Chapter 2
The Topology of Surface Singularities

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Contents

Abstract We consider a reduced complex surface germ (X, p). We do not assume that X is normal at p, and so, the singular locus (Σ, p) of (X, p) could be one dimensional. This text is devoted to the description of the topology of (X, p). By the conic structure theorem (see Milnor, *Singular Points of Complex Hypersurfaces*, Annals of Mathematical Studies 61 (1968), Princeton Univ. Press), (X, p) is homeomorphic to the cone on its link L_X. First of all, for any good resolution $\rho : (Y, E_Y) \to (X, 0)$ of (X, p), there exists a factorization through the normalization $\nu : (\bar{X}, \bar{p}) \to (X, 0)$ (see H. Laufer, *Normal two dimensional singularities*, Ann.

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of Math. Studies **71**, (1971), Princeton Univ. Press., Thm. 3.14). This is why we proceed in two steps.

1. When (X, p) a normal germ of surface, p is an isolated singular point and the link L_X of (X, p) is a well defined differentiable three-manifold. Using the good minimal resolution of (X, p), L_X is given as the boundary of a well defined plumbing (see Sect. 2.2) which has a negative definite intersection form (see Hirzebruch et al., *Differentiable manifolds and quadratic forms*, Math. Lecture Notes, vol 4 (1972), Dekker, New-York and Neumann, *A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves*, Trans. Amer. Math. Soc. **268** (1981), p. 299–344).
2. In Sect. 2.3, we use a suitably general morphism, $\pi : (X, p) \to (\mathbb{C}^2, 0)$, to describe the topology of a surface germ (X, p) which has a 1-dimensional singular locus (Σ, p). We give a detailed description of the quotient morphism induced by the normalization ν on the link $L_{\bar{X}}$ of $(\bar{X}, \bar{p})$ (see also Sect. 2.2 in Luengo-Pichon, *Lê 's conjecture for cyclic covers*, Séminaires et congrès 10, (2005), p. 163–190. Publications de la SMF, Ed. J.-P. Brasselet and T. Suwa).

In Sect. 2.4, we give a detailed proof of the existence of a good resolution of a normal surface germ by the Hirzebruch-Jung method (Theorem 2.4.6). With this method a good resolution is obtained via an embedded resolution of the discriminant of π (see Friedrich Hirzebruch, *Über vierdimensionale Riemannsche Flächen mehrdeutiger analytischer Funktionen von zwei komplexen Veränderlichen*, Math. Ann. 126 (1953) p. 1–22). An example is given Sect. 2.6. An appendix (Sect. 2.5) is devoted to the topological study of lens spaces and to the description of the minimal resolution of quasi-ordinary singularities of surfaces. Section 2.5 provides the necessary background material to make the proof of Theorem 2.4.6 self-contained.

2.1 Introduction

Let I be a reduced ideal in $\mathbb{C}\{z_1, \dots, z_n\}$ such that the quotient algebra $A_X = \mathbb{C}\{z_1, \dots, z_n\}/I$ is two-dimensional. The zero locus, at the origin 0 of $\mathbb{C}^n$, of a set of generators of I is an analytic surface germ embedded in $(\mathbb{C}^n, 0)$. Let $(X, 0)$ be its intersection with the compact ball B_ϵ^{2n} of radius a sufficiently small ϵ, centered at the origin in $\mathbb{C}^n$, and L_X its intersection with the boundary S_ϵ^{2n-1} of B_ϵ^{2n}. Let Σ be the set of the singular points of $(X, 0)$.

As I is reduced Σ is empty when $(X, 0)$ is smooth, it is equal to the origin when 0 is an isolated singular point, it is a curve when the germ has a non-isolated singular locus (in particular we do not exclude the cases of reducible germs).

If Σ is a curve, $K_\Sigma = \Sigma \cap S_\epsilon^{2n-1}$ is the disjoint union of r one-dimensional circles (r being the number of irreducible components of Σ) embedded in L_X. We say that K_Σ is the link of Σ. By the conic structure theorem (see [18]), for a sufficiently small ϵ, $(X, \Sigma, 0)$ is homeomorphic to the cone on the pair (L_X, K_Σ) and to the cone on L_X when $\Sigma = \{0\}$.

On the other hand, thanks to A. Durfee [7], the homeomorphism class of $(X, \Sigma, 0)$ depends only on the isomorphism class of the algebra A_X (i.e. is independent of a sufficiently small ϵ and of the choice of the embedding in $(\mathbb{C}^n, 0)$). We say that the analytic type of $(X, 0)$ is given by the isomorphism class of A_X and, we say that its topological type is given by the homeomorphism class of the pair $(X, 0)$ if 0 is an isolated singular point, and by the homeomorphism class of the triple $(X, \Sigma, 0)$ if the singular locus Σ is a curve.

Definition 2.1.1 **The link of** $(X, 0)$ is the homeomorphism class of L_X if 0 is an isolated singular point (in particular if $(X, 0)$ is normal at 0), and is the homeomorphic class of the pair (L_X, K_Σ) if the singular locus Σ is a curve.

This paper is devoted to the description of the link of $(X, 0)$.

2.1.1 Good Resolutions

Definition 2.1.2 A morphism $\rho : (Y, E_Y) \to (X, 0)$ where $E_Y = \rho^{-1}(0)$ is the exceptional divisor of ρ, is a **good resolution** of $(X, 0)$ if :

1. Y is a smooth complex surface,
2. the total transform $\rho^{-1}(\Sigma) =: E_Y^+$ is a normal crossing divisor with smooth irreducible components.
3. the restriction of ρ to $Y \setminus E_Y^+$ is an isomorphism.

Definition 2.1.3 Let $\rho : (Y, E_Y) \longrightarrow (X, 0)$ be a good resolution of $(X, 0)$.

The **dual graph associated to** ρ, denoted $G(Y)$, is constructed as follows. The vertices of $G(Y)$ represent the irreducible components of E_Y. When two irreducible components of E_Y intersect, we join their associated vertices by edges whose number is equal to the number of intersection points. A dual graph is a **bamboo** if the graph is homeomorphic to a segment and each vertex represents a rational curve.

If E_i is an irreducible component of E_Y, let us denote by e_i the self-intersection number of E_i in Y and by g_i its genus. To obtain the **weighted dual graph associated to** ρ, denoted $G_w(Y)$, we weight $G(Y)$ as follows. A vertex associated to the irreducible E_i of E_Y is weighted by (e_i) when $g_i = 0$ and by (e_i, g_i) when $g_i > 0$.

For example if $X = \{(x, y, z) \in \mathbb{C}^3, z^m = x^k y^l\}$, where m, k and l are integers greater than two and pairwise relatively prime, Fig. 2.1 describes the shape of the dual graph of the minimal good resolution of $(X, 0)$.

Remark 2.1.4 If $(X, 0)$ is reducible, let $(\cup_{1 \leq i \leq r} X_i, 0)$ be its decomposition as a union of irreducible surface germs. Let $\nu_i : (\bar{X}_i, p_i) \to (X_i, 0)$ be the normalization of the irreducible components of $(X, 0)$. The morphisms ν_i induce the normalization morphism on the disjoint union $\coprod_{1 \leq i \leq r} (\bar{X}_i, p_i)$.

Fig. 2.1 $G_w(Y)$ when $X = \{(x, y, z) \in \mathbb{C}^3, z^m = x^k y^l\}$. Here $G(Y)$ is a bamboo. The arrows represent the strict transform of $\{xy = 0\}$. In particular if $m = 12$, $k = 5$ and $l = 11$ the graph has three vertices with $e_1 = -3$, $e_2 = -2$, $e_3 = -3$ (see [16, p. 759])

Remark 2.1.5 First of all, for any good resolution $\rho : (Y, E_Y) \to (X, 0)$ there exists a factorization through the normalization $\nu : (\bar{X}, \bar{p}) \to (X, 0)$ (see [11, Thm. 3.14]). In Sect. 2.3, we describe the topology of normalization morphisms. After that it will be sufficient to describe the topology of the links of normal surface germs.

A good resolution is minimal if its exceptional divisor doesn't contain any irreducible component of genus zero, self-intersection -1 and which meets only one or two other irreducible components. Let $\rho : (Y, E_Y) \to (X, 0)$ be a good resolution and $\rho' : (Y', E_{Y'}) \to (X, 0)$ be a good minimal resolution of $(X, 0)$. Then there exists a morphism $\beta : (Y, E_Y) \to (Y', E_{Y'})$ which is a sequence of blowing-downs of irreducible components of genus zero and self-intersection -1 (see [11, Thm 5.9] or [1, p. 86]). It implies the unicity, up to isomorphism, of the minimal good resolution of $(X, 0)$.

As there exists a factorization of ρ' through ν, $(Y', E_{Y'})$ is also the minimal good resolution of $(\bar{X}, \bar{p})$. Let $\bar{\rho} : (Y', E_{Y'}) \to (\bar{X}, \bar{p})$ be the minimal good resolution of $(\bar{X}, \bar{p})$ defined on $(Y', E_{Y'})$. What we said just above implies that $\rho = \nu \circ \bar{\rho} \circ \beta$, i.e. ρ is the composition of the following three morphisms:

$$(Y, E_Y) \xrightarrow{\beta} (Y', E_{Y'}) \xrightarrow{\bar{\rho}} (\bar{X}, \bar{p}) \xrightarrow{\nu} (X, 0)$$

2.1.2 Link of a Complex Surface Germ

In Sect. 2.2, we describe the topology of a plumbing and the topology of its boundary. We explain how the existence of a good resolution describes the link of a normal complex surface germ as the boundary of a plumbing of disc bundles on oriented smooth compact real surfaces with empty boundary. The boundary of a plumbing is, by definition, a plumbed 3-manifold [10, 20] or equivalently a graph manifold in the sense of Waldhausen [23]. The plumbing given by the minimal good resolution of $(X, 0)$ has a normal form in the sense of Neumann [20] and represents its boundary in a unique way.

It implies that the link of a normal complex surface germ $(X, 0)$ determines the weighted dual graph of its good minimal resolution. In particular, if the link is S^3, then the good minimal resolution of $(X, 0)$ is an isomorphism and $(X, 0)$ is smooth at the origin. This is the famous result obtained in 1961 by Mumford [19]. When the singular locus of $(X, 0)$ is an irreducible germ of curve, its link can be S^3. Lê's conjecture, which is still open (see [14] and [2] for partial results), states that it can only happen for an equisingular family of irreducible curves.

In Sect. 2.3, we use a suitably general projection $\pi : (X, 0) \to (\mathbb{C}^2, 0)$ (as told in Sect. 2.3.1) to describe the topology of the restriction $\nu_L : L_{\bar{X}} \to L_X$ of the normalization ν on the link $L_{\bar{X}}$. We will show that ν_L is a homeomorphism if and only if a general hyperplane section of $(X, 0)$ is locally irreducible at z for all points $z \in (\Sigma \setminus \{0\})$. Otherwise, as stated without a proof in Luengo-Pichon [14], ν_L is the composition of two kind of topological quotients: curlings and identifications. Here, we give detailed proofs. Some years ago, John Milnor asked me for a description of the link of a surface germ with non-isolated singular locus. I hope that Sect. 2.3 gives a satisfactory answer.

In Sect. 2.4 we suppose that $(X, 0)$ is **normal**. We use a finite morphism $\pi : (X, 0) \to (\mathbb{C}^2, 0)$ and its discriminant Δ, to obtain a good resolution $\rho : (Y, E_Y) \to (X, 0)$ of $(X, 0)$. We follow Hirzebruch's method (see [9], see also Brieskorn [5] for a presentation of Hirzebruch's work). The scheme to obtain ρ is as in [15], but our redaction here is quite different. In [15], the purpose is to study the behaviour of invariants associated to finite morphisms defined on $(X, 0)$. Here, we explain in detail the topology of each steps of the construction to specify the behaviour of ρ. Hirzebruch's method uses the properties of the topology of the normalization, presented in Sect. 2.3, and the resolution of the quasi-ordinary singularities of surfaces already studied by Jung. This is why one says that this resolution ρ is the Hirzebruch-Jung resolution associated to π. Then L_X is homeomorphic to the boundary of a regular neighborhood of the exceptional divisor E_Y of $\rho : (Y, E_Y) \to (X, 0)$ which is a plumbing as defined in Sect. 2.2.

Section 2.5 is an appendix which can be read independently of the other sections. We suppose again that $(X, 0)$ is **normal**. We give topological proofs of basic results, already used in Sect. 2.4 on finite morphism $\phi : (X, 0) \to (\mathbb{C}^2, 0)$, in the following two cases:

1. The discriminant of ϕ is a smooth germ of curve. Then, in Lemma 2.5.6, we show that $(X, 0)$ is analytically isomorphic to $(\mathbb{C}^2, 0)$ and that ϕ is analytically isomorphic to the map from $(\mathbb{C}^2, 0)$ to $(\mathbb{C}^2, 0)$ defined by $(x, y) \mapsto (x, y^n)$.
2. The discriminant of ϕ is a normal crossing. By definition $(X, 0)$ is then a quasi-ordinary singularity and its link is a lens space. We prove that the minimal resolution of $(X, 0)$ is a bamboo of rational curves (Proposition 2.5.7).

Section 2.6 is an example of Hirzebruch-Jung's resolution.

2.1.3 Conventions

The boundary of a topological manifold W will be denoted by $b(W)$.

A **disc** (resp. an **open disc**) will always be an oriented topological manifold orientation preserving homeomorphic to $\{z \in \mathbb{C}, |z| \leq 1\}$ (resp. to $\{z \in \mathbb{C}, |z| < 1\}$).

A **circle** will always be an oriented topological manifold orientation preserving homeomorphic to $S = \{z \in \mathbb{C}, |z| = 1\}$. Moreover, for $0 < \alpha$, we use the following notation: $D_\alpha = \{z \in \mathbb{C}, |z| \leq \alpha\}$, and $S_\alpha = b(D_\alpha)$.

2.2 The Topology of Plumbings

In this Section $(X, 0)$ is a **normal** complex surface germ.

The name "plumbing" was introduced by David Mumford in [19]. There, he showed that the topology of a resolution of a normal singularity of a complex surface can be described as a "plumbing".

In [9], Hirzebruch constructed good resolutions of normal singularities. Let $\rho : (Y, E_Y) \to (X, 0)$ be a good resolution of the normal germ of surface $(X, 0)$. Each irreducible component E_i of the exceptional divisor is equipped with its normal complex fiber bundle. With their complex structure the fibers have dimension 1. So, a regular compact tubular neighbourhood $N(E_i)$ of E_i in Y, is a disc bundle. As E_i is a smooth compact complex curve, E_i is an oriented differential compact surface with an empty boundary. Then, the isomorphism class, as differential bundle, of the disc bundle $N(E_i)$ is determined by the genus g_i of E_i and its self-intersection number e_i in Y. The complex structure gives an orientation on Y and on E_i, these orientations induce an orientation on $N(E_i)$ and on the fibers of the disc bundle over E_i.

Remark 2.2.1 By definition $(X, 0)$ is a sufficiently small compact representative of the given normal surface germ. Let k be the number of irreducible components of E_Y, $M(Y) = \cup_{1 \leq i \leq k} N(E_i)$ is a compact neighborhood of E_Y. There exists a retraction by deformation $R : Y \to M(Y)$ which induces a homeomorphism from the boundary of Y, $b(Y) = \rho^{-1}(L_X)$, to the boundary $b(M(Y))$. So, the boundary of $M(Y)$ is the link of $(X, 0)$.

Definition 2.2.2 Let $N(E_i), i = 1, 2$, be two oriented disc bundles on oriented smooth compact differentiable surfaces, with empty boundary, $E_i, i = 1, 2$, and let $p_i \in E_i$. **The plumbing of** $N(E_1)$ **and** $N(E_2)$ **at** p_1 **and** p_2 is equal to the quotient of the disjoint union of $N(E_1)$ and $N(E_2)$ by the following equivalence relation. Let D_i be a small disc neighbourhood of p_i in E_i, and $D_i \times \Delta_i$ be a trivialization of $N(E_i)$ over D_i, $i = 1, 2$. Let $f : D_1 \to \Delta_2$ and $g : \Delta_1 \to D_2$ be two orientation preserving diffeomorphisms such that $f(p_1) = 0$ and $g(0) = p_2$.

For all $(v_1, u_1) \in D_1 \times \Delta_1$, the equivalence relation is $(v_1, u_1) \sim (g(u_1), f(v_1))$.

Remark 2.2.3 The diffeomorphism class of the plumbing of $N(E_1)$ and $N(E_2)$ at (p_1, p_2) does not depend upon the choices of the trivializations nor on the choices of f and g. Moreover, in the plumbing of $N(E_1)$ and $N(E_2)$ at p_1 and p_2:

1. The image of E_1 intersects the image of E_2 at the point p_{12} which is the class, in the quotient, of $(p_1 \times 0) \sim (p_2 \times 0)$.

2. The plumbing is a gluing of $N(E_1)$ and $N(E_2)$ around the chosen neighbourhoods of $(p_1 \times 0)$ and $(p_2 \times 0)$.
3. In the plumbing, $D_1 \times 0 \subset E_1$ is identified, via f, with the fiber $0 \times \Delta_2$ of the disc bundle $N(E_2)$ and the fiber $0 \times \Delta_1$ of $N(E_1)$ is identified, via g, with $D_2 \times 0 \subset E_2$.

Definition 2.2.4 More generally we can perform the plumbing of a family $N(E_i), i = 1, \ldots, n$, of oriented disc bundles on oriented smooth compact differentiable surfaces E_i with empty boundary, at a finite number of pairs of points $(p_i, p_j) \in E_i \times E_j$. Let g_i be the genus of E_i and e_i be the self-intersection number of E_i in $N(E_i)$. The vertices of the **weighted plumbing graph** associated to such a plumbing represent the basis $E_i, i = 1, \ldots, n$, of the bundles. These vertices are weighted by e_i when $g_i = 0$, and by (e_i, g_i) when $0 < g_i$. Each edge which relates (i) to (j), represents an intersection point between the image of E_i and E_j in the plumbing.

In the boundary of the plumbing of the family $N(E_i), i = 1, \ldots, n$, the intersections $b(N(E_i)) \cap b(N(E_j))$ are a union of disjoint tori which is the **family of plumbing tori** of the plumbing.

We can perform a plumbing between $N(E_i)$ and $N(E_j)$ at several pairs of points of $E_i \times E_j$ if and only if every two such pairs of points (p_i, p_j) and (p'_i, p'_j) are such that $p_i \neq p'_i$ and $p_j \neq p'_j$. Let $k_{ij} \geq 0$ be the number of these pairs of points. Obviously, k_{ij} is the number of disjoint tori which form the intersection $b(N(E_i)) \cap b(N(E_j)$ and also the number of edges which relate the vertices associated to E_i and E_j in the plumbing graph associated to the plumbing.

An oriented disc bundle $N(E)$ on a differential compact surface E of genus g and empty boundary is determined as differentiable bundle by g and by the self-intersection number of E in $N(E)$. If two plumbings have the same weighted plumbing graph, there exists a diffeomorphism between the two plumbings such that its restriction on the corresponding disc bundles is an isomorphism of differentiable disc bundles.

Proposition 2.2.5 *Let $\rho : (Y, E_Y) \to (X, 0)$ be a good resolution of the normal germ of surface $(X, 0)$. Then a regular neighbourhood, in Y, of the exceptional divisor E_Y, is diffeomorphic to a plumbing of the disc bundles $N(E_i)$. The plumbings are performed around the double points $p_{ij} = E_i \cap E_j$.The associated weighted plumbing graph coincides with the weighted dual graph $G_w(Y)$ of ρ. To each point $p_{ij} \in (E_i \cap E_j)$ we associate a torus $T(p_{ij}) \subset (b(N(E_i)) \cap b(N(E_j)))$.*

Proof We choose trivializations of the disc bundles $N(E_i)$ and $N(E_j)$ in a small closed neighborhood V of p_{ij}. First, we center the trivializations at $(0, 0) = p_{ij}$ and we parametrize V as disc a bundle

1. over E_i by $V_i = \{(v_i, u_i) \in D_i \times \Delta_i\}$, where $D_i \times 0$ is a disc neighborhood of $(0, 0) = p_{ij}$ in E_i and $v_i \times \Delta_i$ is the normal disc fiber at $v_i \in D_i$.
2. over E_j by $V_j = \{(v_j, u_j) \in D_j \times \Delta_j\}$, where $D_j \times 0$ is a disc neighborhood of $(0, 0) = p_{ij}$ in E_j and $v_j \times \Delta_j$ is the normal disc fiber at $v_j \in D_j$.

As E_Y is a normal crossing divisor, we can parametrize V in such a way that $E_Y \cap V = \{uv = 0\}$ where $v = v_i = u_j$ and $u = v_j = u_i$. These equalities provide the plumbing of $N(E_i)$ and $N(E_j)$ around p_{ij}. By construction , the associated weighted plumbing graph is equal to $G_w(Y)$. □

Definition 2.2.6 The union of disc bundles $M(Y) = \cup_{1 \le i \le k} N(E_i)$ is **the plumbing** associated to $\rho : (Y, E_Y) \to (X, 0)$.

With the above notation, in a neighborhood of p_{ij}, there is a unique connected component of the intersection $(b(N(E_i)) \cap b(N(E_j)))$ which is parametrized by the torus $b(D_i) \times b(\Delta_i)$ which is glued point by point with $b(D_j) \times b(\Delta_j)$.

Definition 2.2.7 The image of $(b(D_i) \times b(\Delta_i)) \sim (b(D_j) \times b(\Delta_j))$ in the boundary of $M(Y)$ is the **plumbing torus** $T(p_{ij})$ **associated to** p_{ij}.

2.3 The Topology of the Normalization

In this Section $(X, 0)$ is the intersection of a reduced complex surface germ, which can have a 1-dimensional singular locus, with the compact ball B^{2n}_ϵ of radius a small ϵ (i.e. where ϵ is as in Milnor's Theorem 2.10 of [18]), centered at the origin in $\mathbb{C}^n$. As in the Introduction (Sect. 2.1), L_X is the intersection of X with the boundary S^{2n-1}_ϵ of B^{2n}_ϵ.

2.3.1 L_X as Singular Covering over S^3

We choose a general projection $\pi : (X, 0) \to (\mathbb{C}^2, 0)$. We denote by Γ the singular locus of π (in particular $\Sigma \subset \Gamma$) and by Δ its discriminant ($\Delta = \pi(\Gamma)$). In fact it is sufficient to choose new coordinates in $\mathbb{C}^n$, $(x, y, w_1, \ldots, w_{n-2}) \in \mathbb{C}^n$, such that the restriction on $(X, 0)$ of the projection

$$(x, y, w_1, \ldots, w_{n-2}) \mapsto (x, y),$$

denoted by π, is finite and such that, for a sufficiently small α with $\alpha < \epsilon$, and all $a \in \mathbb{C}$ with $|a| \le \alpha$, the hyperplanes $H_a = \{x = a\}$ meet transversally the singular locus Γ of π. In particular, $H_0 \cap \Gamma = \{0\}$.

Convention and Notation

Let $D_\alpha \times D_\beta \in \mathbb{C}^2$ be a polydisc at the origin in $\mathbb{C}^2$ where $0 < \alpha < \beta < \epsilon$ are chosen sufficiently small such that the following two points are satisfied:

I) $\mathcal{B} = B^{2n}_\epsilon \cap \pi^{-1}(D_\alpha \times D_\beta)$ is a good semi-analytic neighborhood of $(X, 0)$ in the sense of A. Durfee [7]. Then $(X \cap \mathcal{B}, 0)$ is homeomorphic to $(X, 0)$. In this section $(X, 0)$ is given by $(X \cap \mathcal{B}, 0)$. The link $L_X = X \cap b(\mathcal{B})$ is the link of X. The link of Γ is the link $K_\Gamma = \Gamma \cap b(\mathcal{B})$ embedded in L_X.

II) We have the following inclusion:

$$K_\Delta = \Delta \cap ((S_\alpha \times D_\beta) \cup (D_\alpha \times S_\beta)) \subset (S_\alpha \times D_\beta).$$

In this section, we choose such a K_Δ to represent the link of Δ embedded in the 3-sphere (with corners) $((S_\alpha \times D_\beta) \cup (D_\alpha \times S_\beta))$. Let δ_j, $1 \le j \le r$, be the r branches of the discriminant Δ. Let $N(K_\Delta)$ be a tubular compact neighborhood of K_Δ. So, $N(K_\Delta)$ is a disjoint union of r solid tori. For a sufficiently small $N(K_\Delta)$, the union $N(K_\Gamma)$ of the connected components of $L_X \cap \pi^{-1}(N(K_\Delta))$ which contain a connected component of K_Γ, constitutes a tubular compact neighbourhood of K_Γ in L_X.

Let us denote by $\mathring{N}(K_\Delta)$ the interior of $N(K_\Delta)$. **The exterior M of the link K_Δ** is defined by:

$$M = ((S_\alpha \times D_\beta) \cup (D_\alpha \times S_\beta)) \setminus \mathring{N}(K_\Delta).$$

Moreover, let γ be a branch of the singular locus Γ of π. So, $\pi(\gamma) = \delta$ is a branch of Δ. Let $N(K_\delta)$ (resp. $N(K_\gamma)$) be the connected component of $N(K_\Delta)$ (resp. of $N(K_\Gamma)$) which contains the link K_δ (resp. K_γ).

Remark 2.3.1 The restriction $\pi_L : L_X \to ((S_\alpha \times D_\beta) \cup (D_\alpha \times S_\beta))$ of π to L_X is a finite morphism, its restriction on M is a finite regular covering. If γ is not a branch of the singular locus Σ of X, π_L restricted to $N(K_\gamma)$ is a ramified covering with K_γ as ramification locus. If γ is a branch of Σ, $N(K_\gamma)$ is a singular pinched solid torus as defined in Definition 2.3.13 and π_L restricted to $N(K_\gamma)$ is singular all along K_γ.

2.3.2 Waldhausen Graph Manifolds and Plumbing Graphs

Definition 2.3.2 A Seifert fibration on an oriented, compact 3-manifold is an oriented foliation by circles such that every leaf has a tubular neighbourhood (which is a solid torus) saturated by leaves. A Seifert 3-manifold is an oriented, compact 3-manifold equipped with a Seifert fibration.

Remark 2.3.3

1. A Seifert 3-manifold M can have a non-empty boundary. As this boundary is equipped with a foliation by circles, if $B(M)$ is non-empty it is a disjoint union of tori.
2. Let D be a disc and r be a rotation of angle $2\pi q/p$ where (q, p) are two positive integers prime to each other and $0 < q/p < 1$. Let T_r be the solid torus equipped with a Seifert foliation given by the trajectories of r in the following mapping torus:

$$T_r = D \times [0, 1]/(z, 1) \sim (r(z), 0).$$

In particular, $l_0 = (0 \times [0, 1])/(0, 1) \sim (0, 0)$ is a core of T_r. The other leaves are (q, p)-torus knots in T_r. Let T_0 be $D \times S$ equipped with the trivial fibration by circles $l(z) = \{z\} \times S,\ z \in D$. A solid torus $T(l)$ which is a tubular neighbourhood of a leave l of a Seifert 3-manifold M is either

1) orientation and foliation preserving homeomorphic to T_0. In this case, l is a regular Seifert leave.
2) or, is orientation and foliation preserving homeomorphic to T_r. In this case, l is an exceptional leave of M.

3. The compactness of M implies that the set of exceptional leaves is finite.

Definition 2.3.4 Let M be an oriented and compact 3-manifold. The manifold M is a **Waldhausen graph manifold** if there exists a finite family $\mathcal{T}$, of disjoint tori embedded in M, such that if M_i, $i = 1, \ldots, m$, is the family of the closures of the connected components of $M \setminus \mathcal{T}$, then M_i is a Seifert manifold for all i, $1 \le i \le m$. We assume that it gives us a finite decomposition $M = \cup_{1 \le i \le m} M_i$ into a union of compact connected Seifert manifolds which satisfies the following properties:

1. For each M_i, $i = 1, \ldots, m$, the boundary of M_i is in $\mathcal{T}$ i.e. $b(M_i) \subset \mathcal{T}$.
2. If $i \neq j$ we have the inclusion $(M_i \cap M_j) \subset \mathcal{T}$.
3. The intersection $(M_i \cap M_j)$, between two Seifert manifolds of the given decomposition, is either empty or equal to the union of the common boundary components of M_i and M_j.

Such a decomposition $M = \cup_{1 \le i \le m} M_i$, is **the Waldhausen decomposition of M, associated to the family of tori $\mathcal{T}$**.

Remark 2.3.5 One can easily deduce from Definition 2.2.4, that the family of the plumbing tori gives a decomposition of the boundary of a plumbing as a union of Seifert manifolds because the boundary of a disc bundle is a circle bundle. So, the boundary of a plumbing is a Waldhausen graph manifold.

In [20], W. Neumann shows how to construct a plumbing from a given Waldhausen decomposition of a 3-dimensional oriented compact manifold.

As in Sect. 2.3.1, we consider the exterior $M = ((S_\alpha \times D_\beta) \cup (D_\alpha \times S_\beta)) \setminus \mathring{N}(K_\Delta)$ of the link K_Δ. The following proposition is well known (for example see [8, 17]). Moreover, a detailed description of M, as included in the boundary of the plumbing graph given by the minimal resolution of Δ, is given in [12, p. 147–150].

Proposition 2.3.6 *The exterior M of the link of a plane curve germ Δ is a Waldhausen graph manifold. The minimal Waldhausen decomposition of M can be extended to a Waldhausen decomposition of the sphere $((S_\alpha \times D_\beta) \cup (D_\alpha \times S_\beta))$ in which the connected components of K_Δ are Seifert leaves. Moreover, with such a Waldhausen decomposition, the solid tori connected components of $N(K_\Delta)$ are saturated by Seifert leaves which are oriented circles transverse to $(a \times D_\beta)$, $a \in S_\alpha$. The cores K_Δ of $N(K_\Delta)$ are a union of these Seifert leaves.*

2.3.3 The Topology of L_X When L_X Is a Topological Manifold

If $(X, 0)$ is not normal, let $\nu_L : L_{\bar{X}} \to L_X$ be the normalization of $(X, 0)$ restricted to the link of $(\bar{X}, p)$ (if $(X, 0)$ is normal ν_L is the identity).

Remark 2.3.7 The link of a normal complex surface germ is a Waldhausen graph manifold. Indeed, the composition morphism $\pi_L \circ \nu_L$ is a ramified covering with the link K_Δ as set of ramification values:

$$(\pi_L \circ \nu_L) : L_{\bar{X}} \to ((S_\alpha \times D_\beta) \cup (D_\alpha \times S_\beta)).$$

We can take the inverse image under $\pi_L \circ \nu_L$ of the tori and of the Seifert leaves of a Waldhausen decomposition of $((S_\alpha \times D_\beta) \cup (D_\alpha \times S_\beta))$ in which K_Δ is a union of Seifert leaves, to obtain a Waldhausen decomposition of $L_{\bar{X}}$. Then, the plumbing calculus [20] describes $L_{\bar{X}}$ as the boundary of a plumbing without the help of a good resolution of $(\bar{X}, p)$.

If the singular locus $(\Sigma, 0)$ of $(X, 0)$ is one-dimensional, let $(\sigma, 0)$ be a branch of $(\Sigma, 0)$ and s be a point of the intersection $\sigma \cap \{x = a\}$. Let $\delta = \pi(\sigma)$ be the branch of the discriminant Δ which is the image of σ by the morphism π. Then, $\pi_L(s) = (a, y) \in (S_\alpha \times D_\beta)$. Let $N(K_\delta)$ be a solid torus regular neighbourhood of K_δ in $(S_\alpha \times D_\beta)$ and let $N(K_\sigma)$ be the connected component of $(\pi_L)^{-1}(N(K_\delta))$ which contains s (and K_σ).

Let (C, s) be the germ of curve which is the connected component of $N(K_\sigma) \cap \{x = a\}$ which contains s. For a sufficiently small $\alpha = |a|$, (C, s) is reduced and its topological type does not depend upon the choice of s. In particular, **the number of the irreducible components of** (C, s) is well defined, let us denote this number **by** $k(\sigma)$.

Definition 2.3.8

1. By definition (C, s) is the **hyperplane section germ** of σ at s.
   If $k(\sigma) = 1$, σ is a **branch** of Σ **with irreducible hyperplane sections**. Let $\Sigma = \Sigma_1 \cup \Sigma_+$ where Σ_1 is the union of the branches of Σ with irreducible hyperplane sections and Σ_+ is the union of the branches of Σ with reducible hyperplane sections.
2. Let D_i, $1 \leq i \leq k$ be k oriented discs centered at $0_i \in D_i$. A k**-pinched disc** $k(D)$ is a topological space orientation preserving homeomorphic to the quotient of the disjoint union of the k discs by the identification of their centers in a unique point $\tilde{0}$ i.e. $0_i \sim 0_j$ for all i and j where $1 \leq i \leq k$, $1 \leq j \leq k$. **The center of** $k(D)$ is the equivalence class $\tilde{0}$ of the centers 0_i, $1 \leq i \leq k$.
3. If $h : k(D) \to k(D)'$ is a homeomorphism between two k-pinched discs with $k > 1$, $h(\tilde{0})$ is obviously the center of $k(D)'$. We say that h **is orientation preserving** if h preserves the orientation of the punctured k-pinched discs $(k(D) \setminus \{\tilde{0}\})$ and $(k(D)' \setminus \{\tilde{0}\})$.

Lemma 2.3.9 *Let (C, s) be the germ of curve which is the connected component of $N(K_\sigma) \cap \{x = a\}$ which contains s. Then, C is a $k(\sigma)$-pinched disc centered at s and $N(K_\sigma)$ is the mapping torus of C by an orientation preserving homeomorphism h which fixes the point s.*

Proof As (C, s) is a germ of curve with $k(\sigma)$ branches, up to homeomorphism (C, s) is a $k(\sigma)$-pinched disc with $s = \tilde{0}$.

We can saturate the solid torus $N(K_\delta) = \pi(N(K_\sigma))$ with oriented circles such that K_δ is one of these circles and such that the first return homeomorphism defined by these circles on the disc $\pi(C)$ is the identity. Let γ be one circle of the chosen saturation of $N(K_\delta)$. Then $\pi^{-1}(\gamma) \cap N(K_\sigma)$ is a disjoint union of oriented circles because π restricted to $N(K_\sigma) \setminus K_\sigma$ is a regular covering and $(\pi^{-1}(K_\delta) \cap N(K_\sigma)) = K_\sigma$. So, $N(K_\sigma)$ is equipped with a saturation by oriented circles. The first return map on C along the so constructed circles is an orientation preserving homeomorphism h such that $h(s) = s$ because K_σ is one of the given circles. □

Lemma 2.3.10 *As above, let (C, s) be the hyperplane section germ at $s \in \sigma \cap \{x = a\}$. Let $\bar{\sigma}_j$, $1 \leq j \leq n$, be the n irreducible components of $\nu_L^{-1}(\sigma)$ and let d_j be the degree of ν_L restricted to $\bar{\sigma}_j$. Then, we have*

$$k(\sigma) = d_1 + \cdots + d_j + \cdots + d_n.$$

Proof The normalization ν restricted to $\bar{X} \setminus \bar{\Sigma}$, where $\bar{\Sigma} = \pi^{-1}(\Sigma)$, is an isomorphism. The number n of the irreducible components of $\nu_L^{-1}(\sigma)$ is equal to the number of the connected components of $\nu_L^{-1}(N(K_\sigma))$. So, n is the number of the connected components of the boundaries $b(\nu_L^{-1}(N(K_\sigma)))$ which is equal to the number of the connected components of $b(N(K_\sigma))$. Let τ_j, $1 \leq j \leq n$, be the n disjoint tori which are the boundary of $N(K_\sigma)$. The degree d_j of ν restricted to $\bar{\sigma}_j$ is equal to the number of points of $\nu_L^{-1}(s) \cap (\bar{\sigma}_j)$.

Let (γ_j, s) be an irreducible component of (C, s) such that $m_j = b(\gamma_j) \subset \tau_j$. The normalization ν restricted to $(\nu_L^{-1}(\gamma_j \setminus \{s\}))$ is an isomorphism over the punctured disc $(\gamma_j \setminus \{s\})$. So, the intersection $\nu_L^{-1}(\gamma_j) \cap \bar{\sigma}_j$ is a unique point p_j. As $(\bar{X}, p)$ is normal, p_j is a smooth point of $(\bar{X}, p)$ and then, $\nu_L^{-1}(\gamma_j)$ is irreducible and it is the only irreducible component of $\nu_L^{-1}(C)$ at p_j. By symmetry there is exactly one irreducible component of $\nu_L^{-1}(C)$ at every point of $\nu_L^{-1}(s) \cap (\bar{\sigma}_j)$.

So, d_j is the number of the meridian circles of the solid torus $N(K_{\bar{\sigma}_j})$ obtained by the following intersection $(\nu_L^{-1}(C)) \cap (\nu_L^{-1}(\tau_j))$. But ν restricted to $(\nu_L^{-1}(\tau_j))$ is an isomorphism and d_j is also the number of connected components of $C \cap \tau_j$. So, $d_1 + \cdots + d_j + \cdots + d_n$, is equal to the number of connected components of $b(C) = C \cap b(N(K_\sigma))$ which is the number of irreducible components of (C, s). □

Remark 2.3.11 A well-known result of analytic geometry could be roughly stated as follows: “The normalization separates the irreducible components”. Here, $(X, 0)$ has $k(\sigma)$ irreducible components around $s \in \sigma$. Using only basic topology, Lemma 3.3.4 proves that $(\nu_L^{-1}(s))$ has $k(\sigma) = d_1 + \cdots + d_j + \cdots + d_n$ distinct points and that there is exactly one irreducible component of $\nu_L^{-1}(C)$ at every point of $\nu_L^{-1}(s)$. This gives a topological proof that the normalization ν separates the $k(\sigma)$ irreducible components of (C, s) around $s \in \sigma$.

Proposition 2.3.12 *The following three statements are equivalent:*

1. *L_X is a topological manifold equipped with a Waldhausen graph manifold structure.*
2. *The normalization $\nu : (\bar{X}, p) \to (X, 0)$ is a homeomorphism.*
3. *All the branches of Σ have irreducible hyperplane sections.*

Proof The normalization ν restricted to $\bar{X} \setminus \bar{\Sigma}$, where $\bar{\Sigma} = \pi^{-1}(\Sigma)$, is an isomorphism. The normalization is a homeomorphism if and only if ν restricted to $\bar{\Sigma} = \pi^{-1}(\Sigma)$ is a bijection. This is the case if and only if we have $1 = d_1 + \cdots + d_j + \cdots + d_n$ for all the branches σ of Σ. But, by Lemma 2.3.10, $k(\sigma) = d_1 + \cdots + d_j + \cdots + d_n$. This proves the equivalence of the statements 2 and 3.

Let (C, s) be the hyperplane section germ at $s \in \sigma \cap \{x = a\}$. If L_X is a topological manifold, it is a topological manifold at s and $k(\sigma) = 1$ for all branches σ of Σ. If all the branches of Σ have irreducible hyperplane sections, we already know that the normalization $\nu : (\bar{X}, p) \to (X, 0)$ is a homeomorphism. Then, the restriction ν_L of ν to $L_{\bar{X}}$ is also a homeomorphism. By Remark 2.3.7, $L_{\bar{X}}$ is a Waldhausen graph manifold. In particular, we can equip L_X with the Waldhausen graph manifold structure carried by ν_L. This proves the equivalence of the statements 1 and 3. □

2.3.4 Singular L_X, Curlings and Identifications

In Sect. 2.3.3 (Definition 2.3.8), we have considered the union Σ_+ of the branches of the singular locus Σ of $(X, 0)$ which have reducible hyperplane sections. We consider a tubular neighbourhood $N_+ = \cup_{\sigma \subset \Sigma_+} N(K_\sigma)$ of the link K_{Σ_+} of Σ_+ in L_X. As in the proof of Proposition 2.3.12, the exterior $M_1 = L_X \setminus \mathring{N}_+$, of K_{Σ_+} in L_X, is a topological manifold because ν restricted to $\nu^{-1}(M_1)$ is a homeomorphism. From now on σ is a branch of Σ_+. The definition of Σ_+ implies that L_X is topologically singular at every point of K_σ. In this section, we show that $N(K_\sigma)$ is a singular pinched solid torus. In Lemma 2.3.9, it is shown that $N(K_\sigma)$ is the mapping torus of a $k(\sigma)$-pinched disc by an orientation preserving homeomorphism. But, the homeomorphism class of the mapping torus of a homeomorphism h depends only on the isotopy class of h. Moreover the isotopy class of an orientation preserving homeomorphism h of a k-pinched disc depends only on the permutation induced by h on the k discs. In particular, if $h : D \to D$ is an orientation preserving homeomorphism of a disc D the associated mapping torus

$$T(D, h) = [0, 1] \times D/(1, x) \sim (0, h(x))$$

is homeomorphic to the standard torus $S \times D$.

Definition 2.3.13

1. Let $k(D)$ be the k-pinched disc quotient by identification of their centrum of k oriented and ordered discs D_i, $1 \leq i \leq k$. Let $c = c_1 \circ c_2 \circ \cdots \circ c_n$ be a permutation of the indices $\{1, \ldots, k\}$ given as the composition of n disjoint cycles c_j, $1 \leq j \leq n$, where c_j is a cycle of order d_j. Let $\tilde{h}_c$ be an orientation preserving homeomorphism of the disjoint union of D_i, $1 \leq i \leq k$ such that $\tilde{h}_c(D_i) = D_{c(i)}$ and $\tilde{h}_c(0_i) = 0_{c(i)}$. Then, $\tilde{h}_c$ induces an orientation preserving homeomorphism h_c on $k(D)$. By construction we have $h_c(\tilde{0}) = \tilde{0}$. A **singular pinched solid torus associated to the permutation** c is a topological space orientation preserving homeomorphic to the mapping torus $T(k(D), c)$ of h_c:

   $$T(k(D), c) = [0, 1] \times k(D)/(1, x) \sim (0, h_c(x))$$

   The **core** of $T(k(D), c)$ is the oriented circle $l_0 = [0, 1] \times \tilde{0}/(1, \tilde{0}) \sim (0, \tilde{0})$. A homeomorphism between two singular pinched solid tori is orientation preserving if it preserves the orientation of $k(D) \setminus \{\tilde{0}\}$ and the orientation of the trajectories of h_c in its mapping torus $T(k(D), c)$.
2. A d**-curling** C_d is a topological space homeomorphic to the following quotient of a solid torus $S \times D$:

   $$C_d = S \times D/(u, 0) \sim (u', 0) \Leftrightarrow u^d = u'^d.$$

   Let $q : (S \times D) \to C_d$ be the associated quotient morphism. By definition, $l_0 = q(S \times \{0\})$ is the **core of** C_d.

Example 2.3.14 Let $X = \{(x, y, z) \in \mathbb{C}^3 \; where \; z^d - xy^d = 0\}$. The normalization of $(X, 0)$ is smooth i.e. $\nu : (\mathbb{C}^2, 0) \to (X, 0)$ is given by $(u, v) \mapsto (u^d, v, uv)$. Let $T = \{(u, v) \in (S \times D) \subset \mathbb{C}^2\}$. Let $\pi_x : \nu(T) \to S$ be the projection $(x, y, z) \mapsto x$ restricted to $\nu(T)$. Here the singular locus of $(X, 0)$ is the line $\sigma = (x, 0, 0)$, $x \in \mathbb{C}$. We have $N(K_\sigma) = L_X \cap (\pi_x^{-1}(S)) = \nu(T)$ as a tubular neighbourhood of K_σ. Let $q : T \to C_d$ be the quotient morphism defined above. There exists a well defined homeomorphism $f : C_d \to N(K_\sigma)$ which satisfies $f(q(u, v)) = (u^d, v, uv)$. So, $N(K_\sigma)$ is a d-curling and K_σ is its core. Moreover, f restricted to the core l_0 of C_d is a homeomorphism onto K_σ.

Figure 2.2 shows schematically $\bar{\Gamma} = \nu^{-1}(\Gamma) \subset \bar{X}$ and Δ when Σ is irreducible and $\Gamma \setminus \Sigma$ has two irreducible components.

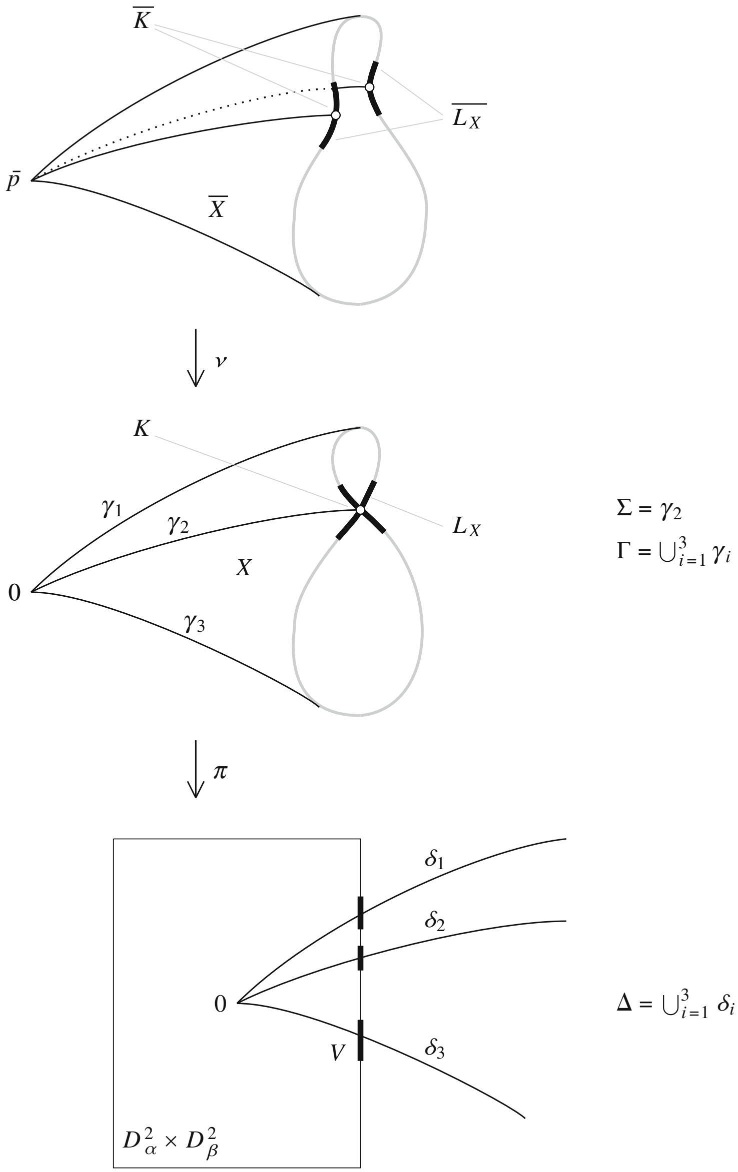

Fig. 2.2 Schematic picture of π and ν when there is a 2-curling on $\Sigma = \gamma_2$

Lemma 2.3.15 *A d-curling is a singular pinched solid torus associated to a d-cycle, i.e. if c is a d-cycle, then* C_d *is homeomorphic to* $T(d(D), c)$.

Proof We use the notation of Example 2.3.14. The model of d-curling obtained in this example is the tubular neighbourhood $N(K_\sigma)$ of the singular knot of the link L_X

of $X = \{(x, y, z) \in \mathbb{C}^3 \; where \; z^d - xy^d = 0\}$. As we work up to homeomorphism, it is sufficient to prove that $N(K_\sigma)$ is a singular pinched solid torus associated to a d-cycle. We can saturate the solid torus T by the oriented circles $l_b = S \times \{b\}, \; b \in D$. The circles $\nu(l_b), b \in D$ also saturate $N(K_\sigma)$ with oriented circles. The fiber $\pi_x^{-1}(a) = (C, (a, 0, 0))$ is a singular fiber of the fibration $\pi_x : \nu(T) \to S_\alpha$. The equation of the curve germ C at $(a, 0, 0)$ is $\{z^d - ay^d = 0\}$, this is a plane curve germ with d branches. So, C is homeomorphic to a d-pinched disc. Moreover, the first return along the circles $\nu(l_b)$ is a monodromy h of π_x which satisfies the conditions given in Definition 3.4.1 to obtain a singular pinched solid torus associated to a d-cycle.

Indeed, $(\pi_x \circ \nu) : T \to S_\alpha$ is a trivial fibration with fiber $\nu^{-1}(C) = \{(\{u_i\} \times D_\beta), u_i^d = a\}$ which is the disjoint union of d ordered meridian discs of T. The first return h_T along the oriented circles l_b is a cyclic permutation of the ordered d meridian discs and $(h_T)^d$ is the identity morphism. Moreover ν restricted to $T \setminus (S \times \{0\})$ is a homeomorphism. As h_T is a lifting of h by ν, the monodromy h determines $N(K_\sigma)$ as a singular pinched solid torus associated to a d-cycle. □

Proposition 2.3.16 *Let σ be a branch of the singular locus of $(X, 0)$ which has a reducible hyperplane section. Let (C, s) be the hyperplane section germ at $s \in \sigma \cap \{x = a\}$. Let $\bar{\sigma}_j$, $1 \le j \le n$, be the n irreducible components of $\nu_L^{-1}(\sigma)$ and let d_j be the degree of ν_L restricted to $\bar{\sigma}_j$. Let c_j be a d_j-cycle and let $c = c_1 \circ c_2 \circ \cdots \circ c_n$ be the permutation of $k(\sigma) = d_1 + \cdots + d_j + \cdots + d_n$ elements which is the composition of the n disjoint cycles c_j. A tubular neighbourhood $N(K_\sigma)$ of K_σ is a singular pinched solid torus associated to the permutation c. Moreover, the restriction of ν to $\coprod_{1 \le j \le n} N(K_{\bar{\sigma}_j})$ is the composition of two quotients: the quotients which define the d_j-curlings followed by the quotient f_σ which identifies their cores.*

Proof Let $N(K_{\bar{\sigma}_j})$, $1 \le j \le n$ be the n connected components of $\nu^{-1}(N(K_\sigma))$. So, $N(K_\sigma) \setminus K_\sigma$ has also n connected components and $(N(K_\sigma))_j = \nu(N(K_{\bar{\sigma}_j}))$ is the closure of one of them. Every $N(K_{\bar{\sigma}_j})$ is a solid torus and the restriction of ν to its core $K_{\bar{\sigma}_j}$ has degree d_j. The intersection $(\nu^{-1}(C)) \cap N(K_{\bar{\sigma}_j})$ is a disjoint union of d_j ordered and oriented meridian discs of $N(K_{\bar{\sigma}_j})$. We can choose a homeomorphism $g_j : (S \times D) \to N(K_{\bar{\sigma}_j})$ such that $(\nu \circ g_j)^{-1}(C) = \{u\} \times D, \; u^{d_j} = 1$.

The model of a d_j-curling gives the quotient $q_j : (S \times D) \to C_{d_j}$. As in Example 2.3.14, there exists a unique homeomorphism $f_j : C_{d_j} \to (N(K_\sigma))_j$ such that $f_j \circ q_j = \nu \circ g_j$. So, $(N(K_\sigma))_j$ is a d_j-curling. In particular, if ν_j is the restriction of ν to $N(K_{\bar{\sigma}_j})$, then $\nu_j = f_j \circ q_j \circ (g_j)^{-1}$. Up to homeomorphism ν_j is equivalent to the quotient which defines the d_j-curling.

But for all $j, 1 \le j \le n$, we have $\nu(K_{\bar{\sigma}_j}) = (K_\sigma)$. Up to homeomorphism, $N(K_\sigma)$ is obtained as the quotient of the disjoint union of the d_j-curlings by the identification of their cores. The disjoint union of the f_j induces a homeomorphism f_σ from

$$N = (\coprod_{1 \le j \le n} C_{d_j}) / q_j(u, 0) \sim q_i(u, 0) \Leftrightarrow \nu(g_j(u, 0)) = \nu(g_i(u, 0))$$

onto $N(K_\sigma)$. Up to homeomorphism, the restriction of ν to $\coprod_{1\leq j\leq n} N(K_{\bar{\sigma}_j})$ is the composition of two quotients: the quotients which define the d_j-curlings followed by the quotient f_σ which identifies their cores. It is sufficient to prove that $N = T(k(\sigma)(D), c)$ where c is the composition of n disjoint cycles c_j of order d_j. By Lemma 2.3.15, $C_{d_j} = T(d_j(D), c_j)$ and it is obvious that the identifications correspond to the disjoint union of the cycles. □

2.4 Hirzebruch-Jung's Resolution of $(X, 0)$

In this section $(X, 0)$ is a normal surface germ.

Let $\pi : (X, 0) \longrightarrow (\mathbb{C}^2, 0)$ be a finite analytic morphism which is defined on $(X, 0)$. For example π can be the restriction to $(X, 0)$ of a linear projection, as chosen in the beginning of Sect. 2.3.1. But the construction can be performed with any finite morphism π. We denote by Γ the singular locus of π and by $\Delta = \pi(\Gamma)$ its discriminant.

Let $r : (Z, E_Z) \to (\mathbb{C}^2, 0)$ be the minimal embedded resolution of Δ, let $E_Z = r^{-1}(0)$ be the exceptional divisor of r, and let $E_Z^+ = r^{-1}(\Delta)$ be the total transform of Δ. The irreducible components of E_Z are smooth complex curves because the resolution r is obtained by a sequence of blowing up of points in a smooth complex surface. Let us denote by E_Z^0 the set of the smooth points of E_Z^+. So, $E_Z^+ \setminus E_Z^0$ is the set of the double points of E_Z^+.

Here, we give a detailed construction of the Hirzebruch-Jung resolution $\rho : (Y, E_Y) \to (X, 0)$ associated to π. This will prove the existence of a good resolution of $(X, 0)$. As the link L_X is diffeomorphic to the boundary of Y, this will describe L_X as the boundary of a plumbing. In particular, we will explain how to obtain the dual graph $G(Y)$ of E_Y when we have the dual graph $G(Z)$ associated to E_Z. Knowing the Puiseux expansions of all the branches of Δ, there exists an algorithm to compute the dual graph $G_w(Z)$ weighted by the self-intersection numbers of the irreducible components of $E(Z)$ (For example see [6] and Chap. 6 and 7 in [17]). Except in special cases, the determination of the self-intersection numbers of the irreducible components of E_Y is rather delicate.

2.4.1 First Step: Normalization

We begin with the minimal resolution r of Δ. The pull-back of π by r is a finite morphism $\pi_r : (Z', E_{Z'}) \to (Z, E_Z)$ which induces an isomorphism from $E_{Z'}$ to E_Z. We denote $r_\pi : (Z', E_{Z'}) \to (X, 0)$, the pull-back of r by π. Figure 2.3 represents the resulting commutative diagram.

In general Z' is not normal. Let $n : (\bar{Z}, E_{\bar{Z}}) \to (Z', E_{Z'})$ be the normalization of Z'.

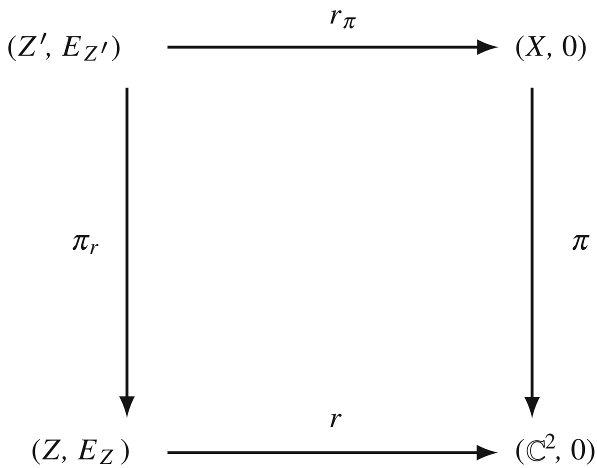

Fig. 2.3 The diagram of the pull-back of the resolution r by π

Remark 2.4.1

1. By construction, the discriminant locus of $\pi_r \circ n$ is included in $E^+_Z = r^{-1}(\Delta)$ which is the total transform of Δ in Z. As, X is normal at 0, $(X \setminus \{0\})$ has no singular points.
2. As the restriction of r to $Z \setminus E_Z$ is an isomorphism, the restriction of r_π to $Z' \setminus E_{Z'}$ is also an isomorphism. We denote by Γ' (resp. $\bar{\Gamma}$) the closure of $(r_\pi)^{-1}(\Gamma \setminus \{0\})$ in $E_{Z'}$ (resp. the closure of $(r_\pi \circ n)^{-1}(\Gamma \setminus \{0\})$ in $E_{\bar{Z}}$). The restriction of r_π to Γ' (resp. $(r_\pi \circ n)$ on $\bar{\Gamma}$) is an isomorphism onto Γ.
3. The singular locus of Z' is included in $E_{Z'}$. The normalization n restricted to $\bar{Z} \setminus E_{\bar{Z}}$ is an isomorphism.

Notation We use the following notations:
$E^+_{Z'} = E_{Z'} \cup \Gamma'$, and $E^0_{Z'}$ is the set of the points of $E_{Z'}$ which belong to a unique irreducible component of $E^+_{Z'}$. Similarly: $E^+_{\bar{Z}} = E_{\bar{Z}} \cup \bar{\Gamma}$, and $E^0_{\bar{Z}}$ is the set of the points of $E_{\bar{Z}}$ which belong to a unique irreducible component of $E^+_{\bar{Z}}$.

Proposition 2.4.2 *Every singular point of $\bar{Z}$ belongs to at least two irreducible components of $E^+_{\bar{Z}}$. The restriction of the map $(\pi_r \circ n)$ to $E_{\bar{Z}}$ induces a finite morphism from $E_{\bar{Z}}$ to E_Z which is a regular covering from $(\pi_r \circ n)^{-1}(E^0_Z)$ to (E^0_Z).*

Proof As X is normal at 0, $(X \setminus \{0\})$ has no singular points. The pull-back construction implies that:

1. The morphism π_r is finite and its generic degree is equal to the generic degree of π. Indeed, π_r restricted to $E_{Z'}$ is an isomorphism. Moreover, the restriction of π_r to $(Z' \setminus E_{Z'})$ is isomorphic, as a ramified covering, to the restriction of π to $(X \setminus \{0\})$. So, the restriction morphism $(\pi_r)_| : (Z' \setminus E_{Z'}) \to (Z \setminus E_Z)$ is a finite ramified covering with ramification locus Γ'.

2. As the restriction of r to $(Z \setminus E_Z)$ is an isomorphism, then the restriction of r_π to $(Z' \setminus E_{Z'})$ is also an isomorphism. So, the restriction of $(r_\pi \circ n)$ to $(\bar{Z} \setminus E_{\bar{Z}})$ is an analytic isomorphism onto the non-singular analytic set $(X \setminus \{0\})$. It implies that $(\bar{Z} \setminus E_{\bar{Z}})$ is smooth.

If $\bar{P} \in E^0_{\bar{Z}}$, then $P = (\pi_r \circ n)(\bar{P})$ is a smooth point of an irreducible component E_i of E_Z. The normal fiber bundle to E_i in Z can be locally trivialized at P. We can choose a small closed neighborhood N of P in Z such that $N = D \times \Delta$ where D and Δ are two discs , $N \cap E_Z = (D \times 0)$ and for all $z \in D$, $z \times \Delta$ are fibers of the bundle in discs associated to the normal bundle of E_i. We choose $\bar{N} = (\pi_r \circ n)^{-1}(N)$ as closed neighborhood of $\bar{P}$ in $\bar{Z}$. But $\bar{Z}$ is normal and the local discriminant of the restriction $(\pi_r \circ n)_| : (\bar{N}, \bar{P}) \to (N, P)$ is included in $D \times 0$ which is a smooth germ of curve. In that case, the link of $(\bar{N}, \bar{P})$ is S^3 (in Lemma 2.5.6, we give a topological proof of this classical result). As $\bar{Z}$ is normal, by Mumford's Theorem [19], $\bar{P}$ is a smooth point of $\bar{Z}$. This ends the proof of the first statement of the proposition.

Now, we know that the morphism $(\pi_r \circ n)_{|\bar{N}} : (\bar{N}, \bar{P}) \to (N, P)$ is a finite morphism between two smooth germs of surfaces with non-singular discriminant locus. Let d be its generic order. By Lemma 2.5.6, such a morphism is locally isomorphic (as an analytic morphism) to the morphism defined on $(\mathbb{C}^2, 0)$ by $(x, y) \mapsto (x, y^d)$. So, $\bar{D} = (\pi_r \circ n)^{-1}(D \times 0)$ is a smooth disc in $E^0_{\bar{Z}}$ and the restriction of such a morphism to $\{(x, 0), x \in \bar{D}\}$ is a local isomorphism.

By definition of E^0_Z, $P \in (E_i \cap E^0_Z)$ is a smooth point in the total transform of Δ. If we take a smooth germ (γ, P) transverse to E_i at P, then $(r(\gamma), 0)$ is not a branch of Δ. The restriction of π to $\pi^{-1}(r(\gamma) \setminus 0)$ is a regular covering. Let k be the number of irreducible components of $\pi^{-1}(r(\gamma))$. The number k is constant for all $P \in E_i \cap E^0_Z$. Let P' be the only point of $(\pi_r)^{-1}(P)$. Remark 2.3.11, which uses Lemma 2.3.10, shows that the k irreducible components of the germ of curve $((\pi_r)^{-1}(\gamma), P')$ are separated by n. So, the restriction of the map $(\pi_r \circ n)$ to $((\pi_r \circ n)^{-1}(E_i \cap E^0_Z))$ is a regular covering of degree k. □

Definition 2.4.3 A germ $(W, 0)$ of complex surface is **quasi-ordinary** if there exists a finite morphism $\phi : (W, p) \to (\mathbb{C}^2, 0)$ which has a normal-crossing discriminant. A **Hirzebruch-Jung singularity** is a quasi-ordinary singularity of normal surface germ.

Lemma 2.4.4 *Let $\bar{P}$ be a point of $E_{\bar{Z}}$ which belongs to several irreducible components of $E^+_{\bar{Z}}$. Then $\bar{P}$ belongs to two irreducible components of $E^+_{\bar{Z}}$. Moreover, either $\bar{P}$ is a smooth point of $\bar{Z}$ and $E^+_{\bar{Z}}$ is a normal crossing divisor around $\bar{P}$, or $\bar{P}$ is a Hirzebruch-Jung singularity of $\bar{Z}$.*

Proof If $\bar{P}$ be a point of $E_{\bar{Z}}$ which belongs to several irreducible components of $E^+_{\bar{Z}}$ then $P = (\pi_r \circ n)(\bar{P})$ is a double point of E^+_Z. Moreover Z is smooth and E^+_Z is a normal crossing divisor. We can choose a closed neighbourhood N of P isomorphic to a product of discs $(D_1 \times D_2)$, and we take $\bar{N} = (\pi_r \circ n)^{-1}(N)$. For a sufficiently small N, the restriction of $(\pi_r \circ n)$ to the pair $(\bar{N}, \bar{N} \cap E^+_{\bar{Z}})$ is a finite ramified morphism over the pair $(\bar{N}, \bar{N} \cap E^+_{\bar{Z}})$ and the ramification locus is included in the normal crossing divisor $(N \cap E^+_Z)$. The pair $(\bar{N}, \bar{P})$ is normal and the link of the pair $(N, N \cap E^+_Z)$ is the Hopf link in S^3. Then the link of $\bar{N}$ is a lens space, and the link of $(\pi_r \circ n)^{-1}(N \cap E^+_Z)$ has two components (Lemma 2.5.4 gives a topological proof of this classical result). So, $E^+_{\bar{Z}}$ has two irreducible components at $\bar{P}$. We have two possibilities:

1. $\bar{P}$ is a smooth point in $\bar{Z}$. Then the link of the pair $(\bar{N}, \bar{N} \cap E^+_{\bar{Z}})$ is the Hopf link in S^3 and $E^+_{\bar{Z}}$ is a normal crossing divisor at $\bar{P}$.
2. $\bar{P}$ is an isolated singular point of $\bar{Z}$. Then, the link of $\bar{N}$ is a lens space which is not S^3. The point $\bar{P}$ is a Hirzebruch-Jung singularity of $\bar{Z}$ equipped with the finite morphism

$$(\pi_r \circ n)_{|\bar{N}} : (\bar{N}, \bar{N} \cap E^+_{\bar{Z}}) \to (N, N \cap E^+_Z)$$

which has the normal crossing divisor $N \cap E^+_Z$ as discriminant. □

The example given in Sect. 2.6 illustrates the following Corollary.

Corollary 2.4.5 *Let $G(\bar{Z})$ be the dual graph of $E_{\bar{Z}}$. Proposition 2.4.2 and Lemma 2.4.4 imply that $(\pi_r \circ n)$ induces a finite ramified covering of graphs from $G(\bar{Z})$ onto $G(Z)$.*

2.4.2 Second Step: Resolution of the Hirzebruch-Jung Singularities

If $\bar{P}$ is a singular point of $\bar{Z}$, then $P = (\pi_r \circ n)(\bar{P})$ is a double point of E^+_Z . In particular, there are finitely many isolated singular points in $\bar{Z}$. The singularities of $\bar{Z}$ are Hirzebruch-Jung singularities. More precisely, let $\bar{P}_i$, $1 \leq i \leq n$, be the finite set of the singular points of $\bar{Z}$ and let $\bar{U}_i$ be a sufficiently small neighborhood of $\bar{P}_i$ in $\bar{Z}$. We have the following result (see [9] for a proof, see also [11, 22] and [13]) and, to be self-contained, we give a proof in Sect. 2.5.3 (Proposition 2.5.7):

Theorem *The exceptional divisor of the minimal resolution of $(\bar{U}_i, \bar{P}_i)$ is a normal crossings divisor with smooth rational irreducible components and its dual graph is a bamboo (it means is homeomorphic to a segment).*

Let $\bar{\rho}_i : (U'_i, E_{U'_i}) \to (\bar{U}_i, \bar{P}_i)$ be the minimal resolution of the singularity $(\bar{U}_i, \bar{P}_i)$. From [13] (corollary 1.4.3), see also [22] (paragraph 4), the spaces U'_i and the maps $\bar{\rho}_i$ can be glued, for $1 \leq i \leq n$, in a suitable way to give a smooth space Y and a map $\bar{\rho} : (Y, E_Y) \to (\bar{Z}, E_{\bar{Z}})$ satisfying the following property (Fig. 2.4).

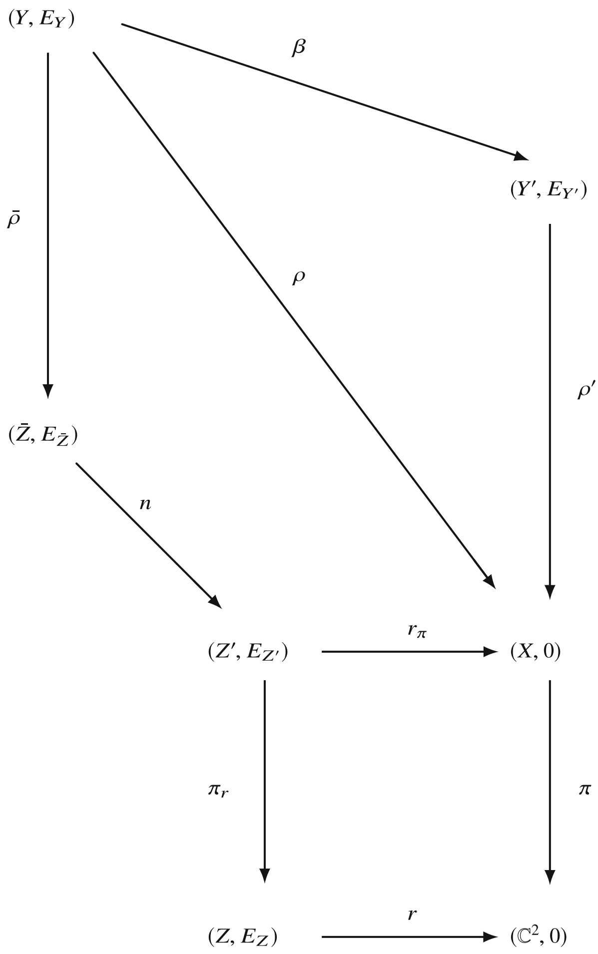

Fig. 2.4 The commutative diagram of the morphisms involved in the Hizebruch-Jung resolution ρ of π. By construction $\rho = r_\pi \circ n \circ \bar{\rho}$

Theorem 2.4.6 *Let us denote* $\rho = r_\pi \circ n \circ \bar{\rho}$*. Then,* $\rho : (Y, E_Y) \to (X, p)$ *is a good resolution of the singularity* (X, p) *in which the total transform* $\rho^{-1}(\Gamma) = E_Y^+$ *of the singular locus* Γ *of* π *is a normal crossings divisor.*

Proof The surface Y is smooth because $\bar{\rho}$ is a resolution of all the singular points of $\bar{Z}$. As proved in Proposition 2.4.2 and Lemma 2.4.4, the only possible singular points of the irreducible components of $E_{\bar{Z}}$ are the double points $\bar{P}_i$ of $E_{\bar{Z}}^+$. These points are resolved by the resolutions $\bar{\rho}_i$. So, the strict transform, by $\bar{\rho}$, of the irreducible components of $E_{\bar{Z}}$ are smooth.

The irreducible components of E_Y created during the resolution $\bar{\rho}$ are smooth rational curves. So, all the irreducible components of E_Y are smooth complex curves.

By Lemma 2.4.4, the only possible points of $E_{\bar{Z}}^+$ around which $E_{\bar{Z}}^+$ is not smooth or a normal crossing divisor are the Hirzebruch-Jung singularities $\bar{P}_i$, $1 \leq i \leq n$.

But as the $\bar{\rho}_i$, $1 \leq i \leq n$, are good resolutions of these singularities, $((\bar{\rho}_i)^{-1}(\bar{U}_i)) \cap (E_Y^+)$, $1 \leq i \leq n$, are normal crossing divisors. □

As ρ is the composition of three well defined morphisms which depend only on the choice of the morphism π and as we follow the Hirzebruch-Jung method, we have the following definition.

Definition 2.4.7 The morphism $\rho : (Y, E_Y) \to (X, 0)$ is the **Hirzebruch-Jung resolution** associated to π.

Corollary 2.4.8 *The dual graph $G(Y)$ of E_Y is obtained from the dual graph $G(\bar{Z})$ of $E_{\bar{Z}}$ by replacing the edges, which represent the Hirzebruch-Jung singular points of $\bar{Z}$, by a bamboo.*

Let $\rho'' : (Y'', E_{Y''}) \to (X, 0)$ be a good resolution of $(X, 0)$. Let E be an irreducible component of the exceptional divisor $E_{Y''}$ and let E^0 be the set of the smooth points of E in $E_{Y''}$. Let us recall that E is a **rupture component** of $E_{Y''}$ if the Euler characteristic of E^0 is strictly negative. Now we can use the following result (for a proof see [11, Theorem 5.9, p.87]):

Theorem *Let $\rho' : (Y', E_{Y'}) \to (X, 0)$ be the minimal resolution of $(X, 0)$. There exists $\beta : (Y, E_Y) \to (\tilde{Y}', E_{Y'})$ such that $\rho' \circ \beta = \rho$ and the map β consists in a composition of blowing-downs of irreducible components, of the successively obtained exceptional divisors, of self-intersection -1 and genus 0, which are not rupture components.*

2.5 Appendix: The Topology of a Quasi-ordinary Singularity of Surface

2.5.1 Lens Spaces

One can find details on lens spaces and surface singularities in [24]. See also [21].

Definition 2.5.1 A **lens space** L is an oriented compact three-dimensional topological manifold which can be obtained as the union of two solid tori $T_1 \cup T_2$ glued along their boundaries. The torus $\tau = T_1 \cap T_2$ is the Heegaard torus of the given decomposition $L = T_1 \cup T_2$.

Remark 2.5.2 If L is a lens space, there exists an embedded torus τ in L such that $L \setminus \tau$ has two connected components which are open solid tori $\mathring{T}_i, i = 1, 2$. Let $T_i, i = 1, 2$, be the two compact solid tori closure of $\mathring{T}_i$ in L. Of course $\tau = T_1 \cap T_2$. In [3], F. Bonahon shows that a lens space has a unique, up to isotopy, Heegaard torus. This implies that the decomposition $L = T_1 \cup T_2$ is unique up to isotopy, it is "the" Heegaard decomposition of L.

A lens space L with a decomposition of Heegaard torus τ can be described as follows. The solid tori $T_i, i = 1, 2$, are oriented by the orientation induced by L.

Let τ_i be the torus τ with the orientation induced by T_i. By definition a meridian m_i of T_i is a closed oriented circle on τ_i which is the boundary of a disc D_i embedded in T_i. A meridian of a solid torus is well defined up to isotopy. A parallel l_i of T_i is a closed oriented curve on τ_i such that the intersection $m_i \cap l_i = +1$ (we also write m_i (resp. l_i) for the homology class of m_i (resp. l_i) in the first homology group of τ_i). The homology classes of two parallels differ by a multiple of the meridian.

We choose on τ_2, an oriented meridian m_2 and a parallel l_2 of the solid torus T_2. As in [24, p. 23], we write a meridian m_1 of T_1 as $m_1 = nl_2 - qm_2$ with $n \in \mathbb{N}$ and $q \in \mathbb{Z}$ where q is well defined modulo n. As m_1 is a closed curve on τ, q is prime to n. Moreover, the class of q modulo n depends on the choice of l_2. So, we can chose l_2 such that $0 \leq q < n$.

Let τ be a boundary component of an oriented compact three-dimensional manifold M. Let T be a solid torus given with a meridian m on its boundary. If γ is a circle embedded in τ there is a unique way to glue T to M by an orientation reversing homeomorphism between the boundary of T and τ which send m to γ. The result of such a gluing is unique up to orientation preserving homeomorphism and it is called the **Dehn filling** of M associated to γ.

Definition 2.5.3 By a Dehn filling argument, it is sufficient to know the homology class $m_1 = nl_2 - qm_2$ to reconstruct L. By definition **the lens space** $L(n, q)$ is the lens space constructed with $m_1 = nl_2 - qm_2$. We have two special cases:

1. $m_1 = m_2$, if and only if L is homeomorphic to $S^1 \times S^2$,
2. $m_1 = l_2$ if and only if L is homeomorphic to S^3.

Lemma 2.5.4 *Let $\phi : (W, p) \to (\mathbb{C}^2, 0)$ be a finite morphism defined on an irreducible surface germ (W, p). If the discriminant Δ of ϕ is included in a normal crossing germ of curve, then the link L_W of (W, p) is a lens space. The link K_Γ of the singular locus Γ of ϕ, has at most two connected components. Moreover, K_Γ is a sub-link of the two cores of the two solid tori of a Heegaard decomposition of L_W as a union of two solid tori.*

Proof After performing a possible analytic isomorphism of $(\mathbb{C}^2, 0)$, Δ is, by hypothesis, included in the two axes i.e. $\Delta \subset \{xy = 0\}$.

Let $D_\alpha \times D_\beta \in \mathbb{C}^2$ be a polydisc at the origin in $\mathbb{C}^2$ where $0 < \alpha < \beta < \epsilon$ are chosen sufficiently small as in Sect. 2.3.1. Then, the restriction ϕ_L of ϕ on the link L_W is a ramified covering of the sphere (with corners)

$$\mathcal{S} = (S_\alpha \times D_\beta) \cup (D_\alpha \times S_\beta)$$

with a set of ramification values included in the Hopf link $K_{xy} = (S_\alpha \times \{0\}) \cup (\{0\} \times S_\beta)$.

Let $N(K_{xy})$ be a small compact tubular neighborhood of K_{xy} in $\mathcal{S}$. Then, $N(K_{xy})$ is the union of two disjoint solid tori $T_y = (S_\alpha \times D_{\beta'}), 0 < \beta' < \beta$, and $T_x = (D_{\alpha'} \times S_\beta), 0 < \alpha' < \alpha$. Then, $\phi_L^{-1}(T_x)$ (resp. $\phi_L^{-1}(T_y)$) is a union of

$r_x > 0$ (resp. $r_y > 0$) disjoint solid tori because the set of the ramification values of ϕ_L is included in the core of T_x (resp. T_y).

Let V be the closure, in $\mathcal{S}$, of $\mathcal{S} \setminus N(K_{xy})$. But, V is a thickened torus which does not meet the ramification values of ϕ_L. Then, $\phi_L^{-1}(V)$ is a union of $r > 0$ disjoint thickened tori. But, L_W is connected because (W, p) is irreducible by hypothesis. The only possibility to obtain a connected space by gluing $\phi_L^{-1}(T_x)$, $\phi_L^{-1}(T_y)$ and $\phi_L^{-1}(V)$ along their boundaries is $1 = r = r_x = r_y$.

So, $\phi_L^{-1}(T_x)$ (resp. $\phi_L^{-1}(T_y)$) which is in L_W a deformation retract of $T_2 = \phi_L^{-1}(S_\alpha \times D_\beta)$ (resp. $T_1 = \phi_L^{-1}(D_\alpha \times S_\beta)$) is a single solid torus. Then $\tau = \phi_L^{-1}(S_\alpha \times S_\beta)$ is a single torus. We have proved that L_W is the lens space obtained as the union of the two solid tori T_1 and T_2 along their common boundary $\tau = \phi_L^{-1}(S_\alpha \times S_\beta)$. So, $T_1 \cup T_2$ is a Heegaard decomposition of L_W as a union of two solid tori.

By hypothesis $K_\Delta \subset (S_\alpha \times \{0\}) \cup (\{0\} \times S_\beta)$. Then, K_Γ is included in the disjoint union of $\phi_L^{-1}(S_\alpha \times \{0\})$ and $\phi_L^{-1}(\{0\} \times S_\beta)$ which are the cores of T_1 and T_2. So, K_Γ has at most two connected components. □

Example 2.5.5 Let n and q be two relatively prime strictly positive integers. We suppose that $q < n$. Let $X = \{(x, y, z) \in \mathbb{C}^3 \ s.t. \ z^n - xy^q = 0\}$. The link L_X of $(X, 0)$ is the lens space $L(n, n - q)$.

Indeed, let $\phi : (X, 0) \to (\mathbb{C}^2, 0)$ be the projection $(x, y, z) \mapsto (x, y)$ restricted to X. The discriminant Δ of ϕ is equal to $\{xy = 0\}$. By Lemma 2.5.4, L_X is a lens space. As in the proof of Lemma 2.5.4, $L_X = \phi^{-1}(\mathcal{S})$ where

$$\mathcal{S} = (S_\alpha \times D_\beta) \cup (D_\alpha \times S_\beta).$$

In the proof of Lemma 2.5.4, it is shown that $T_2 = \phi^{-1}(S_\alpha \times D_\beta)$ and $T_1 = \phi^{-1}(D_\alpha \times S_\beta)$) are two solid tori. Let $(a, b) \in (S_\alpha \times S_\beta)$. As n and q are relatively prime $m_1 = \phi^{-1}(\{a\} \times S_\beta)$ and $m_2 = \phi^{-1}(S_\alpha \times \{b\})$ are connected. So, $m_i, \ i = 1, 2$, is a meridian of T_i.

We choose $c \in \mathbb{C}$ such that $c^n = ab^q$. Let $l_2 = \{z = c\} \cap \phi^{-1}(S_\alpha \times S_\beta)$. On the torus $\tau = \phi^{-1}(S_\alpha \times S_\beta)$, oriented as the boundary of T_2, we have $m_2 \cap l_2 = +1$ and $m_1 = nl_2 - (-q)m_2$. As defined in Definition 2.5.3, we have $L_X = L(n, -q) = L(n, n - q)$.

2.5.2 Finite Morphisms with Smooth Discriminant

Lemma 2.5.6 *Let $\phi : (W, p) \to (\mathbb{C}^2, 0)$ be a finite morphism, of generic degree n, defined on a normal surface germ (W, p). If the discriminant of ϕ is a smooth germ of curve, then $(X, 0)$ is analytically isomorphic to $(\mathbb{C}^2, 0)$ and ϕ is analytically isomorphic to the map from $(\mathbb{C}^2, 0)$ to $(\mathbb{C}^2, 0)$ defined by $(x, y) \mapsto (x, y^n)$.*

Proof After performing an analytic automorphism of $(\mathbb{C}^2, 0)$, we can choose coordinates such that $\Delta = \{y = 0\}$.

Let $D_\alpha \times D_\beta \in \mathbb{C}^2$ be a polydisc at the origin in $\mathbb{C}^2$ where $0 < \alpha < \beta < \epsilon$ are chosen sufficiently small as in Sect. 2.3.1. Then, the restriction ϕ_L of ϕ on the link L_W is a ramified covering of the sphere (with corners)

$$\mathcal{S} = (S_\alpha \times D_\beta) \cup (D_\alpha \times S_\beta)$$

with a set of ramification values included in the trivial link $K_y = (S_\alpha \times \{0\})$.

Here, we satisfy the hypotheses of Lemma 2.5.4. So, $T_2 = \phi_L^{-1}(S_\alpha \times D_\beta)$ and $T_1 = \phi_L^{-1}(D_\alpha \times S_\beta)$ are two solid tori with common boundary $\tau = \phi_L^{-1}(S_\alpha \times S_\beta)$. We take $a \in S_\alpha$ and $b \in S_\beta$.

Let us consider $\mathcal{D}_a = \phi_L^{-1}(\{a\} \times D_\beta) \subset T_2$ and $\mathcal{D}_b = \phi_L^{-1}(D_\alpha \times \{b\}) \subset T_1$.

Here the singular locus of ϕ_L is the core of T_2 and does not meet T_1.

The restriction of ϕ_L to $\phi_L^{-1}(D_\alpha \times \{b\})$ is a regular covering of a disc. Then $\mathcal{D}_b$ is a disjoint union of n discs where n is the general degree of ϕ_L. Let m_1 be the oriented boundary of one of the n discs which are the connected components of $\mathcal{D}_b$. By definition m_1 is a meridian of T_1.

The restriction of ϕ_L to $\mathcal{D}_a$ is a covering of a disc and $(a \times 0)$ is the only ramification value . Then $\mathcal{D}_a$ is a disjoint union of d discs where $d < n$. On τ, the intersection between the circles boundaries of $\mathcal{D}_a$ and $\mathcal{D}_b$ is equal to n because it is given by the (positively counted) n points of $\phi_L^{-1}(a \times b)$. The restriction of ϕ_L to T_1 is a Galois covering of degree n which permutes cyclically the connected components of $\mathcal{D}_a$. So, on the torus $\tau = b(T_1)$, any of the d circles boundaries of the connected components of $\mathcal{D}_a$ intersects any of the n circles boundaries of the connected components of $\mathcal{D}_b$. So computed, the intersection $b(\mathcal{D}_a) \cap b(\mathcal{D}_b)$ is equal to nd. But, $nd = n$ because this intersection is given by the n points of $\phi_L^{-1}(a \times b)$.

So, $d = 1$ and $\mathcal{D}_a$ has a unique connected component. The boundary of $\mathcal{D}_a$ is a meridian m_2 of T_2. As m_1 is the boundary of one of the n connected components of $\mathcal{D}_b$, $m_1 \cap m_2 = +1$ and m_1 can be a parallel l_2 of T_2. This is the case 2) in Definition 2.5.3, so the link L_W of (W, p) is the 3-sphere S^3. As (W, p) is normal, by Mumford [19], (W, p) is a smooth surface germ i.e (W, p) is analytically isomorphic to $(\mathbb{C}^2, 0)$. The first part of Lemma 2.5.6 has been proved.

(*) Moreover $\phi_L^{-1}(S_\alpha \times \{0\}) \cup (\{0\} \times S_\beta)$ is the union of the cores of T_1 and T_2. Then, $(S_\alpha \times \{0\}) \cup (\{0\} \times S_\beta)$ **is a Hopf link in the 3-sphere** L_W.

From now on, $\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is a finite morphism and its discriminant locus is $\{y = 0\}$. Let us write $\phi = (\phi_1, \phi_2)$. The link of the zero locus of the function germ

$$(\phi_1.\phi_2) : (\mathbb{C}^2, 0) \to (\mathbb{C}`0)$$

is the link describe above (see (*)), i.e. it is a Hopf link. The function $(\phi_1.\phi_2)$ reduced is analytically isomorphic to $(x, y) \mapsto (xy)$. But ϕ_1 is reduced because its Milnor fiber is diffeomorphic to $\mathcal{D}_a = \phi_L^{-1}(\{a\} \times D_\beta) \subset T_2$ which is a disc. So, ϕ_1 is isomorphic to x.

The Milnor fiber of ϕ_2 is diffeomorphic to the disjoint union of the n discs $\mathcal{D}_b = \phi_L^{-1}(D_\alpha \times \{b\}) \subset T_1$. When the Milnor fiber of a function germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ has n connected components, n is the $g.c.d.$ of the multiplicities of the irreducible factors of f. Here $\phi_2 = g^n$ where g is an irreducible function germ. We already have seen that ϕ_2 reduced is isomorphic to y. This completes the proof that ϕ_2 is isomorphic to y^n and $\phi = (\phi_1, \phi_2)$ is isomorphic to (x, y^n). □

2.5.3 The Hirzebruch-Jung Singularities

Proposition 2.5.7 *Let (W, p) be a normal surface germ such that there exists a finite morphism $\phi : (W, p) \to (\mathbb{C}^2, 0)$ which has a normal-crossing discriminant $(\Delta, 0)$. Then, (W, p) has a minimal good resolution $\rho : (\tilde{W}, E_{\tilde{W}}) \to (W, p)$ such that:*

I) the exceptional divisor $E_{\tilde{W}}$ of ρ has smooth rational irreducible components and its dual graph is a bamboo. We orient the bamboo from the vertex (1) to the vertex (k). The vertices are indexed by this orientation,

II) the strict transform of $\phi^{-1}(\Delta)$ has two smooth irreducible components which meet $E_{\tilde{W}}$ transversally, one of them at a smooth point of E_1 and the other component at a smooth point of E_k.

Proof After performing an analytic isomorphism of $(\mathbb{C}^2, 0)$, we can choose coordinates such that $\Delta = \{xy = 0\}$. We have to prove that there exists a minimal resolution ρ of (W, p) such that the shape of the dual graph of the total transform of Δ in $\tilde{W}$ looks like the graph drawn in Fig. 2.5 where all vertices represent smooth rational curves.

By Lemma 2.5.4, the link L_W of (W, p) is a lens space. If L_W is homeomorphic to S^3, (W, p) is smooth by Mumford [19], and there is nothing to prove. Otherwise, let n and q be the two positive integers, prime to each other, with $0 < q < n$, such that L_W is the lens space $L(n, n - q)$. By Brieskorn [4] (see also Sect. 2.5 in [24]), the normal quasi-ordinary complex surface germs are taut. It means that any normal quasi-ordinary complex surface germ (W', p') which has a link orientation preserving homeomorphic to $L(n, n - q)$ is analytically isomorphic to (W, p). In

Fig. 2.5 The shape of the dual graph of $G(\tilde{W})$ to which we add an arrow to the vertex (1) to represent the strict transform of $\{x = 0\}$ and another arrow to the vertex (k) to represent the strict transform of $\{y = 0\}$

particular, (W, p) and (W', p') have isomorphic minimal good resolutions. Now, it is sufficient to describe the good minimal resolution of a given normal quasi-ordinary surface germ which has a link homeomorphic to $L(n, n-q)$. As explained below, we can use $(\bar{X}, \bar{p})$ where $\nu : (\bar{X}, \bar{p}) \to (X, 0)$ is the normalization of $X = \{(x, y, z) \in \mathbb{C}^3 \ s.t. \ z^n - xy^q = 0\}$. □

Lemma 2.5.8 *Let n and q be two relatively prime positive integers. We suppose that $0 < q < n$. Let $X = \{(x, y, z) \in \mathbb{C}^3 \ s.t. \ z^n - xy^q = 0\}$. There exists a good resolution $\rho_Y : (Y, E_Y) \to (X, 0)$ of $(X, 0)$ such that the dual graph $G(Y)$ of E_Y is a bamboo and the dual graph of the total transform of $\{xy = 0\}$ has the shape of the graph given in Fig. 2.5.*

Lemma 2.5.8 implies Proposition 2.5.7. Indeed:

1) In Example 2.5.5, we show that the link L_X of $(X, 0)$ is the lens space $L(n, n-q)$. Let $\nu : (\bar{X}, \bar{p}) \to (X, 0)$ be the normalization of $(X, 0)$. The singular locus of $(X, 0)$ is the line $\Sigma = \{(x, 0, 0), x \in \mathbb{C}\}$. For $a \in \mathbb{C}$, the hyperplane section of X at $(a, 0, 0)$ is the plane curve germ $\{z^n - ay^q = 0\}$. As n and q are prime to each other $\{z^n - ay^q = 0\}$ is irreducible. Then, by Proposition 2.3.12, ν is a homeomorphism. So, the link $L_{\bar{X}}$ of $(\bar{X}, 0)$ is the lens space $L(n, n-q)$.
2) Let $\rho_Y : (Y, E_Y) \to (X, p)$ be a good resolution of $(X, 0)$ given as in Lemma 2.5.8, in particular the dual graph $G(Y)$ of E_Y is a bamboo. As any good resolution factorizes through the normalization $\nu : (\bar{X}, \bar{p}) \to (X, 0)$ (see [11, Thm. 3.14]), there exists a unique morphism $\bar{\rho_Y} : (Y, E_Y) \to (\bar{X}, \bar{p})$ which is a good resolution of $(\bar{X}, \bar{p})$. Let $\rho' : (Y', E_{Y'}) \to (\bar{X}, \bar{p})$ be the minimal good resolution of $(\bar{X}, \bar{p})$. Then, (for example see [11, Thm 5.9] or [1, p. 86]), there exists a morphism $\beta : (Y, E_Y) \to (Y', E_{Y'})$ which is a sequence of blowing-downs of irreducible components of genus zero and self-intersection -1. By Lemma 2.5.8, the dual graph $G(Y)$ is a bamboo and the dual graph of the total transform of $\{xy = 0\}$ has the shape of the graph given in Fig. 2.5. So, the morphism of graph $\beta* : G(Y) \to G(Y')$ induced by β, is only a contraction of $G(Y)$ in a shorter bamboo.

Proof (of Lemma 2.5.8) In X, we consider the lines $l_x = \{(x, 0, 0), x \in \mathbb{C}\}$ and $l_y = \{(0, y, 0), y \in \mathbb{C}\}$ and the singular locus of $(X, 0)$ is equal to l_x. We prove Lemma 2.5.8 by a finite induction on $q \geq 1$.

1) If $q = 1$, $X = \{(x, y, z) \in \mathbb{C}^3 \ s.t. \ z^n - xy = 0\}$ is the well-known normal singularity A_{n-1}. The minimal resolution is a bamboo of $(n-1)$ irreducible components of genus zero. Indeed, to construct $\rho_Y : (Y, E_Y) \to (X, 0)$, it is sufficient to perform a sequence of blowing-ups of points (we blow up $n/2$ points when n is even and $(n-1)/2$ points when n is odd). We begin to blow up the origin, this separates the strict transform of the lines l_x and l_y. The exceptional divisor, in the strict transform of $(X, 0)$ by the blowing-up of the origin in $\mathbb{C}^3$, has two irreducible rational components when $n > 2$ and only one irreducible rational component when $n = 2$. If $n > 2$, we continue by the

blowing-up of the intersection point of the two irreducible components of the exceptional divisor.

2) If $1 < q < n$, we state the following points I and II which describe how we proceed, we justify them just below.

   I) As n and q are relatively prime , the remainder r of the division $n = mq + r$ is prime to q and $1 < r < q$. Let $R : Z \to \mathbb{C}^3$ be a sequence of m blowing-ups of the line l_x in $\mathbb{C}^3$ and of its strict transforms in a smooth 3-dimensional complex space. Let Y_1 be the strict transform of X by R. Let $\rho : (Y_1, E) \to (X, 0)$ be R restricted to Y_1 and let $E = \rho{-}1(0) \subset Y_1$. The total transform of $l_x \cup l_y$ by ρ, which is equal to $E^+ = \rho^{-1}(l_x \cup l_y)$, has a dual graph which is a bamboo as in Fig. 2.5 with $k = m$ vertices. Let l_x^1 be the strict transform of l_x by ρ. Then, l_x^1 only meets the irreducible component of E obtained by the last blowing-up of a line. The equation of Y_1 along l_x^1 is $\{z^r - xy^q = 0\}$.

   II) If $r = 1$, Y_1 is smooth and Lemma 2.5.8 is proved i.e. $\rho_Y = \rho$. If $r > 2$, after the division $q = m'r + r'$ with remainder r', we have $r' < r$. As r is prime to q, r' is prime to r and $0 < r'$. Moreover, we have $r' < q$ because $r < q$. Let $R' : Z' \to Z$ be a sequence of m' blowing-ups of the line l_x^1 and of its strict transforms. Let Y_2 be the strict transform of Y_1 by R' and let $\rho' : (Y_2, E') \to (Y_1, E)$ be R' restricted to Y_2. As $r < q$, ρ' is bijective, the dual graph of $\rho'^{-1}(E^+)$ is equal to the dual graph of E^+, which is a bamboo as in Fig. 2.5 with $k = m$ vertices. Moreover, the equation of Y_2, along the strict transform of l_x^1 by ρ', is $\{z^r - xy^{r'} = 0\}$. As $1 \le r' < r$ with relatively prime r and r', Lemma 2.5.8 is proved by induction.

Let us justify the above statements I) and II) by an explicit computation of the blowing-up of l_x. We consider $Z_1 = \{((x, y, z), (v : w)) \in \mathbb{C}^3 \times \mathbb{C}P^1, s.t.\ wy - vz = 0\}$. By definition, the blowing-up of l_x in $\mathbb{C}^3$, $R_1 : Z_1 \to \mathbb{C}^3$, is the projection on $\mathbb{C}^3$ restricted to Z_1.

As in statement I), we consider $X = \{(x, y, z) \in \mathbb{C}^3\ s.t.\ z^n - xy^q = 0\}$ with $q < n$. We have to describe the strict transform Y_{11} of $(X, 0)$ by R_1, the restriction $\rho_1 : (Y_{11}, E) \to (X, 0)$ of R_1 to Y_{11}, $E_1 = \rho_1^{-1}(0)$ and $E_1^+ = \rho_1^{-1}(l_x \cup L_y)$.

I) In the chart $v = 1$, we have $(Z_1 \cap \{v = 1\}) = \{((x, y, wy), (1 : w)) \in \mathbb{C}^3 \times \mathbb{C}P^1\}$. The equation of $R_1^{-1}(0) \cap \{v = 1\}$ and of $E_1 \cap \{v = 1\}$ is $y = 0$.
The equation of $(R_1^{-1}(X) \cap \{v = 1\}) = (Y_{11} \cap \{v = 1\})$ is $\{w^n y^{n-q} - x = 0\}$. So, all the points of $(\{v = 1\} \cap Y_{11})$ are non singular and $(\{v = 1\} \cap \{x \neq 0\} \cap Y_{11})$ doesn't meet E_1.
The strict transform of l_x is not in $Y_{11} \cap \{v = 1\}$. If $x = 0$, we have:

$$E_1 \cap \{v = 1\} = \{((0, 0, 0), (1 : w)), w \in \mathbb{C}\} \subset Y_{11}.$$

In Y_{11}, the strict transform $\tilde{l_y} = \{((0, y, 0), (1 : 0)), y \in \mathbb{C}\}$ of l_y meets E_1 at $((0, 0, 0), (1 : 0))$.

II) In the chart $w = 1$, we have $(Z_1 \cap \{w = 1\}) = \{((x, vz, z), (v : 1)) \in \mathbb{C}^3 \times \mathbb{C}P^1\}$. The equation of $R_1^{-1}(0) \cap \{w = 1\}$ and of $E_1 \cap \{w = 1\}$ is $z = 0$.
The equation of $(Y_{11} \cap \{w = 1\})$ is $\{z^{n-q} - xv^q = 0\}$. So, the strict transform of l_x is equal to

$$\tilde{l_x} = (\{w = 1\} \cap Y_{11} \cap R_1^{-1}(l_x)) = \{((x, 0, 0), (0 : 1)) \in \mathbb{C}^3 \times \mathbb{C}P^1\}.$$

The strict transform $\tilde{l_x}$ meets E_1 at the point $p_1 = E_1 \cap \tilde{l_x} = ((0, 0, 0), (0 : 1)$. Then, $E_1 = ((0, 0, 0) \times \mathbb{C}P^1)$ is included in Y_{11}, moreover, $\tilde{l_x}$ and $\tilde{l_y}$ meet E_1 at two distinct points. The total transform $E_1^+ = \rho_1^{-1}(l_x \cup L_y)$ consists of one irreducible component E_1 and two germs of curves which meet E_1 in two distinct points. Moreover the equation of Y_{11} along its singular locus $\tilde{l_x}$ is $\{z^{n-q} - xy^q = 0\}$. By induction we obtain, as stated in I), the germ $(Y_1, 0)$ defined by $\{z^r - xy^q = 0\}$ with $1 \leq r = n - mq < q$.

To justify statement II), we again consider the blowing-up of l_x, $R_1 : Z_1 \to \mathbb{C}^3$. Let Y_{12} be the strict transform of Y_1 by R_1 and let $\rho_1' : Y_{12} \to Y_1$ be R_1 restricted to Y_{12}. Then, Y_{12} has the equation $\{w^r - xy^{q-r} = 0\}$ in the chart $v = 1$. For all $x \in \mathbb{C}$, the intersection of Y_{12} with $y = 0$ is the only point $((x, 0, 0)), (1 : 0))$. In the chart $w = 1$, Y_{12} has the equation $\{1 - xv^q z^{q-r} = 0\}$ and has empty intersection with $z = 0$. This proves that ρ_1' is bijective and by induction the map $\rho' : (Y_2, E') \to (Y_1, E)$ describe above in II) is also bijective. □

Examples

1) Let us consider $X = \{(x, y, z) \in \mathbb{C}^3 \ s.t. \ z^n - xy^{n-1} = 0\}$. The link of $(X, 0)$ is the lens space $L(n, 1)$. Let $R_1 : Z_1 \to \mathbb{C}^3$ be the blowing-up of the line l_x in $\mathbb{C}^3$. Let Y be the strict transform of X by R_1. The equation of Y along the strict transform of l_x is $\{z - xy^{n-1} = 0\}$. So, Y is non singular and we have obtained a resolution of X. Here the dual graph of the total transform of $l_x \cup l_y$ is as in Fig. 2.5 with only one vertex.
2) Let us consider $X = \{(x, y, z) \in \mathbb{C}^3 \ s.t. \ z^n - xy^{n-2} = 0\}$ with n odd and $3 < n$. The link of $(X, 0)$ is the lens space $L(n, 2)$. Let $R_1 : Z_1 \to \mathbb{C}^3$ be the blowing-up of the line l_x in $\mathbb{C}^3$. The equation of the strict transform Y_1, of X by R_1, along the strict transform of l_x is $\{z^2 - xy^{n-2} = 0\}$. Let $\rho : (Y_1, E) \to (X, 0)$ be R_1 restricted to Y_1. We write $n = 2m + 3$. As proved above, after m blowing-ups of lines, we obtain a surface Y_2 and a bijective morphism $\rho' : (Y_2, E') \to (Y_1, E)$ such that the equation of Y_2 along the strict transform of l_x is $\{z^2 - xy = 0\}$. The exceptional divisor E of ρ (resp. $E'$of $(\rho \circ \rho')$) is an irreducible smooth rational curve. The blowing-up ρ'', of the intersection point between E' and the strict transform of l_x, is a resolution of Y_2 and the exceptional divisor of ρ'' is a smooth rational curve. Then, $\rho \circ \rho' \circ \rho''$ is a resolution of $X = \{(x, y, z) \in \mathbb{C}^3 \ s.t. \ z^n - xy^{n-2} = 0\}$, the dual graph of its exceptional divisor is a bamboo with two vertices.

2.6 An Example of Hirzebruch-Jung's Resolution

We give the Hirzebruch-Jung resolution of the germ of surface in $\mathbb{C}^3$ which satisfies the following equation:

$$z^2 = (x - y + y^3)(x - y + y^2)(y^{34} - (x - y)^{13}).$$

where $\pi : (X, 0) \to (\mathbb{C}^2, 0)$ is the projection on the (x, y)-plane. It is a generic projection. In [15] this example is also explored when π is replaced by a non generic projection.

The discriminant locus of $\pi = (f, g)$ is the curve Δ which has three components with Puiseux expansions given by :

$$x = y - y^2$$
$$x = y - y^3$$
$$x = y + y^{34/13}$$

Notice that the three components of Δ have 1 as first Puiseux exponent and respectively 2, 3, 34/13 as second Puiseux exponent.

The coordinate axes are transverse to the discriminant locus of π. The dual graph $G(Z)$ is in Fig. 2.6.

The dual graph $G(\bar{Z})$ of $E_{\bar{Z}}$ admits a cycle created by the normalization. The irreducible component E'_9 of E_Y is obtained by the resolution $\bar{\rho}$. The irreducible components of the exceptional divisor associated to the vertices of $G(\bar{Z})$ and $G(Y)$ have genus equal to zero (Fig. 2.7).

The minimal good resolution ρ is obtained by blowing down E'_6. Its dual graph is in Fig. 2.8.

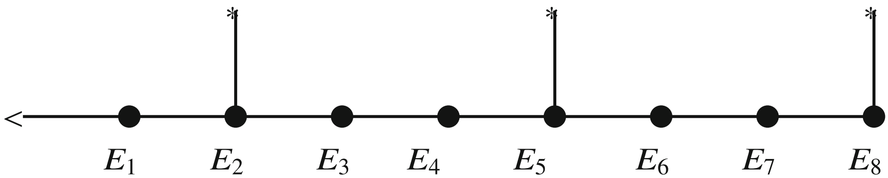

Fig. 2.6 The dual graph of the minimal resolution of Δ. An irreducible component of the strict transform of Δ is represented by an edge with a star. An edge ended by an arrow represents the strict transform of $\{x = 0\}$

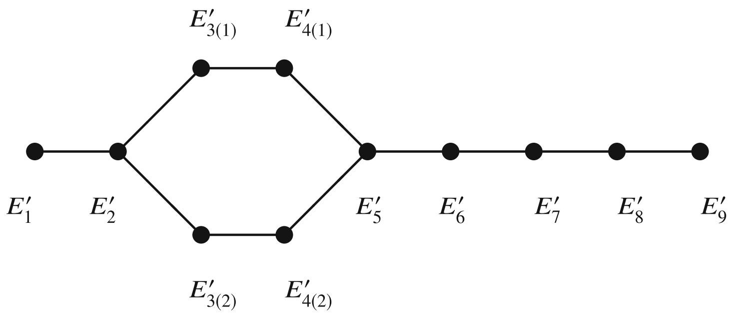

Fig. 2.7 The dual graph $G(Y)$ of the Hirzebruch-Jung resolution associated to π

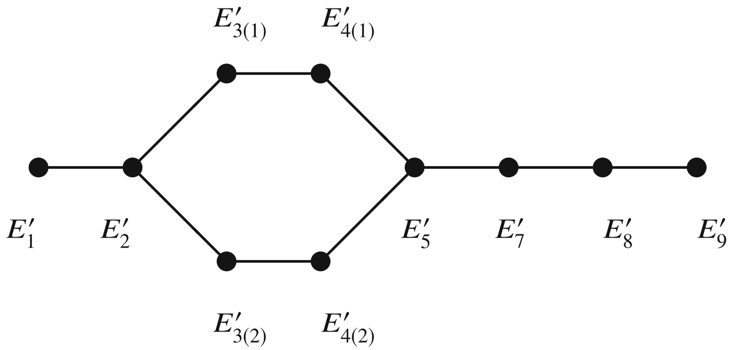

Fig. 2.8 The dual graph $G(Y')$ of the minimal resolution of $(X, 0)$

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