pmatrix}.$$

An anti-self-dual matrix θ is defined as (3) where η is a real number.

Definition 8 Let E_4 be the 4×4 unit matrix and $\hat{F}(x)$ be a 4×4 alternating matrix valued function. Assume that $\det\left[E_4 - \hat{F}(x)\theta\right] \neq 0$, then 4×4 matrix $g\left(\hat{F}(x)\right)$ is defined as

$$g\left(\hat{F}(x)\right) := 2\left(E_4 - \hat{F}(x)\theta\right)^{-1} - E_4. \tag{4}$$

$g\left(\hat{F}(x)\right)$ should be a symmetric matrix because we assume $g\left(\hat{F}(x)\right)$ is a metric.

Lemma 9 *Assume that* $g\left(\hat{F}(x)\right)$ *is a symmetric matrix. If* $g\left(\hat{F}(x)\right) = 2\left(E_4 - \hat{F}(x)\,\theta\right)^{-1} - E_4$ *then,*

$$\hat{F}(x) = \begin{pmatrix} 0 & \hat{F}_{12}(x) & \hat{F}_{13}(x) & \hat{F}_{14}(x) \\ -\hat{F}_{12}(x) & 0 & \hat{F}_{14}(x) & -\hat{F}_{13}(x) \\ -\hat{F}_{13}(x) & -\hat{F}_{14}(x) & 0 & \hat{F}_{34}(x) \\ -\hat{F}_{14}(x) & \hat{F}_{13}(x) & -\hat{F}_{34}(x) & 0 \end{pmatrix}. \tag{5}$$

This lemma is proved by a direct calculation. This means that if $\hat{F}_{12}(x) + \hat{F}_{34}(x) = 0$ *then* $\hat{F}$ *is an anti-self-dual matrix.*

Now that the sufficient condition is "Ricci flat", it is necessary to know how the Ricci curvature is calculated by the gauge field. Before that, $\hat{F}_{\mathbb{C}}(x)$ is defined for convenience.

$$\hat{F}_{\mathbb{C}}(x) := \begin{pmatrix} \hat{F}_{1\bar{1}}(x) & \hat{F}_{1\bar{2}}(x) \\ \hat{F}_{2\bar{1}}(x) & \hat{F}_{2\bar{2}}(x) \end{pmatrix}$$
$$= -\frac{1}{2}\begin{pmatrix} \mathrm{i}\hat{F}_{12}(x) & -\hat{F}_{13}(x) + \mathrm{i}\hat{F}_{14}(x) \\ \hat{F}_{13}(x) + \mathrm{i}\hat{F}_{14}(x) & \mathrm{i}\hat{F}_{34}(x) \end{pmatrix}.$$

Then determinant of the metric matrix $g\left(\hat{F}(x)\right)$ is calculated by the gauge field as below.

Proposition 10 *It is convenient to know* $\det\left[g\left(\hat{F}(x)\right)\right]$ *for calculating the Ricci curvature because of* (1)*. Suppose that the Hermitian matrix* $g\left(\hat{F}(x)\right)$ *satisfies* (4) *and* (5)*.*

$$\det\left[g\left(\hat{F}(x)\right)\right] = 1 + 8\mathrm{i}\eta\,\mathrm{Tr}\left[\hat{F}_{\mathbb{C}}(x)\right] - 32\eta^2\left(\mathrm{Tr}\left[\hat{F}_{\mathbb{C}}(x)\right]\right)^2 + \mathcal{O}\left(\eta^3\right)$$

As is well known, the Ricci curvature is calculated from the determinant of the metric matrix.

Lemma 11 *Suppose that the Hermitian matrix* $g\left(\hat{F}(x)\right)$ *satisfies* (4) *and* (5)*. Then its Ricci curvature(1) is*

$$R_{\bar{j}k}\left(\hat{F}(x)\right) = \eta\partial_{\bar{j}}\partial_k\left(\hat{F}_{12}(x) + \hat{F}_{34}(x)\right) + \mathcal{O}\left(\eta^2\right).$$

Proof

$$\begin{aligned}R_{\bar{j}k}(x) &= \partial_{\bar{j}}\partial_k\left[\log\left\{\det\left[g\left(\hat{F}(x)\right)\right]\right\}\right]\\&= 2\mathrm{i}\eta\partial_{\bar{j}}\partial_k \operatorname{Tr}\left[\hat{F}_{\mathbb{C}}(x)\right] - 8\eta^2\partial_{\bar{j}}\partial_k\left(\operatorname{Tr}\left[\hat{F}_{\mathbb{C}}(x)\right]\right)^2\\&- 16\eta^2 \operatorname{Tr}\left[\hat{F}_{\mathbb{C}}(x)\right]\partial_{\bar{j}}\partial_k \operatorname{Tr}\left[\hat{F}_{\mathbb{C}}(x)\right] + 16\eta^2\left(\partial_{\bar{j}}\operatorname{Tr}\left[\hat{F}_{\mathbb{C}}(x)\right]\right)\left(\partial_k \operatorname{Tr}\left[\hat{F}_{\mathbb{C}}(x)\right]\right)\\&- 32\mathrm{i}\eta^2 \operatorname{Tr}\left[\hat{F}_{\mathbb{C}}(x)\right]\partial_{\bar{j}}\partial_k \operatorname{Tr}\left[\hat{F}_{\mathbb{C}}(x)\right] + \mathcal{O}\left(\eta^3\right)\end{aligned}$$

□

In physics, "the field is almost zero at a far enough distance" is a natural setting.

Definition 12 (Asymptotic to Zero) "$f(z_1, z_2)$ is asymptotically zero" is defined as

$$\lim_{|z_1|^2+|z_2|^2\to\infty} f(z_1, z_2) = 0.$$

The following "Maximum principle" is a famous theorem in Harmonic analysis.

Fact 13 (Maximum Principle) *If a function $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$ satisfies the following condition,*

$$\Delta f(z_1, z_2) = \partial_{\bar{1}}\partial_1 f(z_1, z_2) + \partial_{\bar{2}}\partial_2 f(z_1, z_2) = 0$$

f is called a harmonic function. Harmonic functions satisfy the following maximum principle: if K is a nonempty compact subset of U, then f restricted to K attains its maximum and minimum on the boundary of K. If U is connected, this means that f cannot have local maxima or minima, other than the exceptional case where f is constant.

This fact leads to the following corollary.

Corollary 14 *Assume that $f \in C^\infty\left(\mathbb{C}^2, \mathbb{R}\right)$.If $f(z_1, z_2)$ is asymptotically zero and*

$$\partial_{\bar{j}}\partial_k f(z_1, z_2) = 0$$

for any j and k then

$$f(z_1, z_2) = 0$$

Proof This f is a harmonic function on $\mathbb{C}^2$ because

$$\Delta f(z_1, z_2) = \partial_{\bar{1}}\partial_1 f(z_1, z_2) + \partial_{\bar{2}}\partial_2 f(z_1, z_2) = 0.$$

And f is asymptotically zero hence

$$f(z_1, z_2) = 0.$$

□

Remark 15 Symmetry of $g\left(\hat{F}(x)\right)$ means

$$\hat{F}(x) = \begin{pmatrix} 0 & \hat{F}_{12}(x) & \hat{F}_{13}(x) & \hat{F}_{14}(x) \\ -\hat{F}_{12}(x) & 0 & \hat{F}_{14}(x) & -\hat{F}_{13}(x) \\ -\hat{F}_{13}(x) & -\hat{F}_{14}(x) & 0 & \hat{F}_{34}(x) \\ -\hat{F}_{14}(x) & \hat{F}_{13}(x) & -\hat{F}_{34}(x) & 0 \end{pmatrix}$$

and $\hat{F}_{ij}(x)$ is asymptotically zero and $\partial_{\bar{j}}\partial_k\left(\hat{F}_{12}(x) + \hat{F}_{34}(x)\right) = 0$ means

$$\hat{F}_{12}(x) = -\hat{F}_{34}(x).$$

Theorem 16 (Main Theorem) *If $\hat{F}_{ij}(x)$ is asymptotically zero and $R_{\bar{j}k}(x) \equiv 0 \pmod{\eta^2}$ then $\hat{F}$ is an anti-self-dual matrix.That means*

$$R_{\bar{j}k}\left(\hat{F}(x)\right) \equiv 0 \pmod{\eta^2}, \lim_{|z_1|^2+|z_2|^2\to\infty} \hat{F} = 0$$
$$\Longrightarrow *\hat{F}^-(x) = -\hat{F}^-(x)$$

Proof *If*

$$R_{\bar{j}k}(x) \equiv 0 \pmod{\eta^2}$$

then it means

$$\partial_{\bar{j}}\partial_k\left(\hat{F}_{12}(x) + \hat{F}_{34}(x)\right) = 0$$

and $\hat{F}_{ij}(x)$ is asymptotically zero hence

$$*\hat{F}^-(x) = -\hat{F}^-(x).$$

□

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2-Hom-Associative Bialgebras and Hom-Left Symmetric Dialgebras

Mahouton Norbert Hounkonnou and Gbêvèwou Damien Houndédji

Abstract From the definition and properties of unital hom-associative algebras, and the use of the Kaplansky's construction, we develop algebraic structures called *2-hom-associative bialgebras, 2-hom-bialgebras, and 2-2-hom-bialgebras*. Besides, we define and characterize the hom-associative dialgebras, hom-Leibniz algebra and hom-left symmetric dialgebras, and discuss their main relevant properties. Explicit examples are given to illustrate the developed formalism.

Keywords 2-hom-associative bialgebra · 2-hom-bialgebra · 2-2-hom-bialgebra · Hom-left symmetric dialgebra · Hom-Leibniz algebra

Mathematics Subject Classification (2010) 16T25, 05C25, 16S99, 16Z05

1 Introduction

A theory of 2-associative algebras was developed by J-L. Loday and M. Ronco in [8], where the operad of 2-associative algebras was also introduced as a Koszul operad. The notion of infinitesimal bialgebra was given for the first time by S. Joni and G.-C. Rota in [6]. The basic theory was developed by M. Aguiar in [1] and in [2]. J-L. Loday in [7] also performed a non-antisymmetric version of Lie algebras, called Leibniz algebras, whose the bracket satisfies the Leibniz relation. The Leibniz rule, combined with the antisymmetry property, leads to the Jacobi identity. Therefore, the Lie algebras are anti-symmetric Leibniz algebras. In the same work, Loday formulated an associative version of Leibniz algebras, called diassociative algebras, equipped with two bilinear and associative operations, which satisfy three axioms, all of them being various forms of the associative law. Recently [5], R. Felipe built the left-symmetric dialgebras which include, as a particular case,

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the notion of dialgebras.This gave a new impulse to the construction of Leibniz algebras.

The hom-algebra structures first arose in quasi-deformations of Lie algebras of vector fields. Discrete modifications of vector fields, via twisted derivations, provide hom-Lie and quasi-hom-Lie structures, in which the Jacobi condition is twisted. Other interesting hom-type algebras of classical structures were studied. They include hom-associative algebras, hom-Lie admissible algebras[11], and, more generally, G-hom-associative algebras [10], enveloping algebras of hom-Lie algebras[12], hom-Lie admissible hom-coalgebras and hom-Hopf algebras [9], hom-alternative algebras, hom-Malcev algebras and hom-Jordan algebras [14], L-modules, L-comodules and hom-Lie quasi-bialgebras [3], and Laplacian of hom-Lie quasi-bialgebras [4].

In this paper, we devise a hom-type generalization of 2-associative algebras, 2-bialgebras, 2-associative bialgebras, 2-2-bialgebras and left symmetric dialgebras, leading to the concepts of hom-bialgebras, 2-hom-associative algebras, 2-hom-bialgebras, 2-hom- associative bialgebras, 2-2-hom-bialgebras and hom-left symmetric dialgebras, respectively. The hom-type algebras are usually defined by twisting the defining axioms of a type of algebras by a certain twisting map. When the twisting map happens to be the identity map, we get an ordinary algebraic structure. A hom-counital condition can be given as follows:

$$(\varepsilon \otimes \alpha)\Delta(x) = (\alpha \otimes \varepsilon)\Delta(x) = \alpha^2(x),\ \forall x \in V.$$

This leads to new definitions of counital hom-coassociative coalgebra and unital hom-bialgebra structures. A unital infinitesimal hom-bialgebra condition can be formulated by the relation:

$$\Delta \circ \mu = (\mu \otimes \alpha) \circ (\alpha \otimes \Delta) + (\alpha \otimes \mu) \circ (\Delta \otimes \alpha) - \alpha^2 \otimes \alpha.$$

This unital infinitesimal twisted condition permits to define the unital infinitesimal hom-bialgebra structure. Then, we deal with the concepts of 2-hom-associative bialgebras, 2-hom-bialgebras and 2-2-hom bialgebras. Besides, we provide a hom-algebra version of Kaplansky's construction of hom-bialgebras in order to build unital analogs of 2-hom-associative bialgebras, 2-hom-bialgebras and 2-2-hom-bialgebras. Finally, we define the notion of hom-left symmetric dialgebras generalizing the classical left symmetric dialgebras, and discuss their relevant properties.

The paper is organized as follows. In Sect. 2, we give the definitions of hom-bialgebra, 2-hom-associative algebra, 2-hom-associative bialgebra, 2-hom-bialgebra, 2-2-hom-bialgebra, and derive their main properties. In Sect. 3, we provide a hom-algebra version of Kaplansky's constructions of hom-bialgebras from a unital hom-associative algebra. We show that these constructions induce a large class of 2-hom-bialgebras, 2-hom-associative bialgebras, and 2-2-hom-bialgebras. In Sect. 4, we define and characterize the hom-associative dialgebras, hom-Leibniz algebra and hom-left symmetric dialgebras, and discussed their main relevant properties. Section 5 is devoted to concluding remarks.

2 Definitions of Unital 2-Hom-Associative Bialgebras

2.1 Unital Hom-Bialgebra and Unital Infinitesimal Hom-Bialgebra

Definition 1 ([10]) A hom-associative algebra is a triple (V, μ, α) consisting of a linear space V, a bilinear map $\mu : V \times V \to V$ and a homomorphism $\alpha : V \to V$ satisfying the multiplicativity and hom-associativity properties, i.e.

$$\alpha \circ \mu = \mu \circ \alpha^{\otimes^2} := \mu \circ (\alpha \otimes \alpha), \tag{1}$$

$$\mu \circ (\alpha \otimes \mu) = \mu \circ (\mu \otimes \alpha), \tag{2}$$

respectively.

Definition 2 ([15]) A unital hom-associative algebra is given by a quadruple $(\mathcal{A}, \mu, \alpha, e)$, where $e \in \mathcal{A}$, such that:

- $(\mathcal{A}, \mu, \alpha)$ is a hom-associative algebra,
- $\mu(x, e) = \mu(e, x) = \alpha(x), \forall x \in \mathcal{A}$,
- $\alpha(e) = e$.

Example Let $\mathcal{A}$ be an n-dimensional vector space, ($n = 2, 3$), over a field $\mathcal{K}$ with a basis $\{e_i\}_{i=1,\dots,n}$. The following product μ and linear map α on $\mathcal{A}$ define a unital hom-associative algebra in each of the following cases:

- $\mu(e_1 \otimes e_1) = e_1, \mu(e_2 \otimes e_2) = e_2, \mu(e_1 \otimes e_2) = \mu(e_2 \otimes e_1) = 0, \alpha(e_1) = e_1$ and $\alpha(e_2) = 0$.
- $\mu(e_1 \otimes e_1) = e_1,\ \mu(e_2 \otimes e_2) = e_1,\ \mu(e_1 \otimes e_2) = \mu(e_2 \otimes e_1) = -e_2, \alpha(e_1) = e_1$ and $\alpha(e_2) = -e_2$.
- $\mu(e_1 \otimes e_1) = e_1,\ \mu(e_2 \otimes e_2) = e_2,\ \mu(e_1 \otimes e_3) = -e_3,\ \mu(e_3 \otimes e_1) = -e_3, \mu(e_3 \otimes e_3) = e_1,\ \alpha(e_1) = e_1$ and $\alpha(e_3) = -e_3$.

Definition 3 ([13]) A hom-coassociative coalgebra is a triple (V, Δ, α) consisting of a linear space V, a linear map $\Delta : V \to V \otimes V$, and a homomorphism $\alpha : V \to V$ satisfying

$$\alpha^{\otimes^2} \circ \Delta = \Delta \circ \alpha \text{ (comultiplicativity)} \tag{3}$$

$$(\alpha \otimes \Delta) \circ \Delta = (\Delta \otimes \alpha) \circ \Delta \text{ (hom-coassociativity).} \tag{4}$$

Example Let $\mathcal{A}$ be a 3-dimensional vector space over $\mathcal{K}$ with a basis $\{e_1, e_2, e_3\}$. The following coproduct Δ and linear map α on $\mathcal{A}$ define a hom-coassociative coalgebra:

$$\Delta(e_1) = e_1 \otimes e_1, \quad \Delta(e_2) = e_2 \otimes e_2,$$

$$\Delta(e_3) = \frac{\sqrt{5}-1}{2\sqrt{5}}(e_1 \otimes e_3 + e_3 \otimes e_1) + \frac{1}{\sqrt{5}}(-e_1 \otimes e_1 + e_3 \otimes e_3),$$

$$\alpha(e_1) = e_1, \alpha(e_2) = 0 \text{ and } \alpha(e_3) = e_3.$$

Definition 4 A counital hom-coassociative coalgebra is defined as a quadruple $(V, \Delta, \varepsilon, \alpha)$ such that the triple (V, Δ, α) is a hom-coassociative coalgebra satisfying the hom-counital condition

$$(\varepsilon \otimes \alpha)\Delta(x) = (\alpha \otimes \varepsilon)\Delta(x) = \alpha^2(x), \forall x \in V. \tag{5}$$

Example Let $\mathcal{A}$ be a 3-dimensional vector space over $\mathcal{K}$ with a basis $\{e_1, e_2, e_3\}$. The following coproduct Δ and linear map α on $\mathcal{A}$ define a counital hom- coassociative coalgebra:

$$\Delta(e_1) = e_1 \otimes e_1, \quad \Delta(e_2) = e_2 \otimes e_2,$$

$$\Delta(e_3) = \frac{\sqrt{5}-1}{2\sqrt{5}}(e_1 \otimes e_3 + e_3 \otimes e_1) + \frac{1}{\sqrt{5}}(-e_1 \otimes e_1 + e_3 \otimes e_3),$$

$$\alpha(e_1) = e_1, \alpha(e_2) = 0, \alpha(e_3) = e_3,$$

$$\epsilon(e_1) = 1, \epsilon(e_2) = 1 \text{ and } \epsilon(e_3) = \frac{1+\sqrt{5}}{2}.$$

Definition 5 A unital hom-bialgebra is a system $(V, \mu, \eta, \Delta, \varepsilon, \alpha)$, where $\mu : V \otimes V \to V$(multiplication), $\eta : \mathcal{K} \to V$(unit), $\Delta : V \to V \otimes V$ (comultiplication), $\varepsilon : V \to \mathcal{K}$ (counit), and $\alpha : V \to V$ (endomorphism) are linear maps satisfying the following properties:

(1) the quadruple (V, μ, η, α) is a unital hom-associative algebra;
(2) the quadruple $(V, \Delta, \varepsilon, \alpha)$ is a counital hom-coassociative coalgebra;
(3) the compatibility condition is expressed by the following three identities:

  (a) $\Delta(\mu(x \otimes y)) = \Delta(x) \bullet \Delta(y), \forall x, y \in V$,
  (b) $\alpha^{\otimes^2} \circ \Delta = \Delta \circ \alpha$,
  (c) $\varepsilon(\mu(x \otimes y)) = \varepsilon(x)\varepsilon(y)$,
  (d) $\varepsilon \circ \alpha(x) = \varepsilon(x)$.

Example Let $\mathcal{A}$ be a n-dimensional vector space, ($n = 2, 3$), over a field $\mathcal{K}$ with a basis $\{e_i\}_{i=1,\dots,n}$. The following product μ, coproduct Δ and linear maps α, ε on $\mathcal{A}$ define a unital hom-bialgebra:

- $\mu(e_1 \otimes e_1) = e_1,\ \mu(e_2 \otimes e_2) = 0,\ \mu(e_1 \otimes e_2) = \mu(e_2 \otimes e_1) = e_2,$
  $\alpha(e_1) = e_1,\ \alpha(e_2) = e_2,$
  $\Delta(e_1) = e_1 \otimes e_1,\ \Delta(e_2) = \theta e_2 \otimes e_2,$
  $\varepsilon(e_1) = 1$ and $\varepsilon(e_2) = \frac{1}{\theta}(\theta \neq 0).$
- $\mu(e_1 \otimes e_1) = e_1,\ \mu(e_1 \otimes e_3) = \mu(e_3 \otimes e_1) = e_3,\ \mu(e_2 \otimes e_2) = e_2,$
  $\mu(e_3 \otimes e_3) = e_1 + e_3,$
  $\alpha(e_1) = e_1,\ \alpha(e_2) = 0, \alpha(e_3) = e_3,$
  $\Delta(e_1) = e_1 \otimes e_1,\ \Delta(e_2) = e_2 \otimes e_2,$
  $\Delta(e_3) = \frac{\sqrt{5}-1}{2\sqrt{5}} e_1 \otimes e_1 + \frac{1}{\sqrt{5}}(e_1 \otimes e_3 + e_3 \otimes e_1 - 2e_3 \otimes e_3),$
  $\varepsilon(e_1) = 1,\ \varepsilon(e_2) = 1$ and $\varepsilon(e_3) = \frac{1-\sqrt{5}}{2}.$

It is worth noticing that the conditions (3.a) and (3.b) of Definition 5 do not lead to define a unital hom-bialgebra structure in $(\mathcal{A}, \mu)$ given by $\mu(e_1 \otimes e_1) = e_1,\ \mu(e_2 \otimes e_2) = e_1,\ \mu(e_1 \otimes e_2) = \mu(e_2 \otimes e_1) = -e_2, \alpha(e_1) = e_1,\ \alpha(e_2) = -e_2.$

Definition 6 A unital infinitesimal hom-bialgebra $(V, \mu, \eta, \Delta, \varepsilon, \alpha)$ is a $\mathcal{K}$-vector space V equipped with a unital hom-associative multiplication μ and a counital hom-coassociative comultiplication Δ, which are related by the unital hom-infinitesimal relation

$$\Delta \circ \mu = (\mu \otimes \alpha) \circ (\alpha \otimes \Delta) + (\alpha \otimes \mu) \circ (\Delta \otimes \alpha) - \alpha^2 \otimes \alpha. \tag{6}$$

Note that the unital hom-bialgebras, given in the previous example, are not unital infinitesimal hom-bialgebras. The presence of the term $\alpha^2 \otimes \alpha$ in Eq. (6) complicates the construction of non trivial examples of unital infinitesimal hom-bialgebras, unital 2-hom-associative bialgebras and unital 2-2-hom-bialgebras. Finding such more relevant examples is a task in the core of our current concerns. It deserves further works, which will complete and enrich the present study.

Example Let $\mathcal{A}$ be a 2-dimensional vector space over $\mathcal{K}$ with a basis $\{e_1, e_2\}$. The next product μ, coproduct Δ and linear maps α, ε on $\mathcal{A}$ define a unital hom-bialgebra:

$$\mu(e_1 \otimes e_1) = e_1,\ \mu(e_2 \otimes e_2) = e_2,\ \mu(e_1 \otimes e_2) = \mu(e_2 \otimes e_1) = 0,$$
$$\alpha(e_1) = e_1,\ \alpha(e_2) = 0,$$
$$\Delta(e_1) = e_1 \otimes e_1,\ \Delta(e_2) = 0, \varepsilon(e_1) = 1 \text{ and } \varepsilon(e_2) = 0.$$

2.2 2-Hom-Associative Algebra

The 2-hom-associative algebras generalize the 2-associative algebras in the sense where the associativity laws are twisted.

Definition 7 A 2-hom-associative algebra over $\mathcal{K}$ is a vector space equipped with two hom-associative structures. A 2-hom-associative algebra is said to be unital if there is a unit e, which is a unit for both operations.

Example Let $\mathcal{A}$ be a 3-dimensional vector space over $\mathcal{K}$ with a basis $\{e_1, e_2, e_2\}$. The following multiplications μ_1, μ_2 and linear map α on $\mathcal{A}$ define a unital 2- hom-associative algebra:

$$\mu_1(e_1 \otimes e_1) = e_1,\ \mu_1(e_3 \otimes e_3) = e_3, \mu_1(e_1 \otimes e_3) = \mu_1(e_3 \otimes e_1) = e_3,$$

$$\mu_2(e_1 \otimes e_1) = e_1,\ \mu_2(e_3 \otimes e_3) = e_1 + e_3, \mu_2(e_1 \otimes e_3) = \mu_2(e_3 \otimes e_1) = e_3,$$

$$\mu_2(e_2 \otimes e_2) = e_2 \text{ and } \alpha(e_1) = e_1,\ \alpha(e_2) = 0, \alpha(e_3) = e_3.$$

Definition 8 Let $(V, \mu_1, \mu_2, \alpha)$ and $(V', \mu_1', \mu_2', \alpha')$ be two 2-hom-associative algebras. A liner map $f : V \to V'$ is a morphism of 2-hom-associative algebras if

$$\mu_1' \circ (f \otimes f) = f \circ \mu_1,\ \ \mu_2' \circ (f \otimes f) = f \circ \mu_2 \text{ and } f \circ \alpha = \alpha' \circ f.$$

In particular, the 2-hom-associative algebras $(V, \mu_1, \mu_2, \alpha)$ and $(V', \mu_1', \mu_2', \alpha')$ are isomorphic if f is a bijective linear map such that

$$\mu_1 = f^{-1} \circ \mu_1' \circ (f \otimes f),\ \ \mu_2 = f^{-1} \circ \mu_2' \circ (f \otimes f) \text{ and } \alpha = f^{-1} \circ \alpha' \circ f.$$

Theorem 9 *Let (V, μ_1, μ_2) be a 2-associative algebra, and $\alpha : V \to V$ be an associative algebra endomorphism. Then, $V_\alpha = (V, \alpha \circ \mu_1, \alpha \circ \mu_2, \alpha)$ is a 2-hom-associative algebra. Moreover, suppose that (V', μ_1', μ_2') is another 2-associative algebra and $\alpha' : V' \to V'$ an associative algebra endomorphism. If $f : V \to V'$ is an associative algebra morphism that satisfies $f \circ \alpha = \alpha' \circ f$, then*

$$f : (V, \alpha \circ \mu_1, \alpha \circ \mu_2, \alpha) \to (V', \alpha \circ \mu_1', \alpha \circ \mu_2', \alpha')$$

is a morphism of 2-hom-associative algebra.

2.3 Unital 2-Hom-Associative Bialgebra

We give the notion of unital 2-hom-associative bialgebras generalizing unital 2-associative bialgebras.

Definition 10 A unital 2-hom-associative bialgebra $(V, \mu_1, \mu_2, \eta, \Delta, \varepsilon, \alpha)$ is a vector space V equipped with two multiplications μ_1 and μ_2, a unit η, a comultiplication Δ, a counit ε, and a linear map $\alpha : V \to V$ such that

- $(V, \mu_1, \eta, \Delta, \varepsilon, \alpha)$ is a unital hom-bialgebra, and
- $(V, \mu_2, \eta, \Delta, \varepsilon, \alpha)$ is a unital infinitesimal hom-bialgebra.

Example Let $\mathcal{A}$ be a 2-dimensional vector space over $\mathcal{K}$ with a basis $\{e_1, e_2\}$. The products μ_1, μ_2, the coproduct Δ and the linear maps α and ε given by

$$\mu_1(e_1 \otimes e_1) = e_1,\ \mu_1(e_2 \otimes e_2) = e_2, \mu_1(e_1 \otimes e_2) = \mu_1(e_2 \otimes e_1) = 0,$$

$$\mu_2(e_1 \otimes e_1) = e_1,\ \mu_2(e_2 \otimes e_2) = 0, \mu_2(e_1 \otimes e_2) = \mu_2(e_2 \otimes e_1) = 0,$$

$$\Delta(e_1) = e_1 \otimes e_1,\ \Delta(e_2) = 0,$$

$$\alpha(e_1) = e_1, \alpha(e_2) = 0, \varepsilon(e_1) = 1, \varepsilon(e_2) = 0$$

define a unital 2-hom-associative bialgebra structure on $\mathcal{A}$.

Definition 11 Let $(V, \mu_1, \mu_2, \eta, \Delta, \varepsilon, \alpha)$ and $(V', \mu'_1, \mu'_2, \eta', \Delta', \varepsilon', \alpha')$ be two unital 2-associative hom-bialgebras. A linear map $f : V \to V'$ is a morphism of unital 2-associative hom-bialgebras if:

- $\mu'_1 \circ (f \otimes f) = f \circ \mu_1$,
- $\mu'_2 \circ (f \otimes f) = f \circ \mu_2$,
- $f \circ \eta = \eta'$,
- $(f \otimes f) \circ \Delta = \Delta' \circ f$,
- $\varepsilon' = \varepsilon \circ f$, $f \circ \alpha = \alpha' \circ f$.

2.4 Unital 2-Hom-Bialgebra

Definition 12 A unital 2-hom-bialgebra $(V, \mu_1, \mu_2, \eta, \Delta_1, \Delta_2, \varepsilon_1, \varepsilon_2, \alpha)$ is a vector space V equipped with two multiplications μ_1, μ_2, the unit η, two comultiplications Δ_1, Δ_2, two counits $\varepsilon_1, \varepsilon_2$, and a linear map $\alpha : V \to V$ such that: $(V, \mu_1, \eta, \Delta_1, \varepsilon_1, \alpha)$, $(V, \mu_2, \eta, \Delta_2, \varepsilon_2, \alpha)$, $(V, \mu_1, \eta, \Delta_2, \varepsilon_2, \alpha)$, and $(V, \mu_2, \eta, \Delta_1, \varepsilon_1, \alpha)$ are unital hom-bialgebras.

Example Let $\mathcal{A}$ be a 2-dimensional vector space over $\mathcal{K}$ with a basis $\{e_1, e_2\}$. The products μ_1, μ_2, the coproducts $\Delta = \Delta_1 = \Delta_2$ and the linear maps α and $\varepsilon = \varepsilon_1 = \varepsilon_2$ given by

$$\mu_1(e_1 \otimes e_1) = e_1,\ \mu_1(e_2 \otimes e_2) = e_2, \mu_1(e_1 \otimes e_2) = \mu_1(e_2 \otimes e_1) = 0,$$

$$\mu_2(e_1 \otimes e_1) = e_1,\ \mu_2(e_2 \otimes e_2) = 0, \mu_2(e_1 \otimes e_2) = \mu_2(e_2 \otimes e_1) = 0,$$

$$\Delta(e_1) = e_1 \otimes e_1,\ \Delta(e_2) = 0,$$

$$\alpha(e_1) = e_1, \alpha(e_2) = 0, \varepsilon(e_1) = 1, \varepsilon(e_2) = 0$$

define a unital 2-hom-bialgebra structure on $\mathcal{A}$.

Example Let $\mathcal{A}$ be a 3-dimensional vector space over $\mathcal{K}$ with a basis $\{e_1, e_2, e_3\}$. The products $\mu = \mu_1 = \mu_2$, the coproducts Δ_1, Δ_2 and the linear maps α and $\varepsilon_1, \varepsilon_2$ given by

$$\mu(e_1 \otimes e_1) = e_1, \ \mu(e_2 \otimes e_2) = e_1 + e_2,$$

$$\mu_1(e_1 \otimes e_2) = \mu(e_2 \otimes e_1) = e_2, \mu(e_3 \otimes e_3) = e_3,$$

$$\Delta_1(e_1) = \Delta_2(e_1) = e_1 \otimes e_1, \ \Delta_1(e_3) = \Delta_2(e_3) = e_3 \otimes e_3,$$

$$\Delta_1(e_2) = \frac{\sqrt{5}-1}{2\sqrt{5}}(e_1 \otimes e_2 + e_2 \otimes e_1) + \frac{1}{\sqrt{5}}(-e_1 \otimes e_1 + e_2 \otimes e_2),$$

$$\Delta_2(e_2) = \frac{\sqrt{5}+1}{2\sqrt{5}}(e_1 \otimes e_2 + e_2 \otimes e_1) + \frac{1}{\sqrt{5}}(e_1 \otimes e_1 - e_2 \otimes e_2),$$

$$\alpha(e_1) = e_1, \ \alpha(e_2) = e_2, \ \alpha(e_3) = 0,$$

$$\varepsilon_1(e_1) = \varepsilon_2(e_1) = 1, \ \varepsilon_1(e_3) = \varepsilon_2(e_3) = 1,$$

$$\varepsilon_1(e_2) = \frac{1+\sqrt{5}}{2}, \varepsilon_2(e_2) = \frac{1-\sqrt{5}}{2}$$

define a unital 2-hom-bialgebra structure on $\mathcal{A}$.

The unital 2-hom-bialgebra is called of type (1-1), (resp. of type (2-2)), if the two multiplications and the two comultiplications are identical, (resp. distinct). The unital 2-hom-bialgebra is called of type (1-2), (resp. of type (2-1)), if the two multiplications are identical, (resp. distinct), and the two comultiplications are distinct, (resp. identical).

Proposition 13 *Let $(V, \mu, \eta, \Delta, \varepsilon, \alpha)$ be a unital hom-bialgebra. Then, we have that $(V, \mu, \mu, \eta, \Delta, \Delta, \varepsilon, \alpha)$ and $(V, \mu, \mu^{op}, \eta, \Delta, \Delta^{cop}, \varepsilon, \alpha)$ are unital 2-hom-bialgebras, where $\mu^{op}(x \otimes y) = \mu(y \otimes x)$ and $\Delta^{cop}(x) = \tau \circ \Delta(x)$, with $\tau(x \otimes y) = y \otimes x$. The first unital 2-hom-bialgebra is of type (1-1), and the second one is of type (2-2).*

Proof It comes from a direct computation. □

2.5 Unital 2-2-Hom-Bialgebra

Definition 14 A unital 2-2-hom-bialgebra $(V, \mu_1, \mu_2, \eta, \Delta_1, \Delta_2, \varepsilon_1, \varepsilon_2, \alpha)$ is a vector space V equipped with two multiplications μ_1, μ_2, two comultiplications Δ_1, Δ_2, two counits $\varepsilon_1, \varepsilon_2$, one unit η, and a linear map $\alpha : V \to V$ such that

(1) $(V, \mu_1, \eta, \Delta_1, \varepsilon_1, \alpha)$ and $(V, \mu_2, \eta, \Delta_2, \varepsilon_2, \alpha)$ are unital hom-bialgebras,

(2) $(V, \mu_1, \eta, \Delta_2, \varepsilon_2, \alpha)$ and $(V, \mu_2, \eta, \Delta_1, \varepsilon_1, \alpha)$ are unital infinitesimal hom-bialgebras.

Example Let $\mathcal{A}$ be a 2-dimensional vector space over $\mathcal{K}$ with a basis $\{e_1, e_2\}$. The products μ_1, μ_2, the coproducts $\Delta = \Delta_1 = \Delta_2$ and the linear maps α and $\varepsilon = \varepsilon_1 = \varepsilon_2$ given by

$$\mu_1(e_1, e_1) = e_1,\ \mu_1(e_2, e_2) = e_2, \mu_1(e_1, e_2) = \mu_1(e_2, e_1) = 0,$$
$$\mu_2(e_1, e_1) = e_1,\ \mu_2(e_2, e_2) = 0, \mu_2(e_1, e_2) = \mu_2(e_2, e_1) = 0,$$
$$\Delta(e_1) = e_1 \otimes e_1,\ \Delta(e_2) = 0,$$
$$\alpha(e_1) = e_1, \alpha(e_2) = 0, \varepsilon(e_1) = 1, \varepsilon(e_2) = 0$$

define a unital 2-2-hom-bialgebra structure on $\mathcal{A}$.

A unital 2-2-hom-bialgebra is called of type (1-1), (resp. of type (2-2)), if the two multiplications and the two comultiplications are identical, (resp. distinct). A unital 2-2-hom-bialgebra is called of type (1-2), (resp. of type (2-1)), if the two multiplications are identical, (resp. distinct), and the two comultiplications are distinct, (resp. identical).

The definition of unital 2-2-hom-bialgebra morphism is similar to that of unital 2-hom-bialgebra morphism.

3 Kaplansky's Construction of Hom-Bialgebras

In this section, we give a hom-algebra version of Kaplansky's construction of hom-bialgebras in order to build unital 2-associative hom-bialgebras, unital 2-hom-bialgebras, and unital 2-2-hom-bialgebras. The following statement is in order.

Proposition 15 *Let $\mathcal{A} = (V, \mu, \eta, \alpha)$ be a unital hom-associative algebra, where $e_2 := \eta(1)$ is the unit. Let $\tilde{V}$ be the vector space spanned by V and e_1, $\tilde{V} = \mathrm{span}(V, e_1)$. Then, $\mathcal{K}_1(\mathcal{A}) := (\tilde{V}, \mu_1, \eta_1, \Delta_1, \varepsilon_1, \alpha_1)$ is a unital hom-bialgebra where the multiplication μ_1 is defined by:*

$$\mu_1(e_1 \otimes x) = \mu_1(x \otimes e_1) = \alpha_1(x)\ \forall x \in \tilde{V},$$
$$\mu_1(x \otimes y) = \mu(x \otimes y)\ \ \forall x, y \in V,$$

the unit η_1 *is given by* $\eta_1(1) = e_1$, *while the comultiplication* Δ_1, *the counit* ε_1, *and the linear map* α_1 *are defined by,* $\forall x \in V$*:*

$$\begin{aligned}
\Delta_1(e_1) &= e_1 \otimes e_1 \\
\Delta_1(x) &= \alpha(x) \otimes e_1 + e_1 \otimes \alpha(x) - e_2 \otimes \alpha(x) \\
\varepsilon_1(e_1) &= 1, \varepsilon_1(x) = 0 \\
\alpha_1(e_1) &= e_1, \alpha_1(x) = \alpha(x),
\end{aligned}$$

respectively.

Proof

◇ (V, μ, η, α) is a unital hom-associative algebra. We have: $\forall x \in V$,

$$\mu_1(\alpha_1(x), \mu_1(e_2, e_1)) = \mu_1(\alpha(x), e_2) = \mu(\alpha(x), e_2) = \alpha^2(x), \text{ and}$$
$$\mu_1(\mu_1(x, e_2), \alpha_1(e_1)) = \mu_1(\mu(x, e_2), e_1) = \mu_1(\alpha(x), e_1) = \alpha^2(x).$$

Then, $\mu_1(\alpha_1(x), \mu_1(e_2, e_1)) = \mu_1(\mu_1(x, e_2), \alpha_1(e_1)) = \alpha^2(x)$.

By permuting x, e_1, e_2, we find the same result. Therefore, $\forall x, y, z \in \tilde{V}$, $\mu_1(\alpha_1(x), \mu_1(y, z)) = \mu_1(\mu_1(x, y), \alpha_1(z))$. Hence, $(\tilde{V}, \mu_1, \eta_1, \alpha_1)$ is a unital hom-associative algebra.

◇ We have

$$\begin{aligned}
(\alpha_1 \otimes \Delta_1) \circ \Delta_1(x) &= (\alpha_1 \otimes \Delta_1)(\alpha_1(x) \otimes e_1 + e_1 \otimes \alpha_1(x) - e_2 \otimes \alpha_1(x)) \\
&= \Delta_1(\alpha_1(x)) \otimes e_1 + \Delta_1(e_1) \otimes \alpha_1^2(x) + \Delta_1(e_2) \otimes \alpha_1^2(x) \\
&= (\Delta_1 \otimes \alpha_1) \circ \Delta_1(x),
\end{aligned}$$

and

$$\begin{aligned}
(\varepsilon_1 \otimes \alpha_1) \circ \Delta_1(x) &= 1 \otimes \alpha_1^2(x) = \alpha_1^2(x) \\
(\alpha_1 \otimes \varepsilon_1) \circ \Delta_1(x) &= \alpha_1^2(x) \otimes 1 = \alpha_1^2(x).
\end{aligned}$$

Then, $(\varepsilon_1 \otimes \alpha_1) \circ \Delta_1(x) = (\alpha_1 \otimes \varepsilon_1) \circ \Delta_1(x) = \alpha_1^2(x)$. Hence, we can conclude that $(\tilde{V}, \Delta_1, \varepsilon_1, \alpha_1)$ is a counital hom-coassociative coalgebra.

◇

$$\Delta_1(x) \bullet \Delta_1(y) = \mu_1(\alpha(x), \alpha(y)) \otimes e_1 + e_1 \otimes \mu_1(\alpha(x), \alpha(y))$$
$$- e_2 \otimes \mu_1(\alpha(x), \alpha(y)) = \mu_1 \circ \alpha_1^{\otimes^2}(x \otimes y) \otimes e_1 + e_1 \otimes \mu_1 \circ \alpha_1^{\otimes^2}(x \otimes y)$$
$$- e_2 \otimes \mu_1 \circ \alpha_1^{\otimes^2}(x \otimes y) = \alpha_1(\mu_1(x \otimes y)) \otimes e_1 + e_1 \otimes \alpha_1(\mu_1(x \otimes y))$$
$$- e_2 \otimes \alpha_1(\mu_1(x \otimes y)) = \Delta_1(\mu_1(x \otimes y));$$
$$\Delta_1(\alpha_1(x)) = \alpha_1^2(x) \otimes e_1 + e_1 \otimes \alpha_1^2(x) - e_2 \otimes \alpha_1^2(x)$$
$$= \alpha_1^{\otimes^2}(\alpha_1(x) \otimes e_1 + e_1 \otimes \alpha_1(x) - e_2 \otimes \alpha_1(x)) = \alpha_1^{\otimes^2} \circ \Delta_1(x).$$

Then, Δ is a homomorphism of the hom-associative algebras (V, μ, α) and $(V \otimes V, \bullet, \alpha \otimes \alpha)$.

Therefore, we can conclude that $\mathcal{K}_1(\mathcal{A}) := (\tilde{V}, \mu_1, \eta_1, \Delta_1, \varepsilon_1, \alpha_1)$ is a unital hom-bialgebra. □

Proposition 16 *Let $\mathcal{A} = (V, \mu, \eta, \alpha)$ be a unital hom-associative algebra, where $e_2 := \eta(1)$ is the unit. Let $\tilde{V}$ be the vector space spanned by V and e_1, $\tilde{V} =$* span(V, e_1). *$\mathcal{K}_2(\mathcal{A}) := (\tilde{V}, \mu_2, \eta_2, \Delta_2, \varepsilon_2, \alpha_2)$ is a unital hom-bialgebra, where the multiplication μ_2 is defined by:*

$$\mu_2(e_1 \otimes x) = \mu_2(x \otimes e_1) = \alpha_2(x), \ \forall x \in \tilde{V},$$
$$\mu_2(x \otimes y) = \mu(x \otimes y), \ \ \forall x, y \in V,$$

the unit η_2 is given by $\eta_2(1) = e_1$, while the comultiplication Δ_2, the counit ε_2, and the linear map α_2 are defined as follows:

$$\Delta_2(e_1) = e_1 \otimes e_1,$$
$$\Delta_2(e_2) = e_2 \otimes e_1 + e_1 \otimes e_2 - e_2 \otimes e_2,$$
$$\Delta_2(x) = (e_1 - e_2) \otimes \alpha(x) + \alpha(x) \otimes (e_1 - e_2) \ \ \forall x \in V \backslash \{e_2\},$$
$$\varepsilon_2(e_1) = 1, \varepsilon_2(x) = 0 \ \ \forall x \in V,$$
$$\alpha_2(e_1) = e_1, \alpha_2(x) = \alpha(x), \ \forall x \in V,$$

respectively.

Proof We have:

◇

$$\Delta_2(\alpha_2(x)) = (e_1 - e_2) \otimes \alpha_2^2(x) + \alpha_2^2(x) \otimes (e_1 - e_2)$$
$$= \alpha_2^{\otimes^2}((e_1 - e_2) \otimes \alpha_2(x) + \alpha_2(x) \otimes (e_1 - e_2)) = \alpha_2^{\otimes^2}(\Delta_2(x)).$$

◇

$$\Delta_2(x) \bullet \Delta_2(y) = \mu_2[(e_1 - e_2); (e_1 - e_2)] \otimes \mu_2[\alpha_2(x); \alpha_2(y)]$$
$$+ \mu_2[(e_1 - e_2); \alpha_2(y)] \otimes \mu_2[\alpha_2(x); (e_1 - e_2)]$$
$$+ \mu_2[\alpha_2(x); (e_1 - e_2)] \otimes \mu_2[(e_1 - e_2); \alpha_2(y)]$$
$$+ \mu_2(\alpha_2(x); \alpha_2(y)) \otimes \mu_2[(e_1 - e_2); (e_1 - e_2)]$$
$$= (e_1 - e_2) \otimes \alpha_2(\mu_2(x, y)) + \alpha_2(\mu_2(x, y)) \otimes (e_1 - e_2)$$
$$= \Delta_2(\mu_2(x, y)).$$

◇

$$(\alpha_2 \otimes \Delta_2) \circ \Delta_2(x) = e_1 \otimes e_1 \otimes \alpha_2^2(x) - e_1 \otimes e_2 \otimes \alpha_2^2(x) - e_2 \otimes e_1 \otimes \alpha_2^2(x)$$
$$+ e_2 \otimes e_2 \otimes \alpha_2^2(x) + e_1 \otimes \alpha_2^2(x) \otimes e_1 - e_1 \otimes \alpha_2^2(x) \otimes e_2 - e_2 \otimes \alpha_2^2(x) \otimes e_2$$
$$+ \alpha_2^2(x) \otimes e_1 \otimes e_1 - \alpha_2^2(x) \otimes e_2 \otimes e_1 - \alpha_2^2(x) \otimes e_1 \otimes e_2 + \alpha_2^2(x) \otimes e_2 \otimes e_2$$
$$= (\Delta_2 \otimes \alpha_2) \circ \Delta_2(x).$$

◇ The condition (5) is easily established.

Hence, $\mathcal{K}_2(\mathcal{A}) := (\tilde{V}, \mu_2, \eta_2, \Delta_2, \varepsilon_2, \alpha_2)$ is a unital hom-bialgebra. □

3.1 Construction of Unital 2-Hom-Associative Bialgebras

Here, we construct $(n + 1)$-dimensional unital 2-hom-associative bialgebras from n-dimensional unital hom-associative algebras.

Lemma 17 *Let $\mathcal{A} = (V, \mu, \eta, \alpha)$ be a unital hom-associative algebra. The unital hom-bialgebra $\mathcal{K}_1(\mathcal{A}) = (\tilde{V}, \mu_1, \eta_1, \Delta_1, \varepsilon_1, \alpha_1)$ is a unital infinitesimal hom-bialgebra.*

Proof We know that $\mathcal{K}_1(\mathcal{A})$ is a unital hom-bialgebra. Then, we only have to show the unital hom-infinitesimal condition. For all $x, y \in V$, we have:

$$(\mu \otimes \alpha) \circ (\alpha \otimes \Delta)(x \otimes y) + (\alpha \otimes \mu) \circ (\Delta \otimes \alpha)(x \otimes y) - \alpha^2(x) \otimes \alpha(y)$$
$$= \mu(\alpha(x), \alpha(y)) \otimes e_1 + e_1 \otimes \mu(\alpha(x), \alpha(y)) - e_2 \otimes \mu(\alpha(x), \alpha(y))$$
$$= \alpha(\mu(x, y)) \otimes e_1 + e_1 \otimes \alpha(\mu(x, y)) - e_2 \otimes \alpha(\mu(x, y)) = \Delta(\mu(x, y)).$$

Hence, $\mathcal{K}_1(\mathcal{A})$ is a unital infinitesimal hom-bialgebra. □

Let us point out the following:

(i) Let $\mathcal{A} = (V, \mu, \eta, \alpha)$ be a unital hom-associative algebra. The unital hom-bialgebra $\mathcal{K}_2(\mathcal{A})$ is not a unital infinitesimal hom-bialgebra since the unital hom-infinitesimal condition is not satisfied.
(ii) Let $\mathcal{A}_2 = (V, \mu_1, \mu_2, \eta, \alpha)$ be a unital 2-hom-associative algebra. Then, we have the same hom-coalgebra structure in the associated hom-bialgebra, (or unital infinitesimal hom-bialgebra), related to unital hom-associative algebras (V, μ_1, η, α) and (V, μ_2, η, α).

Proposition 18 *Let $\mathcal{A} = (V, \mu, \eta, \alpha)$ and $\mathcal{A}' = (V, \mu', \eta, \alpha)$ be two unital hom-associative algebras over an n-dimensional vector space V. Let $\mathcal{K}_1(\mathcal{A}) = (\tilde{V}, \mu_1, \eta_1, \Delta_1, \varepsilon_1, \alpha_1)$ and $\mathcal{K}_1(\mathcal{A}') = (\tilde{V}, \mu_1', \eta_1, \Delta_1, \varepsilon_1, \alpha_1)$ be the above defined associated hom-bialgebras. Then, we have that $\mathcal{B}_1 = (\tilde{V}, \mu_1, \mu_1', \eta_1, \Delta_1, \varepsilon_1, \alpha_1)$ is an $(n+1)$-dimensional unital 2-hom-associative bialgebra over the vector space $\tilde{V} = \mathrm{span}(V, e_1)$, where $\eta_1(1) = e_1$.*

Proof From Lemma 17, $\mathcal{K}_1(\mathcal{A}')$ is a unital infinitesimal hom-bialgebra, and $\mathcal{K}_1(\mathcal{A})$ is a unital hom-bialgebra, and hence $\mathcal{B}_1 = (\tilde{V}, \mu_1, \mu_1', \eta_1, \Delta_1, \varepsilon_1, \alpha_1)$ is a unital 2-hom-associative bialgebra. □

Remark 19 Let $(V, \mu, \eta, \Delta, \varepsilon, \alpha)$ be a unital hom-bialgebra. If the comultiplication satisfies the unital hom-infinitesimal condition, then $(V, \mu, \mu, \eta, \Delta, \varepsilon, \alpha)$ is a unital 2-associative-hom-bialgebra.

3.2 Construction of Unital 2-Hom Bialgebras

Proposition 20 *Let V be an n-dimensional vector space over $\mathcal{K}$. Let $\mathcal{A}_1 = (V, \mu_1, \eta_1, \alpha)$ and $\mathcal{A}_2 = (V, \mu_2, \eta_2, \alpha)$ be two unital hom-associative algebras, and $\mathcal{K}_j(\mathcal{A}_i) = (\tilde{V}, \tilde{\mu}_i, \eta, \Delta_i, \varepsilon, \tilde{\alpha})$, $i, j = 1, 2$, the above defined associated hom-bialgebras. Then,*

$$\mathcal{B}_1 = (\tilde{V}, \tilde{\mu}_1, \tilde{\mu}_2, \eta, \Delta_1, \Delta_2, \varepsilon, \tilde{\alpha}) \text{ and } \mathcal{B}_2 = (\tilde{V}, \tilde{\mu}_1, \tilde{\mu}_2, \eta, \Delta_1^{cop}, \Delta_2, \varepsilon, \tilde{\alpha})$$

are two $(n+1)$-dimensional unital 2-hom bialgebras on $\tilde{V} = \mathrm{span}(V, e_1)$, where $\eta(1) = e_1$

Proof From Proposition 13, we establish, by a straightforward computation, that $\mathcal{B}_1$ and $\mathcal{B}_2$ are unital 2-hom-bialgebras. □

The next corollary gives a unital 2-2-hom-bialgebra from two unital hom-associative algebras.

Corollary 21 *Under the above conditions, $\mathcal{B}_1 = (\tilde{V}, \tilde{\mu}_1, \tilde{\mu}_2, \eta, \Delta_1, \Delta_2, \varepsilon, \tilde{\alpha})$ is an $(n+1)$-dimensional unital 2-2-hom bialgebra on $\tilde{V} = \mathrm{span}(V, e_1)$, where $\eta(1) = e_1$.*

4 Hom-Left Symmetric Dialgebras

4.1 Hom-Associative Dialgebra

Definition 22 We call differential hom-associative algebra the quadruple $(\mathcal{A}, \cdot, \alpha, d)$ such that $(\mathcal{A}, \cdot, \alpha)$ is a hom-associative algebra, $d(a \cdot b) = da \cdot b + a \cdot db, \forall a, b \in \mathcal{A},\ d^2 = 0$, and $d \circ \alpha = \alpha \circ d$.

Let us immediately emphasize that this definition has a quite strong condition. Indeed, using the Leibniz rule twice and noting the nilpotency of d leads to $2d(a)d(b) = 0$, for all $a,\ b \in A$. If the ground field has characteristic different from 2, this puts a restriction on the image of d.

Proposition 23 *Let $(\mathcal{A}, \cdot, \alpha, d)$ be a differential hom-associative algebra. Consider the products $\vdash$ and $\dashv$ on $\mathcal{A}$ given by $x \dashv y = \alpha(x)d\alpha(y)$ and $x \vdash y = \alpha(x)d\alpha(y)$. Then, $(\mathcal{A}, \dashv, \vdash, \alpha)$ is a hom-associative dialgebra.*

Proof By hypothesis, $(\mathcal{A}, \cdot, \alpha, d)$ is a differential hom-associative algebra. Hence we have:

- $\alpha(x) \dashv (y \dashv z) = \alpha(x) \dashv (\alpha(y)d\alpha(z)) = \alpha^2(x)d\alpha(\alpha(y)d\alpha(z))$
  $= \alpha^2(x)d\alpha^2(y)d\alpha^2(z) = \alpha^2(x)d[\alpha(d\alpha(y))\alpha^2(z)] = \alpha^2(x) \dashv \alpha[d\alpha(y)\alpha(z)]$
  $= \alpha(x) \dashv (d\alpha(y)\alpha(z)) = \alpha(x) \dashv (y \vdash z)$;
- $(x \vdash y) \dashv \alpha(z) = (d\alpha(x)\alpha(y)) \dashv \alpha(z) = d\alpha^2(x)\alpha^2(y)d\alpha^2(z)$
  $= \alpha(x) \vdash (\alpha(y)d\alpha(z)) = \alpha(x) \vdash (y \dashv z)$;
- $(x \vdash y) \vdash \alpha(z) = (d\alpha(x)\alpha(y)) \vdash \alpha(z) = d[d\alpha^2(x)\alpha^2(y)]\alpha^2(z)$
  $= d(\alpha^2(x)d\alpha^2(y))\alpha^2(z) = (\alpha(x)d\alpha(y)) \vdash \alpha(z) = (x \dashv y) \vdash \alpha(z)$.

Therefore, $(A, \dashv, \vdash, \alpha)$ is a hom-associative dialgebra. □

Theorem 24 *Let $(D, \dashv, \vdash)$ be an associative dialgebra, and $\alpha : D \to D$ be an associative dialgebra endomorphism. Then $D_\alpha = (D, \dashv_\alpha, \vdash_\alpha, \alpha)$, where $\dashv_\alpha = \alpha \circ \dashv$ and $\vdash_\alpha = \alpha \circ \vdash$, is a hom-associative dialgebra. Moreover, suppose that $(D', \dashv', \vdash')$ is another associative dialgebra, and $\alpha' : D' \to D'$ is an associative dialgebra endomorphism. If $f : D \to D'$ is an associative dialgebra morphism that satisfies $f \circ \alpha = \alpha' \circ f$, then $f : D_\alpha \to D'_{\alpha'}$ is a morphism of hom-associative dialgebras.*

Proof We have:

-

$$\alpha(x) \dashv_\alpha (y \dashv_\alpha z) = \alpha(\alpha(x) \dashv (y \dashv_\alpha z)) = \alpha(\alpha(x) \dashv \alpha(y \dashv z))$$
$$= \alpha^2(x \dashv (y \dashv z)) = \alpha^2(x \dashv (y \vdash z)) = \alpha(\alpha(x) \dashv (\alpha(y \vdash z)))$$
$$= \alpha(\alpha(x) \dashv (y \vdash_\alpha z)) = \alpha(x) \dashv_\alpha (y \vdash_\alpha z);$$

- $$(x \vdash_\alpha y) \dashv_\alpha \alpha(z) = \alpha^2((x \vdash y) \dashv z) = \alpha^2(x \vdash (y \dashv z))$$
  $$= \alpha(x) \vdash_\alpha (y \dashv_\alpha z);$$

- $$(x \vdash_\alpha y) \vdash_\alpha \alpha(z) = \alpha^2((x \vdash y) \vdash z) = \alpha^2((x \dashv y) \vdash z)$$
  $$= (x \dashv_\alpha y) \vdash_\alpha \alpha(z);$$

- $$f \circ \dashv_\alpha = f \circ (\alpha \circ \dashv) = (f \circ \alpha) \circ \dashv = (\alpha' \circ f) \circ \dashv = \alpha' \circ (f \circ \dashv)$$
  $$= \alpha' \circ (\dashv' \circ (f \otimes f)) = (\alpha' \circ \dashv') \circ (f \otimes f) = \dashv'_{\alpha'} \circ (f \otimes f),$$
  and we also obtain that $f \circ \vdash_\alpha = \vdash'_{\alpha'} \circ (f \otimes f)$.

Hence, we can conclude that D_α is a hom-associative dialgebra, and f a morphism of hom-associative dialgebras. □

4.2 Hom-Leibniz Algebra

Definition 25 A hom-Leibniz algebra is a triple $(L, [.,.] \alpha)$ consisting of a linear space L, a bilinear product $[.,.] : L \times L \to L$, and a homomorphism $\alpha : L \to L$ satisfying

$$[[x, y], \alpha(z)] = [[x, z], \alpha(y)] + [\alpha(x), [y, z]]. \tag{7}$$

Proposition 26 *Let $(\mathcal{A}, \cdot, \alpha, d)$ be a differential hom-associative algebra. Define the bracket on $\mathcal{A}$ by*

$$[x, y] := \alpha(x) \cdot d\alpha(y) - d\alpha(y) \cdot \alpha(x).$$

Then, the vector space $\mathcal{A}$ equipped with this bracket is a hom-Leibniz algebra.

Proof By direct computation, we obtain:

- $[\alpha(x), [y, z]] = \alpha(x)d\alpha(y)d\alpha(z) - d\alpha(y)d\alpha(z)\alpha(x) - \alpha(x)d\alpha(z)d\alpha(y)$ $+d\alpha(z)d\alpha(y)\alpha(x);$
- $[[x, y], \alpha(z)] = \alpha(x)d\alpha(y)d\alpha(z) - d\alpha(z)\alpha(x)d\alpha(y) - d\alpha(y)\alpha(x)d\alpha(z)$ $+d\alpha(z)d\alpha(y)\alpha(x);$
- $[[x, z], \alpha(y)] = \alpha(x)d\alpha(z)d\alpha(y) - d\alpha(y)\alpha(x)d\alpha(z) - d\alpha(z)\alpha(x)d\alpha(y)$ $+d\alpha(y)d\alpha(z)\alpha(x).$

Then, $[\alpha(x), [y, z]] = [[x, y], \alpha(z)] - [[x, z], \alpha(y)]$. Hence, the pair $(A, [., .])$ is a hom-Leibniz algebra. □

Theorem 27 *Let $(L, [., .])$ be a Leibniz algebra, and $\alpha : L \to L$ be a Leibniz algebra endomorphism. Then, $L_\alpha = (L, [., .]_\alpha, \alpha)$ is a hom-Leibniz algebra. Moreover, suppose that $(L', [., .]')$ is another Leibniz algebra, and $\alpha' : L' \to L'$ a Leibniz algebra endomorphism. If $f : L \to L'$ is a Leibniz algebra morphism satisfying $f \circ \alpha = \alpha' \circ f$, then $L_\alpha \to L'_{\alpha'}$ is a morphism of Leibniz algebras.*

Proof Since

- $[[x, y]_\alpha, \alpha(z)]_\alpha = \alpha([\alpha([x, y]), \alpha(z)]) = \alpha^2([[x, y], z])$
  $= \alpha^2([[x, z], y] + [x, [y, z]]) = [[x, z]_\alpha, \alpha(y)]_\alpha + [\alpha(x), [y, z]_\alpha]_\alpha$,
  and
- $f \circ [., .]_\alpha = f \circ (\alpha \circ [., .]) = (f \circ \alpha) \circ [., .] = (\alpha' \circ f) \circ [., .] = \alpha' \circ (f \circ [., .])$
  $= \alpha' \circ ([., .]' \circ (f \otimes f)) = (\alpha' \circ [., .]') \circ (f \otimes f) = [., .]'_{\alpha'} \circ (f \otimes f)$.

Therefore, we have the results. □

Theorem 28 *Let $(D, \dashv, \vdash, \alpha)$ be a hom-associative dialgebra. Consider a linear map $[., .] : D \otimes D \to D$ defined, for $x, y \in D$, by*

$$[x, y] = x \dashv y - x \vdash y.$$

Then, $(D, [., .], \alpha)$ is a hom-Leibniz algebra.

Proof $(D, \dashv, \vdash, \alpha)$ is a hom-associative dialgebra, then, we have:

$$\alpha(y) \vdash (z \vdash x) = (y \vdash z) \vdash \alpha(x) = (y \dashv z) \vdash \alpha(x);$$

$$(x \dashv z) \dashv \alpha(y) = \alpha(x) \dashv (z \dashv y) = \alpha(x) \dashv (z \vdash y).$$

Therefore, by direct computation, we obtain the identity (7). □

4.3 Hom-Left Symmetric Dialgebras

Now, we generalize the notion of left symmetric dialgebra introduced by R. Felipe, twisting the identities by a linear map, as well as some theorems established in [5].

Definition 29 Let S be a vector space over a field K. Let us assume that S is equipped with two bilinear products $\dashv, \vdash$: $S \otimes S \to S$, and a homomorphism $\alpha : S \to S$ satisfying the identities:

$$\alpha(x) \dashv (y \dashv z) = \alpha(x) \dashv (y \vdash z), \tag{8}$$

$$(x \vdash y) \vdash \alpha(z) = (x \dashv y) \vdash \alpha(z), \tag{9}$$

$$\alpha(x) \dashv (y \dashv z) - (x \dashv y) \dashv \alpha(z) = \alpha(y) \vdash (x \dashv z) - (y \vdash x) \dashv \alpha(z), \quad (10)$$

$$\alpha(x) \vdash (y \vdash z) - (x \vdash y) \vdash \alpha(z) = \alpha(y) \vdash (x \vdash z) - (y \vdash x) \vdash \alpha(z). \quad (11)$$

Then, we say that S is a hom-left symmetric dialgebra (HLSDA), or left disymmetric hom-algebra.

Example Any hom-associative algebra $(\mathcal{A}, \cdot, \alpha)$ is a hom-left symmetric dialgebra with $\vdash = \dashv = \cdot$.

The definition of a hom-left symmetric dialgebra morphism is similar to that of a hom-associative dialgebra morphism. Note also that we can construct a hom-left symmetric dialgebra by the composition method from a classical left-symmetric dialgebra $(D, \dashv, \vdash)$ and an algebra endomorphism α, by considering $(D, \dashv_\alpha, \vdash_\alpha, \alpha)$, where $x \dashv_\alpha y = \alpha(x \dashv y)$ and $x \vdash_\alpha y = \alpha(x \vdash y)$. Let us denote by $\mathcal{HS}$ the set of all hom-left symmetric dialgebras, and $\mathcal{HD}$ the set of all hom-associative dialgebras.

Proposition 30 *Any hom-associative dialgebra is a hom-left symmetric dialgebra. Then,* $\mathcal{HD} \subseteq \mathcal{HS}$.

Proof Let $(D, \dashv, \vdash, \alpha)$ be a hom-associative dialgebra. Then, Eqs. (8) and (9) are satisfied. Since the products $\dashv$ and $\vdash$ are associative, then Eqs. (10) and (11) are established. □

Remark 31 Any hom-left symmetric algebra is a hom-left symmetric dialgebra in which $\vdash = \dashv$. A non associative hom-left symmetric algebra is not a hom-left symmetric dialgebra. Hence, we have $\mathcal{HD} \neq \mathcal{HS}$.

Proposition 32 *A hom-left symmetric dialgebra S is a hom-associative dialgebra if and only if both products of S are hom-associative.*

Proof Let $(S, \dashv, \vdash, \alpha)$ be a hom-left symmetric dialgebra. If S is a hom-associative dialgebra, then the products $\dashv$ and $\vdash$ defined on S are hom-associative. Conversely, suppose that the products $\dashv$ and $\vdash$ are hom-associative. Since S has a hom-left symmetric dialgebra structure, then, from Eq. (10), S is a hom-associative dialgebra. □

Theorem 33 *Let* $(S, \vdash, \dashv, \alpha)$ *be a hom-left symmetric dialgebra. Then, the commutator given by* $[x, y] = x \dashv y - y \vdash x$ *defines a structure of hom-Leibniz algebra on* S*. In other words,* $(S, [.,.], \alpha)$ *is a hom-Leibniz algebra.*

Proof We have:

- $[[x, y], \alpha(z)] = (x \dashv y) \dashv \alpha(z) - \alpha(z) \vdash (x \dashv y) - (y \vdash x) \dashv \alpha(z)$
  $+\alpha(z) \vdash (y \vdash x)$;
- $[[x, z], \alpha(y)] = (x \dashv z) \dashv \alpha(y) - \alpha(y) \vdash (x \dashv z) + (z \vdash x) \dashv \alpha(y)$
  $-\alpha(y) \vdash (z \vdash x)$;
- $[\alpha(x), [y, z]] = \alpha(x) \dashv (y \dashv z) - (y \dashv z) \vdash \alpha(x) - \alpha(x) \dashv (z \vdash y)$
  $+(z \vdash y) \vdash \alpha(x)$.

From Eqs. (10) and (11), we obtain the condition (7). □

Definition 34 Let $(L, [.,.], \alpha)$ be a hom-Leibniz algebra. The pair of bilinear mappings $\triangledown_1, \triangledown_2 : L \times L \to L$ is called an affine hom-Leibniz structure obeying the relations:

$$\triangledown_2(x, y) - \triangledown_1(y, x) = [x, y], \tag{12}$$

$$\begin{aligned} \triangledown_1(\triangledown_1(x, y), \alpha(z)) &= \triangledown_1(\triangledown_2(x, y), \alpha(z)); \\ \triangledown_2(\alpha(x), \triangledown_2(y, z)) &= \triangledown_2(\alpha(x), \triangledown_1(y, z)) \end{aligned} \tag{13}$$

$$\triangledown_2(\alpha(x), \triangledown_2(y; z)) - \triangledown_1(\alpha(y), \triangledown_2(x, z)) = \triangledown_2([x, y], \alpha(z)) \tag{14}$$

and

$$\triangledown_1(\alpha(x), \triangledown_1(y, z)) - \triangledown_1(\alpha(y), \triangledown_1(x, z)) = \triangledown_1([x, y], \alpha(z)) \tag{15}$$

for all $x, y, z \in L$.

Theorem 35 *Let $(L, [.,.], \alpha)$ be a hom-Leibniz algebra, and let $\triangledown_1, \triangledown_2$ define an affine hom-Leibniz structure. Then, L is a hom-left symmetric dialgebra with $\vdash$ and $\dashv$ defined as*

$$x \vdash y = \triangledown_1(x, y); \; x \dashv y = \triangledown_2(x, y). \tag{16}$$

Proof (13) implies (8) and (9). Then, (10) and (11) follow from (14) and (15), respectively. □

5 Concluding Remarks

In this work, from the hom-counital and unital infinitesimal hom-bialgebra conditions, and following Kaplansky's construction based on unital hom-associative algebras, we have built unital 2-hom-associative bialgebras, unital 2-hom-bialgebras, and unital 2-2-hom-bialgebras, and derived their main relevant properties. Finally, we have defined and characterized the hom-associative dialgebras, hom-Leibniz algebra and hom-left symmetric dialgebras generalizing the ordinary left symmetric dialgebras. The study of relevant properties of unital 2-hom-associative bialgebras, unital 2-hom-bialgebras, and unital 2-2-hom-bialgebras will be in the core of our forthcoming works.

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Laguerre–Gaussian Wave Propagation in Parabolic Media

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Abstract We report a new set of Laguerre–Gaussian wave-packets that propagate with periodical self-focusing and finite beam width in weakly guiding inhomogeneous media. These wave-packets are solutions to the paraxial form of the wave equation for a medium with parabolic refractive index. The beam width is defined as a solution of the Ermakov equation associated to the harmonic oscillator, so its amplitude is modulated by the strength of the medium inhomogeneity. The conventional Laguerre–Gaussian modes, available for homogeneous media, are recovered as a particular case.

Keywords Paraxial wave equation · Nonlinear Ermakov equation · Angular momentum of light · Self-focusing of light · Laguerre–Gaussian modes

Mathematics Subject Classification (2010) Primary 35Q60, 78A60; Secondary 35Q40, 81V80

1 Introduction

The study of optical beams having complex structures is a subject of intense activity in current times, mainly because the properties of structured light open new possibilities for the manipulation of individual atoms and small molecules [11, 34].

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This subject represents a feedback pathway between theory and experiment: theoretical advances suggest new experiments while significant experimental results require either new theoretical models or improvements in our understanding of the behavior of light. Remarkably, after realizing that azimuthally phased beams carry angular momentum [1], it was understood that the concept of photon angular momentum is not limited to spin [2], but it may include either extrinsic or intrinsic orbital angular momentum [6] (for a recent discussion on the matter see e.g. [3]). However, it is important to emphasize that, although spin and orbital angular momentum behave quite similar in some instances, "orbital angular momentum has its own distinctive properties and its own distinctive optical components" [25]. Such a subtlety is fundamental in the investigation of light-matter interactions [9, 15, 20]. In this context it is notable that the wavefront structure of the Laguerre–Gaussian beams allows the production of force fields that have no counterpart in conventional optical beams [8, 14]. From the practical point of view, it has been found that Hermite–Gaussian beams with no orbital angular momentum can be transformed into Laguerre–Gaussian beams carrying orbital angular momentum [4, 5, 12]. Thus, one can use either cylindrical lenses [5] or Fork diffractive gratings [4, 12] to produce Laguerre–Gaussian beams in the laboratory (sophisticated spatial light modulators can be used instead). Nevertheless, the propagation of Laguerre–Gaussian beams in free (homogeneous) space implies that the corresponding beam width diverges as the propagation variable increases, which may tie down the usefulness of such beams.

In this paper we address the problem of finding Laguerre-Gauss wave-packets with finite beam width along all the propagation axis. With this aim we solve the paraxial form of the wave equation for a weakly guiding inhomogeneous medium, the refractive index of which is quadratic (parabolic) in the coordinates transverse to the propagation. Our method is based on the approach introduced in [10], where a Gaussian wave-packet is used to solve the Schrödinger equation for time-dependent and nonlinear Hamiltonian operators via complex Riccati equations. The main point in [10] is that the width of the packet is defined as a solution of the Ermakov equation [16] associated to the one-dimensional oscillator. Such approach was already applied to study the propagation of waves in non-homogeneous media [13, 18, 19], were the close relationship between the paraxial wave equation and the Schrödinger equation is successfully exploited to construct Hermite–Gaussian wave-packets for quadratic refractive index optical media. In the present case the beam width is an oscillatory solution of the Ermakov equation that depends on the propagation variable and such that its amplitude is modulated by the strength of the medium inhomogeneity. The Laguerre–Gaussian wave-packets reported here correspond to non-dispersive beams of finite transverse optical power (localized beams) that propagate with periodical self-focusing profile in the medium. Our approach generalizes the methods to define finite beam widths already reported by other authors [7, 22–24, 26, 32], and confirm that the distinctive angular momentum properties of the Laguerre–Gaussian modes are better prepared and exploited if the related beam propagates in parabolic media.

The generalities to construct the above described Laguerre–Gaussian wave-packets are outlined in Sect. 2, where it is also shown that the conventional Laguerre–Gaussian modes arise after turning-off the inhomogeneity of the medium, just as a particular case. In Sect. 3 we summarize our results and provide some directions for future work.

2 Paraxial Wave Equation for Parabolic Media

Consider the z-propagation of waves through a weakly guiding inhomogeneous medium, the refractive index of which is quadratic in the transverse coordinates (the xy-plane). Using polar coordinates to write the position vector transverse to beam propagation as $\boldsymbol{\rho} = (\rho, \theta)$, the refractive index we are dealing with is of the form

$$n^2(\rho) = n_0^2 \left(1 - \Omega^2 \rho^2\right), \quad \Omega^2 \rho^2 \ll 1. \tag{1}$$

Here, n_0 stands for the refractive index at the optical axis and $\Omega \geq 0$ is a parameter that characterizes the focusing properties of the medium. The corresponding paraxial wave equation is given by

$$-\frac{1}{2k_0^2 n_0} \nabla_\perp^2 U + \frac{n_0}{2} \Omega^2 \rho^2 U = \frac{i}{k_0} \frac{\partial U}{\partial z}. \tag{2}$$

The solutions $U = U(\boldsymbol{\rho}, z)$ of (2) describe the transversal amplitude of the electric field in the medium. Hereafter $\nabla_\perp^2$ stands for the transversal component of the Laplacian operator and k_0 is the wave number in free space.

2.1 *Lowest-Order Gaussian Mode*

Following [10, 13, 18], as a fundamental non stationary solution of the paraxial wave equation (2), we propose the Gaussian wave-packet

$$U(\rho, z) = N(z) e^{iS(z)\rho^2}. \tag{3}$$

The straightforward calculation shows that the normalization factor $N(z)$ and the coefficient $S(z)$ are respectively given by

$$N(z) = \frac{N_0}{w(z)} e^{-i\chi(z)}, \quad S(z) = \frac{k_0 n_0}{2} \frac{d}{dz} \ln w(z) + \frac{i}{w^2(z)}, \tag{4}$$

where N_0 is a normalization constant, $w(z)$ is a solution of the Ermakov equation for the one-dimensional harmonic oscillator

$$\frac{d^2w}{dz^2} + \Omega^2 w = \frac{4}{k_0^2 n_0^2 w^3}, \tag{5}$$

and

$$\chi(z) = \frac{2}{k_0 n_0} \int^z \frac{1}{w^2(x)} dx. \tag{6}$$

It is a matter of substitution to verify that after the identification

$$\frac{1}{R(z)} = \frac{d}{dz} \ln w(z), \tag{7}$$

we can rewrite Eq. (3) in the 'canonical' form of the lowest-order Gaussian mode

$$U(\rho, z) = \frac{N_0}{w(z)} \exp\left(\frac{-\rho^2}{w^2(z)}\right) \exp\left(i\left[\frac{k_0 n_0 \rho^2}{2R(z)} - \chi(z)\right]\right). \tag{8}$$

In analogy with the homogeneous case [30, 31], we see that N_0 refers to the maximum electric field strength, $w(z)$ corresponds to the beam width, $R(z)$ to the radius of curvature, and $\chi(z)$ to the Gouy phase. Notice however that the main difference between the wave-packet (8) and the Gaussian mode of the homogeneous case [30, 31] is the z-dependence of the beam width. Indeed, we have already mentioned that $w(z)$ in (8) is a solution of the Ermakov equation (5), so we follow [29] to get

$$w(z) = w_0 \left[\cos^2[\Omega(z - z_0)] + \frac{1}{(\Omega z_R)^2} \sin^2[\Omega(z - z_0)]\right]^{1/2}, \tag{9}$$

with z_0 an integration constant, $w_0 = w(z_0)$, and $z_R = \frac{1}{2} k_0 n_0 w_0^2$.

For $\Omega > 0$ the amplitude of the beam width (9) oscillates with period π/Ω between w_0 and $w_0/(\Omega z_R)$. The latter values are respectively reached at the points $z = \left(n + \frac{1}{2}\right)\frac{\pi}{\Omega} + z_0$ and $z = n\frac{\pi}{\Omega} + z_0$, with $n = 0, 1, 2, \ldots$ Thus, in contraposition to the homogeneous case where $w(z)$ diverges for large values of z (see Sect. 2.1.1), the beam width introduced in (9) is finite over all the z-axis. Moreover, the maximum amplitude reached by $w(z)$ can be adjusted by varying Ω, see Fig. 1a.

Now let us write explicitly the expression for the radius of curvature. From (7) and (9), one obtains

$$R(z) = \frac{z_R^2 \Omega^2 \cot[\Omega(z - z_0)] + \tan[\Omega(z - z_0)]}{\Omega(1 - \Omega^2 z_R^2)}. \tag{10}$$

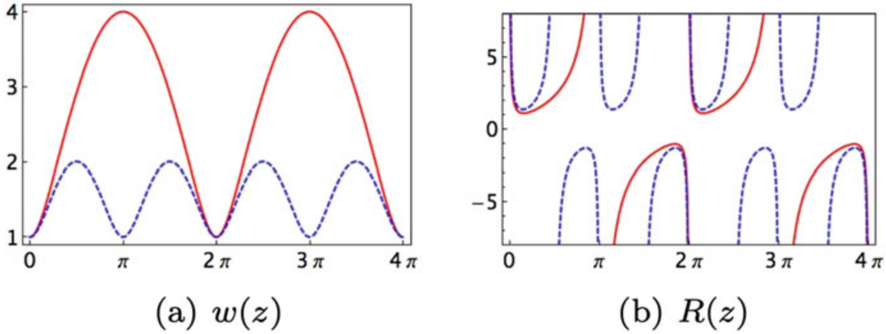

Fig. 1 The beam width defined in (9) and the corresponding radius of curvature (10) for parabolic media, figures (**a**) and (**b**) respectively. In both cases $z_0 = 0$ and $w_0 = 1$, with $\Omega = 0.5$ (red curve) and $\Omega = 1$ (blue-dashed curve). The divergences of $R(z)$ identify the critical values of $w(z)$, and correspond to plane wavefronts

The latter expression diverges at the critical points of $w(z)$, see Fig. 1b. Thus, the wavefront of the Gaussian wave-packet (8) is plane at either the beam waist (minimum beam width) or the beam 'hip' (maximum beam width).

On the other hand, the oscillatory profile of the beam width (9) is inherited to the wave-packet (8). Indeed, the field intensity $|U(\rho, z)|^2$ describes a spot centered at $\rho = 0$ that increases its diameter till a maximum value as the wave-packet propagates along the z-axis. Then, the spot starts to reduce its diameter up to recover its initial configuration. This stretching and squeezing phenomenon is repeated over and over along the propagation axis, see Fig. 2. Such behavior corresponds to the self-focusing of the beam, which is a consequence of the parabolic profile of the refractive index (1). The number of stretching and squeezing of the beam width in a given interval $z \in (a, b) \subset \mathbb{R}$ is determined by the parameter Ω, in complete agreement with the periodical profile of the beam width (9).

2.1.1 Recovering the Results for Homogeneous Media

The description of the propagation in homogeneous media is obtained at the limit $\Omega \to 0^+$. For instance, at such a limit the beam width acquires the well known form [30, 31]:

$$w(z)_{\Omega\to 0^+} = w_{\text{hom}}(z) = w_0\sqrt{1 + \left(\frac{z - z_0}{z_R}\right)^2}. \tag{11}$$

In this case $w_0 = w_{\text{hom}}(z_0)$ corresponds to the beam waist. In turn, $z = z_0$ defines the focal plane and z_R stands for the distance from such plane to the position at which the spot of the beam doubles its size. On the other hand, for large values

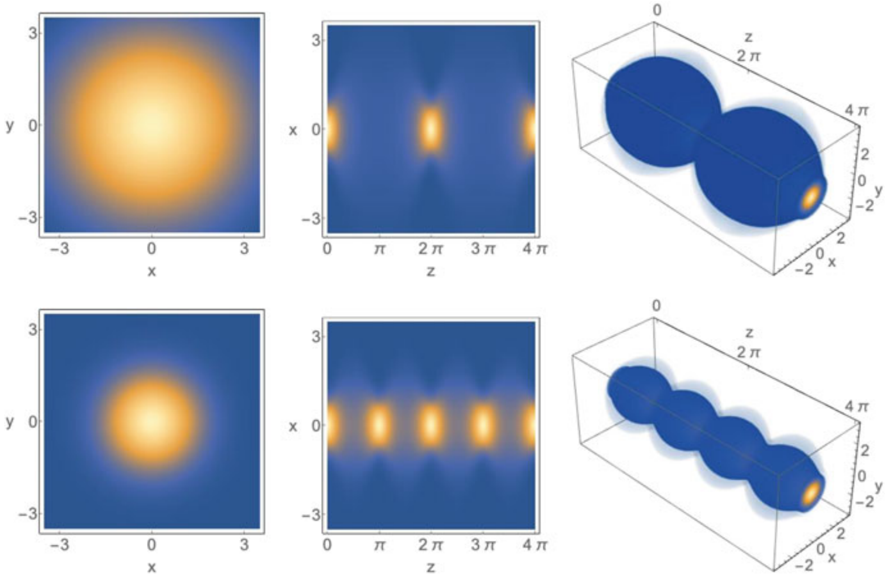

Fig. 2 Field intensity $|U(\rho, z)|^2$ of the Gaussian wave-packet (8) for $z_0 = 0$ and $k_0 = n_0 = w_0 = 1$. The upper and lower rows correspond to $\Omega = 0.5$ and $\Omega = 1$, respectively. From left-to-right the columns show the transversal plane where the beam width reaches the first of its maxima, the longitudinal plane $y = 0$, and the propagation of the wave-packet along the z-axis. Compare with Fig. 1

of z, Eq. (11) can be approximated as $w_{\text{hom}}(z) \sim \frac{w_0}{z_R}(z - z_0)$. Thus, the beam in homogeneous media diverges as $|z| \to \infty$, see Fig. 3a, and describes a cone of half-angle defined by the beam angular divergency $\theta_0 \sim \frac{w_0}{z_R}$ (it is subtended by the gray-dashed line with respect to the horizontal axis in the figure). The corresponding radius of curvature $R_{\text{hom}}(z)$ is depicted in Fig. 3b, where it is compared with the radius of curvature of a spherical wavefront produced by a point source located at the center of the beam waist (gray-dotted line in the figure), and with the function $R(z)$ defined in (10) for $\Omega = 0.3$ (magenta curve in the figure). Notice that $R_{\text{hom}}(z)$ has only a singular point, located at $\rho = 0$, which means that the wavefront of the corresponding wave-packet is plane at the beam waist only, so no self-focusing is predicted for homogeneous media, as expected.

2.2 Laguerre–Gaussian Wave-Packets

We wonder whether there are other wave-packet solutions to the paraxial wave equation associated with a parabolic refractive index. The answer is positive (see, e.g. [26] and [13]) and it may be shown that, using cylindrical coordinates, a

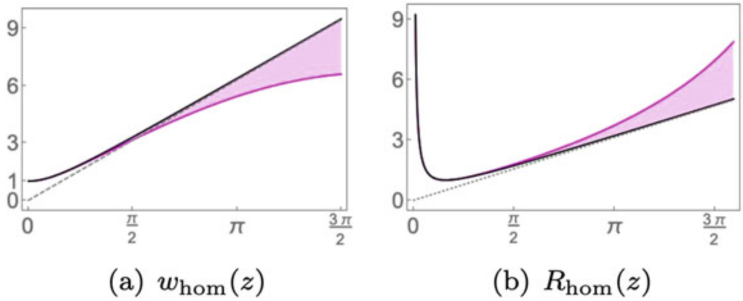

Fig. 3 The beam width $w_{\text{hom}}(z)$ defined in (11) and the corresponding radius of curvature $R_{\text{hom}}(z)$ for homogeneous media, Figures (a) and (b) respectively. In both cases $z_0 = 0$ and $w_0 = 1$. The gray-dashed line in (a) identifies the beam angular divergency θ_0. The gray-dotted line in (b) represents the radius of curvature of a spherical wavefront. The magenta curves respectively correspond to $w(z)$, defined in (9), and to $R(z)$, defined in (10), for $\Omega = 0.3$. Both of them have been included as a reference. The space between $w_{\text{hom}}(z)$ and $w(z)$, and the one between $R_{\text{hom}}(z)$ and $R(z)$, has been filled by the sake of comparison. See also Fig. 1

particularly useful set of solutions can be cast in the form

$$U(\boldsymbol{\rho}, z) = \frac{\mathcal{N}}{w(z)} \exp\left(i\left[\frac{k_0 n_0 \rho^2}{2R(z)} - \beta\chi(z)\right]\right) \Phi(r(\rho, z), \theta), \tag{12}$$

where $\mathcal{N}$ and β are constants to be determined, θ stands for the polar coordinate in the transversal plane, and $r(\rho, z) = \sqrt{2}\frac{\rho}{w(z)}$. The straightforward calculation shows that the function $\Phi(r, \theta)$ satisfies the differential equation

$$-\widetilde{\nabla}_\perp^2 \Phi + r^2\Phi = 2\beta\Phi, \tag{13}$$

where $\widetilde{\nabla}_\perp^2$ is the Laplacian in the (r, θ) plane. Equation (13) resembles the stationary Schrödinger equation for a two dimensional oscillator in the radial variable r. The resemblance is complete if one considers that square integrability in the Hilbert space corresponds to finite transverse optical power P_0 for localized beams in the (r, θ)-plane. That is, we demand the field intensity of the wave-packets (12) to satisfy the condition

$$P_0 = \int_0^{2\pi}\int_0^{\infty} \left|U_\ell^p(\boldsymbol{\rho}, z)\right|^2 \rho d\rho d\theta = 1.$$

The conventional approach used to face stationary problems in quantum mechanics yields the eigenvalues

$$\beta_\ell^p = |\ell| + 2p + 1, \quad \ell \in \mathbb{Z}, \quad p = 0, 1, 2, \ldots, \tag{14}$$

together with the eigenfunctions

$$\Phi_\ell^p(r,\theta) = \left(\frac{\sqrt{2}\rho}{w(z)}\right)^{|\ell|} L_p^{(|\ell|)}\left(\frac{2\rho^2}{w^2(z)}\right)\exp\left(-\frac{\rho^2}{w^2(z)} + i\ell\theta\right), \tag{15}$$

and the constant

$$\mathcal{N}_\ell^p = (-1)^p\sqrt{\frac{2\Gamma(p+1)}{\pi\Gamma(|\ell|+p+1)}}. \tag{16}$$

The introduction of (14)–(16) into (12) gives the expressions $U_\ell^p(\boldsymbol{\rho},z)$ we were looking for. Hereafter they are referred to as Laguerre–Gaussian (LG) wave-packets.

The results discussed in Sect. 2.1 for the non stationary Gaussian wave-packets are recovered from the above formulae after making $p = 0$ and $\ell = 0$, which respectively means the lowest radial parameter and null orbital angular momentum. For $p = 0$ and $\ell \neq 0$ the field intensity $|U_\ell^p(\boldsymbol{\rho},z)|^2$ describes a ring centered at $\rho = 0$, see Fig. 4. In general, for $\ell \neq 0$ and any allowed value of p, the field intensity of the LG wave-packets (12) exhibits a well known ring-shaped distribution. The size of the rings depends on ℓ while p defines the number of nodes in the radial coordinate, see Fig. 5 and compare with Figs. 2 and 4. Thus, controlling the values

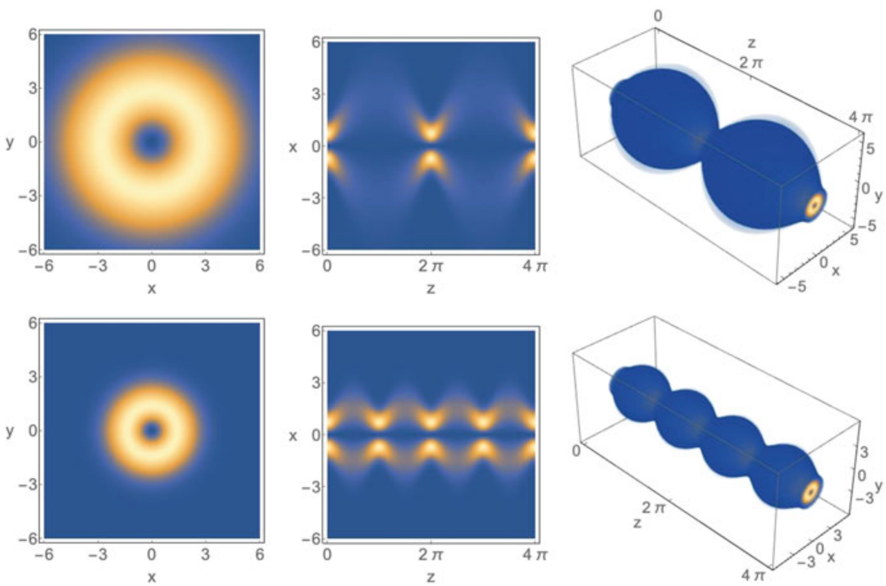

Fig. 4 Field intensity $|U_\ell^p(\boldsymbol{\rho},z)|^2$ of the Laguerre–Gaussian wave-packet (12) for $p = 0$ and $\ell = 1$, with the same parameters and distribution as the panel shown in Fig. 2

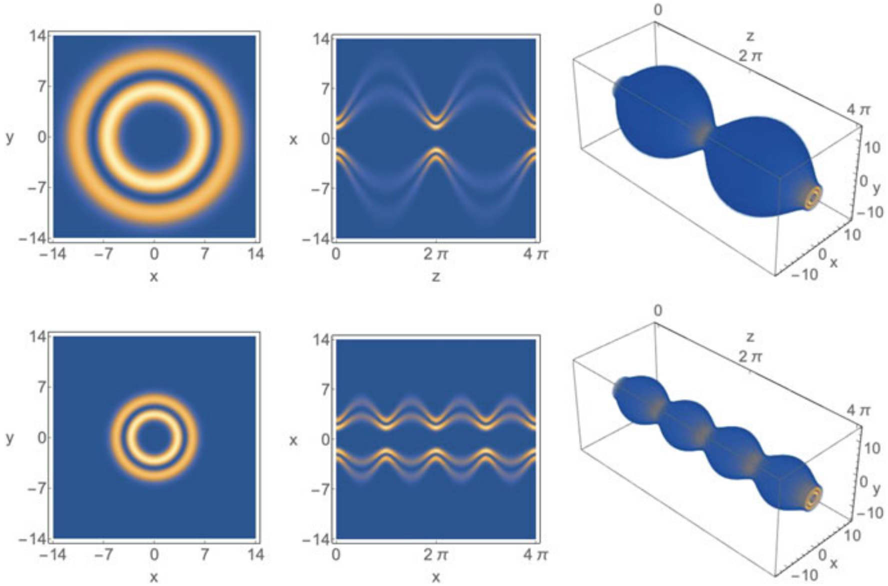

Fig. 5 Field intensity $|U_\ell^p(\boldsymbol{\rho}, z)|^2$ of the Laguerre–Gaussian wave-packet (12) for $p = 1$ and $\ell = 8$, with the same parameters and distribution as the panel shown in Fig. 2

of p, ℓ, and Ω, one can manipulate the profile as well as the collapse-revival (self-focusing) properties of the LG wave-packets (12).

2.2.1 Recovering the Results for Guided Laguerre–Gaussian Modes

If the parameters w_0 and Ω are related as $k_0 n_0 \Omega = 2/w_0^2$, then $\Omega z_R = 1$ and, according to (9), the beam width becomes a constant $w(z) = w_0$, see [19]. In such a case the radius of curvature $R(z)$ diverges and the Gouy phase turns into a linear function of z. Thus, the wavefront of the LG wave-packets is plane. The corresponding field amplitudes, known as guided LG modes, take the form

$$U_\ell^p(\boldsymbol{\rho}, z) = (-1)^p \sqrt{\frac{2\Gamma(p+1)}{\pi w_0^2 \Gamma(|\ell| + p + 1)}} \left(\frac{\sqrt{2}\rho}{w_0}\right)^{|\ell|} L_p^{(|\ell|)}\left(\frac{2\rho^2}{w_0^2}\right) \times \exp\left(-\frac{\rho^2}{w_0^2} - i\beta_\ell^p \Omega (z - z_0) + i\ell\theta\right), \tag{17}$$

and are stationary eigenmodes of the Schrödinger-like operator

$$H = -\frac{1}{2k_0^2 n_0}\nabla_\perp^2 + \frac{n_0}{2}\Omega^2\rho^2, \tag{18}$$

with the propagation constants

$$\varepsilon_\ell^p = \frac{\Omega}{k_0}\beta_\ell^p \equiv \frac{\Omega}{k_0}(|\ell| + 2p + 1), \quad \ell \in \mathbb{Z}, \quad p = 0, 1, 2, \ldots \tag{19}$$

3 Discussion of Results

We have shown that the beam width of the Laguerre–Gaussian wave-packets is finite and periodic along all the propagation axis if it is a solution of the Ermakov equation associated with the one-dimensional harmonic oscillator. The amplitude of the beam width can be modulated by the strength of the medium inhomogeneity. The wave-packets so constructed have finite transverse optical power and propagate with periodical self-focusing in the medium. The conventional Laguerre–Gaussian modes are recovered as a particular case, after turning-off the inhomogeneity.

Since the orbital angular momentum is a consequence of the mode structure of a given beam, one would guess that such a property is also present in single photons [3]. The statement seems to be strengthened by recalling that idealized plane waves with only transverse fields do not carry angular momentum, no matter their degree of polarization [21]. A clue may be found in the phenomenon of parametric-down conversion [27, 28], where the quantum state of down-converted photon pairs is entangled in at least one of their physical variables. As entanglement is a fingerprint of the quantum world, the production of photon pairs entangled in their orbital angular momentum states [17] bets on a quantum nature of structured light. It is then interesting to formulate an approach based on operators in Hilbert spaces to describe the propagation of structured light beams in diverse media. Some initial steps on the matter were given in [33]. The complete operator-like description for the propagation of Hermite–Gaussian wave-packets in parabolic media has been provided in [13, 18, 19], where the Lie group formalism was addressed to construct generalized coherent states as linear superpositions of Hermite–Gaussian modes. The same approach can be applied to the Laguerre–Gaussian wave-packets reported in this paper. Work in this direction is in progress.

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Maximal Surfaces on Two-Step Sub-Lorentzian Structures

Maria Karmanova

Abstract We describe sufficient maximality conditions for the classes of graph surfaces on two-step Carnot groups with sub-Lorentzian structure. In particular, we introduce a non-holonomic notion of variation of the area functional.

Keywords Two-step Carnot group · Contact mapping · Intrinsic measure · Area formula · Maximal surface · Sufficient maximality condition

Mathematics Subject Classification (2010) Primary 58C35; Secondary 49Q20

1 Introduction

The aim of this paper is to describe the classes of maximal graph surfaces in sub-Lorentzian geometry, namely, sufficient maximality conditions. The graph mappings are constructed from mappings of two-step nilpotent graded groups. These groups are a particular case of Carnot–Carathéodory spaces well-known in various problems of pure and applied mathematics; see, e. g., [15] and references therein. We also assume that the image and preimage are both subsets of another nilpotent graded group possessing a sub-Lorentzian structure. This structure is a sub-Riemannian generalization of Minkowski geometry. The main characteristic of this geometry is that the distance between points (x_1, t_1) and (x_2, t_2), with $x_1, x_2 \in \mathbb{R}^n$ and $t_1, t_2 \in \mathbb{R}$, equals

$$\sqrt{(x_1 - x_2)^2 - (t_1 - t_2)^2},$$

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i.e., the squared distance along the time-like direction t is negative, while along every space-like direction $x \in \mathbb{R}^n$ it is positive. If all tangent vectors to a surface in $\mathbb{R}^{n+1}_1$ have only positive lengths then this surface is called space-like and it is locally representable as a graph, where the time-like variable depends on the space-like variables: $t = \psi(x)$ with $x \in \mathbb{R}^n$. Under some additional assumptions it is possible to deduce certain equations describing surfaces of maximal area; it follows that their mean curvature vanishes a. e. According to Nielsen's hypothesis, solutions to Einstein's gravity equations are physically meaningful if and only if they are representable as such surfaces in $\mathbb{R}^{n+1}_1$. For the details concerning the properties, applications and interpretations of Minkowski geometry, see [19] and references: e.g., [20, 21] etc.

Sub-Lorentzian geometry is a relatively young branch of analysis; the first results in this area were obtained in the 1990s; see [2]. Later, series of papers studied some fine properties of geodesics together with their connection to relativity theory problems; see, e. g., [5, 6, 16–18]. New cases of Minkowski geometry with multi-dimensional time were studied recently in [1, 3] etc.

In [9], the author deduced necessary maximality conditions for classes of graph surfaces and, moreover, the equations of maximal surfaces. Here the term "maximal surface" means a surface of maximal area (under the assumption that a solution to the corresponding boundary value problem exists). We emphasize that [9], in view of certain fine properties of non-holonomic geometry, the definition of argument increment of the area functional differs substantially from the classical one. Namely, if the horizontal part of the argument changes arbitrarily to order ε then the other part of the formula that corresponds to degree two fields depending on the horizontal ones involves additional summands of order ε^2. Consequently, when we take the second differential of the area functional to obtain sufficient maximality conditions, some new summands appear, which are absent in Riemannian geometry. Recall that generally in non-holonomic structures the notions "maximal area" and "maximal value of the area functional" are not the same. In the latter case, the functional can take some maximal value but it need not correspond to any mapping defining a surface of this area since the PDE problem may lack solutions.

The result of this paper was announced in [12].

2 Graphs on Carnot Groups

Let us recall necessary notions and results.

Definition 1 (See, e. g., [4]) A *two-step Carnot group* is a connected simply connected stratified Lie group $\mathbb{G}$ with a graded Lie algebra V, that is, $V = V_1 \oplus V_2$ with $[V_1, V_1] = V_2$ and $[V_1, V_2] = \{0\}$. If we replace $[V_1, V_1] = V_2$ by $[V_1, V_1] \subset V_2$ and $[V_2, V_2] = \{0\}$ then $\mathbb{G}$ is called a *two-step nilpotent graded* (Lie) *group*. A basis in V is chosen so that each field belongs either to V_1 or V_2. The vector fields in V_1 are called *horizontal* and their *degree* is equal to one. Otherwise the degree is equal to two.

Definition 2 The derivatives along horizontal vector fields are called *horizontal derivatives*.

The group operation is defined by the Baker–Campbell–Hausdorff formula. Now, introduce the distance corresponding to the group structure.

Definition 3 (See, e. g., [12]) Take $w = \exp\Big(\sum\limits_{i=1}^{N} w_i X_i\Big)(v)$ with $w, v \in \mathbb{G}$. Define $d_2(w, v) = \max\Big\{\Big(\sum\limits_{j:\,\deg X_j=1} w_j^2\Big)^{\frac{1}{2}}, \Big(\sum\limits_{j:\,\deg X_j=2} w_j^2\Big)^{\frac{1}{4}}\Big\}$. The set $\{w \in \mathbb{G} : d_2(w, v) < r\}$ is called the *radius* $r > 0$ *ball in* d_2 *centered at* v and is denoted by $\mathrm{Box}_2(v, r)$.

Definition 4 ([22]; See Also [23] for the General Case) A mapping $\varphi : U \to \widetilde{\mathbb{K}}$, $U \subset \mathbb{K}$, where $\mathbb{K}$ and $\widetilde{\mathbb{K}}$ are nilpotent graded groups, is *hc-differentiable* at $x \in U$ if there exists a horizontal homomorphism $\mathcal{L}_x : \mathbb{K} \to \widetilde{\mathbb{K}}$ such that $d_2(\varphi(w), \mathcal{L}_x\langle w\rangle) = o(d_2(x, w))$, where $U \ni w \to x$. The *hc-differential* $\mathcal{L}_x$ at x is denoted by $\widehat{D}\varphi(x)$.

Definition 5 (See, e. g., [23]) If the horizontal derivatives of φ exist everywhere and are continuous, while the images of horizontal vector fields are horizontal, then φ is called a *mapping of class* C^1_H, or C^1_H*-mapping*.

Let us now give a precise description of the setup. To this end, we consider a mapping $\varphi : \Omega \to \widetilde{\mathbb{G}}$, where:

1. $\Omega \subset \mathbb{G}$ is an open set and $\varphi : \Omega \to \widetilde{\mathbb{G}}$ is a C^1_H-mapping;
2. $\mathbb{G}$ is a Carnot group of topological dimension N with basis vector fields $X_1, \dots, X_N$, Lie algebra $V = V_1 \oplus V_2$, where $X_1, \dots, X_{\dim V_1}$ constitute the basis of V_1, and origin $\mathbf{0}$;
3. each degree two field on $\mathbb{G}$ can be uniquely expressed via the commutators of horizontal fields:

$$X_k = \sum_{i,j=1}^{n} a^k_{i,j}[X_i, X_j],\ i < j,\ k = \dim V_1 + 1, \dots, N \tag{1}$$

   (this enables us to vary the argument arbitrarily; see the details in [10]);
4. $\widetilde{\mathbb{G}}$ is a two-step nilpotent graded group of topological dimension $\widetilde{N}$ with basis fields $\widetilde{X}_1, \dots, \widetilde{X}_{\widetilde{N}}$, Lie algebra $\widetilde{V} = \widetilde{V}_1 \oplus \widetilde{V}_2$, where $\widetilde{X}_1, \dots, \widetilde{X}_{\dim \widetilde{V}_1}$ constitute the basis of $\widetilde{V}_1$, structure constants [4] $\{c_{lmq}\}_{l,m,q}$

$$[\widetilde{X}_l, \widetilde{X}_m] = \sum_{q:\deg \widetilde{X}_q=2} c_{lmq}\widetilde{X}_q, \tag{2}$$

   for $l, m = 1, \dots, \dim \widetilde{V}_1$, and origin $\widetilde{\mathbf{0}}$;

5. $\mathbb{G}, \widetilde{\mathbb{G}} \subset \mathbb{U}$, where $\mathbb{U}$ is a two-step nilpotent graded group of topological dimension $N + \widetilde{N}$, and $\mathbb{G} \cap \widetilde{\mathbb{G}} = \widehat{\mathbf{0}} = (\mathbf{0}, \widetilde{\mathbf{0}})$;
6. the fields $X_1, \dots, X_N$ and $\widetilde{X}_1, \dots, \widetilde{X}_{\widetilde{N}}$ coincide with the restrictions of the basis fields on $\mathbb{U}$ to the groups $\mathbb{G}$ and $\widetilde{\mathbb{G}}$ respectively; moreover, their degrees are equal to those of the corresponding fields on $\mathbb{U}$.

Note that the Cartesian product $\mathbb{G} \times \widetilde{\mathbb{G}}$ is a particular case of $\mathbb{U}$. In general, groups $\mathbb{G}$ and $\widetilde{\mathbb{G}}$ are submanifolds of $\mathbb{U}$ intersecting at their origins. This intersection coincides with the origin $\widehat{\mathbf{0}}$ of $\mathbb{U}$.

The following property is used to obtain the main result.

Theorem 6 ([23]) *Every C^1_H-mapping φ of a Carnot group to a nilpotent graded group is continuously hc-differentiable everywhere, that is, in a neighborhood of each point x it is approximated by a horizontal homomorphism up to $o(d_2(x, \cdot))$. Moreover, the matrix of its hc-differential has a block-diagonal structure with blocks $(\widehat{D}\varphi)_H$ and $(\widehat{D}\varphi)_{H^\perp}$, where the first block corresponds to fields in V_1 and $\widetilde{V}_1$, and the second one, to fields in V_2 and $\widetilde{V}_2$.*

Definition 7 Given φ, the *graph mapping* $\varphi_\Gamma : \Omega \to \mathbb{U}$ assigns to each x the element $\varphi_\Gamma(x) = \exp\Big(\sum\limits_{j=1}^{\widetilde{N}} \varphi_j(x)\widetilde{X}_j\Big)(x)$, where $\exp\Big(\sum\limits_{j=1}^{\widetilde{N}} \varphi_j(x)\widetilde{X}_j\Big)(\widetilde{\mathbf{0}}) = \varphi(x)$.

Straightforward calculations show that the graph mappings of C^1_H-mappings are neither hc-differentiable nor differentiable in the classical sense. Nevertheless, a suitable tool, polynomial hc-differentiability, was created recently in [7, 8]. It enables us to approximate graphs by smooth mappings. The main disadvantage of graph mappings is that the differential of polynomial hc-differential does not have block diagonal structure, which complicates the description of metric properties. The solution is to introduce a new basis [8], called the *intrinsic basis*, close to initial one but ensuring the desired structure of the polynomial hc-differential.

Theorem 8 ([14]) *In a neighborhood of each $\varphi_\Gamma(x)$, where $x \in \Omega$, there exists an intrinsic basis*

$$X_i \mapsto {}^x X_i = X_i + \sum_{k:\deg X_k = 2} a_{ik} X_k + \sum_{l:\deg \widetilde{X}_l = 2} b_{il} \widetilde{X}_l$$

such that the matrix of the differential of polynomial hc-differential has block lower triangle with blocks equal to union of blocks in $\widehat{D}\varphi$ and unit matrices.

3 Sub-Lorentzian Structures

To describe the sub-Lorentzian structure on $\mathbb{U}$, we introduce the following notation. Since we consider non-holonomic generalization of Minkowski geometry with multi-dimensional time, the main idea is to divide basis fields into "positive" and "negative". Here, the squared length of integral curves of "negative" fields is set to be negative.

Definition 9 Put $\{X_1, \dots, X_N, \widetilde{X}_1, \dots, \widetilde{X}_{\widetilde{N}}\} = \{Y_1, \dots, Y_{\widehat{N}}\}$, where $\widehat{N} = N + \widetilde{N}$. Moreover, let the Lie algebra $\widehat{V}$ on $\mathbb{U}$ be equal to $\widehat{V}_1 \oplus \widehat{V}_2$ with

$$[\widehat{V}_1, \widehat{V}_1] \subset \widehat{V}_2,$$

$$(X_1, \dots, X_{\dim V_1}, \widetilde{X}_1, \dots, \widetilde{X}_{\dim \widetilde{V}_1}) = (Y_1, \dots, Y_{\dim \widehat{V}_1^+}, Y_{\dim \widehat{V}_1^+ +1}, \dots, Y_{\dim \widehat{V}_1}),$$

$$\{Y_{\dim \widehat{V}_1^+ +1}, \dots, Y_{\dim \widehat{V}_1}\} = \{\widetilde{X}_1, \dots, \widetilde{X}_{\dim \widetilde{V}_1}\},$$

$$(X_{\dim V_1+1}, \dots, X_N, \widetilde{X}_{\dim \widetilde{V}_1+1}, \dots, X_{\widetilde{N}}) \\ = (Y_{\dim \widehat{V}_1+1}, \dots, Y_{\dim \widehat{V}_1+\dim \widehat{V}_2^+}, Y_{\dim \widehat{V}_1+\dim \widehat{V}_2^+ +1}, \dots, Y_{\widehat{N}}),$$

where $(Y_{\dim \widehat{V}_1+\dim \widehat{V}_2^+ +1}, \dots, Y_{\widehat{N}}) = (\widetilde{X}_{\dim \widetilde{V}_1+1}, \dots, X_{\widetilde{N}})$. Denote $\dim \widehat{V}_2 = \widehat{N} - \dim \widehat{V}_1$, $\dim \widehat{V}_1^- = \dim \widehat{V}_1 - \dim \widehat{V}_1^+ (= \dim \widetilde{V}_1)$ and $\dim \widehat{V}_2^- = \dim \widehat{V}_2 - \dim \widehat{V}_2^+ (= \dim \widetilde{V}_2)$.

Definition 10 For a vector field $T = \sum\limits_{j=1}^{\widehat{N}} y_j Y_j$ with constant coefficients, set the *squared sub-Lorentzian norm* to be

$$\mathbf{d}_2^{SL^2}(T) = \max\Bigg\{ \sum_{j=1}^{\dim \widehat{V}_1^+} y_j^2 - \sum_{k=1}^{\dim \widehat{V}_1^-} y^2_{\dim \widehat{V}_1^+ +k}, \\ \operatorname{sgn}\Bigg(\sum_{j=1}^{\dim \widehat{V}_2^+} y^2_{\dim \widehat{V}_1+j} - \sum_{k=1}^{\dim \widehat{V}_2^-} y^2_{\dim \widehat{V}_1+\dim \widehat{V}_2^+ +k}\Bigg) \\ \times \Bigg| \sum_{j=1}^{\dim \widehat{V}_2^+} y^2_{\dim \widehat{V}_1+j} - \sum_{k=1}^{\dim \widehat{V}_2^-} y^2_{\dim \widehat{V}_1+\dim \widehat{V}_2^+ +k}\Bigg|^{1/2}\Bigg\}.$$

If $w = \exp(T)(v)$ then the *squared sub-Lorentzian distance* $\mathfrak{d}_2^2(v, w)$ equals $\mathbf{d}_2^{SL^2}(T)$. The $\mathfrak{d}_2^2$-*ball* of radius r centered at v is $\operatorname{Box}_{\mathfrak{d}_2^2}(v, r) = \{x \in \mathbb{U} : \mathfrak{d}_2^2(v, x) < r^2\}$.

The *intrinsic squared distance* ${}^x\mathfrak{d}_2^2(v, w)$ is defined similarly with Y_j replaced by ${}^xY_j$ for $j = 1, \dots, \widehat{N}$.

Definition 11 For each $x \in \varphi_\Gamma(\Omega)$, consider a neighborhood $\mathcal{U}(\varphi_\Gamma^{-1}(x)) \subset \Omega$ where $o(1)$ from the definition of hc-differentiability is sufficiently small. Consider

$\delta_0 > 0$ such that each ball in Ω of radius $r < T\delta_0$ lies in at least one of these neighborhoods (since we study local property, we may assume without loss of generality that $\Omega \subset \mathbb{G}$ is a compact neighborhood), where T satisfies

$$\frac{1}{T} d_2(v_j, w) \leq {}^{v_j}\mathfrak{d}_2^2(\varphi_\Gamma(v_j), \varphi_\Gamma(w))^{1/2} \leq T d_2(v_j, w).$$

Define *the intrinsic measure* ${}^{\mathrm{SL}}\mathcal{H}_\Gamma^\nu$ on $S \subset \varphi_\Gamma(\Omega)$ as

$$\omega_{\dim V_1}\omega_{\dim V_2} \lim_{\delta\to 0} \inf\Big\{ \sum_{j\in\mathbb{N}} r_j^\nu : \bigcup_{j\in\mathbb{N}} {}^{\varphi_\Gamma^{-1}(x_j)}\mathrm{Box}_{\mathfrak{d}_2^2}(x_j, r_j) \cap \varphi_\Gamma\big(\mathcal{U}(\varphi_\Gamma^{-1}(x_j))\big) \supset S, \quad x_j \in S,\ r_j < \delta < \delta_0,\ j \in \mathbb{N}\Big\}. \tag{3}$$

To this end, rows $1, \dots, \dim \widetilde{V}_1$ of the matrix of the hc-differential together are denoted by $(\widehat{D}\varphi)_H(x)$. Assume that the squares of its column lengths are at most $\frac{1}{2\dim V_1^2} - c$ with $c > 0$. The block starting from row $\dim \widetilde{V}_1 + 1$ is denoted by $(\widehat{D}\varphi)_{H^\perp}(x)$ and we assume that the squares of its column lengths are at most $\frac{1}{\dim V_2} - c$ with $c > 0$.

Remark 12 The above restrictions guarantee the space-like property of the surface $\varphi_\Gamma(\Omega)$; see the details in [14].

One of the main results of [14] is the following area formula for the graphs of C_H^1-mappings defined on a two-step Carnot group with values in a two-step nilpotent graded group. We formulate it for our case.

Theorem 13 *The surface $\varphi_\Gamma(\Omega)$ is space-like and its ${}^{\mathrm{SL}}\mathcal{H}_\Gamma^\nu$-measure is*

$$\int_\Omega {}^{\mathrm{SL}}\mathcal{J}(\varphi, v)\, d\mathcal{H}^\nu(v) = \int_{\varphi_\Gamma(\Omega)} d\, {}^{\mathrm{SL}}\mathcal{H}_{\mathfrak{b}}^\nu(y), \tag{4}$$

where the sub-Lorentzian Jacobian ${}^{\mathrm{SL}}\mathcal{J}(\varphi, v)$ equals

$$\sqrt{\det\big(E_{\dim V_1} - (\widehat{D}\varphi)_H^*(\widehat{D}\varphi)_H\big)}\sqrt{\det\big(E_{\dim V_2} - (\widehat{D}\varphi)_{H^\perp}^*(\widehat{D}\varphi)_{H^\perp}\big)}$$

and ${}^{\mathrm{SL}}\mathcal{H}_{\mathfrak{b}}^\nu$ is defined the same way as ${}^{\mathrm{SL}}\mathcal{H}_\Gamma^\nu$, where $\omega_{\dim V_1}\omega_{\dim V_2} r_j^\nu$ is replaced by $\mathfrak{b}(x_j, r_j, \nu)$, $j \in \mathbb{N}$; see details in [13]. If the matrix of $D(\widehat{D}_P\varphi_\Gamma)$ has block diagonal structure everywhere then ${}^{\mathrm{SL}}\mathcal{H}_{\mathfrak{b}}^\nu = {}^{\mathrm{SL}}\mathcal{H}_\Gamma^\nu$.

The following notions are important for our description of the main properties of maximal surfaces.

Definition 14 (cf. [11]) The *area functional* $S(\varphi)$ defined on the class of graph mappings constructed from C^1_H-mappings is

$$\int\limits_{\Omega} \sqrt{\det\bigl(E_{\dim V_1} - (\widehat{D}\varphi)^*_H (\widehat{D}\varphi)_H\bigr)} \sqrt{\det\bigl(E_{\dim V_2} - (\widehat{D}\varphi)^*_{H^\perp} (\widehat{D}\varphi)_{H^\perp}\bigr)}\, d\mathcal{H}^\nu. \tag{5}$$

The *area functional increment* on $\xi : \Omega \to \mathbb{R}^{\dim \widetilde{V}_1}$ with $\xi = (\xi_1, \dots, \xi_{\dim \widetilde{V}_1})$ equals $S(\varphi, \xi, \varepsilon) - S(\varphi)$, where $S(\varphi, \xi, \varepsilon)$ is the integral over Ω of

$$\sqrt{\det\bigl(E_{\dim V_1} - \bigl((\widehat{D}\varphi)_H + \varepsilon D_H \xi\bigr)^* \bigl((\widehat{D}\varphi)_H + \varepsilon D_H \xi\bigr)\bigr)}$$
$$\times \sqrt{\det\bigl(E_{\dim V_2} - \bigl((\widehat{D}\varphi)_{H^\perp} + \varepsilon P_1 + \varepsilon^2 P_2\bigr)^* \bigl((\widehat{D}\varphi)_{H^\perp} + \varepsilon P_1 + \varepsilon^2 P_2\bigr)\bigr)},$$

D_H denotes differentiation along the horizontal fields only, $P_1(x)\langle X_k\rangle$ and $P_2(x)\langle X_k\rangle$ are equal to

$$2 \sum_{i,j=1}^{\dim V_1} a^k_{i,j} \sum_{q > \dim \widetilde{V}_1} \sum_{l,m=1}^{\dim \widetilde{V}_1} \Bigl((\widehat{D}\varphi(x))_{li} X_j \xi_m(x) - (\widehat{D}\varphi(x))_{lj} X_i \xi_m(x)\Bigr) c_{lmq} \widetilde{X}_q, \tag{6}$$

and

$$\sum_{i,j=1}^{\dim V_1} a^k_{i,j} \sum_{q > \dim \widetilde{V}_1} \sum_{l,m=1}^{\dim \widetilde{V}_1} \Bigl(X_i \xi_l(x) X_j \xi_m(x) - X_i \xi_m(x) X_j \xi_l(x)\Bigr) c_{lmq} \widetilde{X}_q, \tag{7}$$

respectively, $a^k_{i,j}$ are from (1), c_{lmq} are from (2), $i, j = 1, \dots, \dim V_1$ with $i < j$, $k, l, m = 1, \dots, \dim \widetilde{V}_1$, $q = \dim \widetilde{V}_1 + 1, \dots, \widetilde{N}$ (see the details in [10] and [11]).

Definition 15 Take $\Omega \subset \mathbb{G}$, $\xi_1, \dots, \xi_{\dim \widetilde{V}_1} \in C^1_H(\Omega, \mathbb{R})$, and $m \in \mathbb{N}$. Define the norm $\|\xi\|_m$ for ξ as

$$\|\xi\|_m = \biggl(\int\limits_{\Omega} \sum_{k=1}^{\dim \widetilde{V}_1} |\xi_k(x)|^m + \sum_{\beta:\, |\beta| = m} |\widehat{\xi}(x)^\beta|\, d\mathcal{H}^\nu(x)\biggr)^{\frac{1}{m}},$$

and the (semi)norm $\|\xi\|_{H,m}$ for $\xi = (\xi_1, \dots, \xi_{\dim \widetilde{V}_1})$ as

$$\|\xi\|_{H,m} = \biggl(\int\limits_{\Omega} \sum_{\beta\colon |\beta|=m} |\widehat{\xi}(x)^{\beta}|\, d\mathcal{H}^{\nu}(x)\biggr)^{\frac{1}{m}},$$

where $\widehat{\xi} = (X_1\xi_1, \dots, X_1\xi_{\dim \widetilde{V}_1}, X_2\xi_1, \dots, X_{\dim V_1}\xi_{\dim \widetilde{V}_1})$.

Definition 16 The domain $\Omega \subset \mathbb{G}$ is called *horizontally attainable* if each interior point of it can be connected to a boundary point by a curve consisting of a finite number of integral lines of horizontal vector fields.

Theorem 17 *The area functional* (5) *is differentiable twice with respect to the norm* $\|\cdot\|_{\max\{6\dim V_1, 12\dim V_2\}}$. *If* Ω *is horizontally attainable then* $\|\cdot\|_{H,\max\{6\dim V_1, 12\dim V_2\}}$ *is a norm, and* (5) *is also differentiable twice with respect to it.*

The proof follows the scheme of [10, Theorem 5] almost verbatim with obvious changes. The main idea is to deduce the expression of the third derivative of $\sqrt{f_1(\varepsilon)}\sqrt{f_2(\varepsilon)}$ at ε and then to estimate the maximal degree of $X_1\xi, \dots, X_{\dim V_1}\xi$ in

$$\sqrt{\det\bigl(E_{\dim V_1} - \bigl((\widehat{D}\varphi)_H + \varepsilon D_H\xi\bigr)^*\bigl((\widehat{D}\varphi)_H + \varepsilon D_H\xi\bigr)\bigr)}$$

and

$$f_2(\varepsilon) = \sqrt{\det\bigl(E_{\dim V_2} - \bigl((\widehat{D}\varphi)_{H^\perp} + \varepsilon P_1 + \varepsilon^2 P_2\bigr)^*\bigl((\widehat{D}\varphi)_{H^\perp} + \varepsilon P_1 + \varepsilon^2 P_2\bigr)\bigr)}$$

as well as their derivatives at ε.

Theorem 18 *Assume that* $a^k_{i,j}$ *in* (1), c_{lmq} *in* (2), $i, j = 1, \dots, \dim V_1$ *with* $i < j$, $k, l, m = 1, \dots, \dim \widetilde{V}_1$, $q = \dim \widetilde{V}_1 + 1, \dots, \widetilde{N}$, *are sufficiently small. If there exists* $K > 0$ *such that*

$$\int\limits_{\Omega} \|D_H\xi\|^2\Biggl(\frac{\sqrt{\det\bigl(E_{\dim V_2} - (\widehat{D}\varphi)^*_{H^\perp}(\widehat{D}\varphi)_{H^\perp}\bigr)}}{\sqrt{\det\bigl(E_{\dim V_1} - (\widehat{D}\varphi)^*_H(\widehat{D}\varphi)_H\bigr)}} + \frac{\sqrt{\det\bigl(E_{\dim V_1} - (\widehat{D}\varphi)^*_H(\widehat{D}\varphi)_H\bigr)}}{\sqrt{\det\bigl(E_{\dim V_2} - (\widehat{D}\varphi)^*_{H^\perp}(\widehat{D}\varphi)_{H^\perp}\bigr)}}\Biggr)\, d\mathcal{H}^{\nu}(x) \geq K\|\xi\|^2_{\max\{6\dim V_1, 12\dim V_2\}},$$

and the necessary maximality condition

$$\int\limits_{\Omega} \mathcal{D}_1(\varphi,\xi)\frac{\sqrt{\det\bigl(E_{\dim V_2}-(\widehat{D}\varphi)^*_{H^\perp}(\widehat{D}\varphi)_{H^\perp}\bigr)}}{\sqrt{\det\bigl(E_{\dim V_1}-(\widehat{D}\varphi)^*_{H}(\widehat{D}\varphi)_{H}\bigr)}}\,d\mathcal{H}^\nu + \int\limits_{\Omega} \mathcal{D}_2(\varphi,\xi)\frac{\sqrt{\det\bigl(E_{\dim V_1}-(\widehat{D}\varphi)^*_{H}(\widehat{D}\varphi)_{H}\bigr)}}{\sqrt{\det\bigl(E_{\dim V_2}-(\widehat{D}\varphi)^*_{H^\perp}(\widehat{D}\varphi)_{H^\perp}\bigr)}}\,d\mathcal{H}^\nu = 0 \qquad (8)$$

holds (*cf.* [11]), *where* $\mathcal{D}_1(\varphi,\xi,x)$ *and* $\mathcal{D}_2(\varphi,\xi,x)$ *are equal to*

$$\sum_{i=1}^{\dim V_1}\sum_{j=1}^{\dim V_1}\bigl\langle D_H\xi_i(x),(\widehat{D}\varphi_j)_H(x)\bigr\rangle\bigl(E_{\dim V_1}-(\widehat{D}\varphi)^*_H(x)(\widehat{D}\varphi)_H(x)\bigr)_{ij} + \sum_{i=1}^{\dim V_1}\sum_{j=1}^{\dim V_1}\bigl\langle(\widehat{D}\varphi_i)_H(x),D_H\xi_j(x)\bigr\rangle\bigl(E_{\dim V_1}-(\widehat{D}\varphi)^*_H(x)(\widehat{D}\varphi)_H(x)\bigr)_{ij},$$

and

$$\sum_{i=1}^{\dim V_2}\sum_{j=1}^{\dim V_2}\bigl\langle(P_1)_i(x),\bigl((\widehat{D}\varphi)_{H^\perp}\bigr)_j(x)\bigr\rangle\bigl(E_{\dim V_2}-(\widehat{D}\varphi)^*_{H^\perp}(x)(\widehat{D}\varphi)_{H^\perp}(x)\bigr)_{ij} + \sum_{i=1}^{\dim V_2}\sum_{j=1}^{\dim V_2}\bigl\langle\bigl((\widehat{D}\varphi)_{H^\perp}\bigr)_i(x),(P_1)_j(x)\bigr\rangle\bigl(E_{\dim V_2}-(\widehat{D}\varphi)^*_{H^\perp}(x)(\widehat{D}\varphi)_{H^\perp}(x)\bigr)_{ij},$$

respectively, then (5) *takes maximal value at* φ *on its neighborhood. For a horizontally attainable domain* Ω*, we may use* $\|\cdot\|_{H,\max\{6\dim V_1,12\dim V_2\}}$ *instead of* $\|\cdot\|_{\max\{6\dim V_1,12\dim V_2\}}$.

Proof This statement is actually a reformulation of the following condition of strong positivity of the area functional: *if the functional* F *is differentiable twice, its first variation at* ζ^* *equals zero, and the second variation is strongly positive in the sense that there exists* $K>0$ *such that* $\delta^2F(\zeta^*,\delta\zeta)\geq K\|\delta\zeta\|^2$ *then* F *has minimum at* ζ^*. The necessary condition (8) is deduced in the same way as in [10, Theorem 6]. To describe sufficiency estimates, put

$$f_1(\varepsilon)=\det\bigl(E_{\dim V_1}-\bigl((\widehat{D}\varphi)_H+\varepsilon D_H\xi\bigr)^*\bigl((\widehat{D}\varphi)_H+\varepsilon D_H\xi\bigr)\bigr),$$
$$f_2(\varepsilon)=\det\bigl(E_{\dim V_2}-\bigl((\widehat{D}\varphi)_{H^\perp}+\varepsilon P_1+\varepsilon^2P_2\bigr)^*\bigl((\widehat{D}\varphi)_{H^\perp}+\varepsilon P_1+\varepsilon^2P_2\bigr)\bigr).$$

Then

$$\big(\sqrt{f_1(\varepsilon)}\sqrt{f_2(\varepsilon)}\big)'' = \frac{f_1''\sqrt{f_2}}{2\sqrt{f_1}} + \frac{f_2''\sqrt{f_1}}{2\sqrt{f_2}} + \frac{f_1'f_2'}{2\sqrt{f_1}\sqrt{f_2}} - \frac{(f_1')^2\sqrt{f_2}}{4f_1^{3/2}} - \frac{(f_2')^2\sqrt{f_1}}{4f_2^{3/2}} \leq \frac{f_1''\sqrt{f_2}}{2\sqrt{f_1}} + \frac{f_2''\sqrt{f_1}}{2\sqrt{f_2}}.$$

Consequently, it suffices to estimate the values f_1'' and f_2'' in terms of $D_H\xi$. For f_1'', we see that it coincides with the sum of determinants of the modified matrices $E_{\dim V_1} - (\widehat{D}\varphi)^*_H(\widehat{D}\varphi)_H$ where row k is replaced by $-2X_k\xi \cdot D_H\xi$ or rows i and j with $i \neq j$ are replaced by $-X_i\xi \cdot (\widehat{D}\varphi)_H - X_i\varphi \cdot D_H\xi$ and $-X_j\xi \cdot (\widehat{D}\varphi)_H - X_j\varphi \cdot D_H\xi$ respectively, for $i, j, k = 1, \dots, \dim \widetilde{V}_1$. Here $X_i\varphi$ stands for row i of $(\widehat{D}\varphi)^T_H$.

Applying, if necessary, orthogonal transformations $O_H = O_H(x)$ and $O_{H^\perp} = O_{H^\perp}(x)$, where $x \in \Omega$, we may assume without loss of generality that $(\widehat{D}\varphi)^*_H(\widehat{D}\varphi)_H$ and $(\widehat{D}\varphi)^*_{H^\perp}(\widehat{D}\varphi)_{H^\perp}$ are diagonal matrices. Note that this transformation corresponds to orthogonal transformation of bases within $V_1(x)$ and within $V_2(x)$, thus, all lengths and scalar products are the same at x. Fix $x \in \Omega$. The assumption on the column lengths of these matrices implies that the (diagonal) elements $1 - a_1, \dots, 1 - a_{\dim V_1}$ of $E_{\dim V_1} - (\widehat{D}\varphi)^*_H(\widehat{D}\varphi)_H$ are positive and strictly separated from 0 everywhere in Ω. Thus, if we replace row k of $E_{\dim V_1} - (\widehat{D}\varphi)^*_H(\widehat{D}\varphi)_H$ by $-2X_k\xi \cdot D_H\xi$ then the corresponding determinant equals

$$-2\langle X_k\xi, X_k\xi\rangle \prod_{m:m\neq k}(1 - a_m) = -2\prod_{m:m\neq k}(1 - a_m)\|X_k\xi\|^2 < 0,$$

since $\max\limits_{j=1,\dots,\dim V_1}\{a_j\} \leq \frac{1}{3\dim V_1} - c$ with $c > 0$ for $k = 1, \dots, \dim \widetilde{V}_1$. Next, consider the first group of $\dim V_1(\dim V_1 - 1)$ determinants. Each of them equals the sum of four determinants of the modified matrix $E_{\dim V_1} - (\widehat{D}\varphi)^*_H(\widehat{D}\varphi)_H$, where rows i and j with $i \neq j$ are replaced by only one term. Consider the corresponding cases and estimate each value.

Case 1 Rows i and j are replaced by $-X_i\xi \cdot (\widehat{D}\varphi)_H$ and $-X_j\xi \cdot (\widehat{D}\varphi)_H$. Then, the determinant is estimated as

$$\prod_{m:m\neq i,j}(1 - a_m)\langle X_i\xi \cdot (\widehat{D}\varphi)_H, X_j\xi \cdot (\widehat{D}\varphi)_H\rangle \leq \frac{1}{2}\prod_{m:m\neq i,j}(1 - a_m)\|(\widehat{D}\varphi)_H\|^2(\|X_i\xi\|^2 + \|X_j\xi\|^2).$$

Case 2 Rows i and j are replaced by $-X_i\varphi \cdot D_H\xi$ and $-X_j\xi \cdot (\widehat{D}\varphi)_H$. The estimate is

$$\prod_{m:m\neq i,j}(1-a_m)\langle X_i\varphi \cdot D_H\xi, X_j\xi \cdot (\widehat{D}\varphi)_H\rangle \le \frac{1}{2}\prod_{m:m\neq i,j}(1-a_m)\Big(a_i\sum_{q=1}^{\dim V_1}\|X_q\xi\|^2 + \|(\widehat{D}\varphi)_H\|^2\|X_j\xi\|^2\Big).$$

Case 3 If rows i and j are replaced by $-X_i\xi \cdot (\widehat{D}\varphi)_H$ and $-X_j\varphi \cdot D_H\xi$, then the estimate equals $\frac{1}{2}\prod\limits_{m:m\neq i,j}(1-a_m)\Big(a_j\sum\limits_{q=1}^{\dim V_1}\|X_q\xi\|^2 + \|(\widehat{D}\varphi)_H\|^2\|X_i\xi\|^2\Big)$.

Case 4 If rows i and j are replaced by $-X_i\varphi \cdot D_H\xi$ and $-X_j\varphi \cdot D_H\xi$ then we have $\frac{1}{2}\prod\limits_{m:m\neq i,j}(1-a_m)\Big(a_i\sum\limits_{q=1}^{\dim \widetilde{V}_1}\|X_q\xi\|^2 + a_j\sum\limits_{q=1}^{\dim \widetilde{V}_1}\|X_q\xi\|^2\Big)$.

Fix i and recall that $\|(\widehat{D}\varphi)_H\|^2 = \sum\limits_{q=1}^{\dim V_1} a_q$ as the trace of $\widehat{D}_H\varphi^*\widehat{D}_H\varphi$. We infer that the coefficient at $\|X_i\xi\|^2$ is equal to

$$\sum_{q=1}^{\dim V_1} a_q\sum_{j:j\neq i}\prod_{m:m\neq i,j}(1-a_m) + \sum_{q=1}^{\dim V_1}\sum_{j:j\neq q}\prod_{m:m\neq q,j}(1-a_m)a_q - 2\prod_{m:m\neq i}(1-a_m) \le -\prod_{m:m\neq i}(1-a_m) - \widehat{c} < 0$$

with $\widehat{c} > 0$ since $0 < \max\limits_{q=1,\ldots,\dim V_1}\{a_q\} \le \frac{1}{2\dim V_1^2} - c$ with $c > 0$ for $i = 1,\ldots,\dim V_1$.

Consider now f_2'' and its estimates. It coincides with the sum of determinants of the modified matrices $E_{\dim V_1} - (\widehat{D}\varphi)^*_{H^\perp}(\widehat{D}\varphi)_{H^\perp}$ where row k is replaced by $-2(P_2^*)_k \cdot (D\varphi)_{H^\perp} - 2(P_1^*)_k P_1 - 2\big((\widehat{D}\varphi)_{H^\perp}\big)_k P_2$, or rows i and j with $i \neq j$ are replaced by $-(P_1^*)_i \cdot (\widehat{D}\varphi)_{H^\perp} - (\widehat{D}\varphi_i)_H \cdot P_1$ and $-(P_1^*)_j \cdot (\widehat{D}\varphi)_{H^\perp} - (\widehat{D}\varphi_j)_H \cdot P_1$ respectively, for $i, j, k = 1,\ldots,\dim V_2$. In contrast to the horizontal case, each summand depending on the horizontal derivatives of ξ has coefficients depending on the entries of $\widehat{D}_H\varphi$. Thus, the absolute value of the coefficient at $\|D_H\xi\|^2$ can be considered strictly less than $\prod\limits_{m:m\neq i}(1-a_m)$. Since each summand also contains products of $a_{i,j}^k$ and c_{lmq}, we can easily see that if they are sufficiently small then

$$|f_2''| \le \|D_H\xi\|^2 \cdot \prod_{m:m\neq i}(1-a_m),$$

and finally

$$\left(\sqrt{f_1(\varepsilon)}\sqrt{f_2(\varepsilon)}\right)'' \le \frac{f_1''\sqrt{f_2}}{2\sqrt{f_1}} + \frac{f_2''\sqrt{f_1}}{2\sqrt{f_2}} \le -\widehat{c}\|D_H\xi\|^2\left(\frac{\sqrt{f_2}}{\sqrt{f_1}} + \frac{\sqrt{f_1}}{\sqrt{f_2}}\right), \ \widehat{c} > 0.$$

Thus, the functional $-S(\varphi)$ has minimum since its second variation is strongly positive; and therefore the area functional (5) has maximum at φ. The theorem follows. □

Remark 19 We may replace strong restrictions on $|a_{i,j}^k|$ and $|c_{lmq}|$ by adding some restrictions to $\|(\widehat{D}\varphi)_H\|$ since all coefficients at the horizontal derivatives of ξ contain the horizontal derivatives of φ. Moreover, it is possible to deduce restrictions on $(\widehat{D}\varphi)_H$ basing on the given values of $a_{i,j}^k$ and c_{lmq} for $i, j = 1, \dots, \dim V_1$ with $i < j$ and $k, l, m = 1, \dots, \dim \widetilde{V}_1$, $q = \dim \widetilde{V}_1 + 1, \dots, \widetilde{N}$.

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Following the Trail of the Operator Geometric Mean

Jimmie D. Lawson and Yongdo Lim

Abstract This paper traces the development of the theory of the matrix geometric mean in the cone of positive definite matrices and the closely related operator geometric mean in the positive cone of a unital C^*-algebra. The story begins with the two-variable matrix geometric mean, moves to the n-variable matrix setting, then to the extension to the positive cone of the C^*-algebra of operators on a Hilbert space, and even to general unital C^*-algebras, and finally to the consideration of barycentric maps on the space of integrable probability measures on the positive cone. Besides expected tools from linear algebra and operator theory, one observes a substantial interplay with operator monotone functions, geometrical notions in metric spaces, particularly the notion of nonpositive curvature, some probabilistic theory of random variables with values in a metric space of nonpositive curvature, and the appearance of related means such as the inductive and power means.

Keywords Geometric mean · Operator mean · Operator monotone function · Nonpositively curved metric spaces · Contractive barycentric map

Mathematics Subject Classification (2010) Primary 47A64; Secondary 46L05, 47B65, 47L07

1 Introduction

Positive definite matrices have become fundamental computational objects in many areas of engineering, statistics, quantum information, and applied mathematics.

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They appear as "data points" in a diverse variety of settings: covariance matrices in statistics, elements of the search space in convex and semidefinite programming, kernels in machine learning, observations in radar imaging, and diffusion tensors in medical imaging, to cite only a few. A variety of computational algorithms have arisen for approximation, interpolation, filtering, estimation, and averaging. Our interest focuses on the last named, the process of finding an average or mean, which is again positive definite.

A simple computation would be to take the arithmetic mean of a given finite set of positive definite matrices. However, researchers have learned that to find a mean or average that performs well and exhibits desirable properties, one needs to take into account the underlying geometric structure of Ω_n, the space of $n \times n$-positive definite matrices.

Formally a *mean of order n*, or *n-mean* for short, on a set X is a function $\mu : X^n \to X$ satisfying the idempotency condition $\forall x \in X$, $\mu(x, x, \ldots, x) = x$. It is frequently assumed in the definition of a mean that a mean is invariant under any permutation of variables; we call these *symmetric means*. The mean $\mu : X^n \to X$ is *continuous* or a *topological mean* if X is a topological space and μ is continuous. Typically a mean represents some type of averaging operator.

The subject of (binary) means for positive numbers or line segments has a rich mathematical lineage dating back into antiquity. The Greeks, motivated by their interest in proportions and musical ratios, defined at least eleven different means (depending on how one counts), the arithmetic, geometric, harmonic, and golden being the best known. A geometric construction for the geometric mean $\sqrt{ab}$ of $a, b > 0$ is given by Euclid in Book II in the form of "squaring the rectangle," i.e., constructing a square of the same area as a given rectangle of sides a and b. The study of various means and their properties on the positive reals has remained an active area of investigation up to the present day.

2 Positive Definite Matrices

Let $\mathcal{M}_n(\mathbb{C})$, or simply $\mathcal{M}_n$, denote the set of $n \times n$ complex matrices. We may identify $\mathcal{M}_n$ with the set of linear operators on $\mathbb{C}^n$, where we consider $\mathbb{C}^n$ to be a complex Hilbert space of column vectors with the usual Hermitian inner product. Denoting the conjugate transpose of $A \in \mathcal{M}_n$ by A^*, we recall that A is *Hermitian* if $A = A^*$ and *unitary* if $A^* = A^{-1}$. The Hermitian matrix A is *positive definite* if $\forall u \neq 0$, $\langle u, Au \rangle > 0$. These notions readily generalize to $\mathcal{B}(H)$, the algebra of operators on an arbitrary Hilbert space.

The following are well-known equivalences for a Hermitian matrix A to be positive definite:

1. $\langle Ax, x \rangle > 0$ for all $0 \neq x$, where $\langle \cdot, \cdot \rangle$ is the Hilbert space inner product on $\mathbb{C}^n$.
2. $A = BB^*$ for some invertible B.
3. A has all positive eigenvalues.

4. $A = \exp B = \sum_{n=0}^{\infty} B^n/n!$ for some (unique) Hermitian B.
5. $A = UDU^*$ for some unitary U and diagonal D with positive diagonal entries.

The positive definite $n \times n$-matrices form an open cone in $\mathbb{H}_n$, the $n \times n$ Hermitian matrices, with closure the positive semidefinite matrices (equivalently, $\langle Ax, x\rangle \geq 0$ for all x). We denote the open cone of positive definite matrices by Ω (or Ω_n if we need to distinguish the dimension).

We define a partial order (sometimes called the *Loewner order*) on the vector space $\mathbb{H}_n$ of Hermitian matrices by $A \leq B$ if $B - A$ is positive semidefinite. We note $0 \leq A$ iff A is positive semidefinite and write $0 < A$ if $A \in \Omega$ iff A is positive definite. The matrix A is sometimes called *strictly positive* in this setting.

Every positive definite (Hermitian) matrix operator has a unique *spectral decomposition*

$$A = \sum_{i=1}^{n} \lambda_i E_i,$$

where $\lambda_i > 0$ ($\lambda_i \in \mathbb{R}$) is the i^{th}-eigenvalue and E_i is the orthogonal projection onto the eigenspace of λ_i. One then has

$$A^k = \sum_{i=1}^{n} \lambda_i^k E_i,$$

from which one can easily deduce that every positive definite matrix has a unique positive definite kth-root.

The arithmetic and harmonic means readily extend from $\mathbb{R}^{>0}$ to the set of positive definite matrices:

$$\mathcal{A}(A, B) = \frac{1}{2}(A + B); \qquad \mathcal{H}(A, B) = 2(A^{-1} + B^{-1})^{-1}.$$

The geometric mean is not so obvious (e.g., $\sqrt{AB}$ need not be positive definite for A, B positive definite). One approach is to rewrite the equation $x^2 = ab$ (which has positive solution the geometric mean of a and b) in its appropriate form in the noncommutative setting:

$$\begin{aligned} XA^{-1}X &= B \\ A^{-1/2}XA^{-1/2}A^{-1/2}XA^{-1/2} &= A^{-1/2}BA^{-1/2} \\ A^{-1/2}XA^{-1/2} &= (A^{-1/2}BA^{-1/2})^{1/2} \\ X &= A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}. \end{aligned}$$

We write $A\#B(= A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2})$ for the matrix geometric mean. Other connections between the matrix geometric mean and the one for positive real numbers may be found in [8].

3 Operator Monotone Functions and the Kubo–Ando Theorem

For $M_1, M_2 \subseteq \mathbb{R}$ and a mapping $f : M_1 \to M_2$, we define a function on the set of all Hermitian A with spectrum contained in M_1 by $f(A) = \sum_{i=1}^{n} f(\lambda_i)E_i$, where $A = \sum_{i=1}^{n} \lambda_i E_i$ is the spectral decomposition (functions constructed in this way are called *primary matrix functions* and provide a simple example of the functional calculus). A continuous function $f : M_1 \to M_2$ is *operator monotone* if $f(A) \leq f(B)$ whenever $A \leq B$. Operator monotone functions defined on some interval are continuous, monotone (nondecreasing), and concave.

If $M\mu(A, B)M^* = \mu(MAM^*, MBM^*)$ for all invertible M, the mean $\mu : \Omega \times \Omega \to \Omega$ is said to be *invariant under congruence transformations*. The mean μ is *monotonic* if $A_1 \leq A_2,\ B_1 \leq B_2$ implies $\mu(A_1, B_1) \leq \mu(A_2, B_2)$. The next result is a major 1980 result of F. Kubo and T. Ando [6].

Theorem *Every operator monotone function* $f : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ *with* $f(1) = 1$ *gives rise to a congruence-invariant, monotonic mean* μ *defined by*

$$\mu(A, B)[= A^{\frac{1}{2}}\mu(I, A^{-1/2}BA^{-1/2})A^{\frac{1}{2}}] = A^{\frac{1}{2}} f(A^{-1/2}BA^{-1/2})A^{\frac{1}{2}}.$$

The association $f \to \mu_f$ *is a bijection between the operator monotone functions and the congruence-invariant, monotonic continuous means.* (*For the converse, one defines* f *from* $\mu(I, \lambda I) = f(\lambda)I$.)

To illustrate we apply the Kubo–Ando roadmap for passing from numeric to matrix means for certain important examples:

1. The Geometric Mean $A\#B$ and Weighted Geometric Mean $A\#_t B$:

$$\begin{aligned}\gamma(a, b) = \sqrt{ab} &\to f(x) = \gamma(1, x) = x^{1/2}\\ &\to \mathcal{G}(A, B) = A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}\\ \gamma_t(a, b) = a^{1-t}b^t &\to \mathcal{G}_t(A, B) = A\#_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}\end{aligned}$$

2. The Arithmetic Mean:

$$\begin{aligned}\alpha(a, b) = (a + b)/2 \quad &\to \quad f(x) = \alpha(1, x) = (1/2)(1 + x)\\ &\to \mathcal{A}(A, B) = A^{1/2}((1/2)(I + A^{-1/2}BA^{-1/2}))A^{1/2}\\ &= (1/2)(A + B)\end{aligned}$$

The Kubo–Ando Theorem provided the foundation for the rapid development of the theory of matrix and operator means of two variables. However, no such analogous theorem has been discovered for multivariable means, even when the extension is known for the case of the positive reals. Some means, such as the arithmetic and harmonic, admit rather obvious extensions to the n-variable matrix case. But the problem of extending the geometric mean to the multivariable matrix setting remained unsolved for a number of years. An important step along the way was the gradual realization that the geometric matrix mean of two variables had an important alternative geometric/metric characterization, apparently first appearing in print in an article of the authors in 2001 [8].

4 Means of Several Variables and NPC-Spaces

In 2004 Ando et al. [2] gave the first extension of the binary geometric mean to n-variables, which came to be called the ALM mean. They listed desirable axiomatic properties for such an n-variable extension g and showed they were satisfied by their extension. The proofs typically involved extending from the known case of $n = 2$ by induction.

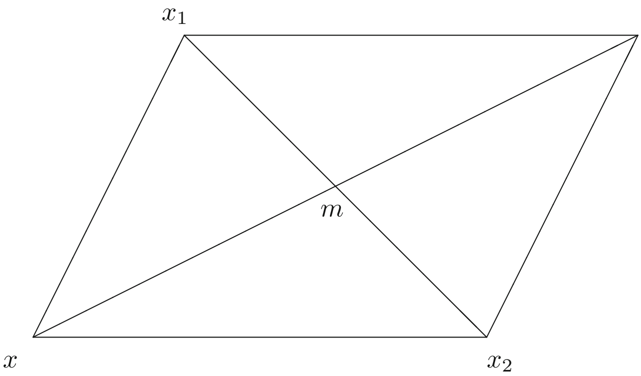
Let $\mathbb{A} = (A_1, \ldots, A_n), \mathbb{B} = (B_1, \ldots, B_n) \in \Omega^n$.

(P1) (Consistency with scalars) $g(\mathbb{A}) = (A_1 \cdots A_n)^{1/n}$ if the A_i's commute;
(P2) (Joint homogeneity) $g(a_1 A_1, \ldots, a_n A_n) = (a_1 \cdots a_n)^{1/n} g(\mathbb{A})$;
(P3) (Permutation invariance) $g(\mathbb{A}_\sigma) = g(\mathbb{A})$, where $\mathbb{A}_\sigma = (A_{\sigma(1)}, \ldots, A_{\sigma(n)})$;
(P4) (Monotonicity) If $B_i \leq A_i$ for all $1 \leq i \leq n$, then $g(\mathbb{B}) \leq g(\mathbb{A})$;
(P5) (Continuity) g is continuous;
(P6) (Congruence invariance) $g(M\mathbb{A}M^*) = Mg(\mathbb{A})M^*$ for M invertible, where $M(A_1, \ldots, A_n)M^* = (MA_1M^*, \ldots, MA_nM^*)$;
(P7) (Joint concavity) $g(\lambda\mathbb{A} + (1-\lambda)\mathbb{B}) \geq \lambda g(\mathbb{A}) + (1-\lambda)g(\mathbb{B})$ for $0 \leq \lambda \leq 1$;
(P8) (Self-duality) $g(A_1^{-1}, \ldots, A_n^{-1})^{-1} = g(A_1, \ldots, A_n)$;
(P9) (Determinantal identity) $\operatorname{Det} g(\mathbb{A}) = \prod_{i=1}^n (\operatorname{Det} A_i)^{1/n}$; and
(P10) (AGH mean inequalities) $n(\sum_{i=1}^n A_i^{-1})^{-1} \leq g(\mathbb{A}) \leq \frac{1}{n}\sum_{i=1}^n A_i$.

But a better candidate soon appeared. To understand it, we need some background. The parallelogram law in Hilbert spaces is given by

$$\text{sum of 2 diagonals squared} = \text{sum of 4 sides squared}$$

$$d^2(x_1, x_2) + 4d^2(x, m)(= (2d(x, m))^2) = 2d^2(x, x_1) + 2d^2(x, x_2)$$



Replacing the equality by an inequality in a general metric space yields the more general *semiparallelogram law*: for all $x_1, x_2 \in X$, there exists $m \in X$ such that for any $x \in X$,

$$d^2(x_1, x_2) + 4d^2(x, m) \leq 2d^2(x, x_1) + 2d^2(x, x_2) \tag{NPC}$$

One can show that $m = m(x_1, x_2)$ is unique and is the unique metric midpoint between x_1 and x_2.

(Global) NPC-spaces are complete metric spaces satisfying the semiparallelogram law (NPC). They have been intensely studied in recent years, often under alternative names such as CAT(0)-spaces or Hadamard spaces.

Condition (NPC) is a metric version of *NonPositive Curvature*, since the distance metric of a simply connected Riemannian manifold satisfies (NPC) iff the Riemannian metric has nonpositive curvature in the usual sense.

Example The open cone Ω_n of $n \times n$ positive definite matrices becomes a Riemannian manifold when equipped with the trace Riemannian metric: $\langle X, Y \rangle_A = \mathrm{tr} A^{-1} X A^{-1} Y$, where $A \in \Omega_n$ and X, Y are $n \times n$ Hermitian matrices. The corresponding distance metric on Ω_n is given by $d(A, B) = \| \log(A^{-1/2} B A^{-1/2} \|_2$, where $\| \cdot \|_2$ is the Frobenius (or Hilbert-Schmidt) norm. The cone Ω_n equipped with the metric d is an NPC-space. Furthermore, the unique midpoint between $A, B \in \Omega_n$ is the geometric mean $A \# B$.

Let (M, d) be a metric space. Given a *weight* $\mathbf{w} = (w_1, \ldots, w_n)$ (each $w_i \geq 0$ and $\sum_{i=1}^n w_i = 1$), the *weighted least squares mean* $\Lambda(\mathbf{w}; a_1, \ldots, a_n)$ of $(a_1, \ldots, a_n) \in M^n$ is defined as the solution to the optimization problem of minimizing the weighted sum of distances squared:

$$\underset{x \in M}{\operatorname{argmin}} \sum_{i=1}^n w_i \delta^2(x, a_i),$$

provided the solution uniquely exists. This is the case for NPC spaces, since the function defined by $x \mapsto \sum_{i=1}^{n} w_i \delta^2(x, a_i)$ is uniformly convex in this case. E. Cartan considered such "barycenters" in the case of Riemannian manifolds, where they uniquely exist for the ones of nonpositive curvature, and M. Fréchet considered them in more general metric spaces. Thus the least squares mean is also called the Cartan mean or Fréchet mean.

First M. Moakher [13] and independently R. Bhatia and J. Holbrook [3, 4] studied the uniformly weighted least squares mean for the set of positive definite matrices Ω equipped with the trace metric as a multivariable generalization of the two-variable geometric mean. They established its (unique) existence and verified several of the axiomatic properties (P1)–(P10) satisfied by the Ando–Li–Mathias geometric mean: consistency with scalars, joint homogeneity, permutation invariance, congruence invariance, and self-duality (the last two being true since congruence transformations and inversion are isometries). Further, based on computational experimentation, Bhatia and Holbrook conjectured monotonicity for the least squares mean, but this was left as an open problem.

5 Monotonicity, Probability, and the Inductive Mean

One other mean will play an important role in what follows, one that we shall call the *inductive mean*, following the terminology of K.-T. Sturm [15]. It appeared elsewhere in the work of M. Sagae and K. Tanabe [14] and Ahn et al. [1]. It is defined inductively for NPC spaces (or more generally for metric spaces with weighted binary means $x\#_t y$) for each $k \geq 2$ by $S_2(x, y) = x\#y$ and for $k \geq 3$, $S_k(x_1, \ldots, x_k) = S_{k-1}(x_1, \ldots, x_{k-1})\#_{\frac{1}{k}} x_k$. (Here $x\#_t y$ is the unique point z such that $d(x, y) = (1-t)d(x, z) + td(y, z)$ for $0 \leq t \leq 1$.) Note that this mean at each stage is defined from the previous stage by taking the appropriate two-variable weighted mean, which is monotone. Thus the inductive mean is monotone.

Let (X, d) be an NPC metric space, $\{x_1, \ldots, x_m\} \subseteq X$. Set $\mathbb{N}_m = \{1, 2, \ldots, m\}$ and assign to $k \in \mathbb{N}_m$ the probability w_k, where $0 \leq w_k \leq 1$ and $\sum_{k=1}^{m} w_i = 1$. For each $\omega \in \prod_{n=1}^{\infty} \mathbb{N}_m$, define a sequence $\sigma = \sigma_\omega$ in X by $\sigma(1) = x_{\omega(1)}$, $\sigma(k) = S_k(x_{\omega(1)}, \ldots, x_{\omega(k)})$, where S_k is the inductive mean. (The sequence σ_ω may be viewed as a "walk" starting at $\sigma(1) = x_{\omega(1)}$ and obtaining $\sigma(k)$ by moving from $\sigma(k-1)$ toward $x_{\omega(k)}$ a distance of $(1/k)d(\sigma(k-1), x_{\omega(k)})$.) The following is a special case of Sturm's main results in [15].

Theorem 1 (Sturm's Theorem) *Giving $\prod_{n=1}^{\infty} \mathbb{N}_m$ the product probability, the set*

$$\{\omega \in \prod_{n=1}^{\infty} \mathbb{N}_m : \lim_n \sigma_\omega(n) = \Lambda(\mathbf{w}; x_1, \ldots, x_m)\}$$

has measure 1*, i.e.,* $\sigma_\omega(n) \to \Lambda(\mathbf{w}; x_1, \ldots, x_m)$ *for almost all* ω.

More generally, Sturm establishes a version of the Strong Law of Large Numbers for random variables into an NPC metric space, with limit the least squares mean.

Using Sturm's Theorem, the authors were able to show (2011) [8]:

Theorem 2 *Let Ω be the open cone of positive definite matrices of some fixed dimension n.*

(1) *The least squares mean Λ on Ω is monotone: $A_i \leq B_i$ for $1 \leq i \leq n$ implies $\Lambda(A_1, \ldots, A_n) \leq \Lambda(B_1, \ldots B_n)$.*
(2) *All ten of the ALM axioms hold for Λ.*
(3) *In a natural way Λ can be extended to a weighted mean, and appropriate weighted versions of the ten properties hold.*

Note The ALM mean is typically distinct from the least squares mean for $n \geq 3$. Thus the ALM axioms do not characterize a mean. The latter fact had already been noted by Bini et al. [5], who introduced a much more computationally efficient variant of the ALM mean [5].

6 The Karcher Equation

The uniform convexity of the trace metric d on Ω yields that the least squares mean is the unique critical point for the function $X \mapsto \sum_{k=1}^{n} d^2(X, A_k)$. The least squares mean is thus characterized by the vanishing of the gradient, which is equivalent to its being a solution of the following *Karcher equation*:

$$\sum_{i=1}^{n} w_i \log(X^{-1/2} A_i X^{-1/2}) = 0. \tag{1}$$

The Karcher equation (1) can be used to *define* a mean on the cone Ω of positive invertible bounded operators on an infinite-dimensional Hilbert space (where one no longer has an NPC-space), called the *Karcher mean*. As we just previously noted, restricted to the matrix case it yields the least squares mean.

Power means for positive definite matrices were introduced by Lim and Palfia [11].

Theorem 3 *Let $A_1, \ldots, A_n \in \Omega$ and let $\mathbf{w} = (w_1, \ldots, w_n)$ be a weight. Then for each $t \in (0, 1]$, the following equation has a unique positive definite solution $X = P_t(\mathbf{w}; A_1, \ldots, A_n)$, called the weighted power mean:*

$$X = \sum_{i=1}^{n} w_i (X \#_t A_i).$$

When restricted to the positive reals, the power mean reduces to the usual power mean

$$P_t(\mathbf{w}; a_1, \ldots, a_n) = \left(w_1 a_1^t + \cdots + w_n a_n^t\right)^{\frac{1}{t}}.$$

In 2014 the authors showed [9] that the preceding notion of power mean extended to the setting of bounded operators on a Hilbert space [9] and established that the power means are decreasing, $s < t$ implies $P_s(\cdot\,;\ \cdot) \le P_t(\cdot\,;\ \cdot)$. Using power means we were able to establish the existence and uniqueness of the Karcher mean in the C^*-algebra of bounded operators on a Hilbert space.

Theorem 4 *In the strong operator topology*

$$\Lambda(\cdot\,;\ \cdot) = \lim_{t\to 0^+} P_t(\cdot\,;\ \cdot) = \inf_{t>0} P_t(\cdot\,;\ \cdot),$$

where Λ *is the Karcher mean, the unique solution of the Karcher equation*

$$X = \Lambda_n(A_1, \ldots, A_n) \Leftrightarrow \sum_{i=1}^{n} \log(X^{-1/2} A_i X^{-1/2}) = 0.$$

Via this machinery many of the axiomatic properties of the least squares mean in the finite-dimensional setting were extended to the corresponding Karcher mean in the infinite-dimensional setting.

Recent work by Lim and Palfia [12] and independently by Lawson [7] shows that the preceding constructions and results remain valid for the open cone of positive invertible elements in any unital C^*-algebra.

7 Barycenters

A *Borel probability measure* on a metric space (X, d) is a countably additive non-negative measure μ on the Borel algebra $\mathcal{B}(X)$, the smallest Σ-algebra containing the open sets, such that $\mu(X) = 1$. We denote the set of all probability measures on $(X, \mathcal{B}(X))$ by $\mathcal{P}(X)$. Let $\mathcal{P}_0(X)$ be the set of all uniform finitely supported probability measures, i.e., all $\mu \in \mathcal{P}(X)$ of the form $\mu = \frac{1}{n}\sum_{j=1}^{n} \delta_{x_j}$ for some $n \in \mathbb{N}$, where δ_x is the point measure of mass 1 at x.

A measure $\mu \in \mathcal{P}(X)$ is said to be *integrable* if

$$\int_X d(x, y)\, d\mu(y) < \infty.$$

The set of integrable measures is denoted by $\mathcal{P}^1(X)$.

The *Wasserstein distance* (alternatively Kantorovich–Rubinstein distance) d^W on $\mathcal{P}^1(X)$ is a standard metric for probability measures. It is known that d^W is a complete metric on $\mathcal{P}^1(X)$ whenever X is a complete metric space and that $\mathcal{P}_0(X)$ is d^W-dense in $\mathcal{P}^1(X)$.

One can view the Karcher mean $(A_1, \ldots, A_n) \mapsto \Lambda(A_1, \ldots, A_n)$ on Ω, the open cone of positive invertible operators, alternatively as yielding a barycenter for the probability measure with weight $1/n$ at each A_k. It turns out that this barycentric map is contractive from $\mathcal{P}_0(\Omega)$ to Ω, and hence extends uniquely to a contractive barycentric map $\Lambda : \mathcal{P}^1(\Omega) \to \Omega$. We call this extended map the *Karcher barycentric map*. It is characterized by

$$X = \Lambda(\mu) \Leftrightarrow \int_\Omega \log(X^{-1/2}AX^{-1/2})d\mu(A) = 0.$$

The existence and basic theory and properties of the Karcher barycentric map can be found in [10]. We note that from its definition it extends the Karcher mean.

8 Summary

In the preceding we have attempted to trace out how the matrix/operator geometric mean has strikingly developed over the past 15 years from a two-variable mean to a multivariable matrix mean (the least squares mean) to an operator mean in unital C^*-algebras (the Karcher mean) to a barycentric map on integrable Borel probability measures. Whatever future developments may hold, it is clear that a substantial theory has already emerged.

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On Hom-Lie–Rinehart Algebras

Ashis Mandal and Satyendra Kumar Mishra

Abstract We describe the notion of hom-Lie–Rinehart algebras as an algebraic analogue of hom-Lie algebroids. We consider modules (left and right) over this hom-structure and describe homology and cohomology complexes by considering coefficient modules. In the sequel, we consider some special classes of hom-Gerstenhaber algebras and their relationship with hom-Lie algebroids by Mandal and Mishra (J Geom Phys 133:287–302, 2018; Commun Algebra 46(9):3722–3744, 2018).

Keywords Lie–Rinehart algebras · Hom-algebras · Lie algebroids

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1 Introduction

The notion of Lie–Rinehart algebras plays an important role in many branches of mathematics. The idea of this notion goes back to the work of N. Jacobson to study certain field extensions. Over the years, this notion appeared with different names in several areas which include differential geometry and differential Galois theory. J. Huebschmann described Lie–Rinehart algebras as an algebraic counterpart of Lie algebroids in [2] and developed systematically through a series of papers. There is also a growing interest in twisted algebraic structures or hom-algebraic structures. The first appearance of a hom-algebra was the notion of hom-Lie algebra, in the context of some particular deformation called q-deformations of Witt and Virasoro algebra of vector fields. Later on, many essential results on hom-Lie algebras and

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hom-associative algebras followed in the works of A. Makhlouf, Y. Sheng, D. Yau and coauthors. In [3], C. Laurent-Gengoux and J. Teles introduced the notion of hom-Lie algebroids through a formulation of hom-Gerstenhaber algebras, following the one-one correspondence between Lie algebroid structures on a vector bundle and Gerstenhaber algebra structures on the exterior algebra of multisections of the vector bundle. In fact, this one-one correspondence is an outcome of a more general categorical result for the algebraic counterpart, namely the existence of adjoint functors between the category of Lie–Rinehart algebras and the category of Gerstenhaber algebras. This adjunction leads us to define the notion of hom-Lie–Rinehart algebras and to construct a pair of adjoint functors between the category of hom-Gerstenhaber algebras and the category of hom-Lie–Rinehart algebras (for details see [4, 5]).

2 Hom-Lie–Rinehart Algebras

Let R be a commutative ring with unity, A be an associative commutative R-algebra, and $\phi : A \to A$ be an algebra endomorphism.

Definition 1 A hom-Lie–Rinehart algebra over (A, ϕ) is given by a tuple $(A, L, [-,-], \phi, \alpha, \rho)$ where L is an A-module, $[-,-] : L \times L \to L$ is a skew symmetric bilinear map, $\alpha : L \to L$ is a ϕ-function linear map satisfying $\alpha([x, y]) = [\alpha(x), \alpha(y)]$, and the map $\rho : L \to \mathrm{Der}_\phi A$ is a ϕ-function linear map such that following conditions hold.

1. The triplet $(L, [-,-], \alpha)$ is a hom-Lie algebra.
2. (ρ, ϕ) is a hom-Lie algebra representation of $(L, [-,-], \alpha)$ on A.
3. $[x, a.y] = \phi(a)[x, y] + \rho(x)(a)\alpha(y)$ for all $a \in A,\ x, y \in L$.

A hom-Lie–Rinehart algebra $(A, L, [-,-], \phi, \alpha, \rho)$ is said to be regular if the map $\phi : A \to A$ is an algebra automorphism and $\alpha : L \to L$ is a bijection.

Example

1. Let L be a Lie–Rinehart algebra over an associative commutative algebra A, and (α, ϕ) be an endomorphism of L, then the tuple $(A, L, [-,-]_\alpha, \phi, \alpha, \rho_\phi)$ is a hom-Lie–Rinehart algebra, where $[-,-]_\alpha := \alpha \circ [-,-]$, and $\rho_\phi := \phi \circ \rho$.
2. Hom-Lie–Rinehart algebra associated to the space of ϕ-derivations: Let $\phi : A \to A$ be an algebra automorphism. Then $(\mathrm{Der}_\phi A, [-,-]_\phi, \alpha_\phi)$ is a hom-Lie algebra, with $\alpha_\phi(D) = \phi \circ D \circ \phi^{-1}$ and the bracket $[D_1, D_2]_\phi = \phi \circ D_1 \circ \phi^{-1} \circ D_2 \circ \phi^{-1} - \phi \circ D_2 \circ \phi^{-1} \circ D_1 \circ \phi^{-1}$, for any $D_1, D_2, D \in \mathrm{Der}_\phi(A)$. In fact, the tuple $(A, \mathrm{Der}_\phi A, [-,-]_\phi, \alpha_\phi, \alpha_\phi)$ is a hom-Lie–Rinehart algebra over (A, ϕ) with the anchor $\rho = \alpha_\phi$.
3. The hom-Lie–Rinehart algebras associated to a Poisson algebra equipped with an automorphism is described in [4].

2.1 Homomorphisms of Hom-Lie–Rinehart Algebras

Definition 2 Let $(A, L, [-,-]_L, \phi, \alpha_L, \rho_L)$ and $(B, L', [-,-]_{L'}, \psi, \alpha_{L'}, \rho_{L'})$ be hom-Lie–Rinehart algebras, then a hom-Lie–Rinehart algebra homomorphism is a pair of maps (g, f), satisfying the following identities:

1. $f \circ \alpha_L = \alpha_{L'} \circ f,\ g \circ \phi = \psi \circ g$;
2. $f([x, y]) = [f(x), f(y)]_{L'}$ for all $x, y \in L$;
3. $g(\rho_L(x)(a)) = \rho_{L'}(f(x))(g(a))$ for all $x \in L,\ a \in A$,

where the map $g : A \to B$ is a R-algebra homomorphism and $f : L \to L'$ is a g-function linear map

Let us denote by hLR the category of hom-Lie-Rinehart algebras and by hGR the category of hom-Gerstenhaber algebras.

Theorem 3 *There are adjoint functors between the categories* hLR *and* hGR.

3 Modules of Hom-Lie–Rinehart Algebras

Let $(\mathcal{L}, \alpha) := (A, L, [-,-], \phi, \alpha, \rho)$ be a hom-Lie–Rinehart algebra over (A, ϕ). Also, let M be an A-module and $\beta : M \to M$ be a ϕ-function linear map.

3.1 Left Modules Over Hom-Lie–Rinehart Algebras

Definition 4 A pair (M, β) is said to be a left module over a hom-Lie Rinehart algebra $(\mathcal{L}, \alpha)$ if the following conditions hold for all $X \in L,\ a \in A,\ m \in M$.

- There is a map $\theta : L \otimes M \to M$, such that the pair (θ, β) is a hom-Lie algebra representation of $(L, [-,-], \alpha)$ on M. Let $\{x, m\} := \theta(x, m)$;
- $\{a.X, m\} = \phi(a)\{X, m\}$;
- $\{X, a.m\} = \phi(a)\{X, m\} + \rho(X)(a).\beta(m)$.

Example

1. If $\alpha = Id_L$ and $\beta = Id_M$ then $(\mathcal{L}, \alpha)$ is a Lie–Rinehart algebra and M is a left Lie–Rinehart algebra module over L.
2. The pair (A, ϕ) is a left module over $(\mathcal{L}, \alpha)$.

Let $(\mathcal{L}, \alpha)$ be a regular hom-Lie–Rinehart algebra. We define a cochain complex $(Alt_A(\mathcal{L}, M), \delta)$, where $Alt_A(\mathcal{L}, M) := \bigoplus_{n \geq 0} Hom_A(\wedge_A^n L, M)$. The coboundary map $\delta : Alt_A^n(\mathcal{L}, M) \to Alt_A^{n+1}(\mathcal{L}, M)$ is defined as follows:

$$\begin{aligned}
&\delta f(x_1, \cdots, x_{n+1})\\
&\qquad = \sum_{i=1}^{n+1} (-1)^{i+1} \theta(\alpha^{-1}(x_i))(f(\alpha^{-1}(x_1), \cdots, \hat{x_i}, \cdots, \alpha^{-1}(x_{n+1}))\\
&\quad + \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} \beta(f(\alpha^{-2}([x_i, x_j]), \alpha^{-1}(x_1), \cdot, \hat{x_i}, \cdot, \hat{x_j}, \cdot, \alpha^{-1}(x_{n+1}))
\end{aligned}$$

for $f \in Alt_A^n(\mathcal{L}, M)$, and $x_i \in L$, for $1 \leq i \leq n+1$. The cohomology of a regular hom-Lie–Rinehart algebra $(\mathcal{L}, \alpha)$ with coefficients in the left module (M, β) is given by the associated cohomology of the cochain complex $(Alt_A(\mathcal{L}, M), \delta)$.

Theorem 5 *Let $(\mathcal{L}, \alpha)$ be a regular hom-Lie–Rinehart algebra over (A, ϕ). If L is a projective A-module of rank n, then there is a bijective correspondence between right $(\mathcal{L}, \alpha)$-module structures on (A, ϕ) and left $(\mathcal{L}, \alpha)$-module structures on $(\wedge_A^n L, \tilde{\alpha})$.*

Corollary 6 *Let $(\mathcal{L}, \alpha)$ be a regular hom-Lie–Rinehart algebra over (A, ϕ). If L is a projective A-module of rank n, then there is a bijective correspondence between left $(\mathcal{L}, \alpha)$-module structures on $(\wedge_A^n L, \tilde{\alpha})$ and exact generators of the hom-Gerstenhaber algebra bracket on $\wedge_A^* L$.*

3.2 Right Modules Over Hom-Lie–Rinehart Algebras

Definition 7 The pair (M, β) is a right module over a hom-Lie Rinehart algebra $(\mathcal{L}, \alpha)$ if the following conditions hold for all $X \in L,\ a \in A,\ m \in M$.

- There is a map $\theta : M \otimes L \to M$ such that the pair (θ, β) is a hom-Lie algebra representation of $(L, [-, -], \alpha)$ on M. Let $\{m, X\} := \theta(m, X)$;
- $\{a.m, X\} = \{m, a.X\} = \phi(a).\{m, X\} - \rho(X)(a).\beta(m)$.

Remark 8 There is no canonical right module structure on (A, ϕ).

For $n \geq 0$, we take $C_n(\mathcal{L}, M) := M \otimes_A \wedge_A^n L$ and define a boundary map $d : C_n(\mathcal{L}, M) \to C_{n-1}(\mathcal{L}, M)$ as

$$\begin{aligned}
&d(m \otimes (x_1 \otimes \cdots \otimes x_n))\\
&\qquad = \sum_{i=1}^{n} (-1)^{i+1} \{m, x_i\} \otimes (\alpha(x_1) \otimes \cdots \otimes \alpha(\hat{x_i}) \otimes \cdots \otimes \alpha(x_n))\\
&+ \sum_{i<j} (-1)^{i+j} \beta(m) \otimes ([x_i, x_j], \alpha(x_1) \otimes \cdots \otimes \alpha(\hat{x_i}) \otimes \cdots \otimes \alpha(\hat{x_j}) \otimes \cdots \otimes \alpha(x_n)).
\end{aligned}$$

Then it follows that $(C_*(\mathcal{L}, M), d)$ is a chain complex. The homology of $(\mathcal{L}, \alpha)$ with coefficient in the right module (M, β) is given by $H_*^{\mathrm{hLR}}(L; M) := H_*(C_*(\mathcal{L}, M))$.

Example If $\alpha = Id_L$ and $\beta = Id_M$, then $H_*^{\mathrm{hLR}}(L; M)$ is the Lie–Rinehart algebra homology with coefficients in M. For $A = R$, the pair (θ, β) is a representation of hom-Lie algebra $(L, [-,-], \alpha)$ on M and $H_*^{\mathrm{hLR}}(L; M)$ is the homology of a hom-Lie algebra with coefficients in M.

Theorem 9 *Let $(\mathcal{L}, \alpha)$ be a hom-Lie–Rinehart algebra over (A, ϕ). Then there is a bijective correspondence between right $(\mathcal{L}, \alpha)$-module structures on (A, ϕ) and exact generators of the associated hom-Gerstenhaber algebra bracket on $\wedge_A^* L$.*

Corollary 10 *The homology $H_*^{\mathrm{hLR}}(\mathcal{L}, A)$ is isomorphic to the homology of the chain complex associated to the hom-Gerstenhaber algebra structure on $\wedge_A^* L$.*

4 Representation of a Hom-Lie Algebroid

In this section, we consider hom-Lie algebroids as a particular case of hom-Lie–Rinehart algebras. Let $\mathcal{A} := (A, \phi, [-,-], \rho, \alpha)$ be a hom-Lie algebroid and (E, ϕ, β) be a hom-bundle over a smooth manifold M. A bilinear map $\nabla : \Gamma A \otimes \Gamma E \to \Gamma E$, denoted by $\nabla_x(s) := \nabla(x, s)$, is a representation of $\mathcal{A}$ on the hom-bundle (E, ϕ, β) if it satisfies the following properties:

1. $\nabla_{f.x}(s) = \phi^*(f).\nabla_x(s)$ for all $x \in \Gamma A,\ s \in \Gamma E$ and $f \in C^\infty(M)$;
2. $\nabla_x(f.s) = \phi^*(f).\nabla_x(s) + \rho(x)[f].\beta(s)$ for all $x \in \Gamma A,\ s \in \Gamma E$ and $f \in C^\infty(M)$;
3. The pair (∇, β) is a hom-Lie algebra representation of $(\Gamma A, [-,-], \alpha)$ on ΓE.

Example

1. Let $\mathcal{A} = (A, \phi, [-,-], \rho, \alpha)$ be a hom-Lie algebroid over M. Then ∇^{ϕ^*} is a canonical representation of $\mathcal{A}$ on the hom-bundle $(M \times \mathbb{R}, \phi, \phi^*)$, given by $\nabla^{\phi^*}(x, f) = \rho(x)[f]$ for all $x \in \Gamma A$ and $f \in C^\infty(M)$.
2. Let $\mathcal{A} = (A, \phi, [-,-], \rho, \alpha)$ be a hom-Lie algebroid over M and (E, ϕ, β) be a hom-bundle over M, where E is a trivial line bundle over M with $s \in \Gamma E$, a nowhere vanishing section of E over M such that $\beta(s) = c.s$ for some $c \in \mathbb{K}$. Define a map $\nabla : \Gamma A \otimes \Gamma E \to \Gamma E$ by $\nabla(x, f.s) = \rho(x)[f].\beta(s)$ for all $x \in \Gamma A$ and $f \in C^\infty(M)$. Then the map ∇ is a representation of $\mathcal{A}$ on (E, ϕ, β).

Proposition 11 ([5]) *Let $\mathcal{A} = (A, \phi, [-,-], \rho, \alpha)$ be a regular hom-Lie algebroid. Then there is a one-one correspondence between representations of $\mathcal{A}$ on the hom-bundle $(\wedge^n A, \phi, \tilde{\alpha})$ and exact generators of the associated hom-Gerstenhaber algebra $\mathfrak{A} := (\oplus_{k \geq 0} \Gamma \wedge^k A^*, \wedge, [-,-]_{\mathcal{A}}, \tilde{\alpha})$ (here, $\tilde{\alpha}$ is extension of the map α to higher degree elements).*

Example Cohomology of regular hom-Lie algebroids: Let $\mathcal{A} := (A, \phi, [-,-], \rho, \alpha)$ be a regular hom-Lie algebroid over M and the map ∇ be a representation of $\mathcal{A}$ on the hom-bundle (E, ϕ, β). Then we define a cochain complex $(C^*(\mathcal{A}; E), d_{A,E})$ for $\mathcal{A}$ with coefficients in this representation as follows: $C^*(\mathcal{A}; E) := \oplus_{n\geq 0}\Gamma(Hom(\wedge^n A, E))$, and the coboundary map $d_{A,E}$ is defined as follows for $\Xi \in \Gamma(Hom(\wedge^n A, E)), x_i \in \Gamma A$ and $1 \leq i \leq n+1$.

$$
\begin{aligned}
&(d_{A,E}\Xi)(x_1, \cdots, x_{n+1}) \\
&\quad = \sum_{i=1}^{n+1}(-1)^{i+1}\nabla_{(\alpha^{-1}(x_i))}(\Xi(\alpha^{-1}(x_1), \cdots, \hat{x_i}, \cdots, \alpha^{-1}(x_{n+1}))) \\
&+ \sum_{1\leq i<j\leq n+1}(-1)^{i+j}\beta(\Xi(\alpha^{-2}([x_i, x_j]), \alpha^{-1}(x_1), \cdots, \hat{x_i}, \cdots, \hat{x_j}, \cdots, \alpha^{-1}(x_{n+1}))).
\end{aligned}
$$

- We denote the cohomology of the resulting cochain complex $(C^*(\mathcal{A}; E), d_{A,E})$ by $H^*(\mathcal{A}, E)$.
- If $\alpha = Id_A$ and $\phi = Id_M$, then $\mathcal{A}$ is a Lie algebroid and $H^*(\mathcal{A}, E)$ is the usual de-Rham cohomology of the Lie algebroid $\mathcal{A}$ with coefficients in the representation on the vector bundle E.

Theorem 12 (Dual Description of a Hom-Lie Algebroid, see [1, 5]) *Let (A, ϕ, α) be a regular hom-bundle over M, i.e. the map $\phi : M \to M$ is a diffeomorphism and $\alpha : \Gamma A \to \Gamma A$ is an invertible map. Then a hom-Lie algebroid structure $\mathcal{A} := (A, \phi, [-, -], \rho, \alpha)$ on the hom-bundle (A, ϕ, α) is equivalent to a $(\hat{\alpha}, \hat{\alpha})$-differential graded commutative algebra on $\oplus_{n\geq 0}\Gamma(\wedge^n A^*)$, where the map $\hat{\alpha} : \Gamma(\wedge^n A^*) \to \Gamma(\wedge^n A^*)$ is defined by $\hat{\alpha}(\xi)(x_1, \cdots, x_n) = \phi^*(\xi(\alpha^{-1}(x_1), \cdots, \alpha^{-1}(x_n)))$ for $\xi \in \Gamma(\wedge^n A^*)$, and $x_i \in \Gamma A$, for $1 \leq i \leq n$.*

5 Strong Differential Hom-Gerstenhaber Algebras

A hom-Gerstenhaber algebra $\mathfrak{A} := (\oplus_{i\in\mathbb{Z}}\mathcal{A}_i, \wedge, [-, -], \alpha)$ is called a differential hom-Gerstenhaber algebra if it is equipped with a degree 1 map $d : \mathfrak{A} \to \mathfrak{A}$ such that

1. the map d is a (α, α)-derivation of degree 1 with respect to the graded commutative and associative product $\wedge$, i.e. for $X, Y \in \mathfrak{A}$,

$$d(X \wedge Y) = d(X) \wedge \alpha(Y) + (-1)^{|X|}\alpha(X) \wedge d(Y).$$

2. $d^2 = 0$ and the map d commutes with α, i.e. $d \circ \alpha = \alpha \circ d$.

The hom-Gerstenhaber algebra $\mathfrak{A}$ is said to be a **strong differential hom-Gerstenhaber algebra** if d also satisfies the equation:

$$d[X, Y] = [dX, \alpha(Y)] + [\alpha(X), dY]$$

for $X, Y \in \mathfrak{A}$. A strong differential hom-Gerstenhaber algebra $\mathfrak{A}$ is called regular if the map $\alpha : \mathfrak{A} \to \mathfrak{A}$ is an invertible map.

Example

1. For any hom-Poisson manifold, the hom-Gerstenhaber algebra associated to the tangent hom-Lie algebroid (and to the cotangent hom-Lie algebroid) is a strong differential hom-Gerstenhaber algebra.
2. For any purely hom-Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, the associated hom-Gerstenhaber algebras are equipped with a strong differential.
3. Given a Gerstenhaber algebra $(\mathcal{A}, [-, -], \wedge)$ with a strong differential d and an endomorphism $\alpha : (\mathcal{A}, [-, -], \wedge) \to (\mathcal{A}, [-, -], \wedge)$ satisfying $d \circ \alpha = \alpha \circ d$, the tuple $(\mathcal{A}, \wedge, [-, -]_\alpha = \alpha \circ [-, -], \alpha, d_\alpha = \alpha \circ d)$ is a strong differential hom-Gerstenhaber algebra.

Theorem 13 *The tuple* $(\bigoplus_{i \in \mathbb{Z}_+} \Gamma(\wedge^i A), \wedge, [-, -], \alpha, d)$ *is a strong differential regular hom-Gerstenhaber algebra if and only if* (A, A^*) *is a hom-Lie bialgebroid (see [1, 5]).*

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One Step Degeneration of Trigonal Curves and Mixing of Solitons and Quasi-Periodic Solutions of the KP Equation

Atsushi Nakayashiki

To the memory of Victor Enolski

Abstract We consider certain degenerations of trigonal curves and hyperelliptic curves, which we call one step degeneration. We compute the limits of corresponding quasi-periodic solutions using the Sato Grassmannian. The mixing of solitons and quasi-periodic solutions is clearly visible in the obtained solutions.

Keywords KP equation · Soliton · Quasi-periodic solution · Sato Grassmannian · Trigonal curve

Mathematics Subject Classification (2010) 37K40, 35C08, 14H70

1 Introduction

The aim of this paper is to compute explicitly the limits of quasi-periodic solutions of the KP (Kadomtsev-Petviashvili) equation according to certain degenerations of trigonal and hyperelliptic curves, which we call one step degeneration.

The KP equation is the 2 + 1 dimensional equation given by

$$3u_{t_2t_2} + (-4u_{t_3} + 6uu_{t_1} + u_{t_1t_1t_1})_{t_1} = 0, \tag{1}$$

where (t_1, t_2) and t_3 are space and time variables respectively. It can be rewritten in the Hirota bilinear form:

$$(D_{t_1}^4 - 4D_{t_2}D_{t_3} + 3D_{t_2}^2)\tau \cdot \tau = 0, \tag{2}$$

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where D_{t_i}'s are Hirota derivatives defined by

$$f(t+s)g(t-s)=\sum_{n=0}^{\infty} D_t^n f\cdot g\frac{s^n}{n!}.$$

For a solution τ of (2) $u=2(\log\tau)_{t_1t_1}$ gives a solution of (1). The KP hierarchy is the infinite system of differential equations which contains the KP equation (2) as its first member [6]. It is given by

$$\int \tau(t-s-[z^{-1}])\tau(t+s+[z^{-1}])\mathrm{e}^{-2\sum_{j=1}^{\infty} s_j z^j}\frac{dz}{2\pi i}=0, \tag{3}$$

where $t=(t_1,t_2,\ldots)$, $s=(s_1,s_2,\ldots)$, $[z^{-1}]=[z^{-1},z^{-2}/2,z^{-3}/3,\ldots]$ and the integral signifies taking the coefficient of z^{-1} in the series expansion of the integrand. Expanding (3) by s we get differential equations for $\tau(t)$ in the Hirota bilinear form. A solution $\tau(t)$ is sometimes called a tau function. The introduction of the infinitely many variables is indispensable to the Sato theory which we use in this paper.

The KP hierarchy has a variety of solutions. Among them soliton solutions and algebro-geometric solutions are relevant to us. Soliton solutions are the solutions expressed by exponential functions given as follows (see [11] for example). Take positive integers $N<M$, non-zero parameters κ_j, $1\le j\le M$ and an $N\times M$-matrix $A=(a_{i,j})$. Then soliton solution is given by

$$\tau(t)=\sum_{I=(i_1<\cdots<i_N)}\Delta_I A_I \mathrm{e}^{\eta(\kappa_{i_1})+\cdots+\eta(\kappa_{i_M})}, \tag{4}$$

$$\Delta_I=\prod_{p<q}(\kappa_{i_q}-\kappa_{i_p}),\quad A_I=\det(a_{p,i_q})_{1\le p,q\le N},\quad \eta(\kappa)=\sum_{i=1}^{\infty}t_i\kappa^i.$$

Recently it was discovered that the shapes of soliton solutions form various web patterns and that they are related with the geometry of Grassmann manifolds, cluster algebras (see [11] and references therein).

Quasi-periodic solutions, which is also called algebro-geometric solutions, constitute a class of solutions expressed by theta functions of algebraic curves with positive genus. Periodic solutions are contained in this class. Soliton solutions can be considered as the limits of quasi-periodic solutions when periods go to infinity. In terms of curves soliton solutions are the genus zero limits of quasi-periodic solutions. Our original motivation of the research was to take these limits and compare the structure of the quasi-periodic solutions and that of solitons described in [11]. However in the course of study [2] we come to the recognition that the limits to positive genus solutions are more fundamental. Anyhow the difficulty here is that to take a limit of a theta function or, in other words, to take a limit of the period matrix of an algebraic curve, is not very easy.

In [2, 18, 19] we have demonstrated that the Sato Grassmannian (UGM) approach to this kind of problem is very effective. The reason, roughly speaking, is explained as follows. There is a one to one correspondence between points of UGM and solutions of the KP-hierarchy up to constants. Using UGM an algebro-geometric solution can be described as a series whose coefficients are constructed from some rational functions on an algebraic curve. In this way the difficult problem on taking limits of period matrices reduces to much easier problem on taking limits of rational functions. In this paper we develop the UGM approach further.

We consider the following degeneration of algebraic curves, which we call one step degeneration, given by

$$y^m = \prod_{j=1}^{mn+1} (x-\alpha_j) \longrightarrow y^m = (x-\alpha)^m \prod_{j=1}^{m(n-1)+1} (x-\alpha_j), \tag{5}$$

for $m = 2, 3$. Fix m and denote by C_n the non-singular curve before taking the limit. We define some canonical tau function $\tau_{n,0}(t)$ (see (35)) corresponding to the curve C_n. Then we express the limit of $\tau_{n,0}(t)$ in terms of $\tau_{n-1,0}(t)$ with the variable t being appropriately shifted. Then a solitonic structure can be seen clearly in the degeneration of the algebro-geometric solution $\tau_{n,0}(t)$. This is another crucial idea in this paper.

The results are as follows. For $m = 2$, that is, the case of a hyperelliptic curve, we have (Theorem 24),

$$\begin{aligned} \lim \tau_{n,0}(t) &= C \mathrm{e}^{-2\sum_{l=1}^{\infty} \alpha^l t_{2l}} \\ &\times \left(\mathrm{e}^{\eta(\alpha^{1/2})} \tau_{n-1,0}(t - [\alpha^{-1/2}]) + (-1)^n \mathrm{e}^{\eta(-\alpha^{1/2})} \tau_{n-1,0}(t - [-\alpha^{-1/2}]) \right), \end{aligned} \tag{6}$$

for some constant C. It is observed that the soliton factors $\mathrm{e}^{\eta(\pm\alpha^{1/2})}$ pop out from $\tau_{n,0}(t)$. Then the solution (6) looks like a mixture of solitons and quasi-periodic solutions. Using the formula repeatedly and noting that $\tau_{0,0}(t) = 1$ if $\alpha_1 = 0$ we get well known soliton solutions of the KdV equation.

For $m = 3$ we have (Theorem 18)

$$\begin{aligned} \lim \tau_{n,0}(t) &= \mathrm{e}^{-6\sum_{l=1}^{\infty} \alpha^l t_{3l}} \\ &\times \sum_{0\le i<j\le 2, 0\le k\le 2} \frac{\partial}{\partial \beta} \Big(\tilde{C}_{i,j,k}(\alpha,\beta) \mathrm{e}^{\eta(z_i(\alpha)^{-1}) + \eta(z_j(\alpha)^{-1}) + \eta(z_k(\beta)^{-1})} \\ &\times \tau_{n-1,0}(t - [z_i(\alpha)] - [z_j(\alpha)] - [z_k(\beta)]) \Big)\Big|_{\beta=\alpha}, \\ & z_i(\alpha) = \omega^{-i}\alpha^{-1/3}, \quad \omega = \mathrm{e}^{2\pi i/3}, \end{aligned} \tag{7}$$

for some constants $\tilde{C}_{i,j,k}(\alpha, \beta)$. A new feature in this case is the appearance of the derivative with respect to the parameter β. This corresponds to the fact that the limit of $\tau_{n,0}$ to genus zero curve in this case is not a soliton but a generalized soliton [18]. The constants $\tilde{C}_{i,j,k}(\alpha, \beta)$ should be expressed by some derivatives of the sigma function. The explicit formulas for them are important for the further analysis of the solutions.

We remark that the formula of the forms (6), (7) can be generalized for $m \geq 4$ in (5). They should be treated in a subsequent papers. A generalization of the results in this paper to other class of curves such as that treated in [1] is also interesting.

The paper is organized as follows. In Sect. 2 we first review the theory of the Sato Grassmannian (UGM). Then we explain how to embed the space of functions on an algebraic curve to UGM. Next we apply the general theory to our concrete examples and define the frame $\tilde{\xi}_n$ of a point of UGM corresponding to the space of regular rational functions on $C_n \backslash \{\infty\}$. Then we study the degeneration of $\tilde{\xi}_n$ and define the frame ξ_n as a gauge transformation of $\tilde{\xi}_n$. In order to express ξ_n by an object associated with the curve C_{n-1} we study the frame associated with the space of rational functions on $C_{n-1} \backslash \{\infty\}$ which are singular at three points. Decomposing some rational functions we derive the degeneration formula of the tau function $\tau(t; \tilde{\xi}_n)$ corresponding to $\tilde{\xi}_n$ in terms of some tau functions associated with the curve C_{n-1} in the final subsection of Sect. 2. In Sect. 3 we first review the sigma function of a so called (N, M) curve. Then we recall the sigma function expression of $\tau(t; \tilde{\xi}_n)$. Next we express the tau function corresponding to the space of functions with additional singularities as a shift of $\tau(t; \tilde{\xi}_n)$. By substituting these formulas to the degeneration formula derived in Sect. 2 we express the limit of $\tau(t; \tilde{\xi}_n)$ in terms of the shift of $\tau(t; \tilde{\xi}_{n-1})$. In Sect. 4 we derive a similar degeneration formula for hyperelliptic curves based on the results of [2].

2 Sato Grassmannian and τ-Function

In this section we briefly recall the definition and basic properties of the Sato Grassmannian.

2.1 Sato Grassmannian

Let $V = \mathbb{C}((z))$ be the vector space of Laurent series in the variable z and $V_\phi = \mathbb{C}[z^{-1}]$, $V_0 = z\mathbb{C}[[z]]$ two subspaces of V. Then V is isomorphic to $V_\phi \oplus V_0$. Let $\pi : V \longrightarrow V_\phi$ be the projection map. Then the Sato Grassmannian UGM is defined as the set of subspaces U of V such that the restriction $\pi|_U$ has the finite dimensional kernel and cokernel whose dimensions coincide.

To an element $\sum a_n z^n \in V$ we associate the infinite column vector $(a_n)_{n\in\mathbb{Z}}$. Then a frame of a point U of UGM is expressed by a $\mathbb{Z}\times\mathbb{N}_{\le 0}$ matrix $\xi = (\xi_{i,j})_{i\in\mathbb{Z},j\in\mathbb{N}_{\le 0}}$, where columns, and therefore a basis of U, are labeled by the set of non-positive integers $\mathbb{N}_{\le 0}$. A frame ξ is written in the form

$$\xi = \begin{pmatrix} & \vdots & \vdots \\ \cdots & \xi_{-1,-1} & \xi_{-1,0} \\ \cdots & \xi_{0,-1} & \xi_{0,0} \\ - - - & - - - & - - - \\ \cdots & \xi_{1,-1} & \xi_{1,0} \\ \cdots & \xi_{2,-1} & \xi_{2,0} \\ & \vdots & \vdots \end{pmatrix} \tag{8}$$

It is always possible to take a frame satisfying the following condition, there exists a negative integer l such that

$$\xi_{i,j} = \begin{cases} 1 & \text{if } j < l \text{ and } i = j \\ 0 & \text{if } (j < l \text{ and } i < j) \text{ or } (j \ge l \text{ and } i < l). \end{cases} \tag{9}$$

In the sequel we always take a frame which satisfies this condition, although it is not unique.

A Maya diagram $M = (m_j)_{j=0}^{\infty}$ is a sequence of decreasing integers such that $m_j = -j$ for all sufficiently large j. For a Maya diagram $M = (m_j)_{j=0}^{\infty}$ the corresponding partition is defined by $\lambda(M) = (j + m_j)_{j=0}^{\infty}$. By this correspondence the set of Maya diagrams and the set of partitions bijectively correspond to each other.

For a frame ξ and a Maya diagram $M = (m_j)_{j=0}^{\infty}$ define the Plücker coordinate by

$$\xi_M = \det(\xi_{m_i,j})_{-i,j\le 0}$$

Due to the condition (9) and the condition of the Maya diagram M this infinite determinant can be computed as the finite determinant $\det(\xi_{m_i,j})_{k\le -i,j\le 0}$ for sufficiently small k.

Define the elementary Schur function $p_n(t)$ by

$$e^{\sum_{n=1}^{\infty} t_n \kappa^n} = \sum_{n=0}^{\infty} p_n(t)\kappa^n.$$

The Schur function [13] corresponding to a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ is defined by

$$s_\lambda(t) = \det(p_{\lambda_i - i + j}(t))_{1\le i,j\le l}.$$

Assign the weight j to the variable t_j. Then it is known that $s_\lambda(t)$ is homogeneous of weight $|\lambda| = \lambda_1 + \cdots + \lambda_l$. To a point U of UGM take a frame ξ and define the tau function by

$$\tau(t;\xi) = \sum_M \xi_M s_{\lambda(M)}(t). \tag{10}$$

If we change the frame ξ $\tau(t;\xi)$ is multiplied by a constant. We call $\tau(t;\xi)$, for any frame ξ of U, a tau function corresponding to U. So tau functions of a point of UGM differ by constant multiples to each other.

Then

Theorem 1 ([24]) *The tau function $\tau(t;\xi)$ is a solution of the KP-hierarchy. Conversely for a formal power series solution $\tau(t)$ of the KP-hierarchy there exists a point U of UGM such that $\tau(t)$ coincides with a tau function of U.*

The point U of UGM corresponding to a solution $\tau(t)$ in Theorem 1 is given as follows [10, 16, 23, 24].

Let $\Psi^*(t;z)$ be the adjoint wave function [6] corresponding to $\tau(t)$ which is defined by

$$\Psi^*(t;z) = \frac{\tau(t+[z])}{\tau(t)} e^{-\sum_{i=1}^\infty t_i z^{-i}}. \tag{11}$$

Define $\Psi_i^*(z)$ by the following expansion

$$\begin{aligned}&\left(\tau(t)\Psi^*(t;z)\right)|_{t=(x,0,0,0,\ldots)}\\ &\qquad = \tau((x,0,0,0,\ldots)+[z])e^{-xz^{-1}} = \sum_{i=0}^\infty \Psi_i^*(z)x^i.\end{aligned} \tag{12}$$

Then

$$U = \sum_{i=0}^\infty \mathbb{C}\Psi_i^*(z). \tag{13}$$

By this correspondence between points of UGM and tau functions the following property follows. Let U be a point of UGM, $\tau(t)$ be a tau function corresponding to U and $f(z) = e^{\sum_{i=1}^\infty a_i \frac{z^i}{i}}$ be an invertible formal power series. Then $f(z)U$ belongs to UGM and the corresponding tau function is given by

$$e^{\sum_{i=1}^\infty a_i t_i}\tau(t). \tag{14}$$

It is sometimes called the gauge transformation of $\tau(t)$.

2.2 Embedding of Algebro-Geometric Data to UGM

In this section we recall the construction of points of UGM from algebraic curves (see [14, 19] for more details).

Let C be a compact Riemann surface of genus g, p_∞ a point on it, z a local coordinate around p_∞. For $m \geq 0$ and points p_i, $1 \leq i \leq m$, on C, such that $p_j \neq \infty$ for any j, we denote by

$$H^0(C, \mathcal{O}(\sum_{j=1}^{m} p_j + *p_\infty)) \tag{15}$$

the vector space of meromorphic functions on C which have a pole at each p_j of order at most 1 and have a pole at p_∞ of any order. By expanding functions in the local coordinate z we can consider $H^0(C, \mathcal{O}(\sum_{j=1}^{m} p_j + *p_\infty))$ as a subspace of $V = \mathbb{C}((z))$. Then

Proposition 2 ([14, 19]) *The subspace* $z^{g-m} H^0(C, \mathcal{O}(\sum_{j=1}^{m} p_j + *p_\infty))$ *belongs to UGM.*

Remark 3 This Proposition was proved in [19] from the general results [14], for $m \leq g$. But the case $m > g$ can be proved in the same way.

2.3 Tau Function Corresponding to Zero Point Space

For $n \geq 1$ and mutually distinct complex numbers $\{\alpha_i\}_{i=1}^{3n}$ consider the compact Riemann surface C_n corresponding to the algebraic curve defined by the equation

$$y^3 = \prod_{j=1}^{3n+1} (x - \alpha_j). \tag{16}$$

The genus of C_n is $g = 3n$ and there is a unique point on C_n over $x = \infty$ which we denote by ∞.

Consider the space $H^0(C_n, \mathcal{O}(*\infty))$ which corresponds to $m = 0$ in (15). It is the space of meromorphic functions on C which are regular on $C_n \backslash \{\infty\}$. It can be easily proved that it coincides with the vector space $\mathbb{C}[x, y]$ of polynomials in x, y. A basis of this vector space is given by

$$x^i, \quad x^i y, \quad x^i y^2 \quad i \geq 0. \tag{17}$$

We take the local coordinate z around ∞ such that

$$x = z^{-3}, \qquad y = z^{-(3n+1)} F_n(z), \qquad F_n(z) = \left(\prod_{j=1}^{3n+1} (1-\alpha_j z^3)\right)^{1/3}. \tag{18}$$

In the following we denote by z this local coordinate unless otherwise stated. The function $F_n(z)$ is considered as a power series in z by the Taylor expansion at $z = 0$.

By Proposition 2 $z^g H^0(C_n, \mathcal{O}(*\infty))$ determines a point of UGM. Writing (17) in terms of z and multiplying them by z^g we get a basis of it,

$$z^{3n-3i}, \qquad z^{-1-3i} F_n(z), \qquad z^{-3n-2-3i} F_n(z)^2 \qquad i \geq 0. \tag{19}$$

We define the frame $\tilde{\xi}_n$ from this basis as follows.

For an element $v(z) = \sum_{n \leq i} a_i z^i$, $a_n \neq 0$, define the order of $v(z)$ to be $-n$ and write $\mathrm{ord}\, v(z) = -n$.

Definition 4 Label the elements of (19) by $\tilde{v}_i$, $i \leq 0$, in such a way that $\mathrm{ord}\, \tilde{v}_0 < \mathrm{ord}\, \tilde{v}_{-1} < \mathrm{ord}\, \tilde{v}_{-2} < \cdots$ and define the frame $\tilde{\xi}_n$ of $z^g H^0(C_n, \mathcal{O}(*\infty))$ by

$$\tilde{\xi}_n = (\ldots, \tilde{v}_{-2}, \tilde{v}_{-1}, \tilde{v}_0). \tag{20}$$

By the construction of $\tilde{\xi}_n$ the tau function $\tau(t; \tilde{\xi}_n)$ has the following expansion (see [16])

$$\tau(t; \tilde{\xi}_n) = s_{\lambda^{(n)}}(t) + \text{h.w.t}, \tag{21}$$

where h.w.t means the higher weight terms, $\lambda^{(n)}$ is the partition determined from the gap sequence $w_1 < \cdots < w_g$ at ∞ of C_n and is given by

$$\lambda^{(n)} = (w_g - (g-1), \ldots, w_2 - 1, w_1).$$

Example 5 $\lambda^{(1)} = (3, 1, 1), \lambda^{(2)} = (6, 4, 2, 2, 1, 1), \lambda^{(3)} = (9, 7, 5, 3, 3, 2, 2, 1, 1)$.

2.4 Degeneration

Let us take a complex number α which is different from α_i, $1 \leq i \leq 3n-2$ and consider the limit

$$\alpha_{3n+1}, \alpha_{3n}, \alpha_{3n-1} \to \alpha, \tag{22}$$

which means that the curve C_n degenerates to

$$y^3 = (x-\alpha)^3 \prod_{j=1}^{3n-2} (x-\alpha_j). \tag{23}$$

which we call one step degeneration of C_n.

In the limit

$$F_n(z) \longrightarrow (1-\alpha z^3)F_{n-1}(z),$$

and the basis (19) tends to

$$z^{3n-3i},\ z^{-1-3i}(1-\alpha z^3)F_{n-1}(z),\ z^{-3n-2-3i}(1-\alpha z^3)^2F_{n-1}(z)^2,\ i \geq 0. \tag{24}$$

Let W_n be the point of UGM generated by this basis. Multiply (24) by $(1-\alpha z^3)^{-2}$ we have

$$\frac{z^{3n-3i}}{(1-\alpha z^3)^2}, \quad \frac{z^{-1-3i}}{(1-\alpha z^3)}F_{n-1}(z), \quad z^{-3n-2-3i}F_{n-1}(z)^2 \qquad i \geq 0. \tag{25}$$

By taking linear combinations we have

Lemma 6 *The following set of elements gives a basis of* $(1-\alpha z^3)^{-2}W_n$.

$$\begin{aligned} &z^{3n-6-3i}, \quad z^{-4-3i}F_{n-1}(z), \quad z^{-3n-2-3i}F_{n-1}(z)^2, \quad i \geq 0, \\ &\frac{z^{3n}}{(1-\alpha z^3)^2}, \quad \frac{z^{3n-3}}{1-\alpha z^3}, \quad \frac{z^{-1}}{1-\alpha z^3}F_{n-1}(z). \end{aligned} \tag{26}$$

We arrange the basis elements of this lemma according as their orders and define the frame ξ_n as follows.

Definition 7 Define the frame ξ_n of W_n by

$$\xi_n = (\ldots, v_{-2}, v_{-1}, v_0),$$

with

$$\begin{aligned} v_0 &= \frac{z^{3n}}{(1-\alpha z^3)^2}, \\ v_{-1} &= \frac{z^{3n-3}}{1-\alpha z^3}, \\ v_{-(2+i)} &= z^{3n-6-3i}, \qquad\qquad 0 \leq i \leq n-2, \end{aligned}$$

$$
\begin{aligned}
v_{-(n+1)} &= \frac{z^{-1}}{1-\alpha z^3} F_{n-1}(z), && \\
v_{-(n+2+2i)} &= z^{-3-3i}, && 0 \le i \le n-1, \\
v_{-(n+3+2i)} &= z^{-4-3i} F_{n-1}(z), && 0 \le i \le n-1, \\
v_{-(3n+2+3i)} &= z^{-3n-2-3i} F_{n-1}(z)^2, && i \ge 0, \\
v_{-(3n+3+3i)} &= z^{-3n-3-3i}, && i \ge 0, \\
v_{-(3n+4+3i)} &= z^{-3n-4-3i} F_{n-1}(z), && i \ge 0.
\end{aligned}
$$

Since we have the expansion

$$
\log(1-\alpha z^3)^{-2} = 6 \sum_{l=1}^{\infty} \alpha^l \frac{z^{3l}}{3l},
$$

the following relation holds by (14),

$$
\tau(t;\xi_n) = \mathrm{e}^{6\sum_{l=1}^{\infty} \alpha^l t_{3l}} \lim \tau(t;\tilde{\xi}_n), \tag{27}
$$

where the lim signifies taking the limit (22).

2.5 Three Point Insertion

Consider the curve C_{n-1} defined by (16) where n is replaced by $n-1$. The genus of C_{n-1} is $g' = 3n-3 = g-3$. Let

$$
Q_j = (c_j, Y_j), \quad j = 0, 1, 2, \tag{28}
$$

be points on C_{n-1}. We assume $c_j \neq \alpha_i$ for any i, j. Define φ_j by

$$
\varphi_j = \frac{y^2 + Y_j y + Y_j^2}{x - c_j}.
$$

The pole divisor of this function is $Q_j + (2g'-1)\infty$. Consider the space $H^0(C_{n-1}, \mathcal{O}(Q_0 + Q_1 + Q_2 + *\infty))$. A basis of it is given by

$$
x^i, \quad x^i y, \quad x^i y^2, \quad \varphi_j, \quad i \ge 0, \quad j = 0, 1, 2.
$$

Write this basis in terms of the local coordinate z and multiply it by $z^{g'-3}$ we have

$$z^{3n-6-3i},\ z^{-4-3i}F_{n-1}(z),\ z^{-3n-2-3i}F_{n-1}(z)^2,\ z^{3n-6}\varphi_j,\ i \geq 0,\ j = 0, 1, 2. \tag{29}$$

By Proposition 2 $z^{g'-3}H^0(C_{n-1}, \mathcal{O}(Q_0 + Q_1 + Q_2 + *\infty))$ is a point of UGM and the set of functions (29) is a basis of it. Using this basis define the frame of $z^{g'-3}H^0(C_{n-1}, \mathcal{O}(Q_0 + Q_1 + Q_2 + *\infty))$ by

$$\begin{aligned} &\xi_{n-1}(Q_0, Q_1, Q_2) \\ &\qquad = (\ldots, v_{-(n+3)}, v_{-(n+2)}, v_{-n}, \ldots, v_{-2}, z^{3n-6}\varphi_0, z^{3n-6}\varphi_1, z^{3n-6}\varphi_2), \end{aligned}$$

where v_j is the same as that in ξ_n.

2.6 Degeneration Formula in Algebraic Form

Corresponding to the parameter α in (22) let $P_i(\alpha) = (\alpha, \omega^i y_0(\alpha))$, $i = 0, 1, 2$ be points on C_{n-1}, where $\omega = e^{2\pi i/3}$. Take $Q_j = P_j(\alpha)$ in (28) and denote the function φ_j by $\varphi_j(\alpha)$. Then

$$\varphi_j(\alpha) = \frac{y^2 + (\omega^j y_0(\alpha))y + (\omega^j y_0(\alpha))^2}{x - \alpha}.$$

Lemma 8 *For* $0 \leq i \leq 2$ *we have*

$$\frac{y^i}{x - \alpha} = \frac{1}{3y_0(\alpha)^{2-i}} \sum_{j=0}^{2} \omega^{(i+1)j}\varphi_j(\alpha)$$

The lemma can be verified by direct computation. From these relations we have

$$v_{-1} = \frac{z^{3n-3}}{1 - \alpha z^3} = \frac{1}{3y_0(\alpha)^2} \sum_{i=0}^{2} \omega^i z^{3n-6}\varphi_i(\alpha) \tag{30}$$

$$v_{-(n+1)} = \frac{z^{-1}F_{n-1}(z)}{1 - \alpha z^3} = \frac{1}{3y_0(\alpha)} \sum_{i=0}^{2} \omega^{2i} z^{3n-6}\varphi_i(\alpha) \tag{31}$$

$$v_0 = \frac{z^{3n}}{(1 - \alpha z^3)^2} = \frac{\partial}{\partial\beta}\left(\frac{1}{3y_0(\beta)^2} \sum_{i=0}^{2} \omega^i z^{3n-6}\varphi_i(\beta)\right)\Bigg|_{\beta=\alpha}. \tag{32}$$

The third equation is obtained by differentiating the first equation in α.

Let λ be a partition and consider the Plücker coordinate of $(\xi_n)_\lambda$. Substitute the above expression to the definition of $(\xi_n)_\lambda$ of ξ_n. Then Eqs. (30)–(32) mean that each of the column vectors of ξ_n corresponding to v_0, v_{-1}, $v_{-(n+1)}$ is a sum of vectors. So we have

$$(\xi_n)_\lambda = \frac{(-1)^n}{27y_{n-1,0}(\alpha)^5} \sum_{0\le i<j\le 2, 0\le k\le 2} \omega^{i+k+2j}(1-\omega^{i-j}) \times \frac{\partial}{\partial\beta}\left(\xi_{n-1}(P_i(\alpha), P_j(\alpha), P_k(\beta))_\lambda\right)|_{\beta=\alpha}.$$

Multiplying this equation by $s_\lambda(x)$ and summing up in λ we get

$$\tau(t;\xi_n) = \frac{(-1)^n}{27y_{n-1,0}(\alpha)^5} \sum_{0\le i<j\le 2, 0\le k\le 2} \omega^{i+k+2j}(1-\omega^{i-j}) \times \frac{\partial}{\partial\beta}\left(\tau(t;\xi_{n-1}(P_i(\alpha), P_j(\alpha), P_k(\beta)))\right)|_{\beta=\alpha}.$$

Finally using (27) we obtain

Theorem 9 *Consider the limit (22). Then the limit of the tau function of the frame $\tilde{\xi}_n$ defined by (20) is given by the following formula:*

$$\lim \tau(t;\tilde{\xi}_n) = \frac{(-1)^n}{27y_{n-1,0}(\alpha)^5} e^{-6\sum_{l=1}^{\infty}\alpha^l t_{3l}} \sum_{0\le i<j\le 2, 0\le k\le 2} \omega^{i+k+2j}(1-\omega^{i-j}) \times \frac{\partial}{\partial\beta}\left(\tau\left(t;\xi_{n-1}(P_i(\alpha), P_j(\alpha), P_k(\beta))\right)\right)\Big|_{\beta=\alpha}.$$

Remark 10 The new feature of the trigonal case compared with the hyperelliptic case studied in [2] (see Theorem 20) is the existence of a derivative in the parameter β. In [18] the degeneration to genus zero curve in the trigonal case was directly studied. The obtained solutions are not solitons but generalized solitons. The appearance of the derivative corresponds to this phenomenon.

3 Analytic Expression of Tau Functions

In this section we derive the analytic expression of tau functions appeared in Theorem 9 in terms of the multivariate sigma function [3–5, 15, 16]. The fundamental idea behind constructing the expression is due to Krichever [12].

3.1 The Sigma Function of an (N, M) Curve

We consider the general (N, M)-curve [5] defined by $f(x, y) = 0$ with

$$f(x, y) = y^N - x^M - \sum_{Ni+Mj<NM} \lambda_{ij} x^i y^j, \tag{33}$$

where N, M are relatively prime integers such that $1 < N < M$. We assume that the curve is non singular. We denote the corresponding compact Riemann surface by C. Then the genus of C is given by $g = 1/2(N-1)(M-1)$. There is one point on C over $x = \infty$ which is also denoted by ∞. Here we recall several necessary facts related with the curve C. See [15, 16] for details.

We assign the order $Ni + Mj$ to the monomial $x^i y^j$, $i, j \geq 0$, and define f_i, $i \geq 1$, to be the i-th monomial in this order. For example $f_1 = 1$, $f_2 = x$. Then the set of differentials

$$du_i = -\frac{f_{g+1-i}dx}{f_y}, \quad 1 \leq i \leq g$$

constitutes a basis of holomorphic one forms. We choose an algebraic fundamental form $\widehat{\omega}(p_1, p_2)$ on $C \times C$ as in [15]. It has the decomposition of the form

$$\widehat{\omega}(p_1, p_2) = d_{p_2}\Omega(p_1, p_2) + \sum_{i=1}^{g} du_i(p_1)dr_i(p_2),$$

where $\Omega(p_1, p_2)$ is a certain meromorphic one form on $C \times C$ and $dr_i(p)$ is a certain differential of the second kind on C with a pole only at ∞ (see [15] for more precise form of $\widehat{\omega}$, Ω, dr_i). Taking a symplectic basis $\{\alpha_i, \beta_i\}_{i=1}^{g}$ of the homology group of C we define the period matrices ω_k, η_k, $k = 1, 2$, Π by

$$2\omega_1 = \left(\int_{\alpha_j} du_i\right), \qquad 2\omega_2 = \left(\int_{\beta_j} du_i\right),$$

$$-2\eta_1 = \left(\int_{\alpha_j} dr_i\right), \qquad -2\eta_2 = \left(\int_{\beta_j} dr_i\right),$$

and $\Pi = \omega_1^{-1}\omega_2$. Define Riemann's theta function by

$$\theta[\epsilon](z, \Pi) = \sum_{m \in \mathbb{Z}^g} \mathrm{e}^{\pi i^t(m+\epsilon')\Pi(m+\epsilon')+2\pi i^t(m+\epsilon')(z+\epsilon'')},$$

where $\epsilon = {}^t(\epsilon', \epsilon'') \in \mathbb{R}^{2g}$, $\epsilon', \epsilon'' \in \mathbb{R}^g$. Let $\Pi\delta' + \delta''$, $\delta', \delta'' \in (1/2)\mathbb{Z}^g$, be a representative of Riemann's constant with respect to the choice of the base point ∞ and $\{\alpha_i, \beta_i\}_{i=1}^g$, and $\delta = {}^t(\delta', \delta'') \in (1/2)\mathbb{Z}^{2g}$.

Let $(w_1, \ldots, w_g)$, $w_1 < \cdots < w_g$, be the gap sequence of the curve C at ∞ (see [7, 15] for example). Define the partition $\lambda^{(N,M)}$ by

$$\lambda^{(N,M)} = (w_g - (g-1), \ldots, w_2 - 1, w_1).$$

By the definition $\lambda^{(n)} = \lambda^{(3,3n+1)}$.

Definition 11 The sigma function is defined by

$$\sigma(u) = C e^{\frac{1}{2}{}^t u \eta_1 \omega_1^{-1} u} \theta[-\delta]((2\omega_1)^{-1} u, \Pi),$$
$$u = {}^t(u_1, \ldots, u_g)$$

for some constant C.

Assign the weight w_i to u_i. Then the constant C is specified by the condition that $\sigma(u)$ has the expansion of the form

$$\sigma(u) = s_{\lambda^{(N,M)}}(t)|_{t_{w_i} = u_i} + \text{h.w.t.}$$

It is known that C is explicitly expressed by some derivatives of the Riemann's theta function [17, 20]. The sigma function satisfies the following quasi-periodicity property:

$$\sigma(u + \sum_{i=1}^{2} 2\omega_i m_i) = (-1)^{{}^t m_1 m_2 + 2({}^t\delta' m_1 - {}^t\delta'' m_2)} e^{{}^t(\sum_{i=1}^{2} 2\eta_i m_i)(u + \sum_{i=1}^{2} \omega_i m_i)} \sigma(u). \tag{34}$$

3.2 Sigma Function Expression of Tau Functions

Here we derive sigma function expressions for the tau functions corresponding to the spaces in Proposition 2 in the case of (N, M) curves.

We take the local coordinate z around ∞ such that

$$x = z^{-N}, \qquad y = z^{-M}(1 + O(z)).$$

Expand du_i, $\widehat{\omega}$ in z as

$$du_i = \sum_{j=1}^{\infty} b_{i,j} z^{j-1},$$

$$\widehat{\omega}(p_1, p_2) = \left(\frac{1}{(z_1 - z_2)^2} + \sum_{i,j \geq 1} \widehat{q}_{i,j} z_1^{i-1} z_2^{j-1} \right) dz_1 dz_2,$$

where $z_i = z(p_i)$. The differential du_g has a zero of order $2g-2$ at ∞ and has the expansion of the form

$$du_g = z^{2g-2}(1 + \sum_{j=2g}^{\infty} b_{g,j} z^{j-2g+1}) dz.$$

Define c_i by the expansion

$$\log \left(\sqrt{z^{-2g+2} \frac{du_g}{dz}} \right) = \sum_{i=1}^{\infty} c_i \frac{z^i}{i}.$$

In [16] there is a misprint, $c_i z^i$ should be $c_i z^i / i$ as above. Define $g \times \mathbb{N}$ matrix B and the quadratic form $\widehat{q}$ by

$$B = (b_{i,j})_{1 \leq i \leq g, j \geq 1}, \qquad \widehat{q}(t) = \sum_{i,j=1}^{\infty} \widehat{q}_{i,j} t_i t_j.$$

The following theorem is proved in [16].

Theorem 12 ([16]) *A tau function corresponding to* $z^g H^0(C, \mathcal{O}(*\infty))$ *is given by*

$$\tau_0(t) := \mathrm{e}^{-\sum_{i=1}^{\infty} c_i t_i + \frac{1}{2}\widehat{q}(t)} \sigma(Bt). \tag{35}$$

It has the expansion of the form

$$\tau_0(t) = s_{\lambda(N,M)}(t) + \text{h.w.t.} \tag{36}$$

Remark 13 In [16] it is proved that $\tau_0(t)$ defined by (35) is a solution of the N-reduced KP-hierarchy [6].

More generally the tau function corresponding to the m-point space with $m \geq 1$ given by Proposition 2 is described in terms of the shift of $\tau_0(t)$.

Theorem 14 *Let p_i, $1 \le i \le m$, be points on $C \backslash \{\infty\}$ and $z_i = z(p_i)$. A tau function corresponding to $z^{g-m} H^0(C, \mathcal{O}(\sum_{i=1}^m p_i + *\infty))$ is given by*

$$\tau(t|p_1, \ldots, p_m) := \mathrm{e}^{\sum_{i=1}^{\infty} \eta(z_i^{-1})} \tau_0(t - \sum_{i=1}^{m} [z_i]), \tag{37}$$

where $\eta(\kappa) = \sum_{i=1}^{\infty} t_i \kappa^i$, $[w] = [w, w^2/2, w^3/3, \ldots]$.

By (14) and by that the KP-hierarchy is the system of autonomous equations, if $\tau(t)$ is a solution of the KP-hierarchy, so is $\mathrm{e}^{\sum_{i=1}^{\infty} \gamma_i t_i} \tau(t + \zeta)$ for any set of constants $\{\gamma_i\}$ and a constant vector ζ. Therefore $\tau(t|p_1, \ldots, p_m)$ is a solution of the KP-hierarchy.

Then the theorem is proved by calculating the adjoint wave function using (13). To this end we need some notation.

Let $E(p_1, p_2)$ be the prime form [8] (see also [10]). Define $E(z_1, z_2)$, $E(q, p)$ with $z_i = z(p_i)$ and q being a fixed point on C by

$$E(p_1, p_2) = \frac{E(z_1, z_2)}{\sqrt{dz_1}\sqrt{dz_2}}, \qquad E(q, p) = \frac{E(z(q), z(p))}{\sqrt{dz(p)}}.$$

Define $\tilde{E}(q, p)$ for q fixed by

$$\tilde{E}(q, p) = E(q, p)\sqrt{du_g(p)}\mathrm{e}^{\frac{1}{2}\int_q^p {}^t du (\eta_1 \omega_1^{-1}) \int_q^p du},$$
$$du = {}^t(du_1, \ldots, du_g).$$

In [15] two variables $\tilde{E}(p_1, p_2)$ and one variable $\tilde{E}(\infty, p)$ were introduced and studied. It should be noticed that $\tilde{E}(q, p)$ is a multiplicative function of p while $E(q, p)$ is a $-1/2$ form. Similarly to the case of $\tilde{E}(\infty, p)$ in [15] the following lemma can be proved.

Lemma 15

(i) *The function $\tilde{E}(q, p)$ has the expansion in $z = z(p)$ near ∞ of the form*

$$\tilde{E}(q, p) = (z - z(q)) z^{g-1}(1 + O(z)).$$

(ii) *Let γ be an element of $\pi_1(C, \infty)$ and its Abelian image be $\sum_{i=1}^{g}(m_{1,i}\alpha_i + m_{2,i}\beta_i)$. Then*

$$\tilde{E}(q, \gamma(p))/\tilde{E}(q, p)$$
$$= (-1)^{{}^t m_1 m_2 + 2({}^t\delta' m_1 - {}^t\delta'' m_2)} \mathrm{e}^{{}^t(\sum_{i=1}^2 2\eta_i m_i)(\int_q^p du + \sum_{i=1}^2 \omega_i m_i)}, \tag{38}$$

where $m_i = {}^t(m_{i,1}, \ldots, m_{i,g})$.

By (i) of this lemma $\tilde{E}(\infty, p)$ has a zero of order g at ∞.

Let $d\tilde{r}_i$ be the normalized differential of the second kind with a pole only at ∞, that is, it satisfies

$$\int_{\alpha_j} d\tilde{r}_i = 0, \quad 1 \le j \le g, \qquad d\tilde{r}_i = d(z^{-i} + O(1)).$$

Define

$$d\hat{r}_i = d\tilde{r}_i + \sum_{j,k=1}^{g} b_{j,i}(\eta_1\omega_1^{-1})_{j,k} du_k.$$

By the construction their periods can be computed as (Lemma 5 in [16])

$$\int_{\alpha_j} d\hat{r}_i = \left({}^t(2\eta_1)B\right)_{j,i}, \qquad \int_{\beta_j} d\hat{r}_i = \left({}^t(2\eta_2)B\right)_{j,i}. \tag{39}$$

In Lemma 5 of [16] there is a misprint: the right hand side is not the (i, j) component but the (j, i) component.

Proof of Theorem 14 The adjoint wave function (11) corresponding to the tau function (37) is computed as

$$\Psi^*(t, z) = C(z_1, \ldots, z_m) z^{g-m} \frac{\tilde{E}(\infty, p)^{m-1} \sigma\left(\int_\infty^p du - \sum_{i=1}^m \int_\infty^{p_i} du + Bt\right)}{\prod_{i=1}^m \tilde{E}(p_i, p)\ \sigma\left(-\sum_{i=1}^m \int_\infty^{p_i} du + Bt\right)} \times e^{-\sum_{i=1}^\infty t_i \int^p d\hat{r}_i},$$

$$C(z_1, \ldots, z_m) = (-1)^m \left(\prod_{i=1}^m z_i\right) e^{\frac{1}{2}\sum_{i=1}^m \int_\infty^{p_i} {}^t du (\eta_1\omega_1^{-1}) \int_\infty^{p_i} du}.$$

By Lemma 15 and (39) we can check that $z^{-g+m}\Psi^*(t, z)$ is, as a function of $p \in C$, $\pi_1(C, \infty)$ invariant. Then the same is true for any expansion coefficient of $\Psi^*(t, z)$ in t. Expansion coefficients in t are regular except p_i, $1 \le i \le m$, ∞ and have at most a simple pole at p_i. Therefore the point U of UGM corresponding to $\tau(t|p_1, \ldots, p_m)$ is contained in $z^{g-m} H^0(C, \mathcal{O}(\sum_{i=1}^m p_i + *\infty))$. Since a strict inclusion relation is impossible for two points of UGM [2, Lemma 4.17], these two points of UGM coincide. □

3.3 Degeneration Formula in Analytic Form

In this section we apply the results in the previous section to the curves C_n, C_{n-1} and associated tau functions in Theorem 9. So, in this section $\tau_{n,0}(t)$ denotes the function defined by (35) for the curve C_n.

Lemma 16 *We have*

$$\tau(t;\tilde{\xi}_n) = \tau_{n,0}(t). \tag{40}$$

Proof Since $\tilde{\xi}_n$ is a tau function corresponding to $z^g H^0(C_n, \mathcal{O}(*\infty))$, we have, by Theorem 12,

$$\tau(t;\tilde{\xi}_n) = C\tau_{n,0}(t),$$

for some constant C. Comparing the expansions (21) and (36) we have $C = 1$. □

Next we consider tau functions appearing in the right hand side of the equation in Theorem 9. We need a point $(\alpha, y_0(\alpha))$ of C_{n-1}. To specify $y_0(\alpha)$ is equivalent to specify one value of z such that $z^{-3} = \alpha$, that is, $\alpha^{-1/3}$. In fact, if $z = \alpha^{-1/3}$ is given the value of $y_0(\alpha)$ is determined by (18) as

$$y_0(\alpha) = \alpha^{n-1}\alpha^{1/3}F_{n-1}(\alpha^{-1/3}). \tag{41}$$

Since $P_i(\alpha) = (\alpha, \omega^i y_0(\alpha))$, we have

$$z(P_i(\alpha)) = \omega^{-i}\alpha^{-\frac{1}{3}}. \tag{42}$$

For simplicity we set

$$z_i(\alpha) = \omega^{-i}\alpha^{-\frac{1}{3}}. \tag{43}$$

Since, in general $\xi_{n-1}(Q_0, Q_1, Q_2)$ is a frame of the point

$$z^{g'-3}H^0(C_{n-1}, \mathcal{O}(\sum_{i=0}^{2} Q_i + *\infty)) \in \mathrm{UGM}$$

we have, by Theorem 14,

$$\begin{aligned}\tau(t;\xi_{n-1}(P_i(\alpha), P_j(\alpha), P_k(\beta))) &= C_{i,j,k}(\alpha,\beta)\mathrm{e}^{\eta(z_i(\alpha)^{-1})+\eta(z_j(\alpha)^{-1})+\eta(z_k(\beta)^{-1})}\\ &\quad\times \tau_{n-1,0}(t-[z_i(\alpha)]-[z_j(\alpha)]-[z_k(\beta)]),\end{aligned} \tag{44}$$

for some constant $C_{i,j,k}(\alpha, \beta)$.

Remark 17 The explicit forms of the constants $C_{i,j,k}(\alpha, \beta)$ are not yet determined. They should be calculated by comparing the Schur function expansions and are expected to be expressed by some derivatives of the sigma function.

Substituting (40), (44) into the relation in Theorem 9 we get

Theorem 18 *Let $\tau_{n,0}(t)$ be defined by the right hand side of (35) for the curve C_n and $z_i(\alpha)$ defined by (43). Then, in the limit $\alpha_j \to \alpha$ for $j = 3n, 3n \pm 1$, we have*

$$\begin{aligned}
&\lim \tau_{n,0}(t) \\
&\quad = \frac{(-1)^n}{27 y_0(\alpha)^5} \mathrm{e}^{-6\sum_{l=1}^{\infty} \alpha^l t_{3l}} \sum_{0 \le i < j \le 2, 0 \le k \le 2} \omega^{i+k+2j}(1-\omega^{i-j}) \\
&\qquad \times \frac{\partial}{\partial \beta}\Big(C_{i,j,k}(\alpha, \beta) \mathrm{e}^{\eta(z_i(\alpha)^{-1})+\eta(z_j(\alpha)^{-1})+\eta(z_k(\beta)^{-1})} \\
&\qquad\qquad \times \tau_{n-1,0}(t - [z_i(\alpha)] - [z_j(\alpha)] - [z_k(\beta)]) \Big)\Big|_{\beta=\alpha}. \qquad (45)
\end{aligned}$$

for the constants $C_{i,j,k}(\alpha, \beta)$ in (44), where $y_0(\alpha)$ is given by (41).

Remark 19 In the right hand side of (45) the exponential factor, which is characteristic to soliton solutions, is clearly visible. Since it can be shown that $\tau_{0,0} = 1$ for the genus zero curve $y^3 = x$ which corresponds to the case $\alpha_1 = 0$, using repeatedly the formula (18) we obtain the formula which contains only exponential functions and their derivatives with respect to parameters. The formulas for them were computed in [18] independently of Theorem 18, where all constants are explicitly given as functions of $\{\alpha_j\}$. These solutions are called generalized solitons in [18].

4 The Case of Hyperelliptic Curves

In this section, based on the results of [2], we derive the corresponding formula to (45) in the case of hyperelliptic curve X_g defined by

$$y^2 = \prod_{j=1}^{2g+1} (x - \alpha_j) \qquad (46)$$

and its degeneration

$$\alpha_{2g+1}, \alpha_{2g-1} \to \alpha, \qquad (47)$$

where $\alpha \neq \alpha_j$ for $1 \leq j \leq 2g-2$. The curve X_g has the unique point over $x=\infty$ which we also denote by ∞. We take the local coordinate z around ∞ such that

$$x = z^{-2}, \qquad y = z^{-2g-1}F_g(z), \quad F_g(z) = \left(\prod_{j=1}^{2g+1}(1-\alpha_i z^2)\right)^{1/2}. \tag{48}$$

Let

$$\mu^{(g)} = (g, g-1, \ldots, 1)$$

be the partition and $\tilde{\xi}_g$ a frame of $z^g H^0(X_g, \mathcal{O}(*\infty))$ such that the corresponding tau function has the expansion of the form

$$\tau(t;\tilde{\xi}_g) = s_{\mu^{(g)}}(t) + \text{h.w.t.} \tag{49}$$

Fix one of the square root $\alpha^{-1/2}$ and define y_0 by

$$y_0 = \alpha^{g-1/2}F_{g-1}(\alpha^{-1/2}). \tag{50}$$

Then (α, y_0) is a point of X_{g-1}. Set

$$p_\pm = (\alpha, \pm y_0). \tag{51}$$

Then the values of the local coordinates of $p_\pm$ are

$$z(p_\pm) = \pm\alpha^{-1/2}.$$

Let $\xi_{g-1}(p_\pm)$ be a frame of $z^{g-2}H^0\left(X_{g-1}, \mathcal{O}(p_\pm + *\infty)\right)$ such that their tau functions have the following expansions

$$\tau(t;\xi_{g-1}(p_\pm)) = s_{\mu^{(g-2)}}(t) + \text{h.w.t.} \tag{52}$$

The following theorem is proved in [2] in a similar way to Theorem 9.

Theorem 20 ([2]) *The following relation holds.*

$$\lim \tau(t;\tilde{\xi})$$
$$= (-1)^{g-1}(2y_0)^{-1}\mathrm{e}^{-2\sum_{l=1}^{\infty}\alpha^l t_{2l}}\left(\tau(t;\xi_{g-1}(p_+)) - \tau(t;\xi_{g-1}(p_-))\right), \tag{53}$$

where lim *in the left hand side means the limit taking α_{2g+1}, α_{2g} to α.*

Let $\tau_{g,0}(t)$ denote the function defined by the right hand side of (35) for X_g.

Lemma 21

(i) $\tau(t;\tilde{\xi}_g)=\tau_{g,0}(t)$.
(ii) *For some constant* $C_\epsilon(\alpha)$

$$\tau(t;\xi_{g-1}(p_\epsilon))=C_\epsilon(\alpha)\mathrm{e}^{\sum_{l=1}^{\infty}(\epsilon\alpha^{-1/2})^{-l}t_l}\tau_{g-1,0}(t-[\epsilon\alpha^{-1/2}]),\quad \epsilon=\pm.$$

Proof

(i) Both $\tau(t;\tilde{\xi}_g)$ and $\tau_{g,0}(t)$ are tau functions corresponding to $z^g H^0(X_g,\mathcal{O}(*\infty))$. By comparing the expansions (36) and (49) we get the result.
(ii) Since the right hand side and the left hand side without $C_\epsilon(\alpha)$ of the equation in the assertion are the tau functions corresponding to $z^{g-2}H^0(X_{g-1},\mathcal{O}(p_\epsilon+*\infty))$ by the definition of $\xi_{g-1}(p_\epsilon)$ and Theorem 14, the assertion follows.

□

This lemma is proved in [2] in a different form. The explicit form of the constant $C_\epsilon(\alpha)$ can be extracted from there. Let us give the formula.

Let $m^{(g)}=\left[\frac{g+1}{2}\right]$. Define the sequence $A^{(g)}$ and $s^{(g)}\in\{\pm 1\}$ by

$$A^{(g)}=(a_1^{(g)},\ldots,a_{m^{(g)}}^{(g)})=(2g-1,2g-5,2g-9,\ldots),$$
$$s^{(g)}=(-1)^{(g-1)m^{(g)}}.$$

Example 22 $A^{(1)}=(1)$, $A^{(2)}=(3)$, $A^{(3)}=(5,1)$, $A^{(4)}=(7,3)$.
$s^{(1)}=1$, $s^{(2)}=-1$, $s^{(3)}=1$, $s^{(4)}=1$.

The following property of $A^{(g)}$ is known [20, 22],

$$|A^{(g)}|:=\sum_{j=1}^{m^{(g)}}a_j^{(g)}=\frac{1}{2}g(g+1). \tag{54}$$

Denote the sigma function of X_{g-1} by $\sigma^{(g-1)}(u)$. Set

$$b_i=(a_i^{(g-2)}+1)/2\in\{1,2,\ldots,g-2\},\quad 1\le i\le m^{(g-2)},$$

and define

$$\sigma_{A^{(g-2)}}^{(g-1)}(u)=\frac{\partial^{m^{(g-2)}}}{\partial u_{b_1}\cdots\partial u_{b_{m^{(g-2)}}}}\sigma^{(g-1)}(u).$$

Then, by Theorem 4.14 of [2], we can deduce that

$$C_\epsilon(\alpha) = s^{(g-2)}\sigma^{(g-1)}_{A^{(g-2)}}(-\int_\infty^{p_\epsilon} du)^{-1}, \quad du = {}^t(du_1, \ldots, du_g). \tag{55}$$

Lemma 23 *The following relation is valid.*

$$C_-(\alpha) = (-1)^{g-1}C_+(\alpha). \tag{56}$$

Proof It is known that the sigma function satisfies the following relation [15, 22]

$$\sigma^{(g-1)}(-u) = (-1)^{\frac{1}{2}g(g-1)}\sigma^{(g-1)}(u).$$

By differentiating it we get

$$\sigma^{(g-1)}_{A^{(g-2)}}(-u) = (-1)^{\frac{1}{2}g(g-1)+m^{(g-2)}}\sigma^{(g-1)}(u). \tag{57}$$

We can easily verify that

$$\frac{1}{2}g(g-1) + m^{(g-2)} = g-1 \quad \text{mod.2}. \tag{58}$$

For the hyperelliptic curve X_{g-1} the following relation holds,

$$\int_\infty^{p_-} du = -\int_\infty^{p_+} du. \tag{59}$$

The assertion of the lemma follows from (55), (57), (58), (59). □

Substituting the equations of (i), (ii) in Lemma 21 into (53) and using (56) we get

Theorem 24 *Let $\tau_{g,0}(t)$ be given by the right hand side of (35) for the hyperelliptic curve X_g defined by (46). Then in the limit $\alpha_{2g+1}, \alpha_{2g} \to \alpha$ we have the following formula,*

$$\lim \tau_{g,0}(t) = (-1)^g(2y_0)^{-1}C_+(\alpha)\mathrm{e}^{-2\sum_{l=1}^\infty \alpha^l t_{2l}}$$
$$\times \left(\mathrm{e}^{\eta(\alpha^{1/2})}\tau_{g-1,0}(t-[\alpha^{-1/2}]) + (-1)^g\mathrm{e}^{\eta(-\alpha^{1/2})}\tau_{g-1,0}(t-[-\alpha^{-1/2}])\right),$$

where y_0, $p_\pm$, $C_+(\alpha)$ are given by (50), (51), (55) respectively.

Remark 25 The tau function $\tau_{g,0}(t)$ gives a solution of the KdV hierarchy (see Remark 13). Again it can be shown that $\tau_{0,0} = 1$ for the genus zero curve $y^2 = x$ which corresponds to $\alpha_1 = 0$. Using the formula repeatedly we get the well known soliton solution [9, 21]. For $\alpha_1 \neq 0$ we can show that $\tau_{0,0}(t) = \mathrm{e}^{L(t)+Q(t)}$, where $L(t)$ and $Q(t)$ are certain linear and quadratic functions of t.

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Fock Quantization of Canonical Transformations and Semiclassical Asymptotics for Degenerate Problems

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Abstract The aim of this work is to explain the role played by the Fock quantization of canonical transformations in the construction of the global semiclassical (high-frequency) asymptotic approximation. This role may well pass unnoticed as long as one deals with nondegenerate differential equations. However, the situation is different for some classes of equations with degeneration, where the Fock quantization of canonical transformations becomes instrumental in the construction of asymptotic solutions.

Keywords Semiclassical asymptotics · Canonical transformation · Quantization · Degenerate equation · Maslov's canonical operator

Mathematics Subject Classification (2010) Primary 81Q20; Secondary 35L80, 81S10, 53D12, 53D22

1 Introduction

Maslov's canonical operator [14, 15] is a powerful tool for constructing global semiclassical asymptotics of solutions of differential equations with a small parameter multiplying the derivatives. The asymptotic solutions produced by this operator have the form of sums of *WKB elements*[1] in coordinate and momentum representations, with the $1/h$-Fourier transform $\mathcal{F}^{1/h}_{p\to x}$ applied to the latter to make them functions

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[1] See Sect. 4 for more details.

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of the coordinate rather than the momentum.[2] The operator $\mathcal{F}^{1/h}_{p\to x}$ is actually the Fock quantization of the rotation by an angle of $\pi/2$ in the phase plane; however, this, in a sense, works behind the scenes, and one may not know that but still apply the canonical operator successfully to problems with nondegenerate characteristics. Things become more difficult when one deals with degenerate operators to which the standard scheme of the canonical operator does not apply. In that case, to desingularize the problem, one may need more complicated canonical transformations than mere rotations by $\pi/2$, and then the Fock quantization rule gives the right recipe of what to do with the WKB elements arising in the new variables and how to construct a modified canonical operator suitable for the degeneration in question.

This is exactly what happens for the class of operators with boundary degeneration arising in the linear theory of run-up of long waves on a shallow beach [19, 21]. The theory of global semiclassical asymptotics for this class of problems has been developed in the recent years by the authors and their colleagues [1, 2, 5–10, 16, 17]. The aim of the present note is to explain how the Fock quantization of canonical transformations enters the construction of semiclassical asymptotics. As an example, we use the simplest problem of this class in dimension 1, that is, a problem for an ordinary differential equation (ODE).

2 Degenerate Boundary Value Problem

Let $D(x) \in C^\infty([-1,1])$ be a function such that $D(x) > 0$ for $x \in (-1,1)$, $D(-1) = D(1) = 0$, $D'(-1) > 0$, and $D'(1) < 0$. Further, consider the operator

$$L_0 = -\frac{d}{dx}D(x)\frac{d}{dx} \quad \text{with domain} \quad \mathcal{D}(L_0) = C_0^\infty((-1,1))$$

in the space $L^2([-1,1])$. The operator L_0 degenerates at the endpoints of the interval $(-1,1)$, and hence one cannot define any self-adjoint extensions of L_0 with the use of classical boundary conditions such as the Dirichlet or Neumann conditions [18]. Thus, one has to use "generalized boundary conditions." Define the operator L in $L^2([-1,1])$ as the Friedrichs extension [3, Sec. 10.3] of L_0, which is equivalent to the finiteness of the energy integral [22, Sec. 33.1]. Consider the *eigenvalue problem*

$$L\eta = \lambda\eta, \tag{1}$$

[2]For simplicity, we only deal here with the case of one spatial variable x (i.e., $x \in \mathbb{R}^1$); if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, then the construction also involves partial Fourier transforms (Fourier transforms with respect to part of the variables).

which naturally arises in the approximation given by the linearized shallow water equations as the one-dimensional model of harmonic water waves (such as seiches) in a basin of variable depth $D(x)$. Here $\eta(x)e^{i\omega t}$, $\omega = \sqrt{\lambda}$, has the meaning of the free surface elevation at the point x at time t. The motion of water is assumed to be potential, and we use a system of units in which the acceleration due to gravity has the value $g = 1$.

We will be interested in the behavior of solutions of this eigenvalue problem with large λ. One defines an *asymptotic series* of solutions as a sequence $\lambda_n \to \infty$ of numbers (called *asymptotic eigenvalues*) and a sequence of functions $\eta_n \in \mathcal{D}(L)$ such that $\|\eta_n\| \geq C > 0$ (where the norm is taken in $L^2([-1, 1])$) and these functions are *almost eigenfunctions* in the sense that $\|L\eta_n - \lambda_n\eta_n\| = O(1)$ as $n \to \infty$. By the well-known estimates for the resolvent of a self-adjoint operator, an asymptotic series satisfies the relation $\operatorname{dist}(\lambda_n, \sigma(L)) = O(1)$, where $\sigma(L)$ is the spectrum of L, and has other useful properties.

Equation (1) is an ODE with singular points, and there is a vast literature concerning the theory of such equations (e.g., see the books by Fedoryuk [11] and Slavyanov [20] and references therein). Needless to say, problem (1) can be solved by methods of that theory; for example, one can use the method of standard equations with the Bessel equation serving as a standard equation (see [6, Sec. 2]). However, these methods have a drawback in that they cannot be transferred to the multidimensional case automatically; for us, Eq. (1) only serves as a simple example, and we will use an approach is free from this drawback. This approach is based on the geometry of the characteristics of the problem and extends Maslov's canonical operator.

3 Quantization of Canonical Transformations

The idea of quantization of canonical transformations is apparently due to Dirac, who wrote [4, Sec. 26]:

> ...for a quantum dynamic system that has a classical analogue, unitary transformation in the quantum theory is the analogue of contact transformation in the classical theory.

The definition of quantization of canonical transformations was given by Fock [12]. Since then, there have been an extensive literature on the topic. In particular, a comprehensive theory including global aspects and featuring far-reaching generalizations was developed by Karasev and Maslov [13]. We will need the simplest local version essentially defined by Fock himself. In this paper, we restrict ourselves to the one-dimensional case. Consider a canonical transformation $g : \mathbb{R}^2_{(x,p)} \to \mathbb{R}^2_{(y,q)}$. The quantized canonical transformation is given by

$$T(g) \colon L^2(\mathbb{R}_y) \longrightarrow L^2(\mathbb{R}_x), \qquad [T(g)u](x) = \int K_g(x, y)u(y)\, dy,$$

where the kernel K_g depends on the small parameter $h > 0$ and is defined via the *generating function* of g as follows.

1. If g is defined by a generating function $\Phi(x, y)$ by the formulas $q = -\Phi_y(x, y)$, $p = \Phi_x(x, y)$, where, by definition, $\Phi''_{xy}(x, y) \neq 0$, then the kernel is given by

$$K_g(x, y) = \left(\frac{-i}{2\pi h}\right)^{1/2} e^{\frac{i}{h}\Phi(x,y)} \sqrt{\Phi''_{xy}(x, y)}, \qquad \arg i = \frac{\pi}{2}.$$

2. If g is defined by a generating function $\Phi(x, q)$ by the formulas $y = \Phi_q(x, q)$, $p = \Phi\ x(x, q)$, where, by definition, $\Phi''_{xq}(x, q) \neq 0$, then

$$K_g(x, y) = \frac{1}{2\pi h} \int e^{\frac{i}{h}(\Phi(x,q)-yq} \sqrt{\Phi''_{xq}(x, q)}\, dq.$$

The choice of the argument of the radicand is irrelevant to our discussion.

Let us present two examples.

1. Let $\Phi(x, y) = -xy$, so that $p = \Phi_x = -y$, $q = -\Phi_y = x$, and the transformation is the counterclockwise rotation by $\pi/2$. Then the quantized transformation has the kernel

$$K_g(x, y) = \left(\frac{-i}{2\pi h}\right)^{1/2} e^{-\frac{i}{h}xy}, \quad [T(g)u](x) = \left(\frac{-i}{2\pi h}\right)^{1/2} \int e^{-\frac{i}{h}xy} u(y)\, dy;$$

   thus, $T(g) = \mathcal{F}^{1/h}_{y\to x}$ is the $1/h$-Fourier transform.
2. Now let $\Phi(x, q) = qf(x)$ (where $f'(x) \neq 0$); then $y = f(x)$, $p = (f'(x))^{-1}q$ is the classical canonical transformation associated with a change of variables. The kernel has the form

$$K_g(x, y) = \frac{\sqrt{f'(x)}}{2\pi h} \int e^{\frac{i}{h}q(f(x)-y)}\, dq = \sqrt{f'(x)}\delta(y - f(x)),$$

   and the transformation $T(g)$ itself is the same change of variables in a function followed by the multiplication by a factor ensuring the unitarity of $T(g)$ in L^2.

4 Semiclassical Asymptotics

The semiclassical theory deals with equations of the form $\widehat{H}u = 0$, where $\widehat{H} = H(x, \widehat{p})$, $\widehat{p} = -ih\frac{\partial}{\partial x}$, is a differential operator with a small parameter $h > 0$ multiplying the derivatives. Semiclassical asymptotic theory provides rapidly oscillating asymptotic solutions of the equation $\widehat{H}u = 0$ as $h \to 0$. Let us recall the standard construction of the canonical operator [14, 15], again sticking to the case of $n = 1$. To define the canonical operator, we need a Lagrangian manifold $\Lambda \subset \mathbb{R}^2_{(x,p)}$

with a smooth measure $d\mu$ (volume form) on it. The canonical operator K^h_Λ takes smooth functions on Λ to rapidly oscillating functions on $\mathbb{R}_x$. The manifold Λ must be compact (or at least the projection $\Lambda \to \mathbb{R}_x$ must be proper).

The function $K^h_\Lambda\phi$ is pasted together from local elements corresponding to parts of Λ with "good" projection onto one of the coordinate axes. There can be two possible cases:

(i) Assume that the projection of $\operatorname{supp}\phi \subset \Lambda$ onto the x-axis is good. Then $[K^h_\Lambda\phi](x)$ is the WKB element

$$[K^h_\Lambda\phi](x) = \exp\left(\frac{iS(x)}{h}\right)\phi(x)\left(\frac{d\mu}{dx}\right)^{1/2}, \quad \text{where } \Lambda = \left\{p = \frac{\partial S}{\partial x}(x)\right\}.$$

(ii) Assume that the projection of $\operatorname{supp}\phi \subset \Lambda$ onto the p-axis is good. Then we can in a similar way define the WKB element

$$\exp\left(\frac{i\widetilde{S}(p)}{h}\right)\phi(p)\left(\frac{d\mu}{dp}\right)^{1/2}, \quad \text{where } \Lambda = \left\{x = -\frac{\partial \widetilde{S}}{\partial p}(p)\right\},$$

but we cannot make it the value of the canonical operator, because it depends on the wrong variable! To obtain a function of x, we transpose the axes by rotating the picture by an angle of $\pi/2$. The Fock quantization of this rotation gives the Fourier transform, and we obtain

$$[K^h_\Lambda\phi](x) = \left(\frac{i}{2\pi h}\right)^{1/2}\int \exp\left(\frac{i(\widetilde{S}(p)+px)}{h}\right)\phi(p)\left(\frac{d\mu}{dp}\right)^{1/2} dp.$$

Now, to define $K^h_\Lambda\phi$ for an arbitrary compactly supported smooth function ϕ on Λ, one uses a partition of unity to split ϕ into a sum of terms each of which can be treated with the use of (i) or (ii). The consistency of (i) and (ii) in case they both apply is ensured by additional unimodular factors; in turn, these can be chosen consistently if Λ satisfies the quantization conditions (see [14, 15]).

5 Solution of the Degenerate Problem

5.1 Geometric Construction

We rewrite problem (1) in the semiclassical form

$$\widehat{H}\eta = \eta, \qquad \widehat{H} = \widehat{p}D(x)\widehat{p},$$

with Hamiltonian $H(x, p) = D(x)p^2$. The semiclassical asymptotics is associated with a Lagrangian manifold Λ_0 contained in the set $\{(x, p) \colon H(x, p) = 1\}$. In the one-dimensional case, this set is a curve, and the Lagrangian manifold necessarily coincides with it. The difficulty is that the Lagrangian manifold is singular (namely, the projection onto the base is improper). The solution is to extend the phase space.

The geometric construction was suggested in [16] based on the idea in [23] that one should proceed from the momentum variable p to its reciprocal, $1/p$. The natural next step (which however was not made in [23]) is to accompany this transformation with a transformation of the variable x so as to obtain a canonical transformation. This was done in [16]. The desired change of variables in the phase space $T^*((-1, 1))$ over a neighborhood of the left end $x = -1$ of the interval $(-1, 1)$ has the form

$$\theta = p^2(x + 1), \quad q = -\frac{1}{p} \qquad \Leftrightarrow \qquad x = q^2\theta - 1, \qquad p = -\frac{1}{q}. \tag{2}$$

This transformation is canonical, $dp \wedge dx = dq \wedge d\theta$. We add the open half-line $\{q = 0, \theta > 0\}$ to this chart of the phase space in the new coordinates and carry out a similar construction near the right end $x = 1$. The resulting new phase space Φ is diffeomorphic to a plane with two deleted points, $\Phi \simeq R^2 \setminus \{(-1, 0), (1, 0)\}$. The closure Λ of the manifold Λ_0 in the phase space Φ is obtained by the addition of two points; it is a smooth Lagrangian manifold diffeomorphic to a circle. To construct asymptotic eigenfunctions, we must define the canonical operator on Λ in the vicinity of the newly added points.

5.2 Modified Canonical Operator

Consider a neighborhood of a point in $\Lambda \setminus \Lambda_0$. This point is projected into one of the endpoints of $[-1, 1]$ and is defined by the equation $q = 0$ in the corresponding new coordinates. Thus, the endpoints are a special kind of caustic. To define the canonical operator near these points, we use the same idea as earlier for the "standard" canonical operator. Namely, we write a WKB element that is a function of q and then define a function of the variable x by applying the Fock quantized canonical transformation corresponding to the classical canonical transformation (2). To be definite, consider a neighborhood of the left endpoint $x = -1$. Then the canonical transformation (2) can be defined by the generating function $\Phi(x, q) = -x/q$, and accordingly the quantized canonical transformation is

$$[T(g)u](x) = \int K(x, \theta)u(\theta)\, d\theta,$$

where

$$K(x,\theta)=\frac{1}{2\pi h}\int_{-\infty}^{\infty}e^{-\frac{i}{h}\left(\frac{x}{q}+\theta q\right)}\frac{dq}{q}=-\frac{i}{h}J_0\left(\frac{2\sqrt{x\theta}}{h}\right)$$

and $J_0(z)$ is the Bessel function of the first kind and zero order. Thus, we have the Hankel transform instead of the usual Fourier transform in the definition of the canonical operator. In other words, the canonical operator in a neighborhood of the boundary point acts as an application of the Hankel transform (composed with the Fourier transform) to a WKB element. The corresponding integral formulas can be found in [17]; the kernels of these integrals are products of $K(x,\theta)$ by certain rapidly oscillating exponentials. Computing these Bessel type integrals according to [5], we arrive at the form of the modified canonical operator given in [1]. In the one-dimensional case, these formulas do not contain any integrals and hence express the asymptotic solution in closed form. We refer the reader for the general formulas to [1, 6] and restrict ourselves in the present paper to the solution formulas for our specific problem.

5.3 Formulas for the Asymptotic Eigenfunctions

The final answer in problem (1) reads [6, Eq. (1.6)]

$$\eta_n(x)\asymp\begin{cases}\sqrt{2\pi\omega_n}\,J_0(\omega_n S(-1,x))\left(\dfrac{S(-1,x)}{c(x)}\right)^{1/2}, & x\in[-1,1-\varepsilon],\\ (-1)^n\sqrt{2\pi\omega_n}\,J_0(\omega_n S(x,1))\left(\dfrac{S(x,1)}{c(x)}\right)^{1/2}, & x\in[-1+\varepsilon,1],\end{cases}$$

where $\varepsilon>0$ is fixed,

$$c(x)=\sqrt{D(x)},\qquad S(x_0,x)=\int_{x_0}^{x}\frac{d\xi}{c(\xi)},\qquad -1\le x_0,x\le 1,$$

and

$$\omega_n=\frac{\pi}{S(-1,1)}\left(n+\frac{1}{2}\right),\qquad n=1,2,\dots,$$

are the asymptotic eigenvalues of the problem.

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Some Recent Results on Contact or Point Supported Potentials

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Abstract We introduced some contact potentials that can be written as a linear combination of the Dirac delta and its first derivative, the δ-δ' interaction. After a simple general presentation in one dimension, we briefly discuss a one dimensional periodic potential with a δ-δ' interaction at each node. The dependence of energy bands with the parameters (coefficients of the deltas) can be computed numerically. We also study the δ-δ' interaction supported on spheres of arbitrary dimension. The spherical symmetry of this model allows us to obtain rigorous conclusions concerning the number of bound states in terms of the parameters and the dimension. Finally, a δ-δ' interaction is used to approximate a potential of wide use in nuclear physics, and estimate the total number of bound states as well as the behaviour of some resonance poles with the lowest energy.

Keywords Contact potentials · Periodic potentials · Nuclear potentials · Atomic potentials

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1 Introduction

Contact potentials are interactions supported on manifolds of lower dimension than the dimension of the overall space [1, 3, 11, 16]. Along the present manuscript, we shall consider the time independent one dimensional Schrödinger equation and

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contact potentials supported on isolated points (this is why we shall also use the term of point interactions to refer to them) or on lower dimensional varieties. The simplest case of a one dimensional contact potential is the Dirac delta interaction $\delta(x)$ supported at a point. In this case the Schrödinger equation comes from a one dimensional Hamiltonian of the form $H = -d^2/dx^2 + V(x)$, where $V(x)$ accounts for the contact potential. This study is important in quantum mechanics and here are a few reasons:

- Many of these models are exactly solvable and are very suitable to study scattering properties [2, 21, 31]. In particular, they are good toy models to study resonances and antibound states and their properties [4, 5].
- They may serve to model point defects in materials, topological insulators [10, 28] and heterostructures, which may be represented by abrupt mass changes [20, 30].
- In nanophysics: to mimic sharply peaked impurities inside quantum dots.
- In scalar QFT on a line: used to show the influence of impurities and external singular backgrounds [33].
- Point interactions of the type Dirac delta, $\delta(x)$ or $\delta'(x)$, can be understood as perturbations of a free kinetic Schrödinger Hamiltonian, but they could be also combined with other type of interactions such as the harmonic oscillator, a constant electric field, the infinite square well, the conical oscillator, etc. [19, 22, 23, 27, 39].
- Double δ-δ' barriers have been used to study the Casimir effect [8, 17, 24, 35].
- Chains of periodic δ-δ' interactions have been considered in order to analyze a solvable Kronig-Penney model in solid state, where the behaviour of band spectrum has been thoroughly analyzed in order to obtain a better comprehension of dielectric and conducting phenomena [15, 25].
- Although in principle we focused our attention in one dimensional non-relativistic problems, work has been done also in the study of contact potentials in higher dimensions [36], or as perturbations of the Dirac equation or the Salpeter Hamiltonian [18]. There is a wide range of problems in this field that will be studied in a near future.

In one dimension, it has been proven the existence of families depending on four real parameters of contact potentials at each point compatible with the self-adjointness of the Hamiltonian. There are some discussion on the physical meaning of these families that are obtained through the formalism of self-adjoint extensions of symmetric operators on Hilbert spaces.

Along this presentation, we shall consider the following forms for $V(x)$:

- $V(x) = -a\delta(x) + b\delta'(x)$, where a and b are real numbers with $a > 0$.
- The Kronig-Penney model $V(x) = \sum_{n=-\infty}^{\infty}(V_0\delta(x - na) + aV_1\delta'(x - na))$.
- The radial potential $V(r) = a\delta(r - r') + b\delta'(r - r')$ with a and b real.
- An application to nuclear physics, considering the previous radial potential plus a finite spherical well $V_0[\theta(r - R) - 1]$.

2 A δ-δ' Perturbation of the One Dimensional Free Hamiltonian

We start with the one dimensional Hamiltonian of the form

$$H = H_0 + V(x) = \frac{p^2}{2m} - a\delta(x) + b\delta'(x)\,, \text{ with } a > 0\,, b \in \mathbb{R}\,, \tag{1}$$

where $H_0 = p^2/(2m)$ and $V(x) := -a\delta(x) + b\delta'(x)$. Here, we need a definition of the potential $V(x)$ such that the Hamiltonian H in (1) be self-adjoint. While a perturbation of the type $-\delta(x)$ is well defined on H_0, the point is to add the term containing the $\delta'(x)$. There is not a unique definition for perturbation of this kind, but we need one compatible with the term on $\delta(x)$. This is sometimes called the *local* $\delta'(x)$ and the interaction $V(x)$ has to be defined via the self-adjoint extensions of symmetric (Hermitian) operators.

A self-adjoint determination of the Hamiltonian (1) can be provided through the theory of self-adjoint extensions of symmetric (Hermitian) operators with equal deficiency indices. First of all, we define the domain of the "free" operator $H_0 = -d^2/dx^2$ as the Sobolev space $W_2^2(\mathbb{R}\backslash\{0\})$ of absolutely continuous functions $\psi(x) : \mathbb{R}\backslash\{0\} \longmapsto \mathbb{C}$, on the real line excluded the origin, such that:

(1) The first derivative $\psi'(x)$ is absolutely continuous on $\mathbb{R}\backslash\{0\}$ (note that an absolutely continuous function admits derivative at almost all points);
(2) Both $\psi(x)$ and $\psi''(x)$ are square integrable:

$$\int_{-\infty}^{\infty} \{|\psi(x)|^2 + |\psi''(x)|^2\}\, dx < \infty\,. \tag{2}$$

(3) $\psi(0) = \psi'(0) = 0$.

With this domain, H_0 is a symmetric operator with deficiency indices (2, 2), which means that it has a set of self-adjoint extensions depending on 4 real parameters. Note that Conditions (1) and (2) give the domain of the adjoint, $H_0^\dagger$, of H_0. Self-adjoint extensions of H_0 have domains included in the domain of $H_0^\dagger$ and are characterized by matching conditions at the origin. They have been classified in [9, 31]. In our case, we propose for $V(x) = -a\,\delta(x) + b\,\delta'(x)$ the following matching conditions:

$$\begin{pmatrix} \psi(0^+) \\ \psi'(0^+) \end{pmatrix} = \begin{pmatrix} \dfrac{\hbar^2 + mb}{\hbar^2 - mb} & 0 \\ \dfrac{-2\hbar^2 am}{\hbar^4 - m^2 b^2} & \dfrac{\hbar^2 - mb}{\hbar^2 + mb} \end{pmatrix} \begin{pmatrix} \psi(0^-) \\ \psi'(0^-) \end{pmatrix}, \tag{3}$$

where $f(0^+)$ and $f(0^-)$ are the right and left limits, respectively, of the function $f(x)$ at the origin. The corresponding Schrödinger equation for $H = H_0 + V(x)$ is

$$-\frac{\hbar^2}{2m}\,\psi''(x) - a\,\delta(x)\psi(x) + b\,\delta'(x)\,\psi(x) = E\,\psi(x)\,. \tag{4}$$

Since neither the functions $\psi(x)$ in the domain of H nor their first derivatives are continuous at the origin, we need to give a determination of the products $\delta(x)\psi(x)$ and $\delta'(x)\,\psi(x)$ that replace the usual ones and that were somehow compatible with (3). Following [31], we propose

$$\delta(x)\psi(x) := \frac{\psi(0^+) + \psi(0^-)}{2}\,\delta(x)\,, \tag{5}$$

$$\delta'(x)\,\psi(x) := \frac{\psi(0^+) + \psi(0^-)}{2}\,\delta'(x) - \frac{\psi'(0^+) + \psi'(0^-)}{2}\,\delta(x)\,. \tag{6}$$

Some conclusions will be presented next. This includes bound states and scattering coefficients.

2.1 Bound States and Scattering Coefficients

It is well known that the Hamiltonian (1) has a bound state for $b = 0$, since $-a$ is negative. When $b \neq 0$, it is easy to prove that a bound state must exist. Furthermore, we can find its energy and its wave function by solving the Schrödinger equation (4). Note that outside the origin, this is the Schrödinger equation for the free particle, so its solution should be of the form

$$\psi(x) = \alpha\, e^{\kappa x}\,\theta(-x) + \beta\, e^{-\kappa x}\,\theta(x)\,, \quad \kappa = \sqrt{-2mE/\hbar^2}\,, \tag{7}$$

with $E < 0$, $\theta(x)$ is the Heaviside step function, $\alpha = \psi(0^-)$ and $\beta = \psi(0^+)$. In addition, the function $\psi(x)$ in (7) must belong to the domain of the Hamiltonian (1), so that it must satisfy the matching conditions (3). Taking into account (3), the final form of (7) is

$$\psi(x) = \frac{\sqrt{ma}\,\hbar}{\hbar^4 + m^2b^2}[(\hbar^2 - mb)\,e^{\kappa x}\,\theta(-x) + (\hbar^2 + mb)\,e^{-\kappa x}\,\theta(x)]\,. \tag{8}$$

Note that the function (8) is square integrable and, therefore, represents the wave function for the unique bound state of the system. Then, we plug (8) into the Schrödinger equation (4), which after some algebra gives the energy value for the unique bound state,

$$E = -\frac{1}{2}\,\frac{ma^2\hbar^6}{(\hbar^4 + b^2m^2)^2}\,. \tag{9}$$

It is a simple task to obtain the scattering coefficients. Assume that a monochromatic wave e^{ikx}, $k = \sqrt{2mE}/\hbar^2$, $E \geq 0$, comes from the left to the right. After scattering with the potential $V(x)$, the resulting wave function has different forms on the regiones $x < 0$ or $x > 0$, which are given by

$$\text{for } x < 0:\ \psi(x) = e^{ikx} + R\,e^{-ikx}\,; \quad \text{for } x > 0:\ \psi(x) = T\,e^{ikx}, \tag{10}$$

where R and T are the reflection and transmission coefficients, respectively. These coefficients are easily obtained by using matching conditions (3), where we now choose $\hbar = 1$ for simplicity:

$$\begin{pmatrix} T \\ \\ ikT \end{pmatrix} = \begin{pmatrix} \dfrac{1+mb}{1-mb} & 0 \\ \dfrac{-2am}{1-m^2b^2} & \dfrac{1-mb}{1+mb} \end{pmatrix} \begin{pmatrix} 1+R \\ \\ ik(1-R) \end{pmatrix}, \tag{11}$$

so that,

$$R(k) = \frac{-(am + 2mbki)}{am + (1+m^2b^2)ki}\,, \qquad T(k) = \frac{(1-m^2b^2)ki}{am + (1+m^2b^2)ki}\,, \tag{12}$$

where i is the imaginary unit. Note that $|R(k)|^2 + |T(k)|^2 = 1$. At the exceptional values $b = \pm 1/m$, there is no transmission. This case will not be treated in the sequel, but it was carefully considered in [24, 34].

3 The Dirac δ-δ' Comb

The correspondence between boundary conditions and surface interactions in quantum field theory was established by Symanzik some time ago [38]. One the most interesting examples of these surface interactions is given by the Casimir effect [14]. It was in [34] where an interpretation of the Casimir effect using a δ-δ' type of potential was proposed. The idea in [34] was mimicking the plates in the Casimir effect as two point interactions, so that the Hamiltonian becomes

$$H = H_0 + V(x) = -\frac{\hbar^2}{2m}\,\frac{d^2}{dx^2} + a_1\,\delta(x+q) + b_1\,\delta'(x+q) + a_2\,\delta(x-q) + b_2\,\delta'(x-q)\,, \tag{13}$$

where $q > 0$ and the meaning of H_0 and $V(x)$ is obvious.

A generalization of the Hamiltonian (13) is given by the Dirac δ-δ' comb. This is a modification of the Kronig-Penney model, which is an exactly solvable periodic potential, used in Solid State Physics, which describes electron motion in a periodic array of rectangular barriers. The most obvious generalization of the Kronig-Penney model is to replace the rectangular barriers by Dirac deltas of the same amplitude, something that can be obtained by a formal limit procedure. Now, the one dimensional Hamiltonian $H = H_0 + V(x)$ is given by a periodic potential of the form $V(x) = V_0 \sum_{n=-\infty}^{\infty} \delta(x - na)$, with $V_0 > 0$ and $a > 0$.

Inspired in the above mentioned analysis of the Casimir effect, we propose the study of the Dirac δ-δ' comb, in which the potential takes the form:

$$V_1(x) = \sum_{n=-\infty}^{\infty} \left(V_0\, \delta(x - na) + a\, V_1\, \delta'(x - na)\right), \quad a, V_0 > 0\,, \;\; V_1 \in \mathbb{R}\,. \tag{14}$$

so that it is a second generalization of the Kronig-Penney model. From the point of view of physics, this chain may model a periodic array of charges and dipoles. The objective is to solve the one dimensional Schrödinger equation using (14) as potential.

Now, we operate on a neighbourhood of the origin, see Fig. 1. If we call $\psi_I(x)$ and $\psi_{II}(x)$ to the wave functions in the region I (left) and II (right), respectively, they have the following form ($k = \frac{\sqrt{2mE}}{\hbar} > 0$):

$$\begin{aligned} \psi_I(x) &= A_I\, e^{ikx} + B_1\, e^{-ikx}\,, \quad \psi_{II}(x) = A_{II}\, e^{ikx} + B_{II}\, e^{-ikx}\,, \\ \psi_I'(x) &= ik\, A_I\, e^{ikx} - ik\, B_1\, e^{-ikx}\,, \quad \psi_{II}'(x) = ik\, A_{II}\, e^{ikx} - ik\, B_{II}\, e^{-ikx}\,. \end{aligned} \tag{15}$$

Equations (15) can be written in simplified matrix form as

$$\boldsymbol{\psi}_J(x) := \begin{pmatrix} \psi_J(x) \\ \psi_J'(x) \end{pmatrix} = \mathbb{KM}_x \begin{pmatrix} c A_J \\ B_J \end{pmatrix}, \quad J = I, II\,, \tag{16}$$

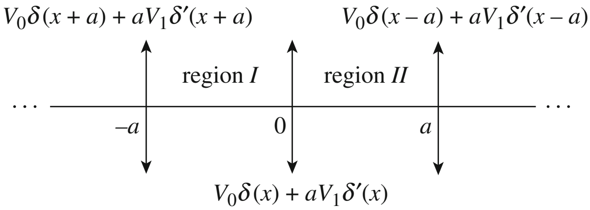

Fig. 1 Periodic potential (14) near the origin

with

$$\mathbb{K} = \begin{pmatrix} 1 & 1 \\ ik & -ik \end{pmatrix}, \qquad \mathbb{M}_x = \begin{pmatrix} e^{ikx} & 0 \\ 0 & e^{-ikx} \end{pmatrix}. \tag{17}$$

In order to include the perturbation of the form δ-δ' at the origin, we have to use the matching conditions, as before. The resulting equation has the form $\boldsymbol{\psi}_{II}(0^+) = \mathbb{T}_U\, \boldsymbol{\psi}_I(0^-)$, with

$$\mathbb{T}_U = \begin{pmatrix} \dfrac{1+U_1}{1-U_1} & 0 \\ \dfrac{2U_0/a}{1-U_1^2} & \dfrac{1-U_1}{1+U_1} \end{pmatrix}, \qquad U_0 = \frac{maV_0}{\hbar^2}, \quad U_1 = \frac{maV_1}{\hbar^2}. \tag{18}$$

Again, (18) is valid provided that $V_1 \neq \pm\hbar^2/(ma)$, otherwise the origin becomes opaque. After some algebra, we finally arrive to the following relation between the coefficients of the wave function to both sides of the origin:

$$\begin{pmatrix} A_{II} \\ B_{II} \end{pmatrix} = \mathbb{K}^{-1}\mathbb{T}_U\mathbb{K} \begin{pmatrix} A_I \\ B_I \end{pmatrix}. \tag{19}$$

Then, we use the periodicity properties of the potential in order to obtain the wave function over all the real line $\mathbb{R}$ and some other properties. First of all, the Floquet-Bloch theorem imposes the following condition ($x \in (-a, a)$):

$$\psi(x+a) = e^{iqa}\, \psi(x) \Longrightarrow \psi'(x+a) = e^{iqa}\, \psi'(x)\,, \tag{20}$$

where q is a constant called the quasi-momentum and it is a characteristic of the periodic potential given, and a is the distance between the nodes or points supporting the contact potential. We may write relation (20) in matrix form, which for $x \in (-a, 0)$ is

$$\boldsymbol{\psi}_{II}(x+a) = e^{iqa}\, \boldsymbol{\psi}_I(x) \Longrightarrow \mathbb{K}\mathbb{M}_x\mathbb{M}_a \begin{pmatrix} A_{II} \\ B_{II} \end{pmatrix} = e^{iqa}\, \mathbb{K}\mathbb{M}_x \begin{pmatrix} A_I \\ B_I \end{pmatrix}. \tag{21}$$

From (17), the matrices $\mathbb{M}_x$ and $\mathbb{K}$ are invertible, so that (21) implies that

$$[\mathbb{M}_a\mathbb{K}^{-1}\mathbb{T}_U\mathbb{K} - e^{iqa}\,\mathbb{I}] \begin{pmatrix} A_I \\ B_I \end{pmatrix} = \mathbf{0} \Leftrightarrow \det[\mathbb{T}_U - e^{iqa}\, \mathbb{K}\mathbb{M}_a^{-1}\mathbb{K}^{-1}] = 0\,, \tag{22}$$

where $\mathbb{I}$ is the 2×2 identity matrix. The cancellation of the determinant in (22) has some important consequences. With the definitions $\widetilde{q} = aq$ and $\widetilde{k} = ka$, Eq. (22) gives

$$\cos \widetilde{q} = f(U_1) \left[\cos \widetilde{k} + U_0\, g(U_1)\, \frac{\sin \widetilde{k}}{\widetilde{k}} \right],\ f(U_1) = \frac{1+U_1^2}{1-U_1^2},\ g(U_1) = \frac{1}{1+U_1^2}, \tag{23}$$

and U_0 and U_1 are as in (18). The first equation in (23) is often known as the *secular band equation* and determines the *dispersion relation* in each energy band $\tilde{k} = \tilde{k}_n(q)$. It is an even function of U_1, or equivalently, of $a V_1$ the coefficient of δ'. The main interest of the dispersion relation comes from the fact that that it provides the band spectrum of the Hamiltonian (14). The case $U_1 = 0$, i. e., no δ' term is present, has been previously studied. If $U_1 \neq 0$, the δ' term appears and the structure of the band spectrum changes drastically and must be obtained numerically. The graphical results can be seen in Fig. 2.

3.1 A Two Species Dirac δ-δ′ Comb

Let us now consider a Hamiltonian of the form $H = H_0 + V_1(x) + V_2(x)$, where $H_0 = -\hbar^2/(2m)\, d^2/dx^2$, $V_1(x)$ is as in (14) and $V_2(x)$ is given by

$$V_2(x) = \sum_{n=-\infty}^{\infty} \big(W_0\, \delta(x - na - b) + a\, W_1\, \delta'(x - na - b) \big),\ a > 0,\ W_0,\ W_1 \in \mathbb{R}.$$

We called this model the two species Dirac δ-δ' comb in comparison with the model discussed just above in relation to the Hamiltonian with periodic potential $V_1(x)$. The objective is again to study the band spectrum. Now, the discussion is quite similar to the precedent one, albeit a bit more complicated, but it is carried out under the same premises. We arrive to a band secular equation of the form

$$\cos(qa) = F(k; a, b, W_0, W_1, U_0, U_1)\,, \tag{24}$$

where the explicit form of the function F is rather complicated and has been obtained in [25]. A numerical analysis gives the behaviour of the band spectrum. There are interesting differences in the behaviour of band spectrum as compared with this band spectrum for the one species Dirac δ-δ' comb. Now the band shape is completely deformed and, for certain values of the parameters U_1 and W_w, the band shape is the reverse of what is for the one species comb. See details in [25]. This effect is particularly notorious for high values of $|U_1|$ and $|W_1|$. In addition, there are critical values of the parameters, typically $U_1 = \pm 1$ and $W_1 = \pm 1$, for which impenetrable barriers appear.

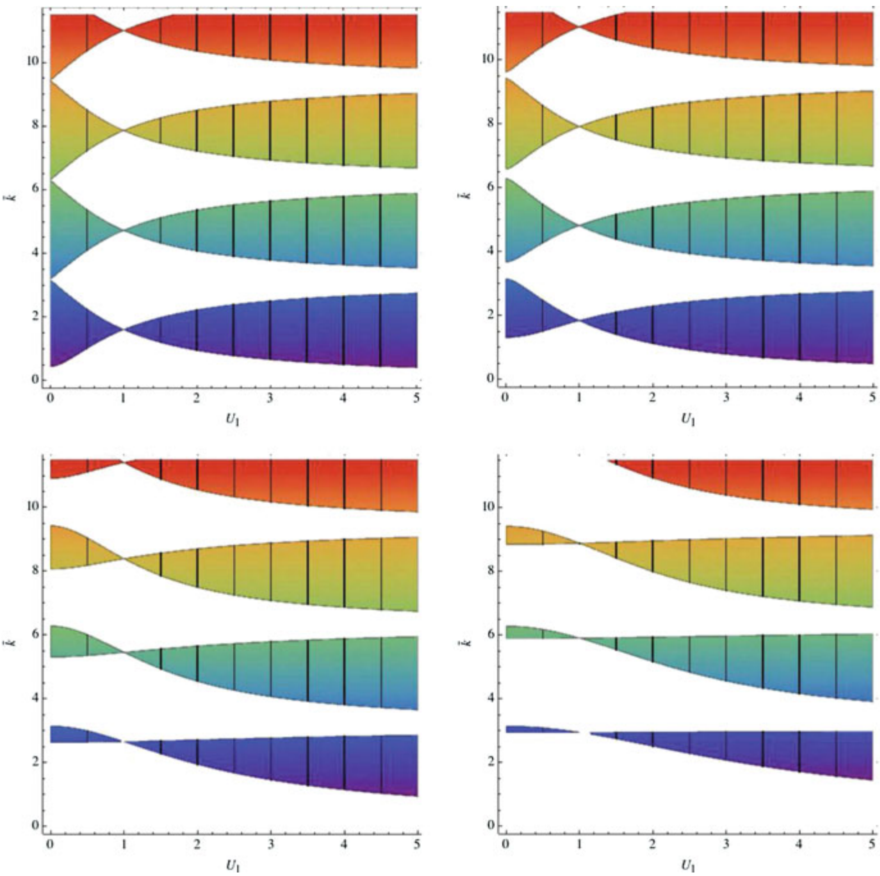

Fig. 2 Band structure for different values of U_0. From left to right $U_0 = 0.1$, $U_0 = 1$, $U_0 = 10$ and $U_0 = 30$. In all the cases, the band structure of the standard Dirac comb corresponds to $U_1 = 0$

4 Hyperspherical δ-δ'

One of the most obvious generalizations of the Dirac δ-δ' potentials is a homogeneous d-th dimensional potential supported on a hull sphere of radius r_0. Due to the symmetry of this model, this potential would be equivalent to a one dimensional contact potential at the point $r = r_0 > 0$ plus an impenetrable barrier at the origin. Let us pose the problem from the very beginning and consider the d-th dimensional Hamiltonian of the form [35]

$$H := -\frac{\hbar^2}{2m}\,\widehat{\boldsymbol{\Delta}}_d + \widehat{V}(\mathbf{x})\,, \tag{25}$$

with

$$\widehat{V}(\mathbf{x}) = a\,\delta(x - x_0) + b\,\delta'(x - x_0), \quad x = |\mathbf{x}|. \tag{26}$$

Here it is convenient to introduce the following dimensionless quantities:

$$\mathfrak{h} := \frac{2}{mc^2}\,H \quad w_0 := \frac{2a}{\hbar c}\,, \quad w_1 := \frac{bm}{\hbar^2}\,, \quad r := \frac{mc}{\hbar}\,|\mathbf{x}|\,, \tag{27}$$

where c is the speed of light in the vacuum. After (27), the new Hamiltonian reads:

$$\mathfrak{h} = -\Delta_d + w_0\,\delta(r - r_0) + 2w_1\,\delta'(r - r_0) = -\Delta_d + V(r)\,. \tag{28}$$

Here, Δ_d is the d-dimensional Laplace operator, which expressed in hyperspherical coordinates, $(r, \Omega_d := \{\theta_1, \theta_2, \dots, \theta_{d-2}, \phi\})$ reads:

$$\Delta_d = \frac{1}{r^{d-1}}\,\frac{\partial}{\partial\, r}\left(r^{d-1}\,\frac{1}{r^{d-1}}\right) + \frac{\Delta_{S^{d-1}}}{r^2}\,, \tag{29}$$

$\Delta_{S^{d-1}}$ being the Laplace-Beltrami operator on functions defined on the hull hypersphere S^{d-1} with dimension $d-1$. This operator satisfies the identity $\Delta_{S^{d-1}} = -\mathbf{L}_d^2$, where $\mathbf{L}_d$ is the generalized d-dimensional angular momentum operator.

The eigenvalue equation for $\mathfrak{h}$ is separable, so that there are factorizable solutions of the form $\psi_{\lambda\ell}(r, \Omega_d) = R_{\lambda\ell}(r)\,Y_\ell(\Omega_d)$, where $R_{\lambda\ell}(r)$ is the radial wave function and $Y_\ell(\Omega_d)$ are the hyperspherical harmonics. These are eigenfunctions of the Laplace-Beltrami operator $\Delta_{S^{d-1}}$ with eigenvalues $\chi(d, \ell) = -\ell(\ell + d - 2)$ [29]. The radial wave function is given by

$$\left[-\frac{d^2}{dr^2} - \frac{d-1}{r}\,\frac{d}{dr} + \frac{\ell(\ell + d - 2)}{r^2} + V(r)\right] R_{\lambda\ell}(r) = \lambda\, R_{\lambda\ell}(r)\,, \tag{30}$$

where $V(r)$ was defined in (28).

Next, we introduce the reduced radial function,

$$u_{\lambda\ell}(r) := r^{\frac{d-1}{2}}\, R_{\lambda\ell}(r)\,. \tag{31}$$

The effect of this change of indeterminate is to remove the term with the first derivative in (30). The resulting equation is

$$(h_0 + V(r))\, u_{\lambda\ell}(r) = \lambda_\ell\, u_{\lambda\ell}(r)\,, \tag{32}$$

where,

$$h_0 = -\frac{d^2}{dr^2} + \frac{(d + 2\ell - 3)(d + 2\ell - 1)}{4r^2}\,. \tag{33}$$

In order to define the potential $V(r)$ using the theory of self-adjoint extensions of symmetric operators, we need to define a domain for h_0, in which h_0 be symmetric with equal deficiency indices $(2, 2)$. Then, the domain $\mathcal{D}(h_0)$ is the space of functions $\varphi(r) \in L^2(\mathbb{R}^+)$ with the following properties:

1. Any $\varphi(r) \in \mathcal{D}(h_0)$ is in the Sobolev space $W_2^2(\mathbb{R}^+)$ of absolutely continuous functions with absolutely continuous first derivative and which second derivative is in $L^2(\mathbb{R}^+)$.
2. They vanish at the origin, $\varphi(0) = 0$.
3. At the point $r = r_0$, they satisfy the property: $\varphi(r_0) = \varphi'(r_0) = 0$.

The domain $\mathcal{D}(h_0^\dagger)$ of the adjoint, $h_0^\dagger$, of h_0 is the space verifying some changes in the above conditions: in Condition (1), we replace $W_2^2(\mathbb{R}^+)$ by $W_2^2(\mathbb{R}^+\backslash\{r_0\})$, which is the space satisfying the same properties, except that its functions and their first derivatives have finite jumps at r_0 and, then, Condition (3) is not fulfilled. The domain $\mathcal{D}(h_0 + V(r))$ that makes the operator $h_0 + V(r)$ self-adjoint is the space of all functions $\varphi(r)$ in $\mathcal{D}(h_0^\dagger)$ satisfying the following matching conditions at r_0:

$$\begin{pmatrix} \varphi(r_0^+) \\ \varphi'(r_0^+) \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \beta & \alpha^{-1} \end{pmatrix} \begin{pmatrix} \varphi(r_0^-) \\ \varphi'(r_0^-) \end{pmatrix}, \tag{34}$$

where $\varphi(r_0^\pm)$ are the right $(+)$ and left $(-)$ limits of $\varphi(r)$ at $r = r_0$. Also,

$$\alpha = \frac{1 + w_1}{1 - w_1}, \qquad \beta = \frac{w_0}{1 - w_1^2}. \tag{35}$$

These matching conditions determine the boundary conditions that should be verified by the radial wave functions $R_{\lambda\ell}(r)$. In fact, (31) and (34) give:

$$\begin{pmatrix} R_{\lambda\ell}(r_0^+) \\ R'_{\lambda\ell}(r_0^+) \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \widetilde{\beta} & \alpha^{-1} \end{pmatrix} \begin{pmatrix} R_{\lambda\ell}(r_0^-) \\ R'_{\lambda\ell}(r_0^-) \end{pmatrix}, \tag{36}$$

with

$$\widetilde{\beta} := \beta - \frac{(\alpha^2 - 1)(d - 1)}{2\alpha r_0} = \frac{\widetilde{w}_0}{1 - w_1^2}, \quad \widetilde{w}_0 = \frac{2(1 - d)w_1}{r_0} + w_0. \tag{37}$$

These matching conditions are well defined, except at the exceptional values $w_1 = \pm 1$. These two cases have to be treated separately, see [24, 31].

4.1 Bound States

Here, we present some results concerning the existence of bound states for the model under consideration. The eigenvalue equation for bound states is (30) with $\lambda < 0$. Then, it is convenient to use the parameter $\kappa > 0$ with $\lambda = -\kappa^2$. The general solution of (30) is

$$R_{\kappa\ell}(r) = \begin{cases} A_1\,\mathcal{I}_\ell(\kappa r) + B_1\,\mathcal{K}_\ell(\kappa r) & \text{if } r \in (0, r_0)\,, \\ \\ A_2\,\mathcal{I}_\ell(\kappa r) + B_2\,\mathcal{K}_\ell(\kappa r) & \text{if } r \in (r_0, \infty)\,. \end{cases} \tag{38}$$

Then, $R_{\kappa\ell}(r)$ can be written in terms of modified hyperspherical Bessel functions of the first ($I_\ell(z)$) and second ($K_\ell(z)$) kind, respectively, where,

$$\mathcal{I}_\ell(\kappa r) = \frac{1}{(\kappa r)^\nu}\, I_{\ell+\nu}(\kappa \nu)\,, \quad \mathcal{K}_\ell(\kappa \nu) = \frac{1}{(\kappa r)^\nu}\, K_{\ell+\nu}(\kappa \nu)\,, \quad \nu := \frac{d-2}{2}\,.$$

The form of the solution in terms of the functions $u_{\kappa\ell}(r)$ defined in (31) comes straightforwardly from (38). The square integrability condition of the radial wave function for bound states imposes that $A_2 = 0$. Furthermore, the term multiplied by B_1 is not square integrable, except for zero angular momentum in two and three dimensions. In three dimensions, the condition $u_{\kappa\ell}(0) = 0$ implies that $B_1 = 0$. There are other type of arguments that show that in two dimensions, we also have $B_1 = 0$ [26]. After these considerations, (36) can be written as

$$B_2 \begin{pmatrix} \mathcal{K}_\ell(\kappa r_0) \\ \\ \kappa\,\mathcal{K}'_\ell(\kappa r_0) \end{pmatrix} = A_1 \begin{pmatrix} \alpha & 0 \\ \\ \widetilde{\beta} & \alpha^{-1} \end{pmatrix} \begin{pmatrix} \mathcal{I}_\ell(\kappa r_0) \\ \\ \kappa\,\mathcal{I}'_\ell(\kappa r_0) \end{pmatrix}. \tag{39}$$

If we divide the identity obtained with the lower component of (39) with that found with the first component, we get the following expression called the *secular equation*:

$$\alpha \frac{d}{dr} \log \mathcal{K}_\ell(\kappa r)|_{r=r_0} = \widetilde{\beta} + \alpha^{-1} \frac{d}{dr} \log \mathcal{I}_\ell(\kappa r)|_{r=r_0}. \tag{40}$$

Solutions for $\kappa > 0$ of (40) give the energies for the bound states of the model under consideration. If we denote by $y_0 = \kappa r_0$, (40) takes the form

$$F(y_0) = -y_0 \left(\frac{I_{\nu+\ell-1}(y_0)}{I_{\nu+\ell}(y_0)} + \frac{\alpha K_{\nu+\ell-1}(y_0)}{K_{\nu+\ell}(y_0)} \right) - (\alpha - \alpha^{-1})\ell = 2\nu(\alpha - \alpha^{-1}) + \widetilde{\beta} r_0\,.$$

Observe that the right hand side is independent on the energy and the angular momentum. This equation cannot be solved analytically. However, it may be used to obtain some properties concerning the number of bound states, $N_\ell^d = n_\ell^d \deg(d, \ell)$, that exist for given values of d and ℓ. Here n_ℓ^d is the number of negative energy eigenvalues and $\deg(d, \ell)$ the degeneracy associated with ℓ in d dimensions. We listed here below these results without proofs that may be found in [26]:

1. In the d-dimensional quantum system described by the Hamiltonian (28), the number n_ℓ^d defined above is at most one. This is, $n_\ell^d \in \{0, 1\}$.
2. The d-dimensional quantum system described by the Hamiltonian (28) admits bound states with angular momentum ℓ if and only if

$$\ell_{\max} \neq L_{\max}\,, \quad \text{and} \quad \ell \in \{0, 1, \ldots, \ell_{\max}\}\,, \quad \ell_{\max} > -1\,, \tag{41}$$

with

$$\ell_{\max} := \lfloor L_{\max} \rfloor\,, \quad L_{\max} := \frac{w_1 - r_0 w_0/2}{w_1^2 + 1} + \frac{2-d}{2}\,, \tag{42}$$

where $\lfloor A \rfloor$ denotes the integer part of the real number A. In addition, if $\lambda_\ell = -\kappa_\ell^2$ is the energy of the bound state with angular momentum ℓ, the following inequality holds:

$$\lambda_\ell < \lambda_{\ell+1} < 0\,, \qquad \ell \in \{0, 1, \ldots, \ell_{\max} - 1\}\,. \tag{43}$$

3. The quantum Hamiltonian (28) admits a bound state for any $\omega_0 > 0$, only if $d = 2$ and $\ell = 0$.

5 An Application to Nuclear Physics

The δ-δ' is an approximation that serves to obtain interesting results concerning realistic models in physics. Next, we want to introduce one of these examples coming from nuclear physics. Let us consider a model for atomic nuclei based on a mean field potential with volume, surface and spin orbits parts, for which the Hamiltonian is given by

$$H(\mathbf{r}) = -\frac{\hbar^2}{2\mu}\nabla_{\mathbf{r}}^2 + U_0(r) + U_{SO}(r)(\mathbf{L}\cdot\mathbf{S}) + U_q(r)\,, \tag{44}$$

where $r = |\mathbf{r}|$, μ is the reduced mass and the terms $U_0(r)$, $U_{SO}(r)$ and $U_q(r)$ have their origin in the Wood-Saxon potential:

$$U_0(r) = -V_0\, f(r) := -V_0\,\frac{1}{1 + e^{(r-R)/a}}\,, \tag{45}$$

$$U_{SO}(r) = \frac{V_{SO}}{\hbar^2}\, f'(r) = -\frac{V_{SO}}{a\hbar^2}\, \frac{e^{(r-R)/a}}{(1+e^{(r-R)/a})^2}\,, \tag{46}$$

$$U_q(r) = V_q\, fi''(r) = -\frac{V_q}{a^2}\frac{e^{(r-R)/a}\,(1-e^{(r-R)/a})}{(1+e^{(r-R)/a})^3}\,. \tag{47}$$

Here, V_0, V_{SO} and V_q are positive constants, R is the nuclear radius and a is the thickness of the nuclear surface.

The kinetic term in (44) can be written in terms of the orbital angular momentum $\mathbf{L}$ as

$$-\frac{\hbar^2}{2\mu}\,\nabla^2_{\mathbf{r}} = -\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\mathbf{L}^2/\hbar^2}{r^2}\right]. \tag{48}$$

Then, there exist factorizable solutions for the Schrödinger equation associated to (48). This factorization is of the form,

$$\psi(\mathbf{r}) = \frac{u_{n\ell j}(r)}{r}\,\mathcal{Y}_{\ell jm}(\theta,\phi)\,, \tag{49}$$

where the angular part, satisfies the following relations:

$$\mathbf{L}^2\,\mathcal{Y}_{\ell jm}(\theta,\phi) = \hbar^2\,\ell(\ell+1)\,\mathcal{Y}_{\ell jm}(\theta,\phi)\,, \tag{50}$$

and

$$(\mathbf{L}\cdot\mathbf{S})\mathcal{Y}_{\ell jm}(\theta,\phi) = \hbar^2\xi_{\ell,j}\mathcal{Y}_{\ell jm}(\theta,\phi),\ \text{with}\ \xi_{\ell,j} := \begin{cases} \frac{\ell}{2} & \text{for } j=\ell+\frac{1}{2}, \\ -\frac{\ell+1}{2} & \text{for } j=\ell-\frac{1}{2}. \end{cases}$$

Note that $\ell\in\mathbb{N}\cup\{0\}$. The functions denoted as $\mathcal{Y}_{\ell jm}(\theta,\phi)$ are linear combination of spherical harmonics $Y_\ell^m(\theta,\phi)$, which are simultaneous eigenfunctions of the operators $\mathbf{L}^2$, $\mathbf{S}^2$ and $\mathbf{J}^2 = (\mathbf{L}+\mathbf{S})^2$. The radial part of the three dimensional Schrödinger equation has the form

$$H(r)\,u_{n\ell j}(r) = E_{n\ell j}\,u_{n\ell j}(r)\,, \tag{51}$$

where,

$$H(r) = -\frac{\hbar^2}{2\mu}\left[\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2}\right] - V_0\,f(r) + V_{SO}\,\xi_{\ell,j}\,f'(r) + V_q\,fi''(r)\,. \tag{52}$$

Our approximation can be obtained by taking the limit $a \to 0^+$ in the potential terms. This limit makes proper mathematical meaning in a distributional sense. From this point of view, we have that ($r \geq 0$)

$$\lim_{a\to 0^+} U_0(r) = V_0[\theta(r-R)-1]\,, \tag{53}$$

$$\lim_{a\to 0^+} V_{SO}(r) = -V_{SO}\,\xi_{\ell,j}\,\delta(r-R)\,, \tag{54}$$

$$\lim_{a\to 0^+} U_q(r) = -V_q\,\delta'(r-R)\,, \tag{55}$$

where $\theta(x)$ in (53) is the Heaviside step function. After this limit procedure, we finally obtain our model, which is given by the following radial Hamiltonian with contact potential:

$$H_c = -\frac{\hbar^2}{2\mu}\left[\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2}\right] + V_0[\theta(r-R)-1] - V_{SO}\xi_{\ell,j}\delta(r-R) - V_q\delta'(r-R). \tag{56}$$

The advantage of the Hamiltonian in (56) over that in (48) is that the Schrödinger equation, $H_s(r)\,u_{n\ell j}(r) = E_{n\ell j}\,u_{n\ell j}$, associated to the former can be exactly solved for all values of ℓ and j. If we use, $\alpha := (2\mu/\hbar^2)V_{SO}\xi_{\ell,j}$ and $\beta = (2\mu/\hbar^2)V_q$, this Schrödinger equation becomes, were we omit the subindices in $u(r)$ for simplicity:

$$\frac{d^2u(r)}{dr^2} + \left\{\frac{2\mu E}{\hbar^2} - \frac{2\mu V_0}{\hbar^2}[\theta(r-R)-1] + \alpha\,\delta(r-R) + \beta\,\delta'(r-R) - \frac{\ell(\ell+1)}{r^2}\right\}u(r) = 0\,.$$

Square integrable solutions inside the nucleus are

$$u_\ell(r) = A_\ell\,\sqrt{\gamma r}\,J_{\ell+\frac{1}{2}}(\gamma r)\,, \qquad \gamma = \frac{\sqrt{2\mu(V_0+E)}}{\hbar}\,, \qquad r\in[0,R)\,, \tag{57}$$

and outside the nucleus,

$$u_\ell = D_\ell\,\sqrt{\kappa r}\,K_{\ell+\frac{1}{2}}(\kappa r)\,, \qquad \kappa = \frac{\sqrt{2\mu|E|}}{\hbar}\,, \qquad r\in(R,\infty)\,. \tag{58}$$

Then, we impose the condition that the above solution be in the domain of the Hamiltonian (52). To do it, we need to find a relation between the coefficients A_ℓ and D_ℓ such that (57) and (58) verify the precise matching relations at $r = R$ so that (52) be self-adjoint. These relations are

$$\begin{pmatrix} u_\ell(R^+) \\ u'_\ell(R^+) \end{pmatrix} = \begin{pmatrix} \dfrac{2-\beta}{2-\beta} & 0 \\ \dfrac{4\alpha}{4-\beta^2} & \dfrac{2-\beta}{2+\beta} \end{pmatrix} \begin{pmatrix} u_\ell(R^-) \\ u'_\ell(R^-) \end{pmatrix} . \tag{59}$$

This gives a system of two equations, which permits to find a relation which is independent of the coefficients A_ℓ and D_ℓ and is

$$\varphi(\chi) := \frac{\chi\, J_{\ell+3/2}(\chi)}{J_{\ell+1/2}(\chi)} = \frac{(2+\beta)^2}{(2-\beta)^2}\frac{\sigma\, K_{\ell+3/2}(\sigma)}{K_{\ell+1/2}(\sigma)} - \frac{8\beta(\ell+1)}{(2-\beta)^2} + \frac{w_0}{(2-b)^2} =: \phi(\sigma), \tag{60}$$

with

$$\chi := v_0\sqrt{1-\varepsilon}\,, \quad \sigma := v_0\sqrt{\epsilon}\,, \quad \epsilon := \frac{|E|}{V_0} \in (0,1)\,, \tag{61}$$

$$v_0 = \sqrt{\frac{2\mu R^2 V_0}{\hbar^2}} > 0\,, \qquad w_0 = \frac{8\mu V_{SO}\, \xi_{\ell,j}\, R}{\hbar^2}\,. \tag{62}$$

Equation (60) if often called the *secular equation*. It is useful in order to obtain results concerning bound states. These results have been derived and proven in [26]. Here, we listed some of which we consider the most interesting:

1. If for any value $\ell \in \mathbb{N}_0$ such that $\ell \leq \ell_{max}$ the following inequality holds

$$w_0 > -\left((\beta-2)^2 + 2\ell\left(\beta^2+4\right)\right), \tag{63}$$

   there exists one, and only one, energy level with relative energy

$$\varepsilon_s \in \left(1 - \frac{j^2_{\ell+1/2,s}}{v_0^2}, 1 - \frac{j^2_{\ell+3/2,s-1}}{v_0^2}\right) \subset (0,1), \qquad s \in \mathbb{N}. \tag{64}$$

   For $w_0 \in \mathbb{R}$ the final number of bound states, $N_\ell = (2\ell+1)n_\ell$, is determined by

$$n_\ell = M + m_1 - m_2, \tag{65}$$

   where M is

$$M = \min\{s \in \mathbb{N}_0 \mid j_{\ell+1/2,s+1} > v_0\}, \tag{66}$$

and, using the functions $\varphi(\chi)$ and $\phi(\sigma)$ defined in (60), we obtain

$$m_1 = \begin{cases} 1 & \text{if } \varphi(v_0) > \phi(0^+), \\ 0 & \text{if } \varphi(v_0) < \phi(0^+) \ \text{ or } \ v_0 = j_{\ell+1/2,M}, \end{cases} \qquad m_2 = \begin{cases} 1 & \text{if } 0 > \phi(v_0), \\ 0 & \text{if } 0 < \phi(v_0). \end{cases}$$

2. The quantum system governed by the Hamiltonian (56) does not admit bound states with angular momentum $\ell > \ell_{\max}$, where

$$\ell_{max} := \max\{\ell \in \mathbb{N}_0 \mid j_{\ell+1/2,1} < v_0 \text{ or } \varphi(v_0) > \phi(0^+)\}.$$

   If there exist $s_0 \in \mathbb{N}$ and $\ell_0 \in \mathbb{N}_0$ such that $v_0 = j_{\ell_0+1/2,s_0}$ the second condition in the previous set cannot be evaluated. Nonetheless, it is not necessary since the existence of at least one bound state for ℓ_0 is guaranteed.
3. If there exist bound states with relative energies $\varepsilon_{n\ell_j}, \varepsilon_{(n+1)\ell_j}, \varepsilon_{n(\ell+1)_j}$ for $n, \ell \in \mathbb{N}_0$ the following inequalities hold:

$$(a)\ \varepsilon_{n\ell_j} > \varepsilon_{(n+1)\ell_j}, \quad (b)\ \varepsilon_{n\ell_j} > \varepsilon_{n(\ell+1)_j}, \quad (c)\ \varepsilon_{n\ell_{\ell+1/2}} > \varepsilon_{n\ell_{\ell-1/2}}.$$

   Notice that the second inequality only applies for $j = \ell + 1/2$.
4. There are two special cases, in which $\beta = \pm 2$. Now, the contact potential at $r = R$ becomes opaque in the sense that the transmission coefficient is equal to zero. Here, we expect the existence of bound states alone, without resonances or scattering states. This specific problem has been discussed in [26], where the proposed nuclear model is tested with experimental and numerical data in the double magic nuclei ^{132}Sn and ^{208}Pb with an additional neutron.

5.1 *Resonances*

Apart from bound states, we may analyze scattering states or the possibility of the existence of resonances or even antibound states. Here, we briefly discuss the existence of resonances, which are unstable quantum states [12, 13]. Contrary to the case of bound states, wave functions (usually called Gamow functions) for unstable quantum states are not square integrable. Moreover, in the coordinate representation, they show an asymptotically exponential grow at the infinity. In our case, this have the following consequence: Although for consistency reasons, we should keep the expression (57) for the wave function inside the nucleus ($r < R$), we should use the complete solution for the Schrödinger equation outside the nucleus, i.e., in the region $r > R$. This is

$$u_\ell(r) = \sqrt{\kappa r}\left(C_\ell\, H^{(1)}_{\ell+\frac{1}{2}}(\kappa r) + D_\ell\, H^{(2)}_{\ell+\frac{1}{2}}(\kappa r)\right), \quad \kappa := \frac{\sqrt{2\mu E}}{\hbar}, \quad E > 0, \tag{67}$$

where $H^{(i)}(\kappa r)$ are the Hänkel functions of first (1) and second (2) kind, respectively, and C_ℓ and D_ℓ are coefficients depending solely on κ. In the search for resonances, the knowledge of the asymptotic forms of the Hänkel functions for large values of r is essential. These are:

$$H^{(1)}_{\ell+\frac{1}{2}}(\kappa r) \approx \sqrt{\frac{2}{\pi\kappa r}}\, e^{-i(\kappa r-(\ell+1)\pi/2)}, \quad H^{(2)}_{\ell+\frac{1}{2}}(\kappa r) \approx \sqrt{\frac{2}{\pi\kappa r}}\, e^{i(\kappa r-(\ell+1)\pi/2)}. \tag{68}$$

These asymptotic forms show that $H^{(1)}_{\ell+\frac{1}{2}}(\kappa r)$ is an *outgoing* wave function while $H^{(2)}_{\ell+\frac{1}{2}}(\kappa r)$ is an *incoming* wave function. Resonances are determined by the often called *purely outgoing boundary conditions*, which assumes that only the outgoing wave function survives. This implies that $D_\ell(\kappa) = 0$, and this is a transcendental equation for which the solutions coincide with the resonance poles of the S-matrix [13]. The determination of D_ℓ comes after the use of the matching conditions (59) and the expression (57) for the wave function inside the nucleus, where without lack of generality we may choose $A_\ell = 1$. This gives $D_\ell(\kappa) = 0$. The latter is a complicated transcendental equation, which depends on Hänkel and Bessel functions with different indices, see [26]. The solutions of this equation should be classified in three categories:

1. Simple solutions on the positive imaginary axis correspond to bound states.
2. Simple solutions on the negative imaginary axis correspond to virtual states also called antibound states.
3. Pairs of solutions on the lower half plane, symmetrically located with respect to the imaginary axis that correspond to resonances. Both members of each pair determine the same resonance and must have the same multiplicity. Usually, this multiplicity is one, although models with resonance poles with multiplicity two have been constructed [7, 32].

This model shows resonance poles. Due to the complexity of the relation $D_\ell(\kappa) = 0$ these resonances can only be obtained numerically in most of cases. It is important to remark that the imaginary part of the resonance poles is always negative. This implies that the asymptotic form on r of the first expression in (68) is exponentially growing, as previously noted.

General arguments [37] show that the number of resonance poles should be infinite. In order to give an idea on how these poles look like, we show a few in Fig. 3. Resonance poles lie at the intersection of two curves. Here, we have chosen the following values of the parameters: $\ell = 0$, $v_0 = 5$, $w_0 = 10$ and $\beta = 1$. Observe that resonance poles are rather close to the real axis, so that their imaginary part is rather small. Since the mean life of an unstable quantum state is related with the inverse of the imaginary part of its resonance pole, this means that the unstable states corresponding to the poles shown in Fig. 3 are rather stable. Some other cases with $\ell = 1, 2, 3, 4$ have been also considered and we have seen a similar pattern for resonance poles [26]. Exact analytical results were also obtained.

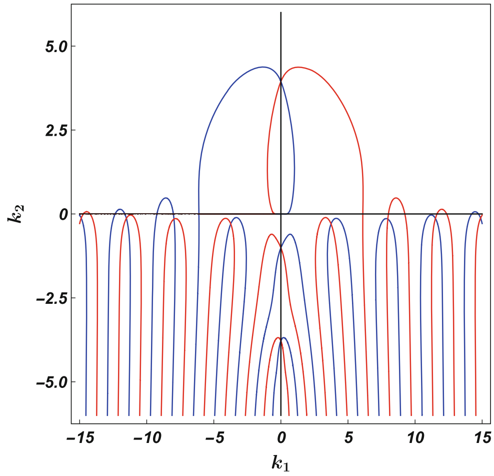

Fig. 3 Resonance poles are located at the intersection of curves below the real axis. here, $\ell = 0$, $v_0 = 5$, $w_0 = 10$ and $\beta = 1$

5.2 A Comment on the Self-adjointness of the Hamiltonian (56)

Take the Hamiltonian $H_c(r)$ in (56) and fix for simplicity $\hbar^2/2\mu = 1$, which shall not alter our results. Then, write $H_c(r) = H_\ell + V(r)$, with

$$H_\ell := -\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + V_0[\theta(r-R)-1]\,, \quad V(r) = a\delta(r-R) + b\delta'(r-R)\,. \tag{69}$$

We study the cases $\ell = 0$ and $\ell \neq 0$ separately. Let us discuss $\ell = 0$, first. To begin with, take $H_r := -d^2/dr^2$ with domain, $\mathcal{D}_c$, the subspace of functions $f(r) \in L^2[0,\infty)$ such that: (1) $f(r)$ is absolutely continuous with absolutely continuous first derivative; (2) The second derivative $fi''(r) \in L^2[0,\infty)$ is square integrable; (3) For all functions $f(r)$ in this domain, either $f(0) + cf'(0) = 0$ for some fixed real number c or $f'(0) = 0$. Each of these choices gives a self-adjoint determination of H_r.

Next, define the subdomain $\mathcal{D}_c(H_r)$ of all $f(r) \in \mathcal{D}_c$ such that $f(R) = f'(R) = 0$. Choosing $\mathcal{D}_c(H_r)$ as domain of H_r, we conclude that H_r is symmetric (Hermitian) with deficiency indices (2, 2). When H_r is define on this domain, then the domain of the adjoint of H_r, $\mathcal{D}_c(H_r^\dagger)$, is the space of functions $f(r)$ fulfilling conditions 1 and 2 above with one modification: they and their first derivatives have arbitrary although finite jumps at $r = R$. Self-adjoint extensions of H_r are given by imposing the functions $f(r) \in \mathcal{D}_c(H_r^\dagger)$ the matching conditions (59) at $r = R$. The exceptional cases $\beta = \pm 2$ also give respective self-adjoint extensions. These extensions determine self-adjoint operators of the form $-d^2/dr^2 + a\delta(r-R) + b\delta'(r-R)$. Since the term $V_0[\theta(r-R)-1)$ is bounded, adding it does not change anything.

Let us consider now the case $\ell \neq 0$. In this case, we do not need to establish boundary conditions at the origin of the type $f(0) = cf'(0)$, since the Hamiltonian H_ℓ in (69) with $\ell \neq 0$ is already essentially self-adjoint when its domain is the Schwartz space of functions supported on $\mathbb{R}^+ \equiv [0, \infty)$, $S(\mathbb{R}^+)$, for which we always have that $f(0) = f'(0) = 0$. In this case $-d^2/dr^2 + \ell(\ell+1)/r^2$ is essentially self-adjoint on the mentioned domain [6] and the same condition for H_ℓ comes trivially, since $V_0[\theta(r - R) - 1$ is bounded.

Then for any $\ell \neq 0$, let us define a domain $\mathcal{D}_{\ell,0}$ of functions $f(r) \in L^2(\mathbb{R}^+)$ fulfilling the following conditions:

1. $f(r)$ and $f'(r)$ are absolutely continuous;
2. The function $-fi''(r) = [\ell(\ell + 1)/r^2]f(r)$ belongs to $L^2(\mathbb{R}^+)$;
3. $f(0) = 0$;
4. $f(R) = f'(R) = 0$.

The conclusion is that H_ℓ on $\mathcal{D}_{\ell,0}$ is symmetric with deficiency indices $(2, 2)$.

In order to add to H_ℓ the perturbation $V(r) = a\delta(r - R) + b\delta'(r - R)$, we define the domain of the adjoint of H_ℓ on $\mathcal{D}_{\ell,0}$ as the subspace of $L^2(\mathbb{R}^+)$ satisfying the above conditions 1, 2 and 3 and replacing 4 by: 4′ $f(r)$ and $f'(r)$ have finite discontinuities at $r = R$. Then, imposing the matching conditions (59) to these functions, we obtain the domain in which $H_c(r) = H_\ell + V(r)$ is self-adjoint for any value of a and b. For $\ell \neq 0$, the subindex c is irrelevant. This completes our discussion on the self-adjoint of the Hamiltonian.

6 Concluding Remarks

Contact potentials are quite interesting in quantum mechanics because they provide of simple models to analyze the behaviour of quantum systems. Along this presentation, we were concerned with perturbations of the type $a\delta(x - x_0) + b\delta'(x - x_0)$ either in one dimension or in arbitrary dimensions with spherical symmetry, so that the model could be projected to a one dimensional one. This is what we call δ-δ' interactions.

In the first place, we have introduced a very simple one-dimensional model with a unique δ-δ' interaction on the free Hamiltonian. This interaction can be easily studied and serves as a basis for more complicated models. The contact interaction can be mathematically well defined using the theory of self-adjoint extensions of symmetric operators with equal deficiency indices. The possible existence of a bound state is investigated and scattering coefficients are determined.

This is used for the construction of a sort of Kronig-Pennery model in which rectangular barriers are replaced by δ-δ' interactions with identical coefficients, so that the resulting potential is periodic. The behaviour of the energy bands can be studied in terms of the variations of the coefficients of the delta and the delta prime. We have also considered an hybrid potential with two types of δ-δ' interactions. The study of the energy bands requires powerful numerical estimations and the use of the software Mathematica. A detailed description of this model, which could be interesting in Condensed Matter, can be just briefly summarized in this short review and has been published in [25].

Spherically symmetric models in quantum mechanics are often studied as one dimensional models with an infinite barrier at the origin, after separation of radial and angular variables. This is also the case of the δ-δ' interactions supported on hull spheres of arbitrary dimensions. Here, we have determined matching conditions that make the Hamiltonian with this type of interaction self-adjoint and have obtained some results concerning the number of bound states. These results depend on the dimension as well as the angular momentum.

Finally, we have used one type of δ-δ' interaction as an approximation of a mean field potential of wide use in nuclear physics. The objective is double. In one side, we have obtained results concerning the existence and number of bound states in the considered model in terms of the given parameters. For two exceptional cases, the model shows no transmission through the δ-δ' barrier, so that the number of bound states is infinite. Otherwise this number is finite. Outside the exceptional cases, the model shows resonances that are manifested as pairs of poles of the analytic continuation to the complex plane of the S-matrix, $S(k)$, in the momentum representation. These resonance poles can be obtained numerically as solutions of a transcendental equation. There is an infinite in number, so that in Fig. 2, we have depicted some resonance poles with the lowest real part. We have also discussed the construction of a self-adjoint Hamiltonian for such purpose.

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$2D$ Yang–Mills Theory and Tau Functions

Aleksandr Yu. Orlov

This article is dedicated to the 80th anniversary of Vladimir
E. Zakharov

Abstract We present tau functions of the multi-component KP hierarchy whose integral is equal to the correlation function of the Wilson loops of the two-dimensional Yang–Mills (YM) model on a surface Σ (orientable case). By adding the simplest BKP tau to the integration measure we get the YM model on the non-orientable surface. The higher times of the tau functions are related to random matrices and also source matrices; the latter play the role of free parameters, and the mentioned integrals are integrals over ensembles of random matrices. We study the cases where the integral of the tau function is a tau function again—the tau function of different hierarchies—two-component KP and one-component BKP.

Keywords Wilson loops · Random matrices · Tau functions · BKP hierarchy · Schur polynomials · Hypergeometric functions · Random partitions · 2D YM

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1 Introduction

In the work [41] it was shown that integrals of tau functions, namely the expectation values of tau functions in ensembles of random matrices, are equal to the special sums over partitions (Young diagrams) which can be related to the correlation functions of the Wilson loops W_i^*, $i = 1, \dots, V$ of quantum $2D$ gauge theory on a Riemann surface Σ, see [55, 62]:

$$\mathbb{E}_{\mathbb{U}_N^{\otimes n_1}} \mathbb{E}_{\mathbb{GL}_N^{\otimes n_2}} \left[\mathbf{L}_U^{I_1} \mathbf{L}_Z^{I_2} \left(\tau(\mathbf{p}(W_1), \dots, \mathbf{p}(W_{2\mathrm{D}})) \right) \right]$$

$$= 1 + \sum_{d>0} \sum_{\lambda \vdash d} \frac{(d!)^{n_2}}{N^{nd}} \mathbf{A}(\lambda) \left(\dim_{GL} \lambda \right)^{-n_1} \left(\dim_S \lambda \right)^{-n_2} \prod_{i=1}^{\mathrm{V}} s_\lambda \left(W_i^* \right) \qquad (1)$$

Here, $\mathbb{E}_{\mathbb{U}_N^{\otimes n_1}} \mathbb{E}_{\mathbb{GL}_N^{\otimes n_2}} [\tau]$ denotes the expectation value of the tau function in a certain ensemble of random matrices, $s_\lambda(W_i^*)$ is the $\mathbb{GL}_N$ ($\mathbb{U}_N$) character, and $W_i^* \in \mathbb{GL}_N$ (or. respectively, $W_i^* \in \mathbb{U}_N$). The details and notations will be explained in the next section.

The right hand side also generates Hurwitz numbers with completed cycles [42] H_{E} for covers of the base surface Σ where the $n_1 + n_2 - \mathrm{V}$ is the Euler characteristic of Σ.

The goal of the present note is to present a short review of results of [41, 42] and to pick up all cases where the right hand side of (1) is also a tau function. The results of the work are presented in Sect. 3. We show that integrals of tau functions are also integrable in the sense of [24] and may be related to quantum models as two-dimensional Yang–Mills theory on orientable [31, 55, 62] and non-orientable [62] surfaces.

The works [40–42] and the present one appear as a result of the cross study of three topics: integrable hierarchies [14, 22, 23, 32, 33, 49, 53, 54, 56, 58–60, 63–65], matrix models [17, 20, 24, 25, 38, 45, 66, 67] products of random matrices [2–4, 12] and Hurwitz numbers in case of orientable surfaces [15, 18, 43, 44], [7, 8, 13, 21, 26, 34–37] and also of non orientable ones [5, 6, 10, 11, 39, 40]. See also [9, 16, 19],

2 Random Matrices, Graphs and Tau Functions

2.1 Notations and Preliminaries

Mixed Ensembles of Random Matrices $\mathbb{E}_{\mathbb{U}_N^{\otimes n_1}} \mathbb{E}_{\mathbb{GL}_N^{\otimes n_2}} [f]$ is the notation for the expectation values of a function f that depends on the entries of matrices

$Z_1, \dots, Z_{n_1} \in \mathbb{GL}_N(\mathbb{C})$ and of matrices $U_1, \dots, U_{n_2} \in \mathbb{U}_N$, which are defined as

$$\mathbb{E}_{\mathbb{U}_N^{\otimes n_1}} \mathbb{E}_{\mathbb{GL}_N^{\otimes n_1}}[f] = \int f(U_1, \dots, U_{n_2}, Z_1, \dots, Z_{n_1}) \prod_{i=1}^{n_2} d_* U_i \prod_{i=1}^{n_1} d\mu(Z_i) \tag{2}$$

where d_*U_i $(i = 1, \dots, n_2)$ is the Haar measure on $\mathbb{U}_N$ and $d\mu(Z_i) = c \prod_{a,b} e^{-|(Z_i)_{a,b}|^2} d^2(Z_i)_{a,b}$ is the Gauss measure. Each set of Z_i and of $d\mu(Z_i)$ is called complex Ginibre ensemble, and the whole set $Z_1, \dots, Z_{n_1}$, $d\mu(Z_1), \dots, d\mu(Z_{n_1})$ is called n_1 independent complex Ginibre ensemble. The set $U_1, \dots, U_{n_2}, d_*U_1, \dots, d_*U_{n_2}$ is called n_2 independent circular ensemble. We assume each $\int d_*U = \int d\mu(Z_i) = 1$. The ensemble given by (2) we call the mixed ensemble.

Partitions: Schur Functions, Characters Then, $\lambda = (\lambda_1, \dots, \lambda_\ell)$ denotes a partition of d (this fact is written as $\lambda \vdash d$), that is $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$ are natural numbers and $\lambda_1 + \cdots + \lambda_\ell := |\lambda| = d$. The number of non-vanishing parts is called the length of the partition: $\ell = \ell(\lambda)$.

The sum in the right hand side of (1) ranges over set of all partitions $\ell(\lambda) \leq N$.

Given λ and a matrix $X \in \mathbb{GL}_N(\mathbb{C})$, the multi-variable polynomial

$$s_\lambda(X) = \frac{\det\left(x_i^{\lambda_i - i + N}\right)_{i,j}}{\det\left(x_i^{-i+N}\right)_{i,j}} \tag{3}$$

where $x_1, \dots, x_N$ denote the eigenvalues of X, is called the Schur function, or the Schur polynomial, indexed by λ. Here we suppose $\ell(\lambda) \leq N$, otherwise we put $s_\lambda(X) = 0$. The polynomial (3) is also known as the character of the group $\mathbb{GL}_N$ and also the character of the unitary group in case $X \in \mathbb{U}_N$. The Schur polynomial is the polynomial also in the entries of X:

$$s_\lambda(\mathbf{p}(X)) := s_\lambda(X) = \dim_S \lambda \sum_{\Delta \vdash d} \varphi_\lambda(\Delta) \mathbf{p}_\Delta(X) \tag{4}$$

where the sum ranges over all partitions of $d = |\Delta|$, $\Delta = (\Delta_1, \Delta_2, \dots)$, then we consider the sets and the variables (the so-called power sums [29])

$$\mathbf{p} = (p_1, p_2, \dots), \quad \mathbf{p}_\Delta := p_{\Delta_1} p_{\Delta_2} \cdots$$

and the notation $p_\Delta(X)$ means

$$p_\Delta(X) = (p_1(X), p_2(X), \dots), \quad p_m(X) = \sum_{a=1}^{N} x_a^m = \mathrm{tr}\left(X^m\right) \tag{5}$$

or, the same

$$p_\Delta(X) = \mathrm{tr}\left(X^{\Delta_1}\right)\mathrm{tr}\left(X^{\Delta_1}\right)\cdots$$

which is a polynomial function in the entries of the matrix X. The variables $p_m(X)$ are also known as Newton sums.

The notation $\dim_S \lambda$ serves for the dimension of the irrep λ of the symmetric group S_d, it is known that

$$\dim_S \lambda = d! s_\lambda(\mathbf{p}_\infty), \quad \mathbf{p}_\infty := (1, 0, 0, \dots)$$

Lastly, $\varphi_\lambda(\Delta)$ are certain rational numbers which are the normalized irrep characters of the group S_d indexed by λ and evaluated on the cycle class Δ, more precisely,

$$\varphi_\lambda(\Delta) = \frac{|C_\Delta|}{\dim_S \lambda}\chi_\lambda(\Delta)$$

where $|C_\Delta|$ is the cardinality of the class Δ (here, let me recall that each element of the symmetric group is the product of the non-intersecting cycles of lengths $\Delta_1 \geq \Delta_2 \geq \cdots$) and $\chi_\lambda(\Delta)$ is the irreducible character evaluated on any element of C_Δ. The cardinality of C_Δ can be written as follows

$$|C_\Delta| = \frac{|S_d|}{z_\Delta}, \quad z_\Delta = \prod_{i\geq 1} i^{m_i} m_i! \tag{6}$$

where m_i is the number of times the number i occurs in the partition (and sometimes, partitions are written as $\Delta = 1^{m_1}2^{m_2}3^{m_3}\cdots$), $|S_d| = d!$.

The relation (4) is also known as the character map relation because it relates the characters of linear groups to the characters of the symmetric group. The dimension of the representation of the linear group indexed by λ is

$$\dim_{GL} \lambda = s_\lambda(\mathbb{I}_d) = \dim_S \lambda \prod_{(i,j)\in\lambda}(N - j + i)$$

where $\mathbb{I}_N$ is the identity $N \times N$ matrix.

Tau Functions In the seminal work [64, 65], Vladimir Zakharov and Alexei Shabat discovered a way to build integrable $(2+1)$-dimensional systems together with their representations of Lax type. This work has opened a huge field in science. Integrable systems found unexpected applications in mathematics, physics and in technology. The main example was the Kadomtsev-Petviashvili (KP) equation, which naturally arises immediately along with the infinite number of different "higher KP equations" compatible with each other, and this fact can be reformulated as the existence of higher commuting flows; the group parameters of these flows were later named higher KP times and we denote the infinite set of times $\mathbf{p} = (p_1, p_2, p_3, \dots)$. In applications of the KP equation to the sea waves, the parameters p_1, p_2 and p_3 play the role of horizontal coordinates x, y and of time, respectively. Later, starting with the works of S.P. Novikov, the important role of the whole set of commuting flows of higher KP emerged (see chapter 2 in [63]). An important next step was taken by Sato and his school where the higher KP times were crucial in the vertex-operator approach to the theory of solitons [22, 56]. Matrix generalizations of the KP equation were also obtained in [64, 65] as part of a general approach. In the works of the Kyoto school, such systems were called multi-component KP hierarchies. The sets of the higher times of a multi-component hierarchy, I denote $\mathbf{p}^1, \ldots, \mathbf{p}^k$, where k is the number of the components. In a series of works by the Sato school, the tau-function approach was developed. The tau function is a kind of universal potential that describes the entire set of higher multi-component equations. A well-known example of a tau function is the Riemann theta function, where flows are flows on Jacobian.

For our purpose we choose special tau functions which we call *round dance tau functions*.

Example: Round Dance Tau Functions Let us use Frobenius coordinates $\alpha_1 > \cdots > \alpha_\kappa > 0$ and $\beta_1 > \cdots > \beta_\kappa > 0$ for partitions $\lambda = (\alpha|\beta) = (\alpha_1, \ldots, \alpha_\kappa|\beta_1, \ldots, \beta_\kappa)$. (We recall that Frobenius coordinates α_i and β_i are the lengths of respectively arms and legs of the hooks on the Young diagram λ where the corner of the i-th hook is the i-th node on the main diagonal, and $i = 1, \ldots, \kappa$, where κ is the number of the nodes on the main diagonal of λ, see [29]. Frobenius coordinates can also be viewed as the pairs of strict partitions (namely partitions with strictly decreasing parts) of the same length $\ell(\alpha) = \ell(\beta) = \kappa$, this length being the length of the main diagonal of the Young diagram $\lambda = (\alpha|\beta)$).

Let me introduce the *round dance tau function*. The round dance tau function is the special tau function of the multi-component KP hierarchy that can be written as follows:

$$\tau(\mathbf{p}^1, \ldots, \mathbf{p}^{2\mathrm{D}}) = 1 + \sum_{\kappa>0} \sum_{\substack{\alpha^1,\ldots,\alpha^{2\mathrm{D}} \\ \ell(\alpha^1)=\cdots=\ell(\alpha^{2\mathrm{D}})=\kappa}} \prod_{j=1}^{2\mathrm{D}} s_{(\alpha^j|\alpha^{j+1})}(\mathbf{p}^j) \prod_{i=1}^{\kappa} f_j(\alpha_i^j) \tag{7}$$

where the sum ranges over $2\mathrm{D}$ sets of strict partitions: $\alpha^j = \left(\alpha_1^{(j)}, \ldots, \alpha_\kappa^{(j)}\right)$, $\alpha^{(j_1)} > \cdots > \alpha_\kappa^{(j)} > 0$, and where the prime above the sum means that we set $\alpha^{2\mathrm{D}+1} = \alpha^1$ (for me, this necklace resembles a painting by Matisse called "Dance" [30], although there are 5 dancers, which is not an even number as in my case). We choose an even number of components, $2\mathrm{D}$, for further purposes. Let us note that this tau function is also the tau function for the k-component KP hierarchy where $k \leq 2\mathrm{D}$. To get it one just "freezes" extra higher times, keeping only the k sets, say, $\mathbf{p}^1, \ldots, \mathbf{p}^k$ as the variables.

The round dance tau function generalizes the so-called *hypergeometric tau function* [27, 49]

$$e^{-\sum_{m>0}\frac{1}{m}p_m^{(1)}p_m^{(2)}} = 1 + \sum_{\kappa=1}^{\infty} \sum_{\substack{\alpha^1,\alpha^2 \\ \ell(\alpha^1)=\ell(\alpha^2)=\kappa}} (-1)^{|\alpha^1|+|\alpha^2|+\kappa}$$

$$\times\, s_{(\alpha^1|\alpha^2)}(\mathbf{p}^1) s_{(\alpha^2|\alpha^1)}(\mathbf{p}^2) \prod_{i=1}^{\kappa} f_1(\alpha_i^1) f_2(\alpha_i^2) \qquad (8)$$

(that is the case $\mathrm{D} = 1$ in (7)) where the sum range over the pairs of strict partitions α^1 and α^2 are the Frobenius coordinates of the partition $\lambda = (\alpha^1|\alpha^2)$ and $|\alpha^1| + |\alpha^2| + \kappa = |\lambda|$ is the total number of nodes in the Young diagram $\lambda = (\alpha^1|\alpha^2)$.

In turn, the hypergeometric tau function generalizes the simplest non-trivial two-component KP tau function

$$e^{\sum_{m>1}\frac{1}{m}p_m^{(1)}p_m^{(2)}} = \sum_{\lambda} s_\lambda(\mathbf{p}^1)s_\lambda(\mathbf{p}^2) = \sum_{\Delta} \frac{1}{z_\Delta}\mathbf{p}_\Delta^1\mathbf{p}_\Delta^2 \qquad (9)$$

where the last two sums over, respectively, partitions λ and Δ range over the set of all partitions. The first equality is also known as Cauchy-Littlewood equality.

BKP Tau Functions We need also the simplest non-trivial hypergeometric BKP tau function which is [53]

$$\tau^B(\mathbf{p}) = 1 + \sum_{\kappa=1}^{\infty} \sum_{\alpha^1,\alpha^2} s_{(\alpha^2|\alpha^1)}(\mathbf{p}) \prod_{i=1}^{\kappa} f_1(\alpha_i^1) f_2(\alpha_i^2) \qquad (10)$$

which generalizes

$$\tau_1^B(\mathbf{p}) := 1 + \sum_{\kappa=1}^{\infty} \sum_{\alpha^1,\alpha^2} s_{(\alpha^2|\alpha^1)}(\mathbf{p}) = e^{\sum_{m>0}\frac{1}{2m}p_{2m}^2 + \frac{1}{2m-1}p_{2m-1}} \qquad (11)$$

Here we have in mind the BKP hierarchy of Kac-van de Leur introduced in [23]. In [61], it was shown that the BKP hierarchy can be treated as the orthogonal reduction of the two-component KP hierarchy which implies that the square of a BKP tau function is equal to a certain tau function of the two-component KP.

Remark 1 The series over partitions can be considered as formal ones according to the grading $\deg s_\lambda = |\lambda|$. However, there are open domains of convergency in the space of higher times $\mathbf{p}^i = (p_1^{(i)}, p_2^{(i)}, \dots)$, $i = 1, \dots, 2\mathrm{D}$. Notice that thanks to the infinite number of these parameters one can write

$$s_{\lambda^i}(\mathbf{p}^i) = 0 \qquad \text{if} \qquad p_m^{(i)} = \pm \sum_{j=1}^{k} x_j^m$$

for $\ell(\lambda) > k$ in the case of "+" factor and for $\lambda_1^{(i)} > k$ in the case of "−" factor. For tau functions of multi-component KP and BKP equations there exist recursion equations.

The fermionic expression for the tau functions (7) and for that of BKP (11) (and (10) below) one can find in Appendix B.

Content Product For a given number x and a given Young diagram λ the content product is defined as the product

$$(x)_\lambda := \prod_{(i,j)\in\lambda} (x + j - i) \tag{12}$$

The number $j - i$, which is the distance of the node with coordinates (i, j) to the main diagonal of the Young diagram λ is called the *content* of the node. For one-row λ, the content product is the Pochhammer symbol $(a)_{\lambda_1}$. For a given function of one variable r, we define the *generalized content product* (the generalized Pochhammer symbol) as

$$r_\lambda(x) = \prod_{(i,j)\in\lambda} r(x + j - i) \tag{13}$$

The content product plays an important role in the representation theory of the symmetric groups. Let us note that (13) may be re-written in Frobenius coordinated of $\lambda = (\alpha^1|\alpha^2)$ and yields $\prod_{i=1}^{\kappa} f_1(\alpha_i^1) f_2(\alpha_i^2)$ where f_i, $i = 1, 2$ are defined in terms of $r(x)$. It was used in [49] to define the family of tau functions which we called hypergeometric tau functions, see (8) and BKP hypergeometric tau functions [53] (10)

Content Products in Terms of the Schur Functions Evaluated at Special Points Examples from this paragraph were widely used in [21, 45, 47, 49, 52]. The example of the generalized content product may be constructed purely in terms of the Schur functions: if we choose

$$r(x) = \prod_i \left(\frac{1 - \mathtt{t}_i \mathtt{q}_i^x}{1 - \mathtt{q}_i^x} \right)^{\mathtt{d}_i} \tag{14}$$

where $\mathtt{t}_i, \mathtt{q}_i, \mathtt{d}_i$ are parameters, we obtain

$$r_\lambda(x) = \prod_i \left(\frac{s_\lambda(\mathbf{p}(\mathtt{t}_i, \mathtt{q}_i))}{s_\lambda(\mathbf{p}(0, \mathtt{q}_i))} \right)^{\mathtt{d}_i} \tag{15}$$

One can take a limit (14) to a rational function and obtain

$$r_\lambda(x) = \frac{\prod_{i=1}^p (\mathtt{a}_i)_\lambda}{\prod_{i=1}^q (\mathtt{b}_i)_\lambda} = \prod_{i=1}^p \frac{s_\lambda(\mathbf{p}(\mathtt{a}_i))}{s_\lambda(\mathbf{p}_\infty)} \prod_{i=1}^q \frac{s_\lambda(\mathbf{p}_\infty)}{s_\lambda(\mathbf{p}(\mathtt{b}_i))} \tag{16}$$

Above we used the following special notations:

$$\mathbf{p}_\infty = (1, 0, 0, \dots), \quad \mathbf{p}(\mathtt{a}) = (\mathtt{a}, \mathtt{a}, \mathtt{a}, \dots) \tag{17}$$

$$p_m(\mathtt{t}, \mathtt{q}) = \frac{1 - \mathtt{t}^m}{1 - \mathtt{q}^m} \tag{18}$$

Actually, any reasonable content product can be interpolated by expressions (16).

Remark 2 Notice, that $\mathbf{p}(\mathtt{q}^N, \mathtt{q}) = 1 + \mathtt{q} + \cdots + \mathtt{q}^{N-1}$ which allows to interpret it as $\mathbf{p}(X)$ where $1, \dots, \mathtt{q}^{N-1}$ are eigenvalues of X.

Let us also write down the known formulae [29]

$$\begin{aligned} s_\lambda(\mathbf{p}(a)) &= s_\lambda(\mathbf{p}_\infty) \prod_{(i,j)\in\lambda} (a + j - i)\,, \\ s_\lambda(\mathbf{p}_\infty) &= \frac{\dim_S \lambda}{d!} = \frac{\prod_{i<j}^N (h_i - h_j)}{\prod_{i=1}^N h_i!} \end{aligned} \tag{19}$$

$$\begin{aligned} s_\lambda(\mathbf{p}(\mathtt{t}, \mathtt{q})) &= s_\lambda(\mathbf{p}(0, \mathtt{q})) \prod_{(i,j)\in\lambda} \left(1 - \mathtt{t}\mathtt{q}^{j-i}\right), \\ s_\lambda(\mathbf{p}(0, \mathtt{q})) &= \mathtt{q}^{n(\lambda)} \frac{\prod_{i<j}^N (1 - \mathtt{q}^{h_i - h_j})}{\prod_{i=1}^N (\mathtt{q}; \mathtt{q})_{h_i}} \end{aligned} \tag{20}$$

where $n(\lambda) := \sum_{i=1}^{\ell(\lambda)} (i-1)\lambda_i$

$$(a; \mathtt{q})_n := (1 - a)(1 - a\mathtt{q}) \cdots (1 - a\mathtt{q}^{n-1}) \tag{21}$$

and where $d = |\lambda| := \sum_{i=1}^N \lambda_i$ is the weight of λ, and $h_i = \lambda_i - i + N,\ i = 1, \dots, N$ are called the *shifted parts* of the partition λ, we imply $N \geq \ell(\lambda)$.

Remark 3 Formulae (19) and (20) possess the saturation property: they do not depend on the choice of N if N is large enough.

2.2 Graphs and Words

This subsection summarizes the joint works [41] and [42].

Decorated Embedded Graph: Words and Dual Words Consider an alphabet consisting of the characters A_i and B_i, $i = 1, \ldots, n$. Symbols A_i, B_i we call dual pair for each $i = 1, \ldots, n$, and we call this alphabet the alphabet of pairs.

Next, we consider the oriented compact surface Σ without boundary with a given embedded graph with F faces, n edges and V vertices. We assume that the complement to the graph on Σ is a union of disks, therefore the Euler characteristic of Σ is $\mathrm{E} = \mathrm{F} - n + \mathrm{V}$.

Let us decorate the graph as follows. First, we number each edge, and place A_i and B_i from both sides of the edge number i in any fixed way. Second, we number all faces by $c = 1, \ldots, \mathrm{F}$. Let us go around the boundary of the face numbered c in the clockwise direction and assign to each boundary edge, say e_i, either the symbol A_i in case the edge is directed positively, or B_i in case the edge is directed negatively. We get the formal product of symbols written from the left to the right according to the clockwise round trip. This product defined up to the cyclic permutations of the characters in it we call the *word* W_c. Thus, each symbol of the collection $\{A_i, B_i, \ i = 1, \ldots, n\}$ is assigned to an edge and each word is assigned to a face.

Notice that each symbol of the alphabet of pairs enters once and only once in the set of words.

Such graph we call *decorated* and is denoted $(\Gamma, W_1, \ldots, W_{\mathrm{F}})$. The given full set of words $W = (W_1, \ldots, W_{\mathrm{F}})$ gives rise to the set of *dual words* $W^* = (W_1^*, \ldots, W_{\mathrm{V}}^*)$ as follows:

Let us enumerate the vertices $i = 1, \ldots, \mathrm{V}$. Let us go in the counterclockwise direction around a given vertex and (from the left to the right) write down symbols which we meet prior to each outgoing edge. This product defined up to cyclic permutations of the characters in the product we call dual word W_i^* assigned to the vertex number i.

We recall that the graph Γ^* dual to Γ has V faces, n edges and F vertices, where each face of Γ^* contains a single vertex of Γ and each face of Γ contains a single vertex of Γ^*. Each edge of Γ, say e_i, crosses a single edge of Γ^*, denoted as e_i^*. We assign the orientation to each e_i^* in a way that e_i crosses it form the left to the right. The rule to assign the word to the dual graph is the same, however now we write the words from the right to the left. The collection of the words of the dual graph obtained in such a way coincides with $W_1^*, \ldots, W_{\mathrm{V}}^*$ presented above. We have one-to-one correspondence

$$W_1, \ldots, W_{\mathrm{F}} \quad \leftrightarrow \quad W_1^*, \ldots, W_{\mathrm{V}}^*$$

The decorated graph dual to $(\Gamma, W_1, \ldots, W_{\mathrm{F}})$ is denoted $\left(\Gamma^*, W_1^*, \ldots, W_{\mathrm{V}}^*\right)$.

One can start from the *alphabet of pairs* $A_i, B_i,\ i = 1, \dots, n$ and the set of products $W_1, \dots, W_{\mathrm{F}}$ where each letter is used only once. Then one can consider the set of F polygons: the word $W_c,\ c = 1, \dots, \mathrm{F}$ gives rise to the polygon numbered c whose edges are labeled in clockwise direction by symbols of the word read from the left. Then, by gluing each pair of edges labeled by a pair $A_i, B_i,\ i = 1, \dots, n$ we obtain the decorated embedded graph $(\Gamma, W_1, \dots, W_{\mathrm{F}})$ and also $\left(\Gamma^*, W_1^*, \dots, W_{\mathrm{V}}^*\right)$.

We call sets of words isomorphic if one can be obtained from another by transpositions of dual pairs $a_i \leftrightarrow b_i$, by a re-enumeration of edges and by re-enumeration of faces.

Chord Diagrams and Decorated Chord Networks In the previous paragraphs, words were introduced with the help of a decorated graph drawn on a Riemann surface Σ.

In this paragraph, we "forget" about the surface and the embedded graphs and treat a word simply as a product of characters where each product is defined up to cyclic permutations. As before, we consider the alphabet of pairs, which consists of n pairs of dual characters $a_i, b_i,\ i \in \mathbf{I}$, and require that each character enter only once in the word set $w = (W_1, \dots, W_{\mathrm{F}})$. We call the set of words *connected* if there is no subset of words which contains only characters from a sub-alphabet of pairs.

A set of words can be drawn on paper in a natural way as a set of F oriented polygons with the total number of edges equal to $2n$, where symbols are assigned to edges of the polygons: each word is obtained by going clockwise around the polygon related to the word and multiplying from left to right all the characters along the way. We draw lines which we call *chords* whose endpoints are placed on the edges with dual symbols.

For further purposes, it is convenient to draw not single chords, but *dual chords*: two directed arrows that together with arrows A_i and B_i form (topologically) a 4-polygon as follows. Let the arrow A_i start at the point 1 and end at the point 2, and the arrow B_i start at point 3 and end at the point 4, then the chord A_i^* starts at the point 1 and ends at the point 4 and the chord B_i^* starts at the point 3 and ends at the point 2. Notice that arrows-characters and arrows-chords are directed oppositely on the polygon 1234: both characters are directed positively and both chords are directed negatively.

Let us call this set decorated set of polygons.

We recall that a *chord diagram* is an oriented circle S^1 with a certain number of pairs of points connected by lines called chords. A *network* of chord diagrams is a set of oriented loops and a set of lines, also called chords, each chord connecting a pair of points belonging to any circle. We call a chord internal if its endpoints belong to the same circle, and otherwise we call it external.

Decorated sets of polygons (networks) we call isomorphic if they correspond to isomorphic sets of words.

Below we consider only connected networks. Thus, we also ask the set of words to be connected (which means that there is no subset of words constructed with a sub-alphabet of pairs).

Consider a network with F loops and n chords. One can naturally identify such network with the set of F polygons with n pairs of edges, where edges of polygons are segments of loops which contains endpoints of the chords.

Let us number the chords and decorate the network assigning matrices a_i and b_i to the segments of loops containing the endpoints of the i-th chord. Then, up to the $\mathbb{Z}_2^n$ action $A_i \leftrightarrow b_i$, $i = 1, \ldots, n$ we have a one-to-one correspondence between the set of words $w = (W_1, \ldots, W_{\mathrm{F}})$ and the decorated network or, the same, the decorated set of polygons.

A well-known fact (see, for instance [28]) is that any set of oriented polygons with the total number of (the oriented) edges equal to $2n$ gives rise to an embedded graph drawn on a Riemann surface. It is obtained by the identification of the pair of oriented edges in such a way that the origin of one oriented edge coincides with the end point of the other. These identified oppositely oriented edges of polygons turn out to be edges of the embedded graph.

A connected set of words gives rise to a connected decorated graph drawn on the connected Riemann surface and, therefore to the dual set of words:

$$W = (W_1, \ldots, W_{\mathrm{F}}) \to (\Gamma, W_1, \ldots, W_{\mathrm{F}}) \leftrightarrow (\Gamma^*, W_1^*, \ldots, W_{\mathrm{V}}^*) \tag{22}$$

Operations $\mathfrak{m}_i$ and H(i) with Words and Networks Consider the decorated network as the symmetric tensor product of $W_1 \otimes \cdots \otimes W_{\mathrm{F}}$. Select any pair A_i, B_i, $i \in \mathbf{I}$. This pair is either in different words, say $W_a = A_i X$ and $W_b = B_i Y$, or in one word, say $W_c = A_i X_i B_i Y$. Introduce the involutive map $\mathfrak{m}_i$ which acts on the tensor products of words: it acts identically on all words except these (or this, according to the subscript for W) that contain(s) symbols A_i and B_i as follows:

$$\mathfrak{m}_i : \quad A_i X \otimes B_i Y \ \to \ A_i X B_i Y \tag{23}$$

in the first case, and as

$$\mathfrak{m}_i : \quad A_i X B_i Y \to A_i X \otimes B_i Y \tag{24}$$

in the second case. One should pay attention to the coordination of the order of factors on the left and right sides of the maps (23) and (24) and remember that thanks to the fact that words are defined up to a cyclic permutations of the characters, the left hand side of (23) can be also written as $XA_i \otimes YB_i = A_i X \otimes YB_i = XA_i \otimes B_i Y$, and the left hand side of (24) can be written as $YA_i XB_i = B_i YA_i X = XB_i YA_i$.

We see that $\mathfrak{m}_i^2$ is the identity map. One can check that $\mathfrak{m}_i \left(\mathfrak{m}_j(W)\right) = \mathfrak{m}_j\left(\mathfrak{m}_i(W)\right)$, $i, j = 1, \ldots n$.

We recall that having the set of words W and polygons we construct the Riemann surface with the decorated graph in a unique way, and then we have the geometric construction for the dual set of words W^*, see (22). It was the geometric construction of dual words.

The algebraic construction is given by

Proposition 1

$$\left(\prod_{i=1}^{n} \mathfrak{m}_i\right) W_1 \otimes \cdots \otimes W_{\mathrm{F}} = W_1^* \otimes \cdots \otimes W_{\mathrm{V}}^*$$

As for the decorated network of chord diagrams, each operation $\mathfrak{m}_i$, $i \in \mathbf{I}$ means the following. We represent the chord and a pair of directed edges, on which it rests, in the form of the letter H, where the middle line denotes the chord, and the directionality of the edges is depicted as $\frac{1}{2} \downarrow - \uparrow \frac{3}{4}$. Operation $\mathfrak{m}_i$ means the transposition of the endpoints 2 and 3: now new directed edges connect not points 12 and 43, but points 13 and 42 (notice that points 1 and 4 are origins of directed edges in both cases). And these new edges are connected by the new chord (the middle line of the letter H lying on its side). One may call it H-rotation, or, H(i)-rotation having in mind that the chord and the edges are numbered by i.

The proof of the Proposition 1 is based on the realization of the fact that composition of H(1),...,H(n) describes the passage from the decorated graph Γ to the dual decorated graph Γ^*. It is clear because each vertex of the ribbon graph is the face of the graph formed by pointed arrows, while the faces of the ribbon graph Γ are the vertices of the dual graph.

Example 1 $W = A_1A_2B_1B_2 \cdots A_{n-1}A_nB_{n-1}B_n$. Then

$$W^* = A_2A_1B_2B_1 \cdots A_nA_{n-1}B_nB_{n-1}.$$

Example 2 $W = A_1B_1A_2B_2 \cdots A_nB_n$. Then

$$W^* = A_1 \otimes \cdots \otimes A_n \otimes B_nB_{n-1} \cdots B_1.$$

Example 3 $W = A_1 \cdots A_nB_n \cdots B_1$. Then

$$W^* = A_n \otimes B_nA_{n-1} \cdots \otimes B_3A_2 \otimes B_2A_1 \otimes B_1.$$

The Surface $\Sigma_{\mathrm{H,M}}$. Let us remove H pairs of faces of the decorated embedded graph Γ and glue each pair by a handle.

Next, we remove M faces of Γ and glue Möbius strips to the boundary of these faces. The surface of the Euler characteristic $\mathrm{E} - 2\mathrm{H} - \mathrm{M}$ obtained from Σ by the manipulation described above we denote $\Sigma_{\mathrm{H,M}}$. The complement to the graph Γ of this surface is a union of $\mathrm{F} - 2\mathrm{H} - \mathrm{M}$ disks, H cylinders and M Möbius strips.

We will denote the words on the boundary of cylinders as W_i^+, W_i^-, $i = 1, \ldots, \mathrm{H}$, where i is the number of the handle. The words on the boundary of Möbius we will denote W_i', $i = 1, \ldots, \mathrm{M}$.

Thus, in our new notations the set of all words of the decorated Γ consists of W_i^+, W_i^-, $i = 1, \ldots, \mathrm{H}$, W_i', $i = 1, \ldots, \mathrm{M}$ and the set $W_1, \ldots, W_{\mathrm{F}-2\mathrm{H}-\mathrm{M}}$ which corresponds to the faces which we do not remove.

Dressing and Dressed Words Consider any function f of matrices A_i, B_i, $i = 1, \ldots, n$. One can split the index set $\mathbf{I} = \{i = 1, \ldots, n\}$ into two groups: $\mathbf{I} = \mathbf{I}_1 \cup \mathbf{I}_2$. One defines *dressed* function which we denote $\mathbf{L}_U^{\mathbf{I}_1}\mathbf{L}_Z^{\mathbf{I}_2}[f]$, $c = 1, \ldots, \mathrm{F}$ by the following replacement:

$$A_i, B_i \;\to\; U_i A_i, U_i^\dagger B_i \quad \text{if} \quad i \in \mathbf{I}_1$$

and

$$A_i, B_i \;\to\; Z_i A_i, Z_i^\dagger B_i \quad \text{if} \quad i \in \mathbf{I}_2$$

in f. In particular, dressed words are denoted $\mathbf{L}_U^{\mathbf{I}_1}\mathbf{L}_Z^{\mathbf{I}_2}\,(W_c)$ $c = 1, \ldots, \mathrm{F}$.

Integrals of Products of Schur Functions

Proposition 2 ([41, 42]) *Consider a set of partitions $\lambda^1 = \lambda, \lambda^2, \ldots, \lambda^{\mathrm{F}}$ and the set of words $W = (W_1, \ldots, W_{\mathrm{F}})$. Suppose the set $1, \ldots, n$ is split into two sets I_1 and I_2, $|I_i| = n_i$, $i = 1, 2$. We get*

$$\mathbb{E}_{\mathbb{U}_N^{\otimes n_1}}\mathbb{E}_{\mathbb{GL}_N^{\otimes n_2}}\left(\mathbf{L}_U^{I_1}\mathbf{L}_Z^{I_2}\left[\prod_{i=1}^{\mathrm{F}} s_{\lambda^i}\,(W_i)\right]\right)$$
$$= \delta_\lambda \frac{(|\lambda|!)^{n_2}}{N^{n|\lambda|}}\,(\dim_{\mathbb{GL}}\lambda)^{-n_1}\,(\dim_S\lambda)^{-n_2}\prod_{i=1}^{\mathrm{V}} s_\lambda\left(W_i^*\right) \tag{25}$$

and

$$\mathbb{E}_{\mathbb{U}_N^{\otimes n_1}}\mathbb{E}_{\mathbb{GL}_N^{\otimes n_2}}\left(\mathbf{L}_U^{I_1}\mathbf{L}_Z^{I_2}\left[\prod_{i=1}^{\mathrm{V}} s_{\lambda^i}\left(W^*{}_i\right)\right]\right)$$
$$= \delta_\lambda \frac{(|\lambda|!)^{n_2}}{N^{n|\lambda|}}\,(\dim_{\mathbb{GL}}\lambda)^{-n_1}\,(\dim_S\lambda)^{-n_2}\prod_{i=1}^{\mathrm{F}} s_\lambda\,(W_i) \tag{26}$$

There are two ways to prove the Proposition. The first one is based on integration of the characters, see [41] and few introductory facts in the Appendix A. The other way [42] is to use the combinatorics of the Feynman diagrams of the matrix integrals and the relation of this combinatorics to Hurwitz numbers.

3 The Meaning of Integrals of Tau Functions: Integrals of Tau Functions as Tau Functions

3.1 Tau Functions as Integrands: The Meaning of the Integrals—Orientable Case

We recall that we start with the embedded graph Γ with V vertices, n edges and F faces homeomorphic to discs. Imagine that F coincides with the number of arguments of the round dance tau function (7): $\mathrm{F} = 2\mathrm{D}$. Then we shall use the following notation

$$\tau(W_1, \ldots, W_{2\mathrm{D}}) := \tau(\mathbf{p}^1, \ldots, \mathbf{p}^{2\mathrm{D}}) \tag{27}$$

where, in the right hand side, we put the round dance tau function given by (7) and where

$$\mathbf{p}^i = (-1)^{i+1}\mathbf{p}(W_i) = \left((-1)^{i+1} p_1(W_i), (-1)^{i+1} p_2(W_i), \ldots\right), \quad i = 1, \ldots, 2\mathrm{D} \tag{28}$$

Proposition 3 *Consider tau function (7) where higher times are given by*

$$\begin{aligned} \mathbf{p}^i &= \mathbf{p}^i(W_i) = \left(p_1^i(W_i), p_2^i(W_i), p_3^i(W_i), \ldots\right), \\ p_m^{(i)} &= \mathrm{tr}\left((W_i)^m\right) \quad i = 1, \ldots, 2\mathrm{D} \end{aligned} \tag{29}$$

Then

$$\begin{aligned} &\mathbb{E}_{\mathbb{U}_N^{\otimes n_1}} \mathbb{E}_{\mathbb{GL}_N^{\otimes n_2}} \left(\mathbf{L}_U^{I_1} \mathbf{L}_Z^{I_2} \left[\tau(W_1, \ldots, W_{2\mathrm{D}})\right]\right) \\ &\quad = 1 + \sum_{d>0} \sum_{\substack{\lambda \\ |\lambda|=d}} \frac{(d!)^{n_2}}{N^{nd}} \mathbf{A}(\lambda) \left(\dim_{\mathbb{GL}} \lambda\right)^{-n_1} \left(\dim_S \lambda\right)^{-n_2} \prod_{i=1}^{\mathrm{V}} s_\lambda\left(W_i^*\right), \end{aligned} \tag{30}$$

where

$$\mathbf{A}(\lambda) = \prod_{i=1}^{\kappa} \prod_{j=1}^{2\mathrm{D}} f_{2j-1}(\alpha_i) f_{2j}(\beta_i)$$

For the proof one should use Proposition 2, the explicit form of the tau function (7) where the higher times are chosen by (28) and the known relation

$$s_{(\alpha|\beta)}(-\mathbf{p}) = (-1)^{|\alpha|+|\beta|+\kappa} s_{(\beta|\alpha))}(\mathbf{p}) \tag{31}$$

see [29]).

In what follows we will consider two different special choices of the sets of functions $f_j,\ j = 1, \dots, 2\text{D}$.

(A)

$$f_1 = f_1(t_1, t_2, \dots) = e^{\sum_{m>0} t_m\left[\left(\alpha_i^1+\frac{1}{2}\right)^m - \left(-\alpha_i^2-\frac{1}{2}\right)^m\right]}$$
$$= e^{\sum_{m>0} t_m\left(h_i+\frac{1}{2}\right)^m}, \quad f_j = 1,\ j > 1 \tag{32}$$

(B)

$$f_1 = f_1(t_1^*, t_2^*, \dots) = e^{\sum_{m>0} t_m^*\left(q^{m\left(\alpha_i+\frac{1}{2}\right)} - q^{m\left(-\beta_i-\frac{1}{2}\right)}\right)}$$
$$= e^{\sum_{m>0} t_m^* q^{m\left(h_i+\frac{1}{2}\right)}}, \quad f_j = 1,\ j > 1, \tag{33}$$

where, in both cases, $h_i = \lambda_i - i + N,\ i = 1, \dots, N$ are shifted parts of the partition $\lambda = (\alpha|\beta)$. (the second equality in both (32) and (33) was used in [49] the proof is easy). The sets $\mathbf{t} = (t_1, t_2, \dots)$ and $\mathbf{t}^* = (t_1^*, t_2^*, \dots)$ are the sets of free parameters.

Remark 4 Let us consider the surface $\Sigma_{\text{D},0}$ in the notation of Sect. 2.2, denoted Σ' below. We recall that it is constructed with the help of the Riemann surface Σ and the graph Γ that has n edges, V vertices and F faces homeomorphic to discs. In the case considered in Proposition 3, we choose $\text{F} = 2\text{D}$. Then, we glue up all faces by $\text{H} = \text{D}$ handles. The Euler characteristic of Σ' is equal to $\text{E}' = n - \text{V}$. In [42] we presented the generating function for general Hurwitz numbers $H_{\text{E}'}\left(\tilde{\Delta}^1, \dots, \tilde{\Delta}^{\text{V}}\right)$ with base surface Σ' with Euler characteristic $\text{V} - n$ and ramification profiles $\tilde{\Delta}^1, \dots, \tilde{\Delta}^{\text{V}}$ and this function is (30), the case (A) where $t_m = 0$:

$$1 + \sum_{d>0} \sum_{\substack{\lambda \\ |\lambda|=d}} \frac{(d!)^n}{N^{nd}} (\dim_S \lambda)^{-n} \prod_{i=1}^{\text{V}} s_\lambda\left(W_i^*\right)$$
$$= 1 + \sum_{\tilde{\Delta}^1, \dots, \tilde{\Delta}^{\text{V}}} H_{\text{V}-n}\left(\tilde{\Delta}^1, \dots, \tilde{\Delta}^{\text{V}}\right) \prod_{p=1}^{\text{V}} \mathbf{p}_{\tilde{\Delta}^p}\left(W_p^*\right) \tag{34}$$

It is reasonable to write down a more general case where $\mathrm{F} < 2\mathrm{D}$ (we recall that F is the number of faces and words, and $2\mathrm{D}$ is the number of sets of higher times in the round dance tau function, or the number of components in the multi-component KP hierarchy):

Proposition 4 *Consider tau function (7) where* $2\mathrm{D} = \mathrm{F} + \mathrm{H}$ *and where higher times are chosen as follows: according to*

$$\tau\left(\mathbf{p}^1, W_1, \mathbf{p}^2, W_2, \ldots, \mathbf{p}^{\mathrm{F}-2\mathrm{H}}, W_{\mathrm{F}-2\mathrm{H}}, W_{\mathrm{F}-2\mathrm{H}+1}, W_{\mathrm{F}-2\mathrm{H}+2}, \ldots, W_{\mathrm{F}}\right) \tag{35}$$

where an argument of the tau function W_k *on an even place (if one counts arguments from the left to the right) denotes* $-\mathbf{p}(W_k)$ *and the argument* W_i *on an odd place denotes* $\mathbf{p}(W_i)$. *Here* $\mathbf{p}^i$, $i = 1, \ldots, \mathrm{F} - 2\mathrm{H}$ *are free parameters. Then*

$$\begin{aligned}
&\mathbb{E}_{\mathbb{U}_N^{\otimes n_1}} \mathbb{E}_{\mathbb{GL}_N^{\otimes n_2}} \Big(\mathbf{L}_U^{I_1} \mathbf{L}_Z^{I_2} \big[\tau\big(\mathbf{p}^1, W_1, \mathbf{p}^2, W_2, \ldots, \mathbf{p}^{\mathrm{F}-2\mathrm{H}}, \\
&\qquad W_{\mathrm{F}-2\mathrm{H}}, W_{\mathrm{F}-2\mathrm{H}+1}, W_{\mathrm{F}-2\mathrm{H}+2}, \ldots, W_{\mathrm{F}}\big)\big]\Big) \\
&= 1 + \sum_{d>0} \sum_{\substack{\lambda \\ |\lambda|=d}} \frac{(d!)^{n_2}}{N^{nd}} \mathbf{A}(\lambda) \left(\dim_{\mathbb{GL}} \lambda\right)^{-n_1} \left(\dim_S \lambda\right)^{-n_2} \\
&\qquad \times \prod_{i=1}^{\mathrm{F}-2\mathrm{H}} s_\lambda\left(\mathbf{p}^i\right) \prod_{i=1}^{\mathrm{V}} s_\lambda\left(W_i^*\right),
\end{aligned} \tag{36}$$

where

$$\mathbf{A}(\lambda) = \prod_{i=1}^{\kappa} \prod_{j=1}^{2\mathrm{D}} f_{2j-1}(\alpha_i) f_{2j}(\beta_i)$$

Remark 5 Now we get the surface $\Sigma_{\mathrm{H},0}$ in notations of Sect. 2.2 and, now, $\mathrm{H} < \mathrm{D}$. We recall that it is constructed with the help of the Riemann surface Σ, whose Euler characteristic is $\mathrm{E} = \mathrm{F} - n + \mathrm{V}$, and the graph Γ that has n edges, V vertices and F faces homeomorphic to discs. In the case considered in Proposition 4, we choose $\mathrm{F} - \mathrm{H} = \mathrm{D}$. Then, we glue up randomly selected $2\mathrm{H}$ faces by H handles. The Euler characteristic of $\Sigma_{\mathrm{H},0}$ is equal to $\mathrm{E}' = \mathrm{E} - 2\mathrm{H}$. In [42] we presented the generating function for general Hurwitz numbers $H_{\mathrm{E}'}\left(\Delta^1, \ldots, \Delta^{\mathrm{F}-2\mathrm{H}}, \tilde{\Delta}^1, \ldots, \tilde{\Delta}^{\mathrm{V}}\right)$ with base surface $\Sigma_{\mathrm{H},0}$ with Euler characteristic $\mathrm{E}' = \mathrm{F} - n + \mathrm{V} - 2\mathrm{H}$ and ramification profiles $\Delta^1, \ldots, \Delta^{\mathrm{F}-2\mathrm{H}}$ and $\tilde{\Delta}^1, \ldots, \tilde{\Delta}^{\mathrm{V}}$ and this generating function is (36) where $\mathbf{A}(\lambda) = 1$:

$$\mathbb{E}_{\mathbb{GL}_N^{\otimes n}}\Big(\mathbf{L}_Z\big[\tau\big(\mathbf{p}^1, W_1, \mathbf{p}^2, W_2, \ldots, \mathbf{p}^{\mathrm{F}-2\mathrm{H}}, W_{\mathrm{F}-2\mathrm{H}}, W_{\mathrm{F}-2\mathrm{H}+1}, W_{\mathrm{F}-2\mathrm{H}+2}, \ldots, W_{\mathrm{F}}\big)\big]\Big)$$
$$= 1 + \sum_{d>0} \sum_{\substack{\lambda \\ |\lambda|=d}} \frac{(d!)^n}{N^{nd}} (\dim_S \lambda)^{-n} \prod_{i=1}^{\mathrm{F}-2\mathrm{H}} s_\lambda\left(\mathbf{p}^i\right) \prod_{i=1}^{\mathrm{V}} s_\lambda\left(W_i^*\right)$$
$$= 1 + \sum_{\substack{\Delta^1,\ldots,\Delta^{\mathrm{F}-2\mathrm{H}} \\ \tilde{\Delta}^1,\ldots,\tilde{\Delta}^{\mathrm{V}}}} H_{\mathrm{E}'}\left(\Delta^1, \ldots, \Delta^{\mathrm{F}-2\mathrm{H}}, \tilde{\Delta}^1, \ldots, \tilde{\Delta}^{\mathrm{V}}\right) \prod_{p=1}^{\mathrm{F}-2\mathrm{H}} \mathbf{p}_{\Delta^p} \prod_{p=1}^{\mathrm{V}} \mathbf{p}_{\tilde{\Delta}^p}\left(W_p^*\right) \tag{37}$$

Remark 6 Put $\mathbf{p}^i = \mathbf{p}^i(\tilde{W}_i)$, $i = 1, \ldots, \mathrm{F}-2\mathrm{H}$, where $\tilde{W}_i \in \mathbb{U}_N$ are given matrices. In the case of the choice (A), $n_2 = 0$, and $t_m = t_2\delta_{m,2}$, the right hand side of (36) coincides with the correlation function of the Wilson loops $\tilde{W}_i = P \exp \oint_{x_i} Adz \in \mathbb{U}_N$ and $W_i^* = P \exp \oint_{y_i} Adz \in \mathbb{U}_N$ around the points x_i, $i = 1, \ldots, \mathrm{F} - 2\mathrm{H}$ and y_i, $i = 1, \ldots, \mathrm{V}$ in the two-dimensional gauge theory on the Riemann surface $\Sigma_{\mathrm{H},0}$ (with the same Euler characteristic E' as in Remark 5) found in [55, 62]

$$\left\langle \tilde{W}_1, \ldots, \tilde{W}_{\mathrm{F}-2\mathrm{H}}, W_1^*, \ldots, W_{2\mathrm{D}}^* \right\rangle_{\Sigma_{\mathrm{H},0}}^{2D-YM} =$$
$$1 + \sum_{d>0} \sum_{\substack{\lambda \\ |\lambda|=d}} \frac{1}{N^{nd}} e^{t_2\left(h_i+\frac{1}{2}\right)^2} (\dim_{\mathbb{U}} \lambda)^{-n} \prod_{i=1}^{\mathrm{F}-2\mathrm{H}} s_\lambda\left(\tilde{W}_i\right) \prod_{p=1}^{\mathrm{V}} s_\lambda\left(W_p^*\right) \tag{38}$$

(In gauge theory, the parameter t_2 has a meaning of $-\rho e^2$ where ρ is the area of $\Sigma_{\mathrm{H},0}$ and e is the coupling constant, see [62]).

In case (A) and $t_m \neq 0$, $m > 0$ the right hand side of (36) was considered as certain generalization of $2D$ YM theory, see [62]. In this case the Lagrangian is not quadratic in the curvature $F = dA + A \wedge A$ any more.

The choice (B) is related to q-deformations of $2D$ YM theory considered in some papers, e.g., [1, 57].

Remark 7

(i) Consider the choice (A) and $n_1 = 0$, $\mathrm{D} = 1$, $\mathrm{H} = 0$. Let us take $t_m = t_2\delta_{m,2}$, then, the right hand side of (36) coincides with the generating function for the double Hurwitz numbers found by A. Okounkov in [43].

(ii) In the previous case where however $t_m \neq 0$, $m > 0$, the right hand side of (36) gives rise to the Hurwitz numbers evaluated on complete cycles found by Okounkov and Pandparihander in [44]:

$$1+\sum_{d>0}\sum_{\substack{\lambda\\|\lambda|=d}}\frac{(d!)^{n_2}}{N^{nd}}e^{\sum_{m>0}t_m\left(h_i+\frac{1}{2}\right)^m}\left(\dim_S\lambda\right)^{-n_2}s_\lambda(\mathbf{p}^1)s_\lambda\left(W_1^*\right) \tag{39}$$

(iii) The case (B) where $n_1=0$, $\mathrm{D}=1$, $\mathrm{H}=0$ and where $t_m^*=\delta_{m,1}$ was also considered as the generating function for complete cycles.

3.2 Integrals of Tau Functions as Tau Functions

Proposition 5 *Consider the case* $\mathrm{E}'=\mathrm{E}=2$ *(*E' *as in Remark 5).*

(1) *If each* W_p^* *except two, say* W_1^*, W_2^**, is proportional to the identity matrix* $\mathbb{I}_N$*, and each of* $\mathbf{p}^i$ *is fixed in form (17) with arbitrary parameters, then (36) is the hypergeometric two-component KP tau function (see (8)) where* $\mathbf{p}(W_i^*)$, $i=1,2$ *are higher times and* N *is the discrete time of the two-component KP hierarchy*

(2) *If each* W_p^* *except a single one, say* W_1^**, is proportional to the identity matrix* $\mathbb{I}_N$ *and each of* $\mathbf{p}^i$ *except a single one, say* $\mathbf{p}^1$*, is fixed in form (17) with arbitrary parameters, then (36) is the hypergeometric two-component KP tau function (see (8)) where* $\mathbf{p}(W_1^*)$ *and* $\mathbf{p}^1$ *are higher times, and where the matrix size* N *has a meaning of discrete time of the two-component KP hierarchy*

(3) *If each* W_p^**, is proportional to the identity matrix* $\mathbb{I}_N$*, and each of* $\mathbf{p}^i$ *except two, say* $\mathbf{p}^i$ *,* $i=1,2$ *is fixed in form (17) with arbitrary parameters, then (36) is the hypergeometric two-component KP tau function (see (8)) where* $\mathbf{p}^i$ *,* $i=1,2$ *are higher times and* N *is the discrete time of the two-component KP hierarchy.*

Let us recall that (8) solves not only bilinear Hirota equations (see appendix section "Hirota Equation for the TL and for the Two-Component KP Tau Functions") but also linear differential equations, see [49–51].

Sketch of proof. It follows from the fact that $\frac{s_\lambda(\mathbb{I}_N)}{\dim_S\lambda}$ is the content product, see (19) and can be presented in form (8).

Proposition 6 *Consider case (A) and* $\mathrm{E}'=\mathrm{E}=2$.

(1) *If each* W_p^* *is proportional to the identity matrix* $\mathbb{I}_N$*, and each of* $\mathbf{p}^i$ *except, say,* $\mathbf{p}^1$ *has form (17), then the sum (36) is the infinite-soliton tau function of the two-component KP equation where the sets of parameters* $\mathbf{t}$ *and* $\mathbf{p}^1$ *play the role of the KP higher times and the matrix size* N *is the discrete time*

(2) *If each* W_p^**, except a single one, say,* W_1^* *is proportional to the identity matrix* $\mathbb{I}_N$*, and each of* $\mathbf{p}^i$ *has form (17), then the sum (36) is the infinite-soliton tau*

function of the two-component KP equation where the sets of parameters $\mathbf{t}$ *and* $\mathbf{p}(W_1^*)$ *play the role of the KP higher times and the matrix size* N *is the discrete time.*

The proof follows from the analysis given in [46, 48].

Proposition 7 *Consider case (B) and* $\mathrm{E}' = \mathrm{E} = 2$.

(1) *If each* W_p^* *is proportional to the identity matrix* $\mathbb{I}_N$*, and each of* $\mathbf{p}^i$ *except two, say,* $\mathbf{p}^i$, $i = 1, 2$*, has form (17), and* $\mathbf{p}^2$ *is given by (18) with* q *as in the condition (B), then the sum (36) is the infinite-soliton tau function of the two-component KP equation where the sets of parameters* $\mathbf{t}^*$ *and* $\mathbf{p}^1$ *play the role of the KP higher times and the matrix size* N *is the discrete time*

(2) *If each* W_p^* *except a single one, say,* W_1^**, is proportional to the identity matrix* $\mathbb{I}_N$*, and each of* $\mathbf{p}^i$ *except a single one, say,* $\mathbf{p}^1$*, has form (17), and* $\mathbf{p}^1$ *is given by (18) with* q *as in the condition (B), then the sum (36) is the infinite-soliton tau function of the two-component KP equation where the sets of parameters* $\mathbf{t}^*$ *and* $\mathbf{p}(W_1^*)$ *play the role of the KP higher times and the matrix size* N *is the discrete time*

(3) *If each* W_p^**, except a single one, say* W_1^**, is proportional to the identity matrix* $\mathbb{I}_N$*, and*

$$\mathrm{Spect}\, W_1^* = 1, q, q^2, \ldots, q^{N-1}$$

and each of $\mathbf{p}^i$ *except a single one, say,* $\mathbf{p}^1$*, has form (17), then the sum (36) is the infinite-soliton tau function of the two-component KP equation where the sets of parameters* $\mathbf{t}^*$ *and* $\mathbf{p}^1$ *play the role of the KP higher times and the matrix size* N *is the discrete time*

(4) *If each* W_p^**, except two, say* W_i^*, $i = 1, 2$*, is proportional to the identity matrix* $\mathbb{I}_N$*, and*

$$\mathrm{Spect}\, W_1^* = 1, q, q^2, \ldots, q^{N-1}$$

and each of $\mathbf{p}^i$ *has form (17), then the sum (36) is the infinite-soliton tau function of the two-component KP equation where the sets of parameters* $\mathbf{t}^*$ *and* $\mathbf{p}(W_2^*)$ *play the role of the KP higher times and the matrix size* N *is the discrete time*

The proof follows from Remark 2 and the results of work [46, 48].

3.3 Non-orientable Case

In this subsection we shall use notations of the previous subsections.

Consider the simplest nontrivial tau function of the one component BKP which is

$$\tau_1^B(W) = \sum_\lambda s_\lambda(\mathbf{p}(W)) \tag{40}$$

see (11). Then, we get

Proposition 8 *Consider the tau function (7) where* $2\mathrm{D} = \mathrm{F} + \mathrm{H} - 1$ *and where higher times are chosen as follows:*

$$\tau\left(\mathbf{p}^1, W_1, \mathbf{p}^2, W_2, \ldots, \mathbf{p}^{\mathrm{F}-2\mathrm{H}-1}, W_{\mathrm{F}-2\mathrm{H}-1}, W_{\mathrm{F}-2\mathrm{H}}, W_{\mathrm{F}-2\mathrm{H}+1}, \ldots, W_{\mathrm{F}-1}\right) \tag{41}$$

where an argument of the tau function W_k *on an even place (if one counts arguments from the left to the right) denotes* $-\mathbf{p}(W_k)$ *and the argument* W_i *on an odd place denotes* $\mathbf{p}(W_i)$*. Here* $\mathbf{p}^i$, $i = 1, \ldots, \mathrm{F} - 2\mathrm{H} - 1$ *are free parameters. Then*

$$\begin{aligned} &\mathbb{E}_{\mathbb{U}_N^{\otimes n_1}} \mathbb{E}_{\mathbb{GL}_N^{\otimes n_2}} \Big(\mathbf{L}_U^{I_1} \mathbf{L}_Z^{I_2} \big[\tau\big(\mathbf{p}^1, W_1, \mathbf{p}^2, W_2, \ldots, \mathbf{p}^{\mathrm{F}-2\mathrm{H}-1}, \\ &\qquad W_{\mathrm{F}-2\mathrm{H}-1}, W_{\mathrm{F}-2\mathrm{H}}, W_{\mathrm{F}-2\mathrm{H}+1}, \ldots, W_{\mathrm{F}-1}\big)\tau_1^B(W_\mathrm{F})\big]\Big) \\ &= \sum_{\substack{\lambda \\ \ell(\lambda)\le N}} \frac{(|\lambda|!)^{n_2}}{N^{n|\lambda|}} A(\lambda)\, (\dim_{\mathbb{GL}} \lambda)^{-n_1}\, (\dim_S \lambda)^{-n_2} \prod_{i=1}^{\mathrm{F}-2\mathrm{H}-1} s_\lambda(\mathbf{p}^i) \prod_{i=1}^{\mathrm{V}} s_\lambda\left(W_i^*\right) \end{aligned} \tag{42}$$

where $\mathbf{A}(\lambda)$ *is the same as in Proposition 3.*

This sum can be considered in a way similar to the orientable case.

Remark 8 ([41, 42]) Now we get the surface $\Sigma_{\mathrm{H},1}$ in notations of Sect. 2.2. We recall that it is constructed with the help of the Riemann surface Σ, whose Euler characteristic is $\mathrm{E} = \mathrm{F} - n + \mathrm{V}$, and the graph Γ that has n edges, V vertices and F faces homeomorphic to discs. In the case considered in Proposition 8, we choose $2\mathrm{F} - 2\mathrm{H} - 1 = 2\mathrm{D}$. Then, we glue up randomly selected $2\mathrm{H} + 1$ faces by H handles and one Möbius strip, and the Euler characteristic of $\Sigma_{\mathrm{H},1}$ is equal to $\mathrm{E}' = \mathrm{E} - 2\mathrm{H} - 1$. In [42] we presented the generating function for general Hurwitz numbers $H_{\mathrm{E}'}\left(\Delta^1, \ldots, \Delta^{\mathrm{F}-2\mathrm{H}-1}, \tilde{\Delta}^1, \ldots, \tilde{\Delta}^{\mathrm{V}}\right)$ with base surface $\Sigma_{\mathrm{H},1}$ with Euler characteristic E' and ramification profiles $\Delta^1, \ldots, \Delta^{\text{F}-2\text{H}-1}$ and $\tilde{\Delta}^1, \ldots, \tilde{\Delta}^{\text{V}}$, and this generating function is (42) where $\mathbf{A}(\lambda) = 1$:

$$\mathbb{E}_{\mathbb{GL}_N^{\otimes n}}\Big(\mathbf{L}_Z\big[\tau\big(\mathbf{p}^1, W_1, \mathbf{p}^2, W_2, \ldots, \mathbf{p}^{\text{F}-2\text{H}-1}, W_{\text{F}-2\text{H}-1}, W_{\text{F}-2\text{H}}, W_{\text{F}-2\text{H}+1}, \ldots, W_{\text{F}-1}\big)\tau_1^B(W_{\text{F}})\big]\Big)$$

$$= 1 + \sum_{d>0}\sum_{\substack{\lambda\\|\lambda|=d}} \frac{(d!)^n}{N^{nd}} (\dim_S \lambda)^{-n} \prod_{i=1}^{\text{F}-2\text{H}-1} s_\lambda\left(\mathbf{p}^i\right) \prod_{i=1}^{\text{V}} s_\lambda\left(W_i^*\right)$$

$$= 1 + \sum_{\substack{\Delta^1,\ldots,\Delta^{\text{F}-2\text{H}-1}\\ \tilde{\Delta}^1,\ldots,\tilde{\Delta}^{\text{V}}}} H_{\text{F}-n+\text{V}-2\text{H}-1}\left(\Delta^1, \ldots, \Delta^{\text{F}-2\text{H}-1}, \tilde{\Delta}^1, \ldots, \tilde{\Delta}^{\text{V}}\right) \times \prod_{p=1}^{\text{F}-2\text{H}-1} \mathbf{p}_{\Delta^p} \prod_{p=1}^{\text{V}} \mathbf{p}_{\tilde{\Delta}^p}\left(W_p^*\right) \qquad (43)$$

Remark 9 Put $\mathbf{p}^i = \mathbf{p}^i(\tilde{W}_i)$, $i = 1, \ldots, \text{F} - 2\text{H} - 1$, where $\tilde{W}_i \in \mathbb{U}_N$ are given matrices. In case of the choice (A), $n_2 = 0$, and $t_m = t_2\delta_{m,2}$, the right hand side of (36) coincides with the correlation function of the Wilson loops $\tilde{W}_i = P\exp\oint_{x_i} Adz \in \mathbb{U}_N$ and $W_i^* = P\exp\oint_{y_i} Adz \in \mathbb{U}_N$ around points x_i, $i = 1, \ldots, \text{F} - 2\text{H} - 1$ and y_i, $i = 1, \ldots, \text{V}$ in two-dimensional gauge theory on the Riemann surface $\Sigma_{\text{H},0}$ (with the same Euler characteristic E' as in Remark 8) found in [55, 62]

$$\left\langle \tilde{W}_1, \ldots, \tilde{W}_{\text{F}-2\text{H}-1}, W_1^*, \ldots, W_{2\text{D}}^* \right\rangle_{\Sigma_{\text{H},1}}^{2D-YM}$$

$$= 1 + \sum_{d>0}\sum_{\substack{\lambda\\|\lambda|=d}} \frac{1}{N^{nd}} e^{t_2\left(h_i+\frac{1}{2}\right)^2} (\dim_{\mathbb{U}} \lambda)^{-n} \prod_{i=1}^{\text{F}-2\text{H}-1} s_\lambda\left(\tilde{W}_i\right) \prod_{p=1}^{\text{V}} s_\lambda\left(W_p^*\right) \qquad (44)$$

The right hand side of (42) generates both (1) Hurwitz numbers with non-orientable base surface $\Sigma_{\text{H},1}$ (and one can replace it by $\Sigma_{\text{H}-k,1+2k}$ with the same result) with Euler characteristic $\text{E}' = \text{F} - n + \text{V} - 2\text{H} - 1$ and (2) correlation function for Wilson loops for the gauge theory on $\Sigma_{\text{H},1}$ (and/or on $\Sigma_{\text{H}-k,1+2k}$), see [62].

By results of [53] we can show that there are similar cases where the right hand side of (42) is equal to the hypergeometric function of BKP equation (10) which generates projective Hurwitz numbers as it was shown in [40].

Proposition 9 *Consider the case* $\mathrm{E}' = \mathrm{E} = 1$ *(*E' *as in Remark 8).*

(1) *If each* W_p^* *except a single one, say* W_1^**, is proportional to the identity matrix* $\mathbb{I}_N$*, and each of* $\mathbf{p}^i$ *is fixed in form (17) with arbitrary parameters, then (36) is the hypergeometric BKP tau function (see (10)) where* $\mathbf{p}(W_1^*)$ *is the set of higher times and* N *is the discrete time of the BKP hierarchy*
(2) *If each* W_p^* *is proportional to the identity matrix* $\mathbb{I}_N$*. and each of* $\mathbf{p}^i$ *except a single one, say* $\mathbf{p}^1$ *is fixed in form (17) with arbitrary parameters, then (42) is the hypergeometric BKP tau function (see (10)) where* $\mathbf{p}^1$ *is the set of higher times, and where the matrix size* N *has a meaning of discrete time of the BKP hierarchy*

Proposition 10 *Consider the case (A) and* $\mathrm{E}' = \mathrm{E} = 1$.

(1) *If each* W_p^* *is proportional to the identity matrix* $\mathbb{I}_N$*, and each of* $\mathbf{p}^i$ *has form (17), then the sum (42) is the infinite-soliton tau function of the two-component KP equation where the set of parameters* $\mathbf{t}$ *plays the role of the BKP higher times and the matrix size* N *is the discrete time*
(2) *If each* W_p^**, except a single one, say,* W_1^* *is proportional to the identity matrix* $\mathbb{I}_N$*, and each of* $\mathbf{p}^i$ *has form (17), then the sum (42) is the infinite-soliton tau function of the two-component KP equation where the set of parameters* $\mathbf{p}(W_1^*)$ *plays the role of the BKP higher times and the matrix size* N *is the discrete time*

Proposition 11 *Consider the case (B) and* $\mathrm{E}' = \mathrm{E} = 1$.

(1) *If each* W_p^* *is proportional to the identity matrix* $\mathbb{I}_N$*, and each of* $\mathbf{p}^i$ *except a single one, say,* $\mathbf{p}^1$*, has form (17), and* $\mathbf{p}^1$ *is given by (18) where* q *is the same as written down in the condition (B), then the sum (42) is the infinite-soliton tau function of the BKP equation where the set of parameters* $\mathbf{t}^*$ *plays the role of the BKP higher times and the matrix size* N *is the discrete time*
(2) *If each* W_p^**, except a single one, say* W_1^**, is proportional to the identity matrix* $\mathbb{I}_N$*, and*

$$\mathrm{Spect}\ W_1^* = 1, q, q^2, \ldots, q^{N-1}$$

and each of $\mathbf{p}^i$ *has form (17), then the sum (42) is the infinite-soliton tau function of the two-component KP equation where the set of parameters* $\mathbf{t}^*$ *plays the role of the BKP higher times and the matrix size* N *is the discrete time*

The proof follows from the consideration in [53].

Let us recall that (10) solves not only bilinear Hirota equations (see [23]) but also linear differential equations, see [53].

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Appendix A: Integration of the Schur Functions

The Expectation of $s_\lambda(ZAZ^\dagger B)$ **and of** $s_\lambda(ZA)s_\nu(Z^\dagger B)$

Lemma 1 *For any* $N \times N$ *complex matrices* A *and* B *we have*

$$\mathbb{E}_{\mathbb{GL}_N}\left(s_\lambda(ZAZ^\dagger B)\right) = N^{-d}\frac{s_\lambda(A)s_\lambda(B)}{s_\lambda(\mathbf{p}_\infty)} \tag{A.1}$$

where $d = |\lambda|$.

In the case when A and B are Hermitian matrices, the first relation (A.1) is well-known and written down in textbooks, see for instance Example 5 in [29, Section VII]. (The only difference with the well-known formula is the factor N^{-d} which results from the fact that we replace the Gauss weight $e^{-\operatorname{tr} ZZ^\dagger}$ (see [29]) in the definition of the measure $d\mu$ by $e^{-N \operatorname{tr} ZZ^\dagger}$, see (2). Then, the factor N^{-d} is obtained by the rescaling of the Schur function s_λ which is the homogeneous polynomial of the weight $d = |\lambda|$). However, thanks to the fact that we can present Schur functions in form (4) where each $\mathbf{p}_\Delta = \mathbf{p}_\Delta(X)$ is a polynomial in the entries of matrix X. Then, the both sides of (A.1) are analytic functions in the entries of the matrices and, therefore, (A.1) is true for $A, B \in GL_N(\mathbb{C})$.

Lemma 2 *For any* $N \times N$ *complex matrices* A *and* B *the following equality is correct:*

$$\mathbb{E}_{\mathbb{GL}_N}\left(s_\lambda(ZA)s_\nu(Z^\dagger B)\right) = N^{-d}\delta_{\lambda\nu}\frac{s_\lambda(AB)}{s_\lambda(\mathbf{p}_\infty)} \tag{A.2}$$

where $d = |\lambda|$.

The Expectation of $s_\lambda(UAU^\dagger B)$ **and of** $s_\lambda(UA)s_\nu(U^\dagger B)$

Proposition 12 *For any* $N \times N$ *complex matrices* A *and* B *the following two equalities are correct:*

$$\mathbb{E}_{\mathbb{U}_N}\left(s_\lambda(UAU^\dagger B)\right) = \frac{s_\lambda(A)s_\lambda(B)}{s_\lambda(\mathbb{I}_N)} \tag{A.3}$$

where

$$\mathbb{E}_{\mathbb{U}_N}\left(\mathbf{p}^*_\Delta(UAU^\dagger B)\right) = z_\Delta\, N^{-\ell(\Delta)} \sum_{\Delta^a,\Delta^b} {}^U H_{\mathbb{CP}^1}(\Delta, \Delta^a, \Delta^b)\, \mathbf{p}^*_{\Delta^a}(A)\, \mathbf{p}^*_{\Delta^b}(B) \tag{A.4}$$

where the summation in the right hand side ranges over all partitions Δ^a, Δ^b of the weight $d = |\Delta|$ and where

$$ {}^U H_{\mathbb{CP}^1}(\Delta, \Delta^a, \Delta^b) := \sum_{\substack{\lambda \\ |\lambda|=d}} \left(\frac{\dim_{\mathbb{U}} \lambda}{d!} \right)^2 \frac{\varphi_\lambda(\Delta)\varphi_\lambda(\Delta^a)\varphi_\lambda(\Delta^b)}{((N)_\lambda)^3} \tag{A.5} $$

Lemma 3 *For any $N \times N$ complex matrices A and B the following two equalities are correct and equivalent:*

$$ \mathbb{E}_{\mathbb{U}_N} \left(s_\lambda(UA) s_\nu(U^\dagger B) \right) = \delta_{\lambda\nu} \frac{s_\lambda(AB)}{s_\lambda(\mathbb{I}_N)} \tag{A.6} $$

where

$$ \mathbb{E}_{\mathbb{U}_N} \left(\mathbf{p}_{\Delta^a}(UA)\, \mathbf{p}_{\Delta^b}(U^\dagger B) \right) = z_{\Delta^a} z_{\Delta^b}\, N^{-\ell(\Delta^a)-\ell(\Delta^b)} \sum_{\Delta} H_{\mathbb{CP}^1}(\Delta^a, \Delta^b, \Delta)\, \mathbf{p}_\Delta(AB) \tag{A.7} $$

where the summation in the right hand side ranges over all partitions Δ of the weight $d = |\Delta^a| = |\Delta^b|$ and where $H_{\mathbb{CP}^1}(\Delta, \Delta^a, \Delta^b)$ is the same three-point Hurwitz number with the base $\mathbb{CP}^1$.

The Sketch of Proofs

We use

$$ \int_{\mathbb{C}^{N^2}} s_\lambda(AZBZ^+) e^{-N \operatorname{tr} ZZ^+} \prod_{i,j=1}^{N} d^2 Z_{ij} = \frac{s_\lambda(A) s_\lambda(B)}{s_\lambda(N\mathbf{p}_\infty)} \tag{A.8} $$

and

$$ \int_{\mathbb{C}^{N^2}} s_\Delta(AZ) s_\lambda(Z^+ B) e^{-N \operatorname{tr} ZZ^+} \prod_{i,j=1}^{N} d^2 Z_{ij} = \frac{s_\lambda(AB)}{s_\lambda(N\mathbf{p}_\infty)} \delta_{\Delta,\lambda}\,. \tag{A.9} $$

These relations are used for step-by-step integration (Gaussian in the case of complex matrices).

As we can see, these relations perform the procedure of cutting and joining loops in a network of chord diagrams, and also create edges of embedded graph (each edge is a coupled pair of conjugate random matrices). Namely, the equation (A.8) performs the splitting of the loop $AZBZ^\dagger$ into two loops, A and B, for complex Ginibre ensembles (the resulting equation performs the union of two loops A and B for complex Ginibre ensembles. Every time we apply some of the relations (A.8)–(A.9), we get the factor (the "propagator" of the edge of the embedded graph), which is $\frac{1}{s_\lambda(N\mathbf{p}_\infty)}$ in the case of complex Ginibre ensemble.

Hirota Equation for the TL and for the Two-Component KP Tau Functions

The TL tau function was introduced in [22] and may be defined by

$$\tau_n^{\mathrm{TL}}(t,\bar{t}) = \langle n|e^{\sum_{i>0} t_i\alpha_i} g^{\mathrm{TL}} e^{-\sum_{i>0}\bar{t}_i\alpha_{-i}}|n\rangle \tag{A.10}$$

This tau function solves Hirota equation, [22, 60]

$$\oint \frac{dz}{2\pi i} z^{n'-n} e^{\mathrm{V}(t'-t,z)} \tau_{n'}^{\mathrm{TL}}\left(t'-[z^{-1}],\bar{t}'\right)\tau_n^{\mathrm{TL}}\left(t+[z^{-1}],\bar{t}\right)$$
$$= \oint \frac{dz}{2\pi i} z^{n'-n} e^{V\left(\bar{t}'-\bar{t},z^{-1}\right)} \tau_{n'+1}^{\mathrm{TL}}\left(t',\bar{t}'-[z]\right)\tau_{n-1}^{\mathrm{TL}}\left(t,\bar{t}+[z]\right) \tag{A.11}$$

(see [22, 60]) which includes

$$\frac{\partial^2\tau_n^{\mathrm{TL}}}{\partial t_1\partial\bar{t}_1}\tau_n^{\mathrm{TL}} - \frac{\partial\tau_n^{\mathrm{TL}}}{\partial t_1}\frac{\partial\tau_n^{\mathrm{TL}}}{\partial\bar{t}_1} = -\tau_{n+1}^{\mathrm{TL}}\tau_{n-1}^{\mathrm{TL}} \tag{A.12}$$

The two-component KP tau function

$$\tau_n^{\mathrm{2KP}}(t,\bar{t}) = \langle n,-n|e^{\sum_{i>0}\left(t_i'\alpha_i^{(1)}+\bar{t}_i'\alpha_i^{(2)}\right)} g^{\mathrm{2KP}}|0\rangle \tag{A.13}$$

solves Hirota equation

$$\oint \frac{dz}{2\pi i}(-)^{-n'-n} z^{n'-n} e^{\mathrm{V}(t'-t,z)} \tau_{n'}^{\mathrm{2KP}}\left(t'-[z^{-1}],\bar{t}'\right)\tau_n^{\mathrm{2KP}}\left(t+[z^{-1}],\bar{t}\right)$$
$$= \oint \frac{dz}{2\pi i} z^{n-n'-2} e^{V\left(\bar{t}'-\bar{t},z\right)} \tau_{n'+1}^{\mathrm{2KP}}\left(t',\bar{t}'-[z^{-1}]\right)\tau_{n-1}^{\mathrm{2KP}}\left(t,\bar{t}+[z^{-1}]\right) \tag{A.14}$$

which up to the sign factor $(-)^{n+n'}$ in the first integral is (A.11) if we change $z \to z^{-1}$ in the second integral in (A.14).

Appendix B: Fermionic Formulae for Tau Functions

Details may be found in [49, 53]. Let $\{\psi_i,\ \psi_i^\dagger,\ i\ \in\ \mathbb{Z}\}$ be Fermi creation and annihilation operators that satisfy the usual anticommutation relations and vacuum annihilation conditions

$$[\psi_i,\ \psi_j]_+ = \delta_{i,j}\,, \qquad \psi_i|n\rangle = \psi_{-i-1}|n\rangle = 0\,, \qquad i < n\,.$$

In contrast to the DKP hierarchy introduced in [22], for the BKP hierarchy introduced in [23], we need an additional Fermi mode ϕ which anticommutes with all the other Fermi operators except itself, for which $\phi^2 = 1/2$, and $\phi|0\rangle = |0\rangle/\sqrt{2}$ [23]. Then the hypergeometric BKP tau function introduced in [53] may be written as

$$\begin{aligned} &g(n)\tau_r^{\mathrm{BKP}}(N, n, \mathbf{p}) \\ &= \left\langle n \left| e^{\sum_{m>0}\frac{1}{m}J_m p_m} e^{\sum_{i<0} U_i\psi_i^\dagger\psi_i - \sum_{i\geq 0} U_i\psi_i\psi_i^\dagger} e^{\sum_{i>j}\psi_i\psi_j - \sqrt{2}\phi\sum_{i\in\mathbb{Z}}\psi_i} \right| n - N \right\rangle \\ &= \sum_{\substack{\lambda \\ \ell(\lambda)\leq N}} e^{-U_\lambda(n)} s_\lambda(\mathbf{p}) = g(n) \sum_{\substack{\lambda \\ \ell(\lambda)\leq N}} r_\lambda(n) s_\lambda(\mathbf{p})\,, \end{aligned} \tag{B.1}$$

where $J_m = \sum_{i\in\mathbb{Z}} \psi_i\psi_{i+m}^\dagger$, $m > 0$, $U_\lambda(n) = \sum_i U_{h_i+n}$, $r(i) = e^{U_{i-1}-U_i}$, and

$$g(n) := \left\langle n \left| e^{\sum_{i<0} U_i\psi_i^\dagger\psi_i - \sum_{i\geq 0} U_i\psi_i\psi_i^\dagger} \right| n \right\rangle = \begin{cases} e^{-U_0+\cdots-U_{n-1}} & \text{if } n > 0\,, \\ 1 & \text{if } n = 0\,, \\ e^{U_{-1}+\cdots U_n} & \text{if } n < 0\,. \end{cases} \tag{B.2}$$

In (B.1) the summation runs over all partitions whose lengths do not exceed N.

Remark 10 Note that, without the additional Fermi mode ϕ, the summation range in (B.1) does include partitions with odd partition lengths. One can avoid this restriction by introducing a pair of DKP tau functions, which seems unnatural.

Apart from (B.1), the same series without the restriction $\ell(\lambda) \leq N$ gives the BKP tau function. However, it is related to the single value $n = 0$. The n-dependence destroys the simple form of this tau function [53].

Round Dance Tau Function The fermionic representation for the round trip tau functions is as follows

$$\tau(\mathbf{p}^1,\ldots,\mathbf{p}^{2\mathrm{D}}) = \langle 0| \prod_{i=1}^{2\mathrm{D}} e^{\sum_{m>0}\frac{1}{m}J_m^j p_m^{(j)}} e^{\sum (-1)^i f^{(1)}(i)\psi_i^{(1)}\psi_{-i-1}^{\dagger(2)}} \\ \ldots e^{\sum_{i\in\mathbb{Z}}(-1)^i f^{(2\mathrm{D})}(i)\psi_i^{(2\mathrm{D})}\psi_{-i-1}^{\dagger(1)}}|0\rangle \qquad \text{(B.3)}$$

where $\psi_i^{(j)} = \psi_{2\mathrm{D}i+j}$ and $\psi_i^{\dagger(j)} = \psi^{\dagger(j)}_{2\mathrm{D}i+j}$ and $J_m^j = \sum_{i\in\mathbb{Z}}\psi_i^{(j)}\psi_{i+m}^{\dagger(j)}$, $m>0$.

The round dance tau function solves Hirota equations for the multicomponent KP hierarchy which generalize Hirota equations for the two-component case written down in appendix section "Hirota Equation for the TL and for the Two-Component KP Tau Functions", see [22].

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Many-Particle Schrödinger Type Finitely Factorized Quantum Hamiltonian Systems and Their Integrability

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Abstract We develop G. A. Goldin and D. H. Sharp's quantum current algebra approach to many-particle Hamiltonian operators. We demonstrate its deep relationship to the Hamiltonian operators' factorized structure. We investigate this for completely integrable spinless systems, showing the connection with the classical Bethe ansatz ground state representation. The quantum Hamilton operators are considered for integrable delta-potential and oscillatory Caloger-Moser-Sutherland models.

Keywords Fock space · Current algebra symmetry representations · Quantum delta-potential Schrödinger type operator · Oscillatory Calogero-Moser-Sutherland model · Quantum integrability · Quantum symmetries

Mathematics Subject Classification (2010) 17B68, 17B80, 58J70, 58J72

1 Introduction

In this work we develop investigations of local quantum current algebra symmetry representations in suitably renormalized representation Hilbert spaces, suggested and proposed before by G. A. Goldin and D. H. Sharp[18, 19, 21–23] and their collaborators, having further applied their results to constructing the related factorized operator representations for secondly-quantized many-particle integrable Hamil-

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tonian systems. The main technical ingredient of the current algebra symmetry representation approach consists in the weak equivalence of the initial many-particle quantum Hamiltonian operator to a suitably constructed quantum Hamiltonian operator in the factorized form, strictly depending only on its ground state vector. The latter makes it possible to reconstruct the initial quantum Hamiltonian operator in the case of its strong equivalence to the related factorized Hamiltonian operator form, thereby constructing, as a by-product, the corresponding N-particle ground state vector for arbitrary $N \in \mathbb{Z}_+$. Being uniquely defined by means of the Bethe ground state vector representation in the Hilbert space, the analyzed factorized operator structure of quantum completely integrable many-particle Hamiltonian systems on the axis proves to be closely related to their quantum integrability by means of the quantum inverse scattering transform.

The analytical studies in modern mathematical physics are strongly based on the exactly solvable physical models which are of great help in the understanding of their mathematical and often hidden physical nature, the solvable models are of great importance in quantum many particle physics, among which one can single out such as the oscillatory systems and Coulomb systems, modeling phenomena in plasma physics, the well known Calogero-Moser and Calogero-Moser-Sutherland models, describing system of many particles on an axis, interacting pair wise through long range potentials, modeling both some quantum-gravity and fractional statistics effects. As examples we have studied in detail the factorized structure of Hamiltonian operators, describing such quantum integrable spatially one-dimensional models as the delta-potential and the generalized oscillatory Calogero-Moser-Sutherland dynamical Schrödinger type quantum systems of spinless Bose-particles.

2 Preliminaries: The Representation Fock and Hilbert Spaces and Nonrelativistic Quantum Currents Algebra Symmetry Structure

2.1 The Fock Space and Nonrelativistic Quantum Current Algebra

Let us consider the canonical Fock space [6–9, 14, 16, 18, 28–30, 35], that is the direct sum

$$\Phi_F = \oplus_{n \in \mathbb{Z}_+} \Phi_{(s)}^{\otimes n}, \tag{1}$$

where subspaces $\Phi_{(s)}^{\otimes n}$, $n \in \mathbb{Z}_+$, are the symmetrized tensor products of a Hilbert space $\mathcal{H} \simeq L_2(\mathbb{R}^m; \mathbb{C})$. If a vector $\varphi := (\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots) \in \Phi_F$, its norm

$$\|\varphi\|_{\Phi_F} := \left(\sum_{n \in \mathbb{Z}_+} \|\varphi_n\|_n^2 \right)^{1/2}, \tag{2}$$

where $\varphi_n \in \Phi_{(s)}^{\otimes n} \simeq L_{2,(s)}((\mathbb{R}^m)^{\otimes n}; \mathbb{C})$ and $\| \ldots \|_n$ is the corresponding norm in $\Phi_{(s)}^{\otimes n}$ for all $n \in \mathbb{Z}_+$. Note here that there holds the corresponding rigging [6, 7] of the Hilbert spaces $\Phi_{(s)}^{\otimes n}$, $n \in \mathbb{Z}_+$, that is

$$\mathcal{D}_{(s)}^n \subset \Phi_{(s),+}^{\otimes n} \subset \Phi_{(s)}^{\otimes n} \subset \Phi_{(s),-}^{\otimes n} \tag{3}$$

with some suitably chosen dense and separable topological spaces of symmetric functions $\mathcal{D}_{(s)}^n$, $n \in \mathbb{Z}_+$, allowing to describe both point wise particle objects in $\mathbb{R}^m$ by means of the corresponding generalized positive $\Phi_{(s),+}^{\otimes n}$ the adjoint negative Hilbert spaces $\Phi_{(s),-}^{\otimes n}$ and the corresponding Hamiltonian operator $\mathrm{H} : \Phi \rightarrow \Phi$, governing the quantum states and their evolution. Concerning expansion (1) one obtains by means of projective and inductive limits [3–7] the quasi-nucleus rigging of the Fock space Φ_F exactly in the form (3).

Consider now any vector $|(\alpha)_n) \in \Phi_{(s)}^{\otimes n}$, $n \in \mathbb{Z}_+$, which can be written [7, 9, 12, 16, 30] in the following canonical Dirac ket-form:

$$|(\alpha)_n) := |\alpha_1, \alpha_2, \ldots, \alpha_n), \tag{4}$$

where, by definition,

$$|\alpha_1, \alpha_2, \ldots, \alpha_n) := \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} |\alpha_{\sigma(1)}) \otimes |\alpha_{\sigma(2)}) \ldots |\alpha_{\sigma(n)}) \tag{5}$$

and vectors $|\alpha_j) \in \Phi_{(s)}^{\otimes 1}(\mathbb{R}^m; \mathbb{C}) \simeq \mathcal{H}$, $j \in \mathbb{Z}_+$, are bi-orthogonal to each other, that is $(\alpha_k|\alpha_j)_{\mathcal{H}} = \delta_{k,j}$ for any $k, j \in \mathbb{Z}_+$. The corresponding scalar product of base vectors (5) is given as follows:

$$\begin{aligned} ((\beta)_n|(\alpha)_n) &:= (\beta_n, \beta_{n-1}, \ldots, \beta_2, \beta_1|\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n) \\ &= \sum_{\sigma \in S_n} (\beta_1|\alpha_{\sigma(1)})_{\mathcal{H}} \ldots (\beta_n|\alpha_{\sigma(n)})_{\mathcal{H}} := per\{(\beta_i|\alpha_j)_{\mathcal{H}}\}_{i,j=\overline{1,n}}, \end{aligned}$$

where "*per*" denotes the permanent of matrix and $< \cdot|\cdot >$ is the corresponding scalar product in the Hilbert space $\mathcal{H}$. Based now on the representation (4) one can define an operator $a^+(\alpha) : \Phi_{(s)}^{\otimes n} \longrightarrow \Phi_{(s)}^{\otimes(n+1)}$ for any $|\alpha) \in \mathcal{H}$ as follows:

$$a^+(\alpha)|\alpha_1, \alpha_2, \ldots, \alpha_n) := |\alpha, \alpha_1, \alpha_2, \ldots, \alpha_n), \tag{6}$$

which is called the "*creation*" operator in the Fock space Φ. The adjoint operator $a(\beta) := (a^+(\beta))^* : \Phi_{(s)}^{\otimes(n+1)} \longrightarrow \Phi_{(s)}^{\otimes n}$ with respect to the Fock space Φ_F (1) for any $|\beta\rangle \in \mathcal{H}$, called the "*annihilation*" operator, acts as follows:

$$a(\beta)|\alpha_1, \alpha_2, \ldots, \alpha_{n+1}) := \sum_{j=\overline{1,n+1}} (\beta|\alpha_j)|\alpha_1, \alpha_2, \ldots, \alpha_{j-1}, \hat{\alpha}_j, \alpha_{j+1}, \ldots, \alpha_{n+1}), \tag{7}$$

where the "*hat*" over a vector denotes that it should be omitted from the sequence.

It is easy to check that the commutator relationship

$$[a(\alpha), a^+(\beta)] = (\alpha|\beta)_{\mathcal{H}} \tag{8}$$

holds for any vectors $|\alpha) \in \mathcal{H}$ and $|\beta) \in \mathcal{H}$. Expression (8), owing to the quasi-nucleus [6, 7] rigged Fock space (1), can be naturally extended to the general case, when vectors $|\alpha)$ and $|\beta) \in \mathcal{H}_-$, where $\mathcal{H}_-$ denotes the corresponding *negative* Hilbert space of generalized functions, conserving its usual form. In particular, taking $|\alpha) := |\alpha(x)) = \frac{1}{\sqrt{2\pi}} e^{i\langle\lambda|x\rangle} \in \mathcal{H}_- := L_{2,-}(\mathbb{R}^m; \mathbb{C})$ for any $x \in \mathbb{R}^m$, one easily gets from (8) that

$$[a(x), a^+(y)] = \delta(x - y) \tag{9}$$

where we put, by definition, $< \cdot|\cdot >$ the usual scalar product in the m-dimensional Euclidean space $(\mathbb{R}^m; < \cdot|\cdot >)$, $a^+(x) := a^+(\alpha(x))$ and $a(y) := a(\alpha(y))$ for all $x, y \in \mathbb{R}^m$ and denoted by $\delta(\cdot)$ the classical Dirac delta-function.

The construction above makes it possible to observe that if there exists the so called unique vacuum vector $|0) \in \Phi_{(s)}^{\otimes 0}$, such that for any $x \in \mathbb{R}^m$

$$a(x)|0) = 0, \tag{10}$$

and the set of vectors

$$\left(\prod_{j=1}^{n} a^+(x_j)\right)|0) \in \Phi_{(s)}^{\otimes n} \tag{11}$$

is total in $\Phi_{(s)}^{\otimes n}$, that is their linear integral hull over the functional spaces $\mathcal{H}^{\otimes s}$ is dense in the Hilbert space $\Phi_{(s)}^{\otimes n}$ for every $n \in \mathbb{Z}_+$. This means that for any vector $\varphi \in \Phi$ the following canonical representation

$$\varphi = \sum_{n\in\mathbb{Z}_+}^{\oplus} \frac{1}{\sqrt{n!}} \int_{\mathbb{R}^{m\times n}} \varphi_n(x_1, \ldots, x_n) a^+(x_1) a^+(x_2) \ldots a^+(x_n)|0) \tag{12}$$

holds with the Fourier type coefficients $\varphi_n \in \Phi_{(s)}^{\otimes n}$ for all $n \in \mathbb{Z}_+$ with $\Phi_{(s)}^{\otimes 1} \simeq \mathcal{H} = L_2(\mathbb{R}^m; \mathbb{C})$. The latter is naturally endowed with the Gelfand type quasi-nucleus rigging, dual to

$$\mathcal{D} \subseteq \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- \subseteq \mathcal{D}', \tag{13}$$

where $\mathcal{D} := \mathcal{S}(\mathbb{R}^m; \mathbb{R})$ is the Schwartz space on $\mathbb{R}^m$, making it possible to construct a quasi-nucleus rigging of the dual Fock space $\Phi := \oplus_{n\in\mathbb{Z}_+} \Phi_{(s)}^{\otimes n}$, where $\mathcal{H}_+$ denotes the corresponding *positive* Hilbert space of testing functions. Thereby, the chain (13) generates the dual Fock space quasi-nucleus rigging

$$\mathcal{D}_F \subset \Phi_{F,+} \subset \Phi_F \subset \Phi_{F,-} \subset \mathcal{D}'_F \tag{14}$$

with respect to the central Fock Φ_F, easily following from (1) and (13).

Construct now the following self-adjoint operator $\rho(x) : \Phi_F \to \Phi_F$ as

$$\rho(x) := a^+(x)a(x), \tag{15}$$

called the density operator at a point $x \in \mathbb{R}^m$, satisfying the commutation properties:

$$\begin{aligned} [\rho(x), \rho(y)] &= 0, \\ [\rho(x), a(y)] &= -a(y)\delta(x-y), \\ [\rho(x), a^+(y)] &= a^+(y)\delta(x-y) \end{aligned} \tag{16}$$

for any $x, y \in \mathbb{R}^m$.

Choose now a many-particle quantum Hamilton operator $\mathrm{H} : \Phi_F \longrightarrow \Phi_F$ in the following secondly quantized [9, 10, 18, 21, 28, 35] representation

$$\mathrm{H} := \frac{1}{2}\int_{\mathbb{R}^m} \left\langle \nabla_x a^+(x) | \nabla_x a(x) \right\rangle dx + \mathrm{V}(\rho), \tag{17}$$

where the sign "∇_x" means the usual gradient operation with respect to $x \in \mathbb{R}^m$ in the Euclidean space $\mathbb{R}^m \simeq (\mathbb{R}^m; \langle\cdot|\cdot\rangle)$ and, by definition,

$$\begin{aligned} \mathrm{V}(\rho) &:= \int_{\mathbb{R}^m} dx V_1(x)\rho(x) + \frac{1}{2!}\int_{\mathbb{R}^m} dx_1 dx_2 V_2(x_1, x_2) : \rho(x_1)\rho(x_2) : \\ &\quad + \frac{1}{3!}\int_{\mathbb{R}^m} dx_1 dx_2 dx_3 V_3(x_1, x_2, x_3) : \rho(x_1)\rho(x_2)\rho(x_3) : + \cdots, \end{aligned} \tag{18}$$

is the potential energy operator with suitably determined a one-particle interaction potential $V_1 : \mathbb{R}^m \to \mathbb{R}$, a two-particle interaction symmetric potential $V_2 : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$, a three-particle interaction symmetric potential $V_3 : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ and so on.

Based on the Fock space Φ_F, defined by (1) and generated by the creation-annihilation operators (6) and (7), one can easily construct the local current operator $J(x) : \Phi_F \longrightarrow \Phi_F^m$, $x \in \mathbb{R}^m$, as

$$J(x) = \frac{1}{2i}[a^+(x)\ \nabla_x a(x) - \nabla_x a^+(x)\ a(x)], \tag{19}$$

satisfying jointly with the density operator $\rho(x) : \Phi_F \longrightarrow \Phi_F$, $x \in \mathbb{R}^m$, defined by (15), the following quantum current symmetry algebra [2, 10, 18–23, 29] relationships:

$$\begin{aligned} &[J(g_1), J(g_2)] = iJ([g_{1,}g_2]), \quad [\rho(f_1), \rho(f_2)] = 0, \\ &[J(g_1), \rho(f_1)] = i\rho(\langle g_1|\nabla f_1\rangle), \end{aligned} \tag{20}$$

holding for all $f_1, f_1 \in F$ and $g_1, g_2 \in F^m$, where we put, by definition,

$$[g_{1,}g_2] := \langle g_1|\nabla\rangle\, g_2 - \langle g_2|\nabla\rangle\, g_1, \tag{21}$$

being the usual commutator of vector fields $\langle g_1|\nabla\rangle$ and $\langle g_2|\nabla\rangle$ on the configuration space $\mathbb{R}^m$.

2.2 The Nonrelativistic Quantum Current Algebra and Its Hilbert Space Cyclic Representation

If the energy spectrum density of the Hamiltonian operator (17) in the Fock space Φ_F is bounded from below, the expression (17) can be rewritten algebraically as

$$\mathrm{H} = \frac{1}{2}\int_{\mathbb{R}^m} \Big\langle \mathrm{K}^+(x)|\rho^{-1}(x)\mathrm{K}(x)\Big\rangle\, dx + \mathrm{V}(\rho), \tag{22}$$

being equivalent [18, 19, 21–23] in the corresponding current algebra symmetry representation Hilbert space Φ, modulo the ground state energy eigenvalue, to the positive definite gauge type operator form

$$\hat{\mathrm{H}} = \frac{1}{2}\int_{\mathbb{R}^m} \Big\langle (\mathrm{K}^+(x) - \mathrm{A}(x;\rho))|\rho^{-1}(x)(\mathrm{K}(x) - \mathrm{A}(x;\rho))\Big\rangle\, dx, \tag{23}$$

satisfying conditions (30) and (31), where $\mathrm{A}(x;\rho) : \Phi \to \Phi^m$, $x \in \mathbb{R}^m$, is some specially constructed [26, 27] linear self-adjoint operator, satisfying the condition

$$\mathrm{K}(x)|\Omega) = \mathrm{A}(x;\rho)|\Omega) \tag{24}$$

for any $x \in \mathbb{R}^m$ and the ground state $|\Omega) \in \Phi$, corresponding to the chosen above potential operators $\mathrm{V}(\rho) : \Phi \longrightarrow \Phi$, defined by (18).

Remark 1 Here we mention that the operator $\mathrm{K}(x) : \Phi \longrightarrow \Phi^m$, $x \in \mathbb{R}^m$, defined by (24), relates to that from the work [18, 21, 26] via scaling $\mathrm{K}(x) \rightarrow \mathrm{K}(x)/2$, $x \in \mathbb{R}^m$.

The operator $\mathrm{K}(x) : \Phi \longrightarrow \Phi^m$, acting already on the representation Hilbert space Φ, is given by the expression

$$\mathrm{K}(x) := \nabla_x \rho(x)/2 + iJ(x), \tag{25}$$

where the self-adjoint current operator $J(x) : \Phi \longrightarrow \Phi^m$, acting now on the Hilbert space Φ, can be naturally defined (but non-uniquely) from the continuity equality

$$\partial\rho/\partial t = -i[\mathrm{H}, \rho(x)] = - < \nabla | J(x) >, \tag{26}$$

holding for all $x \in \mathbb{R}^m$. Such an operator $J(x) : \Phi \longrightarrow \Phi^m$, $x \in \mathbb{R}^m$, exists owing to the commutation condition $[\hat{\mathrm{H}}, \hat{\mathrm{N}}] = 0$, giving rise to the continuity relationship (26), if additionally to take into account that supports suppρ of the density operator $\rho(x) : \Phi \longrightarrow \Phi$, $x \in \mathbb{R}^m$, can be chosen arbitrarily owing to the independence of the evolution relationship (26) on the potential operator $V(\rho) : \Phi \longrightarrow \Phi$, but its strict dependence on the existence of the corresponding Hilbert space representation (24).

It is easy to observe that the current algebra (20) is the Lie algebra $\mathcal{G}$, corresponding to the Banach Lie group $G = \mathrm{Diff}(\mathbb{R}^m) \ltimes F$, the semidirect product of the Banach Lie group of diffeomorphisms $\mathrm{Diff}(\mathbb{R}^m)$ of the m-dimensional space $\mathbb{R}^m$ and the abelian group F subject to the multiplicative operation Banach group of smooth functions F. Its properties can be effectively studied by means of the corresponding continuous unitary representations [17, 19, 21–23, 31] of the unitary density family $\mathcal{U} := \{\exp[i\rho(f) : f \in F\}$ in a suitable Hilbert space Φ generated by a cyclic vector $|\Omega) \in \Phi$. Then we can put, by definition,

$$\mathcal{L}(f) := (\Omega|\mathrm{U}(f)|\Omega) \tag{27}$$

for any $f \in F$ and observe that functional (27) is continuous on F owing to the continuity of the representation. Therefore, this functional is the generalized Fourier transform of a cylindrical measure μ on F' :

$$(\Omega|\mathrm{U}(f)|\Omega) = \int_{\mathcal{S}'} \exp[i\eta(f)]d\mu(\eta). \tag{28}$$

From the spectral point of view, there is an isomorphism between the Hilbert spaces Φ and $L_2^{(\mu)}(F;\mathbb{C})$, defined by $|\Omega) \longrightarrow \Omega(\eta) = 1$ and $\mathrm{U}(f)|\Omega) \longrightarrow \exp[i\eta(f)]$ and the extended by linearity upon the whole Hilbert space Φ. Thus, having constructed the nonlinear functional (27) in an exact analytical form, one can retrieve the representation of the unitary family $\mathcal{U}$ in the corresponding Hilbert space Φ. The cyclic vector $|\Omega) \in \Phi$ can be, in particular, obtained as the ground state vector of some unbounded self-adjoint positive definite Hamilton operator $\hat{\mathrm{H}} : \Phi \longrightarrow \Phi$, commuting in addition with the self-adjoint non-negative particle number operator

$$\mathrm{N} := \int_{\mathbb{R}^m} \rho(x) dx, \tag{29}$$

that is $[\hat{\mathrm{H}}, \mathrm{N}] = 0$. Moreover, the conditions

$$\hat{\mathrm{H}}|\Omega) = 0 \tag{30}$$

and

$$\inf_{\varphi \in D_{\hat{\mathrm{H}}}} (\varphi|\hat{\mathrm{H}}|\varphi) = (\Omega|\hat{\mathrm{H}}|\Omega) = 0 \tag{31}$$

hold for the operator $\hat{\mathrm{H}} : \Phi \longrightarrow \Phi$, where $D_{\hat{\mathrm{H}}}$ denotes its dense in Φ domain of definition.

To find the functional (27), which is called the generating Bogolubov type functional for moment distribution functions

$$F_n(x_1, x_2, \ldots, x_n) := (\Omega| : \rho(x_1)\rho(x_2)\ldots\rho(x_n) : |\Omega), \tag{32}$$

where $x_j \in \mathbb{R}^m$, $j = \overline{1, n}$, and the *normal ordering operation* $: \cdot :$ is defined [6, 9, 10, 19–23] as

$$: \rho(x_1)\rho(x_2)\ldots\rho(x_n) : = \prod_{j=1}^{n} \left(\rho(x_j) - \sum_{k=1}^{j-1} \delta(x_j - x_k) \right), \tag{33}$$

Remark 2 The self-adjointness of the operator $\mathrm{A}(g; \rho) : \Phi \longrightarrow \Phi$, $g \in F$, can be stated, following schemes from works [2, 10, 18], under the additional existence of such a linear anti-unitary mapping $\mathrm{T} : \Phi \longrightarrow \Phi$ that the following invariance conditions hold:

$$\mathrm{T}\rho(x)\mathrm{T}^{-1} = \rho(x), \qquad \mathrm{T}\, J(x)\, \mathrm{T}^{-1} = -J(x), \qquad \mathrm{T}|\Omega) = |\Omega) \tag{34}$$

for any $x \in \mathbb{R}^m$. Thereby, owing to conditions (34), the following equalities

$$\mathrm{K}(x)|\Omega) = \mathrm{A}(x; \rho)|\Omega) \tag{35}$$

hold for any $x \in \mathbb{R}^m$, giving rise to the self-adjointness of the operator $\mathrm{A}(g; \rho) : \Phi \longrightarrow \Phi$, $g \in F$.

It is easy to observe that the time-reversal condition (34) imposes the real value relationship for the ground state $\Omega_N = \overline{\Omega}_N \in \Phi_N \simeq L_2^{(s)}(\mathbb{R}^{m\times N};\mathbb{C})$ of the canonically represented N-particle Hamiltonian $\mathrm{H}_N : \Phi_N^{(s)} \to \Phi_N^{(s)}$ for arbitrary $N \in \mathbb{Z}_+$, existing [9, 31] owing to the commutativity condition $[\mathrm{H},\mathrm{N}] = 0$. Moreover, taking into account the relationship (35), one can easily observe that on the invariant Fock subspace $\Phi_N^{(s)} \subset \Phi_F$ the operator $K(x) : \Phi_F \longrightarrow \Phi_F$ is representable as

$$K_N(x) = \sum_{j=\overline{1,N}} \delta(x-x_j)\frac{\partial}{\partial x_j}, \tag{36}$$

entailing the following expression for the related operator $A_N(x;\rho) : \Phi_N^{(s)} \to \Phi_N^{(s)}$ on the subspace $\Phi_N^{(s)} \subset \Phi_F$:

$$(A_N(x;\rho) = \sum_{j=\overline{1,N}} \delta(x-x_j)\nabla_{x_j} \ln|\Omega_N(x_1,x_2,\ldots,x_N)|. \tag{37}$$

The latter makes it possible to derive its secondly quantized [7, 9] expression as

$$\begin{aligned}\mathrm{A}(x;\rho) = \int_{\mathbb{R}^{m\times N}} dx_2 dx_3 \ldots dx_N : \rho(x)\rho(x_2)\rho(x_3) \times \cdots \\ \times\, \rho(x_{N-1})\rho(x_N) : \nabla_x \ln|\Omega_N(x_1,x_2,\ldots,x_N)|\end{aligned} \tag{38}$$

which holds for any $x \in \mathbb{R}^m$ and arbitrary $N \in \mathbb{Z}_+$. Being interested in the infinite particle case when $N \to \infty$, the expression (38) can be naturally decomposed [11, 15, 27] as

$$\begin{aligned}\mathrm{A}(x;\rho) = \sum_{n\in\mathbb{Z}_+} \frac{1}{n!}\int_{\mathbb{R}^{m\times n}} dy_1 dy_2 \cdots dy_n : \rho(x)\rho(y_1)\rho(y)\rho(y_3) \times \cdots \\ \times\, \rho(y_{n-1})\rho(y_n) : \nabla_x W_{n+1}(x;y_1,y_2,\ldots,y_n),\end{aligned} \tag{39}$$

where the corresponding real-valued coefficients $W_n \in H_2^{(1)}(\mathbb{R}^{m\times n};\mathbb{R})$ should be such functions that the series (39) were convergent in a suitably chosen representation Fock space Φ_F, for which the resulting ground state $\lim_{N\to\infty} \Omega_N \simeq |\Omega) \in \Phi_F$ is necessarily cyclic and normalized.

3 The Density Operator Representation of the Nonrelativistic Local Current Algebra and the Factorized Structure of Quantum Integrable Many-Particle Hamiltonian Systems

3.1 The Calogero-Moser-Sutherland Quantum Model: The Density Operator Current Algebra Symmetry Representation, the Hamiltonian Reconstruction and Integrability

The Calogero-Moser-Sutherland quantum bosonic model on the axis $\mathbb{R}$ is governed by the N-particle Hamiltonian

$$H_N := -\sum_{j=\overline{1,N}} \left(\frac{\partial^2}{\partial x_j^2} - \omega^2 x_j^2\right) + \sum_{j\neq k=\overline{1,N}} \frac{\beta(\beta-1)}{(x_j - x_k)^2} \tag{40}$$

in the symmetric Hilbert space $L_2^{(s)}(\mathbb{R}^N;\mathbb{C})$, where $N \in \mathbb{Z}_+$ and the interaction constants $\omega \geq 0$, $\beta \in \mathbb{R}$ are such that $\beta(\beta-1) \geq -1/4$, when the model (40) is stable and has no bound eigenstates. As it was stated in a very interesting and highly speculative works [25, 32], there exists linear differential operators

$$\mathcal{D}_j := \frac{\partial}{\partial x_j} - \omega x_j - \beta \sum_{k=\overline{1,N}, k\neq j} \frac{1}{x_j - x_k} \tag{41}$$

for $j = \overline{1,N}$, such that the Hamiltonian (40) is factorized as the bounded from below symmetric operator

$$H_N = \sum_{j=\overline{1,N}} \mathcal{D}_j^+ \mathcal{D}_j + E_N, \tag{42}$$

where

$$E_N = \frac{1}{3}\beta^2 N \bar{\rho}^2 + \frac{\omega}{2}\bar{\rho} N \tag{43}$$

is the ground state energy of the Hamiltonian operator (40) with the average particle density per length unite $\bar{\rho} > 0$, that is there exists such a vector $|\Omega_N) \in L_2^{(s)}(\mathbb{R}^N;\mathbb{C})$, satisfying for $N \to \infty_+$ the eigenfunction condition

$$H_N|\Omega_N) = E_N|\Omega_N) \tag{44}$$

and equals

$$|\Omega_N) = \prod_{j<k=\overline{1,N}} (x_j - x_k)^\beta \exp(-\sum_{j=\overline{1,N}} \frac{\omega}{2} x_j^2), \tag{45}$$

coinciding at $\omega = 0$ with the corresponding Bethe ansatz representation [24, 33] for the ground state of the standard quantum Calogero-Moser-Sutherland model.

Being interested additionally in proving the quantum integrability of the generalized Calogero-Moser-Sutherland model (40), we will proceed to its second quantized representation [7, 9, 10, 28, 33–35] and studying it by means of the density operator representation approach to the current algebra, described above in Sect. 2 and devised before in [18, 19, 21–23, 26, 27].

The secondly quantized form of the generalized Calogero-Moser-Sutherland Hamiltonian operator (40) looks as

$$\mathrm{H} = \int_0^l dx \left[\psi_x^+(x)\psi_x(x) + \omega^2 x^2 \psi^+(x)\psi(x)\right] + \beta(\beta-1)\int_0^l dx \int_0^l dy \frac{\psi^+(x)\psi^+(y)\psi(y)\psi(x)}{(x-y)^2}, \tag{46}$$

acting in the corresponding Fock space $\Phi_F := \oplus_{n\in\mathbb{Z}_+}\Phi_n^{(s)}$, $\Phi_n^{(s)} \simeq L_2^{(s)}(\mathbb{R}^n;\mathbb{C})$, $n \in \mathbb{Z}_+$. To proceed to the current algebra representation of the Hamiltonian operator (46), it would useful to recall the factorized representation (42) and construct preliminarily the following singular Dunkl type [1, 13, 25, 32] symmetrized differential operator

$$D_N(x) := \sum_{j=\overline{1,N}} \delta(x - x_j)(\frac{\partial}{\partial x_j} - \omega x_j) - \frac{1}{2}\sum_{j\neq k=\overline{1,N}} \beta \sum_{k=\overline{1,N},k\neq j} (\frac{\delta(x-x_j)}{x_j - x_k} + \frac{\delta(x-x_k)}{x_k - x_j} \tag{47}$$

in the Hilbert space $L_2^{(s)}(\mathbb{R}^N;\mathbb{C})$, $N \in \mathbb{Z}_+$, parameterized by a running point $x \in \mathbb{R}$. The corresponding secondly quantized representation of the operator (47) looks as for any $x \in \mathbb{R}$, or in the density operator $\rho : \Phi_F \to \Phi_F$ representation form as

$$\mathrm{D}(x) = \nabla_x \rho(x)/2 + iJ(x) - \omega x \rho(x) - \frac{\beta}{2}\int_{\mathbb{R}} dy \left(\frac{:\rho(x)\rho(y):}{x-y} - \frac{:\rho(y)\rho(x):}{y-x}\right), \tag{48}$$

which is equivalently representable in a suitable current algebra symmetry representation Hilbert space Φ, as

$$\mathrm{D}(x) = K(x) - \omega x\rho(x) - \int_{\mathbb{R}} dy \, \frac{\beta}{(x-y)} : \rho(x)\rho(x) : . \tag{49}$$

Now, based on the operator (48), one can formulate the following proposition.

Proposition 1 *The secondly quantized Calogero-Moser-Sutherland Hamiltonian operator (46) in a suitable current algebra symmetry representation Hilbert space Φ is weakly equivalent to the factorized Hamiltonian operator*

$$\hat{\mathrm{H}} = \int_{\mathbb{R}} dx \mathrm{D}^{+}(x)\rho(x)^{-1}\mathrm{D}(x) \tag{50}$$

modulo the ground state energy operator $\mathrm{E} : \Phi \to \Phi$, *where*

$$\mathrm{E} = \frac{1}{3}\beta^2 \mathrm{N}\bar{\rho}^2 + \frac{\omega}{2}\bar{\rho}\mathrm{N}, \tag{51}$$

where, as before,

$$\mathrm{N} := \int_{\mathbb{R}} \rho(x)dx \tag{52}$$

is the particle number operator, and satisfies the determining conditions

$$(\mathrm{H} - \mathrm{E})|\Omega) = 0, \ \mathrm{D}(x)|\Omega) = 0 \tag{53}$$

on the suitably renormalized ground state vector $|\Omega) \in \Phi$ for all $x \in \mathbb{R}$. Moreover, for any integer $N \in \mathbb{Z}_+$ the corresponding projected vector $|\Omega_N) := |\Omega)|_{\Phi_N}$ exactly coincides with the related Bethe ground state vector for the generalized N-particle Calogero-Moser-Sutherland model (40) and satisfies the following eigenfunction relationships:

$$\mathrm{N}|\Omega_N) = N|\Omega_N),$$
$$\mathrm{E}|\Omega_N) = \left(\frac{1}{3}\beta^2\mathrm{N}\bar{\rho}^2 + \frac{\omega}{2}\bar{\rho}\mathrm{N}\right)|\Omega_N) := E_N|\Omega_N),$$

ensuing exactly the result (43).

Remark 3 When deriving the expression (1), we have used the identities

$$\begin{aligned} \rho(x)\rho(y) &= :\rho(x)\rho(y): + \rho(y)\delta(x-y), \\ \rho(x)\rho(y)\rho(z) &= :\rho(x)\rho(y)\rho(z): + :\rho(x)\rho(y): \delta(y-z) \\ &+ :\rho(y)\rho(z): \delta(z-x) + :\rho(z)\rho(x): \delta(x-y) + \rho(x)\delta(y-z)\delta(z-x), \end{aligned} \tag{54}$$

which hold [6, 9, 18, 27] for the density operator $\rho : \Phi \to \Phi$ at any points $x, y, z \in \mathbb{R}$.

Observe now that the operator (48) can be rewritten down in Φ as

$$\mathrm{D}(x) = \mathrm{K}(x) - \mathrm{A}(x), \tag{55}$$

where, by definition,

$$\mathrm{K}(x) := \nabla_x \rho(x)/2 + iJ(x),\ \ \mathrm{A}(x) := \omega x + \beta \int_{\mathbb{R}} dy\, \frac{:\rho(x)\rho(y):}{x-y} \tag{56}$$

for all $x \in \mathbb{R}/\{[0, l]\mathbb{Z}\}$. Recalling now the second condition of (53), one can rewrite it equivalently as

$$\mathrm{K}(x)|\Omega) = \mathrm{A}(x)|\Omega) \tag{57}$$

on the renormalized ground state vector $|\Omega) \in \Phi$ for all $x \in \mathbb{R}/\{[0, l]\mathbb{Z}\}$. On the other hand, owing to the expression (50), we obtain the searched for current algebra representation

$$\hat{\mathrm{H}} = \int_{\mathbb{R}} dx (\mathrm{K}^{+}(x) - \mathrm{A}(x))\rho(x)^{-1}(\mathrm{K}(x) - \mathrm{A}(x)) \tag{58}$$

of the Calogero-Moser-Sutherland Hamiltonian operator (40) in the suitably renormalized Hilbert space Φ, as it was already demonstrated in the work [26, 27], using the condition (57) in the form (38).

Let us discuss now shortly the quantum integrability of the Calogero-Moser-Sutherland model (40). Owing to the factorized representation (58) one can easily observe that for any integer $p \in \mathbb{Z}_{+}$ the suitably symmetrized Hamiltonian operator densities $\mathrm{h}(x) := \mathrm{D}^{+}(x)\rho(x)^{-1}\mathrm{D}(x) : \Phi \to \Phi$, $x \in \mathbb{R}$, commute to each other and with the particle number operator $\mathrm{N} : \Phi \to \Phi$, that is

$$[\mathrm{h}(x), \mathrm{h}(y)] = 0,\ \ [\mathrm{h}(x), \mathrm{N}] = 0 \tag{59}$$

for any $x, y \in \mathbb{R}/[0, l]\mathbb{Z}$. As a result of the commutation property (59) one easily obtains that for any integer $p \in \mathbb{Z}_{+}$ the symmetric operators

$$\hat{\mathrm{H}}^{(p)} := \int_{\mathbb{R}} dx\, \mathrm{h}(x)^{p} \tag{60}$$

also commute to each other

$$[\hat{\mathrm{H}}^{(p)}, \hat{\mathrm{H}}^{(q)}] = 0 \tag{61}$$

for all integers $p, q \in \mathbb{Z}_+$, and in particular, commute to the Calogero-Moser-Sutherland Hamiltonian operator (50):

$$[\hat{\mathrm{H}}^{(p)}, \hat{\mathrm{H}}] = 0. \tag{62}$$

Concerning the related N-particle differential expressions for the operators (60), it is enough to calculate their projections on the N-particle Fock subspace $\Phi_N^{(s)} \subset \Phi_F$, $N \in \mathbb{Z}_+$. Namely, let an arbitrary vector $|\varphi_N) \in \Phi_N^{(s)}$ is representable as

$$|\varphi_N) := \int_{\mathbb{R}^N} f_N(x_1, x_2, \ldots, x_N) \prod_{j=\overline{1,N}} dx_j \psi^+(x_j)|0) \tag{63}$$

for some coefficient function $f_N \in L_2^{(s)}([0,l]^N; \mathbb{C})$. Then, by definition,

$$\mathrm{H}^{(p)}|\varphi_N) := |\varphi_N^{(p)}), \tag{64}$$

where

$$|\varphi_N^{(p)}) = \int_{\mathbb{R}^N} (H_N^{(p)} f_N)(x_1, x_2, \ldots, x_N) \prod_{j=\overline{1,N}} dx_j \psi^+(x_j)|0) \tag{65}$$

for a given $p \in \mathbb{Z}_+$ any $N \in \mathbb{Z}_+$. In particular, for $p = 2$, when $\hat{\mathrm{H}}^{(2)} + \mathrm{E} = \mathrm{H} : \Phi_F \to \Phi_F$, one easily retrieves the shifted Calogero-Moser-Sutherland Hamiltonian operator (40):

$$H_N^{(2)} = -\sum_{j=\overline{1,N}} \frac{\partial^2}{\partial x_j^2} + \sum_{j \neq k=\overline{1,N}} \frac{\beta(\beta-1)}{(x_j - x_k)^2} - \left(\frac{\beta^2 \bar{\rho}^2}{3} + \omega\bar{\rho}/2\right) N. \tag{66}$$

Respectively for higher integers $p > 2$ the resulting N-particle differential operator expressions $H_N^{(p)} : L_2^{(s)}(\mathbb{R}^N; \mathbb{C}) \to L_2^{(s)}(\mathbb{R}^N; \mathbb{C})$, $N \in \mathbb{Z}_+$, can be obtained the described above way by means of simple yet well cumbersome calculations, and which will prove to be completely equivalent to those, calculated before at $\omega = 0$ in the work [25].

3.2 Quantum Many-Particle Hamiltonian Dynamical System on Axis with δ-Interaction, Its Quantum Symmetries and Integrability

In this section we will consider a quantum non-relativistic many-particle Bose-system on the axis $\mathbb{R}$, governed by the Hamiltonian operator:

$$H_N := -\sum_{j=\overline{1,N}} \frac{\partial^2}{\partial x_j^2} + \alpha \sum_{j\neq k=\overline{1,N}} \delta(x_j - x_k), \tag{67}$$

where $\alpha \in \mathbb{R}_+$ is an interaction constant, and acting in the symmetric Hilbert space $L_2^{(s)}(\mathbb{R}^N;\mathbb{C})$, $N \in \mathbb{Z}_+$. The corresponding secondly quantized expression [6, 8, 9, 14, 29, 35] for the Hamiltonian operator (67) in the related Fock space $\Phi_F \simeq \sum_{n\in\mathbb{Z}_+}^{\oplus} L_2^{(s)}(\mathbb{R}^n;\mathbb{C})$ equals

$$\mathrm{H} = \int_{\mathbb{R}} dx(\psi_x^+\psi_x + \alpha\psi^+\psi^+\psi\psi), \tag{68}$$

where the creation ψ^+- and annihilation $\psi-$ operators satisfy the canonical commutator relationships

$$\begin{aligned}[][\psi(x), \psi^+(y)] &= \delta(x-y), \\ [\psi^+(x), \psi^+(y)] &= 0 = [\psi(x), \psi(y)]\end{aligned} \tag{69}$$

for any $x, y \in \mathbb{R}$. The Hamiltonian operator (68) via the Heisenberg recipe [9, 28, 35] naturally generates on the creation $\psi^+ : \Phi_F \to \Phi_F$ and annihilation $\psi : \Phi_F \to \Phi_F$ operators the following quantum Schrödinger type evolution flow:

$$\begin{aligned} d\psi/dt &:= \frac{1}{i}[\mathrm{H}, \psi] = -i\psi_{xx} + 2i\alpha\psi^+\psi^2, \\ d\psi^+/dt &:= \frac{1}{i}[\mathrm{H}, \psi^+] = i\psi_{xx}^+ - 2i\alpha(\psi^+)^2\psi \end{aligned}$$

with respect to the temporal parameter $t \in \mathbb{R}$. Subject to the quantum Schrödinger type evolution flow above the particle number operator $\mathrm{N} = \int_{\mathbb{R}} \rho(x)dx$ and the Hamiltonian operator (68) in the Fock space Φ_F are its conservative symmetries, that is

$$\frac{d}{dt}\mathrm{N} = 0, \quad \frac{d}{dt}\mathrm{H} = 0 \tag{70}$$

for any $t \in \mathbb{R}$. The quantum model (68), as is well known [8, 33, 35], presents a completely integrable quantum Schrödinger type dynamical system, possessing an infinite hierarchy of quantum commuting to each other operators in the Fock space Φ_F. This result was proved by means of the quantum inverse scattering transform [33], based on existence of a special so called Lax type quantum operator linearization in the associated operator-valued space $C^\infty(\mathbb{R}; \mathrm{End}\ \Phi_F^2)$. In what follows below we will prove the quantum integrability of the quantum Schrödinger type evolution flow (3.2), making use of the local quantum current algebra representation technique, devised in [18, 19, 21, 23, 26], similar the way this was done in sections above.

Let us define in the Fock space Φ_F the following structural operator:

$$\mathrm{D}^{(\varepsilon)}(x) := K(x) - \alpha \int_{\mathbb{R}} dy \vartheta_\varepsilon(x-y) : \rho(x)\rho(y) :, \tag{71}$$

where for any $\varepsilon > 0$ the expression $\vartheta_\varepsilon(x-y) := \vartheta(x-y-\varepsilon) = \{1, \text{ if } x > y-\varepsilon\} \wedge \{0, \text{ if } x \leq y+\varepsilon\}$ for $x, y \in \mathbb{R}$ denotes the shifted classical Heaviside ϑ-function, and construct the following quantum operator:

$$\hat{\mathrm{H}}^{(\varepsilon)} := \int_{\mathbb{R}} dx \mathrm{D}^{(\varepsilon),+}(x)\rho(x)^{-1}\mathrm{D}^{(\varepsilon)}(x). \tag{72}$$

The next proposition (72) states an equivalence of the quantum Hamiltonian operator (68) and the weak operator limit $\lim_{\varepsilon\to 0} \hat{\mathrm{H}}^{(\varepsilon)}$.

Proposition 2 *The many-particle quantum operator (68) in a suitably chosen Fock space Φ_F is weakly equivalent, as $\varepsilon \to 0$, to the operator expression (72), and satisfies the following regularized limiting relationship:*

$$\operatorname{reg}\lim_{\varepsilon\to 0} \mathrm{H}^{(\varepsilon)} := \lim_{\varepsilon\to 0}\left(\mathrm{H}^{(\varepsilon)} - \alpha^2\mathrm{N}^3/3\right) = \hat{\mathrm{H}}. \tag{73}$$

Proof Having taken into account that $\vartheta_\varepsilon(x-y)\delta(x-y) = 0 = \vartheta_\varepsilon(x-y)\delta'(x-y)$ and $\vartheta'_\varepsilon(x-y) = \delta(x-y-\varepsilon)$ for any $x, y \in \mathbb{R}$, one can calculate the operator expression (72) and obtain:

$$\mathrm{H}^{(\varepsilon)} = \int_{\mathbb{R}} dx \psi_x^+(x)\psi_x(x) + \alpha \int_{\mathbb{R}^2} dx \rho(x)\rho(x-\varepsilon) + \alpha^2\mathrm{N}^3/3. \tag{74}$$

□

Insomuch as from the latter expression (74) one easily ensues that

$$\lim_{\varepsilon\to 0}\left(\mathrm{H}^{(\varepsilon)} - \alpha^2\mathrm{N}^3/3\right) = \mathrm{H}, \tag{75}$$

the weak operator relationship (73) in the Fock space Φ is proved.

It is important to mention now the quasi-local operators $\mathrm{D}^{(\varepsilon),+}(x)\rho(x)^{-1}\mathrm{D}^{(\varepsilon)}(x) : \Phi \to \Phi$, $x \in \mathbb{R}$, owing to their construction, are commuting to each other, that is

$$[\mathrm{D}^{(\varepsilon),+}(x)\rho(x)^{-1}\mathrm{D}^{(\varepsilon)}(x), \mathrm{D}^{(\varepsilon),+}(y)\rho(y)^{-1}\mathrm{D}^{(\varepsilon)}(y)] = 0 \tag{76}$$

for any $x, y \in \mathbb{R}$. The latter makes it possible to construct a countable hierarchy of operators

$$\hat{\mathrm{H}}^{(\varepsilon,p)} := \int_{\mathbb{R}} dx \left(\mathrm{D}^{(\varepsilon),+}(x)\rho(x)^{-1}\mathrm{D}^{(\varepsilon)}(x)\right)^p \tag{77}$$

for $p \in \mathbb{Z}_+$, commuting to each other, that is

$$[\hat{\mathrm{H}}^{(\varepsilon,p)}, \hat{\mathrm{H}}^{(\varepsilon,q)}] = 0 \tag{78}$$

for any $p, q \in \mathbb{Z}_+$. Applying to the hierarchy of operators (77) the standard weak regularization procedure as $\varepsilon \to 0$, one can construct, respectively, a countable hierarchy of quantum operators

$$\hat{\mathrm{H}}^{(p)} := \operatorname{reg} \lim_{\varepsilon \to 0} \hat{\mathrm{H}}^{(\varepsilon,p)} \tag{79}$$

on the Fock space Φ for $p \in \mathbb{Z}_+$, also commuting to each other, that is

$$[\hat{\mathrm{H}}^{(p)}, \hat{\mathrm{H}}^{(q)}] = 0 \tag{80}$$

for any $p, q \in \mathbb{Z}_+$, as this naturally follows from the commutator relationship (78). The latter then means that the quantum Schrödinger type dynamical system (3.2) is integrable, other way confirming the classical result of [34].

Remark 4 It is worthy to mention that the following generalized quantum many-particle Hamiltonian Bose system

$$H_N := -\sum_{j=\overline{1,N}} \frac{\partial^2}{\partial x_j^2} + \alpha \sum_{j \neq k = \overline{1,N}} \delta(x_j - x_k) + i\beta \sum_{j \neq k = \overline{1,N}} \left(\frac{\partial}{\partial x_j} \circ \delta(x_j - x_k) + \delta(x_j - x_k) \circ \frac{\partial}{\partial x_k}\right),$$

where $\alpha, \beta \in \mathbb{R}_+$ are interaction constants, and acting on the symmetric Hilbert space $L_2^{(s)}(\mathbb{R}^N; \mathbb{C})$, $N \in \mathbb{Z}_+$, (that is with $(\alpha\delta + \beta\delta')$-interaction potential) generates the corresponding secondly quantized quantum Hamiltonian system

$$\begin{aligned} d\psi/dt &:= \frac{1}{i}[\mathrm{H}, \psi] = -i\psi_{xx} + 2\alpha\psi^+\psi\psi + 2\beta\psi^+\psi\psi_x, \\ d\psi^+/dt &:= \frac{1}{i}[\mathrm{H}, \psi^+] = i\psi_{xx}^+ - 2i\alpha(\psi^+)^2\psi + 2\beta\psi_x^+\psi\psi \end{aligned} \tag{81}$$

with the quantum Hamiltonian operator

$$\mathrm{H} = \int_{\mathbb{R}} dx[\psi_x^+\psi_x + \alpha\psi^+\psi^+\psi\psi + i\beta(\psi^+\psi^+\psi_x\psi - \psi_x^+\psi^+\psi\psi)] \tag{82}$$

on the Fock space Φ_F, which is completely integrable, as it was proved before in [8, 28, 29] by means of the quantum inverse scattering transform. This fact, eventually, allows us to speculate that there exists a suitable local current algebra cyclic representation space Φ, allowing to construct a related structural operator $\mathrm{D}(x) : \Phi \to \Phi, x \in \mathbb{R}$, factorizing the quantum Hamiltonian operator $\hat{\mathrm{H}} = \int_{\mathbb{R}} \mathrm{D}^+(x)\rho(x)^{-1}\mathrm{D}(x)$, and reducing, up to some renormalizing constant operator, to (82) on the corresponding Fock space Φ_F.

4 Conclusion

In the work we succeeded in developing an effective algebraic scheme of constructing density operator and density functional representations for the local quantum current algebra and its application to quantum Hamiltonian and symmetry operators reconstruction. We analyzed the corresponding factorization structure for quantum Hamiltonian operators, governing spatially many- and one-dimensional integrable dynamical systems. The quantum delta-potential and generalized oscillatory Calogero-Moser-Sutherland models of spin-less Bose-particle systems were analyzed in detail. The central vector of the density operator current algebra representation proved to be the ground vector state of the corresponding completely integrable factorized quantum Hamiltonian system in the classical Bethe ansatz form. The latter makes it possible to classify quantum completely integrable Hamiltonian systems a priori allowing the factorized form and whose ground state is of the Bethe ansatz from. These and related aspects of the factorized and completely integrable quantum Hamiltonian systems are planned to be studied in other place.

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Quantum Master Equation for the Time-Periodic Density Operator of a Single Qubit Coupled to a Harmonic Oscillator

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Abstract The reduced density operator of a single qubit coupled to a quantum oscillator is time-periodic even for stationary Hamiltonians. We construct some quantum master equations for such density operator and show that they can be expressed in the Lindblad form. Although the qubit is treated as an open system that exchanges information with the oscillator (environment), the dynamics of the entire system is unitary and such that no information is lost at any time. The time-evolution of the qubit is therefore developed with balanced gain and loss profile. Some subtleties arise since the appropriate master equation must include not only the decay (diffusion) process but also the excitation of the qubit. The advances reported in this work are addressed to cover the decay process only.

Keywords Quantum master equation · Jaynes–Cummings model · Lindblad superoperator

Mathematics Subject Classification (2010) Primary 81S22; 81R15; Secondary 81V80

1 Introduction

The study of open quantum systems finds a diversity of applications to construct realistic dynamical models in quantum physics [2, 3, 5, 19]. The efforts are addressed to describe the non-unitary behavior that results from the interaction of a

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given quantum system with its environment. Special attention has been payed to the non-equilibrium features of driven quantum systems during the last decades. The latter motivated by the spectacular improvements in the experimental manipulation of atoms and photons [10, 16]. Formal approaches to the dynamics of open quantum systems use the construction of master equations for the related density operator of either Markovian (the characteristic times of the system are no longer than those of the environment) or non-Markovian systems [2, 19]. In the former case the environment may be represented by complete positive trace-preserving maps acting on the system to describe the time-evolution in Lindblad form [7, 11, 12]. Complete positivity ensures unitary time-evolution for the "system + environment" entity while positivity guarantees the statistical interpretation of the density matrix. For non-Markovian systems the Lindblad form of the related master equation is not granted (one of the reasons is that positivity can be violated, see e.g. [14]) and, in general, it is more difficult to find solutions of the dynamics, even numerically. As a matter of fact, the number of exactly solvable models of open quantum systems is small since the exchange of information between system and environment usually involves a large number of degrees of freedom.

The purpose of this contribution is to gain an understanding of the construction of exactly solvable master equations for time-periodic density operators where the existence of a unique generator for the stroboscopic dynamics is a major problem (see e.g. [4, 8, 13, 17]). With this aim we shall focus on the simplest quantum optical example: a two-level atom embedded in a cavity with a single-mode quantized electromagnetic field. Although the simplicity of the model, the coupling between a qubit and a harmonic oscillator, it is commonly used to study many atomic systems in quantum optics (for instance, interesting effects can be found in ion traps and cavity QED experiments [10, 16]). As the "qubit + oscillator" system is exactly solvable by unitary time-evolution, we already know the state of the qubit by tracing over the degrees of freedom of the oscillator (environment). The reduced density operator is therefore time-dependent and defined by periodic functions that rule the population inversion of the energy levels. Considered as an open system, using dipole and rotating-wave approximations, and assuming weak atom-field couplings, the interaction of the qubit with its environment must be associated with a master equation. In the sequel we provide a diversity of master equations for the related time-periodic density operator and show that they can be written in the Lindblad form.

The paper is organized as follows. In Sect. 2 some generalities concerning the Lindblad master equation are provided and the problem we are dealing with is defined. Section 3 deals with the explicit derivation of the master equation, some concrete realizations are provided as examples and the corresponding solutions are analyzed. A summary of our results and some concluding remarks are given in Sect. 4.

2 Statement of the Problem

Let $\mathcal{S}$ and $\mathcal{H}$ respectively denote a given quantum system and the corresponding Hilbert space. The quantum state of $\mathcal{S}$ is described by a positive operator in $\mathcal{H}$, denoted ρ and called density operator, which has trace one [15]. For closed quantum systems the density operator at time $t \geq t_0$ arises from the unitary transformation of the initial state $\rho(t) = U(t, t_0)\rho(t_0)U^\dagger(t, t_0)$. In general, to study the dynamics of the system, one may consider an additional Hilbert space $\mathcal{M}_\mathcal{H}$ in which the operators serve as vectors and the inner product is defined as $A \cdot B = \mathrm{Tr}(A^\dagger B)$ [19]. In this context, a *superoperator* is a map from $\mathcal{M}_\mathcal{H}$ into itself. Of special interest, the superoperator $\Lambda(t, t_0)$ that maps the density operator at initial time $t = t_0$ into the density operator at time $t \geq t_0$,

$$\rho(t) = \Lambda(t, t_0)\,[\rho(t_0)]\,, \tag{1}$$

satisfies the initial condition $\Lambda(t_0, t_0) : \rho(t_0) \to \rho(t_0)$. The time-evolution superoperator $\Lambda(t, t_0)$ is generated by the Liouville superoperator $\mathcal{L}$, which is defined through the generalized Liouville–von Neumann equation:

$$\frac{d}{dt}\rho = \mathcal{L}(\rho). \tag{2}$$

For closed (non-dissipative) quantum systems the superoperator $\Lambda(t, t_0)$ defines a unitary time-evolution while the Liouville superoperator takes its Hamiltonian form

$$\mathcal{L}_H(\rho) = -i[H, \rho], \tag{3}$$

with H the Hamiltonian of the system and $[A, B]$ the commutator between A and B.

For open (dissipative) quantum systems it is necessary to impose the Markov property

$$\Lambda(t, t_1)\Lambda(t_1, t_0) = \Lambda(t, t_0), \quad 0 < t_0 < t_1 < t. \tag{4}$$

Equation (4) means that the time-evolution of ρ in the interval (t_0, t) can be constructed by evolving it in the intervals (t_0, t_1) and (t_1, t), where t_1 is an arbitrary point between t_0 and t. In other words, the time-evolution superoperator $\Lambda(t, t_0)$ forms a two-parameter semi-group and defines a dynamical map of the system [19].

The linear operator differential equation (2) is usually called *master equation* and constitutes a standard tool to describe the dynamics of open quantum systems. All the information about the time-evolution of the system is encoded in the Liouville superoperator $\mathcal{L}$, which generates infinitesimal changes in ρ and may be decomposed into its Hamiltonian and purely dissipative parts $\mathcal{L} = \mathcal{L}_H + \mathcal{L}_D$ [7, 11], with $\mathcal{L}_H$ given in (3) and

$$\mathcal{L}_D(\rho) = \sum_{i,j=1}^{d^2-1} g_{i,j}(t) \left(F_i \rho F_j^\dagger - \frac{1}{2} \left\{ F_j^\dagger F_i, \rho \right\} \right). \tag{5}$$

Here d stands for the dimension of $\mathcal{H}$, $\{F_i\}_{i=1}^{d^2}$ is a set of orthonormal operators $F_i \cdot F_j = \delta_{i,j}$ that are traceless for $i \neq d^2$, and $F_{d^2} = \frac{1}{\sqrt{d}}\mathbb{I}$.

Clearly, $\mathcal{L}_H$ is characterized by the Hamiltonian H while $\mathcal{L}_D$ is determined by the positive semidefinite matrix $G(t) = [g_{i,j}(t)]$ [7, 11]. Using a diagonalized form of $G(t)$, the jump operators F_j account for relevant elementary dissipative processes having a relaxation time $T_j \sim g_{jj}^{-1}(t)$, where $g_{jj}(t) \geq 0 \, \forall t \geq t_0$ [18]. Note that the spectrum of $\mathcal{L}_H$ is pure imaginary, defined by differences between the eigenvalues of the Hamiltonian, so $\mathcal{L}_H$ generates unitary time-evolution. On the other hand, the spectrum of $\mathcal{L}_D$ includes negative real eigenvalues that are associated with decay (dissipation) as the system evolves in time (see e.g. [1]).

Equation (2) can be faced in two different forms. Assume first that the effect of the environment has been properly encoded in the dynamical generator $\mathcal{L}$. That is, the matrix $G(t)$ and the operators F_i are already known (without explicit reference to a given model) or they have been guessed to construct an effective model. Only a very small number of this kind of problems can be solved in closed form and numerical resources are required in general. Other option considers the construction of the Liouville superoperator $\mathcal{L}$ from a model that includes information of the system, the environment and the corresponding interaction, after tracing over the environment degrees of freedom. However, there is not a general rule to define neither the operators F_j nor the matrix $G(t)$.

2.1 The Model

Hereafter we assume that the whole system $\mathcal{S}$ is closed and divided into a subsystem $\mathcal{S}_A$ and its environment $\mathcal{S}_E$. Let H_A and H_E be respectively the Hamiltonians of $\mathcal{S}_A$ and $\mathcal{S}_E$. As they act on their corresponding Hilbert spaces $\mathcal{H}_{A,E}$, we promote them to act on the Hilbert space $\mathcal{H}$ of the entire system $\mathcal{S}$ by means of the tensor products $H_A \otimes \mathbb{I}_E$ and $\mathbb{I}_A \otimes H_E$, with $\mathbb{I}_{A,E}$ the identity operator in $\mathcal{H}_{A,E}$ [6]. We also assume that the initial state is separable $\rho(t_0) = \rho_A(t_0) \otimes \rho_E(t_0)$, and that the subsystem $\mathcal{S}_A$ interacts with its environment $\mathcal{S}_E$ at time $t \geq t_0$ through the interaction Hamiltonian H_I, which acts on $\mathcal{H}$. Hence, the Hamiltonian of the entire system is given by $H = H_A \otimes \mathbb{I}_E + \mathbb{I}_A \otimes H_E + H_I$.

Provided an explicit form of H we are able to obtain the states $\rho(t)$ of the entire system $\mathcal{S}$ in closed form. To analyze the dynamics of $\mathcal{S}_A$ we trace over the environment degrees of freedom to get the reduced operator $\rho_A = \mathrm{Tr}_E(\rho)$.

Similarly, $\rho_E = \mathrm{Tr}_A(\rho)$ represents the state of the environment. After evaluating $\frac{d}{dt}\rho_A(t)$, we shall look for the Liouville superoperator $\mathcal{L}$ such that the following equation holds

$$\frac{d}{dt}\rho_A = \mathcal{L}(\rho_A). \tag{6}$$

Thus, we face the *inverse problem* of an open quantum system: given the state $\rho_A(t)$, we look for the dynamical generator $\mathcal{L}$ that produces the time-evolution $\rho_A(t_0) \to \rho_A(t)$.

It is clear that the solution of the inverse problem described above is not unique, so we address the model to the identification of the simpler forms of $\mathcal{L}$ that include the state $\rho_A(t)$ as a solution of the master equation (6).

We shall consider that ρ_A represents the state of a system $\mathcal{S}_A$ interacting with a controlled environment $\mathcal{S}_E$. That is, as the time-evolution of the entire (closed) system $\mathcal{S} = \mathcal{S}_A + \mathcal{S}_E$ is unitary, the information interchanged between $\mathcal{S}_A$ and $\mathcal{S}_E$ is never lost. The excitations and decays of $\mathcal{S}_A$ are correlated with the decays and excitations of $\mathcal{S}_E$, and vice versa.

3 Effect of Environment Interactions

The simplest model to study is a two-level system that interacts with a single-mode quantized electromagnetic field in an isolated QED cavity. Using notation free of units we write $H_A = \frac{1}{2}\sigma_3$ and $H_E = a^\dagger a + \frac{1}{2}$ for the qubit and field Hamiltonians, respectively. Note that the atomic transition frequency and the field frequency are on resonance. Hereafter, $a^\dagger$ and a stand for the conventional ladder boson operators while

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{7}$$

are respectively the Pauli z-matrix and ladder operators:

$$[\sigma_+, \sigma_-] = \sigma_3, \;\; [\sigma_3, \sigma_\pm] = \pm 2\sigma_\pm, \;\; \sigma_3^2 = \mathbb{I}, \;\; \sigma_+^2 = \sigma_-^2 = 0, \;\; \sigma_+^\dagger = \sigma_-. \tag{8}$$

In the dipole and rotating-wave approximations the interaction Hamiltonian reads $H_I = \gamma\left(\sigma_+ \otimes a + \sigma_- \otimes a^\dagger\right)$, with γ the atom-field coupling constant. The dynamics of the entire system is therefore governed by the Jaynes–Cummings model [9].

3.1 Environment with Definite Energy

Let us consider that the qubit interacts with a field of definite energy. That is, the state of the environment is represented by a Fock state at any time. Let us take $|+\rangle \otimes |r\rangle \equiv |+, r\rangle$ as the initial state of $\mathcal{S}$, where $|+\rangle$ is the excited state of the qubit and $|r\rangle$ the Fock state representing the field with exactly r photons. The straightforward calculation shows that, at time $t \geq 0$, the state of the entire system $\mathcal{S}$ is given by

$$\rho(t) = \cos^2(\gamma t\sqrt{r+1})|+, r\rangle\langle +, r| + \sin^2(\gamma t\sqrt{r+1})|-, r+1\rangle\langle -, r+1| \\ + i\sin(\gamma t\sqrt{r+1})\cos(\gamma t\sqrt{r+1})\left[|+, r\rangle\langle -, r+1| - |-, r+1\rangle\langle +, r|\right].$$

So we arrive at the time-periodic density operator

$$\rho_A(t) = \cos^2(\gamma t\sqrt{r+1})|+\rangle\langle +| + \sin^2(\gamma t\sqrt{r+1})|-\rangle\langle -|, \tag{9}$$

which has period $\tau = \frac{\pi}{\gamma\sqrt{r+1}}$, $\rho_A(t+\tau) = \rho_A(t)$. Thus, although the Jaynes-Cummings Hamiltonian H is not time-dependent, the unitary evolution of $\rho(t)$ provides a concrete form for the time-dependence of $\rho_A(t)$ that is defined by the parameters of the field (the number of photons r) and the atom-field coupling (the constant γ). From (9) it is immediate to get the population inversion

$$\langle\sigma_3\rangle(t) = \cos(\tfrac{2\pi}{\tau}t), \tag{10}$$

which is also of period τ. Thus, the first transition $\rho_+ \to \rho_-$ is produced at $t = \tau/2$, with $\rho_+ = |+\rangle\langle +|$ and $\rho_- = |-\rangle\langle -|$. Given r we have $\tau \to \infty$ as $\gamma \to 0$, meaning that the above transition is almost improbable for very weak couplings. On the other hand, considering that H_A does not depend on time, one finds that the periodicity of $\rho_A(t)$ is a consequence of the controlled exchange of information between $\mathcal{S}_A$ and $\mathcal{S}_E$ (no information is lost in $\mathcal{S}$ at any time!). The dynamical evolution of $\rho_A(t)$ is therefore a periodic transition between ρ_+ and ρ_- where information is transferred to, and taken from, the environment with no loss.

Using (9) we obtain a periodic equation of motion

$$\frac{d}{dt}\rho_A(t) = f(t)\sigma_3, \tag{11}$$

where the time-dependent function

$$f(t) = -\frac{\pi}{\tau}\sin(\tfrac{2\pi}{\tau}t) \tag{12}$$

is of period τ. It is remarkable that the time-periodicity of Eqs. (11)–(12) is not provided by the Hamiltonian H_A, which is time-independent, but by the interaction of $\mathcal{S}_A$ with $\mathcal{S}_E$ after the partial trace.

To express the dynamical equation (11)–(12) in the form of the master equation (6) we first notice that $[H_A, \rho_A(t)] = 0$, so the Hamiltonian part $\mathcal{L}_H$ of the dynamical generator $\mathcal{L}$ is equal to zero. Thus, only the purely dissipative part of $\mathcal{L}$ contributes to the time-evolution of the subsystem $\mathcal{S}_A$. On the other hand, from the relationships (8) one realizes that the set $\{\sigma_3, \sigma_\pm\}$ is appropriate to construct $\mathcal{L}_D$. We make the identification $F_1 = \frac{1}{\sqrt{2}}\sigma_3$, $F_2 = \sigma_+$, $F_3 = \sigma_-$, and $F_4 = \frac{1}{\sqrt{2}}\mathbb{I}$. In this form the dissipative generator $\mathcal{L}_D$ includes three elementary processes: two finite temperature amplitude damping (represented by the jump operators F_2 and F_3), and a dephasing in the σ_3 basis (represented by the jump operator F_1) [18]. Using the above identification, for the state $\rho_A(t)$ given in (9) one finds the time-dependent dissipative matrix

$$G(t) = \begin{pmatrix} 0 & g_{12}(t) & g_{13}(t) \\ -g_{12}(t) & g_{22}(t) & 0 \\ -g_{13}(t) & 0 & g_{33}(t) \end{pmatrix}, \tag{13}$$

where $g_{12}(t)$ and $g_{13}(t)$ are functions to be determined, and

$$g_{22}(t)\sin^2(\tfrac{\pi}{\tau}t) - g_{33}(t)\cos^2(\tfrac{\pi}{\tau}t) = -\frac{\pi}{\tau}\sin(\tfrac{2\pi}{\tau}t). \tag{14}$$

Depending on the nontrivial matrix elements $g_{ij}(t)$, the eigenvalues of $G(t)$ are either all-real or one complex (together with its complex-conjugate) and one real. In the former case one can write $G(t) = \mathrm{diag}[\lambda_1(t), \lambda_2(t), \lambda_3(t)]$ to get a diagonalized version of the dissipative generator (5). However, although we can pay attention to the elementary dissipative processes associated with the diagonalized form of $G(t)$, the eigenvalues λ_j are time-dependent in general and they could even acquire negative values. The time-dependence of λ_j, as well as its possible time-periodicity, arises from the controlled exchange of information between $\mathcal{S}_A$ and $\mathcal{S}_E$ via the matrix elements $g_{ij}(t) \neq 0$ (compare with [4, 8, 13, 17]).

3.1.1 Diagonalized form of the Dynamical Generator

For all-real eigenvalues, without loss of generality, let us consider the following cases:

- **Case I.** If $g_{12} = g_{13} = 0$ the dissipative matrix (13) is diagonal with eigenvalues $\lambda_1 = 0$, $\lambda_2 = g_{22}$, and $\lambda_3 = g_{33}$. The dynamical generator is therefore

$$\mathcal{L}_D^{(\mathrm{I})}(\rho) = g_{22}\left[\sigma_+\rho\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+, \rho\}\right] + g_{33}\left[\sigma_-\rho\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho\}\right]. \tag{15}$$

To gain additional insight into the physics of this case one may consider the mean-field equations of the Hamiltonian $H_A = \frac{1}{2}\sigma_3$ and the Pauli ladder operators $\sigma_\pm$. The straightforward calculation leads to

$$\frac{d}{dt}\langle H_A\rangle = -2\,(g_{22}+g_{33})\,\langle H_A\rangle + g_{22}\cos^2(\tfrac{\pi}{\tau}t) - g_{33}\sin^2(\tfrac{\pi}{\tau}t), \tag{16}$$

and

$$\frac{d}{dt}\langle\sigma_\pm\rangle = -\frac{1}{2}\,(g_{22}+g_{33})\,\langle\sigma_\pm\rangle. \tag{17}$$

Remark that $\langle H_A\rangle$ is equal to half the population inversion (10).

- **Case II.** For $g_{13} = 0$ the dynamical generator is given by

$$\mathcal{L}_D^{(\mathrm{II})}(\rho) = \mathcal{L}_D^{(\mathrm{I})}(\rho) + g_{12}\left[\sigma_3\rho\sigma_- - \sigma_+\rho\sigma_3 - \frac{1}{2}\{\sigma_-\sigma_3,\rho\} + \frac{1}{2}\{\sigma_3\sigma_+,\rho\}\right]. \tag{18}$$

In turn, the eigenvalues of the dissipative matrix (13) are given by

$$\lambda_3 = g_{33}, \quad \lambda_\pm = \frac{1}{2}\left[g_{22} \pm \sqrt{g_{22}^2 - 4g_{12}^2}\right]. \tag{19}$$

The simplest root $g_{12} = 0$ reduces the above results to those of Case I.

- **Case III.** For $g_{12} = 0$ the dynamical generator is of the form

$$\mathcal{L}_D^{(\mathrm{III})}(\rho) = \mathcal{L}_D^{(\mathrm{I})}(\rho) + g_{13}\left[\sigma_3\rho\sigma_+ - \sigma_-\rho\sigma_3 - \frac{1}{2}\{\sigma_+\sigma_3,\rho\} + \frac{1}{2}\{\sigma_3\sigma_-,\rho\}\right]. \tag{20}$$

The eigenvalues of the dissipative matrix (13) are in this case

$$\lambda_3 = g_{22}, \quad \lambda_\pm = \frac{1}{2}\left[g_{33} \pm \sqrt{g_{33}^2 - 4g_{13}^2}\right]. \tag{21}$$

The simplest root $g_{13} = 0$ reduces the above results to those of Case I.

As indicated above, the time-dependence (and possible time-periodicity) of the eigenvalues λ_j is determined by the analytic expressions of the matrix elements $g_{ij}(t) \neq 0$. Some insights are given below, where we discuss concrete realizations of the dissipative matrix (13).

3.1.2 Examples

Let us analyze the already diagonalized form of $\mathcal{L}_D$ given in (15). We distinguish four immediate cases:

- **Case 1.** Making $g_{22} = g_0 = \text{const}$ in Eq. (14) one gets the time-periodic eigenvalue

$$g_{33}(t) = \frac{2\pi}{\tau} \tan\left(\tfrac{\pi}{\tau}t\right) + g_0 \tan^2\left(\tfrac{\pi}{\tau}t\right), \tag{22}$$

  which is of period τ, $g_{33}(t+\tau) = g_{33}(t)$, and acquires negative values as $t \to \tau^-$. Integrating (17) one gets an expression for $\langle\sigma_+(t)\rangle$ that is nonnegative (and represents decay) in the time-interval $(0, \frac{1}{2}\tau)$, just the values of t for which the eigenvalue (22) is positive in one cycle. In $(\frac{1}{2}\tau, \tau)$ both functions acquire negative values. That is, $\langle\sigma_+(t)\rangle$ and $g_{33}(t)$ are not useful as the dipole moment and eigenvalue of the dissipative matrix $G(t)$ in the time-interval $(\frac{1}{2}\tau, \tau)$.
- **Case 2.** For $g_{33} = g_1 = \text{const}$, we arrive at the expression

$$g_{22}(t) = g_1 \cot^2(\tfrac{\pi}{\tau}t) - \frac{2\pi}{\tau} \cot(\tfrac{\pi}{\tau}t). \tag{23}$$

  Here $g_{22}(t+\tau) = g_{22}(t)$. In this case (17) produces a decaying function $\langle\sigma_+(t)\rangle$ that diverges at $t = 0$ and is undefined at $t = \tau$. In turn, the eigenvalue $g_{22}(t)$ acquires negative values in the cycle. For $g_1 = 0$ the function $\langle\sigma_+(t)\rangle$ is nonnegative in the cycle but the eigenvalue $g_{22}(t)$ is negative in the time-interval $(0, \frac{1}{2}\tau)$.
- **Case 3.** Assuming $g_{22}(t) = g_{33}(t) \neq \text{const}$ we obtain the time-periodic function

$$g_{22}(t) = g_{33}(t) = \frac{\pi}{\tau} \tan(\tfrac{2\pi}{\tau}t), \tag{24}$$

  which is of period $\frac{1}{2}\tau$. In this case the eigenvalues $g_{22}(t)$ and $g_{22}(t)$ are nonnegative in $(0, \frac{1}{4}\tau)$, where $\langle\sigma_+(t)\rangle$ is real and nonnegative.
- **Case 4.** If $g_{22}(t)$ and $g_{33}(t)$ are both time-dependent and $g_{22}(t) \neq g_{33}(t)$, then

$$g_{22}(t) = -\tfrac{\pi}{\tau} \cot\left(\tfrac{\pi}{\tau}t\right), \quad g_{33}(t) = \frac{\pi}{\tau} \tan\left(\tfrac{\pi}{\tau}t\right). \tag{25}$$

  The time-periodic functions (24) have the period τ. The function $\langle\sigma_+(t)\rangle$ is real and nonnegative in $(0, \frac{1}{2}\tau)$, where $g_{22}(t)$ and $g_{33}(t)$ are negative and positive, respectively.

4 Discussion

Despite the simplicity of the (Jaynes–Cummings) model used in the previous sections to describe the entire (closed) system $\mathcal{S} = \mathcal{S}_A + \mathcal{S}_E$, the derivation of the master equation for the qubit $\mathcal{S}_A$-interacting with the quantized single-mode field $\mathcal{S}_E$-is nontrivial. For the system discussed in this work the Liouville superoperator $\mathcal{L}$ contains only the purely dissipative part $\mathcal{L}_D$.

In contrast with the dissipative models where the environment permanently takes energy from the system under study, our approach is such that the environment $\mathcal{S}_E$ takes energy from $\mathcal{S}_A$ at some intervals of time while it supplies energy to $\mathcal{S}_A$ at the complementary time-intervals. The time-evolution of $\mathcal{S}_A$ is therefore described by a time-periodic density operator $\rho_A(t + \tau) = \rho_A(t)$, which implies a balanced gain (acceptor) and loss (donor) profile since the entire system $\mathcal{S}$ is closed. However, some subtleties must be attended in order to get a completely satisfactory approach. To be concrete, the construction of the Lindblad superoperator presented here is useful to describe the transition $\rho_+ \to \rho_-$ only, which is developed during half the period of one cycle (recall that we have taken $\rho_+ = |+\rangle\langle+|$ as the initial state of the qubit). At the present stage we would like to emphasize that the weak coupling $\gamma << 1$ could be useful since the period τ acquires large values. In general, in order to cover the entire period, the excitation process $\rho_- \to \rho_+$ must be also included. Then, it is necessary an additional construction of the master equation in which the information is taken from the environment by the system. It is expected that both constructions will be complementary to describe the entire transition $\rho_+ \to \rho_- \to \rho_+$, developed in one-cycle evolution (the approach has been successfully applied to study Floquet-like open quantum systems [4, 8, 13, 17]). Work in this direction is in progress and will be reported elsewhere.

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On the Construction of Non-Hermitian Hamiltonians with All-Real Spectra Through Supersymmetric Algorithms

Kevin Zelaya, Sara Cruz y Cruz, and Oscar Rosas-Ortiz

Abstract The energy spectra of two different quantum systems are paired through supersymmetric algorithms. One of the systems is Hermitian and the other is characterized by a complex-valued potential, both of them with only real eigenvalues in their spectrum. The superpotential that links these systems is complex-valued, parameterized by the solutions of the Ermakov equation, and may be expressed either in nonlinear form or as the logarithmic derivative of a properly chosen complex-valued function. The non-Hermitian systems can be constructed to be either parity-time-symmetric or non-parity-time-symmetric.

Keywords Non-Hermitian Hamiltonians · Supersymmetric quantum mechanics · PT-symmetry · Darboux transformations · Ermakov equation

Mathematics Subject Classification (2010) Primary 81Q60; Secondary 81Q12

1 Introduction

The supersymmetric formulation of quantum mechanics is a subject of intense activity in contemporary physics. It is addressed to analyze the spectral properties of exactly solvable potentials as well as to construct new integrable quantum models

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[5, 12, 21]. Sustained by the factorization method [2, 20], the supersymmetric approach is basically algebraic [3] and permits the pairing between the spectrum of a given (well-known) Hamiltonian H_0 to the spectrum of a second (generally unknown) Hamiltonian H_1. In terms of differential operators, it has been found that the factorization of either H_0 or H_1 is not unique [20] and that the pairing of H_0 with H_1 is ruled by a Darboux transformation [1], which was introduced in 1882 [13] (see historical details in e.g. [21, 28]). The keystone is a solution u (not necessarily normalizable) of the eigenvalue equation $H_0 u = \epsilon u$ that is used to generate the Darboux transformation $V_1(x) = V_0(x) + 2\frac{d}{dx}\beta(x)$ [2, 20], where $\beta(x) = -\frac{d}{dx}\ln u(x)$ is called superpotential and ϵ the factorization energy. Remarkably, not only Hermitian but also non-Hermitian Hamiltonians H_1 can be produced as supersymmetric partners of a given exactly solvable (either Hermitian or non-Hermitian) Hamiltonian H_0. Indeed, depending on the properties of $V_0(x)$ and $\beta(x)$, the new potential $V_1(x)$ may be either real or complex-valued. In any case, the spectrum of the new Hamiltonian H_1 includes either all-real eigenvalues or a combination of real and complex eigenvalues, see e.g. [4, 6–8, 10, 11, 16, 17, 19, 23–27].

Quite recently, a complex-valued superpotential defined by the nonlinear expression

$$\beta(x) = -\frac{d}{dx}\ln\alpha(x) + i\frac{\lambda}{\alpha^2(x)}, \quad \lambda \in \mathbb{R}, \tag{1}$$

has been provided to produce new classes of non-Hermitian Hamiltonians H_1 with all-real spectra [25]. The function $\alpha(x)$ is a solution of the Ermakov equation [15]:

$$-\frac{d^2}{dx^2}\alpha(x) + V_0(x)\alpha(x) = \epsilon\alpha(x) + \frac{\lambda^2}{\alpha^3(x)}, \tag{2}$$

which is reduced to the eigenvalue equation $H_0\alpha = \epsilon\alpha$ for $\lambda = 0$. The eigenfunctions of the resulting non-Hermitian Hamiltonians H_1 satisfy some properties of interlacing of zeros that permit the study of the related systems as if they were Hermitian [19]. Indeed, a bi-orthogonal basis can be introduced to facilitate the construction of coherent states for such a class of systems [27]. Moreover, the factorization energy ϵ can be placed at any arbitrary position in the spectrum of H_1 [10]. Notedly, the eigenvalues of the non-Hermitian Hamiltonians H_1 are all-real regardless of whether H_1 is parity-time-symmetric [9] or not.

In this communication we briefly revisit the method developed in [10, 19, 25, 27] and show that the nonlinear superpotential (1) can be also expressed in the 'canonical form' $\beta(x) = -\frac{d}{dx}\ln u(x)$, where u is an eigenfunction of H_0 with very concrete profile. The results presented here generalize the approach introduced in [11], where it is guessed that a complex linear-combination of eigenfunctions of H_0 may be useful to construct complex-valued potentials $V_1(x)$. We provide a pair of examples where the new potentials are either parity-time-symmetric or non-parity-time-symmetric.

2 Factorization Method and Non-Hermitian Hamiltonians

Consider an initial Hamiltonian

$$H_0 = -\frac{d^2}{dx^2} + V_0(x), \tag{3}$$

with $V_0(x)$ a real-valued potential defined in $\operatorname{Dom} V_0 \subseteq \mathbb{R}$. We assume that the energy eigenvalues $E^{(0)} \in \mathbb{R}$ and eigenfunctions $\phi(x)$ of the related eigenvalue equation $H_0\phi(x) = E^{(0)}\phi(x)$ are already known. In particular, the bounded solutions $\phi_n(x)$ belong to the discrete eigenvalues $E_n^{(0)}$, $n = 0, 1, \ldots$ Let us introduce a pair of non-mutually adjoint operators, A and B, such that

$$H_0 = AB + \epsilon, \quad A = -\frac{d}{dx} + \beta(x), \quad B = \frac{d}{dx} + \beta(x), \tag{4}$$

where $\beta(x)$ is in general a complex-valued function and ϵ is a real constant. After comparing (4) with (3) one arrives at the Riccati equation

$$-\beta' + \beta^2 = V_0(x) - \epsilon, \quad \beta' = \frac{d\beta}{dx}. \tag{5}$$

Provided a solution of (5), reversing the order of the factors in (4) gives

$$H_1 = BA + \epsilon = -\frac{d^2}{dx^2} + V_1(x), \quad V_1(x) = V_0(x) + 2\beta'(x). \tag{6}$$

Notice that the new operator H_1 is not self-adjoint since V_1 is complex-valued in general. Indeed, $H_1^\dagger = A^\dagger B^\dagger + \epsilon = -\frac{d^2}{dx^2} + V_1^* \neq H_1$. Nevertheless, the pair H_0 and H_1 satisfies the intertwining relationships

$$BH_0 = H_1 B, \quad H_0 A = AH_1, \tag{7}$$

so that the eigenvalue equation $H_1\psi_n = E_n^{(1)}\psi_n$, $n = 0, 1, \ldots$, is automatically solved by the set

$$\psi_{n+1} = \frac{1}{\sqrt{E_n^{(0)} - \epsilon}} B\phi_n\,, \quad A\psi_0 = 0\,, \quad E_{n+1}^{(1)} = E_n^{(0)}\,, \quad E_0^{(1)} = \epsilon\,. \tag{8}$$

The functions $\psi_n(x)$ are complex-valued and such that the zeros of their real and imaginary parts satisfy some theorems of interlacing [19].

2.1 Complex-Valued Potentials with All-Real Spectra

In the conventional supersymmetric approaches the solution of the Riccati equation (5) is usually taken to be real-valued. However, complex-valued solutions are feasible even for real-valued potentials V_0 and real factorization energies ϵ. Indeed, the real and imaginary parts of Eq. (5) lead to a coupled system which is solved by the complex-valued superpotential (1). Assuming, with no loss of generality, that $\alpha(x)$ is real-valued, it may be shown that the solution of the Ermakov (2) can be written as [25]:

$$\alpha(x) = \left[a u_1^2(x) + b u_1(x) u_2(x) + c u_2^2(x) \right]^{1/2}, \tag{9}$$

where $u_{1,2}$ are solutions of the system

$$-u''_{1,2} + V_0 u_{1,2} = \epsilon u_{1,2}, \quad W(u_1, u_2) = u_1 u'_2 - u'_1 u_2 = W_0, \tag{10}$$

with $W_0 = \text{const}$. The function α is free of zeros in Dom V_0 if the set $\{a, b, c\}$ is integrated by positive numbers that are constrained as follows

$$b^2 - 4ac = -4\lambda^2 / W_0^2. \tag{11}$$

Using the superpotential (1), with α given in (9), the new potential (6) is now given by the nonlinear expression

$$V_1(x) = V_0(x) - 2 (\ln \alpha(x))'' + i \left(\frac{2\lambda}{\alpha^2(x)} \right)', \quad \lambda \in \mathbb{R}. \tag{12}$$

Notice that the results of the conventional supersymmetric approaches [5, 12, 21] are automatically recovered for $\lambda = 0$. On the other hand, it may be shown that the imaginary part of $V_1(x)$ satisfies the *condition of zero total area* [19]:

$$\int_{\text{Dom}\, V_0} \text{Im}\, V_1(x) dx = \left. \frac{2\lambda}{\alpha^2(x)} \right|_{\text{Dom}\, V_0} = 0, \tag{13}$$

so that the total probability is conserved. The latter means that the potentials (12) can be addressed to represent open quantum systems with balanced gain (acceptor) and loss (donor) profile [14].

2.1.1 Parity-Time-Symmetric Potentials

Potentials featuring the parity-time symmetry [9] represent a particular case of the applicability of the condition of zero total area (13). Such potentials are invariant under parity (P) and time-reversal (T) transformations in quantum mechanics, so that a necessary condition for PT-symmetry is $V(x) = V^*(-x)$, where * stands for complex conjugation. For initial potentials $V_0(x)$ such that $V_0(x) = V_0(-x)$, one can show that making $b = 0$ in (9) is sufficient to get $V_1(x) = V_1^*(-x)$. In other words, the parity-time symmetry is a consequence of the condition of zero total area in our approach.

2.1.2 Non-parity-Time-Symmetric Potentials

For $V_0(x) \neq V_0(-x)$ the property $V_1(x) = V_1^*(-x)$ does not hold anymore, so the complex-valued potentials (12) have all-real spectra although they are non-parity-symmetric. Diverse examples have been already discussed in e.g. [10, 19, 25, 27]. Quite recently the pseudo-Hermiticity and supersymmetric approaches have been combined to get new classes of non-parity-time-symmetric potentials with all-real spectra [6]. Interestingly, such potentials can be manipulated to induce phase transitions where conjugate pairs of complex eigenvalues emerge in the spectrum. Similar results have been reported in [18], where the condition of zero total area (13) plays a relevant role. The discussion on the subject is out of the scope of the present work and will be reported elsewhere.

2.2 Recovering the Canonical form of the Superpotential

We wonder if the nonlinear expression (1) can be reduced to the canonical form $\beta = -\frac{d}{dx} \ln u(x)$. Keeping this in mind, we first rewrite (1) as

$$\beta = -\frac{\frac{1}{2}(\alpha^2)' - i\lambda}{\alpha^2}. \tag{14}$$

Using (9) and (11) we factorize the α-function in the form

$$\alpha^2 = \frac{1}{a}\left[au_1 + \left(\frac{b}{2} + i\frac{\lambda}{W_0}\right)u_2\right]\left[au_1 + \left(\frac{b}{2} - i\frac{\lambda}{W_0}\right)u_2\right]. \tag{15}$$

In turn, expanding the numerator of Eq. (14) yields

$$\frac{1}{2}\left(\alpha^2\right)' - i\lambda = au_1u_1' + cu_2u_2' + bu_1'u_2 + \left(\frac{bW_0}{2} - i\lambda\right), \tag{16}$$

where we have used the Wronskian defined in (10). The latter result is now factorized:

$$(C_0u_1' + C_1u_2')(D_0u_1 + D_1u_2). \tag{17}$$

The coefficients C_0, C_1, D_0, D_1 are defined by comparing the expanded version of (17) with (16). One gets

$$\frac{1}{2}\left(\alpha^2\right)' - i\lambda = \frac{1}{a}\left[au_1' + \left(\frac{b}{2} - i\frac{\lambda}{W_0}\right)u_2'\right]\left[au_1 + \left(\frac{b}{2} + i\frac{\lambda}{W_0}\right)u_2\right]. \tag{18}$$

Finally, the substitution of (15) and (18) into (14) produces

$$\beta = -\frac{\alpha'(x)}{\alpha(x)} + i\frac{\lambda}{\alpha^2(x)} = -\frac{d}{dx}\ln\left[au_1 + \left(\frac{b}{2} - i\frac{\lambda}{W_0}\right)u_2\right]. \tag{19}$$

Thus, the function we are looking for is given by the linear superposition

$$u = au_1 + \left(\frac{b}{2} - i\frac{\lambda}{W_0}\right)u_2, \tag{20}$$

where the constants a, b and λ are linked by the condition (11). If $\lambda = 0$ the constraint (11) becomes $b = \pm 2\sqrt{ac}$, so that the coefficients of the superposition (20) are real numbers, $u = \sqrt{a}\left(\sqrt{a}u_1 + \sqrt{c}u_2\right)$, as expected.

The expression (19) shows that the superpotential $\beta(x)$ can be written in either the nonlinear form (1), or as the logarithmic derivative of the function u defined in (20). The latter is a linear superposition of the solutions of (10) with complex coefficients that are uniquely defined by the condition (11). Notice that the derivation of the u-function (20) generalizes the approach introduced in [11], where it is guessed that a linear combination of $u_{1,2}$ would give rise to complex-valued potentials V_1 whenever the appropriate complex coefficients have been included. As an example, in [11] the authors provide the coefficients that produce a family of oscillator-like complex-valued potentials. They also apply their method to study the potential $V_1(x) = -\frac{1}{2}(ix)^N$, $N \geq 2$, introduced in [9], and describe some other potentials that can be studied within their approach. However, no general rule to fix the appropriate complex coefficients is given in [11]. In contrast, the linear superposition (20) is general in the sense that the rule (11) applies for any differentiable and exactly solvable real-valued initial potential $V_0(x)$. Diverse examples have been already provided in [10, 19, 25, 27].

3 Examples and Discussion of Results

As immediate examples let us discuss the regular complex-valued potential $V_1(x)$ generated by the following initial potentials:

- **Free particle.** Given $V_0(x) = 0$, the basis set is $u_1 = e^{ikx}$ and $u_2 = e^{-ikx}$, with $W_0 = -2ik$. To get a real-valued α-function we take $k = i\frac{\kappa}{2}$, with $\kappa > 0$.

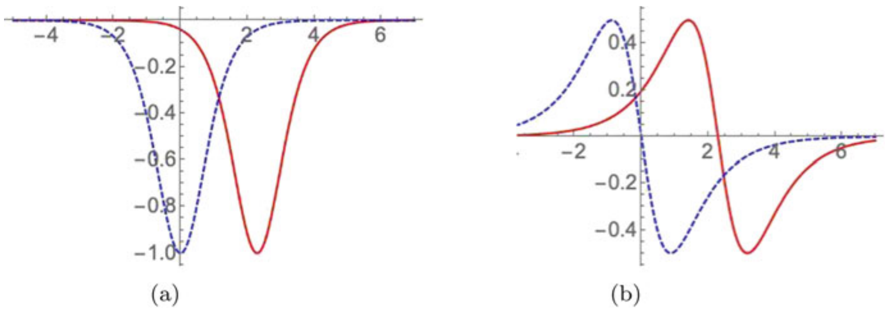

Fig. 1 The real and imaginary parts of the complex-valued potentials with all-real spectra (12) derived from the expressions of the free particle provided in (21) with $b \neq 0$, $a = 1.5$ (red curve), and $b = 0$, $a = 1$ (dotted-blue curve). In all cases $\lambda = \kappa = 1$. (**a**) Re $V_1(x)$. (**b**) Im $V_1(x)$

Without loosing generality we now make $a = c$. Then,

$$\alpha(x) = [2a \cosh \kappa x + b]^{1/2}, \qquad u(x) = a e^{-\kappa x/2} + \left(\frac{b}{2} - i \frac{\lambda}{\kappa} \right) e^{\kappa x/2}. \tag{21}$$

The potentials $V_1(x)$ are depicted in Fig. 1, they are of the Pöschl–Teller type, generalize the well known family of regular (real-valued) supersymmetric partners of the free particle [22], and satisfy the condition of zero total area (13). These potentials include only one bound state of energy $E^{(0)} = -\frac{1}{4}\kappa^2$. The effect of $b \neq 0$ is to slide the potential to the right (red curve in Fig. 1), so that $V_1(x)$ is parity-time-invariant after the appropriate shift. The latter is just because the initial potential $V_0(x) = 0$ satisfies the condition $V_0(x) = V_0(-x)$ and exhibits, at the same time, translational symmetry $V_0(x) = V_0(x + x_0)$. One may say that, in the present case, the translational symmetry is invariant under the Darboux transformations (12).

- **Morse potential.** It is clear that the condition $V_0(x) = V_0(-x)$ cannot be applied on the Morse potential

$$V_0(x) = \Gamma_0 (1 - e^{-\gamma x})^2, \quad x \in \mathbb{R}, \gamma \in \mathbb{R}, \quad \Gamma_0 > 0. \tag{22}$$

Then, the potentials (12) associated to (22) are non-parity-time-symmetric for any values of the set $\{a, b, c\}$. The condition $\Gamma_0 > \gamma^2/2$ ensures that at least one bound state exists. It may be shown [10] that two linear independent solutions of (10) for $\epsilon \in \mathbb{R}$ are given in terms of confluent hypergeometric functions as follows

$$\begin{aligned} u_1(x) &= e^{-y/2} y^{\sigma} {}_1F_1 \left(\sigma + \frac{1}{2} - d; 1 + 2\sigma; y \right), \\ u_2(x) &= e^{-y/2} y^{-\sigma} {}_1F_1 \left(-\sigma + \frac{1}{2} - d; 1 - 2\sigma; y \right), \end{aligned} \tag{23}$$

where

$$y = 2de^{-\gamma x}, \quad d^2 = \frac{\Gamma_0}{\gamma^2}, \quad \sigma^2 = \frac{\Gamma_0 - \epsilon}{\gamma^2}, \quad W_0 = 2\sqrt{\Gamma_0 - \epsilon}. \tag{24}$$

The physical energy eigenvalues are given by

$$E_n = \gamma \left[(2n+1)\sqrt{\Gamma_0} - \gamma(n+1/2)^2 \right], \quad n = 0, 1, \ldots N, \tag{25}$$

where N is given by the floor function $N = \lfloor \frac{\sqrt{\Gamma_0}}{\gamma} - \frac{1}{2} \rfloor$. The related eigenfunctions can be recovered from (23) after substituting E_n for ϵ and the appropriate boundary conditions. In Fig. 2 we show the potential (22) and two of its supersymmetric partners for $\gamma = 1$ and $\Gamma_0 = 4$. In such case, the initial potential admits two bound states with energy eigenvalues $E_0^{(0)} = 7/4$ and $E_1^{(0)} = 15/4$. Notice that, besides the above energies, potentials $V_1(x)$ include the eigenvalue $E_0^{(1)} = \epsilon = 1$ in their spectra. Moreover, they satisfy the condition of zero total area (13).

In summary, the method introduced in [25] and developed in [10, 19, 27] provides complex-valued potentials with all-real spectra that includes the parity-time-symmetric case as a particular result. The keystone of the approach relies on the solutions to the Ermakov equation (2) and the nonlinearity of the imaginary part of the superpotential (1). The latter permits to introduce the constraint (11) as an universal rule to choice the complex parameters that are required in the superposition (20) to get properly defined complex potentials in supersymmetric quantum mechanics.

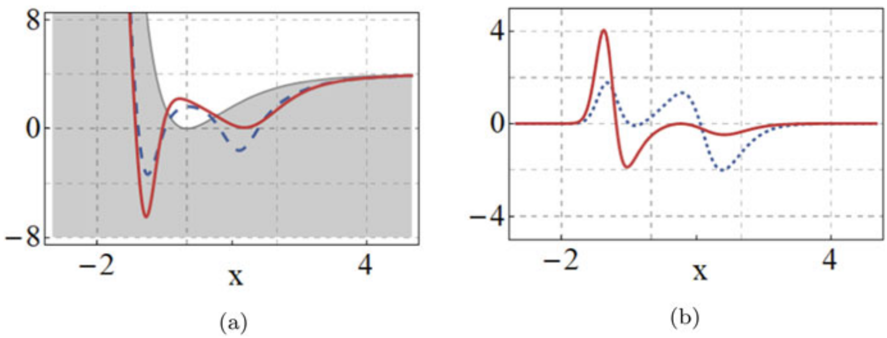

Fig. 2 The real and imaginary parts of the complex-valued potential with all-real spectra (12) derived for the expressions of the Morse potential provided in (23) with $a = c = 1$ (red curve), and $a = 1$, $c = 1/3$, $b = 0$ (dotted-blue curve). In all cases $\lambda = 2$ and $\epsilon = 1$. The gray area delimitates the initial Morse potential. (**a**) Re $V_1(x)$. (**b**) Im $V_1(x)$

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Toeplitz Quantization of an Analogue of the Manin Plane

Stephen Bruce Sontz

Abstract The theory of the Toeplitz quantization of non-commutative algebras, that has been developed in several recent papers by the author, is applied to the specific example of an algebra which is a multi-variable generalization of the Manin plane (sometimes known as the quantum plane). Then a holomorphic sub-algebra is defined, and a sesqui-linear form is used to define a projection of the algebra onto the sub-algebra. Using all of this, Toeplitz operators are introduced with symbols in the algebra. These Toeplitz operators are linear maps from the holomorphic sub-algebra to itself, which is a pre-Hilbert space. Consequently, the Toeplitz operators are densely defined, linear operators in the Hilbert space completion of the sub-algebra. The Toeplitz quantization of the algebra is defined to be the linear mapping of a symbol to its Toeplitz operator. Creation and annihilation operators are defined as Toeplitz operators with certain symbols, and their commutation relations are studied.

Keywords Toeplitz quantization · Toeplitz operator

Mathematics Subject Classification (2010) Primary 47B35; Secondary 81S99

1 Introduction

The topic of this paper concerns a multi-variable analogue of the Toeplitz quantization of the non-commutative complex Manin plane. The approach here is to define and study Toeplitz operators in this new setting. Then this generalizes the material in [3]. While this paper has been motivated by a construction in [2], the method of coherent state quantization used there is distinct from the Toeplitz quantization used here. Further discussion of this topic can be found in [3, 4] and [5].

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2 Main Results

Let $\mathcal{F}$ denote the free algebra on the $2n$ generators $\theta_1, \overline{\theta}_1, \ldots, \theta_n, \overline{\theta}_n$. This is a $*$-algebra, where the $*$-*operation* (or *conjugation*) satisfies $\theta_i^* = \overline{\theta}_i$ and $\overline{\theta}_i{}^* = \theta_i$ for $i = 1, \ldots, n$. We define $\mathcal{A}$ to be the quotient of the free algebra $\mathcal{F}$ divided by the bilateral $*$-ideal $\mathcal{I}$ generated by these elements:

$$\theta_i\theta_j - r_{ij}\theta_j\theta_i \quad 1 \leq i < j \leq n, \tag{1}$$

$$\theta_i\overline{\theta}_j - s_{ij}\overline{\theta}_j\theta_i \quad 1 \leq i \leq j \leq n. \tag{2}$$

for non-zero, complex numbers r_{ij} and s_{ij}. Since $\mathcal{I}$ is a $*$-ideal, it also contains these elements:

$$\overline{\theta}_j\overline{\theta}_i - r_{ij}^*\overline{\theta}_i\overline{\theta}_j \quad 1 \leq i < j \leq n, \tag{3}$$

$$\theta_j\overline{\theta}_i - s_{ij}^*\overline{\theta}_i\theta_j \quad 1 \leq i \leq j \leq n. \tag{4}$$

We let λ^* denote the complex conjugate of the complex number λ. In order to make the relations (2) and (4) consistent, we must require $s_{ii}^* = s_{ii}$, that is $s_{ii} \in \mathbb{R}$. The algebra $\mathcal{A}$ is a multi-variable analogue of the Manin (or quantum) plane studied in [3], where the case $n = 1$ is considered. Then we claim that $\mathcal{A}$ has a vector space (Hamel) basis given by the set $\{\theta^a\overline{\theta}^b\}$. Here

$$a = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$$

is a length n multi-index. Also we put $\theta^a := \theta_1^{a_1}\theta_2^{a_2}\cdots\theta_n^{a_n}$. Similarly, we put $\overline{\theta}^b := \overline{\theta}_1^{b_1}\cdots\overline{\theta}_n^{b_n}$, where $b = (b_1, \ldots, b_n)$ is also a multi-index of length n. From now on all multi-indices will have length n. One point is that the relations allow us to push all occurrence of $\overline{\theta}$'s in a monomial in $\mathcal{A}$ to the right of all occurrences of θ's. Then within each of these two groupings of θ's and $\overline{\theta}$'s we can order the sub-scripts so that they increase when read from left to right. This shows that the set

$$\{\theta^a\overline{\theta}^b = \theta_1^{a_1}\cdots\theta_n^{a_n}\,\overline{\theta}_1^{b_1}\cdots\overline{\theta}_n^{b_n} \mid a, b \in \mathbb{N}^n\}$$

spans the vector space $\mathcal{A}$. To show the linear independence of this set is mildly trickier. Basically, the idea is that each relation in (1)–(4) preserves the number of θ_i's and the number of $\overline{\theta}_j$'s in an expression that is a scalar multiple of a word. So the word $\theta^a\overline{\theta}^b$ minus any linear combination of words of the form $\theta^c\overline{\theta}^d$ with $(c, d) \neq (a, b)$ can not be an element in the ideal $\mathcal{I}$ of relations defining $\mathcal{A}$.

The $*$-operation (or conjugation) on $\mathcal{F}$ passes to the quotient $\mathcal{A}$ and is given on the standard basis elements $\theta^a\overline{\theta}^b$ where $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ by

$$(\theta^a\overline{\theta}^b)^* := \theta_n^{b_n}\cdots\theta_1^{b_1}\overline{\theta}_n^{a_n}\cdots\overline{\theta}_1^{a_1} = \theta^{b^T}\overline{\theta}^{a^T}, \tag{5}$$

where $a^T := (a_n, \dots, a_1)$ is the multi-index that is the *transpose* of the multi-index a and similarly for b^T.

Clearly, $(\theta^a\overline{\theta}^b)^* = C(a,b)\,\theta^b\overline{\theta}^a$, where the constant $C(a,b)$ is non-zero and computable in terms of products of the constants r_{ij}. Then the definition (5) is extended *anti-linearly* to finite linear combinations of these basis elements. Taking $a = b = (0, \dots, 0)$, the zero multi-index of length n, we see that $1^* = 1$.

The space $\mathcal{P}$ of 'holomorphic polynomials' (which is one motivation for using the notation $\mathcal{P}$) is defined to be the sub-algebra (but not $*$-algebra) generated by the elements $\theta_1, \dots, \theta_n$ of $\mathcal{A}$.

Next a sesqui-linear form on $\mathcal{A}$ is defined on pairs of elements of the standard basis and then extended sesqui-linearly to $\mathcal{A}$. Our convention is that a sesqui-linear form is anti-linear in its first entry and linear in its second entry. The definition on these basis elements is given for multi-indices a, b, c, d by

$$\langle \theta^a\overline{\theta}^b, \theta^c\overline{\theta}^d \rangle := w(a+d)\delta_{a+d,b+c} = w(a+d)\delta_{a-b,c-d}, \tag{6}$$

where $w(a) > 0$ is a positive weight defined for every multi-index a. Here $a + d := (a_1+d_1, \dots, a_n+d_n)$ is the usual sum of multi-indices of length n. Similarly for $b+c$ as well as for the differences $a - b$ and $c - d$. Also, $\delta_{a,b}$ is the Kronecker delta. It is straightforward to show that this sesqui-linear form is complex symmetric. The case $n = 1$ is studied in detail in [3], where a necessary and sufficient (and messy, though computable) condition is given for this sesqui-linear form to be non-degenerate. It is expected that a similar result holds here.

Proposition 1 *The identity $\langle f, g\rangle^* = \langle f^*, g^*\rangle$ holds for all $f, g \in \mathcal{A}$ if and only if the identity $w(a) = w(a^T)$ holds for all multi-indices a. In this case the conjugation map $* : \mathcal{A} \to \mathcal{A}$ is anti-unitary.*

Proof It suffices to consider the case $f = \theta^a\overline{\theta}^b$ and $g = \theta^c\overline{\theta}^d$. Then

$$\langle f, g\rangle^* = \langle \theta^a\overline{\theta}^b, \theta^c\overline{\theta}^d\rangle^* = \big(w(a+d)\delta_{a+d,b+c}\big)^* = w(a+d)\delta_{a+d,b+c}$$

by evaluating with (6). Next we have that

$$\begin{aligned}\langle f^*, g^*\rangle &= \langle \theta^{b^T}\overline{\theta}^{a^T}, \langle \theta^{d^T}\overline{\theta}^{c^T}\rangle = w(b^T + c^T)\delta_{(b^T+c^T),(a^T+d^T)} \\ &= w((b+c)^T)\delta_{(b+c)^T,\,(a+d)^T} = w((a+d)^T)\delta_{a+d,b+c},\end{aligned}$$

giving the first result. The second follows from $\langle f, g\rangle^* = \langle g, f\rangle$. □

Of course, for $n = 1$ every multi-index a satisfies $a = a^T$. Next by taking $b = d = (0, \dots, 0)$ in the above definition (6), we find that

$$\langle \theta^a, \theta^c\rangle = w(a)\delta_{a,c}$$

for all multi-indices a and c. This implies that the sesqui-linear form of $\mathcal{A}$ when restricted to $\mathcal{P}$ is a positive definite inner product. This makes $\mathcal{P}$ into a pre-Hilbert space (which is another motivation for using the notation $\mathcal{P}$). Moreover, the set

$$\Phi := \{\varphi_a = w(a)^{-1/2}\theta^a \mid a \in \mathbb{N}^n\}$$

is an orthonormal set in $\mathcal{P}$, which is also a Hamel basis of $\mathcal{P}$. We let $\mathcal{H}$ denote the Hilbert space completion of the pre-Hilbert space $\mathcal{P}$. Then $\mathcal{H}$ plays the role of the Segal-Bargmann space in this example. See [1].

Definition 1 Define the linear map $P : \mathcal{A} \to \mathcal{A}$ by

$$P(f) := \sum_{c \in \mathbb{N}^n} \langle \varphi_c, f \rangle \, \varphi_c \quad \text{for all } f \in \mathcal{A}, \tag{7}$$

provided that this infinite sum makes sense. Using Dirac notation we have $P = \sum_{c \in \mathbb{N}^n} |\varphi_c\rangle\langle\varphi_c|$, a formal expression for the moment.

It remains to show the infinite sum in (7) converges. First we consider the case $f = \theta^a\overline{\theta}^b$ where a, b are multi-indices. Then for any multi-index c we consider the coefficient of φ_c in (7), namely

$$\langle \varphi_c, f \rangle = \langle \varphi_c, \theta^a\overline{\theta}^b \rangle = w(c)^{-1/2}\langle \theta^c, \theta^a\overline{\theta}^b \rangle = w(c)^{-1/2} w(a) \delta_{c,a-b}. \tag{8}$$

So there is at most one value of the multi-index $c \in \mathbb{N}^n$ for which this is non-zero, namely for $c = a - b$ in the case when $a \geq b$. Of course, $a \geq b$ means that $a_j \geq b_j$ for every $j = 1, \ldots, n$. So (7) makes sense in this case. Since any $f \in \mathcal{A}$ is a finite linear combination of the basis elements $\theta^a\overline{\theta}^b$ it follows that only a finite number of terms in the infinite sum (7) are non-zero.

Now we evaluate the projection map P on a basis element $\theta^a\overline{\theta}^b$ using the result in (8):

$$\begin{aligned} P(\theta^a\overline{\theta}^b) &= \sum_c \langle \varphi_c, \theta^a\overline{\theta}^b \rangle \varphi_c = \sum_c w(c)^{-1/2} w(a)\, \delta_{c,a-b}\, \varphi_c \\ &= w(a-b)^{-1/2} w(a) \varphi_{a-b}, \end{aligned} \tag{9}$$

provided that $a \geq b$. Otherwise, $P(\theta^a\overline{\theta}^b) = 0$ if $a \geq b$ does not hold. Clearly, $P(\varphi) \in \mathcal{P}$ holds for all $\varphi \in \mathcal{A}$ and P, when restricted to the pre-Hilbert space $\mathcal{P}$, is the identity. So P is a projection in the sense that $P^2 = P$ holds. But $P : \mathcal{A} \to \mathcal{A}$ is not always an orthogonal projection, since $\mathcal{A}$ is not necessarily a Hilbert space. However, as the reader can check, $\langle f, Pg \rangle = \langle Pf, g \rangle$ does hold for all $f, g \in \mathcal{A}$.

We are now ready for a central definition.

Definition 2 For any $g \in \mathcal{A}$ the *(right) Toeplitz operator with symbol* g, denoted T_g, is defined for all $\varphi \in \mathcal{P}$ by

$$T_g(\varphi) := P(\varphi g) \in \mathcal{P}.$$

Then the linear map $g \mapsto T_g$ is called the *Toeplitz quantization*.

Clearly, $T_g : \mathcal{P} \to \mathcal{P}$ is a linear map. Left Toeplitz operators are defined by $\hat{T}_g(\varphi) := P(g\varphi) \in \mathcal{P}$. They will not be considered here.

We next define $\mathcal{L}(\mathcal{P}) := \{A : \mathcal{P} \to \mathcal{P} \mid A \text{ is linear}\}$. Since $\mathcal{P}$ is only a pre-Hilbert space, and not a Hilbert space, the set $\mathcal{L}(\mathcal{P})$ is not a standard object studied in analysis. However, it is easy to show that with the standard definitions of addition and scalar multiplication it is a vector space over the complex numbers and that it is closed under composition of mappings. In short, $\mathcal{L}(\mathcal{P})$ is an algebra. Notice that the elements of $\mathcal{L}(\mathcal{P})$ are densely defined linear operators in $\mathcal{H}$, the completion of $\mathcal{P}$, and they may be either bounded or unbounded. Of course, $T_g \in \mathcal{L}(\mathcal{P})$ for every symbol $g \in \mathcal{A}$. Whether T_g is bounded or unbounded is a question for further analysis for every symbol $g \in \mathcal{A}$.

Definition 3 For each $j = 1, \ldots, n$ we define the *creation operator* $A_j^\dagger = T_{\theta_j}$ and the *annihilation operator* $A_j = T_{\overline{\theta}_j}$.

We define the *canonical commutation relation (CCR) algebra* to be the sub-algebra of $\mathcal{L}(\mathcal{P})$ generated by the operators $A_j^\dagger, A_j$ for $1 \le j \le n$. The CCR algebra is a quotient of the free algebra on $2n$ non-commuting variables. The two-sided ideal $\mathcal{R}$ of that quotient map is called the *ideal of canonical commutation relations* and any minimal set of generators of the ideal $\mathcal{R}$ is called a *set of canonical commutation relations*.

Theorem 1 *The creation operator $A_j^\dagger$ for $j = 1, \ldots, n$ is given for every multi-index a by*

$$A_j^\dagger(\varphi_a) = R_j(a) \left(\frac{w(a+\varepsilon_j)}{w(a)} \right)^{1/2} \varphi_{a+\varepsilon_j}, \tag{10}$$

where $R_j(a)$ is given as the product

$$R_j(a) = \prod_{l=j+1}^{n} r_{j,l}^{-a_l} \neq 0. \tag{11}$$

Proof We see for $1 \le j \le n$ that

$$\begin{aligned} A_j^\dagger(\varphi_a) &= T_{\theta_j}(\varphi_a) = P(\varphi_a \theta_j) = \varphi_a \theta_j = w(a)^{-1/2}\, \theta^a \theta_j \\ &= R_j(a)\, w(a)^{-1/2}\, \theta^{a+\varepsilon_j} = R_j(a) \left(\frac{w(a+\varepsilon_j)}{w(a)} \right)^{1/2} \varphi_{a+\varepsilon_j}, \end{aligned} \tag{12}$$

where $\varepsilon_j = (0, \dots, 0, 1, 0, \dots, 0)$, the multi-index with zeros in all of its entries except the j-th entry which is equal to 1. Also, the formula for $R_j(a)$ given as in (11) follows by commuting θ_j past the factors $\theta_l^{a_l}$ for $l = j+1, \dots, n$ using the relation (1). □

In our previous paper [3], where the case $n = 1$ was considered, the factor (11) is not written explicitly since it is 1. As the reader can readily check, the formula (12) implies that the kernel of $A_j^\dagger$ is zero. For the case $j = n$ we have $R_n(a) = 1$ and so

$$A_n^\dagger(\varphi_a) = \left(\frac{w(a+\varepsilon_n)}{w(a)}\right)^{1/2} \varphi_{a+\varepsilon_n},$$

which is more in line with results in previous papers such as [3].

Theorem 2 *The annihilation operator A_j for $j = 1, \dots, n$ is given for every multi-index a by*

$$A_j(\varphi_a) = \begin{cases} \left(\dfrac{w(a)}{w(a-\varepsilon_j)}\right)^{1/2} \varphi_{a-\varepsilon_j} & \textit{if } a_j \geq 1, \\ 0 & \textit{if } a_j = 0. \end{cases} \tag{13}$$

Proof We see by using (9) that

$$A_j(\varphi_a) = T_{\overline{\theta}_j}(\varphi_a) = P(\varphi_a \overline{\theta}_j) = w(a)^{-1/2} P(\theta^a \overline{\theta}_j) = w(a)^{-1/2} P(\theta^a \overline{\theta}^{\varepsilon_j})$$
$$= w(a)^{-1/2}\, w(a-\varepsilon_j)^{-1/2}\, w(a)\, \varphi_{a-\varepsilon_j} = \left(\frac{w(a)}{w(a-\varepsilon_j)}\right)^{1/2} \varphi_{a-\varepsilon_j},$$

provided that $a_j \geq 1$. Otherwise $A_j(\varphi_a) = 0$ for $a_j = 0$. □

In particular, $A_j(1) = 0$ for all $j = 1, \dots, n$. Also, for $n \geq 2$ the kernel of A_j is infinite dimensional.

As the reader may have already realized, these computations imply that for $1 \leq j < n$ the operators $A_j^\dagger$ and A_j are not in general adjoints of each other. For example, consider the case $j = 1$ and $n = 2$. Take $f_1 = \theta_2$, $f_2 = \theta_1\theta_2$ and compute

$$\langle f_1, A_1 f_2 \rangle = \langle \theta_2, P(\theta_1\theta_2\overline{\theta}_1) \rangle = \langle P\theta_2, \theta_1\theta_2\overline{\theta}_1 \rangle = \langle \theta_2, \theta_1\theta_2\overline{\theta}_1 \rangle = w\big((1,1)\big),$$

where we used (6) with multi-indices $a = (0, 1)$, $b = (0, 0)$, $c = (1, 1)$ and $d = (1, 0)$. On the other hand

$$\langle A_1^\dagger f_1, f_2 \rangle = \langle P(f_1\theta_1), f_2 \rangle = \langle \theta_2\theta_1, \theta_1\theta_2 \rangle = \langle r_{12}^{-1}\theta_1\theta_2, \theta_1\theta_2 \rangle = r_{12}^{-1} w\big((1,1)\big),$$

where we used (6) with multi-indices $a = (1, 1)$, $b = (0, 0)$, $c = (1, 1)$ and $d = (0, 0)$. So A_1 and $A_1^\dagger$ are not adjoints exactly when $r_{12} \neq 1$, that is when θ_1 and

θ_2 fail to commute. Basically, the same argument applies for $n > 2$ to any pair of non-commuting θ's. However, for $n = 1$ we do have that A_1 and $A_1^\dagger$ are formal adjoints, that is, $A_1^* \supset A_1^\dagger$ as densely defined operators in $\mathcal{H}$.

We now are going to consider the commutation relations of the creation and annihilation operators. It is a straightforward exercise to prove for the annihilation operators that

$$A_j A_k - A_k A_j = 0 \tag{14}$$

for all j and k. This commutation relation is classical in the sense that it says that A_j and A_k commute. The case of creation operators is decidedly more delicate. For starters we do not wish to fall into the trap of thinking that all we have to do is take the adjoint of the relation (14) for annihilation operators.

Theorem 3 *For $j, k = 1, \ldots, n$ the creation operators $A_j^\dagger$ and $A_k^\dagger$ satisfy*

$$A_j^\dagger A_k^\dagger - r_{jk}^{-1} A_k^\dagger A_j^\dagger = 0 \quad \text{for } j < k \tag{15}$$

and

$$A_j^\dagger A_k^\dagger - r_{jk} A_k^\dagger A_j^\dagger = 0 \quad \text{for } j > k, \tag{16}$$

where r_{jk} is the non-zero complex number in (1).

Proof Clearly, explicit computations are called for. Suppose first that $j < k$. Then using (12) twice we see that

$$\begin{aligned} A_j^\dagger A_k^\dagger(\varphi_a) &= R_k(a) \left(\frac{w(a+\varepsilon_k)}{w(a)} \right)^{1/2} A_j^\dagger(\varphi_{a+\varepsilon_k}) \\ &= R_k(a) \left(\frac{w(a+\varepsilon_k)}{w(a)} \right)^{1/2} R_j(a+\varepsilon_k) \left(\frac{w(a+\varepsilon_k+\varepsilon_j)}{w(a+\varepsilon_k)} \right)^{1/2} \varphi_{a+\varepsilon_k+\varepsilon_j} \\ &= R_k(a)\, R_j(a+\varepsilon_k) \left(\frac{w(a+\varepsilon_k+\varepsilon_j)}{w(a)} \right)^{1/2} \varphi_{a+\varepsilon_k+\varepsilon_j}. \end{aligned}$$

Taken in the other order, we calculate again for $j < k$ that

$$\begin{aligned} A_k^\dagger A_j^\dagger(\varphi_a) &= R_j(a) \left(\frac{w(a+\varepsilon_j)}{w(a)} \right)^{1/2} A_k^\dagger(\varphi_{a+\varepsilon_j}) \\ &= R_j(a) \left(\frac{w(a+\varepsilon_j)}{w(a)} \right)^{1/2} R_k(a+\varepsilon_j) \left(\frac{w(a+\varepsilon_j+\varepsilon_k)}{w(a+\varepsilon_k)} \right)^{1/2} \varphi_{a+\varepsilon_j+\varepsilon_k} \\ &= R_j(a)\, R_k(a+\varepsilon_j) \left(\frac{w(a+\varepsilon_j+\varepsilon_k)}{w(a)} \right)^{1/2} \varphi_{a+\varepsilon_j+\varepsilon_k}. \end{aligned}$$

So it comes down to evaluating the four R factors. Note that $R_l(b)$ for a multi-index $b = (b_1, \ldots, b_n)$ depends only on the entries $b_{l+1}, \ldots, b_n$ in a right tail of the multi-index b. Using the hypothesis $j < k$ we find that

$$R_k(a + \varepsilon_j) = R_k(a) \quad \text{and} \quad R_j(a + \varepsilon_k) = r_{jk}^{-1} R_j(a).$$

It follows for $j < k$ that $A_j^\dagger A_k^\dagger - r_{jk}^{-1} A_k^\dagger A_j^\dagger = 0$ which is (15). Multiplying this by $-r_{jk}$ and interchanging subscripts yields (16). □

Equation (15) is to be understood as a classical commutativity relation (which means that the corresponding element in the ideal of CCR in the free algebra on $2n$ generators is homogeneous, in this case of degree 2), even though (15) does not necessarily give the commutativity of the operators $A_j^\dagger$ and $A_k^\dagger$. Notice that in general (15) is *not* the same relation as the defining classical relation for the algebra $\mathcal{A}$, which is

$$\theta_j \theta_k - r_{jk} \theta_k \theta_j = 0 \quad \text{for } j < k.$$

The remaining commutation relations are for A_j and $A_k^\dagger$. Here are the relevant formulas for $j, k \in \{1, \ldots, n\}$ obtained from (10) and (13):

$$A_j A_k^\dagger \varphi_a = R_k(a) \frac{w(a + \varepsilon_k)}{\big(w(a) w(a + \varepsilon_k - \varepsilon_j)\big)^{1/2}} \varphi_{a+\varepsilon_k-\varepsilon_j},$$

and

$$A_k^\dagger A_j \varphi_a = R_k(a - \varepsilon_j) \frac{\big(w(a) w(a + \varepsilon_k - \varepsilon_j)\big)^{1/2}}{w(a - \varepsilon_j)} \varphi_{a+\varepsilon_k-\varepsilon_j}.$$

However, unless the weights satisfy some extra restrictions, there seems to be nothing simple to be said in general about the commutator $[A_j, A_k^\dagger]$ in this case, not even when $j = k$. This is unfortunate since it is precisely with these commutation relations that there is the possibility of finding non-zero 'quantum correction' terms that would include Planck's constant $\hbar$. See [5] for the details on how this is done.

3 Concluding Remarks

While the example in this paper does not satisfy all the properties of the general theory in [3], it still has reasonable Toeplitz operators, including the creation and annihilation operators. So the CCR algebra in this example, while difficult to describe completely, does provide an interesting quantum theory which merits further study. Moreover, by putting all $r_{ij} = 1$ in the relations (1), we get an example more in line with the general theory. But the algebra still is not factorizable as a product of $2n$ sub-algebras, each generated by one element, due to the remaining non-trivial relations (2).

There are many other papers that consider Toeplitz quantization with non-commutative symbols. See [6] for references to that literature. It is worth noting that the Toeplitz quantization presented here does not use a measure. Also, it does not start out with some abstract commutation relations which one tries to realize as operators acting in a Hilbert space. Rather here, the commutation relations arise naturally within the context of the theory.

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The Weyl–Wigner–Moyal Formalism on a Discrete Phase Space

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Abstract A phase space approach to systems with both: classical degrees of freedom and purely quantum discrete ones is discussed. Formulas for the Stratonovich–Weyl quantizer and star product for such systems are proposed. The Wigner function, its properties and the time evolution are presented.

Keywords Quantum mechanics in continuous and discrete phase space · Wigner function · Spin

Mathematics Subject Classification (2010) 81S30

1 Introduction

The Hilbert space formulation of quantum mechanics constitutes mathematical frames for description of physical phenomena in systems localized in configuration space $\mathbb{R}^n$. For such systems effects related to discrete internal degrees of freedom, especially spin, are also modeled in some Hilbert spaces. The crucial difference between the Hilbert space applied to represent quantities with classical counterparts and the space to model internal properties lies in the dimension of space. While position, momentum, angular momentum etc are represented by linear operators acting in infinite dimensional separable Hilbert spaces, spin is represented in a finite dimensional Hilbert space.

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Problems appear when one tries to a phase space counterpart of model of quantum system. Representation of classical degrees of freedom modeled in the Hilbert space formulation on infinite dimensional space is well known (for review see [1, 6]) and there are some indications suggesting what to do with non classical discrete degrees of freedom (see Refs. [5–40] in [3]).

Our contribution is devoted to this question. First we perform construction of lattice $\Gamma^{(s+1)}$ being a phase space analog of $(s+1)$-dimensional Hilbert space $\mathcal{H}^{(s+1)}$. Then we discuss physically acceptable correspondences between operators acting in $\mathcal{H}^{(s+1)}$ and functions on grids $\Gamma^{(s+1)}$ and star products representing multiplication of operators on $\Gamma^{(s+1)}$. Finally we consider representation of states on this lattice building respective Wigner functions.

The current article is based on our paper [4].

2 The Stratonovich–Weyl Quantizer for Systems with Phase Space $\mathbb{R}^{2n}$

In order to deduce a solution of problem of representation quantum internal degrees of freedom on a phase space we will analyse a relationship between phase space description and Hilbert space model for degrees of freedom having classical counterparts first.

Let a Hilbert space of our system be a space isomorphic to the space of square integrable functions $L^2(\mathbb{R})$. As a basis of it one can choose a system of orthonormal generalised vectors

$$\langle q|q'\rangle = \delta(q'-q), \quad q, q' \in \mathbb{R}. \tag{1}$$

In physics these objects can be identified with eigenstates of operator of position but this fact is irrelevant to our construction. Strictly speaking kets $|q\rangle$, $|q'\rangle$ are not elements of space $L^2(\mathbb{R})$. However, it is well known that they may be used in order to decompose functions from $L^2(\mathbb{R})$ i.e.

$$L^2(\mathbb{R}) \ni |\Psi\rangle = \int_{\mathbb{R}} \langle q|\Psi\rangle |q\rangle dq.$$

An alternative decomposition can be done with the use of vectors

$$\begin{aligned} |p\rangle &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ipx}{\hbar}\right) |q\rangle dq, \\ \langle p|p'\rangle &= \delta(p'-p), \quad p, p' \in \mathbb{R}. \end{aligned} \tag{2}$$

Analogously

$$L^2(\mathbb{R}) \ni |\Psi\rangle = \int_{\mathbb{R}} \langle p|\Psi\rangle |p\rangle dp$$

and

$$\langle q|\Psi\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{\mathbb{R}} \exp\left(\frac{ipx}{\hbar}\right) \langle p|\Psi\rangle dp.$$

Collecting operators of projection on all directions $|q\rangle,\ q \in \mathbb{R}$ and $|p\rangle,\ p \in \mathbb{R}$ we introduce self-adjoint operators

$$\hat{q} = \int_{\mathbb{R}} q|q\rangle dq\langle q| \quad , \quad \hat{p} = \int_{\mathbb{R}} p|p\rangle dp\langle p| \tag{3}$$

with the commutation relation

$$[\hat{q}, \hat{p}] = i\hbar\hat{\mathbf{1}}.$$

Applying these operators we build two families of unitary operators:

$$\exp(i\lambda\hat{p}) \quad \text{and} \quad \exp(i\mu\hat{q}), \quad \lambda, \mu \in \mathbb{R} \tag{4}$$

indexed by real parameters λ, μ.

They satisfy the following commutation rule

$$\exp\left(-\frac{i\hbar\lambda\mu}{2}\right)\exp(i\lambda\hat{p})\exp(i\mu\hat{q}) = \exp\left(\frac{i\hbar\lambda\mu}{2}\right)\exp(i\mu\hat{q})\exp(i\lambda\hat{p}). \tag{5}$$

As it is easy to calculate, expressions standing at both sides of equality in formula (5) are equal to the series $\exp\{i(\lambda\hat{p} + \mu\hat{q})\}$. To shorten notation we will simply write

$$\hat{\mathcal{U}}(\lambda, \mu) := \exp\{i(\lambda\hat{p} + \mu\hat{q})\}.$$

One can establish a correspondence between some linear operators acting in the Hilbert space $L^2(\mathbb{R})$ and functions on $\mathbb{R}^2$. This correspondence is of the form

$$f(p, q) = \frac{\hbar}{2\pi} \int_{\mathbb{R}\times\mathbb{R}} d\lambda d\mu \mathcal{P}^{-1}\left(\frac{\hbar\lambda\mu}{2}\right) \exp\{i(\lambda p+\mu q)\}\, \mathrm{Tr}\left\{\widehat{f}\widehat{\mathcal{U}}^{+}(\lambda, \mu)\right\}. \tag{6}$$

As one can see from (6) there exist several correspondences. They are determined by a function $\mathcal{P}\left(\frac{\hbar\lambda\mu}{2}\right)$ which refers to the operator ordering [2, 5]. For example if $\mathcal{P} = 1$ then we deal with the Weyl ordering. For the symmetric ordering $\mathcal{P} = \cos\left(\frac{\hbar\lambda\mu}{2}\right)$.

Since in formula (6) the inverse of $\mathcal{P}\left(\frac{\hbar\lambda\mu}{2}\right)$ appears, it must be different from zero almost everywhere.

Relation (6) indicates that the phase space used for representation of classical degrees of freedom is $\mathbb{R}^2$.

This observation can be easily generalised for the Hilbert space $L^2(\mathbb{R}^n)$ and the phase space $\mathbb{R}^{2n}$. There exists a formula inverse to (6)

$$\widehat{f} = \frac{1}{2\pi\hbar}\int_{\mathbb{R}^2} dp dq f(p,q)\widehat{\Omega}[\mathcal{P}](p,q), \tag{7}$$

where

$$\widehat{\Omega}[\mathcal{P}](p,q) := \frac{\hbar}{2\pi}\int_{\mathbb{R}^2} d\lambda d\mu \mathcal{P}\left(\frac{\hbar\lambda\mu}{2}\right)\exp\{-i(\lambda p + \mu q)\}\hat{\mathcal{U}}(\lambda,\mu) \tag{8}$$

denotes the Stratonovich–Weyl quantizer. Thus indeed for classical degrees of freedom the Hilbert space formulation and the phase space description are equivalent and the explicit form of this equivalence is given by expressions (6) and (7).

Formulas derived in this section can be easily generalised for the Hilbert space of functions $L^2(\mathbb{R}^n)$ and its respective phase space $\mathbb{R}^{2n}$.

3 Representation of Discrete Degrees of Freedom

Let us assume that a quantum system under consideration is characterised also by internal discrete degrees of freedom. Thus in order to deal with effects referring to those internal features we need to introduce a finite dimensional Hilbert space $\mathcal{H}^{(s+1)} \cong \mathbb{C}^{s+1}$, where $s+1,\ s = 0, 1, \ldots$ is equal to the number of degrees of freedom.

Following the pattern introduced in the previous section we construct an orthonormal basis

$$\{|0\rangle, |1\rangle, \ldots, |s\rangle\}, \quad \langle n|n'\rangle = \delta_{nn'}, \quad n, n' = 0, 1, \ldots, s \tag{9}$$

in $\mathcal{H}^{(s+1)}$. But this Hilbert space can be spanned by another system of vectors

$$\begin{gathered} |\phi_m\rangle := \frac{1}{\sqrt{s+1}}\sum_{n=0}^{s}\exp(in\phi_m)|n\rangle, \\ \langle\phi_m|\phi_{m'}\rangle = \delta_{mm'} \quad , \quad m, m' = 0, 1, \ldots, s \end{gathered} \tag{10}$$

with

$$\phi_m = \phi_0 + \frac{2\pi}{s+1}m, \quad m = 0, 1, \ldots, s, \quad \phi_0 \in \mathbb{R}$$

which is also a basis.

Having at disposal two hermitian operators

$$\hat{n} := \sum_{n=0}^{s} n|n\rangle\langle n| \quad , \quad \hat{\phi} := \sum_{m=0}^{s} \phi_m |\phi_m\rangle\langle\phi_m| \tag{11}$$

we construct families of unitary operators:

$$\hat{V} := \exp\left(i\frac{2\pi}{s+1}\hat{n}\right) \tag{12}$$

satisfying property $\hat{V}^{s+1} = \hat{\mathbf{1}}$ and

$$\hat{U} := \exp(i\hat{\phi}) \tag{13}$$

fulfilling the equality $\hat{U}^{s+1} = \exp\Big\{i(s+1)\phi_0\Big\}\hat{\mathbf{1}}$.

These operators obey the commutation relation

$$\exp\left(-i\frac{\pi kl}{s+1}\right)\hat{U}^k\hat{V}^l = \exp\left(i\frac{\pi kl}{s+1}\right)\hat{V}^l\hat{U}^k, \quad k, l \in \mathbb{Z}. \tag{14}$$

In order to simplify formulas we put

$$\hat{\mathcal{D}}(k, l) := \exp\left(i\frac{\pi kl}{s+1}\right)\hat{V}^l\hat{U}^k.$$

Now we are able to assign a function $f(\phi_m, n)$ of two discrete real arguments ϕ_m and n to some operator $\hat{f}$ acting in the Hilbert space $\mathcal{H}^{(s+1)}$ by relation

$$f(\phi_m, n) = \frac{1}{s+1}\sum_{k,l=0}^{s}\mathcal{K}^{-1}(k,l)\exp\left\{i\left(k\phi_m + \frac{2\pi}{s+1}ln\right)\right\} \times \mathrm{Tr}\left\{\hat{f}\,\hat{\mathcal{D}}^{+}(k,l)\right\}. \tag{15}$$

Function $f(\phi_m, n)$ is defined on a discrete phase space (a grid) $\big\{(\phi_m, n)\big\}_{m,n=0}^{s}$ denoted by $\Gamma^{(s+1)}$. Therefore we see that a phase space counterpart of $\mathcal{H}^{(s+1)}$ is a lattice $\Gamma^{(s+1)}$. $\mathcal{K}(k, l)$ plays the same role as $\mathcal{P}\left(\frac{\hbar\lambda\mu}{2}\right)$ in the continuous case and is responsible for the choice of ordering.

The inverse formula is of the form

$$\hat{f} = \frac{1}{s+1}\sum_{m,n=0}^{s} f(\phi_m, n)\hat{\Omega}[\mathcal{K}](\phi_m, n), \tag{16}$$

where the Stratonovich–Weyl quantizer is given by

$$\hat{\Omega}[\mathcal{K}](\phi_m, n) := \frac{1}{s+1}\sum_{k,l=0}^{s} \mathcal{K}(k,l)\hat{\mathcal{D}}(k,l)\exp\left\{-i\left(k\phi_m + \frac{2\pi}{s+1}ln\right)\right\}. \tag{17}$$

Let us discuss some properties of function $\mathcal{K}(k,l)$ which is called a kernel. For simplicity we assume that the kernel depends on the product of numbers k and l so we put $\mathcal{K}\left(\frac{\pi kl}{s+1}\right)$.

Moreover to assure existence of a one to one correspondence between functions and operators there must be

$$\mathcal{K}\left(\frac{\pi kl}{s+1}\right) \neq 0 \;\; \forall k,l \in \{0,\ldots,s\}. \tag{18}$$

For any function f on which depends on one variable i.e. $f = f(\phi_m)$ or $f = f(n)$ the associated operator should be of the form $\widehat{f} = f(\widehat{\phi})$ or $\widehat{f} = f(\widehat{n})$, respectively. Therefore

$$\mathcal{K}(0) = 1. \tag{19}$$

One expects that for any real function $f(p,q,\phi_m,n)$ the corresponding operator $\widehat{f}$ is Hermitian. Thus

$$\mathcal{K}^*\left(\frac{\pi kl}{s+1}\right) = (-1)^{s+1-k-l}\mathcal{K}\left(\frac{\pi(s+1-k)(s+1-l)}{s+1}\right), \quad 1 \leq k,l \leq s \tag{20}$$

and $\mathcal{K}^*(0) = \mathcal{K}(0)$ which is in agreement with condition (19).

Sometimes one adds the constraint

$$\mathrm{Tr}\left\{\widehat{\Omega}[\mathcal{K}](\phi_m,n)\widehat{\Omega}[\mathcal{K}](\phi_{m'},n')\right\} = (s+1)\delta_{mm'}\delta_{nn'}$$

implying

$$\left|\mathcal{K}\left(\frac{\pi kl}{s+1}\right)\right| = 1 \;\; \forall \;\; 0 \leq k,l \leq s. \tag{21}$$

Lets us take a look at some possible choices of the kernel. Please notice that one cannot put $\mathcal{K}\left(\frac{\pi kl}{s+1}\right) = 1$ for all $0 \leq k,l \leq s$.

Thus the simplest acceptable form of the kernel seems to be $\mathcal{K}(0) = 1$ and $\mathcal{K}\left(\frac{\pi kl}{s+1}\right) = \pm 1$ for $kl \neq 0$. These requirements are fulfilled e.g. when $\mathcal{K}\left(\frac{\pi kl}{s+1}\right) = (-1)^{kl}$ for $s+1$ being an odd number. The case when $s+1$ is an even number, is more complicated.

Putting together results obtained for continuous and discrete degrees of freedom one can observe that expressions relating operators from the Hilbert space $L^2(\mathbb{R}) \otimes \mathcal{H}^{(s+1)}$ and functions on the phase space $\mathbb{R} \times \mathbb{R} \times \Gamma^{(s+1)}$ are given by the following formulas

$$f(p, q, \phi_m, n) = \frac{\hbar}{2\pi} \frac{1}{s+1} \sum_{k,l=0}^{s} \int_{\mathbb{R}\times\mathbb{R}} d\lambda d\mu \left(\mathcal{P}\left(\frac{\hbar\lambda\mu}{2}\right) \mathcal{K}\left(\frac{\pi kl}{s+1}\right)\right)^{-1}$$
$$\times \exp\{i(\lambda p + \mu q)\} \exp\left\{ i\frac{2\pi}{s+1}(km + ln)\right\} \mathrm{Tr}\left\{\widehat{f}\widehat{\mathcal{U}}^{+}(\lambda, \mu)\widehat{\mathcal{D}}^{+}(k, l)\right\} \qquad (22)$$

and

$$\widehat{f} = \frac{1}{(2\pi)^2(s+1)^2} \sum_{k,l,m,n=0}^{s} \int_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}} d\lambda d\mu dp dq \mathcal{P}\left(\frac{\hbar\lambda\mu}{2}\right) \mathcal{K}\left(\frac{\pi kl}{s+1}\right) \times$$
$$\exp\{-i(\lambda p + \mu q)\} \exp\left\{ -i\frac{2\pi}{s+1}(km + ln)\right\} f(p, q, \phi_m, n)\widehat{\mathcal{U}}(\lambda, \mu)\widehat{\mathcal{D}}(k, l). \qquad (23)$$

4 Star Product on the Grid $\Gamma^{(s+1)}$

An interesting question is construction of star product on a phase space mixing both continuous and discrete degrees of freedom. As an object of study we choose a spin $\frac{1}{2}$ nonrelativistic particle. The orthonormal basis $\{|n\rangle\}_{n=0,1}$ in $\mathcal{H}^{(2)}$ is spanned by eigenvectors of third component of spin i.e.

$$\widehat{\sigma}_3|0\rangle = 1 \cdot |0\rangle, \quad \widehat{\sigma}_3|1\rangle = -1 \cdot |1\rangle.$$

The phase space representation of our quantum system is the Cartesian product $\mathbb{R}^3 \times \mathbb{R}^3 \times \{(\phi_m, n)\}_{m,n=0,1}$. Therefore now $p = (p_1, p_2, p_3)$, $q = (q_1, q_2, q_3)$, $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and $\mu = (\mu_1, \mu_2, \mu_3)$. A scalar product is denoted by '$\cdot$'.

As kernels we choose

$$\mathcal{P}\left(\frac{\hbar\,\lambda \cdot \mu}{2}\right) = 1, \quad \mathcal{K}\left(\frac{\pi kl}{2}\right) = (-1)^{kl}, \qquad k, l = 0, 1. \qquad (24)$$

Then the $*$- product of two functions f and g has the following form

$$
\begin{aligned}
&(f * g)(p, q, \phi_m, n) \\
&\quad = \frac{1}{16} \sum_{m', n', m'', n''=0}^{1} f(p, q, \phi_{m'}, n') \exp\left\{ \frac{i\hbar}{2} \overleftrightarrow{\mathcal{P}} \right\} g(p, q, \phi''_m, n'') \\
&\quad \times \Big\{ (1 + (-1)^{m'+m''})(1 + (-1)^{n'+n''}) + (-1)^m((-1)^{m'} + (-1)^{m''}) \\
&\quad + (-1)^{m+n}((-1)^{m'+n'} + (-1)^{m''+n''}) + (-1)^n((-1)^{n'} + (-1)^{n''}) \\
&\quad + i\Big[(-1)^m(-1)^{n'+n''}((-1)^{m'} - (-1)^{m''}) + (-1)^{m+n}((-1)^{m''+n'} \\
&\qquad - (-1)^{m'+n''}) + (-1)^n(-1)^{m'+m''}((-1)^{n''} - (-1)^{n'}) \Big] \Big\} . \qquad (25)
\end{aligned}
$$

The continuous component of this product is of course the Moyal product where $\overleftrightarrow{\mathcal{P}}$ denotes the Poisson operator which for systems of 3-dimensional configuration space equals

$$
\overleftrightarrow{\mathcal{P}} := \sum_{j=1}^{3} \left(\frac{\overleftarrow{\partial}}{\partial q_j} \frac{\overrightarrow{\partial}}{\partial p_j} - \frac{\overleftarrow{\partial}}{\partial p_j} \frac{\overrightarrow{\partial}}{\partial q_j} \right).
$$

5 Representation of States on the Discrete Phase Space

In the Hilbert space formulation of quantum mechanics information about a state of system is encoded in a density operator $\widehat{\rho}$. Thus the average value of an observable $\widehat{f}$ is calculated as

$$
\langle \widehat{f} \rangle = \mathrm{Tr}\{\widehat{f}\,\widehat{\rho}\} \qquad (26)
$$

Since the state is determined by an operator we can easily find its phase space analog with the use of correspondence (22).

We define the Wigner function of the state $\widehat{\rho}$ in 3-D space associated to the kernels $(\mathcal{P}, \mathcal{K})$ as

$$
\rho_W[\mathcal{P}, \mathcal{K}](p, q, \phi_m, n) := \frac{1}{(2\pi\hbar)^3(s+1)} \mathrm{Tr}\left\{ \widehat{\rho}\,\widehat{\Omega}[\mathcal{P}, \mathcal{K}](p, q, \phi_m, n) \right\} .
$$

Consequently, the mean value of a function $f(p, q, \phi_m, n)$ can be found from the formula

$$\langle f(p, q, \phi_m, n)\rangle = \sum_{m,n=0}^{s} \int_{\mathbb{R}^3 \times \mathbb{R}^3} dpdqf(p, q, \phi_m, n)\rho_W[\mathcal{P}, \mathcal{K}](p, q, \phi_m, n).$$

Let us mention some properties of Wigner function. Like density of probability, it is a real function

$$\rho_W^*[\mathcal{P}, \mathcal{K}] = \rho_W[\mathcal{P}, \mathcal{K}] \tag{27}$$

Moreover it is normalised in a sense that its trace equals 1

$$\sum_{m,n=0}^{s} \int_{\mathbb{R}^3 \times \mathbb{R}^3} dpdq\rho_W[\mathcal{P}, \mathcal{K}](p, q, \phi_m, n) = \mathrm{Tr}\{\widehat{\rho}\} = 1. \tag{28}$$

Although the Wigner function itself is not a density probability, it leads to marginal distributions being true densities of probability

$$\begin{aligned}
\sum_{m,n=0}^{s} \int_{\mathbb{R}^3} dp\rho_W[\mathcal{P}, \mathcal{K}](p, q, \phi_m, n) &= \mathrm{Tr}\{\widehat{\rho}|q\rangle\langle q|\}, \\
\sum_{m,n=0}^{s} \int_{\mathbb{R}^3} dq\rho_W[\mathcal{P}, \mathcal{K}](p, q, \phi_m, n) &= \mathrm{Tr}\{\widehat{\rho}|p\rangle\langle p|\}, \\
\sum_{m=0}^{s} \int_{\mathbb{R}^3 \times \mathbb{R}^3} dpdq\rho_W[\mathcal{P}, \mathcal{K}](p, q, \phi_m, n) &= \mathrm{Tr}\{\widehat{\rho}|n\rangle\langle n|\}, \\
\sum_{n=0}^{s} \int_{\mathbb{R}^3 \times \mathbb{R}^3} dpdq\rho_W[\mathcal{P}, \mathcal{K}](p, q, \phi_m, n) &= \mathrm{Tr}\{\widehat{\rho}|\phi_m\rangle\langle \phi_m|\}.
\end{aligned} \tag{29}$$

On the other hand, the time evolution for the Wigner function $\rho_W[\mathcal{P}, \mathcal{K}](p, q, \phi_m, n; t)$ reads

$$\frac{\partial \rho_W[\mathcal{P}, \mathcal{K}]}{\partial t} + \frac{1}{i\hbar}\Big(\rho_W[\mathcal{P}, \mathcal{K}] * H - H * \rho_W[\mathcal{P}, \mathcal{K}]\Big) = 0, \tag{30}$$

where the Hamiltonian $H = H(p, q, \phi_m, n)$ is defined as

$$H(p, q, \phi_m, n) = \mathrm{Tr}\left\{\widehat{H}\widehat{\Omega}[\mathcal{P}, \mathcal{K}](p, q, \phi_m, n)\right\}.$$

Formula (30) is known as the Liouville-von Neumann-Wigner equation.

6 Conclusions

We show that it is possible to build phase space version of non relativistic quantum mechanics also for discrete purely microscopic degrees of freedom like spin. But in this case the phase space is not a symplectic differentiable manifold but a lattice. Physical applications of the method can be found in our paper [4], where we calculate the Landau levels, the corresponding Wigner functions for a spin $\frac{1}{2}$ particle and the magnetic resonance for a spin $\frac{1}{2}$ uncharged particle.

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Algebraic Geometric Properties of Spectral Surfaces of Quantum Integrable Systems and Their Isospectral Deformations

Alexander Zheglov

Abstract The aim of this work is to collect all known, recently discovered and also conjectured properties of spectral surfaces of two-dimensional quantum integrable systems and their isospectral deformations. The problem of classification of such systems or the problem of finding new examples of such systems is thus reduced to a problem of finding algebraic projective surfaces with special properties.

Keywords Commuting differential operators · Quantum integrable systems · Moduli space of coherent sheaves

Mathematics Subject Classification (2010) Primary 13N15, 37K20; Secondary 14H70

1 Introduction

In [1] the quantum analogue of the classical definition of an integrable Hamiltonian system was defined. By a Quantum Completely Integrable System (QCIS) on an algebraic variety X the authors understand a pair (Λ, θ), where Λ is an irreducible n-dimensional affine algebraic variety, and $\theta : \mathcal{O}_\Lambda \to D(X)$ is an embedding of algebras (here the algebra $D(X)$ of differential operators on X is the quantum analogue of the Poisson algebra $\mathcal{O}(T^*X)$).

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By definition, a QCIS $S = (\Lambda, \theta)$ is said to be algebraically integrable if it is dominated by another QCIS S' with $\mathrm{rk}(S') = 1$ (see loc. cit.), where the rank of QCIS is the dimension of the space of formal solutions of the system

$$\theta(g)\psi = g(\lambda)\psi, \quad g \in \mathcal{O}_\Lambda$$

near a generic point of X. In [1] these definitions were also generalized to the case of integrable systems on a formal polydisc. Thus, in this case X is $\mathrm{Spec}(K[[x_1, x_2, \dots, x_n]])$ and the symbols $\mathcal{O}_X$, $K(X)$, $D(X)$ denote respectively $K[[x_1, \dots, x_n]]$, $K((x_1, \dots, x_n))$, $\mathcal{O}_X[\partial_1, \dots, \partial_n]$, where $\partial_i = \partial/\partial x_i$.

Recall that for a commutative K-algebra R the filtered ring $D(R)$ is generated by $Der_K(R)$ and R inside the ring $\mathrm{End}_K(R)$

$$(D(R))_0 \subset (D(R))_1 \subset (D(R))_2 \subset \dots, \quad (D(R))_i \cdot (D(R))_j \subset (D(R))_{i+j},$$

where $(D(R))_i$ are defined inductively. So, QCIS for a formal polydisc are just subrings of commuting operators in $\mathcal{O}_X[\partial_1, \dots, \partial_n]$. The case $n = 1$ is well known: the theory of commuting *ordinary differential* operators has a rich history; in mathematical physics it appears as an algebro-geometric tool in the theory of integrating non-linear soliton systems and the spectral theory of periodic finite-zone operators (see [5, 6, 9]). The theory of commuting *partial differential* operators is much more complicated and is not yet completed, though many articles have been published on this theme.

In this paper we briefly review all already known, recently discovered and also some conjectured properties of *spectral surfaces* of *two-dimensional* quantum integrable systems and their isospectral deformations. Unlike the theory in dimension one, there are strong restrictions on the geometry of algebro-geometric spectral data of commutative subrings. Consequently, the problem of classification of QCIS or the problem of finding new examples of such systems can be reformulated as a problem of finding algebraic projective surfaces with special properties.

This short review is based on earlier works on this theme [2, 11, 13, 21–23], and on two works in preparation [12, 19].

Let's recall some important points from these works. Investigating the theory of commuting operators with scalar coefficients, in [21] the author offered an analogue of the Krichever classification theorem for commutative subalgebras in a certain completion $\hat{D}_2$ of the algebra of partial differential operators in two variables. In the approach of [21] the subalgebras in $\hat{D}_2$ appears quite naturally, in particular as isospectral deformations of subalgebras in D_2 (unlike the theory in dimension one, where isospectral deformations still belong to the same ring of differential operators).

However, there are several versions of completion. In most papers cited above a non-symmetric version $\hat{D}_2$ was used. The advantage of this version is the existence of analogues of the Schur theory, an important tool of the classification theory. The classification offered in [21] dealt with subrings $B \subset \hat{D}_2$ satisfying certain mild conditions, and these subrings are classified in terms of certain geometric spectral data, but the conditions and the definition of spectral data are quite technically difficult. With the help of special properties of spectral data investigated in [13, 23], a refined classification of rank one subrings in $\hat{D}_2$ (for an appropriately defined notion of rank) was proved in [22]. In that refinement the definition of spectral data becomes simpler and leads to a purely geometric notion of pre-spectral data.

On the other hand, there is a “symmetric” version of completion introduced in [2, Def. 5.1], which is more convenient in some cases (e.g. for finding explicit iso-spectral deformations, see [2, §6] and Sect. 7), and which contains the non-symmetric one. In this paper we present the following result: the refined classification of rank one subrings can be extended to certain rank one subrings (including the old ones, and satisfying weaker conditions) and even rank r subrings in the symmetric version of completion, see Sect. 7. The final classification is somewhat simpler than in [22, Th. 2.1], however the price is a hidden huge group of units. We expect that this is a final most convenient form of classification theorems. It helps to investigate further properties of spectral surfaces discussed in Sect. 8.

The paper is organized as follows. In Sect. 2 we recall the definition of the “symmetric” version of completion $\hat{D}_n^{\mathrm{sym}}$ of the ring of partial differential operators, and its basic properties. In Sect. 3 we recall the non-symmetric version of completion. In Sect. 4 we recall the notion of spectral module and its basic properties. In Sect. 5 we describe 1-quasi-elliptic subrings—commutative subrings in $\hat{D}_n^{\mathrm{sym}}$ that admit effective description in terms of algebro-geometric spectral data. These rings are subrings in $\hat{D}_n$, but considered up to conjugation by a unity in $\hat{D}_n^{\mathrm{sym}}$. In Sect. 6 we describe basic algebraic properties of quasi-elliptic rings. In Sect. 7 we present new form of the classification theorem of 1-quasi-elliptic rings in $\hat{D}_2^{\mathrm{sym}}$, recall two explicit examples of such rings and their isospectral deformations, and describe the necessary conditions that the spectral data of the rings in D_2 satisfy. Conjecturally, these conditions are also sufficient (see Sect. 7.1). In Sect. 8 we present several results and conjectures about normal forms, i.e. *normal* spectral surfaces from classification Theorem 27.

Everywhere in this paper we assume that K is a field of characteristic zero.

2 The Ring $\hat{D}_n^{\mathrm{sym}}$ and Its Order Function ord

In this subsection we define a symmetric version of completion of the algebra of PDOs $D_n = K[[x_1, \dots, x_n]][\partial_1, \dots, \partial_n]$. It can be thought of as a simple purely algebraic analogue of the algebra of (analytic) pseudodifferential operators on a manifold. It was defined first in [2]. Here we'll use a slightly different notation than in loc. cit.

Denote $\hat{R} := K[[x_1, \dots, x_n]]$. Consider the K-vector space

$$\mathcal{M} := \hat{R}[[\partial_1, \dots, \partial_n]] = \left\{ \sum_{\underline{k} \geq \underline{0}} a_{\underline{k}} \underline{\partial}^{\underline{k}} \;\middle|\; a_{\underline{k}} \in \hat{R} \quad \text{for all } \underline{k} \in \mathbb{N}_0^n \right\},$$

where $\underline{k}$ is the multi-index, $\partial^{\underline{k}} = \partial_1^{k_1} \dots \partial_n^{k_n}$, and $\underline{k} \geq \underline{0}$ means that $k_i \geq 0$ for all $1 \leq i \leq n$.

Let $\upsilon : \hat{R} \to \mathbb{N}_0^n \cup \infty$ be the discrete valuation defined by the unique maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$ of $\hat{R}$. Denote by $|\underline{k}| = k_1 + \dots + k_n$.

Definition 1 For any element $0 \neq P := \sum_{\underline{k} \geq \underline{0}} a_{\underline{k}} \partial^{\underline{k}} \in \mathcal{M}$ we define its *order* to be

$$\mathbf{ord}(P) := \sup \left\{ |\underline{k}| - \upsilon(a_{\underline{k}}) \;\middle|\; \underline{k} \in \mathbb{N}_0^n \right\} \in \mathbb{Z} \cup \{\infty\}, \tag{1}$$

and define $\mathbf{ord}(0) := -\infty$. Define

$$\hat{D}_n^{\mathrm{sym}} := \left\{ Q \in \mathcal{M} \;\middle|\; \mathbf{ord}(Q) < \infty \right\}.$$

Let $P \in \hat{D}_n^{\mathrm{sym}}$. Then we have uniquely determined $\alpha_{\underline{k},\underline{i}} \in K$ such that

$$P = \sum_{\underline{k},\underline{i} \geq \underline{0}} \alpha_{\underline{k},\underline{i}} \, \underline{x}^{\underline{i}} \underline{\partial}^{\underline{k}}. \tag{2}$$

For any $m \geq -d$ we put:

$$P_m := \sum_{|\underline{i}| - |\underline{k}| = m} \alpha_{\underline{k},\underline{i}} \, \underline{x}^{\underline{i}} \underline{\partial}^{\underline{k}}$$

to be the m-th *homogeneous component* of P. Note that $\mathbf{ord}(P_m) = -m$ and we have a decomposition $P = \sum_{m=-d}^{\infty} P_m$.

Remark 2 Note that for a partial differential operator P with *constant* highest symbol the order $\mathbf{ord}(P)$ and the usual order coincide.

Definition 3 Define the *highest symbol* of $P \in \hat{D}_n^{\mathrm{sym}}$ as $\sigma(P) := P_{\mathbf{ord}(P)} = P_{-d}$. We say that $P \in \hat{D}_n^{\mathrm{sym}}$ is *homogeneous* if $P = \sigma(P)$.

Theorem 4 ([2, Th.5.3]) *There are the following properties of* $\hat{D}_n^{\mathrm{sym}}$*:*

1. $\hat{D}_n^{\mathrm{sym}}$ *is a ring (with natural operations* $\cdot, +$ *coming from* D_n*);* $\hat{D}_n^{\mathrm{sym}} \supset D_n$*.*
2. $\hat{R}$ *has a natural structure of a left* $\hat{D}_n^{\mathrm{sym}}$*-module, which extends its natural structure of a left* D_n*-module.*
3. *We have a natural isomorphism of* K*-vector spaces*

$$F := \hat{D}_n^{\mathrm{sym}} / \mathfrak{m} \hat{D}_n^{\mathrm{sym}} \to K[\partial_1, \dots, \partial_n].$$

4. *Operators from* $\hat{D}_n^{\mathrm{sym}}$ *can realize arbitrary endomorphisms of the* K*-algebra* $\hat{R}$ *which are continuous in the* $\mathfrak{m}$*-adic topology.*
5. *There are Dirac delta functions, operators of integration, difference operators.*

Remark 5 Unlike the usual ring of PDOs the ring $\hat{D}_n^{\mathrm{sym}}$ contains zero divisors. There are the following properties of the order function (contained in the proof of [2, Th. 5.3]):

1. $\mathbf{ord}(P \cdot Q) \leq \mathbf{ord}(P) + \mathbf{ord}(Q)$, and the equality holds if $\sigma(P) \cdot \sigma(Q) \neq 0$,
2. $\sigma(P \cdot Q) = \sigma(P) \cdot \sigma(Q)$, provided $\sigma(P) \cdot \sigma(Q) \neq 0$,
3. $\mathbf{ord}(P + Q) \leq \max\{\mathbf{ord}(P), \mathbf{ord}(Q)\}$.

In particular, the function $-\,\mathbf{ord}$ determines a discrete pseudo-valuation on the ring $\hat{D}_n^{\mathrm{sym}}$.

Remark 6 There are other possible ways to define a "symmetric" completion of the ring D_n (see [21, §2.1.5]). E.g. we can define for each sequence in $\mathfrak{m}D_n$, $\{(P_n)_{n\in\mathbb{N}}\}$, such that $P_n(R)$ converges uniformly in $\hat{R}$ (i.e. for any $k > 0$ there is $N > 0$ such that $P_n(\hat{R}) \subseteq \mathfrak{m}^k$ for $n \geq N$) a k-linear operator $P : \hat{R} \rightarrow \hat{R}$ by

$$P(f) = \varinjlim_{n\rightarrow\infty} \sum_{v=0}^{n} P_v(f), \qquad P := \sum_{n} P_n,$$

and define a completion to be the ring consisting of such operators. This completion is bigger, but $\hat{D}_n^{\mathrm{sym}}$ has finer properties sufficient for many aims. In particular, the classification theory from [21] deals with commutative subrings belonging to the more narrow ring $\hat{D}_n^{\mathrm{sym}}$.

3 The Ring $\hat{D}_n$ and Its Order Function ord_n

From technical point of view it is more convenient to deal with a more narrow non-symmetric version of completion $\hat{D}_n$ (it is well adapted for the classification of commutative subrings).

Definition 7 We define $\hat{D}_1 = \hat{D}_1^{\mathrm{sym}}$ and define $\hat{D}_n = \hat{D}_{n-1}^{\mathrm{sym}}[\partial_n]$. Obviously, $\hat{D}_n \subset \hat{D}_n^{\mathrm{sym}}$.

Definition 8 We define the *order function* ord_n on $\hat{D}_n$ as $\mathrm{ord}_n(P) = l$ if $\hat{D}_n \ni P = \sum_{s=0}^{l} p_s \partial_n^s$.

The coefficient p_l is called *the highest term* and will be denoted by $HT_n(P)$ (as the term naturally associated with the function ord_n).

The order function ord_n and the highest term HT_n behave like the **ord**-function and highest symbol. Namely, the following properties obviously hold:

1. $HT_n(P \cdot Q) = HT_n(P) \cdot HT_n(Q)$ provided $HT_n(P) \cdot HT_n(Q) \neq 0$;
2. $\mathrm{ord}_n(P \cdot Q) \leq \mathrm{ord}_n(P) + \mathrm{ord}_n(Q)$, and the equality holds if $HT_n(P) \cdot HT_n(Q) \neq 0$,
3. $\mathrm{ord}_n(P + Q) \leq \max\{\mathrm{ord}_n(P), \mathrm{ord}_n(Q)\}$.

In particular, the function $-\operatorname{ord}_n$ determines a discrete pseudo-valuation on the ring $\hat{D}_n$.

4 Commutative Subrings in $\hat{D}_n^{\mathrm{sym}}$ and Their Spectral Modules

Let $B \subset \hat{D}_n^{\mathrm{sym}}$ be a commutative subring.

Definition 9 The B-module $F = \hat{D}_n^{\mathrm{sym}}/\mathfrak{m}\hat{D}_n^{\mathrm{sym}} \simeq K[\partial_1, \dots, \partial_n]$ is called *spectral module* of the ring B.

Note that F is actually a right $\hat{D}_n^{\mathrm{sym}}$ module. However, since the ring B is commutative, we will view F as a left B-module, having the natural right action in mind. The following proposition explains the term "spectral".

Proposition 10 *Let $B \subset \hat{D}_n^{\mathrm{sym}}$ be a finitely generated commutative subring such that the spectral module F is finitely generated.*

For any character $\chi : B \to K_\chi$, where K_χ is an extension of K, consider the vector space

$$Sol(B, \chi) = \{f \in K_\chi[[x_1, x_2]] \big| \quad Q \circ f = \chi(Q) f \quad \forall Q \in B\}.$$

Then there exists a canonical isomorphism of vector spaces

$$F|_\chi := (B/\ker \chi) \otimes_B F \simeq Sol(B, \chi)^*$$

assigning to a class $\overline{\underline{\partial}^{\underline{p}}} \in F|_\chi$ the linear functional $f \mapsto \frac{1}{\underline{p}!} \frac{\underline{\partial}^{|p|} f}{\partial x_1^{p_1} \dots \partial x_n^{p_n}}\Big|_{(0,0)}$ on the vector space $\mathsf{Sol}(B, \chi)$. In particular, $\dim_K \Big(\mathsf{Sol}(B, \chi)\Big) < \infty$ *for any χ.*

The proof is verbally the same as in [2, Th. 4.5 item 2] (by replacing the rings D and $\mathbb{C}[[x_1, x_2]]$ there with $\hat{D}_n$, $K_\chi[[x_1, x_2]]$ here), cf. [13, Rem. 2.3].

5 Γ-Order and Quasi Elliptic Rings

In this subsection we describe commutative subrings in $\hat{D}_n^{\mathrm{sym}}$ that admit effective description in terms of algebro-geometric spectral data. Below we give a review of the most investigated case, when $n = 1$ or 2. Even in these cases there are many nontrivial open questions.

First we introduce the notion of Γ-order. This order is defined on *some elements* of the ring $\hat{D}_n^{\mathrm{sym}}$.

Definition 11 Let's denote by $\hat{D}_n^{i_1,\dots,i_q}$ the subring in $\hat{D}_n^{\mathrm{sym}}$ consisting of operators *not depending* on $\partial_{i_1},\dots,\partial_{i_q}$. The Γ-order is defined recursively.

We say that a nonzero operator $P \in \hat{D}_n^{2,3,\dots n}$ has Γ-order k_1 if $P = \sum_{s=0}^{k_1} p_s \partial_1^s$, where $0 \neq p_{k_1} \in \hat{R}$.

We say that a nonzero operator $P \in \hat{D}_n^{i+1,i+2,\dots,n}$ has Γ-order $(k_1,\dots,k_i)$ if $P = \sum_{s=0}^{k_i} p_s \partial_i^s$, where $p_s \in \hat{D}_n^{i,i+1,\dots,n}$, and the Γ-order of p_{k_i} is $(k_1,\dots,k_{i-1})$.

We say that a nonzero operator $P \in \hat{D}_n^{\mathrm{sym}}$ *has* Γ*-order*

$$\mathrm{ord}_\Gamma(P) = (k_1,\dots,k_n)$$

if $P = \sum_{s=0}^{k_n} p_s \partial_n^s$, where $p_s \in \hat{D}_n^n$, and the Γ-order of p_{k_n} is $(k_1,\dots,k_{n-1})$.

In this situation we say that the operator P is *monic* if the highest coefficient (defined recursively in analogous way) $p_{k_1,\dots,k_n}$ is 1.

Definition 12 The subring $B \subset \hat{D}_n \subset \hat{D}_n^{\mathrm{sym}}$ of commuting operators is called 1-*quasi elliptic* (or just quasi elliptic for short) if there are n operators $P_1,\dots,P_n$ such that

1. $\mathrm{ord}_\Gamma(P_i) = (0,\dots,0,1,0,\dots,0,l_i)$ for $1 \le i < n$, where 1 stands at the i-th place and $l_i \in \mathbb{Z}_+$;
2. $\mathrm{ord}_\Gamma(P_n) = (0,\dots,0,l_n)$, where $l_n > 0$;
3. For $1 \le i \le n$ $\mathbf{ord}(P_i) = |\,\mathrm{ord}_\Gamma(P_i)|$;
4. P_i are monic.

We call operators $P_1,\dots,P_n$ *formally* 1*-quasi elliptic* if they satisfy the conditions 1–3 above and the highest coefficients of P_i are constants, and we call them monic 1-quasi elliptic, if they satisfy the conditions 1–4 above.

6 Properties of Quasi Elliptic Rings

6.1 Case n = 1: Commuting Ordinary Differential Operators

Immediately from Definition 12 it follows that 1-quasi-elliptic subalgebras in $\hat{D}_1$ are exactly *elliptic* subalgebras of ordinary differential operators.

Definition 13 An ordinary differential operator $P = a_n\partial^n + a_{n-1}\partial^{n-1} + \dots + a_0 \in D_1$ of positive order n is called *(formally) elliptic* if $a_n \in K^*$. A ring $B \subset D_1$ containing an elliptic element is called *elliptic*.

Recall several basic properties of elliptic subrings. The following useful observation is due to Verdier [18, Lemme 1].

Lemma 14 *Let B be a commutative subalgebra of D_1 containing an elliptic element P. Then* all *elements of B are elliptic.*

Proposition 15 ([3, Prop. 3.1]) *Let* $P = a_n\partial^n + a_{n-1}\partial^{n-1} + \cdots + a_0 \in D_1$, *where* $a_n(0) \neq 0$. *Then there exists* $\varphi \in \mathrm{Aut}(D_1)$ *such that*

$$Q := \varphi(P) = \partial^n + b_{n-2}\partial^{n-2} + \cdots + b_0 \tag{3}$$

for some $b_0, \ldots, b_{n-2} \in K[[x]]$.

Theorem 16 *Let* B *be a commutative subalgebra of* D_1.

1. *Then* B *is finitely generated integral domain of Krull dimension one. In particular,* B *determines an integral affine algebraic curve* $C_0 := \mathrm{Spec}(B)$.
2. *Moreover,* C_0 *can be compactified to a projective algebraic curve* C *by adding a* single smooth *point* p, *which is determined by the valuation*

$$\mathsf{val}_p : Quot(B) \to \mathbb{Z}, \quad \frac{P}{Q} \mapsto \frac{\mathrm{ord}(Q) - \mathrm{ord}(P)}{r},$$

where $Quot(B)$ *is the quotient field of* B *and* r *is the rank of* B:

$$r = GCD\{\mathrm{ord}(P),\ P \in B\}$$

Comment to the Proof In the stated form, this result can be found in the articles of Mumford [16, Section 2] and Verdier [18, Proposition 1]. Note that ord $=$ **ord** in this theorem.

Theorem 17 *Let* $B \subset D_1$ *be a commutative subalgebra of rank* r. *Then the spectral module* F *is finitely generated and torsion free over* B. *Moreover,* $Quot(B) \otimes_B F \cong Quot(B)^{\oplus r}$.

Comment to the Proof In the stated form, this result can be found in [18, Proposition 3] and [16, Section 2].

Elliptic commutative subrings admit an effective description in terms of their algebraic-geometric spectral data.

Definition 18 The (one-dimensional) algebraic-geometric spectral data of rank r consist of

- C is an integral projective curve over K;
- $p \in C$ is a closed regular K-point;
- $\mathcal{F}$ is a coherent torsion free sheaf of rank r on C with

$$h^0(C, \mathcal{F}) = h^1(C, \mathcal{F}) = 0;$$

- z is a local coordinate (a formal local parameter) at p;
- $\phi : \hat{\mathcal{F}}_p \simeq (K[[z]])^{\oplus r}$ is a trivialisation (i.e. an $\hat{\mathcal{O}}_p \simeq K[[z]]$-module isomorphism).

There is a naturally defined notion of an isomorphism of spectral data.

Theorem 19 *There is a one-to-one correspondence*

$$[B \subset D_1 \quad \textit{of rank } r] \longleftrightarrow [(C, p, \mathcal{F}, z, \phi) \quad \textit{of rank } r]/\simeq$$

$$[B \subset D_1 \quad \textit{of rank } 1]/\sim \longleftrightarrow [(C, p, \mathcal{F}) \quad \textit{of rank } 1]/\simeq$$

where

- $[B]$ *means a class of equivalent commutative elliptic subrings, where* $B \sim B'$ *iff* $B = f^{-1}B'f$, $f \in D_1^*$.
- $\sim$ *means "up to linear changes of variables"*
- $(C, p, \mathcal{F}, z, \phi)$ *means the algebraic-geometric spectral data of rank r*

Comment to the Proof In the case C is a smooth Riemann surface, the classification theorem has been proven by Krichever [9, 10]. Singular curves and torsion free sheaves which are not locally free were included into the picture by Mumford [16, Section 2] and Verdier [18, Proposition 4]. Mumford's approach was further developed by Mulase [14, Theorem 5.6] and Quandt [17]. We use here the most generic algebraic form of the classification theorem convenient for our presentation.

Remark 20 The major interest concerns those commutative subalgebras of D_1 which belong to the subalgebra $\mathbb{C}\{x\}[\partial]$ of ordinary differential operators, whose coefficients are *convergent* power series. If $P = a_n\partial^n + a_{n-1}\partial^{n-1} + \cdots + a_0$ is such an operator then shifting the variable $x \mapsto x + \varepsilon$ with $\varepsilon \in \mathbb{C}$ such that $|\varepsilon|$ is sufficiently small, we may always achieve that $a_n(0) \neq 0$. Note that this operation can not be extended on the whole D_1. Nonetheless, one can show that all elements of B belong to $\mathbb{C}\{x\}[\partial]$ (this follows for example from Schur's theory) and one can choose a common radius of convergence for all coefficients of all elements of B. According to proposition 15, we can transform P into a normalized elliptic differential operator.

6.2 Case $n = 2$: Commutative Subalgebras in $\hat{D}_2$

In this section we give a description of basic properties of 1-quasi elliptic subrings in $\hat{D}_2$.

Proposition 21 *Let B be a* 1*-quasi elliptic commutative subring in* $\hat{D}_2^{\mathrm{sym}}$. *Then*

1. *B and* $\mathrm{gr}(B)$ *are integral, where* gr *denotes the associated graded ring with respect to the filtration defined by the function* **ord**, *and the function* $-$**ord** *induces a discrete valuation of rank one on B and on its field of fractions* $\mathrm{Quot}(B)$*;*
2. *the* Γ*-order is defined on all elements of B, in particular, the function* $-\mathrm{ord}_\Gamma$ *is a discrete valuation of rank two;*

3. *the natural map*

$$\Phi : gr(\hat{D}_2^{sym}) \to gr(\hat{D}_2^{sym})/\mathfrak{m} gr(\hat{D}_2^{sym}) \simeq K[\xi_1, \xi_2]$$

   induces an embedding of vector spaces on $gr(B)$*;*
4. *the spectral module* F *is torsion free;*
5. *for any* $P \in B$ *holds:* $\mathbf{ord}(P) = k + l$*, where* $(k, l) = \mathrm{ord}_\Gamma(\sigma(P))$.
6. $\mathrm{trdeg}_K(\mathrm{Quot}(B)) = 2$*, the field* $\mathrm{Quot}(B)$ *is finitely generated over* K *and the localisation* $\mathrm{Quot}(B) \cdot F$ *is a finitely generated* $\mathrm{Quot}(B)$*-module.*

Remark 22 Unlike the case $n = 1$ 1-quasi elliptic rings are not necessarily finitely generated. The most simple example is the subring $K[1, \partial_1^i \partial_2^j, i \geq 0, j > 0] \subset \hat{D}_2$. More interesting examples see in [8].

Comment to the Proof The first item follows from [21, Cor. 3.1], the second follows from lemma [23, L. 2.6], items 3–5 are contained in [22, Lemma 2], two first assertions of item 6 are contained in [24, Lemma 2], the last assertion follows from the same arguments as in loc. cit. A more detailed proof will appear in [19].

7 Classification Problem for Commutative Subalgebras in $\hat{D}_2$

Finitely generated quasi elliptic subrings admit a similar, though much more complicated, classification in terms of their spectral data. In [21] such a classification was given for quasi elliptic subrings with an extra property (strongly admissibility). In [22, Th. 1] it was shown that for rank one subrings (see the definition below) this classification can be refined, in particular, the corresponding spectral data become much easier. Here we present the most pleasant refinement of the classification for quasi elliptic subrings whose algebraic and analytical ranks are coincide. Surprisingly the proof needs a significant improvement of the technique used in earlier papers; the details will appear in [19].

Definition 23 Let $B \subset \hat{D}_n$ be a commutative subring. Then we define the *analytical rank* as

$$\mathrm{An.rank}(B) := \mathrm{rk}(F \cdot \mathrm{Quot}(B)) = \dim_{\mathrm{Quot}(B)}(F \cdot \mathrm{Quot}(B)).$$

We define the *algebraic rank* of B as

$$\mathrm{Alg.rank}(B) := GCD\{\mathbf{ord}(P) | \quad P \in B\}$$

We say that B *is of rank* r if $An.rank(B) = Alg.rank(B) = r$ and the spectral module F is finitely generated.

Remark 24 It is not difficult to see that $An.rank(B) \geq Alg.rank(B)$ (cf. [13, Rem. 3.3]). Moreover, if B is a finitely generated ring over K, then the following conditions are equivalent: 1) F is a finitely generated B-module and $\mathrm{An.\,rank}(B) = \mathrm{Alg.\,rank}(B) = r$, 2) $\dim_K B_{rn}/B_{r(n-1)} \sim rn$ for all $n \gg 0$. In case $r = 1$ this was proved in [22, Cor. 1], and in general case the details will appear in [19].

Definition 25 The (two-dimensional) algebraic-geometric spectral data of rank r consist of

1. X is an integral projective algebraic surface over K;
2. C is an integral ample $\mathbb{Q}$-Cartier divisor on X. Moreover, $C^2 = r$.
3. $p \in C$ is a closed K-point, which is regular on C and on X;
4. local K-algebra homomorphism[1]

$$\pi : \widehat{\mathcal{O}}_{X,P} \longrightarrow K[[u,t]]$$

   satisfying the following property. If f is a local equation of the curve C at P, then $\pi(f)K[[u,t]] = t^r K[[u,t]]$ and the induced map $\pi : \widehat{\mathcal{O}}_{C,P} = \widehat{\mathcal{O}}_P/(f) \to K[[u]] = K[[u,t]]/(t)$ is an isomorphism. (The definition of π does not depend on the choice of appropriate f. Besides, from this definition it follows that π is an embedding, $K[[u,t]]$ is a free $\widehat{\mathcal{O}}_{X,P}$-module of rank r with respect to π.)
5. $\mathcal{F}$ is a coherent torsion free sheaf of rank r on X, which is Cohen-Macaulay along C;
6. an $\mathcal{O}_P$-module embedding

$$\phi : \mathcal{F}_P \hookrightarrow K[[u,t]]$$

   subject to the following condition for any $n \geq 0$. By item 2 there is the minimal natural number d such that $C' = dC$ is a very ample divisor on X. Let $\gamma_n : H^0(X, \mathcal{F}(nC')) \hookrightarrow \mathcal{F}(nC')_P$ be an embedding (which is an embedding, since $\mathcal{F}(nC')$ is a torsion free quasi-coherent sheaf on X). Let $\epsilon_n : \mathcal{F}(nC')_P \to \mathcal{F}_P$ be the natural $\mathcal{O}_P$-module isomorphism given by multiplication to an element $f^{nd} \in \mathcal{O}_P$, where $f \in \mathcal{O}_P$ is chosen as in item 4. Let $\tau_n : K[[u,t]] \to K[[u,t]]/(u,t)^{ndr+1}$ be the natural ring epimorphism. We demand that the map

$$\tau_n \circ \phi \circ \epsilon_n \circ \gamma_n \,:\, H^0(X, \mathcal{F}(nC')) \longrightarrow K[[u,t]]/(u,t)^{ndr+1}$$

   is an isomorphism. (These conditions on the map ϕ do not depend on the choice of the appropriate element f.)

[1] Recall that $\widehat{\mathcal{O}}_{X,P} \simeq K[[f,g]]$ by the Cohen structure theorem.

Remark 26 The last three most difficult items of this definition can be replaced by the following items of "more geometric nature" (see details in [19]):

- $\mathcal{F}$ is a coherent torsion free sheaf of rank r on X, which is endowed with a chain of coherent torsion free subsheaves

$$\mathcal{F} = \mathcal{F}_0 \supset \mathcal{F}_{-1} \supset \ldots \supset \mathcal{F}_{-r+1} \supset \mathcal{F}_{-dr} \simeq \mathcal{F}(-C')$$

  such that the sheaves $(\mathcal{F}_i/\mathcal{F}_{i-1})$ have support on C, $(\mathcal{F}_i/\mathcal{F}_{i-1})|_C$ are torsion free sheaves of $\mathcal{O}_C$-modules of rank one, $\chi((\mathcal{F}_i/\mathcal{F}_{i-1})|_C) = 1+i$ for $0 \ge i \ge 1-dr$, and for $n \ge 0$

$$h^0(X, \mathcal{F}(nC')) = \frac{(ndr+1)(ndr+2)}{2},$$

- $\pi : \widehat{\mathcal{O}}_{X,P} \simeq K[[u,t]]$ is an isomorphism of local K-algebras such that t corresponds to a local equation f of C at p, and u corresponds to a local equation of p on C; $\phi : \hat{\mathcal{F}}_p \simeq \hat{\mathcal{O}}_{X,p}^{\oplus r}$ is a trivialisation at p defined up to composition with a permutation isomorphism $\hat{\mathcal{O}}_{X,p}^{\oplus r} \simeq \hat{\mathcal{O}}_{X,p}^{\oplus r}$, which is given by a permutation matrix, i.e. a matrix with only zeros and units as entries and such that each row and each column contains exactly one unit.

There is a naturally defined notion of an isomorphism of spectral data (see [21, Def. 3.11]).

Theorem 27 *There is a one-to-one correspondence*

$$[B \subset \hat{D}_2^{\mathrm{sym}} \quad \textit{of rank } r] \longleftrightarrow [(X, C, p, \mathcal{F}, \pi, \phi) \quad \textit{of rank } r]/\simeq$$

$$[B \subset \hat{D}_2^{\mathrm{sym}} \quad \textit{of rank } 1]/\sim \longleftrightarrow [(X, C, \mathcal{F}) \quad \textit{of rank } 1]/\simeq$$

where

- $[B]$ *means a class of equivalent commutative finitely generated* 1*-quasi-elliptic subrings, where* $B \sim B'$ *iff* $B = f^{-1}B'f$, $f \in \hat{D}_2^*$.
- $[(X, C, p, \mathcal{F}, \pi, \phi)]$ *means a class of isomorphic algebro-geometric spectral data of rank* r*:*
- $\sim$ *in the second row means a stronger equivalence:* $B_1 \sim B_2$ *if there is a linear change of variables* φ *and a unity* $U \in \hat{D}_2^{\mathrm{sym}}$, $\mathbf{ord}(U) = 0$ *such that* $B_1 = U^{-1}\varphi(B_2)U$.
- $[(X, C, \mathcal{F})]$ *means a class of isomorphic triples (simplified spectral data of rank one), where an isomorphism of triples is just an isomorphism of surfaces that induces an isomorphism of corresponding curves and sheaves.*

Remark 28 The geometric part of the spectral data can be easily described: $X \simeq \mathrm{Proj}\, \tilde{B}$, $C \simeq \mathrm{Proj}(\mathrm{gr}\, B)$, $\mathcal{F} \simeq \mathrm{Proj}\, \tilde{F}$, where $\tilde{B}$, $\tilde{F}$ are the Rees ring and Rees module with respect to the filtration defined by the function **ord** (and gr B is the associated graded ring).

We can additionally assume that X is *Cohen-Macaulay* because of the following result:

Proposition 29 *If $B \subset \hat{D}_2$ is a finitely generated 1-quasi-elliptic commutative subring, then there exist a Cohen-Macaulay commutative subring $\tilde{B} \supset B$ with the same properties.*

Moreover, if $B \subset D_2$, then $\tilde{B} \subset D_2$.

Comment to the proof. The proof is based on the following facts. If B is additionally strongly admissible, then both statements follow from [21, Th. 3.2] combined with [21, Th. 4.1]. If $B \subset D_2$ is not strongly admissible, then it becomes strongly admissible after a generic linear change of variables [21, Prop.2.4]. If $B \subset \hat{D}_2$ is not strongly admissible, then the same is true up to conjugation by a unity in $\hat{D}_2^{\mathrm{sym}}$ (as in Theorem 27, item 3), the details will appear in [19].

Remark 30 There is the following analogy with $n = 1$ case. Recall that isospectral deformations of rank one commutative rings of ODOs determine the KP flows on the *Jacobian* (or on its compactification) of the spectral curve, see e.g. [15]. Isospectral deformations of rank one commutative rings of PDOs determine some flows on the *moduli space* M_χ of stable torsion free coherent sheaves on the spectral surface X with fixed Hilbert polynomial $\chi(n) = \frac{(nd+1)(nd+2)}{2}$ with respect to the ample line bundle $\mathcal{O}_X(dC)$, cf. [2, §6], or on the Picard scheme of a formal punctured ribbon, cf. [13, Introduction].

An open subset of this moduli space parametrises *Cohen-Macaulay sheaves* (see [7, Th. 12.2.1]). Cohen-Macaulay sheaves on Cohen-Macaulay surfaces can be effectively described with the help of *matrix-problem approach* due to Burban and Drozd, see [2] and references therein. Then the higher-dimensional version of the *Sato theory* (“algebraic inverse scattering method”, see loc. cit.) is used to obtain explicit examples or explicit deformations of known examples of commuting PDOs.

Below we'll recall some examples obtained with the help of these techniques.

Example This is an example of explicit equations of isospectral deformations and their explicit solution obtained in [21, Ex. 4.2].

Consider a commutative subring $B \subset \hat{D}_2$ generated by 3 operators:

$$P = \partial_2^2 - 2\frac{1}{(1-x_2)^2}\delta_1,$$

$$Q = \partial_1\partial_2 + \frac{1}{1-x_2}\delta_1\partial_1,$$

$$P' = \partial_2^3 - 3\frac{1}{(1-x_2)^2}\delta_1\partial_2 - 3\frac{1}{(1-x_2)^3}\delta_1,$$

where δ_1 is the Dirac delta-function: $\delta_1(f(x_1, x_2)) = f(0, x_2)$ (these operators were obtained in [21, Ex. 4.2] with the help of the higher-dimensional Sato theory starting with the simplest Schur pair). The (projective) spectral surface of the ring $B = K[P, P', Q]$ is a rational singular surface, with normalisation $\mathbb{P}^2$. More precisely, it can be obtained by gluing two lines $2\mathbb{P}^1$ on $\mathbb{P}^2$ (see [20, Ex.30]).

The system of isospectral deformations for these operators is a *modified Parshin system* (cf. [25] or [20, §6.3]; the detailed description will appear in a separate paper)

$$\frac{\partial N}{\partial t_{ij}} = V_N^{ij}, \quad i, j \geq 0$$

where

$$V_N^{ij} = ([(L^i M^j)_+, L], [(L^i M^j)_+, M]),$$

with initial conditions $M = \sqrt{P}$, $L = QP^{-1}$ (these operators belong to an appropriately defined ring of pseudo-differential operators $\hat{E}$, see [21]). This system is equivalent to the modified Sato-Wilson system

$$\frac{\partial S}{\partial t_{ij}} = -(S\partial_1^i \partial_2^j S^{-1})_- S,$$

where $S = 1 + s_1\partial_2^{-1} + s_2\partial_2^{-2} + \ldots \in \hat{E(t)}$, $s_i \in \hat{D}_1$, and $L = S(0)\partial_1 S(0)^{-1}$, $M = S(0)\partial_2 S(0)^{-1}$ ($S(0)$ means $S|_{t_{ij}=0}$), and first equations of this system can be transformed to the following equations:

$$\begin{gathered}\frac{\partial s_1}{\partial t_1} = \frac{1}{4}(s_1)_{x_2x_2x_2} - \frac{3}{2}(s_1)^2_{x_2}, \quad \frac{\partial s_1}{\partial t_2} = -(s_1)_{x_2}(s_1)_{x_1} - \frac{1}{2}(s_1)_{x_2x_2}\partial_1, \\ \frac{\partial s_1}{\partial t_3} = -(s_1)^2_{x_1} - (s_1)_{x_1x_2}\partial_1 - (s_1)_{x_2}\partial_1^2,\end{gathered} \tag{4}$$

where $s_1 = s_1(x_1, x_2, t_1, t_2, t_3)$ is the first coefficient of the operator $S(t)$.

Notably, $s_1(0) = \frac{1}{1-x_2}\delta_1$ is a solution of the equations above (cf. with the rational stationary solution $u(x) = -1/x^2$ of the KdV equation).

Example This is an example of explicit isospectral deformations of the quantum (algebraically) completely integrable Calogero-Moser system obtained in [2, §6].

Consider the Calogero–Moser operator with rational potential

$$H = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) - 2\left(\frac{1}{(x_1 - \xi_1)^2} + \frac{1}{(x_2 - \xi_2)^2}\right),$$

where $(\xi_1, \xi_2) \in \mathbb{C}^2$ is such that $\xi_1\xi_2 \neq 0$. Then (cf. e.g. [4]) H can be included into a large ring of pairwise commuting differential operators $B_H \subset D_2$, where $B_H \simeq A = \mathbb{C}[z_1^2, z_1^3, z_2^2, z_2^3]$, and the isomorphism is given with the help of the *Berest BA-function*:

$$\Psi_{Be} = z_1z_2 + \frac{z_1}{\xi_2 - x_2} + \frac{z_2}{\xi_1 - x_1} + \frac{1}{(\xi_1 - x_1)(\xi_2 - x_2)},$$

i.e. for any $q \in A$ there exists a unique $L_q \in B_H$ s.t.

$$L_q\Psi_{Be} = q\Psi_{Be}.$$

Explicit calculations of the corresponding algebro-geometric spectral data and of the moduli space of coherent torsion free sheaves with fixed Hilbert polynomial leads to the following deformed BA-function (which encodes a dense open part of the moduli space):

$$\Psi(x_1, x_2, z_1, z_2) = \Psi_{Be} + \beta\overline{\Psi},$$

where

$$\begin{aligned}\overline{\Psi} = &\frac{1 + \beta\left(\dfrac{z_1}{\xi_2} + \dfrac{z_2}{\xi_1}\right)}{(\xi_1\xi_2 - \beta)(\xi_1 - x_1)(\xi_2 - x_2)} \\ &+ \frac{1}{(\xi_1 - x_1)(\xi_2 - x_2)\xi_2}\left(\exp(x_1z_1)z_1 + (\xi_1 - x_1)\exp(x_1z_1)z_1^2\right) \\ &\quad + \frac{1}{(\xi_1 - x_1)(\xi_2 - x_2)\xi_2}\left(\exp(x_2z_2)z_2 + (\xi_2 - x_2)\exp(x_2z_2)z_2^2\right).\end{aligned}$$

The simplest deformations of differential operators from B_H can be described as follows. For any $q \in z_1^2z_2^2A$ denote $q'(z_1, z_2) := q/(z_1^2z_2^2)$. Then

$$\hat{D}_2^{\text{sym}} \ni L_q = Sq'(\partial_1, \partial_2)(\partial_1 - \frac{1}{1 - x_1})(\partial_2 - \frac{1}{1 - x_2}), \quad \text{where}$$

$$S = S_0 + \beta T,$$

$$S_0 = \partial_1\partial_2 + \frac{1}{\xi_2 - x_2}\partial_1 + \frac{1}{\xi_1 - x_1}\partial_2 + \frac{1}{(\xi_1 - x_1)(\xi_2 - x_2)},$$

$$T = \frac{1}{(\xi_1 - x_1)(\xi_2 - x_2)} \left(\frac{1}{\xi_2} \left(\delta_2 \partial_1 + (\xi_1 - x_1)\delta_2 \partial_1^2 \right) + \frac{1}{\xi_1} \left(\delta_1 \partial_2 + (\xi_2 - x_2)\delta_1 \partial_2^2 \right) \right) + \frac{1}{(\xi_1\xi_2 - \beta)(\xi_1 - x_1)(\xi_2 - x_2)} \delta_1\delta_2 \left(1 + \beta \left(\frac{\partial_1}{\xi_2} + \frac{\partial_2}{\xi_1} \right) \right)$$

(here, as above, δ_i denote the Dirac delta functions with respect to the variable x_i).

7.1 Spectral Data of Subrings in D_2

Which geometric data describe commutative subrings $B \subset D_2$ of *PDOs*? We give here the answer in the case of rank one rings.

Theorem 31 *If $B \subset D_2$ is a finitely generated 1-quasi-elliptic commutative ring of rank 1 with constant highest symbols, then*

1. *The sheaf $\mathcal{F}$ is coherent Cohen-Macaulay of rank 1;*
2. *The divisor C is a rational curve;*
3. *If $n : \mathbb{P}^1 \to C$ is the normalisation map, then $\mathcal{F}|_C \simeq (n_*(\mathcal{O}_{\mathbb{P}^1}))$.*

Remark 32 By [13, L. 2.1], if quasi-elliptic operators P_i form Definition 12 have constant highest symbols, then all operators from B have constant highest symbols.

Proof By remark 24 we have $\dim_K B_n/B_{n-1} = n + c$ for all $n \gg 0$, where $c \in \mathbb{Z}$ is a constant. Then the ring B must contain two operators P, Q such that the intersection of their characteristic divisors is empty. Indeed, if there are no such operators, then symbols of all operators should belong to a proper ideal generated by a homogeneous polynomial in the polynomial ring $K[\partial_1, \partial_2]$. Since the ring gr B is finitely generated (cf. [21, L. 3.8] or [19, L. 3.5] in general case), there exists some positive integer d such that the Veronese ring $(\text{gr } B)^{(d)}$ is generated by its first homogeneous component over K. Since all elements of this component belong to the proper ideal of the polynomial ring, we must have $\dim_K(\text{gr}_{dn} B) \le dn - n + 1$ for all $n \gg 0$, a contradiction with $\dim_K(\text{gr}_{dn} B) = dn + c$.

Then by [21, L. 2.6, P. 2.4] for almost all linear changes of variables applied to the ring B it will satisfy conditions of theorem 4 in [23], i.e. in combination with Theorem 27 we get item 1 of our theorem.

Item 2 is proved in [13, Th.2.1]. To prove item 3 note that $\mathcal{F}|_C \simeq \text{Proj}(\text{gr } F)$ (as it follows from the proof of Theorem 27, cf. also [20, Corol. 23]). Since gr $B \subset$ gr F in our case, it follows that $\mathcal{F}|_C \simeq (n_*(\mathcal{O}_{\mathbb{P}^1}))$. □

Conjecture 33 *The conditions from theorem are sufficient, i.e. if the spectral triple $(X, C, \mathcal{F})$ satisfies the conditions of theorem, then the corresponding commutative ring B from Theorem 27 will belong to D_2.*

8 Normal Forms

In the matrix problem approach it is important to know what are the Cohen-Macaulay sheaves with special properties on the *normalisation* of the spectral surface. So, it is important to know what are the possible *normal* surfaces X such that a pre-spectral datum $(X, C, \mathcal{F})$ from classification Theorem 27 exists. We'll call such surfaces *normal forms*. In this section we present several results and conjectures about *smooth* normal forms.

Proposition 34 *If X is a smooth normal form of a finitely generated 1-quasi-elliptic commutative ring $B \subset D_2$ of rank 1, then $X \simeq \mathbb{P}^2$ (and then $C \simeq \mathbb{P}^1$, $\mathcal{F} \simeq \mathcal{O}_X$).*

Comment to the Proof If the ring B consists of operators with constant highest symbols (cf. remark 32), then proposition immediately follows from Theorem 31, since all Cohen-Macaulay sheaves on smooth surfaces are locally free. In general case it follows from two facts: first, any commutative ring of PDOs has a non-trivial family of iso-spectral deformations; second, any commutative ring with a smooth spectral surface can not have non-trivial iso-spectral deformation, because $H^1(X, \mathcal{O}_X) = 0$. The details will appear in [12].

Are there smooth normal forms of commutative subrings from $\hat{D}_2$? In [22] the following sufficient conditions were formulated.

Proposition 35 ([22, Cor. 3]) *Assume that K is uncountable and algebraically closed. The following conditions on a smooth projective surface X are sufficient for the existence of a commutative subring of rank one in $\hat{D}_2$:*

1) *there is an ample integral curve C with $C^2 = 1$ and $h^0(X, \mathcal{O}_X(C)) = 1$;*
2) *there is a divisor D with $(D, C)_X = g(C) - 1$ (here $g(C)$ means the arithmetical genus of C), $h^i(X, \mathcal{O}_X(D)) = 0$, $i = 0, 1, 2$, and $h^0(X, \mathcal{O}_X(D + C)) = 1$.*

Remark 36 It would be interesting to clarify whether these conditions are also necessary. In general, the spectral sheaf of a ring $B \subset \hat{D}_2$ is not necessary Cohen-Macaulay, see e.g. [2, Rem 6.2]. However, no examples of not CM spectral sheaves on a smooth spectral surface (i.e. sheaf and surface from definition of spectral data) is known yet.

Remark 37 The condition $h^0(X, \mathcal{O}_X(C)) = 1$ means that we are looking for normal forms of "non-trivial" commutative subrings.

Definition 38 The subring $B \subset \hat{D}_2$ is "trivial", if it contains the operator ∂_1 or the operator ∂_2, i.e. B consists of operators not depending on x_1 or x_2.

The examples of such algebras naturally arise from examples of commuting *ordinary differential operators* just by adding one extra derivation.

Proposition 39 *Let X be a smooth normal form. Then the corresponding commutative subring $B \subset \hat{D}_2$ is "trivial" iff $h^0(X, \mathcal{O}_X(C)) \geq 2$.*

Comment to the proof. The proof essentially follows from the classification Theorem 27. Since X is smooth, the curve C is a Cartier divisor. If B is trivial, then $H^0(X, \mathcal{O}_X(C)) \simeq B_2$ by the classification theorem, and $\dim_K B_2 \geq 2$. Conversely, if $\dim_K B_2 \geq 2$, then by the construction from the proof of theorem B contains either ∂_2 or ∂_1 (up to a linear change of variables B contains ∂_1, cf. [22, Th. 1], [23, Th. 7]).

Theorem 40 ([22, Th. 5]) *Assume that K is algebraically closed. Let $(X, C, \mathcal{F})$ be a pre-spectral data of rank one with a smooth surface X and $g(C) \leq 1$. Then $h^0(X, \mathcal{O}_X(C)) \geq 2$.*

Conjecture 41 *If X is a smooth normal form, then it is either rational (and corresponds to a "trivial" subring) or of general type.*

Theorem 42 ([11]) *There is an eight-dimensional family of pairwise non-isomorphic Godeaux surfaces X such that on each X from this family there are at least 840 different divisors D_j and four curves C_i satisfying the conditions from proposition 35.*

Each of these Godeaux surfaces is a factor of a quintic in $\mathbb{P}^3(\mathbb{C})$ by the group $\mathbb{Z}^5$.

Conjecture 43 *All normal forms have the property $q = H^1(X, \mathcal{O}_X) = 0$. There are no other smooth normal forms of general type corresponding to "non-trivial" subrings.*

According to the last conjecture, the commutative rings of operators corresponding to the smooth normal forms *do not have isospectral deformations!*

On the other hand, we expect there are many non-smooth normal forms:

Conjecture 44 *For any smooth curve C there is a normal cone X (with the only singularity at the cone top) which is a normal form.*

A solution to some of these conjectures is expected to appear in a paper [12].

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Part II
Abstracts of the Lectures at "School on Geometry and Physics"

Soliton Equations and Their Holomorphic Solutions

A. V. Domrin

Abstract We describe all soliton equations of parabolic type on a $(1 + 1)$-dimensional spacetime, give a criterion for solubility of the Cauchy problem in terms of the scattering data of the initial condition, and prove that all local holomorphic solutions are globally meromorphic in the spatial variable.

Keywords Soliton equations · Analytic continuation

Mathematics Subject Classification (2010) Primary 37K15; Secondary 35Q51, 35A01, 32D15, 30D30

The aims of the course were as follows.

First, introduce a large class of $(1+1)$-dimensional soliton equations and systems of parabolic type[1] and simultaneously describe all of their local holomorphic solutions using a local holomorphic version of the inverse scattering method.[2]

[1]Parabolic type includes, for example, Korteweg–de Vries and nonlinear Schrödinger equations, but not sine-Gordon. Presumably all physically relevant equations and systems of parabolic type in dimension $1 + 1$ either belong to the class described in these lectures or can be obtained from elements of this class by a differential-polynomial change of unknown functions. The class of equations and systems referred to is often denoted in the literature by abbreviations like ZS, AKNS, AKNS-ZS, or even AKNS-ZS-D in honour of Zakharov, Shabat, Ablowitz, Kaup, Newell, Segur, and Dubrovin. See, for example, [1, 3, 9].

[2]Here the potentials are germs of holomorphic matrix-valued functions, without any boundary conditions. The scattering data are matrix-valued formal power series in the inverse spectral parameter. The method is based on a Riemann factorization problem extended to matrix-valued formal Laurent series in the spectral parameter.

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Second, discuss a number of special classes of solutions from this point of view (this was omitted from the real lectures for reasons beyond my control).

Third, give a criterion for solubility of the local holomorphic Cauchy problem for such equations in terms of the scattering data of the initial condition and discuss what this criterion says about admissible initial conditions.

Fourth, sketch a proof of the Painlevé property: all local holomorphic solutions of any soliton equation of parabolic type are globally meromorphic and monodromy-free in the spatial variable.

The class of local holomorphic solutions contains (but not reduces to) the class of algebro-geometric (or finite-gap) solutions. Thus the subject of the course is strictly larger (in the sense of equations and solutions) than in the beautiful monograph of Gesztesy and Holden [10] with a similar title, although we do not go in such detail for concrete equations. See [7] for more details and references.

1 Lecture I: Riemann Problem and Soliton Equations

Let $\Omega \subset \mathbb{C}^2_{xt}$ be a simply connected domain (x is the spatial variable, t is the temporal variable) and $U, V : \Omega \to \mathrm{gl}(n, \mathbb{C})$ holomorphic maps. The system

$$E_x = UE, \quad E_t = VE \tag{1}$$

has a holomorphic solution $E : \Omega \to \mathrm{GL}(n, \mathbb{C})$ if and only if

$$U_t - V_x + [U, V] = 0 \quad \text{everywhere in} \quad \Omega. \tag{2}$$

We define a (1+1)-*dimensional soliton equation* as an equation of the form $U_t - V_x + [U, V] = 0$ where $U, V : \Omega \times \mathbb{C}P^1_z \to \mathrm{gl}(n, \mathbb{C})$ are rational functions of an auxiliary variable $z \in \mathbb{C}P^1$ (called the spectral parameter) such that the expression $U_t - V_x + [U, V]$ is independent of z ([9, Part II, Ch. 1, § 3]).

Unknown functions are the entries of the $\mathrm{gl}(n, \mathbb{C})$-valued coefficients of the partial fraction decompositions of U, V. Some of them can be expressed in terms of the others from the condition "$U_t - V_x + [U, V]$ is independent of z". For example, let U and V be polynomials in z with $\deg U = 1$ and $\deg V = m \geq 2$. There is no loss of generality in assuming that

$$U(x, t, z) = az + q(x, t), \quad V(x, t, z) = bz^m + r_1(x, t)z^{m-1} + \cdots + r_m(x, t),$$

where $a, b \in \mathrm{gl}(n, \mathbb{C})$ are diagonal matrices, a has simple spectrum (i.e. all its eigenvalues are distinct), $q : \Omega \to \mathrm{gl}(n, \mathbb{C})$ is holomorphic and off-diagonal, $r_1, \ldots, r_m : \Omega \to \mathrm{gl}(n, \mathbb{C})$ are holomorphic. Then the condition "$U_t - V_x + [U, V]$ is independent of z" determines $r_1, \ldots, r_m$ as differential polynomials $r_j = F_j(q)$

(i.e. polynomials in q and its derivatives with respect to x) uniquely up to diagonal constants of integration $c_1, \dots, c_m \in \mathrm{gl}(n, \mathbb{C})$, and the Eq. (2) with $U = az + q$ and $V = az^m + F_1(q)z^{m-1} + \cdots + F_m(q)$ takes the form

$$q_t = [a, F_{m+1}(q)]. \tag{3}$$

Special cases of (3) with $a = b = \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix}$ and $q(x,t) = \begin{bmatrix} 0 & u(x,t) \\ v(x,t) & 0 \end{bmatrix}$ are the heat equation and its hierarchy $u_t = (\partial_x^m + \gamma_1 \partial_x^{m-1} + \cdots + \gamma_m)u$ (when $v \equiv 0$ and γ_j is the difference of the diagonal entries of c_j), the Korteweg–de Vries equation $u_t - u_{xxx} + 6uu_x = 0$ (when $m = 3$, $v \equiv 1$) and the modified Korteweg–de Vries equation $u_t - u_{xxx} - 6u^2u_x = 0$ (when $m = 3$, $v \equiv u$). Taking $a = \mathrm{diag}(-i/2, i/2)$, $m = 2$ and $v(x,t) = \overline{u(\overline{x}, \overline{t})}$, we obtain the non-linear Schrödinger equation $iu_t + u_{xx} + 2u|u|^2 = 0$ for real x, t. In the last three examples, $c_1 = c_2 = \cdots = 0$.

Let $\mathcal{D}$ be the set of all holomorphic $\mathrm{GL}(n, \mathbb{C})$-valued germs[3] $f(z)$ at ∞ with $f(\infty) = I$. Suppose that $a, b, c_1, c_2, \dots \in \mathrm{gl}(n, \mathbb{C})$ are diagonal matrices and a has simple spectrum. Fix an integer $m \geq 2$ and a point $(x_0, t_0) \in \mathbb{C}^2$.

Theorem 1 ([5, 15, 17]) *For every $f \in \mathcal{D}$ there is an open neighborhood $\Omega(f)$ of (x_0, t_0) in $\mathbb{C}^2$ such that for all $(x, t) \in \Omega(f)$ the function*

$$\gamma(x, t, z) := \exp\{az(x - x_0) + (bz^m + c_1 z^{m-1} + \cdots + c_m)(t - t_0)\} f^{-1}(z)$$

possesses the following property: one can find a holomorphic map $\gamma_+(x, t, \cdot) : \mathbb{C}^1_z \to \mathrm{GL}(n, \mathbb{C})$ and an element $\gamma_-(x, t, \cdot) \in \mathcal{D}$ such that

$$\gamma(x, t, z) = \gamma_-^{-1}(x, t, z)\gamma_+(x, t, z), \quad R_0 < |z| < +\infty. \tag{4}$$

Moreover, the $\mathrm{gl}(n, \mathbb{C})$-valued function

$$q_f(x, t) := \lim_{z \to \infty} z[\gamma_-(x, t, z), a], \qquad (x, t) \in \Omega(f) \tag{5}$$

is off-diagonal, holomorphic and satisfies the soliton equation (3) on $\Omega(f)$.

The proof is based on the fact that $E = \gamma_+$ is a solution of (1). The solution $\gamma_\pm$ of the *Riemann factorization problem* (4) has the following geometrical meaning: $\gamma_+(x, t)$ is a parallel frame field with respect to the flat connection $\nabla(q_f) := d - U(q_f)\,dx - V(q_f)\,dt$ (this is equivalent to (1) for $E = \gamma_+$) and $\gamma_-(x, t)$ is a gauge transformation of the trivial flat connection $\nabla(0) = d - az\,dx - bz^m\,dt$ to the

[3] The elements $f \in \mathcal{D}$ (or rather their equivalence classes modulo right multiplication by diagonal elements of $\mathcal{D}$) will be the scattering data of the potentials $q_f(x, t_0)$ constructed below. Hence the notation $\mathcal{D}$.

connection $\nabla(q_f)$. In other words, putting $\mu = \gamma_-$, we have

$$\mu_x = U(q_f)\mu - \mu U(0), \qquad \mu_t = V(q_f)\mu - \mu V(0), \tag{6}$$

where $U(q) := az + q$ and $V(q) := bz^m + F_1(q)z^{m-1} + \cdots + F_m(q)$.

Examples of the solutions (5) constructed by means of the Riemann problem (4) include the solution of the Cauchy problem for the heat equation by means of the Laplace–Borel transform[4] (put $t = t_0$ for simplicity):

$$f(z) = \begin{bmatrix} 1 & \varphi(z) \\ 0 & 1 \end{bmatrix} \Rightarrow q_f(x) = \begin{bmatrix} 0 & u_f(x) \\ 0 & 0 \end{bmatrix}, \ \varphi(z) = -\int_0^\infty e^{-xz} u_f(x)\, dx, \tag{7}$$

the N-soliton solutions ($N = 1, 2, 3, \dots$) are obtained in the case when $f(z)$ is the product of N non-trivial Blaschke factors

$$B_{\alpha\beta P}(z) := I + \frac{\beta - \alpha}{z - \beta} P = k(z)P + (I - P), \qquad k(z) = \frac{z - \alpha}{z - \beta},$$

where $\alpha, \beta \in \mathbb{C}$ and $P \in \mathrm{gl}(n, \mathbb{C})$ satisfies $P^2 = P$ and, finally, all algebro-geometric solutions are obtained in the case when the columns of $f(z)$ are the eigenvectors of $R(z)$, where $R(z)$ is a rational $\mathrm{gl}(n, \mathbb{C})$-valued function and the matrix $R(\infty)$ has simple spectrum. These three types of examples were considered from our point of view in [5, 15] and [4], respectively.

2 Lecture II: The Local Inverse Scattering Method

Not all local (and even global) holomorphic solutions of (3) are of the form (5).[5] We suggest a generalization of the Riemann problem (4) which gives all local holomorphic solutions of (3) in the form (5). Before doing this, we define the direct scattering transform (that is, a procedure for recovery of f from q_f) using the first equation (6) as a hint.

Let $\mathcal{O}(x_0)^{\mathrm{od}}$ be the set of all $\mathrm{gl}(n, \mathbb{C})$-valued off-diagonal holomorphic germs at a point $x_0 \in \mathbb{C}$. Then for every $q \in \mathcal{O}(x_0)^{\mathrm{od}}$ there is a unique formal series

$$\mu(x, z) = I + \frac{\mu_1(x)}{z} + \frac{\mu_2(x)}{z^2} + \dots \quad \text{with} \quad \mu_j \in \mathcal{O}(x_0) \quad \text{such that}$$

$$\mu_x = (az + q)\mu - \mu a z \quad \text{and the series} \quad \mu(x_0, z) - I \quad \text{is off-diagonal.}$$

---

[4]The initial condition must necessarily be an entire function of exponential type.

[5]For example, the Cauchy problem $u(x, t_0) = u_0(x)$ for the heat equation $u_t = u_{xx}$ has a local holomorphic solution if and only if $u_0 \in \mathcal{O}(\mathbb{C})$ and $|u_0(x)| \le A \exp(B|x|^2)$ for some $A, B > 0$; but a solution of the form (5) exists if and only if $u_0 \in \mathcal{O}(\mathbb{C})$ and $|u_0(x)| \le C \exp(D|x|)$ for some $C, D > 0$.

We define the local scattering data of any potential $q \in \mathcal{O}(x_0)^{\text{od}}$ as the formal power series

$$Lq(z) := \mu(x_0, z) - I = \frac{\mu_1(x_0)}{z} + \frac{\mu_2(x_0)}{z^2} + \cdots .$$

Its radius of convergence will be zero for almost all potentials q. To measure the rate of its divergence, we introduce the so-called Gevrey class α for every $\alpha \geq 0$:

$$\text{Gev}_\alpha := \{\text{all formal power series } \varphi = \sum_{k=1}^{\infty} \frac{\varphi_k}{z^k} \text{ with off-diagonal}$$
$$\varphi_k \in \text{gl}(n, \mathbb{C}) \text{ such that } \sum_{k=1}^{\infty} \frac{|\varphi_k|}{k!^\alpha} A^k < \infty \text{ for some } A > 0\}.$$

Theorem 2 ([6]) *The correspondence* $q_0 \mapsto Lq_0$ *is a one-to-one mapping of* $\mathcal{O}(x_0)^{\text{od}}$ *onto* Gev_1. *Moreover, the Cauchy problem* $q(x, t_0) = q_0(x)$ *for the Eq.* (3) *has a local holomorphic solution* $q(x, t)$ *at the point* $(x_0, t_0) \in \mathbb{C}^2$ *if and only if* $Lq_0 \in \text{Gev}_{1/m}$.

We recall that $m \geq 2$ is the degree of the polynomial V in (1) and (2) or, equivalently, the highest order of derivative with respect to x in (3).

Proof This is only a sketch of proof. We define

$$\text{Ent}_m := \{\text{all holomorphic maps } \Phi : \mathbb{C} \to \text{GL}(n, \mathbb{C}) \text{ such that}$$
$$|\Phi(z)| \leq Ae^{B|z|^m} \text{ for some } A, B > 0\}$$

and consider the following Riemann factorization problem: given any $\varphi \in \text{Gev}_{1/m}$ and $\Phi \in \text{Ent}_m$, find $\psi \in \text{Gev}_{1/m}$ and $\Psi \in \text{Ent}_m$ such that

$$\Phi(z)(I + \varphi(z))^{-1} = (I + \psi(z))^{-1}\Psi(z) \tag{8}$$

as formal Laurent series. If we choose $\Phi(x, t, z) = \exp\{az(x - x_0) + (bz^m + c_1 z^{m-1} + c_m)(t - t_0)\}$ for all $(x, t) \in \mathbb{C}^2$ (as before (4)) and take any element $f(z) = I + \varphi(z) \in I + \text{Gev}_{1/m}$, then (8) takes the form

$$\Phi(x, t, z) f^{-1}(z) = \gamma_-^{-1}(x, t, z)\gamma_+(x, t, z) \tag{9}$$

completely analogous to (4). It has a solution $\gamma_\pm(x, t, z)$ for all (x, t) in some neighborhood $\Omega = \Omega(\varphi)$ of the point (x_0, t_0) in $\mathbb{C}^2$. This is because (9) is equivalent to solving a linear equation $(I + K(x, t))u(x, t) = u_0(x, t)$ in an appropriate Banach space, where $K(x, t)$ is a holomorphic family of bounded linear operators with $K(x_0, t_0) = 0$. Then we define $q_f \in \mathcal{O}(x_0, t_0)^{\text{od}}$ by the formula (5), verify that

$E = \Psi$ is a solution of (1) with $q = q_f$ and deduce (3). The same argument for $m = 1$ and $t = t_0$ yields that the maps $L : \mathcal{O}(x_0)^{\text{od}} \to \text{Gev}_1$ and

$$B : \text{Gev}_1 \to \mathcal{O}(x_0)^{\text{od}}, \qquad B\varphi(x) := \lim_{z\to\infty} z[\gamma_-(x,z), a], \tag{10}$$

are inverse to each other: $B \circ L = \text{Id}$, $L \circ B = \text{Id}$. □

Thus, the larger m, the smaller is the class of initial data for which (3) has a local holomorphic solution.

Using the criterion in the theorem, one can easily construct, for every positive integer M, examples of initial data such that (3) is soluble for $m = 1, \dots, M$, but not for $m = M+1, M+2, \dots$. One can also construct initial data which are not algebro-geometric but make (3) soluble (globally meromorphically in x and t) for every m.

To give a concrete example, take any integers $k, l \geq 0$ and note that

$$q_0(x) = \begin{bmatrix} 0 & (x-x_0)^k \\ (x-x_0)^l & 0 \end{bmatrix} \text{ has } Lq_0 \in \text{Gev}_\alpha \iff \alpha \leq \frac{k+l}{k+l+2}. \tag{11}$$

Hence the Cauchy problem $u(x, 0) = x^k$ for the Korteweg–de Vries equation $u_t + u_{xxx} + uu_x = 0$ or the nonlinear Schrödinger equation $iu_t + u_{xx} + u|u|^2 = 0$ has a local holomorphic solution at the origin of $\mathbb{C}^2$ if and only if $k = 0$ or $k = 1$ (here $l = 0$, $m = 3$ in the first case and $l = k$, $m = 2$ in the second). Another corollary of (11) is the divergence of the Kontsevich-Witten series (a formal power series solution of the whole KdV hierarchy whose coefficients are certain intersection numbers on the moduli space of plane curves) with respect to all time variables except the first one [8].

When the entries of $q_0 \in \mathcal{O}(x_0)^{\text{od}}$ are rational functions with a finite value at infinity (or, more generally, elliptic functions with a common period lattice) and the size n of matrices in question is 2, we encounter a kind of "zero-one law": either $Lq_0 \in \text{Gev}_0$ or $Lq_0 \notin \text{Gev}_\alpha$ whatever $\alpha < 1$. (This can be deduced by combining the theorem in Lecture III and Theorem 4.8 in [11].) The solutions of (3) resulting from the first case are the so-called Calogero-Moser rational solutions. It seems to be an open question whether this zero-one law holds for $n \geq 3$.

3 Lecture III: Painlevé Property

It was known already in the nineteenth century [16] that every local holomorphic solution of the heat equation $u_t = u_{xx}$ is an entire function of x for every fixed t. We say that an equation has holomorphic (resp. meromorphic) extension property (abbreviated to HEP or MEP respectively) if, for every solution $u \in \mathcal{O}(B)$, where $B = \{(x,t) \in \mathbb{C}^2 \mid |x - x_0| < \delta_1, |t - t_0| < \delta_2\}$, there is a holomorphic (resp. meromorphic) function $U \in \mathcal{O}(S)$ with $S = \{(x,t) \in \mathbb{C}^2 \mid |t - t_0| < \delta_2\}$

such that $U = u$ on B. For example, consider the equations

$$u_t = P(\partial_x)u, \quad u_t = u_{xx} + Q(u)u_x, \quad u_t = u_{xx} + R(u_x), \tag{12}$$

$$u_t = u_{xxx} + S(u)u_x, \quad u_t = u_{xxx} + T(u_x) \tag{13}$$

where P, Q, R, S, T are polynomials. These equations possess HEP if and only if they are linear (i.e. P may be arbitrary, Q and S must be constants, and R, T must be of degree ≤ 1). They possess MEP if and only if P is any polynomial, $\deg Q \leq 1$, $\deg R \leq 1$, $\deg S \leq 2$, $\deg T \leq 2$. There are reasons for believing that MEP is characteristic for soliton equations and equations related to them by differential-polynomial changes of unknown functions. See, for example, [1, 2, 12–14] for various approaches to this and other Painlevé-type properties,

The *only* known proof that the Eqs. (13) with $\deg S \leq 2$ and $\deg T \leq 2$ possess MEP is by means of the local inverse scattering method. These equations can be reduced to KdV (when $\deg S = 1$), mKdV (when $\deg S = 2$) or potential KdV (when $\deg T = 2$). Hence MEP for them follows from MEP for the Eqs. (3).

Theorem 3 ([6]) *All equations of the form* (3) *(with any $m \geq 2$) possess MEP. More generally, let $q_0 \in \mathcal{O}(x_0)^{\mathrm{od}}$ be any holomorphic germ with $Lq_0 \in \mathrm{Gev}_\alpha$ for some $\alpha < 1$. Then q_0 extends to a meromorphic* $\mathrm{gl}(n, \mathbb{C})$*-valued function on $\mathbb{C}$ which is monodromy-free in the sense that the first-order system $E_x = (az + q_0(x))E$ has a globally meromorphic fundamental system of solutions $E : \mathbb{C} \to \mathrm{GL}(n, \mathbb{C})$ for every $z \in \mathbb{C}$.*

Proof Let $q(x, t)$ be any local holomorphic solution of (3) near a point $(x_0, t_0) \in \mathbb{C}^2$. Put $q_0(x) = q(x, t_0)$. By the criterion in Lecture II, $Lq_0 \in \mathrm{Gev}_\alpha$ for some $\alpha < 1$ (namely, $\alpha = 1/m$). Consider the Riemann problem (9) with $m = 1$ and $t = t_0$:

$$e^{a(x-x_0)z}(I + Lq_0(z))^{-1} = \gamma_-^{-1}(x, z)\gamma_+(x, z).$$

It is equivalent to solving a linear equation $(I + K(x))u(x) = u_0(x)$ in an appropriate Banach space (see after (9)), but now $K(x)$ is a holomorphic family of *compact* linear operators (we use the inclusion $Lq_0 \in \mathrm{Gev}_\alpha$, $\alpha < 1$) parametrized by $\mathbb{C}^1_x$ with $K(x_0) = 0$. Hence, by the so-called meromorphic Fredholm alternative, there is an entire function $\tau \in \mathcal{O}(\mathbb{C})$ (playing the role of $\det(I + K(x))$) such that $\tau(x_0) = 1$ and

$$I + K(x) \quad \text{is invertible} \qquad \Longleftrightarrow \qquad \tau(x) \neq 0.$$

Moreover, the map $x \mapsto \tau(x)(I + K(x))^{-1}$ extends to $\mathbb{C}$ as a holomorphic operator-valued map. Hence the map $\gamma_-(x, \cdot) = (I + K(x))^{-1}u_0(x)$ is a meromorphic Gev_1-valued map on $\mathbb{C}^1_x$ with denominator $\tau(x)$ and, therefore, the matrix-valued function

$$Q_0(x) := BLq_0(x) = \lim_{z\to\infty} z[\gamma_-(x, z), a]$$

is meromorphic on $\mathbb{C}^1_x$ with denominator $\tau(z)$. By the results in Lecture II (after (10)), $Q_0(x) = q_0(x)$ in a neighborhood of x_0. Hence Q_0 is the desired meromorphic extension of q_0. This proves MEP since t_0 is arbitrary. The zero-monodromy property follows since $E = \gamma_+$ satisfies the first equation in (1). □

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Diffeomorphism Groups in Quantum Theory and Statistical Physics

Gerald A. Goldin

In memory of S. Twareque Ali

Abstract Symmetry groups describe invariances or partial invariances in physical systems under transformations. Locality refers to the association between physical effects and spatial or spacetime regions, with "action at a distance" forbidden. Local symmetry joins these ideas mathematically in the theory of certain infinite-dimensional groups and their representations. This chapter is an extended abstract of lectures by the author, surveying how unitary representations of diffeomorphism groups and the corresponding current algebras provide a unifying framework for understanding or predicting a wide variety of different quantum and statistical systems.

Keywords Cocycles · Configuration spaces · Diffeomorphism groups · Infinite-dimensional groups · Locality · Quantum statistics · Quasi-invariant measures · Statistical physics · Symmetry · Unitary representations

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1 Local Symmetry and Infinite-Dimensional Groups

It is well-known that symmetry groups describe transformations under which the kinematics or dynamics of a physical system may be invariant or partially invariant. Locality is the concept that "action at a distance" does not occur in physics, and that what we measure is always in the "here and now." To express this, we must make use of points or regions in a manifold M (of space or of spacetime). These ideas are joined to describe *local symmetry* by means of local current groups or gauge groups, diffeomorphism groups, their semidirect products, and (sometimes) their extensions. Local current groups (or their extensions) may describe kinematical or dynamical symmetries or partial symmetries; gauge groups typically describe transformations under which the outcomes of measurements remain invariant [1–4].

A local current group or gauge group S associates a Lie group L with points in a smooth manifold M. The group elements are compactly-supported C^∞ functions $f : M \to L$, with the group operation in S defined pointwise by the operation in L. Restricting the support of the functions f to a compact region $B \subset M$ defines a subgroup S_B for each (local) region B, consisting of functions which take the value of the identity in L outside of B. Thus locality is encoded in the group.

The diffeomorphsim group $G = \mathit{Diff}_0(M)$ is the group of compactly-supported, invertible C^∞ maps $\phi : M \to M$ (with C^∞ inverse). The group operation is composition. The support of a diffeomorphism ϕ is defined as the intersection of all closed sets outside of which $\phi(x) = x$, for $x \in M$. A subgroup of G is defined for a compact region $B \subset M$ as the set of diffeomorphisms G_B with support in B. Again, the group G encodes locality.

Both S and G may be endowed with the topology of uniform convergence in all derivatives. Letting M be the manifold of physical space, we consider the natural semidirect product of S with G. We thus have local (spatial) commutativity as a property of the group operation.

2 Quantum Mechanics and Statistical Physics from the Group Representations

These lectures focus on the easiest case, where $L = \mathbf{R}$ (under addition), so the map group S is just the function-space $\mathcal{D}$ of C^∞, compactly-supported functions on M. So we consider the semidirect product $\mathcal{D} \rtimes G$. For $f_1, f_2 \in \mathcal{D}$, and $\phi_1, \phi_2 \in G$, the semidirect product group law is

$$(f_1, \phi_1)\,(f_2, \phi_2) = (f_1 + \phi_1 f_2, \phi_1\phi_2), \tag{1}$$

where $\phi_1 f_2 = f_2 \circ \phi_1$, and $\phi_1\phi_2 = \phi_2 \circ \phi_1$.

A continuous, irreducible unitary representation $U(f)V(\phi)$ of $\mathcal{D} \rtimes G$ in Hilbert space $\mathcal{H}$ describes a quantum system in the physical space M. The diffeomorphism group describes its local kinematical symmetry. The distinct (inequivalent) irreducible unitary representations of $\mathcal{D} \rtimes G$ describe different quantum systems. This leads to a unification describing a wide variety of possible quantum configurations and exchange statistics, as described below.

In (classical) statistical physics, we may have an infinite gas of (non-interacting or interacting) identical point particles in $\mathbf{R}^3$, in equilibrium at a given temperature. There may also be an external potential acting on the positions of individual particles. The probability distribution for the particle locations (canonical ensemble) is described by an ergodic Poisson or Gibbs measure on the configuration space of infinite but locally finite particle configurations. This situation, too, corresponds to a rich class of irreducible unitary representations of the local symmetry group $\mathcal{D} \rtimes G$. In particular, different two-body interaction potentials lead to inequivalent measures, and corresponding inequivalent group representations.

3 Self-Adjoint Generators and Local Current Algebra

The corresponding infinite-dimensional Lie algebra is the semidirect sum of the Lie algebras of $\mathcal{D}$ and of G. Unitary representations of $\mathcal{D} \rtimes G$, satisfying appropriate conditions, determine self-adjoint representations of this Lie algebra by (generally unbounded) operators $\hat{\rho}(f)$ and $\hat{J}(\mathbf{g})$. Here $f \in \mathcal{D}$, while $\mathbf{g} \in vect_0(M)$ is a compactly-supported (tangent) vector field on M. These operators act on a common dense invariant domain of essential self-adjointness, and are the generators of continuous 1-parameter unitary subgroups:

$$U(sf) = \exp i(s/m)\hat{\rho}(f), \quad V(\phi_s^{\mathbf{g}}) = \exp i(s/\hbar)\hat{J}(\mathbf{g}) \quad (s \in \mathbf{R}), \tag{2}$$

where m is a unit mass and $\phi_s^{\mathbf{g}}$ is the flow on M generated by $\mathbf{g}$. The Lie algebra of local currents is:

$$[\hat{\rho}(f_1),\ \hat{\rho}(f_2)] = 0\,, \quad [\hat{\rho}(f),\ \hat{J}(\mathbf{g})] = i\hbar\hat{\rho}(\mathbf{g}\cdot\nabla f), \tag{3}$$

$$[\hat{J}(\mathbf{g}_1),\ \hat{J}(\mathbf{g}_2)] \;=\; -i\hbar\hat{J}([\mathbf{g}_1,\ \mathbf{g}_2])\,, \tag{4}$$

where $\mathbf{g}\cdot\nabla f$ is the Lie derivative of f in the direction of $\mathbf{g}$, and $[\mathbf{g}_1, \mathbf{g}_2] = \mathbf{g}_1\cdot\nabla\mathbf{g}_2 - \mathbf{g}_2\cdot\nabla\mathbf{g}_1$ is the Lie bracket of the vector fields.

This local current algebra is very natural and geometrical. In a representation, $\hat{\rho}(f)$ is interpreted physically as the mass density (spatially averaged with the scalar function f), and $\hat{J}(\mathbf{g})$ as the momentum density (spatially averaged with the vector field $\mathbf{g}$). Formally $\hat{\rho}(x)$ and $\hat{\mathbf{J}}(x)$ are operator-valued distributions over test-function spaces of scalar functions and vector fields (respectively). Free nonrelativistic or (perhaps surprisingly) relativistic Hamiltonians can be expressed explicitly in terms of the local density and current.

For example with $M = \mathbf{R}^d$, a single quantum particle is described by a state Ψ in the Hilbert space $\mathcal{H} = \mathcal{L}^2_{dx}(\mathbf{R}^d)$, and the unitary representation

$$U(f)\Psi(x) = \exp[if(x)]\Psi(x), \;\; V(\phi)\Psi(x) = \Psi(\phi(x))\sqrt{\mathcal{J}_\phi(x)}\,, \tag{5}$$

where $\mathcal{J}_\phi$ is the Jacobian of the diffeomorphism ϕ. The current algebra is represented by the self-adjoint operators,

$$[\hat{\rho}(f)\psi](x) = mf(x)\psi(x), \quad [\hat{J}(\mathbf{g})\psi](x) = (\hbar/2i)\{\mathbf{g}\cdot\nabla\psi + \nabla\cdot[\mathbf{g}\psi]\}(x)\,, \tag{6}$$

where m is the particle mass.

4 Representations Describing Distinct Quantum Systems

A wide variety of quantum-mechanical possibilities are unified in their description by classifying the inequivalent, irreducible unitary representations of $\mathcal{D} \rtimes G$. Some new, unexpected ones were predicted by this method!

Systems with finitely many degrees of freedom include (a) N-particle quantum mechanics, with particles distinguished by their masses; (b) systems of N indistinguishable particles obeying Bose or Fermi exchange statistics (in two or more space dimensions); (c) systems of indistinguishable particles (or excitations) obeying intermediate, anyon statistics in two space dimensions (for given anyonic phase shift under counterclockwise exchange) [5]; (d) systems of distinguishable anyonic particles in two-space with distinct relative phase shifts under counterclockwise exchange; (e) systems of particles obeying parastatistics (in two or more space dimensions); (f) systems of nonabelian anyons in two space dimensions; (g) systems of tightly bound charged particles—point dipoles, quadrupoles, etc.; (h) particles with spin, arranged in spin towers according to representations of the general linear group; and (i) particles with fractional spin, in two space dimensions.

Infinite systems described by unitary representations of the semidirect product group $\mathcal{D} \rtimes G$ include (j) systems of infinitely many particles, in locally finite configurations, corresponding to a free or interacting Bose gas; (k) systems of infinitely many particles obeying Fermi statistics, or (in two-space) exotic statistics; (l) systems of infinitely many particles having accumulation points; (m) quantized vortex systems in incompressible fluids (restricting the group elements to area- or volume-preserving diffeomorphisms), with filaments and patches of vorticity in two space dimensions or ribbons and tubes of vorticity in three dimensions [7]; and (n) configurations of extended quantum objects, including loops and strings, knotted configurations, and configurations of objects with nontrivial topology and/or nontrivial internal symmetry.

By choosing M and redefining G appropriately, one also describes (o) quantum mechanics on physical spaces that themselves are manifolds with boundary, with singularities, or with nontrivial topology. And a certain class of representations leads to (p) a consistent quantum theory of particles with nonlinear time-evolutions, which may be equivalent to linear theories via nonlinear gauge transformations, or else may violate the "no signal" property.

The mathematical theory behind some of the above descriptions is incomplete or only partially developed. There are many unanswered questions, opportunities for new constructions, and (probably) new predictions to be made of a fundamental nature [6].

5 Measures, Cocycles, and Topology

Under very general conditions a unitary representation of G may be written,

$$[V(\phi)\Psi](\gamma) \;=\; \chi_\phi(\gamma)\Psi(\phi\gamma)\sqrt{\frac{d\mu_\phi}{d\mu}(\gamma)}\,, \tag{7}$$

where: γ belongs to a configuration space Γ carrying a group action of G inherited naturally from M; μ is a measure on Γ which is quasi-invariant (i.e., the class of measure zero sets is preserved) under the action of G; and Ψ is a function from Γ to an inner product space $\mathcal{W}$ (a complex Hilbert space), with $\langle\Psi(\gamma), \Psi(\gamma)\rangle_{\mathcal{W}}$ integrable with respect to μ—i.e. $\Psi \in \mathcal{H} = \mathcal{L}^2_{d\mu}(\Gamma, \mathcal{W})$, where the inner product in $\mathcal{H}$ is given by $(\Phi, \Psi) = \int_\Gamma \langle\Phi(\gamma), \Psi(\gamma)\rangle_{\mathcal{W}}\, d\mu(\gamma)$. It may be, of course, that Ψ is just complex-valued. Finally, χ is a unitary 1-cocycle acting on $\Psi \in \mathcal{W}$; i.e., it satisfies the cocycle equation,

$$\chi_{\phi_1\phi_2}(\gamma) \;=\; \chi_{\phi_1}(\gamma)\chi_{\phi_2}(\phi_1\gamma)\,. \tag{8}$$

It is important to note that $\chi_\phi(\gamma)$ is defined only up to sets of μ-measure zero in Γ that can depend on ϕ; also, Eq. (8) holds only outside sets of measure zero that can depend on ϕ_1 and ϕ_2.

The system of Radon–Nikodym derivatives $\alpha_\phi(\gamma) = [d\mu_\phi/d\mu](\gamma)$, which exists due to the quasi-invariance of μ, is likewise a real 1-cocycle. It too is defined and satisfies the cocycle equation (8) only up to measure zero sets.

To represent the full semidirect product group, we also need to associate configurations γ with (continuous) linear functionals on $\mathcal{D}$. For example, N-point configurations may be identified with sums of N δ-functionals at distinct points in M. Then for $f \in \mathcal{D}$, we have $\langle\gamma, f\rangle \in \mathbf{R}$; and Eq. (7) is augmented with

$$[U(f)\Psi](\gamma) = \exp i\langle\gamma, f\rangle\Psi(\gamma)\,. \tag{9}$$

Then $U(f)V(\phi)$ provides the desired representation of $\mathcal{D} \rtimes G$ in $\mathcal{H}$.

Given Γ, the group action of G on Γ and a quasi-invariant measure μ on Γ, we may always set $\mathcal{W} = \mathbf{C}$ and $\chi_\phi(\gamma) \equiv 1$ to obtain a unitary representation of $\mathcal{D} \rtimes G$

on complex-valued wave functions. But inequivalent 1-cocycles describe inequivalent representations. These are constructed by "inducing" from the fundamental group (first homotopy group) of Γ. Thus does the topology of configuration space establish the possibilities for quantum exchange statistics (as well as other exotic possibilities) arising from kinematical symmetry.

The references below are only partial [1–7]; the reader is referred to many additional, important sources for the results described here.

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Position-Dependent Mass Systems: Classical and Quantum Pictures

Oscar Rosas-Ortiz

Abstract The present work is an extended abstract from a series of lectures addressed to introduce elements of the theory of position-dependent mass systems in both, classical and quantum mechanics.

Keywords Position dependent mass systems · Factorization method · Lie algebras · Nonlinear differential equations of motion

Mathematics Subject Classification (2010) Primary 70Hxx, 70Pxx; Secondary 81Q60, 81R15

1 Motivation

The study of systems endowed with position-dependent mass (PDM) is a subject of great interest in many branches of physics. Among others, the examples include dynamical systems in curved spaces with either constant curvature [12, 31] or non-constant curvature [55], geometric optics [61], semiconductor theory [8, 9], motion of rockets [59], raindrop problem [32], variable mass oscillators [27], inversion potential for NH3 in density theory [4], evolution of binary systems [29], effects of galactic mass loss [56], neutrino mass oscillations [10], and the problem of a rigid body against a liquid free surface [54].

Despite the large number of models used to describe the dynamics of PDM systems, the principles of the underlying theory are not fully understood. In the classical picture a position-dependent mass function $m(x)$ gives rise to 'forces quadratic in the velocity' which lead to nonlinear differential equations of motion in the Newtonian approach [19, 36]. In turn, the Hermiticity of the Hamiltonian of a quantum system is part of the problem to solve if the mass depends on the

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position [17]. Nevertheless, the difficulties appearing in both pictures can be faced by using the factorization method together with Lie algebraic tools [16–24, 33, 58].

The present work is an extended abstract from a series of lectures addressed to introduce elements of the theory of position-dependent mass systems in both, classical and quantum mechanics.

2 Classical Picture

The one-dimensional dynamical law for a system with position-dependent mass $m(x) > 0$ that is acted by a force F depending on position x, velocity $\dot{x}$, and time t, may be written as [19]:

$$F(x, \dot{x}; t) = \frac{dp}{dt} = m'(x)\dot{x}^2 + m(x)\ddot{x}, \tag{1}$$

where $p = m(x)\dot{x}$ is the linear momentum. Hereafter $\dot{f}$ and f' stand for time and position derivatives of f, respectively. Let us rewrite the Newton equation (1) in the standard form

$$m(x)\ddot{x} = F_{\text{net}}(x, \dot{x}; t) \equiv F(x, \dot{x}; t) - m'(x)\dot{x}^2. \tag{2}$$

We immediately see that the term quadratic in the velocity corresponds to the thrust of the system, so that Eq. (2) indicates how this term alters the velocity. Indeed, as $\dot{x}^2 \geq 0$, the system is accelerated (decelerated) if the rate $m'(x)$ is negative (positive). Thus, a particle suffering a spatial variation of its mass is acted by a net force F_{net} that results from the combination of the external force F and the thrust $-m'(x)\dot{x}^2$.

Applying the D'Alembert principle and assuming that the external force is derivable from a scalar potential function $\mathcal{V}(x)$ that does not depend on either velocity or time, from (1) we arrive at the Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \widetilde{R}, \qquad L = T - \mathcal{V}, \tag{3}$$

where $\widetilde{R}$ and T are the reacting thrust and kinetic energy, given by

$$\widetilde{R}(x, \dot{x}; t) := -\frac{1}{2}m'(x)\dot{x}^2, \quad T := \frac{1}{2}m(x)\dot{x}^2.$$

In turn, the Hamiltonian $\mathcal{H}$ can be obtained from the Legendre transformation

$$\mathcal{H}(x, p; t) = p\dot{x} - L(x, \dot{x}; t) = \frac{p^2}{2m(x)} + \mathcal{V}(x). \tag{4}$$

However, as the Hamiltonian's time rate of change $\dot{\mathcal{H}} = \widetilde{R}\dot{x}$ is cubic in the velocity, the variable mass system is dissipative [40]. That is, the Hamiltonian (4) is not a constant of motion. Yet, it may be shown [19] that the proper invariant acquires the form

$$I = \frac{p^2}{2m_0} + \int^x \frac{m}{m_0}\left(\frac{\partial \mathcal{V}}{\partial r}\right) dr, \tag{5}$$

where the constant m_0 is in mass units to get I expressed in energy units. Using integration by parts in the latter result we arrive at the expression

$$\left(\frac{m_0}{m}\right) I = \mathcal{H}(x, p; t) - \frac{m_0}{m}\int^x \left(\frac{m}{m_0}\right)' \mathcal{V} dr. \tag{6}$$

Clearly, the second term at the right hand side of (6) is just what the Hamiltonian $\mathcal{H}$ lacks to be a constant of motion. The invariant I can be expressed as a modification of the Hamiltonian (4) due to an effective potential

$$I := \mathcal{H}_{\text{eff}} \equiv \frac{m}{m_0}\left[\frac{p^2}{2m} + \mathcal{V}_{\text{eff}}\right], \tag{7}$$

with

$$\mathcal{V}_{\text{eff}} = \mathcal{V} - \frac{m_0}{m}\int^x \left(\frac{m}{m_0}\right)' \mathcal{V} dr. \tag{8}$$

For the sake of completeness let us introduce a new "mass" term $\mu(x) = m^2(x)/m_0$ as well as a new "momentum" variable $\pi = \mu\dot{x}$. The invariant (7) is simplified to $\mathcal{H}_{\text{eff}} = \frac{\pi^2}{2\mu} + \mathcal{V}_{\text{eff}}$, and the equations of motion can be expressed in conventional form

$$\dot{x} = \{x, \mathcal{H}_{\text{eff}}\}_{x,\pi} = \frac{\partial \mathcal{H}_{\text{eff}}}{\partial \pi}, \quad \dot{\pi} = \{\pi, \mathcal{H}_{\text{eff}}\}_{x,\pi} = -\frac{\partial \mathcal{H}_{\text{eff}}}{\partial x}, \tag{9}$$

where the Poisson bracket

$$\{f, g\}_{x,\pi} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial \pi} - \frac{\partial f}{\partial \pi}\frac{\partial g}{\partial x} \tag{10}$$

defines the time-variation of any smooth function h depending on x and π:

$$\frac{d}{dt}h = \{h, \mathcal{H}_{\text{eff}}\}_{x,\pi} + \frac{\partial h}{\partial t}. \tag{11}$$

In particular, if $h = \mathcal{H}_{\text{eff}}$, from (11) we have $\frac{d}{dt}\mathcal{H}_{\text{eff}} = 0$. Besides, from (10) one gets $\{x, \pi\}_{x,\pi} = 1$, so that x and π are conjugate variables.

To summarize the results of this section let us emphasize that, although the dynamical law for a particle that suffers a spatial variation of its mass includes forces quadratic in the velocity, the Lagrangian can be written in the standard form $L = \frac{p^2}{2m(x)} - \mathcal{V}(x)$, in correspondence with the conditions studied in [40]. The construction of the related Hamiltonian also leads to the standard form $\mathcal{H} = \frac{p^2}{2m(x)} + \mathcal{V}(x)$. That is, a simple description of this kind of systems starts by replacing the (constant) mass m_0 by the appropriate function of the position $m(x)$ in the conventional expressions of L and $\mathcal{H}$. Moreover, although $\mathcal{H}$ is not time-independent, it is possible to construct an energy constant of motion $I = \mathcal{H}_{\text{eff}}$ leading to dynamical equations that have the form of the Hamilton ones. Further details concerning the Lagrangian and Hamiltonian formulations for PDM systems can be consulted in [19, 40]. Additional results on the matter can be found in [26, 37, 44–47].

2.1 Algebraic Approach

The dynamical problem (9) may be studied in two general forms [19]. First, given a specific potential $\mathcal{V}(x)$ acting on the mass $m(x)$, the related phase trajectories are found. Second, given an algebra which rules the dynamical law of the mass, the potential and phase trajectories are constructed in a purely algebraic form. The second approach has been successfully applied in previous works [16, 19–24, 33, 58] and will be revisited in this section. The keystone is to notice that the factorization method introduced in [23, 33] can be extended to the case of PDM classical systems that obey the dynamical law (1). Indeed, working in the (x, π)-plane, the factorization of the Hamiltonian (7) leads in a natural form to the identification of a pair of time-dependent integrals of motion $\mathcal{Q}^{\pm}$ which, in turn, allows the construction of the phase trajectories $(x(t), \pi(t))$ associated to the canonical equations. Following [19], let us look for a couple of complex functions $\mathcal{A}^{+}(x, \pi; t)$, $\mathcal{A}^{-}(x, \pi; t)$, and a constant ϵ such that the Hamiltonian (7) becomes factorized

$$\mathcal{H}_{\text{eff}} = \mathcal{A}^{+}\mathcal{A}^{-} + \epsilon = \mathcal{A}^{-}\mathcal{A}^{+} + \epsilon, \tag{12}$$

where

$$\mathcal{A}^{\pm} = \mp i f(x)\frac{\pi}{\sqrt{2\mu(x)}} + g(x)\sqrt{\gamma\mathcal{H}_{\text{eff}}}. \tag{13}$$

Now, we ask the functions (13) to close a deformed Poisson algebra by demanding that $\{\mathcal{A}^-, \mathcal{A}^+\}$ be expressed in terms of powers of $\sqrt{\gamma \mathcal{H}_{\text{eff}}}$. In the simplest case we have

$$i\left\{\mathcal{A}^-, \mathcal{A}^+\right\} = 2\alpha\sqrt{\gamma\mathcal{H}_{\text{eff}}}, \quad i\left\{\mathcal{H}_{\text{eff}}, \mathcal{A}^\pm\right\} = \pm 2\alpha\sqrt{\gamma\mathcal{H}_{\text{eff}}}\mathcal{A}^\pm, \tag{14}$$

$$\{\mathcal{H}_{\text{eff}}, \mathcal{A}^+\mathcal{A}^-\} = \{\mathcal{H}_{\text{eff}}, \mathcal{A}^-\mathcal{A}^+\} = 0, \tag{15}$$

with

$$g(x) = \begin{cases} \sin\left[\sqrt{2\alpha^2 m_0}\int_c^x J(t)dt\right], & \gamma = 1, \quad \mathcal{H}_{\text{eff}} > 0, \\ \sinh\left[\sqrt{2\alpha^2 m_0}\int_c^x J(t)dt\right], & \gamma = -1, \; \mathcal{H}_{\text{eff}} < 0, \end{cases} \tag{16}$$

and $J(x) = \sqrt{\mu(x)/m_0}$. The function f is obtained from (16) through $f^2(x) = 1 - \gamma g^2(x)$. However, the potential allowing the above equations is not arbitrary since it depends on the g-function as follows

$$\mathcal{V}_{\text{eff}}(x) = \frac{\epsilon}{1-\gamma g^2(x)} = \begin{cases} \dfrac{\epsilon}{\cos^2\left[\sqrt{2\alpha^2 m_0}\int_c^x J(t)dt\right]}, & \mathcal{H}_{\text{eff}} > 0, \\ \dfrac{\epsilon}{\cosh^2\left[\sqrt{2\alpha^2 m_0}\int_c^x J(t)dt\right]}, & \mathcal{H}_{\text{eff}} < 0. \end{cases} \tag{17}$$

In other words, given the mass $\mu(x)$, the Pöschl–Teller potential (17) is such that the factors (13) satisfy the deformed Poisson algebra (14)–(15). Details concerning the time-dependent integrals of motion $\mathcal{Q}^\pm$ as well as further properties of the potentials (17) can be consulted in [19]. For other systems see, e.g. [16, 20, 21, 23, 24, 58].

3 Quantum Picture

Let us consider the one-dimensional Hamiltonian

$$H_a = \frac{1}{2}m^a P m^{2b} P m^a + V \equiv K_a + V, \qquad 2a + 2b = -1 \tag{18}$$

where the mass $m > 0$ and the potential V are functions of the position, K_a is the kinetic term of H_a and P fulfills $[X, P] = i\hbar$, with X the position operator. As indicated above, the Hermiticity of the Hamiltonian H_a is part of the problem to solve. In the present case the parameter a defines the ordering of the mass and momentum operators, so it must be properly chosen [16, 19, 22, 34, 38, 50]. Following [16, 17], the parameter a is kept arbitrary, with no more assumptions on a particular ordering of P and m. In position-representation, the eigenvalue equation

$$H_a \psi(x) = E\psi(x) \tag{19}$$

can be reduced to a simpler form by considering the point transformation

$$\psi(x) = e^{g(x)}\varphi(x), \quad x \mapsto y := s(x), \tag{20}$$

where s stands for a bijection that defines the Jacobian of the transformation $J := s'(x) = \sqrt{m(x)/m_0}$, and

$$\int_{\mathrm{Dom}(H_a)} |\psi(x)|^2 dx = \int_{\mathrm{Dom}(H_a)} |\, e^{g(x)}\varphi(x)|^2 dx < +\infty. \tag{21}$$

The straightforward calculation gives rise to the Hamiltonian

$$H_{\mathrm{eff}}^{(a)}\varphi(y) := \left[-\left(\frac{\hbar^2}{2m_0}\right)\frac{d^2}{dy^2} + V_{\mathrm{eff}}^{(a)}(y)\right]\varphi(y) = E\varphi(y), \tag{22}$$

with an effective potential

$$V_{\mathrm{eff}}^{(a)} := V - \left(\frac{\hbar^2}{2m^3}\right)\left[\left(\frac{1}{4}+a\right)mm'' - \left\{\frac{7}{16}+a(2+a)\right\}(m')^2\right] \tag{23}$$

that depends on the explicit expressions for the mass m and the initial potential V, both of them in the y-representation. Besides,

$$y = s(x) = \int e^{2g(x)}dx + y_0, \quad g(x) = \ln\left[J^{1/2}(x)\right] = \ln\left[\frac{m(x)}{m_0}\right]^{1/4}. \tag{24}$$

At this stage the main simplification is the avoiding of the mass ordering in the kinetic term of the effective Hamiltonian. Then, the techniques used to solve the eigenvalue equation of the constant mass systems may be applied to investigate the spectral problem defined by Eq. (22).

A further simplification is obtained if either (1) $a = -1/4$ or (2) the mass function $m(x)$ is such that $V_{\mathrm{eff}}^{(a)} - V = 0$. As the former case produces the identity $V_{\mathrm{eff}}^{(a)} = V$ for any well defined mass function $m(x)$, one says that the Hamiltonian $H_{\mathrm{eff}}^{(-1/4)}$ is defined by *mass-independent null terms* [17]. On the other hand, when the identity $V_{\mathrm{eff}}^{(a)} = V$ depends explicitly on the mass function $m(x)$, for $a \neq -1/4$ we say that $H_{\mathrm{eff}}^{(-1/4)}$ is defined by *mass-dependent null terms* [17]. In particular, a constant mass $m(x) = m_0$ reduces the effective potential (23) to the initial one V in y-representation.

3.1 Algebraic Approach

Let us factorize the Hamiltonian (18) in the form

$$H_a = AB + \epsilon, \tag{25}$$

with ϵ a constant (in energy units) to be determined,

$$A = -\frac{i}{\sqrt{2}}m^a P m^b + \beta, \qquad B = \frac{i}{\sqrt{2}}m^b P m^a + \beta, \qquad A^\dagger = B, \tag{26}$$

and β a solution of the Riccati equation

$$V - \epsilon = \frac{\hbar}{\sqrt{2m}}\left[2\left(a + \frac{1}{4}\right)\left(\frac{m'}{m}\right)\beta - \beta'\right] + \beta^2. \tag{27}$$

For arbitrary m and β the product between the factorization operators obeys the commutation rule:

$$[A, B] = -\left[\frac{\hbar}{m^{3/2}}\left(a + \frac{1}{4}\right)m'\right]^2 - \sqrt{\frac{2\hbar^2}{m}}\beta'. \tag{28}$$

Demanding the commutator (28) to close a given algebra we are in position of getting concrete realizations of the operators A, B.

In the simplest case one has $[A, B] = -\hbar\omega_0$, so that the β-function is defined in terms of the ordering parameter and the mass function:

$$\beta = \frac{\omega_0}{\sqrt{2}}\int^x m^{1/2}dr - \frac{\hbar}{\sqrt{2}}\left(a + \frac{1}{4}\right)\left(\frac{m'}{m^{3/2}}\right) + \beta_0. \tag{29}$$

Therefore

$$V = \frac{\omega_0^2}{2}\left[\int^x m^{1/2}dr\right]^2 = \frac{m_0\omega_0^2}{2}\left[\int^x J(r)dr\right]^2. \tag{30}$$

Notice that $m(x) = m_0$ produces the harmonic oscillator potential $V(x) = \frac{m_0\omega_0^2}{2}x^2$, with $\beta(x) = \left(\frac{m_0\omega_0^2}{2}\right)^{1/2} x + \beta_0$. Then, up to an additive constant, the operators A and B are reduced to the conventional ladder operators of the harmonic oscillator, as expected. For other forms of the mass function $m(x)$ and the appropriate ordering parameter a, the potential (30) represents a wide family of PDM potentials with the energy spectrum of the harmonic oscillator [17]. On the other hand, the introduction of (29) into (26) generates the ladder operators for such PDM oscillators. The construction of the corresponding generalized coherent states is also feasible [17].

Other PDM quantum systems can be studied through the commutator (28) by identifying the appropriate algebra. For instance, we may look for operators A and B such that the commutator (28) is associated with the su(1, 1) Lie algebra. The potential (30) is in such a case associated with a family of singular oscillators [18]. Additional constructions of coherent states for PDM systems have been reported in Refs. [1–3, 11, 14, 57, 62–64]. Further details concerning the properties of quantum systems with position-dependent mass can be consulted in, e.g. [5–7, 13, 15, 25, 28, 30, 35, 39, 41–43, 48, 49, 51–53, 60].

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Introduction to the Algebraic Bethe Ansatz

N. A. Slavnov

Abstract We introduce the reader to the basic concepts of the Quantum Inverse Scattering Method and the algebraic Bethe ansatz. We describe a method for constructing integrable systems in this framework. In particular, we obtain the Hamiltonian of the XXX Heisenberg spin chain by this method. We also describe a procedure for finding eigenvectors and the spectrum of quantum integrable Hamiltonians.

Keywords Quantum integrable systems · Transfer matrix · Eigenvector

Mathematics Subject Classification (2010) Primary 82B23; Secondary 81R50

1 Introduction

This is a very short introduction to the method of the algebraic Bethe ansatz (ABA). It is based on the lectures [2]. Interested readers are referred to this course of lectures in which a much more detailed and extended description of the ABA can be found. There one also can find references to numerous publications devoted to this method. In the text below, we only formulate the main statements of the ABA. All the proofs can be found in [2].

The ABA is a part of the Quantum Inverse Scattering Method (QISM) developed in the works of the Leningrad School under the leadership of L. D. Faddeev [1]. This method allows one to effectively describe the spectrum of quantum integrable models. The main advantage of the ABA is that this method deals with a special operator algebra describing a rather wide class of quantum systems. Then different physical systems are nothing but different representations of this algebra.

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2 Construction of Integrable Systems

Here we describe a general procedure for constructing quantum integrable systems, which is used within the framework of the QISM. One of the main objects of the QISM is a *monodromy matrix*:

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{1}$$

It's matrix elements are operators. They depend on the complex variable u and act in a Hilbert space $\mathcal{H}$. The latter is called *quantum space*, while the space in which $T(u)$ acts as a 2×2 matrix is called an *auxiliary space*.

The commutation relations between the monodromy matrix entries are given by an *RTT-relation*:

$$R(u, v)\big(T(u) \otimes I\big)\big(I \otimes T(v)\big) = \big(I \otimes T(v)\big)\big(T(u) \otimes I\big)R(u, v). \tag{2}$$

Here $R(u, v)$ is called an *R-matrix*. This is a c-number matrix, acting in the tensor product of the auxiliary spaces.

Let we have some $T(u)$ which satisfies Eq. (2). Then, multiplying (2) with $R^{-1}(u, v)$ from the right and taking the trace with respect to the auxiliary spaces we obtain

$$\mathcal{T}(u)\mathcal{T}(v) = \mathcal{T}(v)\mathcal{T}(u), \tag{3}$$

where

$$\mathcal{T}(u) = \operatorname{tr} T(u) = A(u) + D(u). \tag{4}$$

Let us expand the operator $\mathcal{T}(u)$ into power series over u centered in some point u_0

$$\mathcal{T}(u) = \sum_{k=0}^{\infty} (u - u_0)^k I_k. \tag{5}$$

Here the coefficients I_k are operators acting in the quantum space $\mathcal{H}$. Then it follows from (3) that all these operators commute

$$[I_k, I_n] = 0, \qquad \forall k, n. \tag{6}$$

If we now set one of I_k to be the Hamiltonian of a quantum system, then we get a model with, generally speaking, an infinite set of integrals of motion, that is, an integrable system. This is the general scheme for constructing integrable models in the framework of the QISM.

The operator $\mathcal{T}(u) = \operatorname{tr} T(u)$ is called the *transfer matrix*. The main task of the QISM is to find the eigenvectors of the transfer matrix. These vectors are simultaneously common eigenvectors of the Hamiltonian of the corresponding quantum model and all integrals of motion.

2.1 Yang–Baxter Equation

One might have impression that $R(u, v)$ can be any invertible matrix. However, there are restrictions which follow from the compatibility condition. The RTT-relation allows us to permute two monodromy matrices. We can reorder three or more monodromy matrices using the RTT-relation successively. However, in this case, the result may depend on the order in which the permutations are made. It can be shown that the result is independent of the permutation order if the R-matrix satisfies a relation

$$R_{23}(u_2, u_3)R_{13}(u_1, u_3)R_{12}(u_1, u_2) = R_{12}(u_1, u_2)R_{13}(u_1, u_3)R_{23}(u_2, u_3)\,, \tag{7}$$

which is called *Yang–Baxter equation*. This equation takes place in the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. The subscripts show in which two of the free spaces the R-matrix acts non trivially.

One of the simplest nontrivial solutions to the Yang–Baxter equation (7) is given by

$$R(u, v) = (u - v)\mathbf{1} + cP, \tag{8}$$

where $\mathbf{1}$ is the identity matrix, P is the permutation matrix, and c is a constant.

2.2 Hilbert Space

Recall that the elements of the monodromy matrix act in the quantum space $\mathcal{H}$, which we have not yet discussed. In the framework of the QISM, very small requirements are imposed on this space. Namely, it is necessary that there exists a vacuum vector $|0\rangle \in \mathcal{H}$ such that

$$A(u)|0\rangle = a(u)|0\rangle, \qquad D(u)|0\rangle = d(u)|0\rangle, \qquad C(u)|0\rangle = 0. \tag{9}$$

Here $a(u)$ and $d(u)$ are some functions of u. Their explicit form depends on the specific model. The action of the operator $B(u)$ on the vacuum is free. It is assumed that acting with this operator on $|0\rangle$ we generate all the space $\mathcal{H}$.

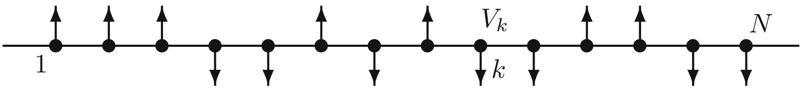

Fig. 1 XXX Heisenberg chain

3 XXX Heisenberg Chain

As we have already mentioned, the main advantage of the ABA is that within this method various physical models can be treated as different representations of the same operator algebra. However, to clarify the main idea of this method, we consider an example of a specific quantum system.

Consider a chain of N spin-$1/2$ particles (see. Fig. 1) and assume that the nearest neighbors in this chain interact with each other. For simplicity, we also assume that the strengths of the interaction along the axis x, y, and z are the same. This model is called the XXX Heisenberg chain (magnet).[1]

To describe this model at the mathematical language we first introduce a Hilbert space of the corresponding Hamiltonian. For this, we associate a *local quantum space* V_k with k-th site of the chain. Each of these spaces is isomorphic to $\mathbb{C}^2$: $V_k \sim \mathbb{C}^2$. The whole quantum space $\mathcal{H}$ of the model is a tensor product of the local spaces V_k: $\mathcal{H} = V_1 \otimes \cdots \otimes V_N$.

The spin-$1/2$ operators are given by the standard Pauli matrices

$$\sigma^x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{10}$$

In the case under consideration, the Pauli matrices corresponding to the particle at the k-th site of the chain have an additional subscript: σ_k^α, where $\alpha = x, y, z$. This means that σ_k^α acts non trivially in V_k only. In other words,

$$\sigma_k^\alpha = \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{k-1} \otimes \sigma^\alpha \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{N-k}, \qquad \alpha = x, y, z. \tag{11}$$

Thus, these operators are the matrices of the size $2^N \times 2^N$.

Then the Hamiltonian of this quantum system can be written in the form

$$H = \sum_{k=1}^{N} \Big(\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \sigma_k^z \sigma_{k+1}^z \Big), \tag{12}$$

[1]If the strengths of the interaction along different axis are different, then the model is called XYZ Heisenberg chain.

where we assumed periodic boundary conditions $\sigma_{N+1}^{x,y,z} = \sigma_1^{x,y,z}$.

The standard problem of the quantum mechanics is to find the spectrum of the Hamiltonian. Since in our case the Hamiltonian is a matrix of the size $2^N \times 2^N$, we are dealing with linear algebra. Therefore, we could find the eigenvalues of H by solving the characteristic equation

$$\det(H - \lambda \mathbf{1}) = 0. \tag{13}$$

Let us, however, assume that the chain under consideration is a model of a one-dimensional macroscopic crystal. This means that the number of sites of the chain N is equal to the number of atoms in this macroscopic crystal, say, $N \sim 10^6$. Then the Hamiltonian is a matrix of the size $2^{10^6} \times 2^{10^6}$, and Eq. (13) is an algebraic equation of degree 2^{10^6}. This number is so huge that there is nothing to compare it with. Clearly, no computer can cope with such a task. We come to the conclusion that the standard methods of the linear algebra in this case are useless, and, therefore, we need to look for some alternative ways. One of these ways is provided by the QISM.

4 *XXX* Chain via QISM

The model of the XXX Heisenberg chain can be formulated within the framework of the QISM. For this, we introduce a set of operator-valued matrices

$$L_k(u) = \begin{pmatrix} u + \frac{c}{2}\sigma_k^z & c\sigma_k^- \\ c\sigma_k^+ & u - \frac{c}{2}\sigma_k^z \end{pmatrix}, \qquad k = 1, \dots, N, \tag{14}$$

where $\sigma_k^{\pm} = \frac{1}{2}(\sigma_k^x \pm i\sigma_k^y)$. Then we define the monodromy matrix as follows:

$$T(u) = L_N(u) \dots L_2(u) L_1(u). \tag{15}$$

This matrix satisfies the RTT-relation (2) with the R-matrix (8). The corresponding transfer matrix $\mathcal{T}(u) = \operatorname{tr} T(u)$ generates a set of commuting operators. In particular, the Hamiltonian of the XXX chain (12) can be written in the form

$$H = c\frac{d\mathcal{T}(u)}{du}\,\mathcal{T}^{-1}(u)\Big|_{u=\frac{c}{2}} - N, \tag{16}$$

where we omitted the identity operator at N for brevity.

It is easy to check that a state with all spins up (ferromagnetic state)

$$|0\rangle = \binom{1}{0}_1 \otimes \cdots \otimes \binom{1}{0}_N \tag{17}$$

satisfies all the properties of the vacuum vector (9). The corresponding functions $a(u)$ and $d(u)$ have the form

$$a(u) = (u + \tfrac{c}{2})^N, \qquad d(u) = (u - \tfrac{c}{2})^N. \tag{18}$$

Thus, we have the complete description of the XXX chain on the language of the QISM.

5 Eigenvectors of the Transfer Matrix

The ABA is a procedure for constructing the eigenvectors of the transfer matrix. We formulate this procedure within the abstract scheme described in Sect. 2. We only need an explicit formula for the R-matrix (8) and the action formulas (9), however, we do not use formulas (14) and (15) for the monodromy matrix.

The main idea of the ABA is that we look for the eigenvectors of the transfer matrix in the form

$$|\Psi(\bar{u})\rangle = \prod_{k=1}^{n} B(u_k)|0\rangle, \qquad n = 0, 1, \dots . \tag{19}$$

In general, the set $\bar{u} = \{u_1, \dots, u_n\}$ can contain an arbitrary number of elements, and the parameters u_k can take arbitrary complex values.

We should act with the transfer matrix $\mathcal{T}(v) = A(v) + D(v)$ on the vector (19) and check whether $|\Psi(\bar{u})\rangle$ is an eigenvector of this operator. We have all necessary tools for this. First of all, the commutation relations between the monodromy matrix entries follow from the RTT-relation. We need only three of them:

$$[B(u), B(v)] = 0, \tag{20}$$

$$\begin{aligned} A(v)B(u) &= f(u,v)B(u)A(v) + g(v,u)B(v)A(u), \\ D(v)B(u) &= f(v,u)B(u)D(v) + g(u,v)B(v)D(u), \end{aligned} \tag{21}$$

where

$$f(u,v) = \frac{u - v + c}{u - v} \qquad \text{and} \qquad g(u,v) = \frac{c}{u - v}. \tag{22}$$

We shall call the first term in the rhs of (21) *the first commutation scheme* (when the operators keep their arguments). The term, in which the operators exchange their arguments, will be called *the second commutation scheme*.

Using these commutation relations we can move the operators A and D through the product of the operators B to the extreme right position. After this, it remains to act with A and D on the vacuum vector via (9).

Let us compute the action of the operator $A(v)$ on the vector $|\Psi(\bar{u})\rangle$. When the operator A is permuted with the operator B, it can either keep its argument, or borrow the argument of the operator B. As a result, we obtain

$$A(v)|\Psi(\bar{u})\rangle = a(v)\Lambda_0(v|\bar{u})|\Psi(\bar{u})\rangle + \sum_{k=1}^{n} a(u_k)\Lambda_k(v|\bar{u})|\Psi(\{v, \bar{u}_k\})\rangle. \tag{23}$$

Here $\Lambda_k(v|\bar{u})$ are some unknown coefficients to be determined and $\bar{u}_k = \bar{u} \setminus u_k$.

Let us compute the coefficient $\Lambda_0(v|\bar{u})$. Obviously, this coefficient arises in (23) if and only if we use only the first commutation scheme when permuting the operators $A(v)$ and $B(u_j)$. Then, after all permutations we obtain a product of the functions $f(u_j, v)$ over u_j, while the operator $A(v)$, approaching the extreme right position and acting on the vacuum, gives the function $a(v)$. As the result we have

$$\Lambda_0(v|\bar{u}) = \prod_{j=1}^{n} f(u_j, v). \tag{24}$$

Let us find now $\Lambda_k(v|\bar{u})$ with $k > 0$. Since, $|\Psi(\bar{u})\rangle$ is symmetric over the set $\bar{u}$ (due to (20)), it is enough to find $\Lambda_1(v|\bar{u})$. Then, in order to obtain a contribution proportional to $a(u_1)$, it is necessary to use the second commutation scheme when permuting $A(v)$ and $B(u_1)$. After this, when moving further to the right, $A(u_1)$ must keep its argument, therefore, we should use the first commutation scheme only. Thus, we arrive at

$$\Lambda_1(v|\bar{u}) = g(v, u_1) \prod_{j=2}^{n} f(u_j, u_1), \tag{25}$$

leading to

$$\Lambda_k(v|\bar{u}) = g(v, u_k) \prod_{\substack{j=1 \\ j\neq k}}^{n} f(u_j, u_k). \tag{26}$$

The action of the D-operator can be computed exactly in the same way, and we find

$$\begin{aligned} \mathcal{T}(v)|\Psi(\bar{u})\rangle &= \left(a(v) \prod_{j=1}^{n} f(u_j, v) + d(v) \prod_{j=1}^{n} f(v, u_j) \right) |\Psi(\bar{u})\rangle \\ &+ \sum_{k=1}^{n} g(v, u_k) \left(a(u_k) \prod_{\substack{j=1 \\ j\neq k}}^{n} f(u_j, u_k) - d(u_k) \prod_{\substack{j=1 \\ j\neq k}}^{n} f(u_k, u_j) \right) |\Psi(\{v, \bar{u}_k\})\rangle. \end{aligned} \tag{27}$$

We see that generically $|\Psi(\bar{u})\rangle$ is not an eigenvector of the transfer matrix because of the presence of new vectors $|\Psi(\{v, \bar{u}_k\})\rangle$ in the second line of (27). However, if we impose conditions

$$\frac{a(u_k)}{d(u_k)} = \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{f(u_k, u_j)}{f(u_j, u_k)}, \qquad k = 1, \dots, n, \tag{28}$$

then the sum in the second line of (27) vanishes. Then the vector $|\Psi(\bar{u})\rangle$ becomes an eigenvector of the operator $\mathcal{T}(v)$ with an eigenvalue

$$\tau(v|\bar{u}) = a(v) \prod_{j=1}^{n} f(u_j, v) + d(v) \prod_{j=1}^{n} f(v, u_j). \tag{29}$$

The system (28) is called the *system of Bethe equations*.

Thus, the transfer matrix eigenvectors have the form (19), where the parameters $\bar{u} = \{u_1, \dots, u_n\}$ are the roots of Bethe equations (28).

Applying this scheme to the XXX Heisenberg chain we should only substitute the functions $a(u)$ and $d(u)$ (18) into the Bethe equations

$$\left(\frac{u_k + \frac{c}{2}}{u_k - \frac{c}{2}}\right)^N = \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{f(u_k, u_j)}{f(u_j, u_k)} = -\prod_{j=1}^{n} \frac{u_k - u_j + c}{u_k - u_j - c}, \qquad k = 1, \dots, n. \tag{30}$$

The Hamiltonian eigenvalues then can be found via (16)

$$E = c \frac{d \log\big(\tau(v|\bar{u})\big)}{dv}\Big|_{v=\frac{c}{2}} - N = \frac{1}{2} \sum_{j=1}^{n} \frac{c^2}{u_j^2 - c^2/4}. \tag{31}$$

Thus, the spectrum of the XXX Heisenberg chain is completely determined by the solutions of the Bethe equations (30). Of course, the Bethe equations admit an exact analytic solution only in exceptional cases. This is to be expected, because having an exact solution to the system (30) would mean that we were able to find an exact solution to the algebraic equation (13). It would be naive to hope so. However, the system (30) is easily amenable to numerical analysis even at n and N large enough. Furthermore, in the limit of the chain of large length, the Bethe equations can be replaced by a linear integral equation for the density of roots, which admits an explicit solution.

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Noncommutative Fiber Bundles

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Abstract In this note, we give a concise introduction to the noncommutative framework for a generalization of the classical concept of compact, locally trivial fiber bundle.

Keywords Noncommutative fiber bundle · Quantum group · Hopf algebra · Noncommutative principal bundle

Mathematics Subject Classification (2010) 46L65, 81R60

1 From Classical to Noncommutative Bundles

Let T, B, F be topological spaces (throughout these notes always assumed compact and Hausdorff), then the structure of a locally trivial fiber bundle $F \longrightarrow T \longrightarrow B$ consists of a continuous surjection $\pi : T \to B$ such that for each point $b \in B$ the fiber $\pi^{-1}(b)$ is homeomorphic with F, and a finite cover of B by open subsets U_j such that each set $\pi^{-1}(U_j)$ is homeomorphic to $U_j \times F$ by a map which carries fibers onto fibers. Instead of purely topological, one may also consider differentiable bundles.

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In order to make transition from the commutative to the noncommutative world, one trades spaces for algebras in the spirit of Gelfand's duality. Thus each space T, B, F is replaced by the corresponding C^*-algebra of continuous, complex-valued functions on it $C(T)$, $C(B)$, $C(F)$. Remembering that Gelfand's duality is a contravariant functor, the projection $\pi : T \to B$ is replaced by a unital embedding $C(B) \subseteq C(T)$. From this embedding of algebras one may also recover the algebra of functions on the fiber $C(F)$, see [9, Remark 3.5]. In order to account for the differentiable structure one considers dense $*$-subalgebras of differentiable (or polynomial) functions $\mathcal{O}(T)$, $\mathcal{O}(B)$, $\mathcal{O}(F)$. A detailed account of how exactly classical theory of differentiable fiber bundles may be developed in the language of algebras rather than spaces is presented in [1]. The next step is to allow for noncommutative algebras and develop an analogous theory.

2 Principal Bundles

In the classical setting, principal bundle arises when the fiber F is a compact group, called the structure group, acting freely on the total space T in such a way that the projection π identifies the base space B with the orbit space for this action. When F is a Lie group, freeness of action automatically implies local triviality of the corresponding bundle.

For applications to noncommutative geometry, we need to deal with structure groups with noncommutative algebras of "functions", that is with compact quantum groups, whose theory was developed in the eighties by Woronowicz, [26], and by the Russian school, [11, 21]. See [17] for a comprehensive account. A compact quantum group G consists of a unital C^*-algebra $C(G)$ (possibly noncommutative!) and a unital $*$-homomorphism

$$\Delta : C(G) \to C(G) \otimes C(G),$$

comultiplication, which is coassociative $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ and such that both $\{(x \otimes 1)\Delta(y) \mid x, y \in C(G)\}$ and $\{(1 \otimes x)\Delta(y) \mid x, y \in C(G)\}$ are linearly dense in the minimal C^*-algebraic tensor product $C(G) \otimes C(G)$. This definition implies, in a non-obvious way, existence of a dense $*$-subalgebra $\mathcal{O}(G) \subseteq C(G)$ carrying the comultiplication Δ and equipped with two linear maps: counit $\epsilon : \mathcal{O}(G) \to \mathbb{C}$ and antipode $S : \mathcal{O}(G) \to \mathcal{O}(G)$ such that

$$m(\epsilon \otimes \mathrm{id})\Delta = \mathrm{id} = m(\mathrm{id} \otimes \epsilon)\Delta \quad \text{and} \quad m(S \otimes \mathrm{id})\Delta = \epsilon = m(\mathrm{id} \otimes S)\Delta.$$

Here $m : \mathcal{O}(G) \otimes \mathcal{O}(G) \to \mathcal{O}(G)$ denotes the algebraic multiplication. Furthermore, there exists a unique faithful state h on $C(G)$, *the Haar measure*, characterized by the identity

$$(h \otimes \mathrm{id})\Delta = h = (\mathrm{id} \otimes h)\Delta.$$

The $*$-algebra $\mathcal{O}(G)$ equipped with the comultiplication Δ, counit ϵ, and antipode S, is a Hopf $*$-algebra, [19]. $\mathcal{O}(G)$ is spanned by matrix elements of irreducible representations of the (quantum) group G.

An action of a compact quantum group G on a unital C^*-algebra A is a unital, injective $*$-homomorphism

$$\rho : A \longrightarrow A \otimes C(G)$$

which is coassociative $(\rho \otimes \mathrm{id})\rho = (\mathrm{id} \otimes \Delta)\rho$, and such that the set $\{\rho(a)(1 \otimes x) \mid a \in A, x \in C(G)\}$ is linearly dense in the minimal C^*-algebraic tensor product $A \otimes C(G)$. Then $A^G := \{a \in A \mid \rho(a) = a \otimes 1\}$ is the fixed point algebra, a unital C^*-subalgebra of A. In the classical case, one defines the corresponding crossed product $A \rtimes_\rho G$ as the C^*-algebra generated by A and the group C^*-algebra $C^*(G)$, which interact non-trivially so that $*$-representations of $A \rtimes_\rho G$ are in one-to-one correspondence with G-covariant representations of A. A similar construction of the crossed product is also possible for actions of compact quantum groups, see [10] for details.

Now, by a noncommutative principal bundle we simply understand a unital C^*-algebra A playing the role of the total space, equipped with a **free** action of a compact quantum structure group G, with the corresponding fixed point algebra A^G playing the role of the base space. Classically freeness means that a group G acts with trivial stabilizers on the primitive ideal space of A. This definition is inapplicable to the noncommutative case, since then the primitive ideal space of A may carry little information. At least two different noncommutative generalizations of freeness have been proposed. First one, due to Rieffel, requires that the fixed point algebra A^G and the crossed product $A \rtimes_\rho G$ be strongly Morita equivalent via a natural imprimitivity bimodule contained in $A \rtimes_\rho G$. This condition is called saturatedness. See [10] for details, and [23] for a discussion of the special case of finite quantum groups (with $C(G)$ finite dimensional). The second one, due to Ellwood, [12], requires that the image of the linear map

$$\chi : A \otimes_{A^G} A \longrightarrow A \otimes C(G), \quad \chi(a_1 \otimes a_2) := (a_1 \otimes 1)\rho(a_2) \tag{1}$$

be dense in the minimal C^*-algebraic tensor product. As shown in [10], these two seemingly different definitions of freeness actually coincide.

In order to account for differentiable structure in noncommutative setting, one needs to consider dense $*$-subalgebras of the ambient C^*-algebras, playing the role of smooth or polynomial functions. To this end, let $H := \mathcal{O}(G)$, a Hopf algebra, and choose an $\mathcal{A} \subseteq A$ invariant under the action, so that $\rho : \mathcal{A} \longrightarrow \mathcal{A} \otimes H$. Let $\mathcal{A}^H$ be the corresponding fixed point subalgebra of $\mathcal{A}$. One says that the inclusion $\mathcal{A}^H \subseteq \mathcal{A}$ is Hopf-Galois if the canonical Galois map $\chi : \mathcal{A} \otimes_{\mathcal{A}^H} \mathcal{A} \longrightarrow \mathcal{A} \otimes H$, defined as in (1), is bijective. If, in addition, $\mathcal{A}$ is a faithfully flat $\mathcal{A}^H$-module then the extension is called principal, [22]. It turns out that principality is equivalent to existence of a strong connection $\ell : H \longrightarrow \mathcal{A} \otimes \mathcal{A}$, that is a unital splitting of the multiplication map m on $\mathcal{A}$, i.e. $m \circ \ell = \epsilon$, satisfying some additional properties, see [4–6, 13].

Finding a strong connection is a good practical way of showing principality of an extension.

In classical geometry, every representation of the structure group of a principal bundle gives rise to an associated bundle for which the corresponding module of sections is projective over the algebra of functions on the base space. There is a perfect analogy of this construction in the noncommutative setting as well. Namely, let V be a left H-comodule, equipped with the coassociative action $\lambda : V \longrightarrow H \otimes V$. The corresponding cotensor product

$$\mathcal{A}\Box_H V := \left\{\sum a_j \otimes v_j \in \mathcal{A} \otimes V \mid \sum \rho(a_j) \otimes v_j = \sum a_j \otimes \lambda(v_j)\right\}$$

plays the role of the module of sections of the associated bundle. In fact, if the extension $\mathcal{A}^H \subseteq \mathcal{A}$ is principal then $\mathcal{A}\Box_H V$ is a projective $\mathcal{A}^H$-module (finitely generated if V is finite dimensional). In particular, $\mathcal{A}$ is a projective $\mathcal{A}^H$-module (typically infinitely generated).

3 Examples and Outlook

3.1 Subgroups

Let G and K be compact quantum groups, and assume there exists a surjective $*$-homomorphism $\phi : C(G) \longrightarrow C(K)$ such that $\Delta_K \phi = (\phi \otimes \phi)\Delta_G$. Thus K is quantum subgroup of G. In that case, $\rho = (\mathrm{id} \otimes \phi)\Delta_G$ yields a principal action of K on $C(G)$. For a concrete example, let $[u_{ij}]$ be the fundamental matrix of the quantum $SU(N)$ group, [28], and let z_i be the standard unitary generators for $C(U(1))$. Then the assignment $u_{ij} \mapsto \delta_{ij} z_i$ extends to such a Δ-commuting homomorphism ϕ : $C(SU_q(N)) \longrightarrow C(U(1)^N)$. In this way, the classical group $U(1)^N$ can be viewed as the maximal torus of the quantum group $SU_q(N)$.

3.2 Gauge Actions on Graph Algebras

Every C^*-algebra $C^*(E)$ of a directed graph E comes equipped with a gauge action of the circle group $U(1)$. This action is principal if and only if the graph has no sinks and no sources, [24]. On the other hand, C^*-algebras of functions on quantum odd-dimensional spheres, [25], are known to be isomorphic to graph algebras, [15]. This leads to construction of noncommutative analogues of classical complex projective spaces, [15, 25], and provides a convenient setting for the quantum Hopf fibration, [14]. It is possible to consider generalized gauge actions as well, with different "coordinates" rotated at different speeds. Fixed point algebras of such actions on the Vaksman-Soibelman quantum spheres give rise to quantum weighted projective spaces, [7]. Restricting the generalized gauge action to cyclic subgroups one obtains a large class of examples of free actions of finite cyclic groups on noncommutative C^*-algebras, [24]. In particular, fixed point algebras for such actions on quantum odd-spheres give rise to quantum lens spaces, [7, 16].

3.3 Generalizations: The Instanton Bundle

Other approaches to noncommutative principal bundles exist in the literature, besides the one described in these notes. In particular, one may talk about principal bundles involving actions ρ that are not multiplicative (i.e. not algebra homomorphisms) but only coassociative, see [6]. An interesting example of such an approach is the construction of the quantum instanton bundle $SU_q(2) \longrightarrow S_q^7 \longrightarrow S_q^4$ in [3] and [2]. Here $SU_q(2)$ is the Woronowicz quantum $SU(2)$ group, [27], and S_q^7 is the Vaksman-Soibelman quantum 7-sphere, [25]. It should be noted that other constructions of quantum instanton bundles have been proposed as well, including the one from [18], where the action ρ is actually multiplicative on the polynomial algebra of the quantum 7-sphere defined therein.

3.4 Sphere Bundles

It is not obvious how to construct a noncommutative analogue of a general locally trivial fiber bundle. We would like to point out two sources of immediate difficulties. Firstly, local triviality is typically defined with reference to points of the base space B of the fiber bundle, that is to characters of the corresponding algebra of functions $C(B)$. This approach is inapplicable in the noncommutative setting due to possible lack of (sufficiently many) characters on noncommutative algebras. Secondly, in order to recover the fiber F from the inclusion $C(B) \subseteq C(T)$ in the classical case, once again one has to resort to characters of $C(B)$.

Recently, a case study of a noncommutative sphere bundle has been undertaken in [8, 9]. Classically the full flag manifold of the $SU(3)$ group has a natural structure of a 2-sphere bundle over the complex projective 2-space. The total space of this bundle admits a natural q-deformation, namely as the fixed point algebra for the natural action of the maximal torus $\mathbb{T}^2$ on the quantum $SU_q(3)$ group of Woronowicz. This algebra $C(SU_q(3)/\mathbb{T}^2)$ contains a copy of the C^*-algebra $C(\mathbb{C}P_q^2)$. It turns out that the inclusion $C(\mathbb{C}P_q^2) \subseteq C(SU_q(3)/\mathbb{T}^2)$ can be viewed as a noncommutative fibration, with a typical fiber quantum complex projective 1-space $\mathbb{C}P_q^1$, i.e. the standard Podleś sphere, [20]. In fact, on the level of polynomial algebras, one obtains the desired cotensor product decomposition mimicking the classical situation:

$$\mathcal{O}(SU_q(3)/\mathbb{T}^2) \cong \mathcal{O}(SU_q(3))\Box_{\mathcal{U}_q(2)}\mathcal{O}(\mathbb{C}P_q^1).$$

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Correction to: Toeplitz Extensions in Noncommutative Topology and Mathematical Physics

Francesca Arici and Bram Mesland

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The original version of the book was inadvertently published without the following corrections. The chapter has now been corrected.

Corrections:

Chapter 1, "Toeplitz Extensions in Noncommutative Topology and Mathematical Physics" was mistakenly published non-open access. This has been amended with the license updated to CC BY 4.0 and the Copyright Holder changed to "The Author(s)". The book has also been updated with this change.

Page iv: The corresponding OA information has been included.

The updated online version of this chapter can be found at
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