Chapter 4
Stratification Theory

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Contents

Abstract This is a survey of stratification theory in the differentiable category from its beginnings with Whitney, Thom and Mather until the present day. We concentrate mainly on the properties of C^∞ stratified sets and of stratifications of subanalytic or definable sets, with some reference to stratifications of complex analytic sets. Brief mention is made of the theory of stratified mappings.

4.1 Stratifications

The idea behind the notion of stratification in differential topology and algebraic geometry is to partition a (possibly singular) space into smooth manifolds with some control on how these manifolds fit together.

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In 1957 Whitney [146] showed that every real algebraic variety V in $\mathbf{R}^n$ can be partitioned into finitely many connected smooth submanifolds of $\mathbf{R}^n$. This he called a *manifold collection*. Such a partition is obtained by showing that the singular part of V is again algebraic and of dimension strictly less than that of V. One obtains thus a filtration of V by algebraic subvarieties,

$$V \supset SingV \supset Sing(SingV) \supset \dots$$

In 1960, Thom [120] replaced the term "manifold collection" of Whitney, by "stratified set", introduced the notions of "stratum" and "stratification", and initiated a theory of stratified sets and stratified maps. Later, in 1964, Thom proposed that a stratification should have the property that transversality of a map $g : \mathbf{R}^m \to \mathbf{R}^n$ to the strata of a stratified set in $\mathbf{R}^n$ be an open condition on maps in $C^\infty(\mathbf{R}^m, \mathbf{R}^n)$, and that there should be some "local triviality" in a neighbourhood of each stratum.

As a result Whitney refined his definition in 2 papers [147, 148] which appeared in 1965, concerning stratifications of real and complex analytic varieties, introducing his conditions (a) and (b). He proved the existence of stratifications satisfying these conditions for any real or complex analytic variety, remarked that Thom's openness of transversality follows from condition (a) and conjectured a local fibration property (known as Whitney's holomorphic fibering conjecture, this was finally proved by Parusinski and Paunescu in 2017 [102] after a partial version obtained by Hardt and Sullivan in 1988 [49]). Thom then developed a theory of C^∞ stratified sets, described in detail in his 1969 paper entitled "*Ensembles et morphismes stratifiés*" [123]. The following year Mather gave a series of lectures at Harvard giving a revised account of Thom's theory of stratified sets and maps, and it is Mather's definitions that have been generally used since then. Mather's 1970 notes of his lectures, which circulated widely via photocopies, were finally published in 2012 [81]. The reader may profitably consult also [36] and [105] for detailed presentations of the theory of stratified sets.

I will now describe what has become the accepted notion of Whitney stratification.

Definition 4.1.1 Let Z be a closed subset of a differentiable manifold M of class C^k. A C^k *stratification* of Z is a filtration by closed subsets

$$Z = Z_d \supset Z_{d-1} \supseteq \cdots \supseteq Z_1 \supseteq Z_0$$

such that each difference $Z_i - Z_{i-1}$ is a differentiable submanifold of M of class C^k and dimension i, or is empty. Each connected component of $Z_i - Z_{i-1}$ is called a *stratum* of dimension i. Thus Z is a disjoint union of the strata, denoted $\{X_\alpha\}_{\alpha \in A}$, and Z is a *stratified set*.

Example 4.1.2 The filtration of a realisation of a simplicial complex defined by skeleta is a C^∞ stratification, where the strata are the open simplices.

We would like our stratifications to "look the same" at different points on the same stratum. This turns out to be possible if "looking the same" is interpreted as having neighbourhoods which are homeomorphic, a kind of equisingularity. Various stronger equisingularity conditions, also called regularity conditions, have been introduced ensuring this. An obvious necessary condition is as follows:

Definition 4.1.3 A stratification $Z = \bigcup_{\alpha \in A} X_\alpha$ satisfies the *frontier condition* if $\forall(\alpha, \beta) \in A \times A$ such that $X_\alpha \cap \overline{X_\beta} \neq \emptyset$, one has $X_\alpha \subseteq \overline{X_\beta}$. As the strata are disjoint this means that either $X_\alpha = X_\beta$ or $X_\alpha \subset \overline{X_\beta} \setminus X_\beta$.

Definition 4.1.4 One says that a stratification is *locally finite* if the number of strata is locally finite.

4.2 Whitney's Conditions (*a*) and (*b*)

The most widely used of the different regularity conditions proposed so as to provide adequate "equisingularity" of a stratification are the conditions (a) and (b) of Whitney.

Definition 4.2.1 Take two adjacent strata X and Y, i.e. two C^1 submanifolds of M such that $Y \subset \overline{X} \setminus X$, so that X is adjacent to Y. The pair (X, Y) is said to satisfy Whitney's condition (a) at $y \in Y$, or to be (a)-regular at y if : for all sequences $\{x_i\} \in X$ with limit y such that, in a local chart at y, $\{T_{x_i} X\}$ tends to τ in the grassmannian $G_{dim X}^{dim M}$, one has $T_y Y \subseteq \tau$.

When every pair of adjacent strata of a stratification is (a)-regular (at each point) then we say that the stratification is (a)-regular.

Definition 4.2.2 The pair (X, Y) is said to satisfy Whitney's condition (b) at $y \in Y$, or to be (b)-regular at y if : for all sequences $\{x_i\} \in X$ and $\{y_i\} \in Y$ with limit y such that, in a local chart at y, $\{T_{x_i} X\}$ tends to τ and the lines $\overline{x_i y_i}$ tend to λ, one has $\lambda \in \tau$.

When every pair of adjacent strata of a stratification is (b)-regular (at each point) then we say that the stratification is (b)-regular.

Definition 4.2.3 Let Z be a closed subset of a differentiable manifold M of class C^1. When $Z = \bigcup_{\alpha \in A} X_\alpha$ is a locally finite (b)-regular stratification satisfying the frontier condition, we say we have a *Whitney stratification* of Z.

Remark 4.2.4 It will be a nontrivial consequence of the theory that the frontier condition is automatically satisfied by pairs of adjacent strata of a locally finite (b)-regular stratification.

Definition 4.2.5 Let $\pi : T_Y \to Y$ be the retraction of a C^1 tubular neighbourhood of Y in M. A pair of adjacent strata (X, Y) is said to be (b^π)-regular if for all sequences $\{x_i\}$ in X such that x_i tends to y and the lines $\overline{x_i \pi(x_i)}$ tend to λ and the tangent planes $T_{x_i} X$ tend to τ, then $\lambda \in \tau$.

When every pair of adjacent strata of a stratification is (b^π)-regular (at each point) then we say that the stratification is (b^π)-regular.

Exercises

1. $(b) \Rightarrow (a)$.
2. $(b) \Leftrightarrow (b^\pi) \quad \forall \pi$.
3. (b) holds if both (a) and (b^π) hold for some π.
4. If (X, Y) is (b)-regular at $y \in Y$, then $\dim Y < \dim X$.
   The following standard example due to Whitney shows that (a) does not imply (b).

Example 4.2.6 Let $Z = Z_2 = \{y^2 = t^2x^2 + x^3\} \subset \mathbf{R}^3$. Set $Z_1 = \{(0, 0, t)|t \in \mathbf{R}\}$ and $Z_0 = \emptyset$. Then $Z_2 \supset Z_1 \supset Z_0 = \emptyset$ is a filtration defining a C^∞ stratification with 4 strata of dimension 2 and one stratum of dimension 1. The strata are defined as follows : $X_1 = (Z_2 - Z_1) \cap \{t > 0\} \cap \{x < 0\}$, $X_2 = (Z_2 - Z_1) \cap \{t < 0\} \cap \{x < 0\}$, $X_3 = (Z_2 - Z_1) \cap \{y < 0\} \cap \{x > 0\}$, $X_4 = (Z_2 - Z_1) \cap \{y > 0\} \cap \{x > 0\}$, $Y = Z_1$. One can check that the pairs of strata (X_3, Y) and (X_4, Y) are (b)-regular, and in fact they form C^∞ manifolds with boundary, while (X_1, Y) and (X_2, Y) are not (b)-regular at $(0, 0, 0)$, although they are (a)-regular. Note that the frontier condition does not hold for (X_1, Y) and (X_2, Y). It is possible to unite X_1 and X_2 into one connected stratum by turning Y into a circle, so that the frontier condition holds. But (b) will still fail.

Next we give examples showing that (b^π) does not imply (a).

Example 4.2.7 Let $Z = \{y^2 = tx^2\} \subset \mathbf{R}^3$; with filtration $Z = Z_2 \supset Z_1 = (Ot) \supset Z_0 = \emptyset$. The stratification is (b^π)-regular if π is the canonical projection onto the t-axis, but it is not (a)-regular, and does not satisfy the frontier condition.

Example 4.2.8 Let $Z = \{x^3 + 3xy^5 + ty^6 = 0\} \subset \mathbf{R}^3$, with filtration $Z = Z_2 \supset Z_1 = (Ot) \supset Z_0 = \emptyset$. Here the stratification is not (a)-regular, but is $(b^\pi)-$regular where π is projection to the t-axis, and satisfies the frontier condition.

Wall [144] conjectured geometric versions of conditions (a) and (b), and these conjectures were proved in [131]. Different proofs were given later by Hajto [43] and by Perkal [104]. Recall that each tubular neighbourhood of a submanifold Y of a manifold M is given by a diffeomorphism ϕ defined on a neighbourhood U of Y. We denote by $\pi_\phi : U \to Y$ the associated retraction and by $\rho_\phi : U \to [0, 1)$ the associated tubular function.

Theorem 4.2.9 (Trotman [131]) *Let X, Y be disjoint C^1 submanifolds in a C^1 manifold M, with $Y \subset \overline{X} \setminus X$. Then X is (b)-regular (resp. (a)-regular) over Y if and only if for every C^1 diffeomorphism ϕ defining a tubular neighbourhood of Y the map $(\pi_\phi, \rho_\phi)|_X$ (resp. $\pi_\phi|_X$) is a submersion.*

The theorem implies that conditions (a) and (b) are C^1 invariants. Examples exist showing that it is not sufficient to take C^2 diffeomorphisms ϕ [62].

One of the main reasons that Whitney stratifications are of interest is because analytic varieties can be Whitney stratified.

Theorem 4.2.10 (Whitney [147, 148]) *Every analytic variety (in* $\mathbf{R}^n$ *or* $\mathbf{C}^n$*) admits a Whitney stratification whose strata are analytic (hence* C^∞*) manifolds.*

The main point in the proof of this theorem is to show that the Whitney conditions are stratifying, i.e. true on an analytic open dense set of a given subspace. This is proved by contradiction using a wing lemma. Also in 1965, Łojasiewicz proved the existence of Whitney stratifications of semi-analytic sets [79]. Hironaka and Hardt proved that the same is true of every subanalytic set [46, 55]. Hironaka's proof uses resolution of singularities. More accessible existence proofs for semialgebraic sets using Whitney's wing lemma method are due to Thom [122] and to Wall [144]. A more elementary proof for subanalytic sets is due to Denkowska, Wachta and Stasica [22, 24]. More generally, existence theorems for Whitney stratifications of definable sets in o-minimal structures [25] have been given by Loi [78], by van den Dries and Miller [26], and by Nguyen, Trivedi and Trotman [95]. Another proof of the o-minimal case follows from the work of Halupczok and Yin [44].

Whitney's theorem above is a pure existence statement, proved by contradiction using Whitney's wing lemma. Teissier in [118] obtained a much more precise result for complex analytic sets: a complex analytic stratification is Whitney (b)-regular if and only if the multiplicities of its polar varieties are constant along strata. So Whitney regularity is equivalent to the constance of a finite set of numerical invariants. The existence theorem follows. Teissier's theorem also implies the existence, for a complex analytic set, of a canonical minimal Whitney stratification of which all others are refinements (see [118]). It also gives rise to the most general Plücker formula, expressing the degree of the dual variety of a projective variety in terms of topological characters of its canonical Whitney stratification and its general plane sections [37]. Another way of characterising Whitney conditions for complex analytic varieties was developed by Gaffney [32] using the integral closure of modules. Gaffney also gives real analogues characterising Whitney (b)-regularity using the real integral closure.

There are other situations where Whitney stratifications arise naturally. The stratification of a smooth manifold by the orbit types of a proper Lie group action is Whitney regular (this was known to Bierstone in the 1970s [10] and was reproved several times, cf. [105]), but in fact a much stronger result holds: it is smoothly locally trivial [35, 145]! Another situation where a natural Whitney stratification turns out to be smoothly local trivial is the partition of a compact smooth manifold into unstable manifolds of a generic Morse function. That this is a Whitney stratification was proved by Nicolaescu [96], while Laudenbach proved the stronger smooth local triviality [74].

One can ask why one should study Whitney's condition (a), as it is strictly weaker than condition (b). One reason is that it is both simple to understand and easy to check. A second reason is that it is a necessary and sufficient condition for transversality to the strata of a stratification to be an *open* property, as we shall see in the next theorem, often cited in the literature.

Definition 4.2.11 We say that a map $f : N \to M$ between C^1 manifolds is *transverse* to a C^1 stratification of a closed set $Z \subset M$, if $\forall x \in N$ such that $f(x) \in Z$, then

$$(df)_x T_x N + T_{f(x)} X = T_{f(x)} M$$

where X is the stratum containing $f(x)$, i.e. the map f is transverse to each stratum of the stratification of Z.

Theorem 4.2.12 (Trotman [130]) *A locally finite C^1 stratification of a closed subset Z of a C^1 manifold M is (a)-regular if and only if for every C^1 manifold N, $\{f \in C^1(N, M) | f$ is transverse to the strata of $Z\}$ is an open set in the Whitney C^1 topology.*

The sufficiency of (a)-regularity here is due to Feldman in 1965 [28]. A simple proof of necessity in Theorem 4.2.12 can be extracted from the proof of a recent relative version of the theorem given by Trivedi and Trotman [125].

A partial version of this theorem in the holomorphic case is due to Trivedi [124]. Let $H(M, N)$ denote the space of holomorphic mappings between two complex analytic manifolds M and N.

Theorem 4.2.13 (Trivedi) *Let M be a Stein manifold and N be an Oka manifold. Let Σ be a stratification of a complex analytic subvariety in N. Let r be the minimum of the dimensions of strata in Σ. If $\dim M = \dim N - r$ and there exists a compact set K in M such that the set of maps $T_K = \{f \in H(M, N) : f \pitchfork_K \Sigma\}$ is open in $H(M, N)$, then Σ is an a-regular stratification.*

Another application of Whitney (a)-regularity is the following.

Theorem 4.2.14 (Kuo-Li-Trotman [70]) *Let X be a stratum of an (a)-regular stratification of a subset Z of $\mathbf{R}^n$. For each $x \in X$ and for every pair of Lipschitz transversals M_1, M_2 to X at x (a Lipschitz transversal is defined to be the graph of a Lipschitz map $N_x X \to T_x X$), there is a homeomorphism*

$$(M_1, Z \cap M_1, x) \to (M_2, Z \cap M_2, x).$$

Such results justify the separate study and verification of (a)-regularity.

A natural geometric operation is to take transverse intersections of geometric objects. Suppose Z and Z' are two closed stratified sets of a manifold M. Denote the set of strata by Σ and Σ' respectively. If Σ and Σ' are transverse, i.e. if for all $X \in \Sigma$, and for all $X' \in \Sigma'$, X and X' are transverse as submanifolds of M, then we can stratify $Z \cap Z'$ by $\Sigma \cap \Sigma' = \{X \cap X' | X \in \Sigma, X' \in \Sigma'\}$. Moreover $Z \cup Z'$ is naturally stratified by adding the complements in Z (resp. Z') $\{(X \setminus X \cap Z') | X \in \Sigma\}$ (resp. $\{(X' \setminus X' \cap Z) | X' \in \Sigma'\}$).

Theorem 4.2.15 *If* (Z, Σ) *and* (Z', Σ') *are Whitney* (b)*-regular (resp.* (a)*-regular), and have transverse intersections in* M*, then* $(Z \cap Z', \Sigma \cap \Sigma')$ *is* (b)*-regularly (resp.* (a)*-regularly) stratified, as is also* $Z \cup Z'$*.*

This can often be useful. The case of (b)-regularity was treated by Gibson in 1976 [36]. A general theorem of this kind was proved by Orro and Trotman in 2002 [98, 99] for a large class of regularity conditions including the (w)-regularity of the next section.

Other useful properties of Whitney stratified sets include stability under taking products, and triangulability.

Products If Z and Z' are Whitney stratified then so is $Z \times Z'$. A similar result is true for the (w)-regular stratified sets of the next section.

Triangulation It is known that compact Whitney stratified sets are triangulable (Goresky [38], Verona [143], Shiota [112]). The non-compact case follows from another result of Shiota [111] stating that every Whitney stratified set is homeomorphic to a subanalytic set, from which triangulability follows by citing the triangulability of subanalytic sets due to Hironaka [56] and Hardt [47].

However there remains an outstanding open question [39]: does a Whitney stratified set (Z, Σ) have a triangulation whose open simplexes are the strata of a Whitney stratification refining Σ ? In other words, does every Whitney stratified set admit a Whitney triangulation ? The existence of Whitney (b)-regular triangulations was proved by Shiota for semialgebraic sets [114], and by Czapla [19] for definable sets in o-minimal structures.

As we want our stratifications to "look the same" at different points of a given stratum one might hope that there is a C^1 diffeomorphism mapping neighbourhoods of a point y_1 on Y to neighbourhoods of another point y_2 on Y. This is not true in general, as illustrated by the following celebrated example.

Example 4.2.16 (Whitney) Let $Z = \{(x, y, t) | xy(x-y)(x-ty) = 0, t \neq 1\} \subset \mathbf{R}^3$, stratified by $Z = Z_2 \supset Z_1 = (Ot)$. This is a family of 4 lines parametrised by t. The stratification is (b)-regular, but there is no C^1 diffeomorphism mapping Z_{t_1} to Z_{t_2} where $Z_t = Z \cap (\mathbf{R}^2 \times \{t\})$, because of the cross-ratio obstruction. (A linear isomorphism of the plane preserving three distinct lines through a point preserves also any fourth line through that point.)

One may observe that Z in the previous example is a union of eight C^1 manifolds-with-boundary, with $(0t)$ the common boundary. Pawłucki [103] proved a general theorem showing this property: if X and Y are subanalytic adjacent strata such that X is (b)-regular over Y, and $\dim X = \dim Y + 1$, then $X \cup Y$ is a finite union of C^1 manifolds-with-boundary with common boundary Y. A generalisation to definable sets in polynomially bounded o-minimal structures was given by Trotman and Valette [135], who show also that this property fails for definable sets in non polynomially bounded o-minimal structures.

Although Example 4.2.16 means we cannot expect to have in general local C^1 triviality of Whitney stratified sets even in the real algebraic case, we can obtain what is known as local topological triviality. The following, together with Whitney's existence Theorem 4.2.10, constitutes the fundamental theorem of stratification theory.

Theorem 4.2.17 (Thom-Mather [81, 123]) *Let (Z, Σ) be a Whitney stratified subset of a C^2 manifold M. Then for each stratum $Y \in \Sigma$ and each point $y_0 \in Y$ there is a neighbourhood U of y_0 in M, a stratified set $L \subset S^{k-1}$ and a homeomorphism*

$$h : (U, U \cap Z, U \cap Y) \to (U \cap Y) \times (B^k, c(L) \cap B^k, 0)$$

such that $p_1 \circ h = \pi_Y$, where $c(L)$ is the cone on the link L with vertex 0, B^k is the k-ball centred at 0, and π_Y is the projection onto $U \cap Y$ of a tubular neighbourhood.

A typical application of this theorem is Fukuda's proof that the number of topological types of polynomial functions $p : \mathbf{R}^n \to \mathbf{R}$ of given degree d is finite [30].

Theorem 4.2.17 applies without any hypothesis of analyticity or subanalyticity. If the strata are assumed to be semialgebraic, Coste and Shiota [18] have shown that the trivialising homeomorphism h may be chosen to be semialgebraic, using real spectrum methods in their proof. See Shiota's book [112] for a more general result applying to definable sets and providing a definable trivialisation. The proof of Mather [81] of Theorem 4.2.17 uses the notion of controlled vector field, and the homeomorphism h resulting from Mather's proof is obtained by integrating such controlled vector fields, so that the resulting homeomorphism h will not in general be semialgebraic even if the strata are semialgebraic.

A (stratified) vector field v on a stratified set (Z, Σ) is defined by a collection of vector fields $\{v_X | X \in \Sigma\}$. It is *controlled* when $(\pi_Y)_* v_X(x) = v_Y(\pi_Y(x))$ and $(\rho_Y)_* v_X(x) = 0$ on a tubular neighbourhood T_Y of Y, where T_Y is part of a set of compatible tubular neighbourhoods called control data. See Mather's notes [81] for details of the theory of control data and of controlled vector fields. It was not until 1996 that a proof was published that these stratified controlled vector fields could be assumed to be *continuous* : given a vector field v_Y on a stratum Y of a Whitney stratified set, or indeed a Bekka stratified set (see Sect. 4.8 below), there exists a continuous controlled stratified vector field $\{v_X\}$ on M extending v_Y (Shiota [112] for Whitney stratified sets, du Plessis [106] for the more general Bekka stratified sets). This result has been used for example by Hamm [45] to simplify some statements in stratified Morse theory [41], and by S. Simon to prove a stratified version of the Poincaré-Hopf theorem [116].

The proof of local topological triviality and conicality of Whitney stratified sets as stated in Theorem 4.2.17 is in fact an easy consequence of the following more general first isotopy lemma of Thom [81, 123]:

Theorem 4.2.18 *Let Z be a Whitney stratified subset of a C^2 manifold M, and let $f : M \to \mathbf{R}^k$ be a C^2 map such that $f|Z$ is proper and the restriction of f to each stratum of Z is a submersion. Then there is a stratum-preserving homeomorphism $h : Z \to \mathbf{R}^k \times (f^{-1}(0) \cap Z)$ which commutes with the projection to $\mathbf{R}^k$, so that the fibres of $f|_Z$ are homeomorphic by a stratum-preserving homeomorphism.*

There is a second isotopy lemma for stratified maps satisfying Thom's (a_f) condition (see Definition 4.4.1 below), a relative version of condition (a) [81, 82, 123]. These two isotopy lemmas were first used in the proof of the difficult topological stability theorem : the space of topologically stable mappings is dense in the space of proper smooth mappings between two smooth manifolds [36, 82, 83, 107, 121, 123]. A recent strengthening of Theorem 4.2.17, obtaining continuity of the tangent spaces to the leaves defined by fixing points in the normal slice, implies the density of strongly topologically stable mappings in the space of proper mappings [89]. Strong topological stability refers to imposing continuity of the commuting homeomorphisms as functions of a varying map.

4.3 The Kuo-Verdier Condition (w)

Condition (a) for (X, Y) says that the distance between the tangent space to X at x and the tangent space to Y at y tends to zero as x tends to y. Kuo and Verdier studied what happens when the rate of vanishing of this distance is $O(|x - \pi_Y(x)|)$ [67, 142].

Definition 4.3.1 Two adjacent strata (X, Y) in a C^1 manifold M are said to be (w)-regular at $y_0 \in Y$, or to satisfy the Kuo-Verdier condition (w) , if there exist a constant $C > 0$ and a neighbourhood U of y_0 in M such that

$$d(T_yY, T_xX) < C||x - y||$$

$\forall x \in U \cap X, \forall y \in U \cap Y.$

Here, for vector subspaces V and W of an inner product space E,

$$d(V, W) = \sup\{\inf\{\sin\theta(v, w)|w \in W^*\}|v \in V^*\}$$

where $\theta(v, w)$ is the angle between v and w.

Note that $d(V, W) = 0 \Leftrightarrow V \subset W$, and that $d(V, W) = 1 \Leftrightarrow \exists v \in V^*, v \perp W$.

Proposition 4.3.2 (Kuo [66]) *For semi-analytic X and Y, $(w) \Rightarrow (b)$.*

Verdier observed that Kuo's proof that (w) implies (b) in [66] (where Kuo takes as hypothesis a weaker condition, that he called the ratio test) works too for subanalytic sets [142], and Loi [78] extended this result to the case of definable sets in o-minimal structures.

So (w)-regularity is a stronger regularity condition than (b) for definable stratified sets (it no longer implies (b) in general for C^∞ stratified sets as shown by Example 4.3.7 below). Moreover it turns out to be generic too, as the following theorem shows.

Theorem 4.3.3 (Verdier [142]) *Every subanalytic set admits a locally finite (w)-regular stratification.*

This is also true for definable sets in arbitrary o-minimal structures as shown by Loi [78]. Other proofs in the subanalytic case are due to Denkowska and Wachta [23], and to Łojasiewicz, Stasica and Wachta [80], both of these proofs avoiding resolution of singularities. Another proof, due to Kashiwara and Schapira [63], follows from the equivalence of (w) and their microlocal condition μ [134].

For complex analytic sets a major result proved in 1982 by Teissier, with a contribution by Henry and Merle, implies the equivalence of (b) and (w) [50, 118]. Real algebraic examples showing that (b) does not imply (w) are common because (b) is a C^1 invariant [131] while (w) is not a C^1 invariant (although it is a C^2 invariant), as shown by the following example.

Example 4.3.4 (Brodersen-Trotman [15]) Let $Z = \{y^4 = t^4x + x^3\} \subset \mathbf{R}^3$. Then the stratification of Z defined by $Z = Z_2 \supset Z_1 = (Ot)$ is (b)-regular but not (w)-regular. Z is actually the graph of the C^1 function $f(x, t) = (t^4x + x^3)^{1/4}$.

Infinitely many real algebraic examples with (b) holding but not (w) may be found in the combined classifications of Juniati, Noirel and Trotman [59, 60, 97, 133]. The first such semialgebraic example was given in [128].

One can characterise (w)-regularity using stratified vector fields as follows.

Proposition 4.3.5 (Brodersen-Trotman [15]) *A stratification is (w)-regular $\Leftrightarrow$ every vector field on a stratum Y extends to a rugose stratified vector field in a neighbourhood of Y.*

Here a stratified vector field $\{v_X : X \in \Sigma\}$ is called *rugose* near y_0, in a stratum Y, when there exists a neighbourhood U of y_0 and a constant $C > 0$, such that for all adjacent strata X, $\forall x \in U \cap X$, $\forall y \in U \cap Y$,

$$\| v_X(x) - v_Y(y) \| \leq C \| x - y \| .$$

This resembles an asymmetric Lipschitz condition, and poses the question of when the extension of a Lipschitz vector field can be chosen to be Lipschitz. This we will discuss in Sect. 4.5.

There is a somewhat weaker version of the Thom-Mather isotopy theorem, due to Verdier [142] in 1976, for his (w)-regular stratified sets. He obtains local topological triviality but not the local conicality of Theorem 4.2.17.

Theorem 4.3.6 (Verdier) *Let (Z, Σ) be a (w)-regular C^2 stratified subset of a C^2 manifold M. Then for each stratum $Y \in \Sigma$ of codimension k in M, and each point $y_0 \in Y$ there is a neighbourhood U of y_0 in M, a stratified set $N \subset B^k$ and a rugose* *homeomorphism*

$$h : (U, U \cap Z, U \cap Y) \to (U \cap Y) \times (B^k, N, 0)$$

such that $p_1 \circ h = \pi_Y$*, where* B^k *is the k-ball centred at* 0*, and* π_Y *is the projection onto* $U \cap Y$ *of a tubular neighbourhood.*

The proof is by integration of rugose vector fields [142]. Another approach to this isotopy theorem was given by Fukui and Paunescu [31].

Example 4.3.7 The topologist's sine curve in $\mathbf{R}^2$, with Z the closure of $\{y = \sin(1/x) : x > 0\}$, provides an example of a (w)-regular stratified set Z which is not Whitney stratified : (b^π) fails at every point of the 1-dimensional stratum on the y-axis. Clearly the conical conclusion of the Thom-Mather isotopy Theorem 4.2.17 fails to hold.

Remark 4.3.8 The homeomorphism obtained in the Thom-Mather isotopy Theorem 4.2.17 is also rugose, because it is controlled, given by integrating controlled vector fields (see [81]).

4.4 Stratified Maps

Knowing that subanalytic sets may be stratified with regularity conditions ensuring local topological control one can ask whether similar structure theorems can be proved for mappings. Hardt [46] proved that every proper real analytic mapping between real analytic manifolds may be stratified, in the sense that one may find Whitney stratifications of source and target such that restricted to each stratum of the source the map is a submersion onto the stratum in the target. However Thom[120] had already observed in 1962 that in a family of polynomial maps the topological type can vary continuously. He proposed a type of regularity condition on maps to avoid this phenomenon [123] as follows.

Definition 4.4.1 A map f defined on a stratified set in a manifold M with Y a stratum is said to satisfy the Thom condition (a_f) at $y \in Y$ when f is of constant rank on each stratum and

$$T_y(Y \cap f^{-1}(f(y))) \subset \lim_{x \to y} T_x(X \cap f^{-1}(f(x))),$$

where, for X a stratum and $x \in X$ tending to y, the limit is taken in the appropriate grassmannian $G^{dim M}_{dim X - k}$ if f restricted to X has rank k.

When the map f is constant on X and Y this is just (a)-regularity.

Thom conjectured in the 1960s that proper stratified maps satisfying (a_f) should be triangulable. This was proved by Shiota in 2000 [113] after an earlier partial result by Verona in his book [143]. See also [115] for the non-proper semialgebraic case.

A striking result [76] by Lê and Saito in complex equisingularity, showing the naturality of Thom's condition, is that constancy of the Milnor numbers of a family of isolated hypersurface singularities defined by

$$F : (\mathbf{C}^n \times \mathbf{C}, 0 \times \mathbf{C}) \to (\mathbf{C}, 0)$$

is equivalent to the map F satisfying (a_F) with respect to the stratification with 3 strata,

$$\mathbf{C}^{n+1} \supset F^{-1}(0) \supset (0 \times \mathbf{C}).$$

If instead of inclusion in Definition 4.4.1 which says the distance between the two tangent spaces goes to zero, one requires that this distance is bounded above by a constant times the distance of x to Y, one then obtains a condition (w_f) generalising the Kuo-Verdier condition (w) of Definition 4.3.1. A study of (w_f) in the complex analytic case with geometric characterisations, analogous to Teissier's study of (w) and (b) in [118] was carried out by Henry, Merle and Sabbah [53]. Gaffney and Kleiman give algebraic versions of (w_f) in the complex case see [33]. In the real C^∞ case (w_f) is one of a family of regularity conditions on maps and spaces studied by Trotman and Wilson in [137].

For subanalytic functions one can always stratify a map so that (w_f) holds (Hironaka [57] for (a_f) and Parusiński [100] for (w_f)). The blowup of a point in the plane provides a counterexample to the existence of a stratification (a_f) when the target space has dimension at least 2.

Associated to Thom maps (stratified maps satisfying (a_f)) there is a second isotopy lemma for which we refer to [81] and [82]. This is important in the study of topological stability of mappings [83], Mather using it to complete the proof of the density of the set of topologically stable mappings between smooth manifolds.

Having seen that conditions (a) and (w) have relative versions (a_f) and (w_f) one may wonder about a possible relative version (b_f) of condition (b). So far there have been 3 different conditions called (b_f) in the literature, introduced and used respectively by Thom [123], by Henry and Merle [51] and by Nakai [90]. There is also a condition (D) due to Goresky [39]. No comparative study of these conditions has been undertaken. However Murolo has recently worked out properties of Goresky's condition (D) [88].

4.5 Lipschitz Stratifications

Mostowski in 1985 [85] introduced conditions (L) on a stratification, which further strengthen the Kuo-Verdier condition (w) and these imply the possibility of extending Lipschitz vector fields and can indeed be characterised by the existence of certain Lipschitz extensions of Lipschitz vector fields (see Theorem 4.5.3 below) [101].

Definition 4.5.1 (Mostowski) Let $Z = Z_d \supset \cdots \supset Z_\ell \neq \emptyset$ be a closed stratified set in $\mathbf{R}^n$. Write $\overset{\circ}{Z}_j = Z_j - Z_{j-1}$. Let $\gamma > 1$ be a fixed constant. A *chain* for a point $q \in \overset{\circ}{Z}_j$ is a strictly decreasing sequence of indices $j = j_1, j_2, \ldots, j_r = \ell$ such that each $j_s (s \geq 2)$ is the greatest integer less than j_{s-1} for which

$$\operatorname{dist}(q, Z_{j_s - 1}) \geq 2\gamma^2 \operatorname{dist}(q, Z_{j_s}).$$

For each $j_s, 1 \leq s \leq r$, choose $q_{j_s} \in \overset{\circ}{Z}_{j_s}$ such that $q_{j_1} = q$ and $|q - q_{j_s}| \leq \gamma \operatorname{dist}(q, Z_{j_s})$.

If there is no confusion we call $\{q_{j_s}\}_{s=1}^r$ a chain of q.

For $q \in \overset{\circ}{Z}_j$, let $P_q : \mathbf{R}^n \to T_q(\overset{\circ}{Z}_j)$ be the orthogonal projection to the tangent space and let $P_q^\perp = I - P_q$ be the orthogonal projection to he normal space $(T_q(\overset{\circ}{Z}_j))^\perp$.

Definition 4.5.2 (Mostowski) A stratification $\Sigma = \{Z_j\}_{j=\ell}^d$ of Z is said to be a *Lipschitz stratification*, or to satisfy the (L)-conditions, if for some constant $C > 0$ and for every chain $\{q = q_{j_1}, \ldots, q_{j_r}\}$ with $q \in \overset{\circ}{Z}_{j_1}$ and each $k, 2 \leq k \leq r$,

$$| P_q^\perp P_{q_{j_2}} \cdots P_{q_{j_k}} | \leq C \mid q - q_{j_2} \mid / d_{j_k - 1}(q) \qquad (L1)$$

and for each $q' \in \overset{\circ}{Z}_{j_1}$ such that $| q - q' | \leq (1/2\gamma)\, d_{j_1 - 1}(q)$,

$$| (P_q - P_{q'}) P_{q_{j_2}} \cdots P_{q_{j_k}} | \leq C \mid q - q' \mid / d_{j_k - 1}(q) \qquad (L2)$$

and

$$| P_q - P_{q'} | \leq C \mid q - q' \mid / d_{j_1 - 1}(q) \qquad (L3).$$

Here $\operatorname{dist}(-, Z_{\ell-1}) \equiv 1$, by convention.

It is not hard to show that for a given Lipschitz stratification $\exists\, C > 0$ such that $\forall x \in \overset{\circ}{Z}_j, \forall y \in \overset{\circ}{Z}_k, k < j$ then

$$|P_x^\perp P_y| \leq \frac{C|x - y|}{\operatorname{dist}(y, Z_{k-1})},$$

and because $|P_x^\perp P_y| = d(T_y \overset{\circ}{Z}_k, \overset{\circ}{Z}_j)$, (w)-regularity follows with a precise estimation for the constant (which can tend to infinity as y approaches Z_{k-1}).

Parusiński has given the following characterisation of Mostowski's Lipschitz conditions in terms of extensions of vector fields.

Theorem 4.5.3 (Parusiński [101]) *A stratification $Z = Z_d \supset Z_{d-1} \supset \cdots \supset Z_0 \supset Z_{-1} = \emptyset$ is Lipschitz if and only if there exists a constant $K > 0$ such that for every subset $W \subset Z$ such that*

$$Z_{j-1} \subset W \subset Z_j$$

for some $j = \ell, \ldots, d$ where ℓ is the lowest dimension of a stratum of Z, each Lipschitz Σ-compatible vector field on W with Lipschitz constant L which is bounded on $W \cap Z_\ell$ by a constant $C > 0$, can be extended to a Lipschitz Σ-compatible vector field on Z with Lipschitz constant $K(L + C)$.

He also proved an existence theorem for subanalytic sets.

Theorem 4.5.4 (Parusiński [101]) *Every subanalytic set admits a Lipschitz stratification. Moreover such Lipschitz stratifications are locally bilipschitz trivial.*

The initial existence theorem for Lipschitz stratifications was for complex analytic sets, due to Mostowski in 1985 [85]. It is not true that definable sets in arbitrary o-minimal structures admit Lipschitz stratifications.

Example 4.5.5 (Parusinski) Let $X(t)$ be the union of the x-axis and the graph $y = x^t (x > 0)$ in $\mathbf{R}^3 = (x, y, t)$. Then the Lipschitz types of $X(t)$ are distinct for all $t > 1$. By Miller's dichotomy every non polynomially bounded o-minimal structure contains this as a definable set.

However we do have an existence theorem in the polynomially bounded case.

Theorem 4.5.6 (Nguyen-Valette [93]) *Every definable set in a polynomially bounded o-minimal structure admits a definable Lipschitz stratification.*

Halupczok and Yin have given another proof of this result [44].

It is clear that the (L)-conditions are much more of a constraint than is (w). Here are some simple examples showing that the two conditions are distinct.

Example 4.5.7 (Mostowski) In $\mathbf{C}^4$ or $\mathbf{R}^4$ let $Z = \{y = z = 0\} \cup \{y = x^3, z = tx\}$. Then (w) holds along the t-axis, but (L) fails.

Example 4.5.8 (Koike-Juniati [61]) In $\mathbf{R}^3$ let $Z = \{y^2 = t^2x^2 + x^3, x \geq 0\}$ and stratify by $Z = Z_2 \supset Z_1 = \langle Ot \rangle$. It is easy to check that (w) holds for this semialgebraic example, while $(L2)$ fails : let $q = q_{j_1} = q_2 = (t^2, \sqrt{2}t^3, t), q' = (t^2, -\sqrt{2}t^3, t), q_{j_2} = q_1 = (0, 0, t)$, as $t \to 0$.

4.5.1 Teissier's Criteria for a Good Equisingularity Condition

In his 1974 Arcata lectures [117] Teissier gave a list of criteria for a good equisingularity condition E on a stratification of a complex analytic set; E-regularity should in particular:

1) be as strong as possible;
2) be generic, i.e. every complex analytic set should possess an E-regular stratification;
3) imply local topological triviality along strata;
4) imply equimultiplicity;
5) be preserved after intersection with generic linear spaces containing a given stratum, locally linearised ($E \Rightarrow E^*$, see below for a precise definition).

Criteria 2) to 5) hold for Whitney (b)-regularity (see Teissier [118]), which turns out to be equivalent to (w) in the complex case as noted above. Criterion 5) is an essential part of the proof of this result via the equimultiplicity of polar varieties. (Recall that (b) does not imply (w) for real algebraic varieties by Example 4.3.4.)

Criterion 4) is a theorem of Hironaka from 1969 [54].

Theorem 4.5.9 (Hironaka) *For a complex analytic Whitney stratified variety V the pointwise multiplicity $m(V, p)$ is constant on each stratum.*

Definition 4.5.10 (E^*-regularity) Let M be a C^2 manifold. Let Y be a C^2 submanifold of M and let $y \in Y$. Let X be a C^2 submanifold of M such that $y \in \overline{X}$ and $Y \cap X = \emptyset$. Let E denote an equisingularity condition (e.g. (b), (w), (L)). Then (X, Y) is said to be $E_{\text{cod}k}$*-regular* at y $(0 \leq k \leq \text{cod}Y)$ if there exists an open dense subset U^k of the grassmannian of codimension k subspaces of T_yM containing T_yY, such that if W is a C^2 submanifold of M with $Y \subset W$ near y, and $T_yW \in U^k$, then W is transverse to X near y, and $(X \cap W, Y)$ is E-regular at y.

One says finally that (X, Y) is E^**-regular* at y if (X, Y) is $E_{\text{cod}k}$-regular for all $k, 0 \leq k < \text{cod}Y$.

Theorem 4.5.11 (Navarro Aznar-Trotman [92]) *For subanalytic stratifications, $(w) \Rightarrow (w^*)$, and if* dim $Y = 1$, $(b) \Rightarrow (b^*)$.

The fact that (b^*)-regular stratifications exist for subanalytic sets allows one to prove that stratified Morse functions (in the sense of Goresky and MacPherson [41]) exist and are generic, using (b_{cod1}). The rapid spiral is an example of a Whitney stratified set for which no (stratified) Morse functions exist [41].

Question: Is it true that $(b) \Rightarrow (b^*)$ for subanalytic stratifications in general, i.e. when dim $Y \geq 2$?

Theorem 4.5.12 (Teissier [118]) *For complex analytic stratifications,* $(b) \Rightarrow (b^*)$.

Theorem 4.5.13 (Juniati-Trotman-Valette [61]) *For subanalytic stratifications,* $(L) \Rightarrow (L^*)$.

According to the 1974 criteria of Teissier [117], Whitney regularity is a good equisingularity condition. Because Mostowski's Lipschitz condition (L) is stronger it may be considered better as it also satisfies Teissier's criteria.

Many results concerning E^*-regularity for different equisingularity conditions E in the complex analytic context are described in [77], including a kind of converse to the Thom-Mather local triviality Theorem 4.2.17, namely that (TT^*) implies (b), where (TT) means local topological triviality along strata.

4.6 Definable Trivialisations

We have seen that Whitney (b)-regularity ensures local topological triviality. Mostowski and Parusiński proved that an (L)-regular stratification of a subanalytic set is locally bilipschitz trivial (Theorem 4.5.4). It is natural to ask if such trivialisations can be chosen to be definable. Or specifically if Z is a semialgebraic set, is there some stratification which is locally semialgebraically trivial ? This was proved by Hardt in 1980 [48]. His method was improved by G. Valette who obtained local semialgebraic bilipschitz triviality [139, 140].

Theorem 4.6.1 (Hardt) *Semialgebraic sets admit locally semialgebraically trivial stratifications.*

Theorem 4.6.2 (Valette) *Semialgebraic sets admit locally semialgebraically bilipschitz trivial stratifications.*

There are also subanalytic versions of these results. For semialgebraic (b)-regular stratifications Coste and Shiota [18] proved a semialgebraic isotopy theorem using real spectrum methods. See the book of Shiota [112] for further details and references.

A recent (2017) very powerful theorem by Parusiński and Paunescu [102], proving the Whitney fibering conjecture of 1965 [147], produces a subanalytic trivialisation of a given stratified analytic variety (real or complex) which is moreover arc-analytic, as is its inverse. The hypothesis on the stratified set is a type of Zariski equisingularity, stronger than (w)-regularity, hence implying Whitney (b)-regularity by Proposition 4.3.2. The relation between this notion of Zariski equisingularity and Mostowski's Lipschitz condition of Definition 4.5.2 is currently being studied in the case of complex analytic varieties by Parusiński and Paunescu. See Parusiński's contribution to this handbook for details of their work.

4.7 Abstract Stratified Sets

One may begin the study of differentiable manifolds in two ways, either by starting with the abstract definition and eventually proving the existence of an embedding into euclidean space [58], or by starting with submanifolds of euclidean space

[42] so that the abstract concept is obtained by taking an equivalence class by diffeomorphisms. In a similar way there is a definition of abstract stratified set, due to Mather [81]. He developed this definition by adapting ideas of Thom [123], who gave a different definition of an abstract stratified set, so that the resulting spaces are called Thom-Mather stratified sets.

Definition 4.7.1 An *abstract stratified set* is a triple $(Z, \Sigma, \mathcal{T})$ satisfying 9 axioms:

A1) Z is a locally compact second countable Hausdorff space, hence metrisable.
A2) Σ is a partition of Z into locally closed subsets, called the strata.
A3) Each stratum is a topological manifold with a differentiable structure of class C^k.
A4) Σ is locally finite.
A5) Σ satisfies the frontier property.
A6) $\mathcal{T}$ is a triple $(\{T_X\}, \{\pi_X\}, \{\rho_X\})$ where for each $X \in \Sigma$, T_X is an open neighbourhood of X in V, $\pi_X : T_X \longrightarrow X$ is a continuous retraction of T_X onto X, and $\rho_X : T_X \longrightarrow [0, \infty)$ is a continuous function. We call T_X the *tubular neighbourhood* of X, π_X the *local retraction* of T_X onto X, and ρ_X the *tubular function* of X.
A7) $X = \{v \in T_X : \rho_X(v) = 0\}$.

Notation For strata X, Y, let $T_{X,Y} = T_X \cap Y$, let $\pi_{X,Y} = \pi_X|_{T_{X,Y}} : T_{X,Y} \longrightarrow X$ and let $\rho_{X,Y} = \rho_X|_{T_{X,Y}} : T_{X,Y} \longrightarrow (0, \infty)$.

A8) For each pair of strata X, Y, $(\pi_{X,Y}, \rho_{X,Y}) : T_{X,Y} \longrightarrow X \times (0, \infty)$ is a C^k submersion (hence $\dim X < \dim Y$ if $T_{X,Y} \neq \emptyset$).
A9) For strata $W < X < Y$ we have $\pi_{W,X} \circ \pi_{X,Y}(z) = \pi_{W,Y}(z)$, and $\rho_{W,X} \circ \pi_{X,Y}(z) = \rho_{W,Y}(z)$. These are called *control conditions* .

Such abstract stratified sets are triangulable, as shown by Goresky [38] and by Verona [143].

Mather's proof in [81] of the first isotopy lemma of Thom for stratified submersions on Whitney stratified sets (Theorem 4.2.18 above) uses Mather's result that every Whitney stratified subset of a manifold admits the structure of an abstract stratified set. He then proves the isotopy lemma in the abstract context.

It is then natural to ask about an embedding theorem for abstract stratified sets, similar to the embedding theorem for smooth manifolds. Teufel [119] and Natsume [91] proved that every abstract stratified set of dimension n can be embedded in $\mathbf{R}^{2n+1}$ as a Whitney stratified set. Noirel [97] improved their statements by showing that the resulting Whitney stratified set may be made subanalytic as may the induced local retractions and tubular functions. Also he showed that the embedded stratification may be made (w)-regular (hence also (b)-regular by the subanalytic version of Proposition 4.3.2). Moreover the embedded set and the induced control data can be made semialgebraic if the set is compact [97].

Note that in the C^∞ category (w)-regular stratified sets do not in general admit the structure of a Thom-Mather abstract stratified set because they are not always locally conical as shown by Example 4.3.7. However they are locally topologically trivial as shown directly by Verdier using integration of rugose stratified vector fields [142].

Much work has been done generalising the differential properties of smooth manifolds to abstract stratified sets in the above sense. See Sect. 4.11 below for some references.

4.8 K. Bekka's (*c*)-Regularity

It can be important to be more precise as to when a stratification is locally topologically trivial in the sense of Theorem 4.2.17, for example when classifying topologically or when studying topological stability (cf. work of Damon, Looijenga, Wirthmüller and the book of du Plessis and Wall [107]). Then one needs the weakest regularity condition on a stratification ensuring local topological triviality. This principle led to the introduction of the following condition.

Definition 4.8.1 (K. Bekka) A stratified set (Z, Σ) in a manifold M is (c)-regular if for every stratum Y of Σ there exists an open neighbourhood U_Y of Y in M and a C^1 function $\rho_Y : U_Y \to [0, \infty)$ such that ${\rho_Y}^{-1}(0) = Y$ and the restriction $\rho_Y|_{U_Y \cap \mathrm{Star}(Y)}$ is a Thom map, where $\mathrm{Star}(Y) = \bigcup\{X \in \Sigma | X \geq Y\}$, i.e. $\forall X \in \mathrm{Star}(Y)$, with $\rho_{XY} = \rho_Y|_X$ and $x \in X$,

$$\lim_{x \to y} T_x({\rho_{XY}}^{-1}(\rho_Y(x))) \supseteq T_y Y \quad \forall y \in Y.$$

Note that $\rho_Y : U_Y \to [0, \infty)$ is defined globally on a neighbourhood of Y. So this is not a local condition. Local (c)-regularity is developed and used by Schürmann [109].

Theorem 4.8.2 (Bekka [4]) *(c)-regular stratifications are locally topologically trivial along strata.*

The proof is by proving the existence of an abstract stratified structure of Thom-Mather which allows the use of Mather's theory of controlled stratified vector fields [81] and implies that the conclusions of Theorems 4.2.17 and 4.2.18 are satisfied. If one only requires constance of homological or cohomological data then one can weaken (c) even further—see chapter 4 of the book of Schürmann [109].

Characterisations of condition (c) are given by Bekka and Koike in [5].

We saw how (w) and (L) are characterised by the existence of appropriate lifts of vector fields. Here is the corresponding result for (c)-regularity.

Theorem 4.8.3 (du Plessis-Bekka [106]) *A stratification is (c)-regular $\Leftrightarrow$ every C^1 vector field on a stratum Y admits a continuous controlled stratified extension to a neighbourhood of Y.*

This means that there exists a family of vector fields $\{v_X | X \in Star(Y)\}$ such that $v = \bigcup v_X$ is continuous (in TM), while being controlled as defined above.

How do (b) and (c) compare ?

I proved [131] that (b) over a stratum Y is equivalent to the property that for every C^1 tubular neighbourhood T_Y of Y the restriction to neighbouring strata of the associated map (π_Y, ρ_Y) is a submersion, where $\pi_Y : T_Y \to Y$ is the canonical retraction and $\rho_Y : T_Y \to [0, 1)$ the canonical distance function (see Theorem 4.2.9).

In comparison, (c) says that there exists *some* C^1 function ρ vanishing on Y (not necessarily associated to a tubular neighbourhood: ρ can be degenerate, e.g. weighted homogeneous, or even flat on Y) such that for every C^1 tubular neighbourhood T_Y of Y the restriction to neighbouring strata of the map (π_Y, ρ) is a submersion [4].

One can prove easily that (b) implies (c) while there are examples showing that the converse is false [4]. See [6] for real algebraic examples. There are complex algebraic examples due to Briançon and Speder [14]: these consist of 1-parameter families of complex hypersurfaces with isolated singularities defined by $F : \mathbf{C}^3 \times \mathbf{C}, 0 \times \mathbf{C} \to \mathbf{C}, 0$ such that $(F^{-1}(0), 0 \times \mathbf{C})$ is (c)-regular (because weighted homogeneous) but not (b)-regular. It is unknown whether topologically trivial complex analytic stratifications are always (c)-regular, or even whether they are (a)-regular (a question of Thom).

Several authors have used (c)-regularity as a means of providing sufficient conditions for the existence of a real Milnor fibration associated to a real analytic map [108, for example].

A recent theorem of Murolo, du Plessis and Trotman [89] states that for Whitney (b)-regular or Bekka (c)-regular stratified sets the Thom-Mather isotopy theorem can be improved so as to provide a smooth form of the Whitney fibering conjecture. One can ensure that the fibres of the trivialising homeomorphism h in the Thom-Mather isotopy Theorem 4.2.17 (or Theorem 4.5.4) for fixed points of $c(L)$ have continuously varying tangent spaces as one goes to the base stratum X, or changes stratum in the star of X. Moreover the associated wings obtained by fixing a point on the link L can be made (c)-regular.

Sandwiched between Whitney (b)-regularity and Bekka's (c)-regularity there is a condition known as weak Whitney regularity. For a pair of adjacent strata (X, Y) we assume that for some choice of local coordinates at a point $y_0 \in Y$ the angle θ between secant lines and the tangent space to X is bounded away from $\pi/2 : \exists \delta > 0$ such that

$$\theta(\overline{xy}, T_x X) < \pi/2 - \delta$$

for all $x \in X$ in some neighbourhood U of y_0.

We call this condition (δ) and the combined condition $(a + \delta)$ (when both (a) and (δ) hold) is known as weak Whitney regularity . The proof that weak Whitney regularity implies (c) (for a standard tubular function ρ_Y associated to a tubular neighbourhood) is in [6]. Real algebraic examples exist showing that the converse is not true. No complex examples are known.

It is a curious fact that weak Whitney regularity for a family of complex hypersurfaces with isolated singularities implies equimultiplicity [136], generalising Hironaka's theorem in this case [54]. It is unknown whether topologically trivial families are equimultiple (a parametrised version of the famous Zariski problem [150] concerning topological invariance of the multiplicity of an isolated complex hypersurface singularity). The examples of Briançon and Speder [14] of μ-constant families of hypersurfaces which are not Whitney regular turn out to be weakly Whitney regular—see [8] and the correction [9]. One can then ask whether μ-constant families of hypersurfaces are always weakly Whitney regular. This would imply topological triviality via (c)-regularity and Bekka's Theorem 4.8.2 [4] that (c) implies topological triviality, and thus extend the Lê-Ramanujam theorem (which uses the h-cobordism theorem) to the missing surface case [75].

We note that weakly Whitney stratified sets in general have similar metric properties to Whitney stratified sets - they are of finite geodesic diameter if compact for example [7]. Also weakly Whitney stratified sets with a smooth singular set of codimension 1 have finite volume. This is not true in general if the singular set has codimension 2 or if the depth is at least 2 [29].

4.9 Condition (t^k)

We return to the first example of Whitney, $Z = \{y^2 = t^2x^2 + x^3\}$. Slice the surface by a plane S transverse to the t-axis at 0. Then the topological type of the germ at 0 of the intersection $Z \cap S$ is constant, i.e. independent of S. Remember that Whitney (a) holds. Thom noticed this and mentioned it to Kuo, who proved the following theorem [68].

Theorem 4.9.1 (Kuo 1978) *If (X, Y) is (a)-regular at $y \in Y$ then (h^∞) holds, i.e. the germs at y of intersections $S \cap X$, where S is a C^∞ submanifold transverse to Y at $y \in S \cap Y$ and $dimS + dimY = dimM$, are homeomorphic.*

It later turned out [132] that one can replace (h^∞) by (h^1), meaning one considers all C^1 transversals S, and weaken (a) to (t^1), defined as follows.

Definition 4.9.2 A pair of strata (X, Y) is (t^k)-regular at $y \in Y$ if for every C^k submanifold S transverse to Y at $y \in Y \cap S$, there is a neighbourhood U of y such that S is transverse to X on $U \cap X$ $(1 \leq k \leq \infty)$.

Clearly (a) implies (t^1). The converse does not hold as first shown in [127, 129]. The converse does hold in the subanalytic case if we allow transversals of arbitrary dimension [132]. In the case of transversals of complementary dimension there are semialgebraic examples with (t^1) but not (a) [132], and there are even complex algebraic examples [34].

Theorem 4.9.3 (Trotman [132]) *If we restrict to transversals of complementary dimension to* Y*,* (t^1) *is equivalent to* (h^1)*.*

Theorem 4.9.4 (Trotman-Wilson [137]) *For subanalytic strata,* (t^k) *is equivalent to the finiteness of the number of topological types of germs at* y *of* $S \cap X$ *for* S *a* C^k *transversal to* Y *(*$k \geq 2$*) of complementary dimension.*

The proofs that I developed with Kuo and with Wilson use the "Grassmann blowup" introduced by Kuo and myself [71]. Let

$$E^{n,d} = \{(L, x) | x \in L\} \subset G^{n,d} \times \mathbf{R}^n$$

for $d < n$, with projection to $G^{n,d}$, denote the canonical d-plane bundle. Let $\beta = \beta_{n,d}$ denote projection to $\mathbf{R}^n$. When $d = 1$ this is the usual blowup of $\mathbf{R}^n$ with centre 0.

Suppose $X, Y \subset \mathbf{R}^n$ and $0 \in Y$ with $d = \operatorname{codim} Y$.

Let $\tilde{X} = \beta^{-1}(X)$ and let $\tilde{Y} = \{(L, 0) | L$ is transverse to Y at $0\}$. The following striking theorem results from work by Kuo and myself [71], completed by work with Wilson [137].

Theorem 4.9.5 *(X, Y) is (t^k)-regular at $0 \in Y$ if and only if $(\tilde{X}, \tilde{Y})$ is (t^{k-1})-regular at every point of $\tilde{Y}$ ($k \geq 1$).*

When $k = 1$, (t^0) is equated with (w), the Kuo-Verdier condition of Definition 4.3.1. So in particular, (w)-regularity is the first in an infinite sequence of (t^k)-regularity conditions !

Now we can see how to prove that (t^1) implies (h^1) by using the Verdier isotopy Theorem 4.3.6 for (w)-regular stratifications in the Grassmann blowup, although this was not the original proof.

The (t^k) conditions were used to characterise jet sufficiency by Trotman and Wilson, generalising theorems of Bochnak, Kuo, Lu and others, and realising part of the early programme of Thom (1964). See [137] for details. Work with Gaffney and Wilson [34] developed an algebraic approach to the (t^k) conditions, using integral closure of modules.

To illustrate the difference between (t^2) and (t^1), and the previous theorem, look at the Koike-Kucharz example [65] given by $Z = \{x^3 - 3xy^5 + ty^6 = 0\} \subset \mathbf{R}^3$ stratified as usual by (X, Y) with Y the t-axis and X its complement $Z - Y$. Then (X, Y) is (t^2) but not (t^1) at 0. It is easy to check that there are 2 topological types of germs at 0 of intersections $S \cap X$ where S is a C^2 submanifold transverse to Y at 0. However the number of topological types of such germs for S of class C^1 is infinite, even uncountable. It is easy to construct similar examples showing (t^k) does not imply (t^{k-1}).

This example arose from the discovery independently by S. Koike and W. Kucharz that the 6-jet $x^3 - 3xy^5$ has infinitely many topological types among its representatives of class C^7, but only finitely many (in fact two) among its representatives of class C^8. Such an example contradicts a conjecture of Thom from [123]. The relation of these properties of jets with stratification theory and the conditions (t^1) and (t^2) was pointed out by Kuo and Lu [69].

On a historical note, condition (t) with no specification on the differentiability of the transversals was first introduced by Thom in 1964 [121], before the appearance of Whitney's conditions (a) and (b). Thom claimed that (t) implies the openness of the set of maps transverse to a stratification [121, 123]. This is true in the semialgebraic case because then (t) implies (a) and one can use Theorem 4.2.12, but is false for C^∞ stratified sets, again using Theorem 4.2.12 and examples with (t) but not (a) [127].

4.10 Density and Normal Cones

We saw above Hironaka's Theorem 4.5.9 (from [54]) that complex analytic Whitney stratifications are equimultiple along strata. What is a real version of this statement ?

The multiplicity $m(V, p)$ at a point p of a complex analytic variety V is the number of points near p in the intersection of V with a generic plane L missing p, of complementary dimension to that of V. This positive integer is equal to the Lelong number, or density $\theta(V, p)$ of V at p defined as the limit as ϵ tends to 0 of the quotient $\frac{\mathrm{vol}(V \cap B_\epsilon(p))}{\mathrm{vol}(P \cap B_\epsilon(p))}$ where P is a plane containing p of the same dimension as V. Kurdyka and Raby showed that the density is well-defined for subanalytic sets, as a positive real number [72]. It is thus natural to conjecture (I did so in 1988) that the density of a subanalytic set is continuous along strata of a subanalytic Whitney stratification, as a generalisation of Hironaka's theorem to the real case. This was partially proved by G. Comte in his thesis (1998) for subanalytic (w)-regular stratifications [17], and more generally for subanalytic (b^*)-regular stratifications. The general conjecture was proved for subanalytic (b)-regular stratifications by G. Valette in 2008 [141]. Valette also showed that the density is a Lipschitz function along strata of a subanalytic (w)-regular stratification. Analogous theorems for the continuity (resp. Lipschitz variation) of Lipschitz-Killing invariants along strata of a definable Whitney (resp. (w)-regular) stratification were proved by Nguyen and Valette in 2018 [94].

For a long time it was thought that Whitney regularity might impose restrictions on the space of limits of tangents to a stratified set. In the case of an isolated singularity this was shown not to be the case by a construction of Kwiecinski and Trotman proving that any continuum (compact connected set) can be realised as the tangent cone or Nash fibre of a Whitney (b)-regular stratified set at an isolated singular point [73].

In the paper [54] about equimultiplicity, Hironaka proved results about the *normal cones* of complex analytic Whitney stratifications that one can generalise to the subanalytic case as follows. Suppose Z is a stratified subset of $\mathbf{R}^n$ and let Y be a stratum. Let π_Y be the projection of a tubular neighbourhood of Y and let

$\mu(v) = \frac{v}{\|v\|}$. The normal cone is defined to be:

$$C_Y Z = \overline{\{(x, \mu(x\pi_Y(x)))|x \in Z - Y\}}|_Y \subset \mathbf{R^n} \times S^{n-1}.$$

Let $p : C_Y Z \to Y$ be the canonical projection.

Theorem 4.10.1 *A (b)-regular subanalytic stratification of a subanalytic set is*

(npf) normally pseudo-flat, i.e. p is an open map, and
(n) for each stratum Y and each point y of Y, the fibre $(C_Y Z)_y$ of the normal cone at y is equal to the tangent cone $C_y(Z_y)$ at y to the special fibre ${\pi_Y}^{-1}(y)$.

The proofs are by integration of vector fields [52, 54, 98].

The result is not true for definable sets in non-polynomially bounded o-minimal structures, as shown by the following examples, together with Miller's dichotomy that an o-minimal structure is polynomially bounded if and only if it does not contain the exponential function as a definable function [84].

Example 4.10.2 Take Z in $\mathbf{R^3}$ to be the graph of the function $f : [0, \infty) \times \mathbf{R} \to \mathbf{R}$ defined by

$$z = f(x, y) = x - \frac{x}{ln(x)} ln(y + (x^2 + y^2)^{\frac{1}{2}}).$$

Stratify Z by $Z_1 = \{0y\} \subset Z$. One checks easily that $(C_Y Z)_0$ is an arc, while $C_0(Z_0)$ is a point so that the criterion (n) above fails. Moreover the example is not normally pseudoflat, nor (b^*)-regular, but it is Whitney (b)-regular (see [138] and [135]).

Example 4.10.3 Consider the closure of the graph in $\mathbf{R}^3$ of the function

$$g(x, y) = y^{x^2+1}$$

defined on $\mathbf{R} \times (0, \infty]$. This is an example of Pawłucki of a definable stratified set which is (b)-regular but not a C^1 manifold with boundary [103]. It is not normally pseudoflat. Also the three dimensional stratified set defined by the span of this graph and the plane $\{z = 0\}$ provides the first example of a definable Whitney stratified set for which the density is not continuous along a stratum. For details see [135].

In [98] real algebraic (a)-regular examples are given showing that (n) does not imply (npf) and conversely.

Example 4.10.4 First let $(0z) = Z_1 \subset Z = \{x(x^2+y^2)z^2-(x^2+y^2)^2+xy^2 = 0\}$. Then (a) and (n) hold but (npf) fails.

Example 4.10.5 Finally look again at $\{y^2 = t^2x^2 + x^3\}$, stratified by the t-axis and its complement. Here (n) fails, because $(C_Y Z)_0$ consists of 2 points while $C_0(Z_0)$ consists of 1 point, but it is normally pseudoflat.

4.11 Algebraic Topology of Stratified Spaces

Because stratified sets are a generalisation of smooth manifolds to singular spaces it is natural to study the analogues of the highly developed theories concerning the algebraic and differential topology of manifolds.

For example Morse theory has been generalised to stratified Morse theory by Goresky and MacPherson [41]. Not all Whitney stratified sets admit Morse functions in their sense, however Morse functions (exist and) are dense on subanalytic sets. See the contribution of Mark Goresky to this handbook for an account of the current state of stratified Morse theory. Also Poincaré duality is a fundamental property of compact smooth manifolds. To provide a suitable generalisation of this duality Goresky and MacPherson developed intersection homology for stratified spaces in 1980 [40].

In his 1976 thesis Goresky developed a geometric theory for homology and cohomology carried by Whitney stratified chains and cochains [39]. He proved that the homology of a compact smooth manifold can be represented by Whitney stratified cycles, and that the cohomology of a compact Whitney stratified set can be represented by Whitney stratified cocycles. Murolo [87] showed how to obtain an isomorphism between the homologies and cohomologies.

A basic theorem in smooth manifold theory is the Poincaré-Hopf theorem equating the Euler characteristic of a compact manifold, possibly with boundary, with the total index of a vector field with isolated zeros. For stratified vector fields on a Whitney stratified set one has to impose restrictions on the vector field, for example to be radial, i.e. exiting from a family of tubes around each stratum, as first defined by M.-H. Schwartz [110]. She used in fact the stronger (w)-regular stratifications in the case of real analytic manifolds with boundary. More general theorems are due to Simon [116] for radial vector fields on (c)-regular stratified sets and to King and Trotman [64] who allow more general stratified sets (including closure orderable subanalytic partitions of a given subanalytic set) and more general vector fields : semi-radial vector fields (which never point orthogonally into a tube) and even arbitrary (generic) vector fields by introducing a notion of virtual index. The very large quantity of results concerning index theorems and Chern classes for singular real and complex analytic varieties up to 2009, almost always using Whitney stratifications, is described by Brasselet, Seade and Suwa in their book [13].

There are versions of the De Rham theorem for stratified spaces and intersection cohomology due to several authors, including Brasselet, Hector and Saralegui [11], also Brasselet and Legrand [12]. Extensive work on the signature of compact stratified pseudomanifolds is due to Albin, Leichtnam, Mazzeo and Piazza [1], related to Melrose's iterated fibration construction. These references are mere examples in a large body of literature.

The study of the topology of Whitney stratified sets is very much alive. Recent work includes a study of their combinatorial properties by Ehrenborg, Goresky and Readdy [27], and a stratum-sensitive approach to homotopy theory in Woolf's transversal homotopy theory [149]. The precise relation of Whitney stratified sets and Thom maps to the deep work of Ayala, Francis, Tanaka and Rozenblyum on local properties of a new class of conically smooth stratified spaces is currently conjectural [2, 3].

4.12 Real World Applications

In so far as (a)-regular stratification is essential in the proof that the space of smooth functions corresponding to the elementary catastrophes is an open set (by Theorem 4.2.12 [126, 130]), so that the properties of the functions are stable, there are hundreds of very varied applications of Whitney stratifications in papers on applications of catastrophe theory to physics (e.g. gravitational lensing), engineering, ship design, economics, urban geography, paleontology, psychology, biology, etc.

Canny used Whitney stratifications to define roadmaps (curves connecting two points in a semialgebraic set) in his prize-winning work on finding simple exponential algorithms for the generalised piano-mover's problem [16] in theoretical robotics. He uses a general position trick to avoid using the doubly exponential algorithm constructing a Whitney stratification of a given algebraic set [86].

More recently Damon, Giblin and Haslinger and Damon with Gasparovic have used extensively Whitney stratifications in their work on the mathematics of natural images and on skeletal structures [20, 21].

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