Chapter 3
Resolution of Singularities: An Introduction

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Contents

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Abstract The problem of resolution of singularities and its solution in various contexts can be traced back to I. Newton and B. Riemann. This paper is an attempt to give a survey of the subject starting with Newton till the modern times, as well as to discuss some of the main open problems that remain to be solved. The main topics covered are the early days of resolution (fields of characteristic zero and dimension up to three), Zariski's approach via valuations, Hironaka's celebrated result in characteristic zero and all dimensions and its subsequent strengthenings and simplifications, existing results in positive characteristic (mostly up to dimension three), de Jong's approach via semi-stable reduction, Nash and higher Nash blowing up, as well as reduction of singularities of vector fields and foliations. In many places, we have tried to summarize the main ideas of proofs of various results without getting too much into technical details.

3.1 Introduction

Let X be a singular irreducible algebraic variety. A **resolution of singularities** of X is a **birational proper morphism**

$$\pi : X' \to X \tag{3.1}$$

such that X' is non-singular.

A morphism $\pi : X' \to X$ is said to be **birational** if there exists a proper algebraic subvariety $Y \subsetneq X$ such that π induces an isomorphism

$$\pi \Big|_{X' \setminus \pi^{-1}(Y)} : X' \setminus \pi^{-1}(Y) \to X \setminus Y \ .$$

The subvariety Y is sometimes called **the center of the blowing up** π and $Y' := \pi^{-1}(Y)$ **the exceptional set of** π.

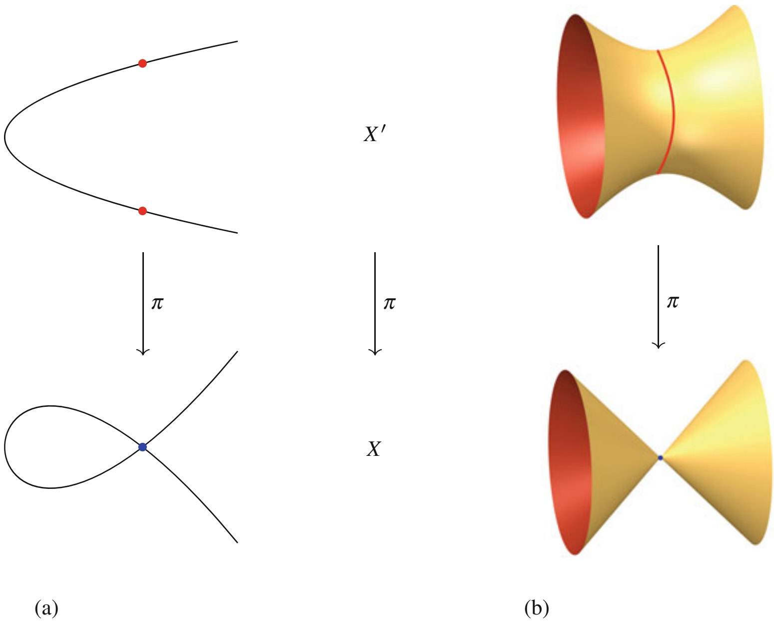

Fig. 3.1 Resolution of singularities. The center of the blowing up is in blue and the exceptional set in red. (**a**) Nodal curve $y^2 - x^2 - x^3 = 0$. (**b**) Non-degenerate quadratic cone $z^2 - x^2 - y^2 = 0$

A morphism π is birational if and only if it induces an isomorphism

$$K(X) \cong K(X')$$

between the fields of rational functions of X and X'.

Figure 3.1 depicts resolution of singularities of the nodal curve and of the non-degenerate quadratic cone.

The equivalence relation induced by all the relations of the form $X \sim X'$ where X' admits a birational morphism (3.1) is called **the birational equivalence relation.**

A very closely related question is resolution of singularities of **analytic** varieties. To state it, replace “algebraic” by “analytic” and “birational” by “bimeromorphic” in the definitions above.

Locally (in the sense of valuation theory explained in detail below) resolution of singularities can be understood as parametrizing wedges of the singular variety X by non-singular algebraic varieties. If X is an (analytically) irreducible curve, resolution of singularities of X is the same as a parametrization of X by a non-singular curve.

The goal of this paper is to give a survey of known results about existence and various constructions of resolution of singularities in cases where it has been achieved as well as discuss the status of this problem in cases when it is still open.

3.1.1 Motivation, Significance and Some Applications of Resolution of Singularities

(1) There are many objects and constructions which can only be defined, or at least are much easier to define and study for non-singular varieties. These include Hodge theory, singular and étale cohomology, the canonical divisor, etc.

(2) The classification problem.

> In any branch of mathematics, there are usually guiding problems, which are so difficult that one never expects to solve them completely, yet which provide stimulus for a great amount of work, and which serve as yardsticks for measuring progress in the field. In algebraic geometry such a problem is the classification problem. In its strongest form, the problem is to classify all algebraic varieties up to isomorphism. We can divide the problem into parts. The first part is to classify varieties up to birational equivalence. As we have seen, this is equivalent to the question of classifying all the function fields (finitely generated extension fields over k) up to isomorphism. The second part is to identify a good subset of a birational equivalence class, such as the nonsingular projective varieties, and classify them up to isomorphism. The third part is to study how far an arbitrary variety is from one of the good ones considered above. In particular, we want to know (a) how much do you have to add to a nonprojective variety to get a projective variety, and (b) what is the structure of singularities, and how can they be resolved to give a nonsingular variety?
>
> Robin Hartshorne, *Algebraic Geometry*, §1.8 What is Algebraic Geometry? [88].

From this point of view, resolution of singularities answers a very natural question: does every birational equivalence class contain a non-singular variety (a **non-singular model**) and, more precisely, is every singular variety X birationally dominated by a non-singular one as in (3.1)? Once this question has been answered affirmatively, one may, on the one hand, look for birational invariants, that is, numbers associated to the given birational equivalence class and defined in terms of some non-singular model, and, on the other hand, address the finer questions about the relation between different non-singular models in the given birational equivalence class and what can be said about the relation between the resolution of singularities and the original singular variety which it dominates. This is a very active area of research, known as the Mori program; it has been the stage of some spectacular recent developments.

(3) **Embedded desingularization** is a somewhat stronger form of resolution of singularities, which is particularly useful for applications. Suppose that X is embedded in a regular variety Z. Embedded desingularization asserts that there exists a sequence $\rho : \tilde{Z} \to Z$ of blowings up along non-singular centers (this notion will be defined precisely below), under which the *total transform* $\rho^{-1}(X)$ of X becomes a *divisor with normal crossings*, which means that all

of its irreducible components are smooth hypersurfaces and locally at each point of $\tilde{Z}$, $\rho^{-1}(X)$ is defined by a monomial with respect to some regular system of parameters. Geometrically, this means that at every point of $\tilde{Z}$ there exists a local coordinate system such that $\rho^{-1}(X)$ looks locally like a union of coordinate hyperplanes, counted with certain multiplicities. Thus divisors with normal crossings locally have a very simple structure. There are many situations in which it is useful to know that every closed subvariety can be turned into a divisor with normal crossings by blowing up. For example, this is used for *compactifying* algebraic varieties (problem (b) mentioned in the passage by R. Hartshorne cited above). Let X be a regular algebraic variety over a field k, embedded in some projective space $\mathbb{P}^n_k$. If X is not closed in $\mathbb{P}^n_k$, we can always consider its Zariski closure $\bar{X}$, which is, by definition, projective over k. The problem is that even though we started with a regular X, $\bar{X}$ may well turn out to be singular. Resolution of singularities, together with its embedded version, assures us that, after blowing up closed subvarieties, disjoint from X, we may embed X in a *regular* projective variety X' such that $X' \setminus X$ is a normal crossings divisor.

(4) Finally, resolution of singularities is useful for studying singularities themselves. Namely, let $\xi \in X$ be a singularity and let $\pi : \tilde{X} \to X$ be a desingularization. We may adopt the following philosophy for studying the singularity ξ. All the regular points are locally the same; every singular point is singular in its own way. We may regard resolution of singularities as a way of getting rid of the local complexity of the singularity ξ and turning it into *global* complexity of the regular variety $\tilde{X}$. Thus some global invariant of $\tilde{X}$ may also be regarded as an invariant of the singularity ξ. For example, if X is a surface and the singularity ξ is isolated, then $\pi^{-1}(\xi)$ is a collection of curves on the regular surface $\tilde{X}$. By embedded resolution for curves, we may further achieve the situation where $\pi^{-1}(\xi)$ is a normal crossings divisor (a resolution of singularities having this property is called **a good resolution**). If $\{E_i\}_{1\le i\le n}$, are the irreducible components of $\pi^{-1}(\xi)$, then the intersection matrix $(E_i.E_j)$ (equivalently, the dual graph of the configuration $\bigcup_{i=1}^{n} E_i$) is an important combinatorial invariant associated to the singularity ξ. A good illustration of the usefulness of replacing local difficulties by global is D. Mumford's theorem that asserts that a normal surface singularity which is topologically trivial is regular. More precisely, given a normal surface singularity $\xi \in X$ over $\mathbb{C}$, one may consider its **link**, which is the intersection of X with a small Euclidean sphere centered at ξ. The link is a real 3-dimensional manifold. Mumford's theorem asserts that if the link is simply connected, then ξ is regular. The idea behind Mumford's proof is that the link is nothing but the boundary of a tubular neighbourhood of the collection $\bigcup_{i=1}^{n} E_i$ of non-singular curves on the non-singular surface $\tilde{X}$. This really helps analyze the link.

3.2 A Brief Early History of the Subject: First Constructions of Resolution of Curve Singularities

3.2.1 Newton Polygon and Newton's Rotating Ruler Method for Resolving Plane Curve Singularities

Resolution of singularities of plane curves is due to Newton and Puiseux.

Consider a polynomial or a power series $f(x, y) = \sum\limits_{i,j\in\mathbb{N}} a_{ji}x^i y^j$, where $a_{ij} \in \mathbb{C}$, $f(0, 0) = 0$ and there exists a strictly positive integer n such that

$$a_{0n} \neq 0 \tag{3.2}$$

(that is, the monomial y^n appears in f with a non-zero coefficient). Newton and Puiseux proved that, viewed as an equation in y to be solved in functions of x, $f(x, y) = 0$ has a solution in **Puiseux series** of x (by definition, in a Puiseux series the exponents are rational numbers with bounded denominators).

Theorem 3.2.1 (Newton 1676, Puiseux 1850) *There exists a strictly positive integer m and a Puiseux series $y(x) = \sum\limits_{i=1}^{\infty} c_i x^{\frac{i}{m}}$ such that $f(x, y(x)) \equiv 0$ as a series in $x^{\frac{1}{m}}$.*

Remark 3.2.2 Let $K = \bigcup\limits_{m=1}^{\infty} \mathbb{C}\left(\left(x^{\frac{1}{m}}\right)\right)$. Theorem 3.2.1 says, in particular, that K is algebraically closed. This was the motivation and the point of view adopted by Puiseux.

Newton Polygon

In order to prove Theorem 3.2.1, Newton introduced the notion of **Newton polygon** which, together with its generalization to higher dimensions called **Newton polyhedron** [60, 94–96, 140] has proved to be one of the most fundamental tools in the theory of resolution of singularities.

Let $\mathbb{R}^2_+$ denote the first quadrant of $\mathbb{R}^2$.

Definition 3.2.3 The **Newton polygon** of f, which we will denote by $\Delta(f, y)$, is the convex hull of the set

$$\bigcup_{\substack{(i,j)\in\mathbb{N}^2 \\ a_{ij}\neq 0}} \left((i, j) + \mathbb{R}^2_+)\right) \subset \mathbb{R}^2.$$

Let n be the smallest strictly positive integer satisfying (3.2).

Definition 3.2.4 The vertex $(0, n)$ is called the **pivotal vertex** of $\Delta(f, y)$. The non-vertical edge of $\Delta(f, y)$ containing $(0, n)$ is called the **leading edge** of $\Delta(f, y)$.

As Newton says, to trace the leading edge we put a vertical ruler through $(0, n)$ and rotate it till it hits another point (i, j) with $a_{ij} \neq 0$ (equivalently, another vertex of $\Delta(f, y)$). Let E denote the leading edge of $\Delta(f, y)$. Let $\text{in}_E f := \sum_{(i,j)\in E} a_{ij}x^i y^j$. The polynomial $\text{in}_E f$ is called **the initial form** of f with respect to E. The leading edge, the pivotal point, the initial form of f with respect to an edge and their generalizations to the higher dimensional context of Newton polyhedra play a crucial role in many constructions of resolution of singularities today.

We give a sketch of Newton's proof of Theorem 3.2.1.

Proof If E is horizontal then $y^n \mid f$, so $y = 0$ is a root of f of multiplicity n. Assume that E is not horizontal. Let α be a root of $\text{in}_E f(1, y)$ and s the multiplicity of the root α.

Write the slope of E as $-\frac{q}{r}$, where q and r are two relatively prime strictly positive integers. There are two cases to be considered.

Case 1. We have $\text{in}_E f \neq a_{0n}\left(y - \alpha x^{\frac{r}{q}}\right)^n$. In other words,

$$s < n. \tag{3.3}$$

Put $x_1 = x^{\frac{1}{q}}$ and $y_1 = \frac{y}{x_1^r} - \alpha$. Make the substitution

$$x = x_1^q \tag{3.4}$$

$$y = y_1 x_1^r + \alpha x_1^r. \tag{3.5}$$

Case 2. We have $\text{in}_E f = a_{0n}\left(y - \alpha x^{\frac{r}{q}}\right)^n$. Note that in this case, by Newton's binomial theorem, we have $\left(n-1, \frac{r}{q}\right) \in E$. This implies that $\frac{r}{q} \in \mathbb{N}$ (in other words, $q = 1$) and $a_{r,n-1} \neq 0$.

Remark 3.2.5 Here we are using in a crucial way the fact that char $\mathbb{C} = 0$. This phenomenon will have important repercussions later when we discuss H. Hironaka's proof of resolution of singularities in characteristic zero and all dimensions, the notions of Tschirnhausen transformation and maximal contact used there and the failure of all them over fields of characteristic $p > 0$.

Put $x_1 = x$ and $y_1 = y - \alpha x_1^r$. Make the substitution

$$x = x_1 \tag{3.6}$$

$$y = y_1 + \alpha x_1^r. \tag{3.7}$$

In both cases, let $f_1(x_1, y_1)$ denote the polynomial or power series, resulting from substituting (3.4)–(3.5) (resp. (3.6)–(3.7)) into f. Let

$$n_1 = n - s \quad \text{in Case 1} \tag{3.8}$$

$$n_1 = n \quad \text{in Case 2.} \tag{3.9}$$

A direct computation shows the following:

(a) the Newton polygon $\Delta(f_1, y_1)$ has $(0, n_1)$ as a vertex
(b) in Case 2, the slope of the leading edge of $\Delta(f_1, y_1)$ is strictly greater than $-\frac{1}{r}$.

Now, iterate the procedure to construct (x_i, y_i) and f_i for $i \in \mathbb{N}$. Since in Case 1 we have $n_1 < n$, Case 1 can occur at most n times. Take $i_0 \in \mathbb{N}$ such that Case 2 occurs for all $i \geq i_0$. For $i > i_0$, let $-\frac{1}{r_i}$ denote the slope of the leading edge of the Newton polyhedron $\Delta(f_i, y_i)$. Our iterative procedure produces $x_i = x_{i_0}$, $y_i = y_{i_0} - \sum\limits_{j=i_0}^{i-1} b_j x_{i_0}^{r_j}$ for suitable $b_j \in \mathbb{C}$. According to statement (b) above, the sequence of integers $(r_i)_i$ is strictly increasing with i, hence goes to ∞ (it may happen that the leading edge of $\Delta(f_i, y_i)$ becomes horizontal for some finite i, in which case we set all the subsequent coefficients b_j to be equal to 0; the procedure will stop here). Let $y_\infty := y_{i_0} - \sum\limits_{j=i_0}^{\infty} b_j x_{i_0}^{r_j}$, substitute $y_{i_0} = y_\infty + \sum\limits_{j=i_0}^{\infty} b_j x_{i_0}^{r_j}$ into f_{i_0} and let f_∞ be the resulting polynomial (resp. power series). The leading edge of $\Delta(f_\infty, y_\infty)$ has slope strictly greater than $-\frac{1}{r_i}$ for all i, hence it is horizontal. Thus $y_\infty^{n_{i_0}} \Bigm| f_\infty$, so $y_{i_0}\left(x_{i_0}\right) := \sum\limits_{j=i_0}^{\infty} b_j x_{i_0}^{r_j}$ is a root of f_{i_0} of multiplicity n_{i_0}.

Let $m := \prod\limits_{j=0}^{i_0} q_i$, $Q := \sum\limits_{j=0}^{i_0} \prod\limits_{\ell=0}^{j} q_\ell$ and $R := \sum\limits_{j=0}^{i_0} \prod\limits_{\ell=0}^{j} r_\ell$. By construction, we have

$$x_{i_0} = x^{\frac{1}{m}} \tag{3.10}$$

and

$$y_{i_0} = yx^{-\frac{R}{Q}} + g\left(x^{\frac{1}{m}}\right), \tag{3.11}$$

where g is a suitable polynomial with complex coefficients. Let $\sum\limits_{i=1}^{\infty} c_i x^{\frac{i}{m}}$ be the Puiseux series $x^{\frac{R}{Q}}\left(\sum\limits_{j=i_0}^{\infty} b_j x_{i_0}^{\frac{r_j}{m}} - g\left(x^{\frac{1}{m}}\right)\right)$. Making the substitution (3.10)–(3.11) back into f_{i_0} and setting $y(x) := \sum\limits_{i=1}^{\infty} c_i x^{\frac{i}{m}}$, we see that $(y - y(x))^{n_{i_0}} \mid f$, that is, $y(x)$ is a root of f of multiplicity n_{i_0}, as desired. □

Remark 3.2.6 Every time Case 1 occurred in Newton's algorithm some choices needed to be made. For example, if Case 1 happens at the first step we had to choose a root α of $\mathrm{in}_E f$. Counted with multiplicity there were $s = n - n_1$ such choices. Starting with the step i_0 we have constructed a root of f of multiplicity n_{i_0}.

Therefore the total number of roots of f obtained by this procedure, counted with multiplicity, is $n_{i_0} + \sum_{j=0}^{i_0-1} \left(n_j - n_{j+1}\right) = n$.

Remark 3.2.7 In the Newton–Puiseux theorem, assume that f is either a polynomial or a convergent power series. It is not hard to show (by estimating the coefficients b_j at each step of the construction) that the Puiseux series produced by Newton's algorithm is also convergent. Assume, in addition, that the plane complex curve $C := \{f(x, y) = 0\}$ is irreducible as an analytic space (in other words, has only one branch near the origin). Then Newton's procedure gives a parametrization of C near the origin by a complex disk with the coordinate x_{i_0}, that is, a resolution of singularities of a suitable neighbourhood of the origin in C. Algebraically, this resolution of singularities is described by the birational, injective ring homomorphism $\mathbb{C}\{x, y\} \hookrightarrow \mathbb{C}\{x_{i_0}\}$, that maps x to $x_{i_0}^m$ and y to $\sum_{i=1}^{\infty} c_i x_{i_0}^i$.

More generally, if the analytic curve C has several branches, parametrizations of each of them are obtained by making suitable choices of roots in Newton's algorithm.

While we are on the subject of resolution of plane curve singularities and Newton polygon, we mention an important work [114] by Monique Lejeune-Jalabert that paved the way to the approach to resolution of singularities and local uniformization via key polynomials (see below).

Global resolution of singularities of analytic curves is due to B. Riemann and was achieved using complex-analytic methods. Indeed, the Riemann surface associated to a complex-analytic curve *is* its resolution of singularities.

Purely algebraic proofs of resolution of curve singularities were given much later by Italian geometers like Albanese [13]. Albanese's proof consists in projecting a singular curve embedded in a projective space of a sufficiently large dimension (more than twice than the degree of the curve) from one of its singular points and showing that this process improves the singularity. Below we will discuss a beautiful one-step procedure defined by O. Zariski [168] that resolves singularities of curves.

3.3 Blowing Up, Multiplicity and the Hilbert–Samuel Function

In this section we introduce one of the main tools for constructing resolution of singularities: blowing up. Blowing up of a variety X along a subvariety Y (more generally, along an ideal sheaf $\mathcal{I}$) is a birational projective morphism $\pi : X' \to X$, defined below, that induces an isomorphism $\pi\big|_{X' \setminus \pi^{-1}(Y)} : X' \setminus \pi^{-1}(Y) \to X \setminus Y$. As we will see, blowing up of a non-singular variety along a non-singular subvariety is again non-singular. Thus a very general idea for constructing a resolution of singularities of a variety X, that we will explain in more detail below, goes as follows.

(1) Embed X in a non-singular variety Z.
(2) Construct a sequence

$$Z \overset{\rho_1}{\longleftarrow} Z_1 \overset{\rho_2}{\longleftarrow} \dots \overset{\rho_i}{\longleftarrow} Z_i \tag{3.12}$$

of blowings up along non-singular centers and study the **strict transform** X_i of X in Z_i (defined below) in the hope of improving and eventually eliminating the singularities of X_i. We now go for precise definitions.

Let X be an affine algebraic variety with coordinate ring A and $I = (f_1, \dots, f_n)$ an ideal of A. As usual, $V(I)$ will denote the zero locus of I.

Definition 3.3.1 The **blowing up** of X along I is the birational projective morphism $\pi : \tilde{X} \to X$, defined as follows. Consider the morphism $\phi : X \setminus V(I) \to X \times_k \mathbb{P}_k^{n-1}$, which sends every $\xi \in X \setminus V(I)$ to $(\xi, (f_1(\xi) : \cdots : f_n(\xi))) \in X \times_k \mathbb{P}_k^{n-1}$. The blowing up $\tilde{X}$ is defined to be the closure $\overline{\phi(X \setminus V(I))} \subset X \times_k \mathbb{P}_k^{n-1}$ in the Zariski topology.

Remark 3.3.2 Since the blowing up $\widetilde{X} = \overline{\phi(X \setminus V(I))} \subset X \times_k \mathbb{P}_k^{n-1}$, the natural projection $X \times_k \mathbb{P}_k^{n-1} \to X$ induces a map $\widetilde{X} \to X$. In particular, $\widetilde{X}$ is projective over X.

The natural map $\pi : \widetilde{X} \to X$ is an isomorphism away from $V(I)$ (the inverse mapping is given by ϕ). This means that the map $\pi : \widetilde{X} \to X$ is birational.

Remark 3.3.3 If X is irreducible (that is, A is an integral domain), then $\widetilde{X}$ is covered by n affine charts U_i, $i \in \{1, \dots, n\}$ with coordinate rings

$$A\left[\frac{f_1}{f_i}, \dots, \frac{f_n}{f_i}\right], 1 \le i \le n, \tag{3.13}$$

where the glueing of the charts is implicit in the notation.

Example 3.3.4

1) Blowing up the plane at a point. Let $X = k^2$ be the affine plane, $A = k[x, y]$ its coordinate ring and $I = (x, y)$ the ideal defining the origin. Let (u_1, u_2) be homogeneous coordinates on $\mathbb{P}_k^1$. We have the map $k^2 \setminus \{0\} \to k^2 \times \mathbb{P}_k^1$ that sends the point (x, y) to the point $(u_1 : u_2) \in \mathbb{P}_k^1$.
The blowing up $\widetilde{X}$ is defined in $k^2 \times_k \mathbb{P}_k^1$ by the equation $xu_2 - yu_1 = 0$. For example, if $k = \mathbb{R}$, then $\widetilde{X}$ is nothing but the Möbius band.
Perhaps the most useful way of thinking about the blowing up $\widetilde{X}$ is that it is a variety glued together from two coordinate charts with coordinate rings $k\left[u_1, \frac{u_2}{u_1}\right]$ and $k\left[u_2, \frac{u_1}{u_2}\right]$, where, again, the glueing is implicit in the notation.

2) More generally, we can blow up the affine n-space at the origin. Let

$$A = k[x_1, \ldots, x_n], \quad I = (x_1, \ldots, x_n).$$

Let $u_1, \ldots, u_n$ denote homogeneous coordinates on $\mathbb{P}_k^{n-1}$. Then $\widetilde{X} \subset k^n \times \mathbb{P}_k^{n-1}$ is the subvariety defined by the equations $x_i u_j - x_j u_i$, $1 \leq i, j \leq n$. Again, $\widetilde{X}$ is covered by n coordinate charts with coordinate rings $k\left[\frac{u_1}{u_i}, \ldots, \frac{u_{i-1}}{u_i}, u_i, \frac{u_{i+1}}{u_i}, \ldots, \frac{u_n}{u_i}\right]$, $1 \leq i \leq n$.

3) Even more generally, the blowing up $\widetilde{X}$ of k^n along $(x_1, \ldots, x_l)$ for $l < n$ is the subvariety of $k^n \times \mathbb{P}^{l-1}$ defined by the equations $x_i u_j - x_j u_i$, $1 \leq i, j \leq l$. The blowing up $\widetilde{X}$ is covered by l coordinate charts with coordinate rings $k\left[\frac{u_1}{u_i}, \ldots, \frac{u_{i-1}}{u_i}, u_i, \frac{u_{i+1}}{u_i}, \ldots, \frac{u_l}{u_i}, u_{l+1}, \ldots, u_n\right]$. Intuitively, we may think of this last construction as first blowing up the origin in k^l and then taking the direct product of the whole situation with k^{n-l}.

3.3.1 The Universal Mapping Property of Blowing Up

We now give a characterization of the blowing up of a variety X along an ideal $I \subset A$ by a universal mapping property (in particular, this characterization makes no reference to any particular ideal base $(f_1, \ldots, f_n)$ of I).

Let $\pi : \widetilde{X} \to X$ be a morphism of algebraic varieties and $\mathcal{I}$ a coherent ideal sheaf on X. Let $\widetilde{X} = \bigcup\limits_{i, j \in \Phi_i} V_{ij}$ and $X = \bigcup\limits_{1 \leq i \leq s} U_i$ be the respective coverings by affine charts, where the Φ_i are certain index sets such that $\pi^{-1}(U_i) = \bigcup\limits_{j \in \Phi_i} V_{ij}$, $1 \leq i \leq s$. Let A_i denote the coordinate ring of U_i and B_{ij} that of V_{ij}. For each i and each $j \in \Phi_i$ we have a homomorphism $A_i \to B_{ij}$. Let $\pi^* \mathcal{I}$ denote the coherent ideal sheaf on $\widetilde{X}$ whose ideal of sections over V_{ij} is $I_i B_{ij}$.

Let X be a scheme and $\mathcal{I}$ a coherent ideal sheaf on X. The idea, which we now explain in detail, is that the blowing up $\pi : \widetilde{X} \to X$ of X along $\mathcal{I}$ is characterized by the universal mapping property with respect to making $\pi^* \mathcal{I}$ invertible (see the Definition below).

Definition 3.3.5 Let I be an ideal in a ring A. The ideal I is said to be **locally principal** if for every maximal ideal $\mathfrak{m}$ of A the ideal $I A_{\mathfrak{m}}$ is principal. The ideal I is said to be **invertible** if for every maximal ideal $\mathfrak{m}$ of A the ideal $I A_{\mathfrak{m}}$ is principal and generated by a non-zero divisor.

Of course, if A is a domain, then invertible and locally principal are the same thing; this case will be our main interest in the present paper.

Definition 3.3.6 An ideal sheaf $\mathcal{I}$ on a variety X is **locally principal** if there exists an affine open cover $X = \bigcup\limits_i U_i$ such that, denoting by A_i the coordinate chart of

U_i, the ideal $\mathcal{I}_{U_i}$ of sections of $\mathcal{I}$ is a principal ideal of A_i for all i. The ideal sheaf $\mathcal{I}$ is said to be **invertible** if each $\mathcal{I}_{U_i}$ is principal and generated by an element which is not a zero divisor.

Again, if X is irreducible then invertible and locally principal are the same thing.

Let the notation be as in (3.13) above. We have

$$IA\left[\frac{f_1}{f_i},\dots,\frac{f_{i-1}}{f_i},\frac{f_{i+1}}{f_i},\dots,\frac{f_n}{f_i}\right]=(f_1,\dots,f_n)A\left[\frac{f_1}{f_i},\dots,\frac{f_{i-1}}{f_i},\frac{f_{i+1}}{f_i},\dots,\frac{f_n}{f_i}\right]=$$
$$=(f_i)A\left[\frac{f_1}{f_i},\dots,\frac{f_{i-1}}{f_i},\frac{f_{i+1}}{f_i},\dots,\frac{f_n}{f_i}\right], \tag{3.14}$$

so that π^*I is invertible on $\widetilde{X}$. Since we are dealing with a *local* property, this statement remains valid even if X is not affine. In other words, if $\pi:\widetilde{X}\to X$ is the blowing up of a coherent ideal sheaf $\mathcal{I}$, then $\pi^*\mathcal{I}$ is invertible.

We now point out that this property is also sufficient to characterize blowing up. Namely, the blowing up π of $\mathcal{I}$ is the smallest (in the sense explained in Theorem 3.3.7 below) projective morphism such that $\pi^*\mathcal{I}$ is invertible. More precisely, we have the following theorem.

Theorem 3.3.7 (The Universal Mapping Property of Blowing Up [88], Proposition II.7.14, p. 164) *Let $\rho:Z\to X$ be a morphism of irreducible algebraic varieties such that $\rho^*\mathcal{I}$ is invertible. Then ρ factors through $\widetilde{X}$ in a unique way.*

Proof We briefly sketch the idea of the proof. Since $\rho^*\mathcal{I}$ is invertible, at each point of Z it must be generated by one of the f_i. Hence Z admits a covering $Z=\bigcup\limits_{i=1}^n V_i$ by affine charts with coordinate rings B_i such that $IB_i=(f_i)B_i$. Then $\frac{f_j}{f_i}\in B_i$, so

$$A\left[\frac{f_1}{f_i},\dots,\frac{f_{i-1}}{f_i},\frac{f_{i+1}}{f_i},\dots,\frac{f_n}{f_i}\right]\hookrightarrow B_i. \tag{3.15}$$

The inclusion (3.15) determines a morphism $\lambda_i:V_i\to U_i$ of affine algebraic varieties, where U_i is as in (3.13). Glueing together the morphisms λ_i, $1\le i\le n$, gives the desired factorization of ρ through $\widetilde{X}$. □

Remark 3.3.8 All of the above definitions, constructions and results can easily be generalized to the case of varieties that may be reducible. We chose to work with irreducible ones to simplify the notation and the exposition.

3.3.2 Strict Transforms

Let Z be an irreducible variety and $\mathcal{I}$ a coherent ideal sheaf on Z. Let $\iota: X\hookrightarrow Z$ be a closed irreducible subvariety of Z with its natural inclusion ι. Let $\pi:\tilde{Z}\to Z$

be the blowing up along $\mathcal{I}$. Let $\widetilde{X} := \overline{\pi^{-1}(X \backslash V(\mathcal{I}))} \subset \tilde{Z}$, where "$\overline{}$" denotes the closure in the Zariski topology.

Definition 3.3.9 The variety $\widetilde{X}$ is called **the strict transform** of X under π.

Of course, $\widetilde{X} \subset \pi^{-1}(X) = \widetilde{X} \cup \pi^{-1}(V(\mathcal{I}))$. To distinguish it from the strict transform, $\pi^{-1}(X)$ is sometimes called **the total transform** of X under π. We state the following useful fact without proof.

Theorem 3.3.10 *The variety $\widetilde{X}$ together with the induced morphism $\rho : \widetilde{X} \to X$ is nothing but the blowing up of the coherent ideal sheaf $\iota^*\mathcal{I}$ on X.*

Example 3.3.11 Let k be a field and u, v—independent variables. Let $Z = k^2$ be the affine plane with coordinate ring $k[u, v]$, $I = (u, v)$ and X—the plane curve $\{u^2 - v^3 = 0\} \subset Z$.

The blowing up $\widetilde{Z}$ of Z along I is covered by two affine charts with coordinate rings

$$k\left[\frac{u}{v}, v\right] \text{ and } k\left[u, \frac{v}{u}\right].$$

Let us denote the coordinates in the first chart U_1 by u_1, v_1, so that $v = v_1$, $u = u_1 v_1$. Let u_2, v_2 be the coordinates in the second chart U_2, so that $u = u_2$, $v = u_2 v_2$.

To calculate the strict transform $\widetilde{X}$ of U_2, we first find its full inverse image. This inverse image is defined by the equation $u^2 - v^3$, but written in the new coordinates:

$$u^2 - v^3 = u_2^2 - u_2^3 v_2^3 = u_2^2(1 - u_2 v_2^3).$$

Here $u_2 = 0$ is the equation of the exceptional divisor. To obtain the strict transform $\widetilde{X}$, we must factor out the maximal power of u_2 out of the equation. In this case, $\widetilde{X} \cap U_2$ is defined by $1 - u_2 v_2^3$. In U_1, we have

$$u^2 - v^3 = u_1^2 v_1^2 - v_1^3 = v_1^2(u_1^2 - v_1).$$

Here $v_1 = 0$ is the equation of exceptional divisor, so that $\widetilde{X} \cap U_1 = V(u_1^2 - v_1)$. In particular, note that although X had a singularity at the origin, $\widetilde{X}$ is non-singular. Thus, in this example we started with a singular variety X with one singular point, blew up the singularity and found that the strict transform of X became non-singular. That is, we obtained a resolution of singularities of X after one blowing up.

3.3.3 Fundamental Numerical Characters of Singularity: Multiplicity and the Hilbert–Samuel Function

We can now elaborate on the very general description of many constructions of resolution of singularities by sequences of blowings up, given at the beginning of this section.

Typically, we embed the variety X we want to desingularize into an ambient non-singular variety Z. Our goal is to successively construct a sequence (3.12) of blowings up along non-singular centers (that is, blowings up that are isomorphic to 3) of Example 1 locally in the classical or étale topology) and study the strict transform X_i of X in Z_i. We want to choose the center of the blowing up ρ_i at each step so as to "improve" the singularities of X_i. The precise meaning of "improve" is the following. Associate to each singular point ξ of X_i a discrete, **upper-semicontinuous** numerical character $d(\xi)$, that is, an element of a fixed well-ordered set, usually a finite string of non-negative integers or a function $\mathbb{N} \to \mathbb{N}$. Improving the singularities of X_i means ensuring that

$$\max\{d(\xi) \mid \xi \in X_{i+1}\} < \max\{d(\xi) \mid \xi \in X_i\}. \tag{3.16}$$

Experience shows that the best bet for achieving the strict inequality (3.16) is to blow up the largest possible centers contained in the maximal stratum of $d(\xi)$.

In this subsection we define the most fundamental numerical characters that usually go into the leading place of $d(\xi)$: multiplicity and its generalization—the Hilbert–Samuel function.

Let k be a field, n a strictly positive integer and $X = V(f)$ an $(n-1)$-dimensional hypersurface in k^n. Write $f = \sum\limits_{\alpha} c_\alpha u^\alpha$, where $c_\alpha \in k$, $u = (u_1, \dots, u_n)$, $\alpha = (\alpha_1, \dots, \alpha_n)$ runs over a finite subset of $\mathbb{N}^n$ and $u^\alpha = \prod\limits_{j=1}^{n} u_j^{\alpha_j}$ is the usual multi-index notation. Further, we will use the notation $|\alpha| = \sum\limits_{j=1}^{n} \alpha_j$.

Definition 3.3.12 The **multiplicity** of f at the origin of k^n is the quantity

$$\mathrm{mult}_0 f := \min\{|\alpha| \mid c_\alpha \neq 0\}.$$

The multiplicity at any other point $\xi = (a_1, \dots, a_n)$ of k^n is defined similarly, but using the expansion of f in terms of $u_i - a_i$ instead of the u_i.

Equivalently, the multiplicity of f at ξ is given by $\mathrm{mult}_\xi f = \max\{n \in \mathbb{N} \mid f \in \mathfrak{m}^n\}$, where $\mathfrak{m} = \left\{\frac{g}{h} \;\middle|\; g, h \in k[u], g(\xi) = 0 \neq h(\xi)\right\}$ is the maximal ideal of the local ring of k^n at ξ.

The only problem with this definition is that it is only valid for hypersurfaces whereas we would like to work with varieties of arbitrary codimension. The generalization of multiplicity that is used in many constructions is the Hilbert–Samuel function [21], which we now define.

Definition 3.3.13 Let (A, m, k) be a local Noetherian ring. The **Hilbert–Samuel function** of A is the function $H_{A,m} : \mathbb{N} \to \mathbb{N}$, defined by $H_{A,m}(n) = length\left(\frac{A}{m^{n+1}}\right)$ (considered as an A-module).

By additivity of length,

$$length\left(\frac{A}{m^{n+1}}\right) = \sum_{i=0}^{n} \dim_k \frac{m^i}{m^{i+1}}, \tag{3.17}$$

where the $\frac{m^i}{m^{i+1}}$ are k-modules, that is, k-vector spaces.

Note that since A is Noetherian, each of m^i is finitely generated, so that all the quantities in (3.17) are finite.

Theorem 3.3.14 (Hilbert–Serre) *The function $H_{A,m}(n)$ is a polynomial for $n \gg 0$. In other words, there exists a polynomial $P(n)$ with rational coefficients, such that*

$$P(n) = H_{A,m}(n) \quad \textit{for}\, n \gg 0.$$

The polynomial $P(n)$ is called **the Hilbert polynomial** of A.

Notation Let $d(A)$ denote the degree of the Hilbert polynomial of A.

Example 3.3.15

1) Let k be a field and $A = k[x_1, \ldots, x_d]$ the polynomial ring in d variables. Let $\mathfrak{m} = (x_1, \ldots, x_d)$ be the maximal ideal corresponding to the origin in k^d. Consider the localization $A_\mathfrak{m}$. The Hilbert-Samuel function of $A_\mathfrak{m}$ is $H_{A_\mathfrak{m},\mathfrak{m}}(n) = length\left(\frac{A}{\mathfrak{m}^{n+1}}\right) = \binom{n+d}{d}$, which is a polynomial in n of degree d. In this case, $H_{A_\mathfrak{m},\mathfrak{m}}(d)$ is a polynomial for *all* n, not merely for n sufficiently large.
2) Let $B := \frac{A_\mathfrak{m}}{(f)}$, where f is a polynomial of multiplicity μ at the origin and let $\mathfrak{n}$ denote the maximal ideal of B. It is not hard to show that

$$\begin{aligned} H_{B,\mathfrak{n}} &= \binom{n+d}{d} && \text{if } n < \mu \\ &= \binom{n+d}{d} - \binom{n+d-\mu}{d} && \text{if } n \geq \mu. \end{aligned} \tag{3.18}$$

Now, $\binom{n+d}{d} - \binom{n+d-\mu}{d}$ is a polynomial of degree $d-1$, whose leading coefficient is $\frac{\mu}{(d-1)!}$. This shows that in the case of hypersurface singularities multiplicity can be recovered from the Hilbert–Samuel function. In fact, in this case multiplicity and the Hilbert–Samuel function are equivalent sets of data.

An important property of multiplicity, the Hilbert–Samuel function and the Hilbert polynomial is that they are **upper semicontinuous**. This means that the stratum of points on an algebraic variety X where the multiplicity (resp. Hilbert–Samuel function, resp. the Hilbert polynomial) is greater than or equal to a given value is a closed algebraic subvariety of X.

3.3.4 Normal Flatness and the Stability of the Hilbert–Samuel Function Under Blowing-Up

In this subsection we provide further details on the above program of resolving the singularities of any algebraic variety by constructing a sequence (3.12) of blowings up that strictly decreases a certain upper semicontinuous numerical invariant $d(\xi)$, $\xi \in X$.

For a point $\xi \in X$, we denote by $\mathcal{O}_{X,\xi}$ the **local ring** of X at ξ, that is, the ring formed by all the rational functions $\frac{g}{h}$ on X whose denominator h does not vanish at ξ. Let $\mathfrak{m}_{X,\xi}$ denote the maximal ideal of $\mathcal{O}_{X,\xi}$; it is the ideal formed by all the $\frac{g}{h}$ such that $g(\xi) = 0$. Write $H_{X,\xi}$ for $H_{\mathcal{O}_{X,\xi},\mathfrak{m}_{X,\xi}}$.

We define the leading component of our numerical invariant $d(\xi)$ to be the Hilbert–Samuel function $H_{X,\xi}$ (resp. $\text{mult}_\xi X$ if X is a hypersurface, where $\text{mult}_\xi X$ denotes the multiplicity at ξ of a local defining equation of X in an ambient non-singular variety Z near ξ).

Let X be an algebraic variety, Y a subvariety of X and ξ a point of Y. Let $\mathcal{I}_Y$ denote the ideal sheaf, defining Y in X. The **normal cone** of Y in X is defined to be the algebraic variety with coordinate ring

$$\bigoplus_{n=0}^{\infty} \frac{\mathcal{I}_Y^n}{\mathcal{I}_Y^{n+1}}.$$

Assume that Y is non-singular.

Definition 3.3.16 (H. Hironaka 1964) We say that X is **normally flat** along Y at ξ if $\bigoplus_{n=0}^{\infty} \frac{\mathcal{I}_{Y,\xi}^n}{\mathcal{I}_{Y,\xi}^{n+1}}$ is a free $\mathcal{O}_{Y,\xi}$-module. We say that X is normally flat along Y if it is normally flat at every point $\xi \in Y$ (equivalently, if $C_{X,Y}$ is flat over Y).

Theorem 3.3.17 (B. Bennett, H. Hironaka) *The variety X is normally flat along Y at ξ if and only if $H_{X,\eta} = H_{X,\xi}$ for all $\eta \in Y$ near ξ (in other words, the Hilbert–Samuel function of X is locally constant on Y near ξ).*

The next theorem (valid over fields of arbitrary characteristic) constitutes the first step of the above program of constructing a resolution of singularities of any algebraic variety by lowering a suitable numerical character $d(\xi)$. Namely, it says that a blowing up along a center Y over which X is normally flat *does not increase* the Hilbert–Samuel function (resp. multiplicity).

Theorem 3.3.18 (H. Hironaka 1964) *Let $Y \subset X$ be a non-singular algebraic subvariety of X over which X is normally flat. Let H denote the common Hilbert–Samuel function $H_{X,\xi}$ for all $\xi \in Y$. Let $\pi : \widetilde{X} \to X$ be the blowing up along Y and $\tilde{\xi} \in \pi^{-1}(Y)$. Then*

$$H_{\tilde{X},\tilde{\xi}} \leq H \tag{3.19}$$

(we compare Hilbert–Samuel functions in the lexicographical order, but in fact all the inequalities we write such as (3.19) hold componentwise, that is, separately for each n).

A subvariety Y as in the Theorem is sometimes referred to as a **permissible center of blowing up** and the blowing π itself as a **permissible blowing up**.

If we can achieve strict inequality in (3.19), our proof of resolution of singularities will be finished by induction. The difficult question is: what to do if equality holds in (3.19)?

3.4 Resolution of Surface Singularities over Fields of Characteristic Zero

Resolution of singularities of surfaces was constructed in late nineteenth—early twentieth century by the Italian school (P. del Pezzo 1892, Beppo Levi 1897 [115, 116], O. Chisini 1921 [51], G. Albanese 1924 [13]) as well as by H.W.E. Jung 1908 [106], followed by the first completely rigorous algebraic proof by R. Walker 1935 [162] and another one by O. Zariski 1939 [169, 171].

Let k be an algebraically closed field of characteristic zero. Below we briefly summarize Beppo Levi's, Jung's and O. Zariski's constructions of resolution of singularities of surface over k, with Beppo Levi's proof valid only for *hypersurfaces*.

3.4.1 Beppo Levi's Method

Let X be an algebraic surface over k, embedded in a smooth threefold Z. For a point $\xi \in X$ let $\text{mult}_\xi X$ denote the multiplicity at ξ of a local defining equation of X in Z near ξ. Beppo Levi's algorithm goes as follows.

1) Let $\mu = \max\left\{\text{mult}_\xi X \;\middle|\; \xi \in X\right\}$.
2) Let $S_\mu = \left\{\xi \in X \;\middle|\; \text{mult}_\xi X = \mu\right\}$. By upper semicontinuity of multiplicity, S_μ is an algebraic subvariety of X, that is, a union of algebraic curves and points.

3) First assume that S_μ is *not* a union of a normal crossings divisor with a finite set of points.
4) The set of points of S_μ where it fails to be a normal crossings divisor is finite. Blow up each of these points, and keep doing so until S_μ becomes a union of a normal crossings divisor with a finite set of points.
5) If S_μ is a union of a normal crossings divisor with a finite set of points, let $\pi : \widetilde{X} \to X$ be a blowing up of an irreducible component of S_μ.
6) By Theorem 3.3.18 (which Beppo Levi proved in the special case of two-dimensional hypersurfaces over fields of characteristic zero), we have

$$\mu \geq \max \left\{ \operatorname{mult}_{\tilde{\xi}} \widetilde{X} \;\middle|\; \tilde{\xi} \in \widetilde{X} \right\}. \tag{3.20}$$

7) If equality holds in (3.20), let $\tilde{S}_\mu = \left\{ \tilde{\xi} \in \widetilde{X} \;\middle|\; \operatorname{mult}_{\tilde{\xi}} \widetilde{X} = \mu \right\}$. Again by Theorem 3.3.18 we have $\tilde{S}_\mu \subset \pi^{-1}(S_\mu)$. Observe that $\tilde{S}_\mu$ is again a union of a normal crossings divisor with a finite set of points (or the empty set).
8) Keep repeating the procedure of 5) until the locus of points of multiplicity μ becomes the empty set. This completes the proof by induction on μ.

Remark 3.4.1 Predictably, Beppo Levi's method of resolution of singularities fails starting with dimension three. Reference [144] gives an example of a threefold X in k^4 all of whose singular points have multiplicity 2. The locus of multiplicity 2 is a normal crossings subvariety consisting of two lines that meet each other at the origin. Blowing up any one of the two lines produces a new threefold whose multiplicity 2 locus is a union of three lines. Blowing up one of those three lines yields a threefold containing a singularity, isomorphic to the origin in X. Thus there exists an infinite sequence of blowings up along non-singular components of the locus of multiplicity 2 which does not resolve the singularities of X.

It was later pointed out by Zariski that none of the proofs of resolution of surfaces by the Italian geometers was complete and some were outright wrong. The first completely rigorous algebraic proof was given by R. Walker in 1935 [162].

3.4.2 Normalization

Before discussing the proofs by Jung and O. Zariski of 1939, we need to introduce the notion of normalization.

Let A be an integral domain with field of fractions K. We may consider the integral closure $\bar{A}$ of A in K (sometimes it is also called **the normalization of** A). If A if of finite type over k, it is the coordinate ring of an irreducible affine algebraic variety X. The inclusion $A \hookrightarrow \bar{A}$ gives rise to the natural birational finite (hence projective) morphism $\pi : \bar{X} \to X$ of irreducible algebraic varieties. The canonical morphism π is called the **normalization** of the variety X. Because of the uniqueness of normalization, even if X is not affine, the separate normalizations of the various affine charts of X glue together in a natural way to yield the normalization of X.

Definition 3.4.2 An integral domain is said to be **normal** if it coincides with its normalization. An algebraic variety is said to be normal if the coordinate rings of all of its affine charts are normal.

The notion of normalization was defined (surprisingly late—in 1939) by Oscar Zariski [168]. This is a great example of the usefulness of the algebraic language in geometry: this notion, extremely important as it turned out to be, did not occur to anyone until the algebraic language was developed. The importance of normalization for resolution of singularities is explained by the following result.

Theorem 3.4.3 (Zariski) *Let A be a one-dimensional Noetherian local ring. Then A is regular if and only if A is normal.*

Corollary 3.4.4 (Zariski) *If X is a normal algebraic variety,*

$$\dim \ Sing(X) \leq \dim X - 2.$$

Geometrically, Theorem 3.4.3 says that normalization resolves the singularities of curves. More generally, it says that for an arbitrary reduced variety normalization resolves the singularities in codimension 1. When normalization was defined, the theorem of resolution of singularities of curves was known for almost a century, yet it was quite a surprise that it had such a simple and elegant proof and that the procedure for desingularization had such a simple description.

We now summarize Jung's and Zariski's methods for the resolution of surfaces.

3.4.3 Jung's Method

1) Fix a projection $\sigma : X \to \mathbb{C}^2$ from our affine singular surface X to a plane and consider the branch locus C of the σ.
2) Apply embedded resolution of plane curve singularities to the curve C, that is, construct a sequence $\rho : W' \to \mathbb{C}^2$ of point blowings up such that the total transform of C under ρ is a normal crossings divisor.
3) Let $X' := X \times_{\mathbb{C}^2} W'$. We obtain a cartesian square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle \sigma'}\downarrow & & \downarrow{\scriptstyle \sigma} \\ W' & \xrightarrow{\rho} & \mathbb{C}^2 \end{array} \tag{3.21}$$

4) Let $\bar{X} \to X'$ be the normalization of X'. The branch locus of $\bar{X}$ over W' is still a normal crossings divisor.

5) Observe that the fact that the branch locus of the normal surface $\bar{X}$ has normal crossings implies that the singularities of $\bar{X}$ are of a very special type, namely, cyclic quotient singularities (that is, singularities obtained from $\mathbb{C}^2$ by taking a quotient by a cyclic group; these are precisely the toric ones among the normal surface singularities).
6) Resolve the cyclic quotient singularities by hand.

Remark 3.4.5 Even though normalization was officially defined by Zariski in 1939, Jung constructs it by hand in this special case. Items 4) and 6) in Jung's proof use complex-analytic and topological methods (namely, the theory of ramified coverings of analytic varieties).

3.4.4 Zariski's Method

Let k be an algebraically closed field of characteristic zero and X an algebraic surface over k. Zariski's method for desingularizing X goes as follows.

1) Let $\bar{X} \to X$ be the normalization of X. According to Corollary 3.4.4, $Sing\left(\bar{X}\right)$ has codimension 2 in $\bar{X}$, that is, is a finite union of isolated points.
2) Let $X' \to X$ be the blowing up of all the singular points of $\bar{X}$.
3) Replace X by X' and go back to step 1). Keep iterating steps 1) and 2) until the singularities are resolved.

Remark 3.4.6 Zariski's algorithm has the virtue of being extremely easy to state. However, proving that it works is technically quite difficult (an improved version of this result was given later by J. Lipman). An intermediate step in the proof is to show that after finitely many iterations the resulting surface $X^{(i)}$ has only **sandwiched singularities** (see the definition below).

Definition 3.4.7 A surface singularity (X, ξ) is said to be **sandwiched** if a neighbourhood of ξ in X admits a birational map to a non-singular surface.

Being sandwiched is quite a strong restriction; in particular, sandwiched singularities are rational.

3.5 Oscar Zariski

The appearance on the scene of O. Zariski and his school marks a completely new era in the study of resolution of singularities. In the earlier section we mentioned the introduction of normalization which gives a one-step procedure for desingularizing curves in all characteristics, as well as Zariski's proof of resolution for surfaces. In the late nineteen thirties and early forties Zariski proposed a completely new approach to the problem using valuation theory (building on some earlier ideas of

Krull). In a nutshell this approach can be summarized as saying that valuation theory provides a natural notion of "local" in birational geometry and allows to state a local version of the resolution problem called Local Uniformization.

3.5.1 Valuations

For a detailed treatment of the basics of valuation theory, we refer the reader to [174] and [155].

Definition 3.5.1 An ordered group is an abelian group Γ together with a subset $P \subset \Gamma$ (here P stands for "positive elements") which is closed under addition and such that

$$\Gamma = P \coprod \{0\} \coprod (-P).$$

Remark 3.5.2 The above decomposition induces a total ordering on Γ:

$$a < b \iff b - a \in P.$$

Thus an equivalent way to define an ordered group would be "a group with a total ordering which respects addition, that is, $a > 0, b > 0 \implies a + b > 0$".

Note that an ordered group is necessarily torsion-free.

Example 3.5.3 The additive groups $\mathbb{Z}$, $\mathbb{R}$ with the usual ordering are ordered groups. Any subgroup $\Gamma \subset \mathbb{R}$ is an ordered group with the induced ordering (more generally, any subgroup of an ordered group is an ordered group). The group $\mathbb{Z}^n$ with the lexicographical ordering is an ordered group.

All the ordered groups that appear in algebraic geometry are subgroups of groups of the form $\bigoplus_{i=1}^{r} \Gamma_i$, where $\Gamma_i \subset \mathbb{R}$ for all i and the total order is lexicographic.

We are now ready to define valuations. Let K be a field, Γ an ordered group. Let K^* denote the multiplicative group of K.

Definition 3.5.4 A valuation of K with value group Γ is a surjective group homomorphism $\nu : K^* \to \Gamma$ such that for all $x, y \in K^*$

$$\nu(x + y) \geq \min\{\nu(x), \nu(y)\}. \tag{3.22}$$

Remark 3.5.5 Let K be a field, ν a valuation of K and x, y non-zero elements of K such that $\nu(x) \neq \nu(y)$. It is a consequence of Definition 3.5.4 that in this case equality must hold in (3.22), that is,

$$\nu(x + y) = \min\{\nu(x), \nu(y)\}. \tag{3.23}$$

Example 3.5.6 Let X be an irreducible algebraic variety, $K = K(X)$ its field of rational functions, $\xi \in X$ such that $\mathcal{O}_{X,\xi}$ is a regular local ring. Let $\mathfrak{m}_{X,\xi}$ be the maximal ideal of $\mathcal{O}_{X,\xi}$. Define $\nu_\xi : K^* \to \mathbb{Z}$ by

$$\nu_\xi(f) = \mathrm{mult}_\xi f = \max\left\{n \;\middle|\; f \in m^n_{X,\xi}\right\}, \ f \in \mathcal{O}_{X,\xi}.$$

The map ν_ξ extends from $\mathcal{O}_{X,\xi}$ to all of K in the obvious way by additivity:

$$\nu_\xi\left(\frac{f}{g}\right) = \nu_\xi(f) - \nu_\xi(g).$$

The map ν_ξ induces a group homomorphism because $\bigoplus \frac{m^n_{X,\xi}}{m^{n+1}_{X,\xi}}$ is an integral domain.

In the above example, note that ξ could be any *scheme-theoretic* point; for example, it could stand for the generic point of an irreducible codimension 1 subvariety. In that case, the condition that $\mathcal{O}_{X,\xi}$ be non-singular holds automatically whenever X is normal (Theorem 3.4.3).

Remark 3.5.7 Let X be an irreducible algebraic variety, A its coordinate ring,

$$K = K(X)$$

the field of fractions of A, $I \subset A$ an ideal. We can generalize the above example as follows. Define

$$\nu_I(f) = \max\{n \mid f \in I^n\}, \quad \text{for } f \in A.$$

In general, ν_I is a pseudo-valuation, which means that the condition of additivity in the definition of valuation is replaced by the *inequality* $\nu_I(xy) \geq \nu_I(x) + \nu_I(y)$. The map ν_I is a valuation if and only if $\oplus \frac{I^n}{I^{n+1}}$ is an integral domain (a condition which always holds if I is maximal and A_I is regular).

Valuations of the form ν_I are called **divisorial**. The reason for this name is that if A is the coordinate ring of an affine algebraic variety X, even if $\dim A_I > 1$, we can always blow up X along I. Let $\pi : \widetilde{X} \to X$ be the blowing up along I. Then $K(X) = K\left(\widetilde{X}\right)$.

The property that $\oplus \frac{I^n}{I^{n+1}}$ is a domain means that the exceptional divisor

$$\tilde{D} := V(\pi^* I) = \pi^{-1}(V(I))$$

is irreducible. Then $\mathcal{O}_{\widetilde{X},\tilde{D}}$ is a regular local ring of dimension 1 and $\nu_I = \nu_{\tilde{D}}$ measures the order of zero or pole of a rational function at the generic point of $\tilde{D}$. This example illustrates an important philosophical point about valuations: a valuation is an object associated to the field K, that is, to an entire birational equivalence class, not to a particular model in that birational equivalence class. Thus to study a given valuation, one is free to perform blowings up until one arrives at a model which is particularly convenient for understanding this valuation.

Valuation Rings

Let K be a field, Γ an ordered group, $\nu : K^* \twoheadrightarrow \Gamma$ a valuation of K. Associated to ν is a local subring $(R_\nu, \mathfrak{m}_\nu)$ of K, having K as its field of fractions:

$$R_\nu = \{x \in K^* \mid \nu(x) \geq 0\} \cup \{0\}$$
$$\mathfrak{m}_\nu = \{x \in K^* \mid \nu(x) > 0\} \cup \{0\}. \tag{3.24}$$

Example 3.5.8 (Divisorial Valuations) Let X be an irreducible algebraic variety, $D \subset X$ a closed irreducible subvariety, ξ the generic point of D.

Assume that $\mathcal{O}_{X,\xi}$ is a regular local ring of dimension 1. Let t be a generator of $\mathfrak{m}_{X,\xi}$. Then $K = (\mathcal{O}_{X,\xi})_t$. Indeed, any element $f \in \mathcal{O}_{X,\xi}$ can be written as $f = t^n u$, where $n \in \mathbb{N}$ and u is invertible. For each $f = t^n u$ as above, we have $\nu_D(f) = n$. Then $R_\nu = \mathcal{O}_{X,D}$.

Definition 3.5.9 Let $(R_1, \mathfrak{m}_1)$, $(R_2, \mathfrak{m}_2)$ be two local domains with the same field of fractions K. We say that R_2 **birationally dominates** R_1, denoted $R_1 < R_2$, if

$$R_1 \subset R_2 \quad \text{and} \tag{3.25}$$
$$\mathfrak{m}_1 = \mathfrak{m}_2 \cap R_1. \tag{3.26}$$

Remark 3.5.10 One of the main examples of birational domination encountered in algebraic geometry is the following. Let X be an irreducible algebraic variety and $\pi : X' \to X$ a blowing up of X. Let $\xi \in X$, $\xi' \in X'$ be such that $\xi = \pi(\xi')$. Then $\mathcal{O}_{X,\xi} < \mathcal{O}_{X',\xi'}$.

Theorem 3.5.11 *Let* $(R, \mathfrak{m})$ *be a local domain with field of fractions* K. *The following conditions are equivalent:*

(1) $R = R_\nu$ *for some valuation* $\nu : K^* \twoheadrightarrow \Gamma$
(2) for any $x \in K^*$, *either* $x \in R$ *or* $\frac{1}{x} \in R$ *(or both)*
(3) the ideals of R *are totally ordered by inclusion*
(4) $(R, \mathfrak{m})$ *is maximal (among all the local subrings of* K*) with respect to birational domination.*

Remark 3.5.12 Although we omit the proof of Theorem 3.5.11, we note that the proof of the implication (3) $\Longrightarrow$ (1) involves reconstructing the valuation ν (in a unique way, modulo the obvious equivalence relation) from the valuation ring. Hence the valuation ring R_ν determines ν up to equivalence.

For future reference, we define two important numerical characters of valuations: rank and rational rank.

Definition 3.5.13 An subgroup Δ of an ordered group Γ is said to be **isolated** if Δ is a segment with respect to the given ordering: if $a \in \Delta$, $b \in \Gamma$ and $-a \leq b \leq a$ then $b \in \Delta$.

The set of isolated subgroups of an ordered group Γ is totally ordered by inclusion.

Definition 3.5.14 Let ν be a valuation with value group Γ. The **rank** of ν, denoted rk ν, is the number of distinct isolated subgroups of Γ. We have rk $\nu = \dim\ R_\nu$.

Definition 3.5.15 The **rational rank** of ν is, by definition, rat.rk $\nu := \dim_{\mathbb{Q}} \Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$.

Theorem 3.5.11 (in particular, its part (4)) paves the way for a geometric interpretation of valuations. This is due to Zariski in the nineteen forties, when he developed valuation theory with the express purpose of applying it to the problem of resolution of singularities. To explain how valuations provide a natural notion of "local" in birational geometry and to give a precise statement of the Local Uniformization Theorem we need the notion of **center of a valuation** and also that of **local blowing up** with respect to a valuation, which we now define.

Definition 3.5.16 Let $(R, \mathfrak{m}, k)$ be a local domain with field of fractions K and ν a valuation of K. We say that ν is **centered** in R if $R < R_\nu$ (this is equivalent to saying that $\nu(R) \geq 0$ and $\nu(\mathfrak{m}) > 0$).

If X is an irreducible algebraic variety with $K = K(X)$ and ξ a point of X, we say that ν is **centered** in ξ (or that ξ is the center of ν on X) if it is centered in the local ring $\mathcal{O}_{X,\xi}$, that is, if $\mathcal{O}_{X,\xi} < R_\nu$.

The center of a given valuation ν on a variety X is uniquely determined by ν.

Let X be an irreducible algebraic variety, ξ a point of X and $\mathcal{I}$ a coherent ideal sheaf on X. Let $\pi : X_1 \to X$ be the blowing up of X along $\mathcal{I}$. Take a point $\xi_1 \in \pi^{-1}(\xi)$. The map π induces a local homomorphism $\sigma : \mathcal{O}_{X,\xi} \to \mathcal{O}_{X_1,\xi_1}$ of local rings.

Definition 3.5.17 A homomorphism of the form $\sigma : \mathcal{O}_{X,\xi} \to \mathcal{O}_{X_1,\xi_1}$, where ξ_1 is a point of $\pi^{-1}(\xi)$, is called a **local blowing up** of $\mathcal{O}_{X,\xi}$ along $\mathcal{I}_{X,\xi}$. Let ν be a valuation, centered at $\mathcal{O}_{X,\xi}$. We say that σ is a **local blowing up with respect to** ν if ν is centered at $\mathcal{O}_{X_1,\xi_1}$, that is,

$$\nu(\mathcal{O}_{X_1,\xi_1}) \geq 0 \left(\mathcal{O}_{X_1,\xi_1}\right) \geq 0$$

and $\nu\left(\mathfrak{m}_{X_1,\xi_1}\right) > 0$.

Let X be an irreducible algebraic variety, ξ a point of X and ν a valuation of $K = K(X)$, centered at ξ. Let $\pi : X' \to X$ be a birational projective morphism. The following theorem is a version of the **valuative criterion of properness**:

Theorem 3.5.18 *There exists a unique point $\xi' \in \pi^{-1}(\xi)$ such that ν is centered in ξ'.*

The notion of center of a valuation together with Theorem 3.5.18 allows us to divide the problem of resolution of singularities into two parts: local and global. The local version of resolution of singularities is called Local Uniformization.

Let X, K and ν be as above and assume that ν is centered at a point ξ of X.

Definition 3.5.19 A **local uniformization of** X with respect to ν is a birational projective morphism $\pi : X' \to X$ such that the center ξ' of ν in $\pi^{-1}(\xi)$ is a regular point of X'.

Zariski proved in 1940 that if X is an algebraic variety over a field of characteristic zero then X admits a local uniformization with respect to any valuation, centered at a point of X [170]. The same question is still open for fields of characteristic $p > 0$ (the papers [130, 131] and [132] show that to prove Local Uniformization in its full generality, it is sufficient to prove it for valuations of rank 1).

Clearly, a resolution of singularities $\pi : X' \to X$ also constitutes a local uniformization simultaneously with respect to every valuation ν, centered at a point of X. The converse, however, is not so clear: assume that local uniformization is known with respect to every valuation. Does this imply the existence of resolution of singularities of X?

To study this question, Zariski introduced what is known today as the Zariski–Riemann space. Let X be an irreducible algebraic variety. Consider the totality of all the birational projective morphisms $X_\alpha \to X$. This set naturally forms a projective system, whose arrows are birational projective morphisms. Indeed, given two such morphisms

$$X_\alpha \to X \quad \text{and} \tag{3.27}$$

$$X_\beta \to X, \tag{3.28}$$

one can construct a new variety $X_{\alpha\beta}$ together with birational projective morphisms

$$\lambda_\alpha : X_{\alpha\beta} \to X_\alpha, \tag{3.29}$$

$$\lambda_\beta : X_{\alpha\beta} \to X_\beta \tag{3.30}$$

which make the diagram

$$\begin{array}{ccc} X_{\alpha\beta} & \xrightarrow{\lambda_\beta} & X_\beta \\ {\scriptstyle\lambda_\alpha}\downarrow & & \downarrow{\scriptstyle\pi_\beta} \\ X_\alpha & \xrightarrow{\pi_\alpha} & X \end{array} \tag{3.31}$$

commute. The variety $X_{\alpha\beta}$ is the unique irreducible component of the cartesian product $X_\alpha \times_X X_\beta$ which maps dominantly onto X, X_α and X_β. More explicitly, since π_α and π_β are birational, there exist non-empty open subvarieties $U \subset X$, $U_\alpha \subset X_\alpha$ and $U_\beta \subset X_\beta$ such that $\pi_\alpha|_{U_\alpha} : U_\alpha \cong U$ and $\pi_\beta|_{U_\beta} : U_\beta \cong U$. Then

$U \cong U_\alpha \times_U U_\beta$ embeds naturally into $X_\alpha \times_X X_\beta$ as an irreducible open set. The variety $X_{\alpha\beta}$ is nothing but the Zariski closure of U in $X_\alpha \times_X X_\beta$. Geometrically, one should think of $X_{\alpha\beta}$ as the graph of the birational correspondence between X_α and X_β.

Let $S := \varprojlim\limits_{\alpha} X_\alpha$.

Theorem 3.5.20 (Zariski) *There is a natural bijection between S and the set*

$$M := \{valuations\ \nu\ of\ K,\ \ centered\ at\ points\ of\ X\}\,.$$

We briefly sketch the proof.

Proof First, fix a valuation ν of K, centered at a point $\xi \in X$. By Theorem 3.5.18, for each $\pi_\alpha : X_\alpha \to X$ in our projective system, there exists a unique $\xi_\alpha \in \pi^{-1}(\xi)$ such that ν is centered at ξ_α. Therefore we can associate to ν a collection $\{\xi_\alpha \in X_\alpha\}_\alpha$, compatible with the morphisms in our projective system, that is, an element of S. This defines a natural map $f : M \to S$.

Conversely, take an element $\{\xi_\alpha \in X_\alpha\}_\alpha \in S$. The local rings $\mathcal{O}_{X_\alpha,\xi_\alpha}$ form a *direct* system, whose arrows are relations of birational domination. It should therefore not come as a surprise that the direct limit $R := \varinjlim\limits_{\to\alpha} \mathcal{O}_{X_\alpha,\xi_\alpha}$ of this system is a local subring of K, maximal with respect to $<$, that is, a valuation ring. To prove this rigorously, a short argument using the equivalence (1)$\Longleftrightarrow$(2) of Theorem 3.5.11 is required. We omit the details.

This defines the map $g : S \to M$. It is routine to check that the maps f and g are inverse to each other. □

Definition 3.5.21 The set S is called the **Zariski–Riemann space** associated to X.

Zariski's original name for this object (in the special case when X was a *projective* variety over k) was **the abstract Riemann surface of the field** K. The thinking was that in the special case when $k = \mathbb{C}$ and $\dim\ X = 1$, the projective system defining S is finite and its inverse limit is nothing but the resolution of singularities of X, that is, a smooth complex projective curve, or a Riemann surface. However, when $\dim\ X \geq 2$, S does not even have a structure of a variety or a scheme, only one of a ringed space. It resembles more John Nash's space of arcs than it does anything like a Riemann surface. This is why the name "Zariski–Riemann space" seems more appropriate.

In order to address the problem of "glueing" the local uniformizations with respect to various valuations, it is useful to introduce a topology on S. Namely, S is naturally endowed with the inverse limit topology (which is usually referred to as the **Zariski topology** on S). By definition of inverse limit, for each X_α in our projective system we have a natural map $\rho_\alpha : S \to X_\alpha$; this map assigns to each valuation ν centered at a point $\xi \in X$ the center of ν in X_α, lying over ξ. A base for the Zariski topology is given by all the sets of the form $\rho_\alpha^{-1}(U)$ where X_α runs over the entire projective system and U over all the Zariski open sets of X_α. In other words, the Zariski topology is the coarsest topology which makes all the maps ρ_α continuous.

Theorem 3.5.22 (Zariski [173], Chevalley) *The topological space S is compact.*

We spell out the main idea of the proof. By definition, S comes with a natural embedding ι into the direct product $\prod\limits_\alpha X_\alpha$. Each X_α is compact with respect to its Zariski topology, hence so is $\prod\limits_\alpha X_\alpha$ by Tychonoff's theorem. If all the topologies in sight were Hausdorff, ι would be a *closed* embedding, and the compactness of S would follow immediately. Indeed, this is how one proves a standard theorem from general topology: an inverse limit of compact Hausdorff spaces is again compact.

Unfortunately, none of the spaces we are working with here are Hausdorff. The next idea is to replace the Zariski topology on the X_α by a finer, Hausdorff topology, pass to the inverse limit and conclude compactness as above, and then observe that the compactness property is preserved by passing from a finer topology to a coarser one. This is, indeed, what Zariski did in the special case of projective varieties over $\mathbb{C}$. He replaced the Zariski topology by the classical Euclidean topology and the proof was completed as above. Finally, Chevalley came up with a proof, which follows roughly the same plan, but is applicable to varieties over fields of any characteristic and even to arbitrary noetherian schemes.

Once Zariski proved the Local Uniformization Theorem in characteristic zero, his plan went as follows. For each valuation $\nu \in S$, let $\pi : X' \to X$ be the local uniformization with respect to ν and let ξ' be the center of ν on X'. Let U denote the preimage in S of the set $Reg(X')$. By definition, U is an open set, containing ν. Furthermore, for every $\nu' \in U$ the map π constitutes local uniformization also with respect to ν'. Conclusion: once we achieve local uniformization with respect to some $\nu \in S$, we automatically achieve it for all the valuations in some open neighbourhood U of ν. Since this can be done for every $\nu \in S$, we obtain an open covering of S by sets U, for each of which there exists a simultaneous local uniformization of all the elements of U. By compactness, this open covering admits a finite subcovering. Finally, we obtain: there exist finitely many birational projective morphisms $\pi_i : X_i \to X$, $1 \le i \le n$, having the following property. Let $\rho_i : S \to X_i$ denote the natural map, given by the definition of projective limit. Then $\bigcup\limits_{i=1}^{n} \rho^{-1}(Reg(X_i)) = S$.

At this point, the problem of resolution of singularities in characteristic zero was reduced to one of "glueing" the n partial desingularizations X_i together to produce a global resolution of singularities. More precisely by induction on n it is sufficient to prove the following:

There exists an algebraic variety X_{12} together with birational projective morphisms

$$\lambda_1 : X_{12} \to X_1 \tag{3.32}$$

$$\lambda_2 : X_{12} \to X_2, \tag{3.33}$$

having the following properties:

1) the diagram

$$\begin{array}{ccc} X_{12} & \xrightarrow{\lambda_2} & X_2 \\ {\scriptstyle\lambda_1}\downarrow & & \downarrow{\scriptstyle\pi_2} \\ X_1 & \xrightarrow{\pi_1} & X \end{array} \tag{3.34}$$

commutes

2) we have $Reg(X_{12}) \supset \lambda_1^{-1}(Reg(X_1)) \cup \lambda_2^{-1}(Reg(X_2))$.

The glueing problem is highly non-trivial because the local uniformization algorithms used to construct the partial resolutions X_i depend on the respective valuations. A priori absolutely nothing is known about the nature of the birational correspondences among the various X_i.

Zariski was able to solve this problem in dimension 2 by proving his famous factorization theorem: a birational morphism between non-singular surfaces is a composition of point blowings up (see [10] for a much more difficult version of this result in higher dimensions). It is also worth mentioning that Zariksi's factorization theorem together with Castelnuovo's criterion for contractibility of rational curves on non-singular surfaces implies the existence of *minimal resolution* for surfaces, that is, a resolution such that every other resolution of singularities factors through it.

With much greater difficulty Zariski advanced to dimension three [172]. This work of Zariski was recently generalized and systematized by O. Piltant [136]. Thanks to this, we now have a general procedure for glueing local uniformizations in dimension three in a much more general context and for much more general objects than just algebraic varieties or schemes.

3.6 Resolution of Singularities of Algebraic Varieties over a Ground Field of Characteristic Zero

Almost twenty-five years have passed after Zariski's proof of his Local Uniformization Theorem until H. Hironaka proved the existence of resolution of singularities in characteristic zero without using valuations or the Zariski–Riemann space. This (next) revolution in the field of resolution of singularities is the subject of the present section.

Theorem 3.6.1 (H. Hironaka [93]) *Every variety X over a ground field of characteristic zero admits a resolution of singularities.*

Hironaka's original proof of this was over 200 pages long. It is one of the most technically difficult and one of the most often quoted results of the twentieth century mathematics. We give a very brief sketch of the main ideas of the proof, as seen from 55 years into the future.

Proof

Step 1. The definition of **normally flat** (see Definition 3.3.16 and Theorem 3.3.17).

Step 2:

Proposition 3.6.2 *Let X be an algebraic variety and Y a smooth subvariety of X. Assume that X is normally flat along Y. Let $\pi : \widetilde{X} \to X$ be the blowing up of X along Y. Take a point $\tilde{\xi} \in \pi^{-1}(Y)$. Then the Hilbert–Samuel function $H_{\widetilde{X},\tilde{\xi}}$ is smaller than or equal to the common Hilbert–Samuel function $H_{X,\xi}$ of all the points $\xi \in X$. In particular, the blowing up π does not increase the maximal value of the Hilbert–Samuel function $H_{X,\xi}$ of all the points $\xi \in X$.* □

To complete the proof of the Theorem, it is sufficient to construct a sequence of blowings up of X that decreases the Hilbert–Samuel function *strictly*.

Step 3. Reduce the problem to the case when X is an n-dimensional hypersurface embedded into k^{n+1}:

$$X = V(f), \quad \text{where } f \in k[x, y], \quad y \text{ is a single variable and } x = (x_1, \dots, x_n). \tag{3.35}$$

This amounts to choosing a Gröbner basis (or a standard basis in Hironaka's terminology) $(f_1, \dots, f_r)$ of the defining ideal I of X having the following properties.

(a) The maximal locus of the Hilbert–Samuel function of X is equal to the intersection of the loci of maximal multiplicity of the polynomials f_i. In particular, a blowing up center Y is permissible for X if and only if it is simultaneously permissible for each of the hypersurfaces $V(f_i)$. This property holds after any permissible sequence of blowings up under which the maximal value of the Hilbert–Samuel function does not decrease.

(b) Let

$$\pi : \widetilde{X} \to X \tag{3.36}$$

be a permissible sequence of blowings up. The sequence π strictly decreases the maximal value of the Hilbert–Samuel function of X if and only if it strictly decreases the maximal multiplicity of a singularity of at least one of the hypersurfaces $V(f_i)$.

Remark 3.6.3 In 1977 H. Hironaka proved that, regardless of the characteristic of the ground field there exists a basis $(f_1, \dots, f_r)$ of I such that (a) and (b) hold [97].

Step 4. From now on, assume that X is a hypersurface as in (3.35). Let $\mu := \text{mult}_0 f$; assume that μ is the greatest multiplicity of a singular point of X. Using the Henselian Weierstrass Preparation Theorem, further reduce the problem to the case when f has the form $f(x, y) = y^\mu + \sum_{i=1}^{\mu} \phi_i(x) y^{\mu-i}$, where $\text{mult}_0 \phi_i \geq i$. This requires replacing X by a suitable étale covering, but we will not dwell on this point here.

Step 5. Make the **Tschirnhausen transformation**, that is, the change of coordinates $y \to y + \frac{1}{\mu}\phi_1(x)$. This amounts to ensuring that in the new coordinates we have

$$\phi_1(x) = 0. \tag{3.37}$$

We will assume that (3.37) holds from now on. In this situation we say that y is a **maximal contact coordinate for** X.

Step 6. The following Proposition is proved by an easy direct calculation.

Proposition 3.6.4

(1) The maximal contact hyperplane $W := \{y = 0\}$ contains all the points of X of multiplicity μ sufficiently close to the origin. In particular, every permissible center Y is contained in W.

(2) Let (3.36) be a permissible blowing up with center Y. Take a point $\tilde{\xi} \in \pi^{-1}(Y)$ and let $\tilde{f}$ be a local defining equation of $\tilde{X}$ near $\tilde{\xi}$. If $\text{mult}_{\tilde{\xi}} \tilde{f} = \mu$ then $\tilde{\xi}$ lies in the strict transform of W. □

This looks like a good setup for induction on dim X. Indeed, on the one hand, we are only interested in blowing up centers Y that are contained in the hyperplane W. On the other hand, the only points we are interested in studying *after* blowing up belong to the strict transform of W. Thus the next idea is to try to define a variety V *strictly contained* in W and relate the problem of desingularizing V to that of desingularizing our original variety X.

Step 7. In fact, instead of a variety V we need to consider a more general object: a scheme, defined by the *idealistic exponent*, associated to f. Precisely, consider the ideal $H := \left(\phi_i^{\frac{\mu!}{i}}\right)_{2 \leq i \leq \mu} \subset k[x]$. After defining the notion of a permissible blowing up center for $V(H)$ and showing that a center Y is permissible for $V(H)$ if and only if it is permissible for X, one can use the induction assumption to construct a sequence (3.36) of permissible blowings up that **monomializes** the ideal H (by this we mean that $\pi^* H$ is principal and generated locally near every point of $\widetilde{X}$ by a single monomial in suitable coordinates; this should be thought of as an embedded resolution of $V(H)$). This is an important feature of Hironaka's construction: in order to construct a resolution of singularities of n-dimensional varieties, we need embedded resolution in dimension $n-1$. Because of this, both resolution and embedded resolution are proved by two simultaneous inductions: embedded resolution in dimension $n-1 \implies$ resolution in dimension $n \implies$ embedded resolution in dimension n.

Step 8. By Step 7, assume that H is generated by a monomial ω. To monomialize f it remains to construct a sequence of blowings up along permissible coordinate subvarieties (3.36) such that at each point of $\widetilde{X}$ one of the monomials ω and $y^{\mu!}$ divides the other. This is a special case of a purely combinatorial problem that has been variously called Hironaka's game, Perron's algorithm and resolution of (not necessarily normal) toric varieties by permissible blowings up. It is the combinatorial skeleton of resolution of singularities that appears, implicitly or explicitly in every desingularization algorithm that consists of a sequence of blowings up along non-singular subvarieties of the ambient regular variety. We refer the reader to [71, 142, 170] and [125] for various solutions of this problem (see [143] for a counterexample to a harder version of Hironaka's game, needed for resolution in characteristic $p > 0$). □

Remark 3.6.5 The assumption char $k = 0$ is used crucially in Step 5. Naively, one sees that $\frac{1}{\mu}$ makes no sense when char $k = p > 0$ and $p \mid \mu$. More seriously, R. Narashimhan [128] gave the following example showing that in positive characteristic there might not exist a non-singular subvariety satisfying (1) of Proposition 3.6.4, that is, containing all the points of multiplicity μ sufficiently near the origin.

Example 3.6.6 Let k be a perfect field of characteristic 2 and consider the hypersurface X defined by $f(x, y) = y^2 + x_1x_2^3 + x_2x_3^3 + x_3x_1^7 = 0$ in k^4. This threefold has multiplicity 2 at the origin and all of its points are either non-singular or have multiplicity 2, so its multiplicity 2 locus coincides with the singular locus. The singular locus $Sing(X)$ is defined by $\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3} = 0$, that is, $x_2^3 + x_3x_1^6 = x_1x_2^2 + x_3^3 = x_2x_3^2 + x_1^7 = 0$. We find that $Sing(X)$ is the parametrized curve $t \to \left(t^7, t^{19}, t^{15}, t^{32}\right)$ and that this curve has embedding dimension 4 at the origin. Thus it is not contained in any proper non-singular subvariety of k^4 passing through the origin. This shows that in this case there does not exist a non-singular variety W satisfying (1) of Proposition 3.6.4.

Much work has been done since 1964 to simplify and better understand resolution of singularities in characteristic zero. We mention [18, 20, 22–35, 41–43, 65, 76, 77, 90, 112, 150–152, 160, 163, 166, 167].

Many of the later proofs (starting with Bierstone–Milman and Villamayor) have the following transparent structure. One defines a discrete, upper semi-continuous numerical character of singularity $d(\xi)$, consisting of the Hilbert–Samuel function followed by a finite string of non-negative integers. We regard the set of possible values of $d(\xi)$ as being totally ordered by the lexicographical ordering. One stratifies the singular variety X according to $d(\xi)$. By upper semi-continuity, the maximal stratum S_{max} of $d(\xi)$ is a closed subvariety of X. One shows that S_{max} is a normal crossings subvariety and chooses one of its coordinate subvarieties Y in a canonical way (discussed and explained below by example). One lets $\pi : \widetilde{X} \to X$ be the blowing up along Y and one shows that for every $\tilde{\xi}$ not belonging to the strict transforms of components of S_{max} other than Y we have $d\left(\tilde{\xi}\right) < d(\xi)$. Repeating this procedure for the other components of S_{max} we strictly lower the maximum value of $d(\xi)$. This completes the proof by induction on $d(\xi)$.

3.6.1 Functorial Properties of Resolution in Characteristic Zero

The later proofs cited above are functorial with respect to smooth morphisms (flat morphisms with non-singular fibers). This means that they produce a functor from the category of varieties and smooth morphisms to the category of non-singular varieties and smooth morphisms that assigns to each variety X its resolution of singularities $\widetilde{X}$. Being a functor simply means that each smooth morphism of varieties $\phi : X \to V$ lifts (necessarily uniquely) to a smooth morphism $\tilde{\phi} : \widetilde{X} \to \tilde{V}$ of their resolutions. In particular, if ϕ is an open embedding (resp. an isomorphism), so is $\tilde{\phi}$. In this way we obtain that our procedure of resolution of singularities is equivariant with respect to automorphisms of X, any group actions on X, etc.

Choosing a Unique Coordinate Subvariety of S_{max} in a Canonical Way
We illustrate the situation by example.

Example 3.6.7 Consider the surface X defined by the equation $z^2 - x^3y^3 = 0$. Its singular locus coincides with its locus of multiplicity 2 and is the union of the x-axis and the y-axis. These two axes play a symmetric role (in fact, they can be carried into each other by an automorphism X). From a naive point of view, blowing up the origin does not seem to improve the singularity, so one is tempted to blow up one of the one-dimensional components of $Sing(X)$. However, there is no way to do this and respect the functoriality described above. Even if one did not care for functoriality in its own right, a desingularization algorithm that involves an arbitrary choice of a branch would present serious problems: after all, there could be a singularity that locally looks like X but such that the two branches of $Sing(X)$ are in fact two branches of the same irreducible curve.

The modern solution to this problem goes as follows. Start by blowing up the origin since it is the only canonical choice that can be made. The multiplicity 2 locus $Sing\left(\widetilde{X}\right)$ of $\widetilde{X}$ now consists of three lines: the respective strict transforms $\tilde{L}_x$ and $\tilde{L}_y$ of the x- and the y-axes and the exceptional divisor E. At first glance the singular points $E \cap \tilde{L}_x$ and $E \cap \tilde{L}_y$ look worse than the singularity at the origin that we started with, and $Sing\left(\widetilde{X}\right)$ is again a union of two lines near each of those points. However, they have one important new advantage: there is a natural ordering on the set of irreducible components of the equimultiple locus, namely, the order of appearance of those components in the history of the resolution process until this point. This settles the difficult issue of which component should be blown up first.

This points to another important feature of all the known resolution procedures by permissible blowings up: the choice of the blowing up center at each step depends not only on our singular variety itself but also on the history of the resolution process up to the given point.

Two recent preprints, [11] and [122], get around this problem by working in the 2-category of excellent Deligne–Mumford stacks instead of varieties or schemes (stacks are beyond the scope of this survey, but a definition of excellent and quasi-excellent schemes is given in the Appendix).

Finally, we mention a construction of resolution of singularities of analytic varieties due to J.M. Aroca, H. Hironaka and J.L. Vicente Cordoba [14–16] as well as the paper [98] by H. Hironaka.

3.7 Resolution of Singularities of Algebraic Varieties over a Ground Field of Positive Characteristic

3.7.1 Resolution in Dimensions 1, 2 and 3

As mentioned above, resolution of curve singularities in arbitrary characteristic was settled in 1939 when Zariski defined normalization: this one-step procedure works equally well in characteristic zero and characteristic $p > 0$.

The first proof for surfaces is due to S. Abhyankar in 1956 [1] with subsequent strengthenings by H. Hironaka [100] and J. Lipman [117] to the case of more general 2-dimensional schemes, with Lipman giving necessary and sufficient condition for a 2-dimensional scheme to admit a resolution of singularities. See also [64].

The next breakthrough came in 1966, again due to S. Abhyankar, who proved resolution of singularities for threefolds except in characteristics 2, 3 and 5. The idea of Abhyankar's proof is the following. The starting point of the proof is an Auxiliary Theorem which says that any d-dimensional variety over an algebraically closed ground field can be birationally transformed to a variety having no e-fold point for any $e > d!$. The proof of this Auxiliary Theorem generalizes an argument used by Albanese [13] in the surface case combined with the Veronese embedding. Since $3! = 6$, in the special case $d = 3$ we obtain that our variety has singularities of multiplicity at most 6. If $p > 6$ then all the singularities have multiplicity strictly smaller than the characteristic of the ground field (this is precisely the reason for the restriction on the characteristic of the ground field in Abhyankar's proof). Roughly speaking, in this situation one can imitate characteristic zero methods to finish the proof. Still, Abhyankar's proof is extremely technical and difficult and comprises a total of 508 pages [2–6]. For a more recent and more palatable proof we refer the reader to [66].

Resolution of singularities for surfaces was reproved by J. Giraud in 1983 [80], using a novel idea that has proved to be very influential for subsequent work (see also [52, 57–59]). Namely, let k be a perfect field of characteristic $p > 0$ and consider the (typical and significant) special case of a surface in k^3, defined by an equation of the form

$$f(x_1, x_2, y) = y^p - g(x_1, x_2) = 0, \tag{3.38}$$

where g is some polynomial in two variables of multiplicity strictly greater than p. If we wanted to imitate characteristic zero methods, we would naturally study the transformation law for g under blowing up and try to relate the resolution problem for f to the problem of monomialization of g. We already mentioned in the previous section that the main obstruction to imitating characteristic zero proof in the case of characteristic $p > 0$ is the non-existence of maximal contact coordinates in the situation when p divides the multiplicity of a defining equation. A natural idea for a replacement of maximal contact coordinates in the case of Eq. (3.38) is to use a transformation of the form

$$y \to y + \phi(x_1, x_2) \tag{3.39}$$

to make sure that no monomials which are p-th powers appear in g. However, unlike maximal contact coordinates in characteristic zero which are stable under coordinate changes in the x variables and under blowings up that do not lower the Hilbert–Samuel function, the above "maximal contact" coordinates in positive characteristic can be destroyed even by the simplest of linear homogeneous coordinate changes, as the following example shows.

Example 3.7.1 Take $g = x_1 x_2^{2p-1}$ in (3.39). Then g is a single monomial that is not a p-th power. However, after a coordinate change $(x_1, x_2) \to (x_1 + x_2, x_2)$ the new equation involves the monomial x_2^p.

Giraud's idea for dealing with this difficulty was to study the behaviour of the differential dg (instead of that of g itself) under permissible blowings up. The point is that the differential dg is stable under coordinate changes of the form (3.39). The drawback of this approach is that the transformation rules of dg under blowing up are much more complicated than those for g itself. In spite of this, Giraud was able to give a new proof of resolution of surface singularities using this idea.

The method of Giraud was systematically exploited by his Ph.D student V. Cossart to give, in his Ph.D thesis [53], a proof of resolution of singularities of threefolds defined by equations of the form $y^p - f(x_1, x_2, x_3) = 0$, which for a long time had been considered to be the basic and significant special case, exhibiting most of the phenomena and difficulties of the general problem.

The same result was obtained independently and by different methods by T.T. Moh [124]. Both works are of a formidable technical difficulty and comprise hundreds of pages.

It was not until much later that V. Cossart and O. Piltant settled the problem of resolution of threefolds in complete generality (their theorem holds for arbitrary quasi-excellent noetherian schemes of dimension three, including the arithmetic case) in a series of three long papers spanning the years 2008 to 2019 [61–63], building on the earlier works [54–56]. The overall method is based on the idea of Giraud mentioned above. The main point is to prove the Local Uniformization Theorem. After that global resolution of singularities becomes an immediate consequence of Piltant's work [136] that axiomatizes Zariski globalization in three dimensions.

3.7.2 Resolution and Local Uniformization in Dimension Four and Higher

In this subsection we briefly mention and discuss known recent partial results, programs and attempts at proofs in arbitrary dimension.

In the paper [153] Michael Temkin proves a version of the Local Uniformization Theorem in which the required desingularization map $\pi : \widetilde{X} \to X$ is *generically finite* instead of being birational (in other words, it induces a *finite extension*

$$\iota : K(X) \hookrightarrow K\left(\widetilde{X}\right)$$

of function fields instead of an isomorphism). In Temkin's proof the extension ι can be taken to be purely inseparable. Among other things, he gives a rigorous proof of a fact that until then was a mere philosophical belief: to prove Local Uniformization (for varieties over fields of characteristic $p > 0$) it is sufficient to prove it for hypersurfaces defined by equations of the form $y^p + g(x_1, \dots, x_n) = 0$.

A similar, though in some sense complementary result was obtained by H. Knaf and F.-V. Kuhlmann [111]: they also prove Local Uniformization after a finite extension ι of function fields, but in their case the extension ι is Galois (combined with a purely inseparable extension of the residue field of the valuation ring in the case of non-perfect residue fields). In the paper [110] the same authors prove Local Uniformization with respect to Abhyankar valuations. A valuation ν is said to be **Abhyankar** if equality holds in Abhyankar's inequality:

$$\text{rat.rk}\ \nu + \text{tr.deg}(k_\nu/k) = \text{tr.deg}(K/k),$$

where k denotes the ground field, $K = K(X)$ is the field of rational functions of the variety X we want to desingularize and k_ν is the residue field of the valuation ring.

It is well known that to prove the Local Uniformization Theorem it is sufficient to prove it for the case of hypersurfaces (since in the case of general varieties one can handle the defining equations one by one). Let X be a hypersurface in k^n defined by an equation $f(u_1, \dots, u_n) = 0$. We would like to construct a local uniformization with respect to a given valuation ν. Consider the extension

$$\theta : k(u_1, \dots, u_{n-1}) \hookrightarrow \frac{k(u_1, \dots, u_{n-1})[u_n]}{(f)} \tag{3.40}$$

of valued fields.

One way of thinking of the main difficulty of constructing a local uniformization of X with respect to ν is in terms of the **defect** δ of the extension θ (this point of view has been promoted by F.-V. Kuhlmann among others, see [113]). Defining defect is beyond the scope of this survey, but we briefly mention some of its properties relevant to us.

Let

$$p = 1 \qquad \text{if char } k_\nu = 0 \tag{3.41}$$

$$= \text{char } k \qquad \text{if char } k_\nu > 0. \tag{3.42}$$

The defect δ is always a power of p, hence is equal to 1 if char $k = 0$. We have $\delta = 1$ as well in the case of Abhyankar valuations (this explains why the characteristic zero case as well as the case of Abhyankar valuations is easier to handle than the case of arbitrary valuations in characteristic $p > 0$). The philosophy that "all the difficulty of local uniformization lies in the defect" has been understood for some time, but we would like to mention two recent works that make the above statement precise: J.-C. San Saturnino [139, Theorem 6.5] and S. D. Cutkosky–H. Mourtada [67].

We mention two papers by B. Teissier, [148] and [149], that propose another possible approach to constructing Local Uniformization using the graded algebra associated to the given valuation ν and trying to interpret this graded algebra as the coordinate ring of an (infinite-dimensional) toric variety that is a deformation of the variety X we want to desingularize, inspired by the case of plane curve singularities [12, 81].

Finally, for the approach to local uniformization via key polynomials we refer the reader to [7, 8, 68, 69, 91, 92, 118–120, 133, 138, 139, 147, 156–159]. J. Decaup's Ph.D. thesis carries out the program of proving a strengthening of the Local Uniformization Theorem over fields of characteristic zero, but with a view to generalizing the result to fields of positive characteristic.

There has also been recent work whose goal is to construct (or at least make progress toward constructing) global resolution of singularities directly, without going through valuation theory and local uniformization, but the jury is still out on how close to or far from a complete proof we are: [19, 36–39, 44, 89, 101–104, 107, 108].

3.8 An Alternative Approach by J. de Jong et al. via Semi-stable Reduction

In 1996 a major event occurred in the field of resolution of singularities: J. de Jong [70] proved the existence of resolution of singularities for varieties over fields of arbitrary characteristic by alterations:

Definition 3.8.1 An **alteration** is a proper surjective morphism

$$\pi : \widetilde{X} \to X \tag{3.43}$$

such that the induced homomorphism $K(X) \hookrightarrow K\left(\widetilde{X}\right)$ of function fields is finite.

Theorem 3.8.2 *Let X be a variety over a ground field k. There exists an alteration (3.43) such that $\widetilde{X}$ is non-singular. In fact, we can choose $\widetilde{X}$ to be a complement of a normal crossings divisor in some regular projective variety X'.*

We briefly summarize his proof which uses the compactification of moduli stacks of curves of genus g by stable curves (in the special case when X is projective).

Proof

Step 1. Take a sufficiently general projection $\rho : X \to Y$ to a variety Y of dimension dim $X - 1$, so that the fibers of ρ are curves.

Step 2. Normalizing X, we may assume, in addition, that X is normal. After further modifying X by a birational transformation, we may choose the fibration morphism to Y to be generically smooth along any component of any fiber.

Step 3. Choose a sufficiently general and sufficiently ample relative divisor H on X over Y. After taking a base change with an alteration $Y' \to Y$, we may assume that H is a union of sections $\sigma_i : Y \to X$:

$$H = \bigcup_{i=1}^{n} \sigma_i(Y)$$

(this is one of the places in the proof where we actually need to use an alteration rather than a birational map).

Step 4. Since H was chosen sufficiently general and sufficiently ample, for every component of every fiber of ρ there are at least three sections σ_i, intersecting it in distinct points of the smooth locus of ρ. Therefore there exists a Zariski open subset $U \subset Y$ such that for each $\eta \in U$ the fiber $\rho^{-1}(\eta)$, together with the points determined by the σ_i, is a stable n-pointed curve of certain genus g. By definition of the moduli stack $\overline{\mathcal{M}}_{g,n}$ of stable curves of genus g with n marked points, we obtain a unique morphism $U \to \overline{\mathcal{M}}_{g,n}$ such that the family

$$\rho\Big|_{\rho^{-1}(U)} : \rho^{-1}(U) \to U \tag{3.44}$$

is the pullback of the universal family of stable n-pointed curves of genus g over $\overline{\mathcal{M}}_{g,n}$. Now, $\overline{\mathcal{M}}_{g,n}$ admits a finite étale covering $\overline{M} \to \overline{\mathcal{M}}_{g,n}$ by a projective scheme $\overline{M}$; the universal family of stable n-pointed curves of genus g can be lifted to $\overline{M}$. Putting $U' := U \times_{\overline{\mathcal{M}}_{g,n}} \overline{M}$, we obtain a cartesian diagram

$$\begin{array}{ccc} U' & \xrightarrow{\theta} & \overline{M} \\ \downarrow \lambda & & \downarrow \\ U & \longrightarrow & \overline{\mathcal{M}}_{g,n} \end{array} \tag{3.45}$$

where λ is an alteration and the pullback of the family of curves (3.44) under λ coincides with the pullback of the universal family by θ (this is the second place in the proof where we genuinely need to use alterations rather than birational morphisms).

Step 5. Let X_U denote the preimage of U in X. Let Y' be the closure of

$$Im(U' \to Y \times \bar{M}) \subset Y \times \bar{M}.$$

Then Y' is a projective variety over k and $Y' \to Y$ is an alteration which is generically étale. The smooth stable n-pointed curve $(X_U, \sigma_1|_U, \dots, \sigma_n|_U) \times U'$ extends to a stable n-pointed curve X' over Y'.

Step 6. Replacing Y by Y' and X by X' we reduce the problem to the case in which there exist a stable n-pointed curve $(C, \tau_1, \dots, \tau_n)$ over Y, a nonempty open subvariety $U \subset Y$ and an isomorphism $\beta : C_U \to X_U$ mapping the section $\tau_i|_U$ to the section $\sigma_i|_U$ (where C_U denotes the preimage of U in C). It can be proved that the rational map β can me be made into a morphism, possibly after base change by a birational projective transformation of Y.

To Summarize the Result of Steps 4–6 We started out with a morphism ρ whose *generic* fiber was a stable n-pointed curve of genus g. We ended up with a morphism ψ, *all* of whose fibers are stable n-pointed curves. In other words, we have reduced the problem to the case where all the fibers of ρ are stable pointed curves (and the generic fiber is non-singular).

Step 7. By induction on dim X, resolve the singularities of Y. Furthermore, by the induction hypothesis in the non-projective case we may assume that the non-smooth locus of the morphism ρ is a normal crossings divisor (note that we are using the induction hypothesis in the non-projective case even to prove the result for projective X).

Step 8. At this point the only singularities of C are given by equations of the form

$$xy = t_1^{n_1} \dots t_d^{n_d}.$$

These are resolved explicitly by hand. □

Now assume that char $k = 0$. Shortly after the appearance of de Jong's theorem on alterations D. Abramovich and J. de Jong [9] took it as a starting point to give a new proof of resolution of singularities by birational morphisms in characteristic zero.

Their proof goes as follows. Fix an alteration $X' \to X$ such that X' is non-singular. We may assume that the corresponding finite extension $K(X) \hookrightarrow K(X')$ of function fields is Galois. Let G denote the Galois group $Gal(K(X')/K(X))$. Then G acts on X' and the quotient of this action birationally dominates X. By induction on dim X we may assume that the subvariety $\left\{\xi \in X' \;\middle|\; g(\xi) = \xi \text{ for some } g \in G\right\}$ of points of X' fixed by at least one $g \in G$ is a normal crossings divisor. A few auxiliary blowups make the quotient X'/G toroidal. Finally, the authors apply the well known result on resolution of toroidal singularities [109, Theorem 11*] to finish the argument.

Another proof of resolution of singularities in characteristic zero based on the same idea but quite different in detail from the Abramovich–de Jong one was given independently by F. Bogomolov and T. Pantev [40].

3.9 Resolving Singularities in Characteristic Zero by Nash and Higher Nash Blowing Up: Results and Conjectures

The goal of this section is define Nash and higher Nash blowing up and to give an overview of both known results and conjectures involving their desingularization properties.

H. Hironaka's proof that every algebraic variety over a field of characteristic zero admits a resolution of singularities provided an inspiration to John Nash for several extremely fruitful ideas, one of the most important being the introduction of Nash blowing up as a conjectural method for constructing a canonical resolution of singularities of varieties in characteristic zero.

Let k be a field and X an affine irreducible algebraic variety of dimension n embedded in k^N.

Definition 3.9.1 The **Gauss map** $\phi : X \setminus Sing(X) \to G := Grass(N, n)$ is the map that sends every non-singular point $\xi \in X$ to its tangent space, viewed as a point of G.

Definition 3.9.2 The **Nash blowing up** NX of X is the closure $\overline{graph(\phi)}$ of $graph(\phi)$ in $X \times G$.

We have a canonical map $\mu : NX \longrightarrow X$ induced by the canonical projection of $X \times G$ onto the first factor. Over $X \setminus Sing(X)$ the variety $\mu^{-1}(X \setminus Sing(X))$ is the graph of the Gauss map, hence isomorphic to $X \setminus Sing(X)$. Thus μ is birational. Since G is a projective variety, the morphism μ is projective.

If X is a complete intersection defined by equations

$$f_1(x_1, \ldots, x_N) = \cdots = f_\ell(x_1, \ldots, x_N) = 0$$

then μ coincides with the blowing up of the Jacobian ideal, that is, the ideal generated by all the $(\ell \times \ell)$-minors of the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}\right)_{\substack{1\le i\le \ell\\ 1\le j\le N}}$. Even if X is not a complete intersection, there is a similar description of Nash blowing up in terms of the Jacobian matrix, though it took mankind much longer to come up with it. Namely, let $r = N - n = codim(X, k^N)$. Let M be a submatrix of the Jacobian matrix formed by r rows that are linearly independent as K-vectors (where, as usual, $K = K(X)$ denotes the field of rational functions of X). Then μ coincides with the blowing up of the ideal generated by all the $(r \times r)$-minors of the matrix M.

The above constructions seem, *a priori*, to depend on the chosen embedding $\iota : X \hookrightarrow \mathbb{C}^N$. We now give two other characterization of Nash blowing up, both of them independent of ι.

This construction of an ideal whose blowing up coincides with the Nash blowing up is a special case of a more general construction of the **determinant** of a module (in this case, the module of Kahler differentials of X) due to Rossi in the analytic case and to O. Villamayor [161] in the general setting. Namely, let R be a domain, K its field of fractions and M and R-module. Let $r := \dim_K(M \otimes_R K)$ denote the generic rank of M.

Definition 3.9.3 The **determinant** of M is $Im\left(\bigwedge^r M \longrightarrow \bigwedge^r M \otimes_R K \cong K\right)$.

We think of the determinant as a fractional ideal, that is, an R-submodule of K. Clearing denominators, we can construct a non-canonical isomorphism of R-modules between a fractional ideal and an honest ideal of R. To obtain an ideal whose blowing up coincides with Nash blowing up, we take the determinant of the module $\Omega^1_{X/\mathbb{C}}$ of Kahler differentials whose generic rank is n.

Finally, Nash blowing up can be characterized by a universal mapping property. Namely, we have the following

Proposition 3.9.4 *Let $\mu : X' \to X$ be the Nash blowing up of X. The following statements hold.*

(1) The $O_{X'}$-module $\frac{\mu^*\Omega^n_{X/\mathbb{C}}}{torsion}$ is locally principal (that is, generated by a single element).

(2) the Nash blowing up μ has the universal mapping property with respect to (1). This means, by definition, that every birational morphism $\lambda : V \to X$ such that $\frac{\lambda^*\Omega^n_{X/\mathbb{C}}}{torsion}$ is locally principal factors through X' in a unique way.

With a view of constructing a resolution of singularities of X, consider the sequence

$$X \overset{\mu_1}{\longleftarrow} X_1 \overset{\mu_2}{\longleftarrow} \dots \overset{\mu_i}{\longleftarrow} X_i \overset{\mu_{i+1}}{\longleftarrow} \dots \tag{3.46}$$

where each μ_i is either a Nash blowing up or a normalized Nash blowing up (that is, a Nash blowing up followed by normalization). The question posed to Hironaka by Nash was: does X_i become non-singular for $i \gg 0$?

An affirmative answer to this question would provide a very simple and natural algorithm for resolving singularities over fields of characteristic zero.

Unfortunately, very little is known about Nash's question, despite considerable effort by many mathematicians. Let us briefly summarize the existing results.

In order to have any hope for the answer to be affirmative, we must at least ensure that no singular variety remains unchanged after Nash blowing up. This is the content of Nobile's Theorem:

Theorem 3.9.5 (Nobile [129]) *The Nash blowing up $\mu : X' \to X$ is an isomorphism if and only if X is non-singular.*

The "if" part of the Theorem is trivial, so its main content is "only if".

Corollary 3.9.6 *If* $\dim X = 1$ *iterating Nash blowing up produces a resolution of singularities.*

Proof Let $\tilde{X} \to X$ be the resolution of singularities of X. As we saw earlier, $\tilde{X}$ is nothing but the normalization of X. In particular, $\mathcal{O}_{\tilde{X}}$ is a finite (hence a noetherian) $\mathcal{O}_X$-module. Now, the sequence of morphisms (3.46) induces a sequence

$$\mathcal{O}_X \xrightarrow{\mu_1^*} \mathcal{O}_{X_1} \xrightarrow{\mu_2^*} \dots \xrightarrow{\mu_i^*} \mathcal{O}_{X_i} \xrightarrow{\mu_{i+1}^*} \dots \tag{3.47}$$

of homomorphisms of rings, with all the $\mathcal{O}_{X_i}$ contained in $\mathcal{O}_{\tilde{X}}$. Since $\mathcal{O}_{\tilde{X}}$ is a noetherian $\mathcal{O}_X$-module, the sequence (3.47) must stabilize after $\mathcal{O}_{X_i}$ for some $i \in \mathbb{N}$. By Nobile's theorem, X_i is non-singular. □

Remark 3.9.7 Assume that char $k = p > 0$, fix a prime number $q \neq 2$ and consider the plane curve $X = \{f(x, y) = y^p + x^q = 0\}$. This is a complete intersection variety whose Jacobian ideal J is principal (since $\frac{\partial f}{\partial y} = 0$). Hence the Nash blowing up $\mu : X' \to X$ is an isomorphism. Thus Nobile's theorem does not hold over fields of positive characteristic. There seems to be little hope to devise a plausible approach to resolution over fields of characteristic $p > 0$ along the lines of Nash blowing up.

Theorem 3.9.8 (Rebassoo [137]) *Iterating Nash blowings up gives resolution of singularities of any surface X defined in $\mathbb{C}^3$ by an equation of the form*

$$z^a - x^b y^c = 0. \tag{3.48}$$

The proof is quite long and technical. One of the difficulties is that after Nash blowing up X stops being a hypersurface, though, as we will see below, it remains a toric variety.

Theorem 3.9.9 (Hironaka [99]) *Starting with a surface X, consider a sequence (3.46) of morphisms such that each μ_i dominates the Nash blowing up of X_{i-1} (that is, μ_i is a composition of Nash blowing up with another birational projective morphism). There exists $i \in \mathbb{N}$ such that the normalization $\bar{X}_i$ of X_i*

dominates a non-singular surface (in other words, $\bar{X}_i$ *has at most sandwiched singularities).*

Using this result as a starting point, M. Spivakovsky proved in 1985 that iterating *normalized* Nash blowings up resolves the singularities of any surface over a field of characteristic zero:

Theorem 3.9.10 ([146]) *Assume that* $\dim X = 2$ *and each* μ_i *in (3.46) is a normalized Nash blowing up. Then* X_i *is non-singular for* $i \gg 0$.

By Hironaka's result, it is enough to prove this Theorem in the case when X has at most sandwiched singularities. Again, the proof is long and technical. The first step is a classification of sandwiched surface singularities, accomplished in [146], building on a classification of valuations in function fields of surfaces [145].

Another important ingredient in the proof is a geometric characterization of Nash blowing up in terms of **polar curves**, inspired by [83, 84].

3.9.1 Nash Blowing Up and the Base Locus of the Polar Curve

Consider a variety X of dimension n embedded in $\mathbb{C}^N$.

Definition 3.9.11 (Lê–Teissier) The **first polar variety** of X is the closure of the critical locus of a generic projection $X \to \mathbb{C}^n$, restricted to $X \setminus Sing(X)$. If X is a surface, the first polar variety is referred to as the **polar curve** of X; it is the critical locus of a generic projection $X \to \mathbb{C}^2$.

One should think of the polar curve as a linear system: as we vary the generic projection, we obtain a family of polar curves, all of them linearly equivalent to each other. In this way, we may talk about the **base locus** of the polar curve. Another way of thinking of polar curves is as zeroes of sections of the sheaf $\Omega^2_{X/\mathbb{C}}$ of Kahler differentials. This is why making this sheaf (modulo torsion) locally principal is equivalent to removing the base locus of the strict transform of the polar curve.

Proposition 3.9.12 ([146]) *Let* X *be a variety of dimension* n.

(1) Consider a birational transformation $\mu : X' \to X$*, dominating the Nash blowing up of* X*. The linear system formed by the strict transforms of the first polar variety has no base points (we say that Nash blowing up resolves the base points of the first polar variety).*
(2) Conversely, assume that μ *resolves the base points of the first polar variety and that* X' *is normal. Then* X' *dominates the Nash blowing up of* X.

This leads to the following method of computing the normalized Nash blowing up of any given normal surface singularity (this method is essentially due to G. Gonzalez-Springberg [83, 84]). Consider the commutative diagram

$$\begin{array}{ccc} Y' & \xrightarrow{\sigma} & Y \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{\mu} & X \end{array} \tag{3.49}$$

where π and π' are the respective minimal resolutions of singularities of X and X', μ is the normalized Nash blowing up and σ the factorization of $\mu \circ \pi'$ through Y given by definition of the minimal resolution Y.

By Zariski's factorization theorem, σ is a sequence of blowings up of points. Now, μ resolves the base points of the polar curve, hence so does $\mu \circ \pi'$. Since, by Theorem 3.9.12, μ is the "smallest" birational transformation with this property, σ is the smallest sequence of point blowings up that resolves the base points of the strict transform of the polar curve of X in Y. The method for studying the desingularization properties of Nash blowing up, inspired by [83–85], consists of computing directly the strict transform of the polar curve in Y, particularly, its base points, and thus deducing information about σ and Y'.

Once we classify sandwiched singularities, we consider a subclass of them called **minimal singularities** (rational singularities of surfaces with reduced fundamental cycle; this includes all the toric surface singularities). In the case of minimal singularities the polar curve, and thus σ and Y', can be computed explicitly. We show that the number of irreducible exceptional curves of π' is at most one half of the number of irreducible exceptional curves of π. Thus, if we let E be the number of irreducible exceptional curves in the minimal resolution of the surface X, the singularities of X are resolved after at most $\log_2 E$ normalized Nash blowings up.

In the case of non-minimal sandwiched singularities our results are much less explicit, but we are able to get enough information about the polar curve to give an indirect proof that if X has at most sandwiched singularities then after finitely many normalized Nash blowings up the resulting surface X_i has at most minimal singularities. This completes the proof.

3.9.2 Nash Blowing Up of Toric Varieties

Recently, there has been a resurgence of interest in resolution of singularities by iterating Nash blowing up, particularly, in the case of (not necessarily normal) toric varieties. We summarize some of the main advances here.

Let n be a strictly positive integer. Consider a semigroup $\Phi \subset \mathbb{Z}^n$ having the following properties:

(1) Φ generates $\mathbb{Z}^n$ as an additive group
(2) the cone C generated by Φ in $\mathbb{R}^n \supset \mathbb{Z}^n$ is strictly convex (this means that C contains no straight lines). Let $\gamma_1, \dots, \gamma_s$ be a set of generators of Φ (not necessarily minimal).

Definition 3.9.13 The **affine toric variety** X determined by Φ is the image of the map

$$\mathbb{C}^n \to \mathbb{C}^s$$

defined by $t \to (t^{\gamma_1}, \dots, t^{\gamma_s})$ (here we are using the multi-index notation: $t = (t_1, \dots, t_n)$, each γ_i is an n-vector and $t^{\gamma_i} = \prod_{j=1}^{n} t_j^{\gamma_{ij}}$).

As everything else related to toric varieties, the Nash blowing up of such a variety can be described combinatorially. More precisely, we can compute the logarithmic Jacobian ideal explicitly in terms of the elements $\gamma_1, \dots, \gamma_s$. This task was accomplished, independently, in [82] and [86] (the latter paper includes the case of reducible toric varieties). Namely, the module $\Omega^n_{X,\mathbb{C}}$ is generated by elements of the form $dt^{\gamma_{i_1}} \bigwedge \cdots \bigwedge dt^{\gamma_{i_n}}$, where $(i_1, \dots, i_n)$ runs over all the n-tuples of distinct elements of $\{1, \dots, s\}$. We have

$$dt^{\gamma_{i_1}} \bigwedge \cdots \bigwedge dt^{\gamma_{i_n}} = \det\left(\gamma_{i_1}, \dots \gamma_{i_n}\right) t^{\sum\limits_{j=1}^{n} \gamma_{i_j} - n} dt_1 \bigwedge \cdots \bigwedge dt_n.$$

Thus the logarithmic Jacobian ideal we must blow up to compute the Nash blowing up is the ideal generated by all the monomials $t^{\sum\limits_{j=1}^{n} \gamma_{i_j} - n}$ as $(i_1, \dots, i_n)$ runs over all the n-tuples of distinct elements of $\{1, \dots, s\}$ satisfying

$$\det\left(\gamma_{i_1}, \dots \gamma_{i_n}\right) \neq 0. \tag{3.50}$$

Picking one of these monomials specifies a coordinate chart of the Nash blowing up. For example, assume that $\det(\gamma_1, \dots, \gamma_n) \neq 0$ and consider the coordinate chart determined by the monomial $t^{\sum\limits_{j=1}^{n} \gamma_j - n}$. The semigroup Φ_1 that determines the corresponding affine toric variety is generated by $\gamma_1, \dots, \gamma_s$ and all the vectors of the form

$$\sum_{j=1}^{n} \gamma_{i_j} - \sum_{j=1}^{n} \gamma_j, \tag{3.51}$$

where $(i_1, \ldots, i_n)$ runs over all the n-tuples of distinct elements of $\{1, \ldots, s\}$ satisfying (3.50). Now, an important special case to be considered is one when there exists $j \in \{1, \ldots, n\}$ such that $i_{j'} = j'$ for all $j' \in \{1, \ldots, n\} \setminus \{j\}$ and $i_j \neq j$. Then the condition (3.50) amounts to saying that

$$\det\left(\gamma_1, \ldots, \gamma_{j-1}, \gamma_{i_j}, \gamma_{j+1}, \ldots, \gamma_n\right) \neq 0. \tag{3.52}$$

One can show that after a permutation of the n-tuple $(i_1, \ldots, i_n)$ we can achieve the situation where condition (3.52) holds for all $j \in \{1, \ldots, n\}$ simultaneously. This shows that Φ_1 is generated by $\gamma_1, \ldots, \gamma_s$ and all the differences of the form

$$\gamma_i - \gamma_j,\ j \in \{1, \ldots, n\},\ i \in \{n+1, \ldots, s\} \quad \text{such that } \det\left(\gamma_1, \ldots, \gamma_{j-1}, \gamma_i, \gamma_{j+1}, \ldots, \gamma_n\right) \neq 0 \tag{3.53}$$

A complete list of coordinate charts on the Nash blowing up of the toric variety X is obtained in this way, after imposing the additional condition that the resulting semigroup determines a strictly convex cone.

One way of thinking of the choice of an affine coordinate chart on the Nash blowing up is in terms of valuations. We saw earlier that by a theorem of Zariski fixing a valuation ν of the rational function field $K(X)$ of X is equivalent to specifying a (scheme-theoretic) point called the center of ν on every blowing up of X. Here we are interested in a less precise version of this statement: specifying the values $\nu(t_1),\ldots,\nu(t_n)$ of the torus variables $t_1,\ldots,t_n$ limits the choice of a coordinate chart to those charts that contain the center of ν. Namely, a coordinate chart as above contains the center of ν if and only if for every pair i, j as in (3.53) we have

$$\nu\left(t^{\gamma_i}\right) \geq \nu\left(t^{\gamma_j}\right). \tag{3.54}$$

In general, even under this restriction the choice of the coordinate chart is not uniquely determined, unless the inequality in (3.54) is strict for all the choices of i, j as in (3.53). This last statement holds whenever the values $\nu(t_1),\ldots,\nu(t_n)$ are $\mathbb{Q}$-linearly independent.

In [72] and [73] it is shown that if $\dim X = 2$ and the rank of the group generated by $\nu(t_1)$ and $\nu(t_2)$ coincides with its rational rank then iterating Nash blowing up resolves the singularities of X in all the coordinate charts compatible with the valuation ν. In [82] the same result is proved for X of arbitrary dimension.

The simplest case of a group whose rank differs from its rational rank is that of rank 1 and rational rank 2. Thus the simplest case in which resolution of singularities of toric varieties by iterating Nash blowing up is not known is the following.

An Open Problem

Let $\Phi = (\gamma_1, \ldots, \gamma_s) \subset \mathbb{Z}^2$ be a semigroup which generates $\mathbb{Z}^2$ as a group, such that the cone generated by it is strictly convex. Let α be an irrational number. Let $L : \mathbb{Z}^2 \to \mathbb{R}$ be the map given by $L(x, y) = x + \alpha y$. Assume that $L(\Phi \setminus \{0\}) > 0$

and that $L(\gamma_1) < L(\gamma_2) < L(\gamma_j)$ for $j > 2$. Let Φ_1 be the semigroup generated by γ_1, γ_2 and all the differences of the form $\gamma_i - \gamma_1$ and $\gamma_j - \gamma_2$ where $\det(\gamma_i, \gamma_2) \neq 0$ and $\det(\gamma_j, \gamma_1) \neq 0$. Replace Φ by Φ_1 (as we explained above, this corresponds to performing a Nash blowing up of our toric surface and picking the unique coordinate chart prescribed by the valuation such that $\nu(t_1) = 1$ and $\nu(t_2) = \alpha$). Question: is it true that after finitely many iterations of this procedure the resulting semigroup Φ_i is generated by two elements?

There is overwhelming computer evidence that the answer to this question is affirmative. Rebassoo's theorem is a special case of this, providing further evidence. On this subject we also mention the paper [17].

3.9.3 Higher Nash Blowing Up

Let $X \subset \mathbb{C}^N$ be an irreducible affine algebraic variety of dimension n and R its coordinate ring. Consider the map $\lambda : R \otimes_k R \to R$ which sends $a \otimes b$ to ab. Let $I = Ker(\lambda)$. We view I as an R-module via the map $R \to R \otimes_k R$, $r \to r \otimes 1$.

For $i \in \mathbb{N}$, $i \geq 2$, the higher Nash blowing up $N_i X$ of X was defined by Oneto and Zatini [134] in terms of the Grassmanian of the i-jet module $\left(\frac{I}{I^{i+1}}\right)^*$ and by Takehiko Yasuda [164] using Hilbert schemes of points of length $\binom{n+i}{n}$, with an alternative, explicit characterization by E. Chavez, D. Duarte and A. Giles in terms of the generalized Jacobian matrix [50]. We summarize the first two constructions here.

For a point $x \in X$. Let $(R_x, \mathfrak{m}_x)$ be the localization of R at the point x and I_x the localization of I. Consider the following $\mathbb{C} = \frac{R_x}{\mathfrak{m}_x}$-vector space:

$$T_x^i X := \left(\frac{I_x}{I_x^{i+1}} \otimes_R \mathbb{C}\right)^*$$

This is a vector space of dimension $L = \binom{i+n}{n} - 1$ whenever x is a non-singular point. Since $X \subset \mathbb{C}^N$, we have $T_x^i X \subset T_x^i \mathbb{C}^N = \mathbb{C}^M$ where $M = \binom{N+i}{N} - 1$, that is, we may view $T_x^i X$ as an element of the Grassmanian $G(M, L)$. Consider the Gauss map:

$$G_i : X \setminus Sing(X) \to G(M, L) \tag{3.55}$$

$$x \to T_x^i X. \tag{3.56}$$

Denote by X_i the Zariski closure of the graph of G_i. Call μ_i the restriction to X_i of the projection of $X \times G(M, L)$ to X.

Definition 3.9.14 ([134, Definition 1.1]) The pair (X_i, μ_i) is called the **Nash blowing up** of X **relative to** $\frac{I}{I^{i+1}}$.

Similarly to the usual Nash blowing up, the Nash blowing up relative to $\frac{I}{I^{i+1}}$ coincides with the blowing up of the determinant of the module $\frac{I}{I^{i+1}}$ [134].

Next, we summarize Yasuda's construction. Consider a $\mathbb{C}$-rational point $x \in X$ and let $\mathfrak{m}$ be the corresponding maximal ideal of R. Let $n = \dim X$. Let $x(i) := Spec\ \frac{R}{\mathfrak{m}^{i+1}}$ be the i-th infinitesimal neighborhood of x. If X is smooth at x, then $x(i)$ is a closed subscheme of X of length $L + 1 = \binom{i+n}{n}$ (that is, $\frac{R}{\mathfrak{m}^{i+1}}$ has length $L + 1$ as an R-module). Therefore, it corresponds to a point $[x(i)] \in Hilb_{L+1}(X)$, where $Hilb_{L+1}(X)$ is the Hilbert scheme of $(L+1)$-points of X (see [127, Definition 1.2]). If $Reg(X)$ denotes the smooth locus of X, we have a map

$$\delta_i : Reg(X) \to Hilb_{L+1}(X) \tag{3.57}$$

$$x \to [x(i)] \tag{3.58}$$

Definition 3.9.15 ([164, Definition 1.2]) The higher Nash blowup of X of order i, denoted by $N_i X$, is the closure of the graph of δ_n in $X \times_k Hilb_{L+1}(X)$ with reduced scheme structure. By restricting the projection $X \times_k Hilb_{L+1}(X) \to X$ to $N_i X$ we obtain a map $\pi_n : N_i X \to X$.

This map is projective, birational, and is an isomorphism over $Reg(X)$.

Proposition 3.9.16 ([164, Proposition 1.8]) *For every variety X and every strictly positive integer i, we have a canonical isomorphism* $(N_i(X), \pi_n) \cong (X_i, \mu_i)$. *In particular,* $N_1(X)$ *is canonically isomorphic to the classical Nash blowup of X.*

Yasuda conjectured that for i large enough, the i-th Nash blowup of X is non-singular [164, Conjecture 0.2]. If the conjecture were true, this construction would give a one-step resolution of singularities. In the same paper, the author proves that the conjecture is true for curves (here we give the statement only for irreducible varieties whereas Yasuda's result is stated and is proved for varieties that may be reducible.):

Theorem 3.9.17 ([164, Corollary 3.7]) *Let X be an irreducible variety of dimension 1. For i large enough the variety* $N_i X$ *is non-singular.*

The proof of this is not trivial and consists of two parts. First, the author shows that for $i \gg 0$ the transformation N_i separates the (analytic) branches of X, that is, X becomes analytically irreducible at every point. Yasuda goes on to show that each branch gets desingularized by N_i for $i \gg 0$. Precisely, he shows the following. Assume that X is analytically irreducible at a certain point ξ. The resolution of singularities of X gives an injection of $\mathcal{O}_{X,\xi}$ into a regular local ring and thus induces a discrete rank 1 valuation ν on $\mathcal{O}_{X,\xi}$. Consider the semigroup $\Phi := \nu(\mathcal{O}_{X,\xi} \setminus \{0\}) \subset \mathbb{N}$ and let $0 = s_0, s_1, s_2, s_3, \ldots$ be the complete list of elements of Φ arranged in an increasing order.

Theorem 3.9.18 ([164, Theorem 3.3]) *For an integer* $i \in \mathbb{N}$ *the curve* $N_i X$ *is non-singular if and only if* $s_{i+1} - 1 \in \Phi$.

Since Φ coincides with $\mathbb{N}$ for $i \gg 0$, Theorem 3.9.18 immediately implies Theorem 3.9.17 in the case of analytically irreducible curves.

Yasuda has stated that the A_3 singularity (that is, the singularity defined by the equation $z^4 - xy = 0$) is probably a counterexample to his conjecture (see [166, Remark 1.5]). Recently Rin Toyama [154] has shown that this is, indeed, the case, building on earlier work by D. Duarte.

Incredibly, the analogue of Nobile's theorem (that is, the statement that a higher Nash blowing up of X is an isomorphism if and only if X is non-singular) is not known for higher Nash blowing up. The best partial results on this subject are due to D. Duarte, who proved it for normal toric varieties [74] and for normal hypersurfaces [75]. It has recently been proved for toric curves [50].

Finally, we mention another conjecture of T. Yasuda about higher Nash blowing up of (analytically) irreducible curves. Let X be an analytically irreducible curve, Φ its associated semigroup and the s_i elements of Φ listed in increasing order, as above.

Conjecture 3.9.19 (Yasuda [165]) Let Φ_i denote the semigroup associated to the analytically irreducible curve $N_i X$. We have $\Phi_i = \{s_\ell - s_j \mid \ell > i, j \leq i\}$.

The paper [50] contains the following results:

(1) a definition of the higher-order Jacobian matrix J of an affine algebraic variety, so that the i-th higher Nash blowing up coincides with the blowing up of an ideal generated by suitable minors of J in a way completely analogous to that of usual Nash blowing up described above
(2) a proof that the higher Nash blowings up of a toric variety are themselves toric varieties
(3) a proof of Conjecture 3.9.19 in the case of toric curves
(4) as an immediate corollary of (3), a proof of the analogue of Nobile's theorem for toric curves
(5) a family of counterexamples to Conjecture 3.9.19 in the general case (namely, the parametrized curves $t \to \left(t^4, t^{4i+2} + t^{4i+3}\right)$ giving a counterexample for each positive integer i).

3.10 Reduction of Singularities of Vector Fields, Foliations by Lines and Codimension One Foliations

Let K be the field of rational functions of a projective algebraic variety M_0 of dimension n over an algebraically closed field k of characteristic zero.

Consider the n-dimensional K-vector space $Der_k K$ of k-derivations from K to itself.

Definition 3.10.1 A **foliation by lines** is a 1-dimensional K-vector subspace

$$\mathcal{L} \subset Der_k K.$$

Take a regular point P on a projective model M of the field K. We know that

$$Der_k \mathcal{O}_{M,P} \subset Der_k K$$

is a free $\mathcal{O}_{M,P}$-module of rank n generated by the partial derivatives $\frac{\partial}{\partial z_i}$, $i \in \{1, 2, \ldots, n\}$, for a regular system of parameters $z_1, z_2, \ldots, z_n$ of the local ring $\mathcal{O}_{M,P}$.

Definition 3.10.2 The free rank one submodule $\mathcal{L}_{M,P} := \mathcal{L} \cap Der_k \mathcal{O}_{M,P}$ of $Der_k \mathcal{O}_{M,P}$ is called the **local foliation induced by** $\mathcal{L}$ at M, P.

Let $\mathfrak{m}_{M,P}$ denote the maximal ideal of $\mathcal{O}_{M,P}$.

Definition 3.10.3 A germ of a vector field $\xi \in Der_k \mathcal{O}_{M,P}$ is said to be **non-singular** if $\xi \notin \mathfrak{m}_{M,P} Der_k \mathcal{O}_{M,P}$. The germ ξ is **elementary** if it is singular and the k-linear endomorphism

$$\xi : \frac{\mathfrak{m}_{M,P}}{\mathfrak{m}^2_{M,P}} \to \frac{\mathfrak{m}_{M,P}}{\mathfrak{m}^2_{M,P}} \tag{3.59}$$

is not nilpotent.

We say that $\mathcal{L}$ is non-singular (resp. elementary) at P if there is a germ $\xi \in \mathcal{L}_{M,P}$ that is non-singular (resp. elementary). If $Y \subset M$ is an irreducible subvariety, we say that $\mathcal{L}$ is non-singular (resp. elementary) at Y if it is so at a generic point of Y. Note that this definition makes sense only if M itself is non-singular at the generic point of Y.

A plane vector field $D = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$, with a, b two relatively prime polynomials in x and y, defines a one-dimensional saturated foliation $\mathcal{F}$ having singularities at the zeroes of D (that is, the common zeroes of a and b). It was proved by Seidenberg in 1968 [141] that after a finite number of point blowings up of the ambient plane we obtain a foliation $\tilde{F}$ which is given locally at each singular point by a vector field $\tilde{D}$ whose linear part has eigenvalues 1 and λ, with $\lambda \notin \mathbb{Q}_+$ (= strictly positive rational numbers); see also [45]. The above singularities may be thought of as final forms in the sense that they are preserved under all subsequent point blowings up. Note also that these singularities are characterized by the fact that they are elementary in the sense of Definition 3.10.3 and remain elementary after the subsequent blowings up. On the other hand, if the eigenvalues are $1, \lambda \in \mathbb{Q}_+$, the linear part of the vector field (cf. (3.59)) will become nilpotent after finitely many blowings up.

This points to a new feature of the desingularization problem for vector fields and foliations: in general, it is not possible to make them non-singular by blowings up, so one must start by defining the final form of the singularity that one is trying to achieve. This is why in this subject we usually talk about *reduction* rather than resolution of singularities. A counterexample by F. Sanz and F. Sancho shows that starting with dimension three it is not possible to arrive at elementary singularities by a sequence of blowings up along non-singular centers (see the Introduction to [135]). Therefore a new notion of a final form of singularities is needed. In higher dimensions a useful and natural notion seems to be that of log-elementary singularities, motivated by the results of [46].

Let the notation be as in the beginning of this section.

Definition 3.10.4 A germ of a vector field $\xi \in Der_k \mathcal{O}_{M,P}$ is said to be **log-elementary** if there is a regular system of parameters $z_1, z_2, \dots, z_n$ of $\mathcal{O}_{M,P}$, and an integer e, $0 \le e \le n$ such that ξ has the form $\xi = \sum_{i=1}^{e} a_i z_i \frac{\partial}{\partial z_i} + \sum_{i=e+1}^{n} a_i \frac{\partial}{\partial z_i}$, where $a_i \in \mathcal{O}_{M,P}$ for $i \in \{1, 2, \dots, n\}$ and $a_j \notin \mathfrak{m}_{M,P}^2$ for at least one index j. We say that $\mathcal{L}$ is log-elementary at P if there is a germ $\xi \in \mathcal{L}_{M,P}$ that is log-elementary. If $Y \subset M$ is an irreducible subvariety, we say that $\mathcal{L}$ is log-elementary at Y if it is so at a generic point of Y.

The following result is the main theorem of [49]:

Theorem 3.10.5 *Assume that $n = 3$. Consider a foliation by lines $\mathcal{L} \subset Der_k K$. There is a birational projective morphism $M \to M_0$ such that $\mathcal{L}$ is log-elementary at all the points of M.*

The general structure of the proof is à la Zariski. First, a local uniformization along any valuation ν of K vanishing on k is established: a sequence of blowings up $M \to M_0$ along non-singular centers is constructed such that $\mathcal{L}$ is log-elementary at the center Y of ν on M. After that Theorem 3.10.5 is deduced from the Piltant–Zariski general globalization procedure in dimension three [136]: one just has to check that Piltant's axioms I–VI hold in this special case. The proof of local uniformization of three-dimensional vector fields is inspired by [46] and [47].

We mention, without giving the details, the following related results on reduction of singularities of foliations and vector fields.

(1) The paper [48] constructs a reduction of singularities of codimension 1 foliations in ambient dimension 3.
(2) The paper [135] accomplishes reduction of singularities of real-analytic vector fields; the real setting is used in an essential way in the proof.
(3) The paper [123] proves reduction of singularities of foliations by curves in ambient dimension 3 to canonical ones (the condition of being canonical is somewhat stronger than being log-elementary), but in the 2-category of Deligne–Mumford stacks.
(4) The papers [78, 79] prove the Local Uniformization theorem for codimension one foliations in all dimensions, under two restrictions on the given valuation ν: $rk\ \nu = 1$ and $k_\nu = \mathbb{C}$.

3.11 Appendix

It is natural to pose the problem of resolution of singularities in the more general context of noetherian schemes.

Definition 3.11.1 Let X be a reduced noetherian scheme. A **resolution of singularities** of X is a blowing up $X' \to X$ along a subscheme of X, not containing any irreducible components of X, such that X' is non-singular.

In this Appendix we address the question of the hypotheses that must be imposed on X in order for resolution of singularities to exist. Let $Reg(X)$ denote the set of regular points of X. It is obvious that the following condition is necessary for the existence of a resolution of singularities of X:

(1) $Reg(X)$ must contain a non-empty Zariski open set.
Furthermore, suppose X admits a resolution of singularities $\pi : X' \to X$ and let

$$\bar{\pi} : \bar{X} \to X$$

denote the normalization of X. Then π must factor through $\bar{X}$. We have $\bar{X} = Spec\ \pi_* \mathcal{O}_{X'}$ and $\pi_* \mathcal{O}_{X'}$ is a coherent sheaf of $\mathcal{O}_X$-modules. This gives another necessary condition for the existence of resolution:

(2) $\bar{X}$ must be finite over X.

Moreover, since the usual methods involve blowing up and induction on dim X, we are led to assume that (1) and (2) hold for every reduced scheme of finite type over X. By Nagata's criterion, (1) then implies that X is a J-2 scheme, that is, for every scheme $\tilde{X}$, reduced and of finite type over X, $Reg(\tilde{X})$ is open.

Grothendieck [87, IV.7.9] proved that if all of the irreducible closed subschemes of X and all of their finite purely inseparable covers admit resolution of singularities, then X must satisfy a somewhat stronger condition than (1)$\wedge$(2) above, called **quasi-excellence**, which we now define. For a point ξ on a scheme we will denote by $\kappa(\xi)$ the residue field of the local ring of that point.

Definition 3.11.2 ([121, Chapter 13, (33.A), p. 249]) Let $\sigma : X \to Y$ be a morphism of noetherian schemes. We say that σ is **regular** if it is flat, and for every $\xi \in Y$ the fiber $X \times_Y Spec\ \kappa(\xi)$ is geometrically regular over $\kappa(\xi)$ (this means that for every finite field extension $\kappa(\xi) \to k'$, the scheme $X \times_Y Spec\ k'$ is regular).

Remark 3.11.3 If $\kappa(\xi)$ is perfect, the fiber $X \times_Y Spec\ \kappa(\xi)$ is geometrically regular over $\kappa(\xi)$ if and only if it is regular.

Remark 3.11.4 It is known that a morphism of finite type is regular in the above sense if and only if it is smooth (that is, flat with smooth fibers).

3.11.1 Quasi-excellent Schemes

Regular morphisms come up in a natural way when one wishes to pass to the formal completion of a local ring at a singularity:

Definition 3.11.5 ([121, (33.A) and (34.A)]) Let R be a noetherian ring. For a maximal ideal $\mathfrak{m}$ of R, let $\hat{R}_{\mathfrak{m}}$ denote the $\mathfrak{m}$-adic completion of R. We say that R is a **G-ring** if for every maximal ideal $\mathfrak{m}$ of R, the natural map $Spec\ \hat{R}_{\mathfrak{m}} \to Spec\ R$ is a regular morphism.

Definition 3.11.6 ([121, (34.A), p. 259]) Let X be a noetherian scheme. We say that X is **quasi-excellent** if the following two conditions hold:

(1) X is J-2, that is, for every scheme $\tilde{X}$, reduced and of finite type over X, $Reg(\tilde{X})$ is open in the Zariski topology.
(2) For every closed point $\xi \in X$, $\mathcal{O}_{X,\xi}$ is a G-ring.

Remark 3.11.7 If $X = Spec\ R$ with R a **local** noetherian ring then (2) $\Longrightarrow$ (1) in the above definition [121].

A scheme is said to be **excellent** if it is quasi-excellent and universally catenary. In general, rings that arise from natural constructions in algebra and geometry are excellent. Complete and complex-analytic local rings are excellent (see [121, Theorem 30.D] for a proof that every complete local ring is excellent and [121, (33.H), Theorem 78, p. 257] for a proof of finiteness of normalization for quasi-excellent schemes). Both excellence and quasi-excellence are preserved by localization and passing to schemes of finite type over X [121, Chapter 13, (33.G), Theorem 77, p. 254]. In particular, every scheme that is essentially of finite type over a field, $\mathbb{Z}$, $\mathbb{Z}_{(p)}$, $\mathbb{Z}_p$, the Witt vectors or any other excellent Dedekind domain, or over a complete or complex-analytic local ring is excellent. See [126, Appendix A.1, p. 203], for some examples of non-excellent rings.

If X is a quasi-excellent scheme then for every $\xi \in X$ the natural map

$$Spec\ \hat{\mathcal{O}}_{X,\xi} \to X$$

is a regular homomorphism (by Definition 3.11.6 (2)). Thus, the passage to the formal completion is a natural operation in the category of quasi-excellent schemes; in particular, it does not change the nature of singularity.

Once local uniformization is proved in a given context, in order to globalize it and to make it canonical (that is, functorial in the category whose objects are quasi-excellent noetherian schemes and whose morphisms are *regular* morphisms), one is interested in local uniformization algorithms determined, locally at every point ξ, by the formal completion $\hat{\mathcal{O}}_{X,\xi}$ of $\mathcal{O}_{X,\xi}$.

Grothendieck's result means that the largest *subcategory* of the category of noetherian schemes, closed under passing to closed subschemes and finite purely inseparable covers, for which resolution of singularities could possibly exist, is that of *quasi-excellent* schemes. In [87, IV.7.9], Grothendieck conjectures that resolution of singularities exists in this most general possible context.

We take this opportunity to mention a recent paper [105] by L. Illusie, Y. Laszlo and F. Orgogozo, based on the ideas of Ofer Gabber.

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