José Luis Cisneros Molina
Dũng Tráng Lê
José Seade *Editors*

Handbook of Geometry and Topology of Singularities I

Springer

Handbook of Geometry and Topology of Singularities I

José Luis Cisneros Molina • Dũng Tráng Lê •
José Seade
Editors

Handbook of Geometry and Topology of Singularities I

Springer

Editors

José Luis Cisneros Molina
Unidad Cuernavaca
Instituto de Matemáticas
Universidad Nacional Autónoma de México
Cuernavaca, Mexico

Unidad Mixta Internacional 2001 CNRS
Laboratorio Solomon Lefschetz
Cuernavaca, Mexico

José Seade
Instituto de Matemáticas
Universidad Nacional Autónoma de México
Mexico City, Mexico

Unidad Mixta Internacional 2001 CNRS
Laboratorio Solomon Lefschetz
Cuernavaca, Mexico

Dũng Tráng Lê
Centre de Mathématiques et Informatique
Université d'Aix-Marseille
Marseille, France

ISBN 978-3-030-53060-0 ISBN 978-3-030-53061-7 (eBook)
https://doi.org/10.1007/978-3-030-53061-7

Mathematics Subject Classification: M11019, M11132, M12198, M21022, M21050, M28027,17B45

© Springer Nature Switzerland AG 2020, corrected publication 2021, 2024
This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.
The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.
The publisher, the authors, and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, expressed or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

This Springer imprint is published by the registered company Springer Nature Switzerland AG.
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Foreword

In the general scientific culture, Mathematics can appear as quite disconnected. One knows about calculus, complex numbers, Fermat's last theorem, convex optimization, fractals, vector fields and dynamical systems, the law of large numbers, projective geometry, vector bundles, the Fourier transform and wavelets, the stationary phase method, numerical solutions of PDEs, etc., but no connection between them is readily apparent. For the mathematician, however, all these and many others are lineaments of a single landscape. Although he or she may spend most of his or her time studying one area of this landscape, the mathematician is conscious of the possibility of traveling to other places, perhaps at the price of much effort, and bringing back fertile ideas. Some of the results or proofs most appreciated by mathematicians are the result of such fertilizations.

I claim that Singularity Theory sits inside Mathematics much as Mathematics sits inside the general scientific culture. The general mathematical culture knows about the existence of Morse theory, parametrizations of curves, Bézout's theorem for plane projective curves, zeroes of vector fields and the Poincaré–Hopf theorem, catastrophe theory, sometimes a version of resolution of singularities, the existence of an entire world of commutative algebra, etc. But again, for the singularist, these and many others are lineaments of a single landscape and he or she is aware of its connectedness. Moreover, just as Mathematics does with science in general, singularity theory interacts energetically with the rest of Mathematics, if only because the closures of non-singular varieties in some ambient space or their projections to smaller dimensional spaces tend to present singularities, smooth functions on a compact manifold must have critical points, etc. But singularity theory is also, again in a role played by Mathematics in general science, a crucible where different types of mathematical problems interact and surprising connections are born.

- Who would have thought in the 1950s that there was a close connection between the classification of differentiable structures on topological spheres and the boundaries of certain isolated singularities of complex hypersurfaces?

- or that Thom's study of singularities of differentiable mappings would give birth to a geometric vision of bifurcation phenomena and of fundamental concepts such as structural stability?
- Who would have thought in the 1970s that there was a relation between the work of Lefschetz comparing the topological invariants of a complex projective variety with those of a general hyperplane section and the characterization of the sequences of integers counting the numbers of faces of all dimensions of simple polytopes?
- Or that one could produce real projective plane curves with a prescribed topology by deforming piecewise linear curves in the real plane?
- Or in the 2000s that properties of the intersections of two curves on a complex surface would lead to the solution of a problem connected with the coloring of graphs?
- Or that the algebraic study of the space of arcs on the simplest singularities ($z^n = 0$ in $\mathbf{C}$, $n \geq 2$) would provide new proofs and generalizations of the Rogers–Ramanujan and Gordon identities between the generating series of certain types of partitions of integers?

These are only a few examples. But to come back to the theory of singularities, I would like to emphasize that what I like so much about it is that not only are surprising connections born there but also very simple questions lead to ideas which resonate in other part of the field or in other fields. For example, if an analytic function has a small modulus at a point, does it have a zero at a distance from that point which is bounded in terms of that modulus? what is a general smooth function on a smooth compact manifold? A Morse function, with very mild singularities! And what happens if you replace the smooth manifold by a space with singularities? And then, given a function, can we measure how far it is from behaving like a general function? Suppose that a holomorphic function has a critical point at the origin. How can we relate the nature of the fiber of the function through this critical point with the geometry or topology of the nearby non-singular fibers? How can we relate it with the geometry of the mappings resolving singularities of this singular fiber? Then again, what is a general map between smooth manifolds? and how do you deform a singular space into a non-singular one in general? Well, that is more complicated. But I hope you get the idea.

The downside is that before he or she can successfully detect and try to answer such apparently simple and natural questions, the student of singularities must become familiar with different subjects and their techniques, and the learning process is long.

And this is why a handbook which presents in-depth and reader-friendly surveys of topics of singularity theory, with a carefully crafted preface explaining their place within the theory, is so useful!

Paris, France
March 2020

Bernard Teissier

Preface

Singularities are ubiquitous in mathematics, appearing naturally in a wide range of different areas of knowledge. They are a meeting point where many areas of mathematics and science in general come together. Their scope is vast, their purpose is multifold.

Singularity theory dates back to I. Newton, É. Bézout, V. Puiseux, F. Klein, M. Noether, F. Severi, and many others. Yet, it emerged as a field of mathematics in itself in the early 1960s, thanks to pioneering work by R. Thom, O. Zariski, H. Whitney, H. Hironaka, J. Milnor, E. Brieskorn, C. T. C. Wall, V. I. Arnold, J. Mather, and many others. Its potential for applications in other areas of mathematics and of knowledge in general is unlimited, and so are its possible sources of inspiration.

As the name suggests, one may naively say that singularity theory studies that which is "singular," that which is different from "most of the rest," different from its surroundings. As basic examples, we may look at the critical points of smooth functions, or at the points where a space loses its manifold structure, at the stationary points of flows and the special orbits of Lie group actions, at bifurcation theory and properties of objects or situations depending on parameters that undergo sudden change under a small variation of the parameters. These are some examples, out of a myriad of possibilities, of how singularities arise. There is great richness in the subject, and the literature is vast, with plenty of different viewpoints, perspectives, and interactions with other areas. That makes this subject fascinating.

That same wideness and amplitude of its scope can make singularity theory hard to grasp for graduate students and researchers in general: what are and what have been the major lines of development in the last decades, what is known and where to find it, what is the current state of the art in its many branches, the various directions into which this theory is flourishing, its interaction with other areas of current research in mathematics. Those are questions that gave birth to this project, the "Handbook of Geometry and Topology of Singularities."

This handbook has the intention of covering a wide scope of singularity theory, presenting articles on various aspects of the theory and its interactions with other areas of mathematics. The authors are world experts; the various articles deal with both classical material and modern developments. They are addressed to graduate students and newcomers into the theory, as well as to specialists that can use these as guidebooks.

Volume I consists of ten articles that cover some of the foundational aspects of the theory. This includes:

- The combinatorics and topology of plane curves and surface singularities.
- An introduction to four classical methods for studying the topology and geometry of singular spaces, namely: resolution of singularities, deformation theory, stratifications, and slicing the spaces *à la* Lefschetz.
- Milnor fibrations and their monodromy.
- Morse theory for stratified spaces and constructible sheaves.
- Simple Lie algebras and simple singularities.

We say below a few words about the content of each chapter. Of course, due to lack of space, many important topics from the geometric study of singularities are missing from this volume. This will be compensated to some extent in the next volumes. Also, the number of possible authors much exceeds the capacity of any project of this kind. We thank our many colleagues that have much contributed to build up singularity theory, and we apologize for our omissions in the selection of subjects. Among the topics we plan to include in later volumes of this Handbook of Geometry and Topology of Singularities are:

- Equisingularity.
- Lipschitz geometry in singularity theory.
- The topology of the complement of arrangements and hypersurface singularities.
- Mixed Hodge structures.
- Analytic classification of singularities of complex plane curves.
- Applications to Lagrangian and Legendrian geometry.
- Contact and symplectic geometry in singularity theory.
- Indices of vector fields and 1-forms on singular varieties.
- Chern classes of singular varieties.
- Tropical geometry and singularity theory.
- Milnor fibrations for real analytic maps.
- Mixed singularities.
- Singularities of map germs. Finite determinacy and unfoldings.
- Relations with moment angle manifolds.
- Invariant algebraic sets in holomorphic dynamics.
- Limits of tangent spaces.
- Invariants of 3-manifolds and surface singularities.
- Zeta functions and the monodromy.

Chapters 1 and 2 of this volume deal with dimensions 1 and 2, respectively. Chapter 1, by Evelia García Barroso, Pedro González Pérez, and Patrick Popescu-Pampu, is entitled "The Combinatorics of Plane Curve Singularities: How Newton Polygons Blossom into Lotuses." In this chapter, the authors discuss classical ways to describe the combinatorics of singularities of complex algebraic curves contained in a smooth complex algebraic surface. In fact, given a smooth complex surface S

and a complex curve C in S with a singular point o, it is customary to study the local structure of (S, C) near o in the following ways:

- By choosing a local parametrization of C. This method dates back to Newton and later Puiseux. The combinatorics in it may be encoded in the Kuo-Lu tree and a Galois quotient of it, the Eggers-Wall tree.
- By blowing up points to obtain an embedded resolution of C. This blow-up process may be encoded in an Enriques diagram and a corresponding weighted dual graph.
- By performing a sequence of toric modifications. The combinatorial data generated during this process can be encoded in a sequence of Newton polygons and Newton fans.
- By looking at the intersection of S and C with a small sphere in some ambient space $\mathbb{C}^n$. One gets a knot (or link) in a 3-sphere. These are all iterated torus knots known as algebraic knots. Their combinatorics is encoded in the Puiseux pairs.

Chapter 1 studies the first three of these methods and explains how the notion of lotus, which is a special type of simplicial complex of dimension 2, allows to think simultaneously about the combinatorics of those three ways of analyzing the curve singularity.

The fourth method mentioned above is actually much related to Chap. 2 in this volume, by Françoise Michel, entitled “The Topology of Surface Singularities.” This chapter surveys the subject of the topology of complex surface singularities. This classical subject dates back to Felix Klein and his work on invariant polynomials for the finite subgroups of the special unitary group SU(2). This gave rise to what today are called Klein singularities, though they have many names, as, for instance, Du Val singularities, rational double points, and simple singularities in Arnold’s classification. If X is a complex surface singularity with base point p in some ambient space $\mathbb{C}^n$, then the intersection $L_X = X \cap \mathbb{S}_e$ with a small sphere centered at p is a 3-dimensional real analytic variety, whose topology is independent of the choice of the embedding of X in $\mathbb{C}^n$ and also independent of the choice of the (sufficiently small) sphere; L_X is called the link of the singularity and it fully describes the topology of X. If X has an isolated singularity at p, then L_X is a 3-manifold. The manifolds one gets in this way are all Waldhausen (or graph) manifolds that can be constructed by plumbing, a technique introduced by John Milnor in all dimensions, in order to construct the first examples of homology spheres. The author also gives an explicit construction of a good resolution of the singularity, and the minimal good resolution by the Hirzebruch–Jung method is described in detail.

Chapters 3 to 9 deal with the four classical ways of studying the geometry and topology of singular spaces mentioned above, namely:

1. Via resolutions of the singularities;
2. Via stratifications;

3. Via deformations, smoothings, and unfoldings; and
4. Taking slices with the fibers of a linear form.

Let us say a few words about each of them.

The problem of resolution of singularities and its solution in various contexts, already discussed for plane curves in Chap. 1 and for surfaces in Chap. 2, can be traced back to Newton and Riemann. Chapter 3, by Mark Spivakovsky, is an introduction to the resolution of singularities. This surveys the subject, starting with Newton till the modern times. It also discusses some of the main open problems that remain to be solved. The main topics covered are the early days of the subject, Zariski's approach via valuations, Hironaka's celebrated result in characteristic zero and all dimensions and its subsequent strengthenings and simplifications, existing results in positive characteristic (mostly up to dimension three), de Jong's approach via semi-stable reduction, Nash and higher Nash blowing up, as well as reduction of singularities of vector field and foliations.

Chapter 4 is an introduction to the stratification theory, by David Trotman. The idea behind the notion of stratification in differential topology and algebraic geometry is to partition a (possibly singular) space into smooth manifolds with some control on how these manifolds fit together. In 1957, Whitney showed that every real algebraic variety V in $\mathbb{R}^n$ can be partitioned into finitely many connected smooth submanifolds of $\mathbb{R}^n$. This he called a manifold collection. In 1960, René Thom replaced the term manifold collection by stratified set and initiated a theory of stratified sets and stratified maps. In this chapter, the author presents in a unifying manner both the abstract theory of stratified sets elaborated by Thom, Whitney, and Mather and the stratification theory of semi-algebraic, subanalytic, or complex analytic sets. In addition, it surveys the relations between several stratifying conditions which are modifications of the Whitney conditions, with an emphasis on the applications to the openness of transversality theorems which are so important in stability problems. The text also explains what remains true of the stratification theory of real algebraic and subanalytic sets in the o-minimal framework.

Chapter 5 by Mark Goresky, entitled "Morse Theory, Stratifications and Sheaves," begins with an introduction to Morse theory for stratified spaces and then moves forward to discussing how stratified Morse theory and the theory of constructible sheaves, introduced by M. Kashiwara and P. Shapira, are two sides of the same coin. A complete and parallel development of the two theories was presented by J. Schürmann. In this chapter, the author provides an intuitive view of this parallel development. The setting presented by Schürmann replaces the subanalytic and Whitney stratified setting with the more general conditions of o-minimal structures and generalized Whitney conditions: w-regularity, d-regularity, and C-regularity. In this chapter, for simplicity, the author remains within the subanalytic and Whitney stratified setting.

Chapter 6, by J. J. Nuño Ballesteros, Lê D. T., and J. Seade, treats a now classical and central subject in singularity theory: the Milnor fibration theorem, which provides the simplest example of a deformation of a singular variety into a smooth one. This fibration theorem, published by John Milnor in 1968, concerns the geometry and topology of analytic maps near their critical points, and it was the culmination of a series of articles by Brieskorn, Hirzebruch, Pham, and others, aimed toward finding complex isolated hypersurface singularities whose link, i.e., its intersection with a small sphere centered at the singular point, is a homotopy sphere. The theorem considers a nonconstant holomorphic map germ $(\mathbb{C}^{n+1}, \underline{0}) \xrightarrow{f} (\mathbb{C}, 0)$ with a critical point at $\underline{0}$, and it can roughly be stated as saying that the local noncritical levels $f^{-1}(t)$ form a locally trivial C^∞ fiber bundle over a sufficiently small punctured disc in $\mathbb{C}$. Notice that one has a flat family F_t of complex manifolds degenerating to the special fiber $f^{-1}(0)$. This is the paradigm of a smoothing, i.e., a flat deformation where all fibers, other than the special one, are non-singular. Milnor's fibration theorem is a cornerstone in singularity theory. It has opened several research fields and given rise to a vast literature. In this chapter, the authors present some of the foundational results about this subject and give proofs of several basic "folklore theorems" which either are not in the literature or are difficult to find. They also glance at the use of polar varieties, developed by Lê and Teissier, for studying the topology of singularities. This springs from ideas by René Thom and relates to the subject mentioned above, of studying singular varieties by slicing them by the fibers of a linear form. The chapter includes a proof of the "attaching-handles" theorem, which is key for Lê–Perron and Massey's theory describing the topology of the Milnor fiber. It also discusses the so-called carousel that allows a deeper understanding of the topology of plane curves (as in Chap. 1) and has several applications in various settings. Finally, two classical open problems in complex dimension two are discussed: Lê's conjecture and the Lê–Ramanujam problem.

Deformation theory, together with the resolution of singularities and stratifications, is one of the fundamental methods for the investigation of singularities. In Chap. 7, entitled, "Deformation and Smoothing of Singularities," Gert-Martin Greuel gives a comprehensive survey of the theory of deformations of isolated singularities and the related question of smoothability. The basic general theory is systematically and carefully presented and the state of the art corresponding to the most important questions is exhaustively discussed. The article contains almost no proofs, but references to the relevant literature, in particular to the textbook of Greuel, Lossen, and Shustin "Introduction to Singularities and Deformations." As in this book, there are some examples treated with Singular, a computer algebra system for polynomial computations. Relations are given between different invariants, such as the Milnor number, the Tjurina number, and the dimension of a smoothing component.

Chapter 8, by Wolfgang Ebeling, gives an introduction to "Distinguished Bases and Monodromy of Complex Hypersurface Singularities," a fundamental topic for understanding the Milnor fibration. The Milnor fibration essentially is a fiber bundle over the circle S^1. Therefore, it is determined by the fiber and by the monodromy map: if we think of S^1 as being obtained from the interval [0, 1] by gluing its end points, then the (geometric) monodromy is a diffeomorphism from the fiber over $\{0\}$ to that over $\{1\}$, telling us how to glue the fibers in order to recover the original bundle. In the isolated singularity case, the fiber F_t (which is the local noncritical level) has the homotopy type of a bouquet of spheres of middle dimension n; the number of such spheres is the aforementioned Milnor number μ. Hence all reduced homology groups of F_t vanish, except $H_n(F)$ which is free abelian of rank μ. The elements in $H_n(F)$ are called vanishing cycles. The geometric monodromy induces an automorphism of $H_n(F)$, known as the monodromy of the map germ f. A natural way to study the monodromy operator is by finding "good" bases for $H_n(F; \mathbb{Z}) \cong \mathbb{Z}^{\mu}$. Such a concept was made precise by Gabrielov in the 1970s, introducing the notion of "distinguished bases." These fundamental concepts and their further developments are discussed in Chap. 8.

One of the basic problems of algebraic geometry is to extract topological information from the equations which define an algebraic variety. The theorem of Lefschetz for hyperplane sections shows that when the base field is the field of complex numbers and the projective variety is non-singular, one can, to some extent, compare the topology of a given projective variety with that of a hyperplane section. In Chap. 9, "Lefschetz Theorem for Hyperplane Sections," by Helmut Hamm and Lê Dũng Tráng, the authors consider different theorems of Lefschetz type. The chapter begins with the classical Lefschetz hyperplane sections theorem on a non-singular projective variety. Then they show that this extends to the cases of a non-singular quasi-projective variety and to singular varieties. They also consider local forms of theorems of Lefschetz type.

As mentioned earlier in this introduction in relation with Chap. 2, Felix Klein studied the action of the finite subgroups G of SU(2) on the complex space $\mathbb{C}^2$ that give rise to the surface singularities $\mathbb{C}^2/G$, which are known nowadays as Klein singularities. Later, in the 1930s, P. Du Val investigated these singularities and proved that the dual graph of their minimal resolution is exactly the Dynkin diagrams of type A_n, D_n, E_6, E_7, and E_8, corresponding to the cyclic groups, the binary dihedral groups, and the binary groups of motions of the tetrahedron, the octahedron, and the icosahedron. This was the first relation found between Kleinian singularities and the simple Lie algebras of type ADE. A natural question was whether this was a coincidence or there was a direct relation between them. Years later, in the 1960s, Brieskorn proved the existence of simultaneous resolutions for Kleinian singularities. After reading Brieskorn's work, Grothendieck conjectured that Kleinian singularities can be obtained from the corresponding simple Lie algebra of type A, D, or E, intersecting its nilpotent variety with a slice transverse to the orbit of a subregular element. The proof of Grothendieck's conjecture was announced by Brieskorn at the ICM in Nice 1970, with a sketch of the proof. In 1976, H. Esnault gave in her PhD thesis a complete proof of this theorem, following Grothendieck's initial ideas. Chapter 10 by José Luis Cisneros Molina and Meral Tosun discusses Brieskorn's theorem and a generalization of this for simple elliptic singularities which are non-hypersurface complete intersections. The chapter gives all the ingredients one needs to understand this beautiful piece of work. It discusses also several more recent developments and related topics, as the McKay correspondence, which describes how to obtain the Dynkin diagrams of type ADE from the irreducible representations of the corresponding finite subgroups of SU(2), giving a one-to-one correspondence between the nontrivial irreducible representations of the group and the components of the exceptional set of the minimal resolution of the associated Kleinian singularity.

So we see that the individual chapters cover a wide range of topics in singularity theory, and at the same time, they are linked to each other in fundamental ways.

Cuernavaca, Mexico — José Luis Cisneros Molina
Marseille, France — Dũng Tráng Lê
Mexico City, Mexico — José Seade
March 2020

Contents

Contributors

José Luis Cisneros Molina Unidad Cuernavaca, Instituto de Matemáticas, Universidad Nacional Autónoma de México, Cuernavaca, Mexico

Unidad Mixta Internacional 2001 CNRS, Laboratorio Solomon Lefschetz, Cuernavaca, Mexico
jlcisneros@im.unam.mx

Wolfgang Ebeling Institut für Algebraische Geometrie, Leibniz Universität Hannover, Hannover, Germany
ebeling@math.uni-hannover.de

Evelia R. García Barroso Departamento de Matemáticas, Estadística e I. O. Sección de Matemáticas, Universidad de La Laguna, Tenerife, Spain
ergarcia@ull.es

Pedro D. González Pérez Instituto de Matemática Interdisciplinar y Departamento de Álgebra, Geometría y Topología, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, Madrid, Spain
pgonzalez@mat.ucm.es

Mark Goresky School of Mathematics, Institute for Advanced Study, Princeton, NJ, USA
goresky@ias.edu

Gert-Martin Greuel Fachbereich Mathematik, Universität Kaiserslautern, Kaiserslautern, Germany
greuel@mathematik.uni-kl.de

Françoise Michel Laboratoire de Mathématiques Emile Picard, Université Paul Sabatier, Toulouse, France
fmichel@picard.ups-tlse.fr

Juan José Nuño Ballesteros Departament de Geometria i Topologia, Facultat de Matemàtiques, Universitat de València, València, Spain
Juan.Nuno@uv.es

Patrick Popescu-Pampu Laboratoire Paul Painlevé, Univ. Lille, CNRS, UMR 8524, Lille, France
patrick.popescu-pampu@univ-lille.fr

José Seade Instituto de Matemáticas, Universidad Nacional Autónoma de México, Mexico City, Mexico

Unidad Mixta Internacional 2001 CNRS, Laboratorio Solomon Lefschetz, Cuernavaca, Mexico
jseade@im.unam.mx

Mark Spivakovsky Institut de Mathématiques de Toulouse, UMR CNRS 5219, Toulouse, Cedex, France

Unidad Mixta Internacional 2001 CNRS, Laboratorio Solomon Lefschetz, Cuernavaca, Mexico

Unidad Cuernavaca, Instituto de Matemáticas, UNAM, Cuernavaca, Mexico
mark.spivakovsky@math.univ-toulouse.fr

Bernard Teissier Institut de Mathématiques de Jussieu-Paris Rive Gauche Bât. Sophie Germain, Paris, Cedex 13, France
bernard.teissier@imj-prg.fr

Meral Tosun Galatasaray University, Çirağan Caddesi, Ortaköy, Istanbul, Turkey
mtosun@gsu.edu.tr

Dũng Tráng Lê Centre de Mathématiques et Informatique, Université d'Aix-Marseille, Marseille, France
ledt@ictp.it

David Trotman Université d'Aix-Marseille, CNRS, Centrale Marseille, Institut Mathématique de Marseille, Marseille, France
david.trotman@univ-amu.fr

Chapter 1
The Combinatorics of Plane Curve Singularities

How Newton Polygons Blossom into Lotuses

Evelia R. García Barroso, Pedro D. González Pérez, and Patrick Popescu-Pampu

This paper is dedicated to Bernard Teissier for his 75th birthday.

Contents

The original version of the chapter has been revised. A correction to this chapter can be found at https://doi.org/10.1007/978-3-030-53061-7_11

E. R. García Barroso
Departamento de Matemáticas, Estadística e I.O. Sección de Matemáticas, Universidad de La Laguna, La Laguna, Tenerife, España
e-mail: ergarcia@ull.es

P. D. González Pérez
Instituto de Matemática Interdisciplinar y Departamento de Álgebra, Geometría y Topología, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, Madrid, España
e-mail: pgonzalez@mat.ucm.es

P. Popescu-Pampu (✉)
Laboratoire Paul Painlevé, Univ. Lille, CNRS, UMR 8524, Lille, France
e-mail: patrick.popescu-pampu@univ-lille.fr

J. L. Cisneros Molina et al. (eds.), *Handbook of Geometry and Topology of Singularities I*, https://doi.org/10.1007/978-3-030-53061-7_1

Abstract This survey may be seen as an introduction to the use of toric and tropical geometry in the analysis of *plane curve singularities*, which are germs (C, o) of complex analytic curves contained in a smooth complex analytic surface S. The *embedded topological type* of such a pair (S, C) is usually defined to be that of the oriented link obtained by intersecting C with a sufficiently small oriented Euclidean sphere centered at the point o, defined once a system of local coordinates (x, y) was chosen on the germ (S, o). If one works more generally over an arbitrary algebraically closed field of characteristic zero, one speaks instead of the *combinatorial type* of (S, C). One may define it by looking either at the Newton-Puiseux series associated to C relative to a generic local coordinate system (x, y), or at the set of infinitely near points which have to be blown up in order to get the minimal embedded resolution of the germ (C, o) or, thirdly, at the preimage of this germ by the resolution. Each point of view leads to a different encoding of the combinatorial type by a decorated tree: an *Eggers-Wall tree*, an *Enriques diagram*, or a *weighted dual graph*. The three trees contain the same information, which in the complex setting is equivalent to the knowledge of the embedded topological type. There are known algorithms for transforming one tree into another. In this paper we explain how a special type of two-dimensional simplicial complex called a *lotus* allows to think geometrically about the relations between the three types of trees. Namely, all of them embed in a natural lotus, their numerical decorations appearing as invariants of it. This lotus is constructed from the finite set of Newton polygons created during any process of resolution of (C, o) by successive toric modifications.

1.1 Introduction

The aim of this paper is to unify various combinatorial objects classically used to encode the equisingularity/combinatorial/embedded topological type of a plane curve singularity. Often, a *plane curve singularity* means a germ (C, o) of algebraic or holomorphic curve defined by one equation in a smooth complex algebraic surface. In this paper we will allow the ambient surface to be any germ (S, o) of smooth complex algebraic or analytic surface, and C to be a formal germ of curve. Using a local formal coordinate system (x, y) on the germ (S, o), the global structure of S disappears completely and one may suppose that C is formally embedded in the affine plane $\mathbb{C}^2$. Usually one analyses in the following ways the structure of this embedding:

- By considering the *Newton-Puiseux series* which express one of the variables (x, y) in terms of the other, whenever the equation $f(x, y) = 0$ defining C is satisfied. Their combinatorics may be encoded in two rooted trees, the *Kuo-Lu tree* and a Galois quotient of it, the *Eggers-Wall tree*.
- By blowing up points starting from $o \in S$, until obtaining an embedded resolution of C, that is, a total transform of C which is a divisor with normal crossings. This blow up process may be encoded in an *Enriques diagram*, and the final total transform of C in a *weighted dual graph*.
- When the singularity C is holomorphic, by intersecting a representative of C with a small enough Euclidean sphere centered at the origin, defined using an arbitrary holomorphic local coordinate system (x, y) on (S, o). This leads to an oriented link in an oriented 3-dimensional sphere. This link is an *iterated torus link*, whose structure may be encoded in terms of another tree, called a *splice diagram*.

Unlike the first two procedures, the third one cannot be applied if the formal germ C is not holomorphic or if one works over an arbitrary algebraically closed field of characteristic zero. For this reason, we do not develop it in this paper. Let us mention only that it was initiated in Brauner's pioneering paper [13], whose historical background was described by Epple in [36]. For its developments, one may consult chronologically Reeve [107], Lê [80], A'Campo [5], Eisenbud & Neumann [34, Appendix to Chap. I], Schrauwen [110], Lê [81], Wall [131, Chap. 9], Weber [132] and the present authors [46, Chap. 5]. Similarly, we will not consider the discrete invariants constructed usually using the topology of the Milnor fibration of a holomorphic germ f, as Milnor numbers, Seifert forms, monodromy operators and their Zeta functions. The readers interested in such invariants may consult the textbooks [15] of Brieskorn and Knörrer and [131] of Wall.

There are algorithms allowing to pass between the Eggers-Wall tree, the dual graph and the Enriques diagram of C. However, they do not allow geometric representations of those passages. Our aim is to represent all these relationships using a single geometric object, called a *lotus*, which is a special type of simplicial complex of dimension at most two.

Our approach for associating lotuses to plane curve singularities is done in the spirit of the papers of Lê & Oka [83], A'Campo & Oka [8], Oka [93], González Pérez [52, Section 3.4], and Cassou Noguès & Libgober [21]. Namely, we use the fact that one may obtain an embedded resolution of C by composing a sequence of *toric* modifications determined by the successive Newton polygons of C or of strict transforms of it, relative to suitable local coordinate systems.

One may construct a lotus using the previous Newton polygons (see Definition 1.5.26). Its one dimensional skeleton may be seen as a dual complex representing the space-time of the evolution of the dual graph during the process of blow ups of points which leads to the embedded resolution. Besides the irreducible components of C and the components of the exceptional divisor, one takes also into account the curves defined by the chosen local coordinate systems. If A and B are two such exceptional or coordinate curves, and them or their strict transforms intersect transversally at a point p which is blown up at some moment of the process, then a two dimensional simplex with vertices labeled by A, B and the exceptional divisor of the blow up of p belongs to the lotus. These simplices are called the *petals* of the lotus (see an example of a lotus with 18 petals in Fig. 1.1). The Eggers-Wall tree, the Enriques diagram and the weighted dual graph embed simultaneously inside the lotus, and the geometry of the lotus also captures the numerical decorations of the weighted dual graph and the Eggers-Wall tree (see Theorem 1.5.29). For instance, the self-intersection number of a component of the final exceptional divisor is the opposite of the number of petals containing the associated vertex of the lotus. The previous lotuses associated to C have also valuative interpretations: they embed canonically in the space of semivaluations of the completed local ring of the germ (S, o) (see Remark 1.5.34).

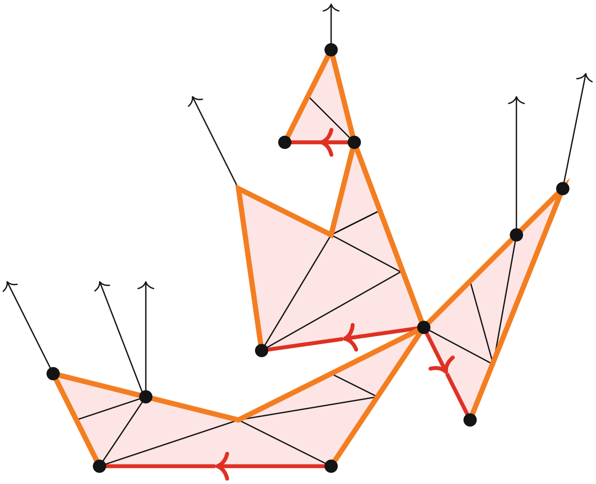


Fig. 1.1 A lotus. It is part of Fig. 1.36, which corresponds to Example 1.5.28

Let us describe the structure of the paper.

In Sect. 1.2 we introduce basic notions about *complex analytic varieties*, *plane curve singularities*, their *multiplicities* and *intersection numbers*, *normalizations*, *Newton-Puiseux series*, *blow ups*, *embedded resolutions of plane curve singularities* and the associated *weighted dual graphs*. The notions of *Newton polygon*, dual *Newton fan* and *lotus* are first presented here on a *Newton non-degenerate* example.

Section 1.3 begins with an explanation of basic notions of *toric geometry*: *fans* and their subdivisions, the associated *toric varieties* and *toric modifications* (see Sects. 1.3.1, 1.3.2 and 1.3.3). In particular, we describe the *toric boundary* of a toric variety—the reduced divisor obtained as the complement of its dense torus—in terms of the associated fan. Then we pass to *toroidal geometry*: we introduce *toroidal varieties*, which are pairs $(\Sigma, \partial\Sigma)$ consisting of a normal complex analytic variety Σ and a reduced divisor $\partial\Sigma$ on it, which are locally analytically isomorphic to a germ of a pair formed by a toric variety and its *boundary divisor*. A basic example of toroidal surface is that of a germ (S, o) of smooth surface, endowed with the divisor $L + L'$, where (L, L') is a *cross*, that is, a pair of smooth transversal germs of curves. A *morphism* $\phi : (\Sigma_2, \partial\Sigma_2) \to (\Sigma_1, \partial\Sigma_1)$ *of toroidal varieties* is a complex analytic morphism such that $\phi^{-1}(\partial\Sigma_1) \subseteq \partial\Sigma_2$ (see Sect. 1.3.4).

In Sect. 1.4 we explain in which way one may associate various morphisms of toroidal surfaces to the plane curve singularity $C \hookrightarrow S$. First, choose a cross (L, L') on (S, o), defined by a local coordinate system (x, y). The *Newton polygon* $\mathcal{N}(f)$ of a defining function $f \in \mathbb{C}[[x, y]]$ of the curve singularity C depends only on C and on the cross (L, L'). Its associated *Newton fan* is obtained by subdividing the first quadrant along the rays orthogonal to the compact edges of the Newton polygon. This fan defines a toric modification of S, the *Newton modification of S defined by C relative to the cross* (L, L') (see Sect. 1.4.1). The Newton modification becomes a toroidal morphism when we endow its target S with the boundary divisor $\partial S := L + L'$ and we define the boundary divisor of its source to be the preimage of $L + L'$. We emphasize the fact that those notions depend only on the objects $(S, C, (L, L'))$, in order to insist on the underlying geometric structures. The strict transform of C by the previous Newton modification intersects the boundary divisor only at smooth points of it, which belong to the exceptional divisor and are smooth points of the ambient surface. If one completes the germ of exceptional divisor into a cross at each such point o_i, then one gets again a triple of the form (*surface*, *curve*, *cross*), where this time the curve is the germ at o_i of the strict transform of C. Therefore one may perform again a Newton modification at each such point, and continue in this way until the strict transform of C defines everywhere crosses with the exceptional divisor. The total transform of C and of all coordinate curves introduced during previous steps define the toroidal boundary $\partial\Sigma$ on the final surface Σ. This non-deterministic algorithm produces morphisms $\pi : (\Sigma, \partial\Sigma) \to (S, \partial S)$ of toroidal surfaces, which are *toroidal pseudo-resolutions* of the plane curve singularity C (see Sect. 1.4.2). The surface Σ has a finite number of singular points, at which it is locally analytically isomorphic to normal toric surfaces. In Sect. 1.4.3 we show how to pass from the toroidal pseudo-resolution π to a *toroidal embedded resolution* by composing π with the minimal resolution of these toric singularities. Finally, we encode the process of successive Newton modifications in a *fan tree*, in terms of the Newton fans produced by the pseudo-resolution process (see Sect. 1.4.4).

In Sect. 1.5 we explain the notion of lotus. A *Newton lotus* associated to a fan encodes geometrically the continued fraction expansions of the slopes of the rays of the fan, as well as their common parts (see Sect. 1.5.2). It is composed of *petals*, and each petal corresponds to the blow up of the base point of a cross. One may clarify the subtitle of the paper by saying that *the collection of Newton polygons appearing during the toroidal pseudo-resolution process blossomed into the associated lotus, each petal corresponding to a blow up operation*. We explain how to associate to the fan tree of the toroidal pseudo-resolution a *lotus*, which is a 2-dimensional simplicial complex obtained by gluing the *Newton lotuses* associated to the Newton fans of the process (see Sects. 1.5.1 and 1.5.3). The lotus of a toroidal pseudo-resolution depends on the choices of crosses made during the process of pseudo-resolution (see Sect. 1.5.4). We explain then how to embed in the lotus the Enriques diagram and the dual graph of the embedded resolution. We conclude the section by defining a *truncation operation* on lotuses, and we explain how it may be used to understand the part of the embedded resolution which does not depend on the supplementary curves introduced during the pseudo-resolution process (see Sect. 1.5.5).

We begin Sect. 1.6 by introducing the notion of *Eggers-Wall tree* of the curve C relative to the smooth germ L (see Sect. 1.6.1) and by expressing the Newton polygon of C relative to a cross (L, L') in terms of the Eggers-Wall tree of $C + L'$ relative to L (see Sect. 1.6.2). Then we explain that the fan tree of the previous toroidal pseudo-resolution process is canonically isomorphic to the Eggers-Wall tree relative to L of the curve obtained by adding to C the projections to S of all the crosses built during the process and how to pass from the numerical decorations of the fan tree to those of the Eggers-Wall tree (see Sect. 1.6.5). As preliminary results, we prove *renormalization formulae* which describe the Eggers-Wall tree of the strict transform of C by a Newton modification, relative to the exceptional divisor, in terms of the Eggers-Wall tree of C relative to L (see Sects. 1.6.3 and 1.6.4).

The final Sect. 1.7 begins by an overview of the construction of a fan tree and of the associated lotus from the Newton fans of a toroidal pseudo-resolution process (see Sect. 1.7.1). Section 1.7.2 describes perspectives on possible applications of lotuses to problems of singularity theory. The final Sect. 1.7.3 contains a list of the main notations used in the article.

Starting from Sect. 1.3, each section ends with a subsection of historical comments. We apologize for any omission, which may result from our limited knowledge. One may also find historical information about various tools used to study plane curve singularities in Enriques and Chisini's book [35], in the first chapter of Zariski's book [134] and in the final sections of the chapters of Wall's book [131].

We tried to make this paper understandable to PhD students who have only a basic knowledge about singularities. Even if everything in this paper holds over an arbitrary algebraically closed field of characteristic zero, we will stick to the complex setting, in order to make things more concrete for the beginner. We accompany the definitions with examples and many figures. Indeed, one of our objectives is to show that lotuses may be a great visual tool for relating the combinatorial objects used to study plane curve singularities. There is a main example, developed throughout the paper starting from Sect. 1.4 (see Examples 1.4.28, 1.4.34, 1.4.36, 1.5.28, 1.5.31, 1.5.36, 1.6.29 and the overview Fig. 1.58). We recommend to study it carefully in order to get a concrete feeling of the various objects manipulated in this paper. We also recommend to those readers who are learning the subject to refer to the Sect. 1.7.1 from time to time, in order to measure their understanding of the geometrical objects presented here.

1.2 Basic Notions and Examples

In this section we recall basic notions about *complex varieties* and *plane curve singularities* (see Sect. 1.2.1), *normalization morphisms* (see Sect. 1.2.2), the relation between *Newton-Puiseux series* and plane curve singularities (see Sect. 1.2.3) and *resolution of such singularities* by iteration of *blow ups of points* (see Sect. 1.2.4). We describe such a resolution for the *semi-cubical parabola* (see Sect. 1.2.5). We give a flavor of the main construction of this paper in Sect. 1.2.6. We show there how to transform the Newton polygon of a certain *Newton non-degenerate* plane curve singularity with two branches into a *lotus*, and how this lotus contains the dual graph of a resolution by blow ups of points.

From now on, $\boxed{\mathbb{N}}$ denotes the set of non-negative integers and $\boxed{\mathbb{N}^*}$ the set of positive integers.

1.2.1 Basic Facts About Plane Curve Singularities

In this subsection we recall basic vocabulary about *complex analytic spaces* (see Definition 1.2.1) and we explain the notions of *plane curve singularity* (see Definition 1.2.5), of *multiplicity* and of *intersection number* (see Definition 1.2.7) for such singularities. Finally, we recall an important way of computing such intersection numbers (see Proposition 1.2.8).

Briefly speaking, a complex analytic space X is obtained by gluing model spaces, which are zero-loci of systems of analytic equations in some complex affine space $\mathbb{C}^n$. One has to prescribe also the analytic “functions” living on the underlying topological space. Those "functions" are elements of a so-called "structure sheaf" $\mathcal{O}_X$, which may contain nilpotent elements. For this reason, they are not classical functions, as they are not determined by their values. For instance, one may endow the origin of $\mathbb{C}$ with the structure sheaves whose rings of sections are the various rings $\mathbb{C}[x]/(x^m)$, with $m \in \mathbb{N}^*$. They are pairwise non-isomorphic and they contain nilpotent elements whenever $m \geq 2$. Let us state now the formal definitions of *complex analytic spaces* and of some special types of complex analytic spaces.

Definition 1.2.1

- A **model complex analytic space** is a ringed space $(X, \mathcal{O}_X)$, where X is the zero locus of I and $\mathcal{O}_X = \mathcal{O}_U/\mathcal{I}$. Here I is a finitely generated ideal of the ring of holomorphic functions on an open set U of $\mathbb{C}^n$, for some $n \in \mathbb{N}^*$, $\mathcal{O}_U$ is the sheaf of holomorphic functions on U and $\mathcal{I}$ is the sheaf of ideals of $\mathcal{O}_U$ generated by I.
- A **complex analytic space** is a ringed space locally isomorphic to a model complex analytic space.
- A complex analytic space is **reduced** if its structure sheaf $\mathcal{O}_X$ is reduced, that is, without nilpotent elements. In this case, one speaks also about a **complex variety**.
- A **complex manifold** is a complex variety X such that any point $x \in X$ has a neighborhood isomorphic to an open set of $\mathbb{C}^n$, for some $n \in \mathbb{N}$. If the non-negative integer n is independent of x, then the complex manifold X is called **equidimensional** and n is its **complex dimension**.
- The **smooth locus** of a complex variety X is its open subspace whose points have neighborhoods which are complex manifolds. Its **singular locus** $\boxed{\mathrm{Sing}(X)}$ is the complement of its smooth locus.
- A **smooth complex curve** is an equidimensional complex manifold of complex dimension one and a **smooth complex surface** is an equidimensional complex manifold of complex dimension two.
- A **complex curve** is a complex variety whose smooth locus is a smooth complex curve and a **complex surface** is a complex variety whose smooth locus is a smooth complex surface.

By construction, the singular locus $\mathrm{Sing}(X)$ of X is a closed subset of X. It is a deep theorem that this subset is in fact a complex subvariety of X (see [66, Corollary 6.3.4]).

Let S be a smooth complex surface. If o is a point of S and $\phi : U \to V$ is an isomorphism from an open neighborhood U of o in S to an open neighborhood V of the origin in $\mathbb{C}^2_{x,y}$, then the coordinate holomorphic functions $x, y : \mathbb{C}^2_{x,y} \to \mathbb{C}$ may be lifted by ϕ to two holomorphic functions on U, vanishing at o. They form a **local coordinate system on the germ** $\boxed{(S, o)}$ **of** S **at** o. By abuse of notations, we still denote this local coordinate system by (x, y), and we see it as a couple of elements of $\boxed{\mathcal{O}_{S,o}}$, the **local ring of** S **at** o, equal by definition to the $\mathbb{C}$-algebra of germs of holomorphic functions defined on some neighborhood of o in S. The local coordinate system (x, y) establishes an isomorphism $\mathcal{O}_{S,o} \simeq \mathbb{C}\{x, y\}$, where $\boxed{\mathbb{C}\{x, y\}}$ denotes the $\mathbb{C}$-algebra of convergent power series in the variables x, y. Denote by $\boxed{\mathbb{C}[[x, y]]}$ the $\mathbb{C}$-algebra of formal power series in the same variables. It is the completion of $\mathbb{C}\{x, y\}$ relative to its maximal ideal $(x, y)\mathbb{C}\{x, y\}$. One has the following fundamental theorem, valid in fact for any finite number of variables (see [66, Corollary 3.3.17]):

Theorem 1.2.2 *The local rings* $\mathbb{C}\{x, y\}$ *and* $\mathbb{C}[[x, y]]$ *are factorial.*

In addition to Definition 1.2.1, we use also the following meaning of the term *curve*:

Definition 1.2.3 A **curve** C **on a smooth complex surface** S is an effective Cartier divisor of S, that is, a complex subspace of S locally definable by the vanishing of a non-zero holomorphic function.

This means that for every point $o \in C$, there exists an open neighborhood U of o in S and a holomorphic function $f : U \to \mathbb{C}$ such that $C \subset U$ is the vanishing locus $\boxed{Z(f)}$ of f and such that the structure sheaf $\boxed{\mathcal{O}_{C|U}}$ of $C \subset U$ is the quotient sheaf $\mathcal{O}_U/(f)\mathcal{O}_U$. In this case, once U is fixed, the defining function f is unique up to multiplication by a holomorphic function on U which vanishes nowhere.

The curve C is called **reduced** if it is a reduced complex analytic space in the sense of Definition 1.2.1. This means that any defining function $f : U \to \mathbb{C}$ as above is square-free in all local rings $\mathcal{O}_{S,o}$, where $o \in U$. For instance, the union C of coordinate axes of $\mathbb{C}^2$ is a reduced curve, being definable by the function xy, which is square-free in all the local rings $\mathcal{O}_{\mathbb{C}^2,o}$, where $o \in C$. By contrast, the curve D defined by the function xy^2 is not reduced.

As results from Definition 1.2.3, a complex subspace C of S is a curve on S if and only if, for any $o \in C$, the ideal of $\mathcal{O}_{S,o}$ consisting of the germs of holomorphic functions vanishing on the germ (C, o) of C at o is *principal*. We would have obtained a more general notion of *curve* if we would have asked C to be a 1-dimensional complex subspace of S in the neighborhood of any of its points. For instance, if $S = \mathbb{C}^2_{x,y}$, and C is defined by the ideal (x^2, xy) of $\mathbb{C}[x, y]$, then set-theoretically C coincides with the y-axis $Z(x)$. But the associated structure sheaf $\mathcal{O}_{\mathbb{C}^2}/(x^2, xy)\mathcal{O}_{\mathbb{C}^2}$ is not the structure sheaf of an effective Cartier divisor. In fact the germ of C at the origin cannot be defined by only one holomorphic function $f(x, y) \in \mathbb{C}\{x, y\}$. Otherwise, we would get that both x^2 and xy are divisible by $f(x, y)$ in the local ring $\mathbb{C}\{x, y\}$. As this ring is factorial by Theorem 1.2.2, we see that f divides x inside this ring, which implies that $(f)\mathbb{C}\{x, y\} = (x)\mathbb{C}\{x, y\}$. Therefore, $(x^2, xy)\mathbb{C}\{x, y\} = (x)\mathbb{C}\{x, y\}$ which is a contradiction, as x is of order 1 and each element of the ideal $(x^2, xy)\mathbb{C}\{x, y\}$ is of order at least 2. The notion of *order* used in the previous sentence is defined by:

Definition 1.2.4 Let $f \in \mathbb{C}[[x, y]]$. Its **order** is the smallest degree of its terms.

For instance, the maximal ideal of $\mathbb{C}[[x, y]]$ consists precisely of the power series of order at least 1. It is a basic exercise to show that the order is invariant by the automorphisms of the $\mathbb{C}$-algebra $\mathbb{C}[[x, y]]$ and by multiplication by the elements of order 0, which are the units of this algebra. Therefore, one gets a well-defined notion of *multiplicity* of a germ of *formal curve* on S:

Definition 1.2.5 A **plane curve singularity** is a germ C of formal curve on a germ of smooth complex surface (S, o), that is, a principal ideal in the completion $\boxed{\hat{O}_{S,o}}$ of the local ring $O_{S,o}$. It is called a **branch** if it is irreducible, that is, if its defining functions are irreducible elements of the factorial local ring $\hat{O}_{S,o}$. The **multiplicity** $\boxed{m_o(C)}$ of C at o is the order of a **defining function** $f \in \hat{O}_{S,o}$ of C, seen as an element of $\mathbb{C}[[x, y]]$ using any local coordinate system (x, y) of the germ (S, o).

Example 1.2.6 Let $\alpha, \beta \in \mathbb{N}^*$ and $f := x^\alpha - y^\beta \in \mathbb{C}[x, y]$. Denote by C the curve on $\mathbb{C}^2$ defined by f. Its multiplicity at the origin O of $\mathbb{C}^2$ is the minimum of α and β. The curve singularity (C, O) is a branch if and only if α and β are coprime. One implication is easy: if α and β have a common factor $\rho > 1$, then $x^\alpha - y^\beta = \prod_{\omega:\, \omega^\rho=1} \left(x^{\alpha/\rho} - \omega y^{\beta/\rho}\right)$, the product being taken over all the complex ρ-th roots ω of 1, which shows that (C, O) is not a branch. The reverse implication results from the fact that, whenever α and β are coprime, C is the image of the parametrization $N(t) := (t^\beta, t^\alpha)$. The inclusion $N(\mathbb{C}) \subseteq C$ being obvious, let us prove the reverse inclusion. Let $(x, y) \in C$. As $N(0) = O$, it is enough to consider the case where $xy \neq 0$. We want to show that there exists $t \in \mathbb{C}^*$ such that $x = t^\beta,\ y = t^\alpha$. Assume the problem solved and consider also a pair $(a, b) \in \mathbb{Z}^2$ such that $a\alpha + b\beta = 1$, which exists by Bezout's theorem. One gets $t = t^{a\alpha+b\beta} = y^a x^b$. Define therefore $t := y^a x^b$. Then:

$$t^\beta = (y^a x^b)^\beta = (y^\beta)^a x^{b\beta} = (x^\alpha)^a x^{b\beta} = x^{a\alpha+b\beta} = x,$$

and similarly one shows that $t^\alpha = y$. This proves that C is indeed included in the image of N.

Let C be a plane curve singularity on the germ of smooth surface (S, o). If $f \in \hat{O}_{S,o}$ is a defining function of C, it may be decomposed as a product:

$$f = \prod_{i \in I} f_i^{p_i}, \tag{1.1}$$

in which the functions f_i are pairwise non-associated prime elements of the local ring $\hat{O}_{S,o}$ and $p_i \in \mathbb{N}^*$ for every $i \in I$. Such a decomposition is unique up to permutation of the factors $f_i^{p_i}$ and up to a replacement of each function f_i by an associated one (recall that two such functions are *associated* if one is the product of another one by a unit of the local ring). If $C_i \subseteq S$ is the plane curve singularity defined by f_i, then the decomposition (1.1) gives a decomposition of C seen as a germ of effective divisor $C = \sum_{i \in I} p_i C_i$, where each curve singularity C_i is a branch. The plane curve singularity C is reduced if and only if $p_i = 1$ for every $i \in I$.

The *intersection number* is the simplest measure of complexity of the way two plane curve singularities interact at a given point:

Definition 1.2.7 Let C and D be two curve singularities on the germ of smooth surface (S, o) defined by functions f and $g \in \hat{O}_{S,o}$ respectively. Their **intersection number** $\boxed{(C \cdot D)_o}$, also denoted $\boxed{C \cdot D}$ if the base point o of the germ is clear from the context, is defined by:

$$C \cdot D := \dim_{\mathbb{C}} \frac{\hat{O}_{S,o}}{(f, g)} \in \mathbb{N} \cup \{\infty\},$$

where (f, g) denotes the ideal of $\hat{O}_{S,o}$ generated by f and g.

If C and D are two curve singularities, then one has that $(C \cdot D)_o \geq m_o(C) m_o(D)$, with equality if and only if the curves C and D are **transversal** (see [131, Lemma 4.4.1]), that is, the tangent plane of (S, o) does not contain lines which are tangent to both C and D.

Seen as a function of two variables, the intersection number is symmetric. It is moreover bilinear, in the sense that if $C = \sum_{i \in I} p_i C_i$, then $C \cdot D = \sum_{i \in I} p_i (C_i \cdot D)$. Therefore, in order to compute $C \cdot D$, it is enough to find $C_i \cdot D$ for all the branches C_i of C.

One has the following useful property (see [66, Lemma 5.1.5]):

Proposition 1.2.8 *Let C be a branch and D be an arbitrary curve singularity on the smooth germ of smooth surface (S, o). Denote by $N : (\mathbb{C}_t, 0) \to (S, o)$ a formal parametrization of degree one of C and $g \in \hat{O}_{S,o}$ be a defining function of D. Then*

$$C \cdot D = \nu_t(g(N(t))),$$

where $\nu_t(h)$ denotes the order of a power series $h \in \mathbb{C}[[t]]$.

Example 1.2.9 Let us consider two curves $C, D \subseteq \mathbb{C}^2_{x,y}$, defined by polynomials $f := x^\alpha - y^\beta$ and $g := x^\gamma - y^\delta$ of the type already considered in Example 1.2.6. Assume that α and β are coprime. This implies, as shown in Example 1.2.6, that the plane curve singularity (C, O) is a branch and that $N(t) := (t^\beta, t^\alpha)$ is a parametrization of degree one of it. By Proposition 1.2.8, if C is not a branch of D, we get:

$$C \cdot D = \nu_t \left((t^\beta)^\gamma - (t^\alpha)^\delta \right) = \nu_t \left(t^{\beta\gamma} - t^{\alpha\delta} \right) = \min\{\beta\gamma, \alpha\delta\}.$$

For more details about intersection numbers of plane curve singularities, one may consult [15, Sect. 6], [113, Vol. 1, Chap. IV.1] and [39, Chap. 8].

The formal parametrizations $N : (\mathbb{C}_t, 0) \to (S, o)$ of degree one of a branch appearing in the statement of Proposition 1.2.8 are exactly the *normalization morphisms of* C whose sources are identified with $(\mathbb{C}, 0)$. Next subsection is dedicated to the general definition of *normal complex variety* and of *normalization morphism* in arbitrary dimension, as we will need them later also for surfaces.

1.2.2 Basic Facts About Normalizations

In this subsection we explain basic facts about *normal rings* (see Definition 1.2.10), *normal complex varieties* (see Definition 1.2.11) and *normalization morphisms* (see Definition 1.2.16) of arbitrary complex varieties. For more details and proofs one may consult [66, Sections 1.5, 4.4] and [58].

The following definition contains *algebraic* notions, concerning extensions of rings:

Definition 1.2.10 Let R be a commutative ring and let $R \subseteq T$ be an extension of R.

1. An element of T is called **integral over** R if it satisfies a monic polynomial relation with coefficients in R.
2. The extension $R \subseteq T$ of R is called **integral** if each element of T is integral over R.
3. The **integral closure** of R is the set of integral elements over R of the total ring of fractions of R.
4. R is called **normal** if it is reduced (without nonzero nilpotent elements) and integrally closed in its total ring of fractions, that is, if it coincides with its integral closure.

The arithmetical notion of *normal ring* allows to define the geometrical notion of *normal variety*:

Definition 1.2.11 Let X be a complex variety in the sense of Definition 1.2.1.

1. If $x \in X$, then the germ (X, x) of X at x is called **normal** if its local ring $\mathcal{O}_{X,x}$ is normal.
2. The complex variety X is **normal** if all its germs are normal.

Normal varieties may be characterized from a more function-theoretical viewpoint as those complex varieties on which holds the following "Riemann extension property": *every bounded holomorphic function defined on the smooth part of an open set extends to a holomorphic function on the whole open set* (see [66, Theorem 4.4.15]).

Recall now the following algebraic regularity condition (see [66, Sect. 4.3]):

Definition 1.2.12 Let $\mathcal{O}$ be a Noetherian local ring, with maximal ideal $\mathfrak{m}$.

1. The **Krull dimension** of $\mathcal{O}$ is the maximal length of its chains of prime ideals.

2. The **embedding dimension** of $\mathcal{O}$ is the dimension of the $\mathcal{O}/\mathfrak{m}$-vector space $\mathfrak{m}/\mathfrak{m}^2$.
3. The local ring $\mathcal{O}$ is called **regular** if its Krull dimension is equal to its embedding dimension.

The Krull dimension of $\mathcal{O}$ is always less or equal to the embedding dimension. The name *embedding dimension* may be understood by restricting to the case where $\mathcal{O}$ is the local ring of a complex space (see [66, Lemma 4.3.5]):

Proposition 1.2.13 *Let (X, x) be a germ of complex space. Then the embedding dimension of its local ring $\mathcal{O}_{X,x}$ is equal to the smallest $n \in \mathbb{N}$ such that there exists an embedding of germs $(X, x) \hookrightarrow (\mathbb{C}^n, 0)$. In particular, $\mathcal{O}_{X,x}$ is regular if and only if (X, x) is smooth, that is, a germ of complex manifold.*

The normal varieties of dimension one are exactly the smooth complex curves because, more generally (see [66, Thm. 4.4.9, Cor. 4.4.10]):

Theorem 1.2.14 *A Noetherian local ring of Krull dimension one is normal if and only if it is regular.*

There is a canonical way to construct a normal variety $\tilde{X}$ starting from any complex variety X (see [66, Sect. 4.4]):

Theorem 1.2.15 *Let X be a complex variety. Then there exists a finite and generically 1 to 1 morphism $N : \tilde{X} \to X$ such that $\tilde{X}$ is normal. Moreover, such a morphism is unique up to a unique isomorphism over X.*

Recall that a morphism between complex varieties is *finite* if it is proper with finite fibers and that it is *generically* 1 *to* 1 if it is an isomorphism above the complement of a nowhere dense closed subvariety of its target space. The existence of a morphism with the properties stated in Theorem 1.2.15 may be proven algebraically by considering the integral closures of the rings of holomorphic functions on the open sets of X, and showing that they are again rings of holomorphic functions on complex varieties which admit finite and generically 1 to 1 morphisms to the starting open sets. This algebraic proof extends to formal germs, by showing that the integral closure in its total ring of fractions of a complete ring of the form $\mathbb{C}[[x_1, \dots, x_n]]/I$, where $n \in \mathbb{N}^*$ and I is an ideal of $\mathbb{C}[[x_1, \dots, x_n]]$, is a direct sum of rings of the same form.

The canonical morphisms characterized in Theorem 1.2.15 received a special name:

Definition 1.2.16 Let X be a complex variety. Then a morphism $N : \tilde{X} \to X$ is called a **normalization morphism** of X if it is finite, generically 1 to 1 and $\tilde{X}$ is a normal complex variety.

Let now (C, o) be a germ of complex variety of Krull dimension one, that is, an **abstract curve singularity**. Its normalization morphisms are of the form: $N : \bigsqcup_{i \in I} (\tilde{C}_i, o_i) \to (C, o)$, where $(C_i, o)_{i \in I}$ is the finite collection of irreducible components of (C, o), and the restriction $N_i : (\tilde{C}_i, o_i) \to (C, o)$ of N to $\tilde{C}_i$ is a normalization of (C_i, o). By Theorem 1.2.14 and Proposition 1.2.13, we see that each germ $(\tilde{C}_i, o_i)$ is smooth, that is, isomorphic to $(\mathbb{C}, 0)$. After precomposing N with such isomorphisms, we see that (C, o) admits a normalization morphism of the form $\bigsqcup_{i \in I}(\mathbb{C}, 0) \to (C, o)$. In particular, if (C, o) is irreducible, its normalization morphism is of the form $N : (\mathbb{C}, 0) \to (C, o)$. The same construction yields a *formal* parametrization when the starting germ (C, o) is formal. This is precisely a *formal parametrization of degree one* as used in the statement of Proposition 1.2.8.

1.2.3 Newton-Puiseux Series and the Newton-Puiseux Theorem

At the end of the previous subsection we explained that normalizations of irreducible germs of complex analytic or formal curves C are holomorphic or formal parametrizations $(\mathbb{C}, 0) \to C$ of degree one. In this subsection we introduce especially nice parametrizations in the case of plane branches, which lead to the notion of *Newton-Puiseux series* (see Definition 1.2.18). The *Newton-Puiseux theorem* (see Theorem 1.2.20) implies that the field of Newton-Puiseux series is algebraically closed. Another consequence of it is stated in Theorem 1.6.1 below.

Let C be a branch on the smooth germ of surface (S, o). Choose an arbitrary system of local coordinates on (S, o). If the branch C is smooth, assume moreover that the germ at o of the y-axis $Z(x)$ is different from C. This means that for any normalization morphism $N : (\mathbb{C}_t, 0) \to (C, o)$ of C, described in this coordinate system as $t \to (\xi(t), \eta(t))$, where $\xi, \eta \in (t)\mathbb{C}[[t]]$, the power series $\xi(t)$ is not identically zero. We have $\xi(t) = t^n \cdot \epsilon(t)$, where $n \in \mathbb{N}^*$ is the order of the power series $\xi(t)$ and $\epsilon(t)$ is a unit in the ring $\mathbb{C}[[t]]$. The series $\epsilon(t)$ has exactly n different n-th roots in $\mathbb{C}[[t]]$, whose constant terms are the n-th roots of $\epsilon(0)$. Pick one of them, denote it by $\epsilon^{1/n}(t)$, and set $\lambda(t) := t\epsilon^{1/n}(t)$. Therefore $\xi(t) = \lambda(t)^n$ and $\nu_t(\lambda(t)) = 1$.

Remark 1.2.17 More generally, if K is an algebraically closed field of characteristic zero, then any unit of $K[[t]]$ has all its n-th roots in $K[[t]]$. This fact is no longer true if K has positive characteristic. For instance, as a direct consequence of the binomial formula, there is no series $\epsilon(t) \in K[[t]]$ such that $\epsilon(t)^p = 1 + t$ when K is of characteristic p. For this reason, the Newton-Puiseux Theorem 1.2.20 below does not always hold in positive characteristic. For more details about the situation in positive characteristic, one may consult [97].

As $\nu_t(\lambda(t)) = 1$, we see that the morphism $(\mathbb{C}_t, 0) \to (\mathbb{C}_u, 0)$, which maps $t \to \lambda(t)$ is an isomorphism of germs of smooth curves. By composing the morphism $N : (\mathbb{C}_t, 0) \to (C, o)$ with its inverse, one gets a new normalization morphism of the form:

$$\begin{array}{rcl} (\mathbb{C}_u, 0) & \to & (C, o) \\ u & \to & (u^n, \zeta(u)) \end{array}$$

where $\zeta(u) \in \mathbb{C}[[u]]$. Therefore, if $f(x, y) \in \mathbb{C}[[x, y]]$ is a defining function of C in the local coordinate system (x, y), we have:

$$f(u^n, \zeta(u)) = 0. \tag{1.2}$$

From the equations $x = u^n$, $y = \zeta(u)$, one may deduce formally that $u = x^{1/n}$, $y = \zeta(x^{1/n})$. Equation (1.2) becomes:

$$f(x, \zeta(x^{1/n})) = 0. \tag{1.3}$$

The composition $\zeta(x^{1/n})$ is a *Newton-Puiseux series* in the following sense:

Definition 1.2.18 The $\mathbb{C}$-algebra $\boxed{\mathbb{C}[[x^{1/\mathbb{N}}]]}$ of **Newton-Puiseux series** consists of all the formal power series of the form $\eta(x^{1/n})$, where $\eta \in \mathbb{C}[[t]]$ and $n \in \mathbb{N}^*$, that is, $\mathbb{C}[[x^{1/\mathbb{N}}]] = \bigcup_{n \in \mathbb{N}^*} \mathbb{C}[[x^{1/n}]]$. Denote by $\boxed{\nu_x} : \mathbb{C}[[x^{1/\mathbb{N}}]] \to [0, \infty]$ the **order function**, which associates to every Newton-Puiseux series the smallest exponent of its terms, where $\nu_x(0) := \infty$.

The function ν_x is a *valuation* of the $\mathbb{C}$-algebra of Newton-Puiseux series, in the following sense:

Definition 1.2.19 A **valuation** on an integral $\mathbb{C}$-algebra A is a function $\nu : A \to \mathbb{R}_+ \cup \{\infty\}$ which satisfies the following conditions:

1. $\nu(fg) = \nu(f) + \nu(g)$, *for all* $f, g \in A$.
2. $\nu(f + g) \geq \min\{\nu(f), \nu(g)\}$, *for all* $f, g \in A$.
3. $\nu(\lambda) = 0$, *for all* $\lambda \in \mathbb{C}^*$.
4. $\nu(f) = \infty$ if and only if $f = 0$.

The basic importance of the ring of Newton-Puiseux series comes from the following *Newton-Puiseux theorem* (see Fischer [39, Chapter 7], Teissier [121, Section 1], [123, Sections 3–4], de Jong & Pfister [66, Section 5.1], Cutkosky [28, Section 2.1] or Greuel, Lossen and Shustin [59, Thm. I.3.3]):

Theorem 1.2.20 (Newton-Puiseux Theorem) *Any non-zero monic polynomial* $f \in \mathbb{C}[[x]][y]$ *such that* $f(0, y) = y^d$ *has* d *roots in the ring* $\mathbb{C}[[x^{1/\mathbb{N}}]]$. *As a consequence, the quotient field of the ring* $\mathbb{C}[[x^{1/\mathbb{N}}]]$ *is the algebraic closure of the quotient field of the ring* $\mathbb{C}[[x]]$.

Proof It is immediate to reduce the proof of the first sentence of the theorem to the case where f is irreducible. Assume that this is the case. By Eq. (1.3), there exists a Newton-Puiseux series $\zeta(x^{1/n})$ which is a root of f. One has necessarily $n = d$. Indeed, by the proof of Eq. (1.3), $u \to (u^n, \zeta(u))$ is a normalization of the formal branch $Z(f)$. Therefore, Proposition 1.2.8 shows that:

$$n = \nu_u(u^n) = Z(f) \cdot Z(x) = Z(x) \cdot Z(f) = \nu_y\left(f(0, y)\right) = d.$$

Consider now the product:

$$F(x, y) := \prod_{\omega : \omega^n = 1} \left(y - \zeta(\omega x^{1/n}) \right) \in \mathbb{C}[[x^{1/n}]][y].$$

It is invariant by the changes of variables $(x^{1/n}, y) \to (\omega x^{1/n}, y)$, where ω varies among the complex n-th roots of 1, which shows that $F(x, y) \in \mathbb{C}[[x]][y]$. As $\zeta(x^{1/n})$ is a root of both $f(x, y)$ and $F(x, y)$ and that $f(x, y)$ is irreducible, we see that f divides F in the ring $\mathbb{C}[[x]][y]$. Both being monic and of the same degree, we get the equality $f = F$. Therefore, all the roots of f belong to $\mathbb{C}[[x^{1/n}]]$.

The second statement of the theorem results from the first statement and from *Hensel's lemma* (see [66, Corollary 3.3.21]), which ensures that a factorisation of $f(0, y) \in \mathbb{C}[y]$ in pairwise coprime factors lifts to an analogous decomposition of $f(x, y) \in \mathbb{C}[[x]][y]$. □

The proof of Theorem 1.2.20 which we have sketched here also shows that the Galois group of the field extension associated to the ring extension $\mathbb{C}[[x]] \subset \mathbb{C}[[x^{1/n}]]$ is isomorphic to the cyclic group of n-th roots of 1, an element ω of this group acting on $\zeta(x^{1/n}) \in \mathbb{C}[[x^{1/n}]]$ replacing it by $\zeta(\omega x^{1/n})$.

Remark 1.2.21 The proof of Theorem 1.2.20 which we have sketched here also shows that the Galois group of the field extension associated to the ring extension $\mathbb{C}[[x]] \subset \mathbb{C}[[x^{1/n}]]$ is isomorphic to the cyclic group of n-th roots of 1, an element ω of this group acting on $\zeta(x^{1/n}) \in \mathbb{C}[[x^{1/n}]]$ replacing it by $\zeta(\omega x^{1/n})$.

Remark 1.2.22 Most proofs of Theorem 1.2.20 use the Newton polygon $\mathcal{N}(f)$ of f (see Definition 1.4.2 below). As explained in Sect. 1.2.5, the restrictions of f to the compact edges of $\mathcal{N}(f)$ allow to find the possible initial terms of the candidate roots $\eta(x)$ of the equation $f(x, y) = 0$ seen as an equation in the single unknown y. Such proofs proceed then by showing that all those terms may be extended to true roots inside $\mathbb{C}[[x^{1/\mathbb{N}}]]$.

Example 1.2.23 Consider coprime integers $\alpha, \beta \in \mathbb{N}^*$ and $f(x, y) := x^\alpha - y^\beta \in \mathbb{C}[[x]][y]$, as in Example 1.2.6. Then the Newton-Puiseux roots of f are the β series $\omega x^{\alpha/\beta}$, where ω varies among the complex β-th roots of 1. If ω' is another such root of 1, it acts on $\omega\, x^{\alpha/\beta}$ by sending it to $(\omega')^\alpha\, \omega\, x^{\alpha/\beta}$.

1.2.4 Blow Ups and Embedded Resolutions of Singularities

In this subsection we explain the notion of *blow up* of $\mathbb{C}^2$ at the origin (see Definition 1.2.24) and more generally of a smooth complex surface at an arbitrary point of it (see Definition 1.2.29), the notion of *embedded resolution* of a curve in a smooth surface (see Definition 1.2.33) and the fact that an embedded resolution may be achieved after a finite number of blow ups of points (see Theorem 1.2.35). We conclude by recalling the notion of *weighted dual graph* of an embedded resolution (see Definition 1.2.36) and the way to compute its weights when this resolution is constructed iteratively by blowing up points.

Look at the complex affine plane $\mathbb{C}^2_{x,y}$ as a complex vector space. Denote by $\boxed{\mathbb{P}(\mathbb{C}^2)_{[u:v]}}$ its **projectivisation**, consisting of its vector subspaces of dimension one, endowed with the projective coordinates $[u : v]$ associated to the cartesian coordinates (x, y) on $\mathbb{C}^2$.

Definition 1.2.24 Consider the **projectivisation map**

$$\begin{array}{rccc} \boxed{\lambda}: & \mathbb{C}^2 & \dashrightarrow & \mathbb{P}(\mathbb{C}^2) \\ & (x, y) & \dashrightarrow & [x : y]. \end{array}$$

associating to each point of $\mathbb{C}^2 \setminus \{O\}$ the line joining it to the origin O of $\mathbb{C}^2$. Let $\boxed{\Sigma}$ be the closure of its graph in the product algebraic variety $\mathbb{C}^2 \times \mathbb{P}(\mathbb{C}^2)$. Then the restriction $\boxed{\pi} : \Sigma \to \mathbb{C}^2$ of the first projection $\mathbb{C}^2 \times \mathbb{P}(\mathbb{C}^2) \to \mathbb{C}^2$ is called the **blow up of $\mathbb{C}^2$ at the origin**. By abuse of language, the surface Σ is also called in this way. The preimage $\pi^{-1}(O)$ of O in Σ is called the **exceptional divisor** of the blow up. The restriction $\boxed{\tilde{\lambda}} : \Sigma \to \mathbb{P}(\mathbb{C}^2)$ to Σ of the second projection $\mathbb{C}^2 \times \mathbb{P}(\mathbb{C}^2) \to \mathbb{P}(\mathbb{C}^2)$ is called the **Hopf morphism**.

The name "*Hopf morphism*" is motivated by the fact that in restriction to the preimage $\pi^{-1}(\mathbb{S}^3)$ of the unit 3-dimensional sphere $\mathbb{S}^3 \subset \mathbb{C}^2$, the morphism $\tilde{\lambda}$ becomes the "Hopf fibration" $\mathbb{S}^3 \to \mathbb{S}^2$, introduced by Hopf in [64, Section 5] (see also [109] for historical details).

The projectivisation *map* restricts to a *morphism* $\lambda : \mathbb{C}^2 \setminus \{O\} \to \mathbb{P}(\mathbb{C}^2)$. This morphism cannot be extended even by continuity to the origin O, because O belongs to the closures of all its level sets, which are the complex lines of $\mathbb{C}^2$ passing through O. Taking the closure of the graph of λ replaces O by the space $\mathbb{P}(\mathbb{C}^2)$ of lines passing through O. This allows the lift of λ to Σ to extend by continuity, and even algebraically, to the whole surface Σ, becoming the Hopf morphism $\tilde{\lambda}$. This morphism is in fact the projection morphism of the total space of a line bundle, as will be shown in Proposition 1.2.25 below. Before proving it, let us explain how to describe using a simple atlas of two charts the blow up surface Σ.

The projective line $\mathbb{P}(\mathbb{C}^2)_{[u:v]}$ is covered by the two affine lines $\mathbb{C}_{u_1}$ and $\mathbb{C}_{v_2}$, where:

$$\boxed{u_1} := \frac{u}{v}, \quad \boxed{v_2} := \frac{v}{u}.$$

Therefore, the product $\mathbb{C}^2 \times \mathbb{P}(\mathbb{C}^2)$ is covered by the two affine 3-folds $\mathbb{C}^3_{x,y,u_1}$ and $\mathbb{C}^3_{x,y,v_2}$.

The surface Σ contained in $\mathbb{C}^2 \times \mathbb{P}(\mathbb{C}^2)$ is the zero locus $Z(xv - yu)$ of a homogeneous polynomial of degree one in the variables u, v. Its intersections with the two previous 3-folds are therefore:

$$\Sigma \cap \mathbb{C}^3_{x,y,u_1} = Z(x - yu_1), \quad \text{and} \quad \Sigma \cap \mathbb{C}^3_{x,y,v_2} = Z(xv_2 - y).$$

One recognizes in each case the equation of the graph of a function of two variables, those pairs of variables being (u_1, y) and (x, v_2) respectively. Therefore, by projecting on the planes of those two pairs of variables, one gets isomorphisms:

$$\Sigma \cap \mathbb{C}^3_{x,y,u_1} \simeq \mathbb{C}^2_{u_1,y}, \quad \text{and} \quad \Sigma \cap \mathbb{C}^3_{x,y,v_2} \simeq \mathbb{C}^2_{x,v_2},$$

which may be thought as the charts of an algebraic atlas of Σ. Let us replace y by $\boxed{u_2}$ in the first chart $\mathbb{C}^2_{u_1,y}$ and x by $\boxed{v_1}$ in the second chart $\mathbb{C}^2_{x,v_2}$. The blow up morphism $\pi : \Sigma \to \mathbb{C}^2$ gets expressed in the following way in the two charts:

$$\begin{cases} x = u_1 u_2 \\ y = \quad u_2, \end{cases} \quad \text{and} \quad \begin{cases} x = v_1 \\ y = v_1 v_2. \end{cases} \tag{1.4}$$

The previous formulae show that the exceptional divisor $\pi^{-1}(O)$ becomes the u_1-axis in the chart $\mathbb{C}^2_{u_1,u_2}$ and the v_2-axis in the chart $\mathbb{C}^2_{v_1,v_2}$.

By composing one such morphism with the inverse of the second one, we see that Σ may be obtained from the two copies $\mathbb{C}^2_{u_1,u_2}$ and $\mathbb{C}^2_{v_1,v_2}$ of $\mathbb{C}^2$ by gluing their open subsets $\mathbb{C}^*_{u_1} \times \mathbb{C}_{u_2}$ and $\mathbb{C}_{v_1} \times \mathbb{C}^*_{v_2}$ respectively using the following inverse changes of variables:

$$\begin{cases} v_1 = u_1 u_2 \\ v_2 = \quad u_1^{-1} \end{cases} \quad \text{and} \quad \begin{cases} u_1 = v_2^{-1} \\ u_2 = v_1 v_2. \end{cases} \tag{1.5}$$

The Hopf morphism $\tilde{\lambda} : \Sigma \to \mathbb{P}(\mathbb{C}^2)$ becomes the morphisms $\mathbb{C}^2_{u_1,u_2} \to \mathbb{C}^1_{u_1}$ and $\mathbb{C}^2_{v_1,v_2} \to \mathbb{C}^1_{v_2}$ if one uses the charts $\mathbb{C}^2_{u_1,u_2}, \mathbb{C}^2_{v_1,v_2}$ for Σ and $\mathbb{C}^1_{u_1}, \mathbb{C}^1_{v_2}$ for $\mathbb{P}(\mathbb{C}^2)$. The fibers of these two morphisms have natural structures of complex lines if one identifies them with the standard complex line $\mathbb{C}$ using the parameters u_2 and v_1 respectively. As the gluing maps (1.5) respect those structures, we get:

Proposition 1.2.25 *The Hopf morphism* $\tilde{\lambda} : \Sigma \to \mathbb{P}(\mathbb{C}^2)$ *is the projection morphism from the total space of a line bundle to its base* $\mathbb{P}(\mathbb{C}^2)$*. Its zero-section is the exceptional divisor* $\pi^{-1}(O)$ *of the blow up morphism* $\pi : \Sigma \to \mathbb{C}^2$.

The fundamental numerical invariant of a complex line bundle over a projective curve, which characterises it up to topological isomorphims in general and up to algebraic isomorphisms if the curve is rational, is its **degree**, defined by:

Definition 1.2.26 The **degree** of a line bundle over a smooth connected projective curve C is the degree of the divisor on C defined by any meromorphic section of the line bundle which is neither constantly 0 nor constantly ∞.

In our case, we have:

Proposition 1.2.27 *The degree of the Hopf line bundle* $\tilde{\lambda} : \Sigma \to \mathbb{P}(\mathbb{C}^2)$ *is equal to* -1.

Proof Let us consider the meromorphic section s of $\tilde{\lambda}$ which appears as the constant function 1 in the charts $\mathbb{C}^2_{u_1,u_2} \to \mathbb{C}^1_{u_1}$. The equation of its graph is $u_2 = 1$. The change of variables (1.5) transform it into $v_1 v_2 = 1$. Therefore, s appears as the rational function v_2^{-1} in the charts $\mathbb{C}^2_{v_1,v_2} \to \mathbb{C}^1_{v_2}$. This shows that the section s has no zeros and a unique pole of multiplicity one. As a consequence, the degree of the divisor defined by s is equal to -1. □

On any smooth complex algebraic or analytic surface S, one may define a notion of *intersection number of two divisors* whenever at least one of them has compact support. This may be done *algebraically*, by considering first the case when one divisor is a reduced compact curve C on S, the intersection number being then the degree of the pullback of the line bundle defined by the second divisor to the normalization of C. Then, one extends this definition by linearity to arbitrary not necessarily reduced or effective divisors. There is also a *topological* definition, obtained by associating a homology class to one divisor, a cohomology class to the second one and then evaluating the cohomology class on the homology class. One may consult [61, Sect. V.1] for the case of algebraic surfaces and [76, Pages 15–20] for the case of analytic surfaces. It turns out that, either by definition or as a theorem, the self-intersection number of the zero-section of a line bundle over a smooth compact complex curve is equal to the degree of the line bundle. Therefore, Proposition 1.2.27 may be also stated as:

Corollary 1.2.28 *The self-intersection number of the zero-section of the Hopf line bundle* $\tilde{\lambda} : \Sigma \to \mathbb{P}(\mathbb{C}^2)$ *is equal to* -1.

Till now, we have discussed in this subsection only the blow up of the origin O of $\mathbb{C}^2$. This operation may be extended to any point o of a smooth complex surface S, by choosing first local coordinates (x, y) in a neighborhood U of that point. This allows to identify U with an open neighborhood of O in $\mathbb{C}^2$. Denote by $\pi_U : \Sigma_U \to U$ the restriction to U of the blow up morphism of O in $\mathbb{C}^2$. This complex analytic morphism is an isomorphism over $U \setminus \{O\}$. Therefore, it allows to glue Σ_U and S along $U \setminus \{O\}$, getting a surface $\tilde{S}$ endowed with a morphism $\tilde{\pi} : \tilde{S} \to S$.

Definition 1.2.29 The morphism $\tilde{\pi} : \tilde{S} \to S$ constructed above is called a **blow up morphism of** S **at the point** o.

It may be shown by a direct computation that the blow up morphism of S at o is independent of the choices of local coordinates and open set U. More precisely, given any two morphisms constructed in this way, there exists a unique isomorphism between their sources above S (see [131, Lemma 3.2.1]). Another way to prove this uniqueness is to characterize such morphisms by a universal property (see [61, Chap. II, Prop. 7.14]):

Proposition 1.2.30 *Let S be a smooth complex surface and $\tilde{\pi} : \tilde{S} \to S$ a blow up morphism of S at its point o. Then for any morphism $f : Y \to S$ such that the ideal sheaf defining o on S lifts to a principal ideal sheaf on Y, there exists a unique morphism $g : Y \to \tilde{S}$ such that $f = \tilde{\pi} \circ g$.*

One may define more generally the blow up of any complex space along a closed subspace, and again this morphism may be characterized using an analogous universal property (see [61, Pages 163–169] for a similar study in the case of schemes).

Returning to the model case of the blow up of $\mathbb{C}^2$ at the origin O, relations (1.4) show that the lift by π to Σ of the maximal ideal (x, y) of $\mathbb{C}[x, y]$ defining O is the principal ideal sheaf defining the exceptional divisor of π. This fact is an algebraic manifestation of the fact that on Σ all the lines of $\mathbb{C}^2$ passing through O get separated: they are simply the fibers of the Hopf morphism $\tilde{\lambda}$. Note that in order to separate them indeed, one does not have to lift them by taking their full preimages by π (called their *total transforms by* π), but only by taking their *strict transforms*. Let us define these notions in greater generality:

Definition 1.2.31 Let $\pi : Y \to X$ be a morphism of complex varieties and $Z \subseteq X$ a closed complex subvariety of X.

1. The morphism π is a **modification** of X if it is proper and bimeromorphic, that is, if it is proper and if there exists a closed nowhere dense subvariety X' of X such that $\pi^{-1}(X')$ is a nowhere dense subvariety of Y and the restriction $\pi : Y \setminus \pi^{-1}(X') \to X \setminus X'$ is an isomorphism.
2. If X' is minimal with the previous properties, then X' is called the **indeterminacy locus** of π^{-1} and $\pi^{-1}(X')$ is called the **exceptional locus** of π.
3. The **total transform** $\boxed{\pi^*(Z)}$ of Z by π is the complex subspace of Y defined by the preimage by π of the ideal sheaf defining Z in X.
4. Assume that no irreducible component of Z is included in the indeterminacy locus X' of π^{-1}. Then the **strict transform** of Z by π is the closure inside Y of $\pi^{-1}(Z \setminus X')$.

The blow up morphisms of surfaces at smooth points are examples of modifications. In the case of the blow up $\pi : \Sigma \to \mathbb{C}^2_{x,y}$ at the origin, the Eqs. (1.4) show that the total transform of a line $Z(y - ax) \subseteq \mathbb{C}^2_{x,y}$, for $a \in \mathbb{C}^*$, may be described as $Z(u_2(1 - au_1)) \subseteq \mathbb{C}^2_{u_1,u_2}$ and $Z(v_1(v_2 - a)) \subseteq \mathbb{C}^2_{u_1,u_2}$ in the two charts covering Σ. As $Z(u_2)$ and $Z(v_1)$ describe the exceptional divisor $\pi^{-1}(O)$ in those two charts, we see that the strict transform of $Z(y - ax)$ is the fiber of $\tilde{\lambda}$ whose equations are $u_1 = a^{-1}$ and $v_2 = a$ in those two charts.

Assume now that C is a finite sum $\sum_{i \in I} L_i$ of such lines L_i passing through the origin in $\mathbb{C}^2$. The strict transform of C by π is the sum of the strict transforms $\tilde{L}_i$ of those lines and the total transform $\pi^*(C)$ is the sum $\pi^{-1}(O) + \sum_{i \in I} \tilde{L}_i$ of the exceptional divisor of π and of the strict transform of C. Therefore, $\pi^*(C)$ is a *normal crossings divisor* in the following sense:

Definition 1.2.32 Let S be a smooth complex surface and D a divisor on it. This divisor is said **to have normal crossings** or to be a **normal crossings divisor** if its support is locally either a smooth curve or the union of two transversal smooth curves.

Coming back to the curve $C = \sum_{i \in I} L_i$ in $\mathbb{C}^2$, the fact that its total transform $\pi^*(C)$ is a normal crossings divisor shows that the blow up morphism $\pi : \Sigma \to \mathbb{C}^2$ is an *embedded resolution* of C, in the following sense:

Definition 1.2.33 Let C be a curve on the smooth complex surface S, in the sense of Definition 1.2.3. An **embedded resolution** of C is a modification $\tilde{\pi} : \tilde{S} \to S$ such that:

1. $\tilde{S}$ is smooth;
2. the total transform $\tilde{\pi}^*(C)$ is a normal crossings divisor;
3. the strict transform $\tilde{C}$ of C by $\tilde{\pi}$ is smooth.

The restriction $\tilde{\pi}_C : \tilde{C} \to C$ of an embedded resolution $\tilde{\pi}$ of C to the strict transform $\tilde{C}$ of C is a *resolution* of C in the following sense:

Definition 1.2.34 Let X be a complex variety. A **resolution** of X is a modification $\pi : \tilde{X} \to X$ such that $\tilde{X}$ is smooth and the indeterminacy locus of π^{-1} is equal to the singular locus of X.

If X is a complex curve, then a resolution of it is the same as a normalization morphism. This is no longer true in higher dimensions, as in each dimension at least 2, there are normal non-smooth complex varieties. For instance, a hypersurface X of $\mathbb{C}^n$ whose singular locus has codimension at least 2 in X is normal (see [1, 92]).

Note that the second condition in Definition 1.2.33 does not imply the third one. For instance, if one takes the folium of Descartes $C \subset \mathbb{C}^2_{x,y}$ defined by the equation $x^3 + y^3 = 3xy$, then C is a normal crossings divisor in $\mathbb{C}^2$ (with a single singular point at the origin), therefore the identity morphism from $\mathbb{C}^2$ to itself satisfies the first two conditions of Definition 1.2.33 but not the last one, because the strict transform of C by it is not smooth, being the curve C itself.

In order to get an embedded resolution of the folium of Descartes, it is enough to blow up $\mathbb{C}^2$ at the origin O. More generally, if C is a curve in a smooth complex surface S such that at each point o of C, the branches of C at o are smooth and pairwise transversal, then the morphism obtained by blowing up S at all the singular points of C is an embedded resolution of C. Conversely, as may be seen by working with the description (1.4) of the blow up morphism at a point in terms of local coordinates, this property of achieving an embedded resolution by blowing up distinct points of S characterizes the previous kind of curves. What about curves with more complicated singularities? It turns out that they also have embedded resolutions, which may be obtained by blowing up points *iteratively* (see [61, Thm. 3.9], [15, Pages 496–497], [66, Thm. 5.4.2], [19, Section 3.7] and [131, Thm. 3.4.4]):

Theorem 1.2.35 *Let C be a curve on the smooth complex surface S. Define $S_0 := S$ and $\pi_0 : S_0 \to S$ to be the identity. Assume that for some $k \geq 0$ one has defined a modification $\pi_k : S_k \to S$ which is not an embedded resolution of C. Denote by $B_k \subset S_k$ the set of points at which either the strict transform of C is not smooth or $\pi_k^*(C)$ is not a normal crossings divisor. Define $\psi_k : S_{k+1} \to S_k$ to be the blow up of S_k at the points of B_k and $\pi_{k+1} := \pi_k \circ \psi_k : S_{k+1} \to S$. Then there exists $k \in \mathbb{N}$ such that π_k is an embedded resolution of C.*

If k is chosen minimal such that π_k is an embedded resolution of C, then π_k is called the **minimal embedded resolution** of C. It may be shown that any other embedded resolution of C factors through it.

The combinatorial structure of the total transform of C on a given embedded resolution $\tilde{\pi} : \tilde{S} \to S$ of C is encoded usually by drawing its *weighted dual graph*:

Definition 1.2.36 Let C be a curve on the smooth complex surface S and $\tilde{\pi} : \tilde{S} \to S$ be an embedded resolution of C. Its **weighted dual graph** is a finite connected graph whose vertices are labeled by the irreducible components of the total transform $\tilde{\pi}^*(C)$, two vertices being connected by an edge whenever their associated curves intersect on Σ. The vertices corresponding to the components of the strict transform of C are drawn arrowheaded. The remaining vertices are weighted by the self-intersection numbers on Σ of the associated irreducible components of the exceptional locus of π.

How to compute the weights of the dual graph of the embedded resolution $\tilde{\pi} : \tilde{S} \to S$? If this resolution is obtained iteratively by the process described in Theorem 1.2.35, then one may compute recursively the self-intersection numbers of the components of the exceptional loci of the modifications π_k using Corollary 1.2.28 and (see [131, Lemma 8.1.6]):

Proposition 1.2.37 *Let C be a compact curve in the smooth complex surface S. Let o be a point of C of multiplicity $m \in \mathbb{N}$. If $\pi : \Sigma \to S$ is the blow up of S at o, then the self-intersection $\tilde{C}^2$ in Σ of the strict transform $\tilde{C}$ of C by π is related to the self-intersection C^2 of C in S by the formula $\tilde{C}^2 = C^2 - m$.*

1.2.5 The Minimal Embedded Resolution of the Semicubical Parabola

In this subsection we show how to achieve the minimal embedded resolution of the *semicubical parabola* using the algorithm described in Theorem 1.2.35 and how to compute its weighted dual graph using Proposition 1.2.37. It is an expansion of [61, Example V.3.9.1].

The **semicubical parabola** is the curve $P \hookrightarrow \mathbb{C}^2_{x,y}$ defined as the vanishing locus of the polynomial $p(x, y) := y^2 - x^3$. The germ of P at the origin O is a branch called sometimes the **standard cusp**. Due to the following *Jacobian criterion* (see

[66, Theorem 4.3.6] for a generalization in arbitrary dimension and codimension), the origin is the only singular point of P.

Theorem 1.2.38 (Jacobian criterion) *Let C be a reduced curve in an open set of $\mathbb{C}^2_{x,y}$, defined by a holomorphic function $f : U \to \mathbb{C}$. Then the singular locus* $\mathrm{Sing}(C)$ *is the zero locus $Z(f, \partial_x f, \partial_y f)$.*

We want to construct a sequence of blow ups which leads to an embedded resolution of P by following the algorithm described in Theorem 1.2.35, whose notations we use. Therefore, denote by $\boxed{\pi_1} : S_1 \to \mathbb{C}^2$ the blow up of the origin $\boxed{O_0} := O$ of $\mathbb{C}^2_{x,y}$, instead of $\pi : \Sigma \to \mathbb{C}^2$ as in Definition 1.2.24. We use the standard charts $\mathbb{C}^2_{u_1,u_2}$ and $\mathbb{C}^2_{v_1,v_2}$ for computations on S_1, the blow up morphism π_1 being then described by the changes of variables (1.4). The total transform $\pi_1^*(P)$ of P by π_1 is defined by the composition $p \circ \pi_1$, which is expressed as follows in the two charts:

$$p(u_1 u_2, u_2) = u_2^2(1 - u_1^3 u_2), \quad p(v_1, v_1 v_2) = v_1^2(v_2^2 - v_1). \tag{1.6}$$

As the curve P is smooth outside the origin, its strict transform $\boxed{P_1}$ by π_1 is also smooth outside the exceptional divisor. This strict transform intersects the exceptional divisor $\pi_1^{-1}(O)$ only in the chart $\mathbb{C}^2_{v_1,v_2}$, because its equations in the two charts are $1 - u_1^3 u_2 = 0$ and $v_2^2 - v_1 = 0$. The second equation is that of a parabola, therefore it defines a smooth curve. This shows that the strict transform P_1 is everywhere smooth. Therefore, the restriction of the morphism π_1 to the curve P_1 is *a resolution of* P, in the sense of Definition 1.2.34. But it is not an *embedded resolution* in the sense of Definition 1.2.33, because the total transform $\pi_1^*(P)$ is not a normal crossings divisor at the origin $\boxed{O_1}$ of the chart $\mathbb{C}^2_{v_1,v_2}$. Indeed, the strict transform $P_1 \cap \mathbb{C}^2_{v_1,v_2} = Z(v_2^2 - v_1)$ and the exceptional divisor $\pi_1^{-1}(O) \cap \mathbb{C}^2_{v_1,v_2} = Z(v_1)$ are tangent at O_1.

Blow up now the point O_1, getting a new surface $\boxed{S_2}$. Let $\boxed{\psi_1} : S_2 \to S_1$ be this blow up morphism. The preimage $\psi_1^{-1}(\mathbb{C}^2_{v_1,v_2})$ of the chart $\mathbb{C}^2_{v_1,v_2}$ of S_1 may be covered by two charts $\mathbb{C}^2_{w_1,w_2}$ and $\mathbb{C}^2_{z_1,z_2}$, in which the morphism ψ_1 is described by the following analogs of Eqs. (1.4):

$$\begin{cases} v_1 = w_1 w_2 \\ v_2 = w_2, \end{cases} \quad \text{and} \quad \begin{cases} v_1 = z_1 \\ v_2 = z_1 z_2. \end{cases} \tag{1.7}$$

In order to cover completely the surface S_2, one needs also the chart $\mathbb{C}^2_{u_1,u_2}$ of S_1, which is left unchanged by the blow up morphism ψ_1 because O_1 does not appear in it.

Denote $\boxed{\pi_2} := \pi_1 \circ \psi_1 : S_2 \to \mathbb{C}^2$. Using Eqs. (1.6) we see that:

$$p \circ \pi_2(w_1, w_2) = w_1^2 w_2^3 (w_2 - w_1), \quad \text{and} \quad p \circ \pi_2(z_1, z_2) = z_1^3(z_1 z_2^2 - 1). \tag{1.8}$$

Therefore, the strict transform $\boxed{P_2}$ of P_1 by π_2 intersects again the exceptional divisor only in one of those charts, namely $\mathbb{C}^2_{w_1,w_2}$. The total transform $\pi_2^*(P) \hookrightarrow S_2$ is still not a normal crossings divisor, because its germ at the origin $\boxed{O_2}$ of $\mathbb{C}^2_{w_1,w_2}$ has three branches: $Z(w_1)$, $Z(w_2)$, $Z(w_2 - w_1)$, as shown by Eq. (1.8). One needs to blow up also this point, getting the morphisms $\boxed{\psi_2} : \boxed{S_3} \to S_2$ and $\boxed{\pi_3} := \pi_2 \circ \psi_2 : S_3 \to \mathbb{C}^2$. The blow up ψ_2 may be described using the following analogs of Eqs. (1.4) above the chart $\mathbb{C}^2_{w_1,w_2}$:

$$\begin{cases} w_1 = s_1 s_2 \\ w_2 = \quad s_2, \end{cases} \quad \text{and} \quad \begin{cases} w_1 = t_1 \\ w_2 = t_1 t_2. \end{cases} \tag{1.9}$$

Composing these changes of variables with the second Eq. (1.8), we get:

$$p \circ \pi_3(s_1, s_2) = s_1^2 s_2^6 (1 - s_1), \quad p \circ \pi_3(t_1, t_2) = t_1^6 t_2^3 (t_2 - 1).$$

In both charts of S_3 the total transform $\pi_3^*(P)$ is a normal crossings divisor. This being the case also in the remaining charts $\mathbb{C}^2_{u_1,u_2}$ and $\mathbb{C}^2_{z_1,z_2}$, we see that π_3 is an embedded resolution of singularities of the semicubical parabola P. By Theorem 1.2.35, it is the minimal such resolution.

We illustrated the previous sequence of blow ups in Fig. 1.2. We drew whenever possible the support of the total transform of P in the chart whose origin is contained in the strict transform of P. In the four charts the strict transforms of P are drawn in orange and the defining polynomial is written near it. We have used systematically the same color for a point O_i which is blown up by a morphism ψ_i, for the exceptional divisor E_i created by this blow up and for its strict transforms $E_{i,j}$ by the next blow ups. Notice that the component $E_{0,2}$ appears on the chart $\mathbb{C}^2_{t_1,t_2}$, but it does not appear on the chart $\mathbb{C}^2_{s_1,s_2}$, represented on the right of Fig. 1.2.

Let us compute now the weighted dual graph of π_3. For every $i \in \{0, 1, 2\}$, denote by $\boxed{E_i} \hookrightarrow S_{i+1}$ the exceptional divisor of the blow up of the point $O_i \in S_i$. If $0 \leq i < j \leq 2$, denote by $\boxed{E_{i,j}}$ the strict transform of E_i on the surface S_{j+1} by the modification $\psi_j \circ \cdots \circ \psi_i : S_{j+1} \to S_i$. By Corollary 1.2.28, one has $E_0^2 =$

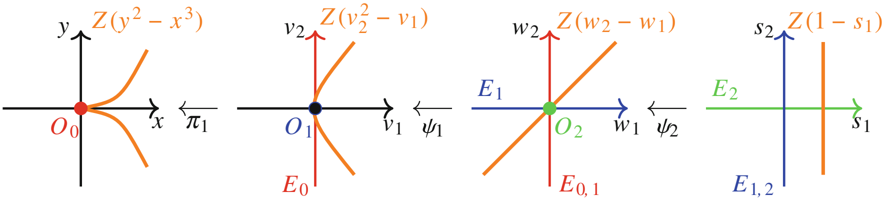


Fig. 1.2 Building iteratively the minimal embedded resolution of the semicubical parabola

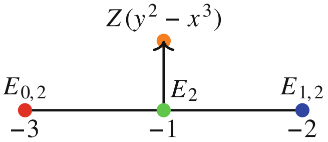


Fig. 1.3 The weighted dual graph of the minimal embedded resolution of the semicubical parabola

$E_1^2 = E_2^2 = -1$. Equations (1.6) and (1.8) imply that $O_1 \in E_0$ and $O_2 \in E_1 \cap E_{0,1}$, because in the chart $\mathbb{C}^2_{v_1,v_2}$ one has $E_{0,1} = Z(v_1)$, $O_1 = (0,0)$ and in the chart $\mathbb{C}^2_{w_1,w_2}$ one has $E_1 = Z(w_2)$, $E_{0,1} = Z(w_1)$, $O_2 = (0,0)$. Using Theorem 1.2.37, we get $E_{0,2}^2 = E_0^2 - 2 = -3$ and $E_{1,2}^2 = E_1^2 - 1 = -2$. Therefore, the weighted dual graph of the minimal embedded resolution $\pi_3 : S_3 \to \mathbb{C}^2$ of the semicubical parabola P is as shown in Fig. 1.3. Near the arrowhead vertex corresponding to the strict transform of P, we have written the defining function of the semicubical parabola.

The previous computations involve many charts, therefore many variables and changes of variables. It is easy to get lost in them. One feels the need of being able to arrive at the final result, the weighted dual graph, without such manipulations. In the next subsection we show how to achieve this goal by a simpler method, without working with charts. We will explain the method using an apparently more complicated example, with two branches. After reading it, we suggest the reader to verify that in the case of the semicubical parabola, the method leads again to the weighted tree of Fig. 1.3.

1.2.6 A Newton Non-degenerate Reducible Example

In this subsection we present on a simple example of *Newton non-degenerate* plane curve singularity the notions of *Newton polygon* and *Newton fan* of a non-zero function $f(x, y) \in \mathbb{C}[[x, y]]$. Then we introduce the associated *lotus* and we show how to construct from it the weighted dual graph of the minimal embedded resolution of the given singularity. These notions are briefly introduced in this section to illustrate our second elementary example and will be revisited formally in Sects. 1.4 and 1.5.

Let $(C, O) \hookrightarrow (\mathbb{C}^2_{x,y}, O)$ be the plane curve singularity defined by the function:

$$f(x, y) := (y^2 - 4x^3)(y^3 - x^7). \tag{1.10}$$

It is the sum of two branches, defined by the equations $y^2 - 4x^3 = 0$ and $y^3 - x^7 = 0$ respectively. Thinking of them as polynomial equations in the unknown y, as explained in Sect. 1.2.3, they have degrees 2 and 3. Their respective sets of roots are $\{\pm 2x^{3/2}\}$ and $\{\omega x^{7/3}\}$, where ω varies among the complex cubic roots of 1. We could express readily in terms of x the roots of the equation $f(x, y) = 0$ seen as a quintic polynomial equation in the variable y, because we knew a factorization of $f(x, y)$ into binomial factors. Is it possible to reach the same objective if one starts instead from the following expanded expression of f?

$$f(x, y) = y^5 - 4x^3y^3 - x^7y^2 + 4x^{10}. \tag{1.11}$$

By the Newton-Puiseux Theorem 1.2.20, we know a priori that the roots of $f(x, y)$ may be expressed as *Newton-Puiseux series*. Newton's fundamental insight was that one may always compute the leading terms of such series only by looking at special terms of f (see the beginning of Sect. 1.4.5). Let us explain this insight in the case of the polynomial (1.11), forgetting its factorization (1.10). Denote by cx^ν the leading term (that is, the term of least degree) of such a series, where $c \in \mathbb{C}^*$ and $\nu > 0$. We have the equality:

$$f(x, cx^\nu + o(x^\nu)) = 0. \tag{1.12}$$

Using formula (1.11), this equality may be rewritten as:

$$(cx^\nu + o(x^\nu))^5 - 4x^3(cx^\nu + o(x^\nu))^3 - x^7(cx^\nu + o(x^\nu))^2 + 4x^{10} = 0,$$

that is, as:

$$\left(c^5x^{5\nu} + o(x^{5\nu})\right) + \left(-4c^3x^{3+3\nu} + o(x^{3+3\nu})\right) + \left(-c^2x^{7+2\nu} + o(x^{7+2\nu})\right) + 4x^{10} = 0. \tag{1.13}$$

The left-hand side of this equation is a sum of four series, whose leading exponents are $5\nu,\ 3 + 3\nu,\ 7 + 2\nu,\ 10$, since $c \neq 0$. The fundamental observation of Newton was that *if the sum* (1.13) *vanishes, then the minimal value of those four exponents is reached at least twice*.

Now, these four exponents may be expressed as the products $(1, \nu) \cdot (a, b) := a + b\nu$, where (a, b) varies among the exponents $(a, b) \in \mathbb{N}^2$ of the monomials $x^a y^b$ appearing in the expanded form (1.11) of $f(x, y)$, that is, as the evaluations of the linear form $l_\nu(a, b) := a + b\nu$ on the *support* $\mathcal{S}(f)$ of the series $f(x, y)$. In our example the support is finite, but it may be infinite if one allows f to be a power series in the variables x, y. It is at this point that *convex geometry* enters into the game, through the following property (which is a consequence of [94, Assertion III.1.5.2]):

Proposition 1.2.39 *Let $\mathcal{S}$ be a subset of $\mathbb{N}^2$. If l is a linear form with non-negative coefficients on $\mathbb{R}^2$, then its restriction to $\mathcal{S}$ achieves its minimum precisely on the subset of $\mathcal{S}$ lying on a face of the convex hull* $\text{Conv}(\mathcal{S} + \mathbb{R}^2_+)$.

Coming back to Eq. (1.13), we see that the linear form $l_\nu(a, b) = a + b\nu$, which computes the leading exponents of the terms appearing in the left-hand side of (1.13), indeed has non-negative coefficients. Therefore,

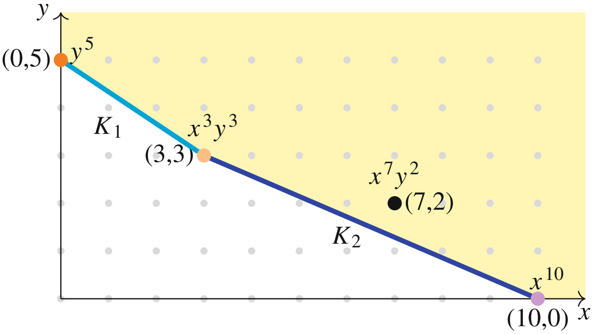


Fig. 1.4 The Newton polygon of the series $f(x, y) = (y^2-2x^3)(y^3-x^7)$

the hypotheses of Proposition 1.2.39 are satisfied. This shows that the minimal value $\min\{5\nu, 3+3\nu, 7+2\nu, 10\}$ is achieved on a face of the convex hull $\text{Conv}(\mathcal{S}(f) + \mathbb{R}_+^2)$. This convex hull, called the *Newton polygon* $\mathcal{N}(f)$ of $f \in \mathbb{C}[[x, y]]$ (see Definition 1.4.2 below), is represented in Fig. 1.4. It has three vertices, which are $(0, 5)$, $(3, 3)$, $(10, 0)$, corresponding to the terms y^5, $-4x^3y^3$ and $4x^{10}$ of the expansion (1.11). It has two compact edges $K_1 := [(0, 5), (3, 3)]$ and $K_2 := [(3, 3), (10, 0)]$. If the minimum is to be achieved at least twice on $\mathcal{S}(f)$, then it must be achieved on one of those two compact edges, because $\nu > 0$. This means that the linear form l_ν must be orthogonal to one of those compact edges. There are therefore two possibilities:

- Either l_ν achieves its minimum on K_1, which means that $(1, \nu)$ is orthogonal to it. In other words $(1, \nu) \cdot (3-0, 3-5) = 0$, that is, $\nu = 3/2$. Writing that the sum of the terms of the left-hand side of Eq. (1.13) whose leading exponents achieve the minimum vanishes, one gets the equation $c^5 - 4c^3 = 0$. As $c \neq 0$, this is equivalent to the equation $c^2 = 4$, hence $c = \pm 2$.
- Or l_ν achieves its minimum on K_2. In other words $(1, \nu) \cdot (10-3, 0-3) = 0$, that is, $\nu = 7/3$. One gets then the equation $-4c^3 + 4 = 0$. That is, c varies now among the cubic roots of 1.

It follows that the possible leading terms of a Newton-Puiseux series η in the variable x such that $f(x, \eta) = 0$ belong to the union $\{\pm 2x^{3/2}\} \cup \{\omega x^{7/3} : \omega^3 = 1\}$. One recognizes the roots from the factorization (1.10). Newton's method shows that *those are the leading terms of the roots* $y(x)$ *of the equation* $g(x, y) = 0$, *for any* $g \in \mathbb{C}[[x, y]]$ *whose Newton polygon is the same as* $\mathcal{N}(f)$, *and whose restrictions to the compact sides of the polygon coincide with the analogous restrictions for* f. Any such function g defines a *Newton non-degenerate* singularity (see Definition 1.4.21 below), because both equations $c^2 = 4$ and $-4c^3 + 4 = 0$ obtained by restricting g to the compact edges of its Newton polygon have simple roots. Variants of Newton's previous line of thought will be followed again in the proofs of Propositions 1.4.11 and 1.4.18 below.

In general, for any series $f(x, y)$, once a first term cx^ν of a potential root of $f(x, y) = 0$ is computed, one may perform a formal change of variables and compute a second term. Newton explained that one could compute as many terms as needed, but it was Puiseux who proved carefully that by pushing this iterative process to its limit, one gets true roots of the equation, which are Newton-Puiseux series. Moreover, he proved that whenever one starts from a convergent function f, one gets only roots of the form $\xi(x^{1/p})$, where $\xi(t) \in \mathbb{C}[[t]]$ is convergent and $p \in \mathbb{N}^*$. This approach leads to a proof of the Newton-Puiseux Theorem 1.2.20, different from the one given above (see Remark 1.2.22).

Let us come back to our example. It turns out that in this *Newton non-degenerate* case, the weighted dual graph of the minimal embedded resolution is determined by the Newton polygon $\mathcal{N}(f)$. In fact, one needs only the *inclinations* of its compact edges. This information is encoded in the associated *Newton fan*, obtained by subdividing the first quadrant along the rays orthogonal to the compact edges of $\mathcal{N}(f)$ (see the left side of Fig. 1.5 and Definition 1.4.9 below). Consider now inside the first quadrant all the triangles with vertices $f_1, f_2, f_1 + f_2$, where (f_1, f_2) is a basis of the ambient lattice $\mathbb{Z}^2$. The edges of those triangles may be drawn recursively by starting from the segment $[e_1, e_2]$ which joins the elements of the canonical basis (e_1, e_2) and, each time a new segment $[f_1, f_2]$ is drawn, by drawing also the segments $[f_1, f_1 + f_2]$ and $[f_2, f_1 + f_2]$. If one performs this construction only whenever the interior of the segment $[f_1, f_2]$ intersects one of the rays of the Newton fan, one gets its associated *lotus*, represented on the right side of Fig. 1.5.

In fact, one needs to attach to it new arrowhead vertices corresponding to the branches of C, as shown in Fig. 1.6. In this figure the lotus was redrawn as an abstract simplicial complex, without representing its precise embedding in the plane $\mathbb{R}^2$. This abstract simplicial structure is sufficient for seeing how it contains the weighted dual graph of the minimal embedded resolution of the curve singularity $Z(xy(y^2 - 4x^3)(y^3 - x^7))$ as part of its boundary. The self-intersection number of an exceptional divisor is simply the opposite of the number of triangles containing the vertex representing this divisor (compare Figs. 1.6 and 1.7).

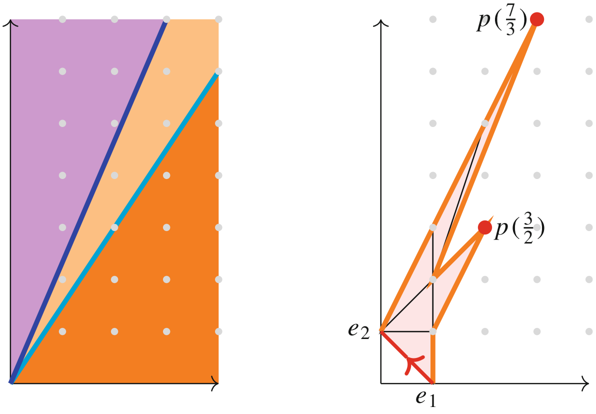


Fig. 1.5 The Newton fan of $f(x, y) = (y^2 - 4x^3)(y^3 - x^7)$ and its associated lotus

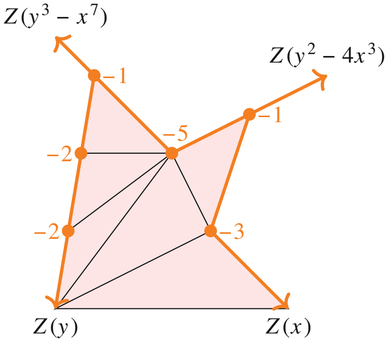


Fig. 1.6 The lotus of $f(x, y) = (y^2 - 4x^3)(y^3 - x^7)$

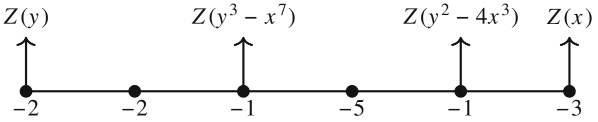


Fig. 1.7 The weighted dual graph of the minimal embedded resolution of $Z(xyf(x, y))$

In the sequel we will associate lotuses to any plane curve singularity C (see Definition 1.5.26). The data needed to construct them will be a finite sequence of Newton polygons generated by a *toroidal pseudo-resolution algorithm* (see Algorithm 1.4.22). We will embed analogously inside them the weighted dual graphs of associated embedded resolutions of *completions* of the curve (see Definition 1.4.15 and Theorem 1.5.29). We will also explain the notions of *fan tree* (see Definition 1.4.33), *Enriques diagram* (see Definition 1.4.31) and *Eggers-Wall tree* (see Definition 1.6.3) of C or of an associated toroidal pseudo-resolution process and we will show that they embed similarly in the corresponding lotus (see Theorem 1.5.29).

1.3 Toric and Toroidal Surfaces and Their Morphisms

In this section we explain basic definitions and intuitions about toric and toroidal varieties and their modifications, which will be used in the subsequent sections in the study of plane curve singularities. Namely, *fans* are introduced in Definition 1.3.3, *affine toric varieties* in Definition 1.3.14, their *boundaries* in Definition 1.3.18, *toric morphisms* in Sect. 1.3.3, in particular the toric description of 2-dimensional blow ups in Example 1.3.27 and the *category of toroidal varieties* in Sect. 1.3.4. Section 1.3.5 contains historical information about the development of toric and toroidal geometry and about its applications to the study of singularities.

1.3.1 Two-Dimensional Fans and Their Regularizations

In this subsection we explain the basic notions of two-dimensional convex geometry needed to define toric varieties in Sect. 1.3.2 and toric morphisms in Sect. 1.3.3: *lattices, rational cones* and *fans*. For more details about toric geometry one may consult the standard textbooks [37, 41, 91] and [26].

A **lattice** is a free $\mathbb{Z}$-module of finite rank. A pair $(a, b) \in \mathbb{Z}^2$ may be seen as an instruction to build two kinds of objects: the Laurent monomial $x^a y^b$ and the parametrized monomial curve $t \to (t^a, t^b)$. The fact that monomials and curves are distinct geometrical objects indicates that it would be good to think also in two ways about the pairs (a, b), that is, as coordinates of vectors relative to bases in two different lattices. Those two lattices are not to be chosen independently of each other. Indeed, given a monomial $x^a y^b$ and a parametrized monomial curve $t \to (t^c, t^d)$, one may substitute the parametrization in the monomial, getting a new monomial, this time in the variable t alone:

$$(x^a y^b) \circ (t^c, t^d) = t^{ac+bd}. \tag{1.14}$$

This indicates that those two lattices should be seen as factors of the domain of definition of the unimodular $\mathbb{Z}$-valued bilinear form $(a, b) \cdot (c, d) := ac + bd$, that is, that they should be *dual* lattices.

In order to distinguish clearly the roles of these two lattices, one denotes them usually by distinct letters, instead of simply writing for instance $\mathbb{Z}^2$ and $(\mathbb{Z}^2)^\vee$. It became traditional after the appearance of Fulton's book [41] to denote by $\boxed{M}$ the lattice whose elements are exponents of monomials in several variables, and by $\boxed{N}$ the dual lattice, whose elements are thought as exponents of parametrized monomial curves in the space of the same variables. It is important to allow for changes of bases of those $\mathbb{Z}$-modules, corresponding to monomial changes of variables of the form $x = u^\alpha v^\gamma$, $y = u^\beta v^\delta$, for which the matrix of exponents is **unimodular**:

$$\begin{vmatrix} \alpha & \gamma \\ \beta & \delta \end{vmatrix} = \pm 1. \tag{1.15}$$

This means that one does not have to fix identifications $M = \mathbb{Z}^2$, $N = \mathbb{Z}^2$, but instead to allow those identifications to depend on the context. Note also that the elements of N may be seen as **weights** for the variables x, y. That is, if $(c, d) \in N$, one gives the weight c to x and the weight d to y, which endows the monomial $x^a y^b$ with the weight $ac + bd$ appearing in the equality (1.14). For this reason, N is called sometimes the **weight lattice** associated to the **monomial lattice** M.

We will call **vectors** the elements of a lattice. Those non-zero vectors which cannot be written as non-trivial integral multiples of other lattice vectors are called **primitive**. Any non-zero lattice vector w may be written uniquely in the form $l_{\mathbb{Z}}(w)\, w'$, with $l_{\mathbb{Z}}(w) \in \mathbb{N}^*$ and w' a primitive lattice vector.

Definition 1.3.1 Let N be a lattice and $w \in N \setminus \{0\}$. The positive integer $\boxed{l_{\mathbb{Z}}(w)}$ is the **integral length** of w. We extend this definition to the whole lattice N by setting $l_{\mathbb{Z}}(0) := 0$. For $w_1, w_2 \in N$, the **integral length** $\boxed{l_{\mathbb{Z}}[w_1, w_2]} \in \mathbb{N}$ of the segment $[w_1, w_2]$ is equal to $l_{\mathbb{Z}}(w_2 - w_1) = l_{\mathbb{Z}}(w_1 - w_2)$.

If N is a lattice, denote by $\boxed{N_{\mathbb{R}}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ the real vector space generated by N. We will say that the elements of N are the **integral points** of the real vector space $N_{\mathbb{R}}$. By a **cone of** N we will mean a convex rational polyhedral cone, that is, a subset of $N_{\mathbb{R}}$ of the form:

$$\boxed{\mathbb{R}_+\langle w_1, \dots, w_k\rangle} := \mathbb{R}_+ w_1 + \cdots + \mathbb{R}_+ w_k,$$

where $w_1, \dots, w_k \in N$. If the cone does not contain a positive dimensional vector subspace of $N_{\mathbb{R}}$, it is called **strictly convex**.

If the lattice N is of rank two, then the strictly convex cones are of three sorts, according to their dimensions:

- The 2-dimensional cones are of the form $\mathbb{R}_+\langle w_1, w_2\rangle$, where $w_1, w_2 \in N$ are non-proportional. In classical geometric terminology, they are strictly convex angles with apex at the origin of $N_{\mathbb{R}}$.
- The 1-dimensional cones are the closed half-lines emanating from the origin; we will call them **rays**.
- There is only one 0-dimensional cone: the origin of N.

As a particular case of a terminology used in any dimension, one speaks about the **faces** of a given cone $\sigma \subseteq N_{\mathbb{R}}$: those are the subsets of σ on which the restriction to σ of a linear form $l \in N_{\mathbb{R}}^{\vee} = M_{\mathbb{R}}$ reaches its minimum. The faces of a strictly convex 2-dimensional cone $\mathbb{R}_+\langle w_1, w_2\rangle$ are the cone itself, its **edges** $\mathbb{R}_+ w_1$, $\mathbb{R}_+ w_2$ and the origin. The faces of a ray are the ray itself and the origin. Finally, the origin has only one face, which is the origin itself.

Endowing the 2-dimensional lattice N with a basis (e_1, e_2) allows to speak of the **slope** $d/c \in \mathbb{R} \cup \{\infty\}$ relative to (e_1, e_2) of any vector $w = c\, e_1 + d\, e_2 \in N_{\mathbb{R}} \setminus \{0\}$ or of the associated ray $\mathbb{R}_+ w$. In terms of the coordinates (c, d), the integral length $l_{\mathbb{Z}}(w)$ of w is equal to the greatest common divisor $\gcd(c, d)$.

Notations 1.3.2 If the basis (e_1, e_2) of N is fixed and clear from the context, we denote by:

$$\boxed{\sigma_0} := \mathbb{R}_+\langle e_1, e_2\rangle \subseteq N_{\mathbb{R}}$$

the cone generated by it. If $\lambda \in \mathbb{Q}_+ \cup \{\infty\}$, we denote by $\boxed{p(\lambda)}$ the unique primitive element of the lattice N contained in the cone σ_0, and which has slope λ.

In the sequel it will be important to work with the following special sets of cones, which are fundamental in toric geometry:

Definition 1.3.3 A **fan** of the lattice N is a finite set of strictly convex cones of N which is closed under the operation of taking faces of its cones and such that the intersection of any two of its cones is a face of each of them. The **support** $\boxed{|\mathcal{F}|}$ of a fan $\mathcal{F}$ is the union of its cones. A fan $\mathcal{F}$ **refines** (or **subdivides**) another fan $\mathcal{F}'$ if they have the same support and if each cone of $\mathcal{F}$ is contained in some cone of $\mathcal{F}'$. A fan **subdivides a cone** σ if it subdivides the fan formed by its faces. We often denote again by σ the fan formed by the faces of a cone σ, by a slight abuse of notation.

Let us complete the previous definition, valid in arbitrary rank, with terminology and notations specific to rank two:

Definition 1.3.4 Let (e_1, e_2) be a basis of the lattice N of rank two and σ_0 be the associated cone $\mathbb{R}_+\langle e_1, e_2\rangle$. Any fan $\mathcal{F}$ subdividing σ_0 is determined by the finite set of slopes $\mathcal{E} \subset \mathbb{Q}_+^*$ of its rays contained in the interior of σ_0. In this case we denote the fan by $\boxed{\mathcal{F}(\mathcal{E})}$ and we call it the **fan of the set** $\mathcal{E}$. We extend the definition of $\mathcal{F}(\mathcal{E})$ to the case where $\mathcal{E}$ contains 0 or ∞, by setting in this case $\mathcal{F}(\mathcal{E}) := \mathcal{F}(\mathcal{E} \setminus \{0, \infty\})$. If $\mathcal{E} = \{\lambda_1, \dots, \lambda_p\}$, we write also $\boxed{\mathcal{F}(\lambda_1, \dots, \lambda_p)}$ instead of $\mathcal{F}(\mathcal{E})$.

Note that $\mathcal{F}(\emptyset)$ is simply the fan consisting of the cone σ_0 and its faces.

Definition 1.3.5 A cone of a lattice N is called **regular** if it can be generated by elements which form a subset of a basis of N. A fan all of whose cones are regular is called **regular**.

It is convenient to set $\mathbb{R}_+\langle\emptyset\rangle := \{0\}$. This implies that $\{0\}$ is also a regular cone.

Assume that a basis (e_1, e_2) of the lattice N is fixed. If $f_1 = \alpha e_1 + \beta e_2$ and $f_2 = \gamma e_1 + \delta e_2$ are two primitive vectors of N, then the cone $\mathbb{R}_+\langle f_1, f_2\rangle$ generated by them is regular if and only if the matrix of the pair (f_1, f_2) in the basis (e_1, e_2) is unimodular, that is, the equality (1.15) holds.

Example 1.3.6 If $\mathcal{E} = \{3/5, 2/1, 5/2\}$, then the rays of the fan $\mathcal{F}(\mathcal{E})$ are represented in Fig. 1.8. On each ray of the fan which is distinct from the edges of the cone σ_0, we indicated by a small red disc the unique primitive element of the lattice N lying on it. That is, on the ray of slope $\lambda \in \mathcal{E}$ is indicated the point $p(\lambda)$. The fan $\mathcal{F}(\mathcal{E})$ contains also 4 cones of dimension 2, which are $\mathbb{R}_+\langle e_1, p\,(3/5)\rangle$, $\mathbb{R}_+\langle p\,(3/5)\,, p\,(2/1)\rangle$, $\mathbb{R}_+\langle p\,(2/1)\,, p\,(5/2)\rangle$, $\mathbb{R}_+\langle p\,(5/2)\,, e_2\rangle$. Using the unimodularity criterion above, we see that $\mathbb{R}_+\langle p\,(2/1)\,, p\,(5/2)\rangle$ is the only 2-dimensional cone of the fan $\mathcal{F}(\mathcal{E})$ which is regular.

The following result is specific for lattices of rank two (see [91, Prop. 1.19]):

Proposition 1.3.7 *If the lattice N is of rank two, any fan relative to N has a minimal regular subdivision, in the sense that any other regular subdivision refines it.*

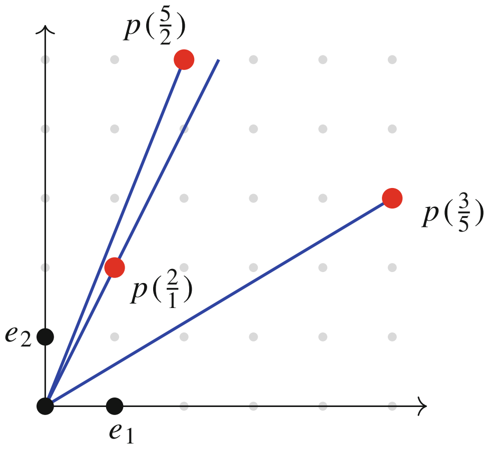


Fig. 1.8 The fan $\mathcal{F}(3/5, 2/1, 5/2)$ and the points $p\,(3/5)\,,\ p\,(2/1)\,,\ p\,(5/2)$

Proposition 1.3.7 motivates the following definition:

Definition 1.3.8 If $\mathcal{F}$ is a 2-dimensional fan, we denote by $\boxed{\mathcal{F}^{reg}}$ its minimal regular subdivision, and we call it the **regularization** of $\mathcal{F}$.

The importance of the regularization operation in our context stems from the fact that it allows to describe combinatorially the minimal resolution of a toric surface (see Proposition 1.3.28 below). The regularization of a 2-dimensional cone may be described in the following way (see [91, Proposition 1.19]):

Proposition 1.3.9 *Let N be a lattice of rank two and let σ be a 2-dimensional strictly convex cone of N. Then the regularization σ^{reg} of the fan of its faces is obtained by subdividing σ using the rays directed by the integral points lying on the boundary of the convex hull of the set of non-zero integral points of σ. If $\mathcal{F}$ is a fan of a lattice of rank two, then its regularization is the union of the regularizations of its cones.*

An alternative recursive description of σ^{reg} was given by Mutsuo Oka in [94, Chap. II.2].

Example 1.3.10 Let us consider again the fan $\mathcal{F}(3/5, 2/1, 5/2)$ of Example 1.3.6. The rays of $\mathcal{F}^{reg}\,(3/5, 2/1, 5/2) = \mathcal{F}(1/2, 3/5, 2/3, 1/1, 2/1, 5/2, 3/1)$ are drawn in green in Fig. 1.9. The thick orange polygonal line, on the right side of this figure, is the union of compact edges of the boundaries of the convex hulls of the sets of non-zero integral points of its 2-dimensional cones.

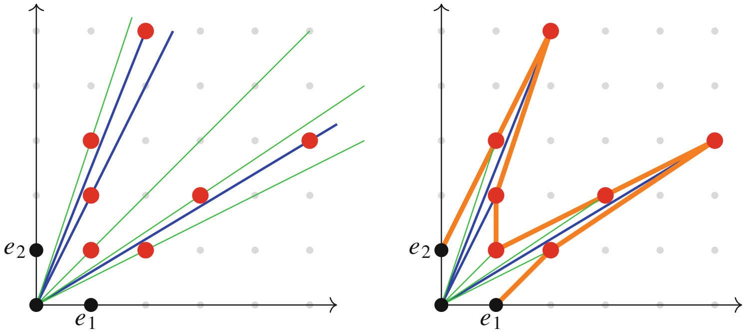


Fig. 1.9 The regularization $\mathcal{F}^{reg}$ $(3/5, 2/1, 5/2)$ of the fan of Fig. 1.8

1.3.2 Toric Varieties and Their Orbits

In this subsection we explain in which way fans determine special kinds of complex algebraic varieties, called *toric varieties*. Namely, every rational polyhedral cone relative to a lattice determines a *monoid algebra* (see Definition 1.3.11), whose maximal spectrum is an *affine toric variety* (see Definition 1.3.14). More generally, every fan determines a toric variety by gluing the affine toric varieties associated to its cones (see Definition 1.3.15).

One associates with a lattice N of rank n the following **complex algebraic torus** of dimension n (that is, an algebraic group isomorphic to $\big((\mathbb{C}^*)^n, \cdot\big)$):

$$\boxed{\mathcal{T}_N} := N \otimes_{\mathbb{Z}} \mathbb{C}^*. \tag{1.16}$$

Here the factors are considered as abelian groups $(N, +)$ and $(\mathbb{C}^*, \cdot)$, therefore they are endowed with canonical structures of $\mathbb{Z}$-modules, relative to which is taken the previous tensor product. This algebraic torus may be also described in terms of the dual lattice M of N, defined by:

$$\boxed{M} := \text{Hom}\ (N, \mathbb{Z}).$$

Namely, one has:

$$\mathcal{T}_N = \text{Hom}(M, \mathbb{C}^*). \tag{1.17}$$

Equations (1.16) and (1.17) allow in turn to give the following interpretations of the lattices N and M in terms of morphisms of algebraic groups:

$$\begin{aligned} N &= \mathrm{Hom}(\mathbb{C}^*, \mathcal{T}_N) = \\ &= \text{the group of } \textbf{one parameter subgroups} \text{ of } \mathcal{T}_N; \\ M &= \mathrm{Hom}(\mathcal{T}_N, \mathbb{C}^*) = \\ &= \text{the group of } \textbf{characters} \text{ of } \mathcal{T}_N. \end{aligned} \tag{1.18}$$

If $w \in N$ is seen as an element of the lattice N, we denote by $\boxed{t^w}$ the same element seen as a morphism of abelian groups from $\mathbb{C}^*$ to $\mathcal{T}_N$.

Let us explain this notation in the case when N has rank 2. If t is viewed as the parameter on the source $\mathbb{C}^*$ and one identifies $\mathcal{T}_N$ with $(\mathbb{C}^*)^2$ using the basis (e_1, e_2) of N, then the morphism becomes the following map from $\mathbb{C}^*$ to $(\mathbb{C}^*)^2$:

$$t \to (t^c, t^d).$$

Here (c, d) denote as before the coordinates of w in the chosen basis (e_1, e_2) of N. One gets therefore a parametrized monomial curve as at the beginning of Sect. 1.3.1. The advantage of seeing it as an element of $\mathrm{Hom}(\mathbb{C}^*, \mathcal{T}_N)$ is that one gets a viewpoint independent of the choice of coordinates for $\mathcal{T}_N$, that is, of bases for M or for N.

It is customary to say that a morphism $t^w \in \mathrm{Hom}\ (\mathbb{C}^*, \mathcal{T}_N)$ is a *one parameter subgroup* of $\mathcal{T}_N$, even when this morphism is not injective. Note that t^w is injective if and only if w is a primitive element of N. In general, when $w \in N \setminus \{0\}$, the map t^w is a cyclic covering of its image, of degree $l_{\mathbb{Z}}(w)$ (see Definition 1.3.1). Note also that t^0 is the constant map with image the unit element 1 of the group $\mathcal{T}_N$.

We introduced the notation t^w in order to be able to distinguish between N seen as an abstract group, and seen as the lattice of one parameter subgroups of $\mathcal{T}_N$. In an analogous way, if $m \in M$, one uses the notation $\boxed{\chi^m} : \mathcal{T}_N \to \mathbb{C}^*$ for its associated character, in order to distinguish between M seen as an abstract group and seen as the lattice of characters of $\mathcal{T}_N$. If one denotes by $\boxed{w \cdot m} \in \mathbb{Z}$ the result of applying the canonical duality pairing $N \times M \to \mathbb{Z}$ to $(w, m) \in N \times M$, then the composite morphism $\chi^m \circ t^w : \mathbb{C}^* \to \mathbb{C}^*$ is simply given by $t \to t^{w \cdot m}$. This is the intrinsic description of the composition performed in formula (1.14).

Let us see more precisely how the choice of basis (e_1, e_2) of N determines an isomorphism $\mathcal{T}_N \simeq (\mathbb{C}^*)^2$. To have such an isomorphism amounts to choosing a special pair (x, y) of regular functions on $\mathcal{T}_N$, which are the pull-backs of the coordinate functions on $(\mathbb{C}^*)^2$. This isomorphism should be not only an isomorphism of algebraic surfaces, but also of groups. As the coordinate functions on $(\mathbb{C}^*)^2$ are characters of $\left((\mathbb{C}^*)^2, \cdot\right)$, that is, elements of $\mathrm{Hom}((\mathbb{C}^*)^2, \mathbb{C}^*)$, we deduce that x, y are also characters, this time of $(\mathcal{T}_N, \cdot)$. It means that they are elements of the lattice M (see the equalities (1.18)). In which way does the basis (e_1, e_2) of N determine a pair of elements of M? Well, this pair is simply the dual basis $\boxed{(\epsilon_1, \epsilon_2)}$ of (e_1, e_2)! Therefore, one has $(x, y) = (\chi^{\epsilon_1}, \chi^{\epsilon_2})$ in terms of the dual basis $(\epsilon_1, \epsilon_2) \in M^2$ of $(e_1, e_2) \in N^2$.

The choice of coordinates (x, y) allows to embed the torus $\mathcal{T}_N$ into the affine plane $\mathbb{C}^2$ with the same coordinates. The coordinate ring of this affine plane is of course $\mathbb{C}[x, y]$. In our context it is important to interpret this ring as the $\mathbb{C}$-algebra of the commutative monoid of monomials with non-negative exponents in the variables x and y. This monoid is isomorphic (using the map $m \to \chi^m$) to the monoid $\mathbb{R}_+\langle \epsilon_1, \epsilon_2 \rangle \cap M$. In turn, the cone $\mathbb{R}_+\langle \epsilon_1, \epsilon_2 \rangle$ is in the following sense the dual cone of $\sigma_0 := \mathbb{R}_+\langle e_1, e_2 \rangle$:

Definition 1.3.11 Let σ be a cone of N. Its **dual** is the cone $\sigma^\vee$ of M defined by:

$$\boxed{\sigma^\vee} := \{m \in M_{\mathbb{R}},\ w \cdot m \geq 0 \text{ for all } w \in \sigma\},$$

and its associated **monoid algebra** is the $\mathbb{C}$-algebra of the abelian monoid $(\sigma^\vee \cap M, +)$:

$$\boxed{\mathbb{C}[\sigma^\vee \cap M]} := \left\{ \sum_{\text{finite}} c_m \chi^m\ ,\ m \in \sigma^\vee \cap M \text{ and } c_m \in \mathbb{C} \right\}.$$

Note that σ is strictly convex if and only if the dimension of $\sigma^\vee$ is equal to the rank of the lattice M. The $\mathbb{C}$-algebra $\mathbb{C}[\sigma^\vee \cap M]$ is finitely generated, since the monoid $(\sigma^\vee \cap M, +)$ is finitely generated by Gordan's Lemma (see [41, Section 1.1, Proposition 1]).

The set $\mathbb{C}^2$ with coordinates (x, y) may now be interpreted in the two following ways:

$$\begin{aligned} \mathbb{C}^2_{x,y} &= \text{the maximal spectrum of the ring } \mathbb{C}[\sigma_0^\vee \cap M] = \\ &= \operatorname{Hom}(\sigma_0^\vee \cap M, \mathbb{C}). \end{aligned} \tag{1.19}$$

The last set of homomorphisms is taken in the category of abelian monoids, where $\mathbb{C}$ is considered as a monoid with respect to multiplication. This interpretation is obtained by looking at the evaluation of the monomials χ^m, with $m \in \sigma_0^\vee \cap M$, at the points of $\mathbb{C}^2$.

The equalities (1.19) may be turned into a general way to associate a complex affine variety to a cone σ of N, in arbitrary dimension:

$$\begin{aligned} \boxed{X_\sigma} &:= \text{the maximal spectrum of } \mathbb{C}[\sigma^\vee \cap M] = \\ &= \operatorname{Hom}(\sigma^\vee \cap M, \mathbb{C}). \end{aligned} \tag{1.20}$$

The equalities (1.19) show that $X_{\sigma_0} = \mathbb{C}^2_{x,y}$, if $x = \chi^{\epsilon_1}$ and $y = \chi^{\epsilon_2}$. Therefore, the affine variety X_{σ_0} is smooth. The following proposition characterizes the cones for which the associated variety is smooth (see [41, Section 2.1, Proposition 1]):

Proposition 1.3.12 *Let σ be a strictly convex cone of the lattice N. Then the affine variety X_σ is smooth if and only if σ is regular in the sense of Definition 1.3.5.*

In the sequel, by a **stratification** of an algebraic variety we mean a finite partition of it into locally closed connected smooth subvarieties, called the **strata** of the stratification, such that the closure of each stratum is a union of strata.

Consider the following stratification of $X_{\sigma_0} = \mathbb{C}^2_{x,y}$:

$$\mathbb{C}^2_{x,y} = \{0\} \sqcup \left(\mathbb{C}^*_x \times \{0\}\right) \sqcup \left(\{0\} \times \mathbb{C}^*_y\right) \sqcup (\mathbb{C}^*)^2_{x,y}. \tag{1.21}$$

One may interpret in the following way its strata in terms of vanishing of monomials whose exponents belong to $\sigma_0^\vee \cap M = \mathbb{N}\langle \epsilon_1, \epsilon_2 \rangle$:

- 0 is the only point of $\mathbb{C}^2_{x,y}$ at which vanish exactly the monomials with exponents in $\left(\sigma_0^\vee \setminus \{0\}\right) \cap M$.
- $\mathbb{C}^*_x \times \{0\}$ is the set of points of $\mathbb{C}^2_{x,y}$ at which vanish exactly the monomials with exponents in $\left(\sigma_0^\vee \setminus \mathbb{R}_+\epsilon_1\right) \cap M$.
- $\{0\} \times \mathbb{C}^*_y$ is the set of points of $\mathbb{C}^2_{x,y}$ at which vanish exactly the monomials with exponents in $\left(\sigma_0^\vee \setminus \mathbb{R}_+\epsilon_2\right) \cap M$.
- $(\mathbb{C}^*)^2_{x,y} = \mathcal{T}_N$ is the set of points of $\mathbb{C}^2_{x,y}$ at which vanish no monomials, that is, at which vanish exactly the monomials with exponents in $\left(\sigma_0^\vee \setminus \sigma_0^\vee\right) \cap M$.

Note that the sets of exponents of monomials appearing in the previous list are precisely those of the form $\left(\sigma_0^\vee \setminus \tau\right) \cap M$, where τ varies among the faces of the cone $\sigma_0^\vee$. It is customary in toric geometry to express them in a dual way, using the following bijection between the faces of σ and of $\sigma^\vee$, valid in all dimensions for (not necessarily rational) convex polyhedral cones σ (see [26, Proposition 1.2.10]):

Proposition 1.3.13 *Let σ be a cone of $N_{\mathbb{R}}$. Then the map $\rho \to \rho^\perp \cap \sigma^\vee$ is an order-reversing bijection from the set of faces of σ to the set of faces of $\sigma^\vee$ (see Fig. 1.10).*

Here $\boxed{\rho^\perp} := \{m \in M_{\mathbb{R}},\ w \cdot m = 0 \text{ for all } w \in \rho\}$ denotes the orthogonal of the cone ρ of N. It is a real vector subspace of $M_{\mathbb{R}}$, which may be characterized as the maximal vector subspace of the convex cone $\rho^\vee$.

The stratification (1.21) of $\mathbb{C}^2$ is a particular case of a stratification of any affine variety of the form X_σ. In order to define it, one associates with each point p of X_σ the subset of $\sigma^\vee \cap M$ formed by the exponents of the monomials vanishing at p. This defines a function from X_σ to the power set of $\sigma^\vee \cap M$, whose levels are precisely the strata of the stratification of X_σ. The set of strata is in bijective correspondence

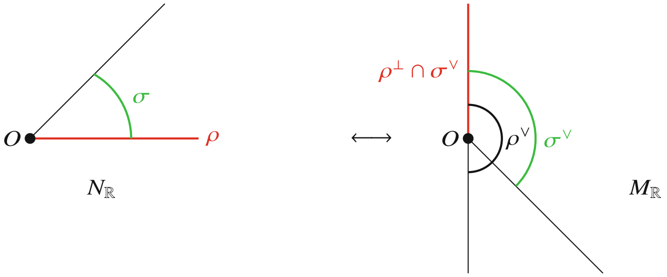


Fig. 1.10 The bijection between the faces of σ and $\sigma^{\vee}$

with the set of faces of σ, the stratum O_{ρ} corresponding to the face ρ of σ being:

$$\boxed{O_{\rho}} := \left\{ p \in \mathrm{Hom}(\sigma^{\vee} \cap M, \mathbb{C}),\ p^{-1}(0) = (\sigma^{\vee} \setminus \rho^{\perp}) \cap M \right\}. \tag{1.22}$$

In particular, $O_{\{0\}} = \mathcal{T}_N$ is the only stratum whose dimension is the same as the dimension of X_{σ}. This shows that the torus $\mathcal{T}_N$ embeds naturally as an affine open set in the affine surface X_{σ}. For this reason, the following vocabulary was introduced:

Definition 1.3.14 If N is a lattice and σ is a strictly convex cone of N, then the variety X_{σ} defined by the equalities (1.20) is called an **affine toric variety**.

Note that for $X_{\sigma_0} = \mathbb{C}^2_{x,y}$, the strata are:

- $O_{\sigma_0} = \{0\}$;
- $O_{\mathbb{R}_+ e_2} = \mathbb{C}^*_x \times \{0\}$;
- $O_{\mathbb{R}_+ e_1} = \{0\} \times \mathbb{C}^*_y$;
- $O_{\{0\}} = (\mathbb{C}^*)^2_{x,y} = \mathcal{T}_N$.

One may feel difficult to remember the second and third equalities, a common error at the time of doing computations being to permute them. A way to remember them is the following: *the orbit corresponding to an edge of a* 2*-dimensional regular cone is the complement of the origin in the axis of coordinates of* $\mathbb{C}^2$ *defined by the vanishing of the dual variable*. In our case, the dual variable of the edge $\mathbb{R}_+ e_1$ is $x = \chi^{\epsilon_1}$, whose 0-locus is the axis of the variable y, and conversely.

The notation O_{ρ} is motivated by the fact that this subset of X_{σ} is an *orbit* of a natural action of the algebraic torus $\mathcal{T}_N$ on X_{σ}. For $X_{\sigma_0} = \mathbb{C}^2_{x,y}$, case in which one may also identify $\mathcal{T}_N$ with $(\mathbb{C}^*)^2_{u,v}$, this action is given by $(u, v) \cdot (x, y) := (ux, vy)$. In general, the action of $\mathcal{T}_N$ on X_{σ} may be described in intrinsic terms by:

$$(M \xrightarrow{\tau} \mathbb{C}^*) \cdot (\sigma^{\vee} \cap M \xrightarrow{p} \mathbb{C}) := (\sigma^{\vee} \cap M \xrightarrow{\tau \cdot p} \mathbb{C}).$$

In the previous equation we used again the interpretations of the points of $\mathcal{T}_N$ and X_σ as morphisms of monoids (see Eqs. (1.17) and (1.20)).

Assume now that $\mathcal{F}$ is a fan of N, in the sense of Definition 1.3.3. Each affine toric variety X_σ, where $\sigma \in \mathcal{F}$, contains the torus $\mathcal{T}_N$ as an affine open set. If σ and τ are two cones of $\mathcal{F}$, then one has a natural identification of their respective tori, and also of their larger Zariski open subsets $X_{\sigma \cap \tau} \subset X_\sigma$ and $X_{\sigma \cap \tau} \subset X_\tau$. If one glues the various affine toric varieties $(X_\sigma)_{\sigma \in \mathcal{F}}$ using the previous identifications, one gets an abstract separated complex algebraic variety $X_{\mathcal{F}}$ which still contains the torus $\mathcal{T}_N$ as an affine open subset (see [91, Theorem 1.4 and 1.5]).

Definition 1.3.15 The **toric variety** associated with a fan $\mathcal{F}$ of a lattice N is the variety $\boxed{X_{\mathcal{F}}}$ constructed above.

Remark 1.3.16 All toric varieties constructed from fans are normal in the sense of Definition 1.2.11 (see [26, Theorem 1.3.5]). One has a more general notion of toric variety, which includes some non-normal varieties as well (see the paper [56] of Teissier and the second author). Those varieties can be described as before by gluing maximal spectra of algebras of not necessarily saturated finite type submonoids of lattices, the normal ones being precisely the toric varieties associated with a fan of Definition 1.3.15.

As a consequence of Proposition 1.3.12, one has a smoothness criterion for toric varieties:

Proposition 1.3.17 *Let $\mathcal{F}$ be a fan of the lattice N. Then the toric variety $X_{\mathcal{F}}$ is smooth if and only if $\mathcal{F}$ is regular in the sense of Definition 1.3.5.*

Let us come back to a fan $\mathcal{F}$ of a weight lattice N. When ρ varies among the cones of $\mathcal{F}$, the actions of the torus $\mathcal{T}_N$ on the affine toric varieties X_ρ glue into an action on $X_{\mathcal{F}}$, whose orbits are still denoted by O_ρ. The conservation of the notation (1.22) is motivated by the fact that in the gluing of X_σ and X_τ, the orbits denoted O_ρ on both sides get identified, for every face ρ of $\sigma \cap \tau$. If ρ is a cone of the fan $\mathcal{F}$, we denote by $\boxed{\overline{O_\rho}}$ the closure in $X_{\mathcal{F}}$ of the orbit O_ρ. The orbit closure $\overline{O_\rho}$ has also a natural structure of normal toric variety (see [41, Chapter 3]).

The torus $\mathcal{T}_N$ is identified canonically with the orbit O_0 corresponding to the origin of $N_{\mathbb{R}}$, seen as a cone of dimension 0. Its complement is the union of all the orbits of codimension at least 1. Let us introduce a special name and notation for this complement:

Definition 1.3.18 Let $X_{\mathcal{F}}$ be a toric variety defined by a fan $\mathcal{F}$. Its **boundary** $\boxed{\partial X_{\mathcal{F}}}$ is the complement of the algebraic torus $\mathcal{T}_N$ inside $X_{\mathcal{F}}$.

The boundary $\partial X_{\mathcal{F}}$ is a reduced Weil divisor inside $X_{\mathcal{F}}$, whose irreducible components are the orbit closures $\overline{O_\rho}$ corresponding to the cones ρ of $\mathcal{F}$ which have dimension 1, that is, to the rays of the fan $\mathcal{F}$.

1.3.3 Toric Morphisms and Toric Modifications

In this subsection we define the notion of *toric morphism* between toric varieties (see Definition 1.3.19) and we explain in which way refining a fan defines a special kind of toric morphism, called a *toric modification* (see Proposition 1.3.21). In Examples 1.3.26 and 1.3.27 we explain how to do concrete computations of toric modifications in dimension two, the second one giving a toric presentation of the blow ups of the origin. Finally, Proposition 1.3.28 explains the combinatorics of the minimal resolution of a normal affine toric surface.

Assume that N_1 and N_2 are two weight lattices, endowed with cones σ_1 and σ_2. Let $\phi : N_1 \to N_2$ be a morphism of lattices which sends the cone σ_1 into the cone σ_2. Using the second interpretation in the equalities (1.20) of the points of affine toric varieties, we see that ϕ induces an algebraic morphism between the associated toric varieties:

$$\begin{array}{rcl} \boxed{\psi^{\sigma_1}_{\sigma_2,\phi}} : X_{\sigma_1} & \to & X_{\sigma_2} \\ p_1 & \to & p_1 \circ \phi^{\vee} \end{array} . \tag{1.23}$$

One sees immediately from the definitions that the adjoint $\phi^{\vee} : M_2 \to M_1$ of ϕ maps $\sigma_2^{\vee}$ into $\sigma_1^{\vee}$, which shows that the composition $p_1 \circ \phi^{\vee}$ belongs indeed to $X_{\sigma_2} = \text{Hom}(\sigma_2^{\vee} \cap M_2, \mathbb{C})$ whenever $p_1 \in X_{\sigma_1} = \text{Hom}(\sigma_1^{\vee} \cap M_1, \mathbb{C})$. The morphism $\psi^{\sigma_1}_{\sigma_2,\phi}$ may be also described using the first interpretation in the equalities (1.20), as the morphism of affine schemes induced by the morphism of $\mathbb{C}$-algebras $\mathbb{C}[\sigma_2^{\vee} \cap M_2] \to \mathbb{C}[\sigma_1^{\vee} \cap M_1]$ which sends each monomial $\chi^{m_2} \in \sigma_2^{\vee} \cap M_2$ to the monomial $\chi^{\phi^{\vee}(m_2)} \in \sigma_1^{\vee} \cap M_1$.

Assume now that N_1 and N_2 are endowed with fans $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively, such that ϕ sends each cone of $\mathcal{F}_1$ into some cone of $\mathcal{F}_2$. We say that ϕ **is compatible with the two fans**. It may be checked formally that the previous morphisms $\psi^{\sigma_1}_{\sigma_2,\phi} : X_{\sigma_1} \to X_{\sigma_2}$, for all the pairs $(\sigma_1, \sigma_2) \in \mathcal{F}_1 \times \mathcal{F}_2$ which verify that $\phi(\sigma_1) \subseteq \sigma_2$, glue into an algebraic morphism: $\boxed{\psi^{\mathcal{F}_1}_{\mathcal{F}_2,\phi}} : X_{\mathcal{F}_1} \to X_{\mathcal{F}_2}$. This morphism is moreover *equivariant* with respect to the actions of $\mathcal{T}_{N_1}$ and $\mathcal{T}_{N_2}$ on $X_{\mathcal{F}_1}$ and $X_{\mathcal{F}_2}$ respectively. For this reason, one uses the following terminology:

Definition 1.3.19 If the morphism of lattices $\phi : N_1 \to N_2$ sends every cone of $\mathcal{F}_1$ into some cone of $\mathcal{F}_2$, then the morphism of algebraic varieties $\psi^{\mathcal{F}_1}_{\mathcal{F}_2,\phi} : X_{\mathcal{F}_1} \to X_{\mathcal{F}_2}$ described above is called the **toric morphism associated with ϕ and the fans $\mathcal{F}_1, \mathcal{F}_2$** .

The toric morphism $\psi^{\mathcal{F}_1}_{\mathcal{F}_2,\phi}$ sends the torus $\mathcal{T}_{N_1} = X_{\mathcal{F}_1} \setminus \partial X_{\mathcal{F}_1}$ into $\mathcal{T}_{N_2} = X_{\mathcal{F}_2} \setminus \partial X_{\mathcal{F}_2}$. This fact implies the following property of toric morphisms relative to the boundaries of their sources and targets, in the sense of Definition 1.3.18:

Proposition 1.3.20 *Let* $\psi : X_{\mathcal{F}_1} \to X_{\mathcal{F}_2}$ *be the toric morphism associated with* ϕ *and the fans* $\mathcal{F}_1$ *and* $\mathcal{F}_2$. *Then* $\psi^{-1}(\partial\, X_{\mathcal{F}_2}) \subseteq \partial\, X_{\mathcal{F}_1}$.

Toric morphisms have the following properties (see [91, Theorems 1.13, 1.15]):

Proposition 1.3.21 *Let* N_1, N_2 *be two lattices and* $\mathcal{F}_1, \mathcal{F}_2$ *be fans of* N_1 *and* N_2 *respectively. Let* $\phi : N_1 \to N_2$ *be a lattice morphism compatible with the two fans. Then:*

1. *The morphism* $\psi^{\mathcal{F}_1}_{\mathcal{F}_2,\phi}$ *is birational if and only if* ϕ *is an isomorphism of lattices.*
2. *The morphism* $\psi^{\mathcal{F}_1}_{\mathcal{F}_2,\phi}$ *is proper if and only if the* $\mathbb{R}$*-linear map* $\phi_{\mathbb{R}} : (N_1)_{\mathbb{R}} \to (N_2)_{\mathbb{R}}$ *sends the support of* $\mathcal{F}_1$ onto *the support of* $\mathcal{F}_2$.

In particular, $\psi^{\mathcal{F}_1}_{\mathcal{F}_2,\phi}$ *is a modification in the sense of Definition 1.2.31 if and only if* ϕ *is an isomorphism and, after identifying* N_1 *and* N_2 *using it, the fan* $\mathcal{F}_1$ *refines the fan* $\mathcal{F}_2$ *in the sense of Definition 1.3.3.*

We will consider most of the time the particular case in which $N_1 = N_2 = N$ is a lattice of rank 2 and ϕ is the identity. Then, if $\sigma \subset \sigma_0$ is a subcone of σ_0, we denote by $\psi^{\sigma}_{\sigma_0}$ the birational toric morphism induced by the identity:

$$\boxed{\psi^{\sigma}_{\sigma_0}} : X_{\sigma} \to X_{\sigma_0} = \mathbb{C}^2_{x,y}. \tag{1.24}$$

When σ varies among all the cones of a fan $\mathcal{F}$ which subdivides the cone σ_0, the morphisms $\psi^{\sigma}_{\sigma_0}$ glue into a single equivariant birational morphism:

$$\boxed{\psi^{\mathcal{F}}_{\sigma_0}} : X_{\mathcal{F}} \to X_{\sigma_0} = \mathbb{C}^2_{x,y}. \tag{1.25}$$

By Proposition 1.3.21, this morphism is also proper, because $\mathcal{F}$ and σ_0 have the same support. Therefore, $\psi^{\mathcal{F}}_{\sigma_0}$ is a modification of $\mathbb{C}^2_{x,y}$.

The strict transform of $\boxed{L} := Z(x)$ (resp. of $\boxed{L'} := Z(y)$) by the modification $\psi^{\mathcal{F}}_{\sigma_0}$ is the orbit closure $\overline{O}_{\mathbb{R}_+ e_1}$ (resp. $\overline{O}_{\mathbb{R}_+ e_2}$) in $X_{\mathcal{F}}$. The preimage of $0 \in \mathbb{C}^2_{x,y}$, called the **exceptional divisor** of $\psi^{\mathcal{F}}_{\sigma_0}$, and the preimage of the sum $L + L'$ of the coordinate axes, which is the total transform of $L + L'$ are:

$$\begin{aligned} (\psi^{\mathcal{F}}_{\sigma_0})^{-1}(0) \qquad &= \overline{O}_{\rho_1} + \cdots + \overline{O}_{\rho_k}, \\ (\psi^{\mathcal{F}}_{\sigma_0})^{-1}(L + L') &= \overline{O}_{\mathbb{R}_+ e_1} + \overline{O}_{\rho_1} + \cdots + \overline{O}_{\rho_k} + \overline{O}_{\mathbb{R}_+ e_2}, \end{aligned} \tag{1.26}$$

where $\rho_1, \ldots, \rho_k$ denote the rays of $\mathcal{F}$ contained in the interior of σ_0, labeled as in Fig. 1.11. Note that $L + L' = \partial X_{\sigma_0}$ and $(\psi^{\mathcal{F}}_{\sigma_0})^{-1}(L + L') = \partial X_{\mathcal{F}}$, which is a particular case of Proposition 1.3.21.

Recall now the following classical notion of (unweighted) *dual graph*, which extends that of Definition 1.2.36 and whose historical evolution was sketched by the third author in [104]:

Definition 1.3.22 A **simple normal crossings curve** is a reduced abstract complex curve whose irreducible components are smooth and whose singularities are normal crossings, that is, analytically isomorphic to the germ at the origin of the union of coordinate axes of $\mathbb{C}^2$. The **dual graph** of a simple normal crossings curve D is the abstract graph whose set of vertices is associated bijectively with the set of irreducible components of D, the edges between two vertices corresponding bijectively with the intersection points of the associated components of D. Each vertex or edge is labeled by the corresponding irreducible component or point of D.

Remark 1.3.23 Let $\sigma = \mathbb{R}_+\langle f_1, f_2\rangle \subset N_{\mathbb{R}}$ be a strictly convex cone of dimension two, not necessarily regular. One may check that the boundary $\partial X_\sigma = \overline{O}_{\mathbb{R}_+ f_1} + \overline{O}_{\mathbb{R}_+ f_2}$ of the affine toric surface X_σ is an abstract simple normal crossings curve, according to Definition 1.3.22.

The dual graph of the total transform $(\psi^{\mathcal{F}}_{\sigma_0})^{-1}(L + L')$ may be embedded in the cone $\sigma_0 \subseteq N_{\mathbb{R}}$:

Proposition 1.3.24 *Let $\mathcal{F}$ be a fan which subdivides the regular cone σ_0. Then the dual graph of the divisor $(\psi^{\mathcal{F}}_{\sigma_0})^{-1}(L + L')$ is a segment with extremities L and L' and with k intermediate points labeled in order by $\overline{O}_{\rho_1}, \dots, \overline{O}_{\rho_k}$ from L to L'. That is, it is isomorphic to the segment $[e_1, e_2] \subset N_{\mathbb{R}}$, marked with its intersection points with the rays of $\mathcal{F}$, the point $[e_1, e_2] \cap \rho_i$ being labeled by the orbit closure $\overline{O}_{\rho_i}$.*

Therefore, the rays of the fan $\mathcal{F}$ correspond bijectively to the irreducible components of the total transform $(\psi^{\mathcal{F}}_{\sigma_0})^{-1}(L + L')$ of $L + L'$. The 2-dimensional cones of $\mathcal{F}$ correspond to the fixed points of the torus action, which are the only possible singular points of the surface $X_{\mathcal{F}}$. The orbit closures $\overline{O}_\rho$ and $\overline{O}_{\rho'}$ intersect at a point $q \in X_{\mathcal{F}}$ if and only if the cone $\rho + \rho'$ is a 2-dimensional cone of $\mathcal{F}$ and then q is the unique orbit $O_{\rho+\rho'}$ of dimension 0 of the affine toric surface $X_{\rho+\rho'} \subset X_{\mathcal{F}}$. The point q is singular on the surface $X_{\mathcal{F}}$ if and only if the cone $\rho + \rho'$ is not regular.

Example 1.3.25 For the fan $\mathcal{F}(3/5, 2/1, 5/2)$ of Fig. 1.8 discussed in Example 1.3.6, the total transform $(\psi^{\mathcal{F}}_{\sigma_0})^{-1}(L + L')$ and its dual graph are represented in Fig. 1.11. The 4 singular points of the total transform are also singular on the surface $X_{\mathcal{F}}$, with the exception of $\overline{O}_{\rho_2} \cap \overline{O}_{\rho_3}$. Indeed, the cone $\rho_2 + \rho_3$ is the only regular 2-dimensional cone of the fan $\mathcal{F}$, as may be seen in Fig. 1.9.

Example 1.3.26 Let us explain how to describe in coordinates the morphism $\psi^{\sigma}_{\sigma_0}$ of (1.24), when σ is a *regular* subcone of σ_0. Denote by f_1, f_2 the primitive generators of the edges of σ, ordered in such a way that the bases (e_1, e_2) and (f_1, f_2) define the same orientation of $N_{\mathbb{R}}$ (see Fig. 1.12). Decompose (f_1, f_2) in the basis (e_1, e_2), writing $f_1 = \alpha e_1 + \beta e_2$ and $f_2 = \gamma e_1 + \delta e_2$. This means that the

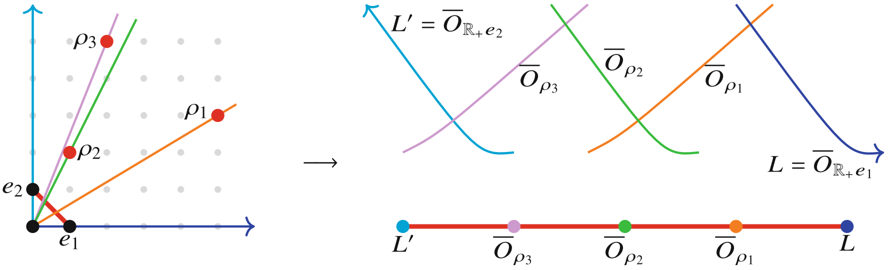


Fig. 1.11 The dual graph of the total transform $(\psi^{\mathcal{F}}_{\sigma_0})^{-1}(L + L')$

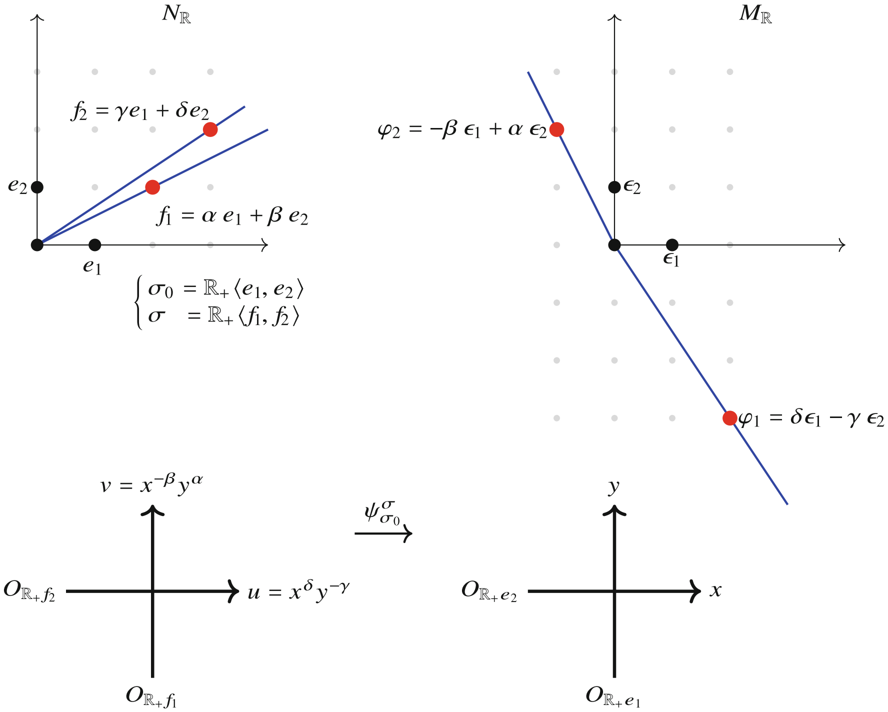


Fig. 1.12 The toric morphism defined by the two regular cones of Example 1.3.26

unimodular matrix of change of bases from (f_1, f_2) to (e_1, e_2) is:

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}. \tag{1.27}$$

Denote by $(\varphi_1, \varphi_2) \in M^2$ the dual basis of (f_1, f_2) and by

$$\begin{cases} u := \chi^{\varphi_1} = x^{\delta} y^{-\gamma} \\ v := \chi^{\varphi_2} = x^{-\beta} y^{\alpha}, \end{cases} \tag{1.28}$$

the associated coordinates. Then, in terms of the identifications $X_\sigma = \mathbb{C}^2_{u,v}$ and $X_{\sigma_0} = \mathbb{C}^2_{x,y}$, the morphism $\psi^{\sigma}_{\sigma_0}$ is given by the following monomial change of coordinates (compare the disposal of exponents with the matrix (1.27)):

$$\begin{cases} x = u^\alpha v^\gamma \\ y = u^\beta v^\delta. \end{cases} \tag{1.29}$$

Note that the system (1.28) implies that the expression of $v = \chi^{\varphi_2}$ as a monomial in x and y is determined only by f_1, being independent of the choice of f_2. This may be explained geometrically. Indeed, as $f_1 \cdot \varphi_2 = 0$, we see that φ_2 belongs to the line $f_1^\perp$ orthogonal to f_1. As φ_2 may be completed into a basis of M, it is primitive, which determines it up to sign. This sign ambiguity is lifted by the constraint that the basis (f_1, f_2) determines the open half-plane bounded by the line $\mathbb{R}f_1$ on which φ_2 has to be positive. Note also that v is a coordinate on the orbit $O_{\mathbb{R}_+ f_1}$ determined by the edge $\mathbb{R}_+ f_1$ of σ. This coordinate determines an isomorphism $O_{\mathbb{R}_+ f_1} \simeq \mathbb{C}^*_v$ of complex tori, and depends only on $\mathbb{R}_+ f_1$, since the orbit $O_{\mathbb{R}_+ f_1}$ can be realized as a subspace of the surface $X_{\mathbb{R}_+ f_1}$ by formula (1.22) above.

Example 1.3.27 In this example we use the explanations given in Example 1.3.26. Let $\mathcal{F}$ be the fan obtained by subdividing $\sigma_0 = \mathbb{R}_+\langle e_1, e_2\rangle$ using the half-line ρ generated by $e_1 + e_2$. It has two cones of dimension 2, denoted $\sigma_1 := \mathbb{R}_+\langle e_1, e_1 + e_2\rangle$ and $\sigma_2 := \mathbb{R}_+\langle e_1 + e_2, e_2\rangle$ (see Fig. 1.13). Then the toric morphism $\psi^{\mathcal{F}}_{\sigma_0}$ may be described by its two restrictions $\psi^{\sigma_1}_{\sigma_0}$ and $\psi^{\sigma_2}_{\sigma_0}$. The matrices of change of bases from $(e_1, e_1 + e_2)$ and $(e_1 + e_2, e_2)$ to (e_1, e_2) respectively are $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Denoting by (u_1, u_2) and (v_1, v_2) the coordinates corresponding to the dual bases of $(e_1, e_1 + e_2)$ and $(e_1 + e_2, e_2)$, the general formulas (1.27) and (1.29) show that

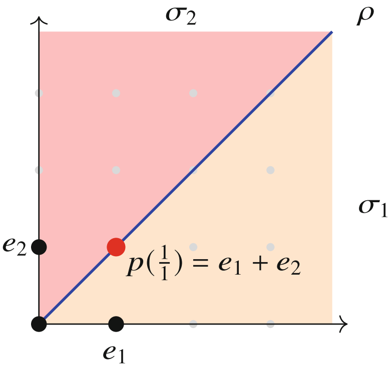


Fig. 1.13 The subdivision of Example 1.3.27, defining the toric blow up of the origin of $\mathbb{C}^2$

the morphisms $\psi_{\sigma_0}^{\sigma_1}$ and $\psi_{\sigma_0}^{\sigma_2}$ are given by the following changes of variables:

$$\begin{cases} x = u_1 u_2 \\ y = \quad u_2, \end{cases} \quad \text{and} \quad \begin{cases} x = v_1 \\ y = v_1 v_2. \end{cases} \tag{1.30}$$

We get the same expressions as in Eqs. (1.4). This shows that $\psi_{\sigma_0}^{\mathcal{F}}$ *is a toric representative of the blow up morphism of* $\mathbb{C}^2_{x,y}$ *at the origin*!

Let σ be a non-regular cone of the weight lattice N of rank two. By Proposition 1.3.12, the affine toric surface X_σ is not smooth. In fact, it has only one singular point, the orbit O_σ of dimension 0. Being of dimension 2, X_σ admits a **minimal resolution**, that is, a resolution through which factors any other resolution (recall that this notion was explained in Definition 1.2.34). It turns out that this minimal resolution may be given by a toric morphism, defined by the regularization of σ in the sense of Definition 1.3.8 (see [91, Proposition 1.19]):

Proposition 1.3.28 *Let* σ *be a non-regular cone of the weight lattice* N *of rank two. Denote by* σ^{reg} *the regularization of the fan formed by the faces of* σ. *Then the toric modification* $\psi_\sigma^{\sigma^{reg}} : X_{\sigma^{reg}} \to X_\sigma$ *is the minimal resolution of* X_σ. *As a consequence, for any fan* $\mathcal{F}$ *of* N, *the toric modification* $\psi_{\mathcal{F}}^{\mathcal{F}^{reg}} : X_{\mathcal{F}^{reg}} \to X_{\mathcal{F}}$ *is the minimal resolution of* $X_{\mathcal{F}}$.

1.3.4 Toroidal Varieties and Modifications in the Toroidal Category

In this subsection we explain analytic generalizations of toric varieties and toric morphisms: the notions of *toroidal variety* and *morphism of toroidal varieties* (see Definition 1.3.29). Then we introduce the notion of *cross* on a smooth germ of surface (see Definition 1.3.31), and we explain how to attach to a cross a canonical oriented regular cone in a two-dimensional lattice (see Definition 1.3.32) and how each subdivision of this cone determines a canonical modification in the toroidal category (see Definition 1.3.33). The toroidal pseudo-resolutions of plane curve singularities introduced in Sect. 1.4.2 below will be constructed as compositions of such toroidal modifications.

Toric surfaces and morphisms are not sufficient for the study of plane curve singularities for the following reasons. One starts often from a germ of curve on a smooth complex surface which does not have a preferred coordinate system. It may be impossible to choose a coordinate system such that the germ of curve gets resolved by only one toric modification relative to the chosen coordinates (if the curve singularity is reduced and such a resolution is possible, then one says that the singularity is *Newton non-degenerate*, see Definition 1.4.21 below). Instead, what may be always achieved is a *morphism of toroidal surfaces*, in the following sense:

Definition 1.3.29 A **toroidal variety** is a pair $(\Sigma, \partial\Sigma)$ consisting of a normal complex variety Σ and a reduced divisor $\partial\Sigma$ on Σ such that the germ of $(\Sigma, \partial\Sigma)$ at any point $p \in \Sigma$ is analytically isomorphic to the germ of a pair $(X_\sigma, \partial X_\sigma)$ at a point of X_σ, where ∂X_σ denotes the boundary of the affine toric variety X_σ in the sense of Definition 1.3.18. Such an isomorphism is called a **toric chart centered at** p of the toroidal variety $(\Sigma, \partial\Sigma)$. The divisor $\partial\Sigma$ is **the boundary of the toroidal variety**.

A **morphism** $\psi : (\Sigma_2, \partial\Sigma_2) \to (\Sigma_1, \partial\Sigma_1)$ between toroidal varieties is a complex analytic morphism $\psi : \Sigma_2 \to \Sigma_1$ such that $\psi^{-1}(\partial\Sigma_1) \subseteq \partial\Sigma_2$. The morphism ψ is a **modification** if the underlying morphism of complex varieties is a modification in the sense of Definition 1.2.31.

Toroidal varieties with their morphisms define a category, called the **toroidal category**.

The previous definition implies that if $(\Sigma, \partial\Sigma)$ is toroidal, then the complement $\Sigma \setminus \partial\Sigma$ is smooth. Indeed, the point p is allowed to be taken outside the boundary $\partial\Sigma$, and the definition shows then that the germ of Σ at p is analytically isomorphic to the germ of a toric variety at a point of the associated torus, which is smooth.

If Σ is of dimension two and if p is a smooth point of $\partial\Sigma$, then p is a smooth point of Σ, since the germ of Σ at p is analytically isomorphic to the germ of a normal toric surface at a point belonging to a 1-dimensional orbit, which is necessarily smooth.

Proposition 1.3.20 implies that a toric morphism $\psi^{\mathcal{F}_1}_{\mathcal{F}_2,\phi} : X_{\mathcal{F}_1} \to X_{\mathcal{F}_2}$ becomes an element of the toroidal category if one looks at it as a complex analytic morphism from the pair $(X_{\mathcal{F}_1}, \partial X_{\mathcal{F}_1})$ to the pair $(X_{\mathcal{F}_2}, \partial X_{\mathcal{F}_2})$, the boundaries being taken in the sense of Definition 1.3.18.

Remark 1.3.30 There exists also a more restrictive notion of *toroidal morphism* $\psi : (\Sigma_2, \partial\Sigma_2) \to (\Sigma_1, \partial\Sigma_1)$ between toroidal varieties. By definition, such a morphism becomes monomial in the neighborhood of any point p of Σ_2, after some choice of toric charts at the source and the target, centered at p and $\psi(p)$ respectively. Toroidal morphisms belong to the toroidal category, but the converse is not true. For instance, take two copies $\mathbb{C}^2_{u,v}$ and $\mathbb{C}^2_{x,y}$ of the complex affine plane and the affine morphism $\psi : \mathbb{C}^2_{u,v} \to \mathbb{C}^2_{x,y}$ defined by $x = u, y = u(1 + v)$. Consider the plane $\mathbb{C}^2_{u,v}$ as a toroidal surface with boundary equal to the union of its coordinate axes, and $\mathbb{C}^2_{x,y}$ as a toroidal surface with boundary equal to the y-axis. As $\psi^{-1}(\partial\mathbb{C}^2_{x,y}) \subseteq \partial\mathbb{C}^2_{u,v}$, ψ is a morphism of toroidal varieties. But it is not a toroidal morphism. Otherwise, it would become the morphism $(u, v) \to (u, u)$ after analytic changes of coordinates in the neighborhoods of the origins of the two planes, which is impossible, because ψ is birational, therefore dominant.

Let us come back to the case of a smooth germ of surface (S, o).

Definition 1.3.31 A **cross** on the smooth germ of surface (S, o) is a pair (L, L') of transversal smooth branches on (S, o). A local coordinate system (x, y) on (S, o) is said **to define the cross** (L, L') if $L = Z(x)$ and $L' = Z(y)$.

We chose the name *cross* by analogy with the denomination *normal crossings divisor* (see Definition 1.2.32). Note the subtle difference between the two notions: the pair (L, L') is a cross if and only if $L + L'$ is a normal crossings divisor, but the knowledge of the divisor does not allow to remember the order of its branches.

Definition 1.3.32 Let (L, L') be a cross on (S, o). We associate with it the two-dimensional lattice $\boxed{M_{L,L'}}$ of integral divisors supported by $L \cup L'$, called the **monomial lattice of the cross** (L, L'). The **weight lattice of the cross** (L, L') is the dual lattice $\boxed{N_{L,L'}}$ of $M_{L,L'}$. Denote by $\boxed{(\epsilon_L, \epsilon_{L'})}$ the basis $\epsilon_L := L, \epsilon_{L'} := L'$ of $M_{L,L'}$, by $\boxed{(e_L, e_{L'})}$ the dual basis of $N_{L,L'}$, and by $\boxed{\sigma_0^{L,L'}}$ the cone $\mathbb{R}_+\langle e_L, e_{L'}\rangle$. When the cross (L, L') is clear from the context, we often write simply $\boxed{(\epsilon_1, \epsilon_2)}$, $\boxed{(e_1, e_2)}$ and $\boxed{\sigma_0}$ instead of $(\epsilon_L, \epsilon_{L'})$, $(e_L, e_{L'})$ and $\sigma_0^{L,L'}$ respectively.

Each time we choose local coordinates (x, y) defining the cross (L, L'), we identify $M_{L,L'}$ with the lattice of exponents of monomials in those coordinates. That is, $a\epsilon_1 + b\epsilon_2$ corresponds to $x^a y^b$. Such a choice of coordinates also identifies holomorphically a neighborhood of o in S with a neighborhood of the origin in $\mathbb{C}^2$ and the cross (L, L') with the coordinate cross in $\mathbb{C}^2$ at the origin. Therefore, any subdivision $\mathcal{F}$ of σ_0 defines an analytic modification $\psi_{L,L'}^{\mathcal{F}} : S_{\mathcal{F}} \to S$ of S. As these modifications are isomorphisms over $S \setminus \{o\}$, it is easy to see that they are independent of the chosen coordinate system (x, y) defining (L, L'), up to canonical analytical isomorphisms above S. Moreover, if we define $\partial S := L + L'$ and $\partial S_{\mathcal{F}} := (\psi_{L,L'}^{\mathcal{F}})^{-1}(L + L')$, the morphism $\psi_{L,L'}^{\mathcal{F}}$ becomes a morphism from the toroidal surface $(S_{\mathcal{F}}, \partial S_{\mathcal{F}})$ to the toroidal surface $(S, \partial S)$.

Definition 1.3.33 If $\mathcal{F}$ is a fan subdividing the cone $\sigma_0 \subset N_{L,L'}$, then the morphism of the toroidal category

$$\boxed{\psi_{L,L'}^{\mathcal{F}} : (S_{\mathcal{F}}, \partial S_{\mathcal{F}}) \to (S, L + L')}$$

associated with $\mathcal{F}$ is the **modification of** S **associated with** $\mathcal{F}$ **relative to the cross** (L, L').

When the fan $\mathcal{F}$ is regular, the morphism $\psi_{L,L'}^{\mathcal{F}}$ between the underlying complex surfaces (forgetting the toroidal structures) is a composition of blow ups of points (see Definition 1.2.29). We will explain the structure of this decomposition of $\psi_{L,L'}^{\mathcal{F}}$ in Sect. 1.5 (see Propositions 1.5.10, 1.5.11).

1.3.5 Historical Comments

Toric varieties were called *torus embeddings* at the beginning of the development of toric geometry in the 1970s, following the terminology of Kempf, Knudsen, Mumford and Saint-Donat's 1973 book [71], as these are varieties into which an algebraic torus embeds as an affine Zariski open subset. The introduction of the book [71] contains information about sources of toric geometry in papers by Demazure, Hochster, Bergman, Sumihiro and Miyake & Oda. Details about the development of toric geometry may be found in Cox, Little and Schenck's 2011 book [26, Appendix A].

The first applications of toric geometry to the study of singularities were done by Kouchnirenko, Varchenko and Khovanskii in their 1976–77 papers [74, 128] and [73] respectively. But one may see in retrospect toric techniques in Puiseux's 1850 paper [106, Sections 20, 23], in Jung's 1908 paper [68], in Dumas' 1911–12 papers [31, 32], in Hodge's 1930 paper [63], in Hirzebruch's 1953 paper [62] and in Teissier's 1973 paper [119]. Indeed, in all those papers, monomial changes of variables more general than those describing blow ups are used in an essential way. For instance, in his paper [62], Hirzebruch described the minimal resolution of an affine toric surface by gluing the toric charts of the resolved surface by explicit monomial birational maps. Toric surfaces appeared in Hirzebruch's paper as normalizations of the affine surfaces in $\mathbb{C}^3$ defined by equations of the form $z^m = x^p y^q$, with $(m, p, q) \in (\mathbb{N}^*)^3$ globally coprime. Interesting details about Hirzebruch's work [62] are contained in Brieskorn's paper [14].

The notion of *toroidal variety* of arbitrary dimension was introduced in a slightly different form in the same book [71] of Kempf, Knudson, Mumford and Saint-Donat. The emphasis was put there on a given complex manifold V, and one looked for partial compactifications of it which were locally analytically isomorphic to embeddings of an algebraic torus into a toric variety. Such partial compactifications $\overline{V}$ were called *toroidal embeddings* of V. Therefore, a toroidal embedding was a pair $(\overline{V}, V)$ such that $(\overline{V}, \overline{V} \setminus V)$ is a toroidal variety in our sense. For more remarks about the toroidal category see [4, Section 1.5].

1.4 Toroidal Pseudo-Resolutions of Plane Curve Singularities

In Sect. 1.4.1 we introduce the notions of *Newton polygon* $\mathcal{N}_{L,L'}(C)$, *tropical function* $\text{trop}^C_{L,L'}$, *Newton fan* $\mathcal{F}^C_{L,L'}$ and *Newton modification* $\psi^C_{L,L'}$ (see Definition 1.4.14) determined by a curve singularity C on the smooth germ of surface (S, o), relative to a cross (L, L'). The strict transform of C by its Newton modification is a finite set of germs. If one completes for each one of them the corresponding germ of exceptional divisor into a cross, one gets again a Newton polygon, a fan and a modification. This produces an *algorithm of toroidal pseudo-resolution* of C (see Algorithm 1.4.22). It leads only to a *pseudo-resolution* morphism, because its source is a possibly singular surface (with toric singularities). In Sect. 1.4.3 we explain how to modify Algorithm 1.4.22 in order to get an algorithm of *embedded resolution* of C. In Sect. 1.4.4 we encode the combinatorics of this algorithm into a *fan tree* (see Definition 1.4.33), which is a rooted tree endowed with a *slope function*, constructed by gluing *trunks* associated with the Newton fans generated by the process. The final Sect. 1.4.5 contains historical information about Newton's and Puiseux's work on plane curve singularities, the resolution of such singularities by iteration of morphisms which are toric in suitable coordinates, and the relations with tropical geometry.

1.4.1 Newton Polygons, Their Tropicalizations, Fans and Modifications

This subsection begins with the definitions of the *Newton polygon* $\mathcal{N}(f)$ (see Definition 1.4.2), the *tropicalization* (see Definition 1.4.4) and the *Newton fan* $\mathcal{F}(f)$ (see Definition 1.4.9) associated with a non-zero germ $f \in \mathbb{C}[[x, y]]$. It turns out that they only depend on the germs L, L', C defined by x, y and f respectively (see Proposition 1.4.13). Therefore, given a cross (L, L') and a plane curve singularity C on the smooth germ (S, o), one has associated Newton polygon, tropicalization and fan. This fan allows to introduce the *Newton modification* of the toroidal germ $(S, L + L')$ determined by C (see Definition 1.4.14).

Assume that a cross (L, L') is fixed on (S, o) (see Definition 1.3.31) and that (x, y) is a local coordinate system defining it. This system allows to see any $f \in \hat{\mathcal{O}}_{S,o}$ as a series in the variables (x, y), that is, in toric terms, as a possibly infinite sum of terms of the form $\boxed{c_m(f)\, \chi^m}$, for $c_m(f) \in \mathbb{C}$ and $m \in \sigma_0^{\vee} \cap M$, where $\boxed{M} := M_{L,L'}$ and $\boxed{\sigma_0} := \sigma_0^{L,L'}$ (see Definition 1.3.32). Denote also $\boxed{N} := N_{L,L'}$. One has canonical identifications $M \simeq \mathbb{Z}^2$, $N \simeq \mathbb{Z}^2$, $\sigma_0 \simeq (\mathbb{R}_+)^2$, and $\sigma_0^{\vee} \simeq (\mathbb{R}_+)^2$.

Definition 1.4.1 Let $f \in \mathbb{C}[[x, y]]$ be a nonzero series. The **support** $\mathcal{S}(f) \subseteq \sigma_0^{\vee} \cap M \simeq \mathbb{N}^2$ of f is the set of exponents of monomials with non-zero coefficients in f. That is, if

$$f = \sum_{m \in \sigma_0^{\vee} \cap M} c_m(f) \chi^m, \tag{1.31}$$

then $\boxed{\mathcal{S}(f)} := \{m \in \sigma_0^{\vee} \cap M,\ c_m(f) \neq 0\}$.

If Y is a subset of a real affine space, then $\boxed{\mathrm{Conv}(Y)}$ denotes its **convex hull**.

Definition 1.4.2 Let $f \in \mathbb{C}[[x, y]]$. Its **Newton polygon** $\mathcal{N}(f)$ is the following convex subset of $\sigma_0^\vee \simeq (\mathbb{R}_+)^2$:

$$\boxed{\mathcal{N}(f)} := \text{Conv}(\mathcal{S}(f) + (\sigma_0^\vee \cap M)).$$

Its **faces** are its vertices, its edges and the whole polygon itself. If K is a compact edge of the boundary $\boxed{\partial\mathcal{N}(f)}$ of $\mathcal{N}(f)$, then the **restriction** $\boxed{f_K}$ **of** f **to** K is the sum of the terms of f whose exponents belong to K.

Remark 1.4.3 In general, the Newton polygon of an element of $\hat{O}_{S,o}$ depends on the choice of local coordinates. For instance, let us consider the change of coordinates $(x, y) = (u, u + v)$. The function $f(x, y) := y^2 - x^3$ becomes $g(u, v) := f(u, u + v) = (u+v)^2 - u^3$. The corresponding Newton polygons are represented in Fig. 1.14. In contrast, if the local coordinate change preserves the coordinate curves, then the Newton polygon remains unchanged (see Proposition 1.4.13 below).

Suppose now that the variables x and y are weighted by non-negative real numbers. Denote by $c \in \mathbb{R}_+$ the weight of x and by $d \in \mathbb{R}_+$ the weight of y. Therefore the pair $w := (c, d)$ may be seen as an element of the weight vector space $N_\mathbb{R} = (N_{L,L'})_\mathbb{R}$. More precisely, one has $w \in (\mathbb{R}_+)^2 \simeq \sigma_0$. Assuming that the non-zero complex constants have weight 0, we see that the weight $w(c_m(f)\chi^m)$ of a non-zero term of f is simply $w \cdot m \in \mathbb{R}_+$. Define then the w**-weight of the series** $f \in \mathbb{C}[[x, y]]$ as the minimal weight of its terms. One gets the function:

$$\boxed{\nu_w} : \begin{array}{ccc} \mathbb{C}[[x, y]] & \to & \mathbb{R}_+ \cup \{\infty\} \\ f & \to & \min\{w \cdot m,\ m \in \mathcal{S}(f)\} \end{array}. \tag{1.32}$$

It is an exercise to show that ν_w is a valuation on the $\mathbb{C}$-algebra $\mathbb{C}[[x, y]]$, in the sense of Definition 1.2.19.

Instead of fixing w and letting f vary, let us fix now a non-zero series $f \in \mathbb{C}[[x, y]]$. Considering the w-weight of f for every $w \in \sigma_0$ leads to the following function:

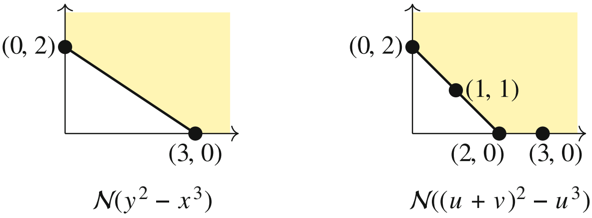


Fig. 1.14 Illustration of Remark 1.4.3

Definition 1.4.4 The **tropicalization** trop^f of $f \in \mathbb{C}[[x, y]] \setminus \{0\}$ is the function:

$$\boxed{\mathrm{trop}^f} : \begin{array}{rcl} \sigma_0 & \to & \mathbb{R}_+ \\ w & \to & \min\{w \cdot m,\ m \in \mathcal{S}(f)\} \end{array}. \tag{1.33}$$

Remark 1.4.5 Let us explain the name of *tropicalization* used in the previous definition (see also Sect. 1.4.5). Consider the set $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$, endowed with the operations $\oplus :=$ min and $\odot := +$. Under both operations, $\overline{\mathbb{R}}$ is a commutative monoid, the product $\odot$ is distributive with respect to addition and the addition $\oplus$ is idempotent, that is, $a \oplus a = a$, for all $a \in \overline{\mathbb{R}}$. One says then that $(\overline{\mathbb{R}}, \oplus, \odot)$ is a *tropical semiring*. Consider now the expression defining trop^f, and compare it with the expansion (1.31) of f as a power series. One sees that one gets formally trop^f from (1.31) by replacing each constant or variable x, y by its weight, and by replacing the usual operations of sum and product by their tropical analogs. For further references see the textbook [84] on *tropical geometry*. Foundations for the tropical study of singularities were written by Stepanov and the third author in the paper [105].

Remark 1.4.6 If A is a subset of a real vector space V, then its *support function* is the function defined on the dual vector space $V^\vee$ and taking values in $\mathbb{R} \cup \{-\infty\}$, which associates to every element of $V^\vee$ seen as a linear form on V, the infimum of its restriction to A. The tropicalization trop^f is the restriction of the support function of the subset $\mathcal{S}(f)$ of the real vector space $M_\mathbb{R}$ to the subset of $M_\mathbb{R}^\vee \simeq N_\mathbb{R}$ on which it does not take the value $-\infty$. The notion of support function is an essential tool in the study of convex polyhedra (see for instance Ewald's book [37]).

For every ray $\rho = \mathbb{R}_+ w$ included in the cone σ_0, consider the following closed half-plane of $M_\mathbb{R}$:

$$\boxed{H_{f,\rho}} := \{m \in M_\mathbb{R},\ w \cdot m \geq \mathrm{trop}^f(w)\}. \tag{1.34}$$

This definition is independent of the choice a generator w of the ray ρ.

The basic reason of the importance of the Newton polygon $\mathcal{N}(f)$ of f in our context is the following strengthening of Proposition 1.2.39:

Proposition 1.4.7 *Let the ray $\rho \subset \sigma_0$ be fixed. Then the closed half-plane $H_{f,\rho}$ of $M_\mathbb{R}$ is a* **supporting half-plane** *of $\mathcal{N}(f)$, in the sense that it contains $\mathcal{N}(f)$ and its boundary $\{m \in M_\mathbb{R},\ w \cdot m = \mathrm{trop}^f(w)\}$ has a non-empty intersection with the boundary $\partial\mathcal{N}(f)$ of $\mathcal{N}(f)$.*

Proof Let w be a generating vector of the ray ρ. The inclusion $\mathcal{N}(f) \subseteq H_{f,\rho}$ is equivalent to the property $w \cdot n \geq \mathrm{trop}^f(w)$, for all $n \in \mathcal{N}(f)$. These inequalities result from Definition 1.4.4 of the tropicalization function $\mathrm{trop}^f(w)$ and from the following basic equality, implied by the hypothesis that $w \in \sigma_0$ (see Proposition 1.2.39):

$$\min\{w \cdot m,\ m \in \mathcal{S}(f)\} = \min\{w \cdot m,\ m \in \mathcal{N}(f)\}.$$

The boundary of the half-plane $H_{f,\rho}$ intersects $\mathcal{N}(f)$ at its points at which the restriction of the linear form $w : M_{\mathbb{R}} \to \mathbb{R}$ to $\mathcal{N}(f)$ achieves its minimum, that is, along its face $\mathcal{N}(f) \cap \{m \in M_{\mathbb{R}},\ w \cdot m = \mathrm{trop}^f(w)\}$. □

As every closed convex subset of a real plane is the intersection of its supporting half-planes, one deduces that the tropicalization trop^f determines the Newton polygon $\mathcal{N}(f)$ in the following way:

$$\mathcal{N}(f) = \{m \in M_{\mathbb{R}},\ w \cdot m \geq \mathrm{trop}^f(w),\ \text{for all } w \in \sigma_0\}. \tag{1.35}$$

Formula (1.35) presents $\mathcal{N}(f)$ as the intersection of an infinite set of closed half-planes. In fact, as a consequence of the previous discussion, a finite number of them suffices:

Proposition 1.4.8 *Let $\mathcal{F}(f)$ be the fan of N obtained by subdividing the cone σ_0 using the rays orthogonal to the compact edges of $\mathcal{N}(f)$. Then:*

1. *The tropicalization* trop^f *is continuous and its restriction to any cone in $\mathcal{F}(f)$ is linear.*
2. *The relative interiors of the cones of $\mathcal{F}(f)$ may be characterized as the levels of the following map from σ_0 to the set of faces of $\mathcal{N}(f)$, in the sense of Definition 1.4.2:*

$$w \to \mathcal{N}(f) \cap \{m \in M_{\mathbb{R}},\ w \cdot m = \mathrm{trop}^f(w)\}.$$

3. *This map realizes an inclusion-reversing bijection between $\mathcal{F}(f)$ and the set of faces of $\mathcal{N}(f)$. If K_σ is the face of $\mathcal{N}(f)$ corresponding to the cone σ of $\mathcal{F}(f)$, then:*

$$\mathrm{trop}^f(w) = w \cdot m,\ \text{for all } w \in \sigma,\ \text{and for all } m \in K_\sigma.$$

4. *The Newton polygon $\mathcal{N}(f)$ is the intersection of the closed half-planes $H_{f,\rho}$ defined by relation (1.34), where ρ varies among the rays of the fan $\mathcal{F}(f)$.*

The fans $\mathcal{F}(f)$ appearing in the previous proposition are particularly important for the sequel, that is why they deserve a name:

Definition 1.4.9 The **Newton fan** $\boxed{\mathcal{F}(f)}$ of $f \in \mathbb{C}[[x, y]] \setminus \{0\}$ is the fan of N obtained by subdividing the cone σ_0 using the rays orthogonal to the compact edges of the Newton polygon $\mathcal{N}(f) \subseteq \sigma_0^\vee$ of f, that is, by the interior normals of the compact edges of $\mathcal{N}(f)$. A **Newton fan** in a weight lattice N and relative to a basis (e_1, e_2) is any fan subdividing the regular cone $\sigma_0 = \mathbb{R}_+\langle e_1, e_2\rangle$.

Example 1.4.10 Consider the series $f \in \mathbb{C}[[x, y]]$ defined by:

$$f(x, y) := -x^{12} + x^{14} + x^7y^2 + 2x^5y^3 - x^{10}y^3 + x^3y^4 + 3x^7y^4 + y^9.$$

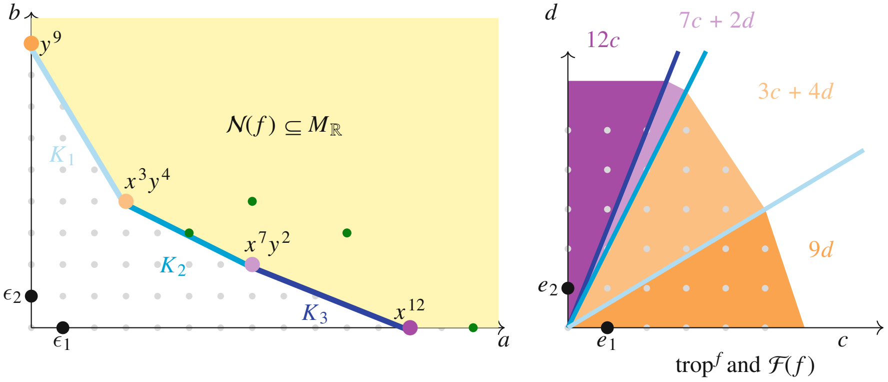


Fig. 1.15 The Newton polygon, the tropicalization and the Newton fan of Example 1.4.10

On the left side of Fig. 1.15 is represented its Newton polygon $\mathcal{N}(f)$, and on the right side are represented its tropicalization trop^f and its Newton fan $\mathcal{F}(f)$. The support of the series f is:

$$\mathcal{S}(f) = \{(12, 0), (14, 0), (7, 2), (5, 3), (10, 3), (3, 4), (7, 4), (0, 9)\}.$$

Among its elements, the vertices of $\mathcal{N}(f)$ are $(12, 0), (7, 2), (3, 4), (0, 9)$. The corresponding monomials are marked on the left of the figure, near the associated vertices. The other elements of $\mathcal{S}(f)$ are marked as green dots. Now, each vertex (a, b) of $\mathcal{N}(f)$ may be seen as the linear function $w = (c, d) \to ac + bd$ on $N_{\mathbb{R}}$. The tropicalization trop^f computes the minimal value of those 4 linear functions at the points of σ_0. The regular cone σ_0 gets decomposed into 4 smaller 2-dimensional subcones, according to the vertex which gives this minimum. On the right side of Fig. 1.15 those subcones are represented in different colors. Each such subcone has the same color as the expression of the associated linear function and the vertex of $\mathcal{N}(f)$ defining it. Each ray separating two successive subcones is orthogonal to a compact edge of $\mathcal{N}(f)$ and both are drawn with the same color. Denoting the compact edges by $K_1 := [(0, 9), (3, 4)]$, $K_2 := [(3, 4), (7, 2)]$, $K_3 := [(7, 2), (12, 0)]$, the associated restrictions of f (see Definition 1.4.2) are:

$$f_{K_1} = x^3y^4 + y^9, \quad f_{K_2} = x^7y^2 + 2x^5y^3 + x^3y^4, \quad \text{and } f_{K_3} = -x^{12} + x^7y^2.$$

The Newton fan of f is $\mathcal{F}(f) = \mathcal{F}(3/5, 2/1, 5/2)$ (see Definition 1.3.4 for this last notation).

If $\alpha \in \mathbb{C}[[t]] \setminus \{0\}$, we denote by $c_{\nu_t(\alpha)}(\alpha)$ the coefficient of $t^{\nu_t(\alpha)}$ in the series α, and we call it the **leading coefficient** of α.

The following proposition shows why it is important to introduce trop^f when studying the germ C defined by f:

Proposition 1.4.11 *Let $f \in \mathbb{C}[[x, y]]$ be a non-zero series. Let $t \to (\alpha(t), \beta(t))$ be a germ of formal morphism from $(\mathbb{C}, 0)$ to $(\mathbb{C}^2, 0)$, whose image is not contained in the union $L \cup L'$ of the coordinate axes. Then one has the inequality:*

$$\nu_t(f(\alpha(t), \beta(t))) \geq \operatorname{trop}^f(\nu_t(\alpha), \nu_t(\beta)),$$

with equality if and only if $f_K(c_{\nu_t(\alpha)}(\alpha), c_{\nu_t(\beta)}(\beta)) \neq 0$, where K is the compact face of $\mathcal{N}(f)$ orthogonal to $(\nu_t(\alpha), \nu_t(\beta)) \in N$, in the sense that its restriction to $\mathcal{N}(f)$ achieves its minimum on this face.

Proof The basic idea of the proof goes back to Newton's method of computing the leading term of a Newton-Puiseux series $\eta(x)$ such that $f(x, \eta(x)) = 0$, which we explained on the example of Sect. 1.2.5, starting from Eq. (1.12).

The hypothesis that the image of $t \to (\alpha(t), \beta(t))$ is not contained in the union of coordinate axes means that both α and β are non-zero series. Therefore, they admit non-vanishing leading coefficients $c_{\nu_t(\alpha)}(\alpha)$ and $c_{\nu_t(\beta)}(\beta)$ (see Definition 1.2.18).

Using the expansion (1.31), we get that $f(\alpha(t), \beta(t))$ is equal to:

$$\sum_{(a,b)\in\mathcal{S}(f)} c_{(a,b)}(f) \left(c_{\nu_t(\alpha)}(\alpha)t^{\nu_t(\alpha)} + o(t^{\nu_t(\alpha)})\right)^a \left(c_{\nu_t(\beta)}(\beta)t^{\nu_t(\beta)} + o(t^{\nu_t(\beta)})\right)^b =$$
$$= \sum_{(a,b)\in\mathcal{S}(f)} c_{(a,b)}(f) \left(c_{\nu_t(\alpha)}(\alpha)\right)^a \left(c_{\nu_t(\beta)}(\beta)\right)^b \left(t^{a\nu_t(\alpha)+b\nu_t(\beta)} + o(t^{a\nu_t(\alpha)+b\nu_t(\beta)})\right). \tag{1.36}$$

As a consequence:

$$\nu_t\left(f(\alpha(t), \beta(t))\right) \geq \min_{(a,b)\in\mathcal{S}(f)} \{a\nu_t(\alpha) + b\nu_t(\beta)\} = \operatorname{trop}^f(\nu_t(\alpha), \nu_t(\beta)),$$

where the last equality follows from Definition 1.4.4. This proves the inequality stated in the proposition.

The case of equality follows from the fact, implied by the computation (1.36), that the coefficient of the term with exponent $\operatorname{trop}^f(\nu_t(\alpha), \nu_t(\beta))$ of the series $f(\alpha(t), \beta(t))$ is $f_K(c_{\nu_t(\alpha)}(\alpha), c_{\nu_t(\beta)}(\beta))$. □

In Proposition 1.4.11, K may be either an edge or a vertex of $\mathcal{N}(f)$. Note that this statement plays with the two dual ways of defining a curve singularity on $(\mathbb{C}^2, 0)$, either as the vanishing locus of a function or by a parametrization.

Consider now the reduced image of the morphism $t \to (\alpha(t), \beta(t))$. The hypothesis that it is not contained in $L \cup L'$ shows that it is a branch on (S, o), different from L and L'. Endow it with a multiplicity equal to the degree of the morphism onto its image, seeing it therefore as a divisor A on (S, o). By Proposition 1.2.8, the orders $\nu_t(\alpha(t))$, $\nu_t(\beta(t))$ which appear in Proposition 1.4.11 may be interpreted as $\nu_t(\alpha(t)) = L \cdot A$, and $\nu_t(\beta(t)) = L' \cdot A$. We get the following corollary of Proposition 1.4.11:

Proposition 1.4.12 *Let (L, L') be a cross on (S, o) and C be a curve singularity on (S, o). Assume that the local coordinate system (x, y) defines the cross (L, L') and that $f \in \hat{O}_{S,o}$ defines C. Then, for every effective divisor A on (S, o) supported on a branch distinct from L and L', one has the inequality:*

$$C \cdot A \geq \operatorname{trop}^f((L \cdot A)e_1 + (L' \cdot A)e_2).$$

Moreover, one has equality when A is generic for fixed values of $L \cdot A$ and $L' \cdot A$.

One may describe the genericity condition involved in the last sentence of Proposition 1.4.12 as follows. As a consequence of the proof of Proposition 1.4.18 below, one has $f_K(c_{\nu_t(\alpha)}(\alpha), c_{\nu_t(\beta)}(\beta)) \neq 0$ (which is equivalent to the equality $C \cdot A = \operatorname{trop}^f((L \cdot A)e_1 + (L' \cdot A)e_2)$) if and only if the strict transforms of A and C by the Newton modification $\psi^C_{L,L'}$ of S defined by C (see Definition 1.4.14 below) are disjoint.

As a consequence of Propositions 1.4.8 (3) and 1.4.12 we have:

Proposition 1.4.13 *Let (L, L') be a cross on (S, o) and C be a curve singularity on (S, o). Assume that the local coordinate system (x, y) defines the cross (L, L') and that $f \in \hat{O}_{S,o}$ defines C. Then the Newton polygon $\mathcal{N}(f)$, the tropicalization* trop^f *and the Newton fan $\mathcal{F}(f)$ do not depend on the choice of the defining functions x, y, f of the curve germs L, L', C.*

By contrast, the support of f depends on the choice of the local coordinate system (x, y) defining a fixed cross, even if $f \in \hat{O}_{S,o}$ is fixed. For instance, the monomial xy becomes a series with infinite support if one replaces simply x by $x(1 + x + x^2 + \cdots)$.

Proposition 1.4.13 implies that the following notions are well-defined:

Definition 1.4.14 Let (L, L') be a cross on (S, o), and let (x, y) be a local coordinate system defining it. Let C be a curve singularity on (S, o), defined by a function $f \in \hat{O}_{S,o}$, seen as a series in $\mathbb{C}[[x, y]]$ using the coordinate system (x, y). Then:

- The **Newton polygon** $\boxed{\mathcal{N}_{L,L'}(C)} \subseteq M_{L,L'}$ **of** C **relative to the cross** (L, L') is the Newton polygon $\mathcal{N}(f)$.
- The **tropical function** $\boxed{\operatorname{trop}^C_{L,L'}} : \sigma_0 \to \mathbb{R}_+$ of C **relative to the cross** (L, L') is the tropicalization trop^f of the series f.
- The **Newton fan** $\boxed{\mathcal{F}_{L,L'}(C)}$ **of** C **relative to the cross** (L, L') is the fan $\mathcal{F}(f)$.
- The **Newton modification** $\boxed{\psi^C_{L,L'}} : (S_{\mathcal{F}_{L,L'}(C)}, \partial S_{\mathcal{F}_{L,L'}(C)}) \to (S, L + L')$ **of** S **defined by** C **relative to the cross** (L, L') is the modification of S associated with $\mathcal{F}_{L,L'}(C)$ relative to the cross (L, L'), that is, $\psi^C_{L,L'} := \psi^{\mathcal{F}_{L,L'}(C)}_{L,L'}$ (see Definition 1.3.33). The strict transform of C by $\psi^C_{L,L'}$ is denoted $\boxed{C_{L,L'}}$.

Note that we consider the Newton modification $\psi^C_{L,L'}$ as a morphism in the toroidal category, by endowing S with the boundary $L + L'$ and the modified surface $S_{\mathcal{F}_{L,L'}(C)}$ with a boundary equal to the reduced total transform of $L + L'$.

1.4.2 An Algorithm of Toroidal Pseudo-Resolution

In this subsection we assume for simplicity that the plane curve singularity C is reduced (see Remark 1.4.27). We explain that, once a smooth branch L is fixed on the germ of smooth surface (S, o), one may obtain a so-called *toroidal pseudo-resolution* of C on (S, o) (see Definition 1.4.15) by completing the smooth branch into a cross (L, L'), by performing the associated Newton modification, and by iterating these steps at every point at which the strict transform of C intersects the exceptional divisor of the Newton modification (see Theorem 1.4.23). The algorithm stops after the first step if and only if C is *Newton non-degenerate* relative to the cross (L, L') (see Definition 1.4.21).

The following definition formulates two notions of possibly partial *resolution of C in the toroidal category*, relative to the ambient smooth germ of surface S:

Definition 1.4.15 Let (L, L') be a cross in the sense of Definition 1.3.31 on the smooth germ of surface (S, o) and let C be a curve singularity on S. Consider a modification $\pi : (\Sigma, \partial\Sigma) \to (S, L + L')$ of $(S, L + L')$ in the toroidal category, in the sense of Definition 1.3.29. It is called, in decreasing generality:

- A **toroidal pseudo-resolution of** C if the following conditions are satisfied:
  1. the boundary $\partial\Sigma$ of Σ contains the reduction of the total transform $\pi^*(C)$ of C by π;
  2. the strict transform of C by π (see Definition 1.2.31) does not contain singular points of Σ.
- A **toroidal embedded resolution of** C if, moreover, the surface Σ is smooth.

If $\pi : (\Sigma, \partial\Sigma) \to (S, L + L')$ is a toroidal pseudo-resolution of C, then the reduction of the image $\pi(\partial\Sigma)$ of $\partial\Sigma$ in S is called the **completion** $\boxed{\hat{C}_\pi}$ **of** C **relative to** π.

Remark 1.4.16 Note that if $\pi : (\Sigma, \partial\Sigma) \to (S, L + L')$ is a toroidal pseudo-resolution of C, then the strict transform of C by π is smooth and $\hat{C}_\pi \supseteq C + L + L'$. If moreover π is an embedded resolution, then the total transform $\pi^*(C)$ is a normal crossings divisor in Σ (see Definition 1.2.32). Note also that if $\pi : (\Sigma, \partial\Sigma) \to (S, L + L')$ is a toroidal embedded resolution of C, then $\pi : \Sigma \to S$ is an embedded resolution of C according to Definition 1.2.33. From now on, we will keep track carefully of the toroidal structures, considering only toroidal embedded resolutions in the sense of Definition 1.4.15.

Remark 1.4.17 If $\pi : (\Sigma, \partial\Sigma) \to (S, L + L')$ is a toroidal pseudo-resolution of C, then the strict transform of C is transversal to the critical locus of π. Our choice of terminology in Definition 1.4.15 is inspired by Goldin and Teissier's paper [51], where an analogous notion of (embedded) *toric pseudo-resolution* of a subvariety of the affine space is considered.

Let us look now at the strict transform $C_{L,L'}$ of C by the Newton modification $\psi^C_{L,L'}$ defined by C relative to the cross (L, L') (see Definition 1.4.14). The following proposition describes its intersection with the boundary $\partial S_{\mathcal{F}_{L,L'}(C)}$:

Proposition 1.4.18 *Assume that neither L nor L' is a branch of C. Then the strict transform $C_{L,L'}$ of C by the Newton modification $\psi^C_{L,L'}$ intersects the boundary $\partial S_{\mathcal{F}_{L,L'}(C)}$ of the toroidal surface $(S_{\mathcal{F}_{L,L'}(C)}, \partial S_{\mathcal{F}_{L,L'}(C)})$ only at smooth points of it. Moreover, if ρ is a ray of the Newton fan $\mathcal{F}_{L,L'}(C)$ different from the edges of σ_0, then $C_{L,L'}$ intersects the corresponding component $\overline{O_\rho}$ of the exceptional divisor of $\psi^C_{L,L'}$ only inside the orbit O_ρ. The number of intersection points counted with multiplicities is equal to the integral length of the edge of the Newton polygon $\mathcal{N}_{L,L'}(C)$ which is orthogonal to the ray ρ.*

Proof We give a detailed proof of this proposition in geometric language, in order to emphasize the roles played by the fundamental combinatorial objects $\mathcal{N}_{L,L'}(C)$, $\text{trop}^C_{L,L'}$ and $\mathcal{F}_{L,L'}(C)$ associated with C relative to the cross (L, L') (see Definition 1.4.14).

The orbit O_ρ is independent of the toric surface containing it, because any two such surfaces contain the affine toric surface $X_\rho \supset O_\rho$ as Zariski open sets. Therefore, in order to compute the intersection of the strict transform of C with O_ρ, we may choose another surface than $X_{\mathcal{F}_{L,L'}(C)}$.

Choose local coordinates (x, y) defining the cross (L, L'). In this way $M_{L,L'}$ gets identified with the lattice of exponents of Laurent monomials in (x, y). Assume that $f_1 := \alpha e_1 + \beta e_2$ is the unique primitive generator of the ray ρ. Let us complete it in a basis (f_1, f_2) of the lattice $N_{L,L'}$, such that the cone $\sigma := \mathbb{R}_+\langle f_1, f_2\rangle$ is contained in one of the two cones of dimension 2 of $\mathcal{F}_{L,L'}(C)$ adjacent to ρ. We are now in the setting of Example 1.3.26. As explained there, if (φ_1, φ_2) is the basis of $M_{L,L'}$ dual to the basis (f_1, f_2) of $N_{L,L'}$ and $u := \chi^{\varphi_1}, v := \chi^{\varphi_2}$, then $v = x^{-\beta} y^{\alpha}$ is a coordinate of the orbit O_ρ. Moreover, it realises an isomorphism of its closure in the affine toric surface $X_\sigma = \mathbb{C}^2_{u,v}$ with the affine line $\mathbb{C}_v$.

Let K_ρ be the edge of the Newton polygon $\mathcal{N}_{L,L'}(C)$ which is orthogonal to the ray ρ. It is parallel to the line $\mathbb{R}\varphi_2$, because by definition $f_1 \cdot \varphi_2 = 0$. Orient K_ρ by the vector φ_2 and denote then its vertices by m_0 and m_1, such that K_ρ is oriented from m_0 to m_1. This means that $m_1 - m_0 = L_\rho\, \varphi_2$, where L_ρ denotes the integral length of the segment K_ρ, in the sense of Definition 1.3.1. Moreover, the points of $K_\rho \cap M$ are precisely those of the form:

$$m := m_0 + k\, \varphi_2, \ \text{for}\ k \in \{0, 1, \dots, L_\rho\}. \tag{1.37}$$

Consider the smooth toric surface $X_\sigma = \mathbb{C}^2_{u,v}$. The orbit O_ρ is its pointed v-axis $\mathbb{C}^*_v$. Therefore, one may compute the intersection of the strict transform of C with this orbit by taking the lift $(\psi^\sigma_{\sigma_0})^* f$ of a defining function f of C to $\mathbb{C}^2_{u,v}$, by simplifying by the greatest monomial in $\sigma^\vee \cap M$ which divides it, and then by setting $u = 0$. Let therefore

$$f := \sum_{m \in \mathcal{S}(f)} c_m(f) \chi^m \in \mathbb{C}[[x, y]]$$

be a defining function of C. As the bases (f_1, f_2) and (φ_1, φ_2) are dual of each other, we have the relation $m = (f_1 \cdot m)\varphi_1 + (f_2 \cdot m)\varphi_2$. This implies that $\chi^m = u^{f_1 \cdot m}\, v^{f_2 \cdot m}$. As the ray $\rho = \mathbb{R}_+ f_1$ is orthogonal to the edge K_ρ of the Newton polygon $\mathcal{N}_{L,L'}(C) = \mathcal{N}(f)$, we know that:

$$f_1 \cdot m \geq h_\rho \text{ for all } m \in \mathcal{S}(f),$$

where $h_\rho := \mathrm{trop}^f(f_1)$, with equality if and only if $m \in K_\rho$. Therefore, the highest power of u which divides

$$(\psi^\sigma_{\sigma_0})^* f = \sum_{m \in \mathcal{S}(f)} c_m(f)\, u^{f_1 \cdot m}\, v^{f_2 \cdot m}$$

is u^{h_ρ}, and it is achieved only on the edge K_ρ of $\mathcal{N}(f)$. Moreover, the linear form $m \to f_2 \cdot m$ achieves its minimum (at least) at the vertex m_0 of $\mathcal{N}(f)$, by the hypothesis that σ is contained in one of the two 2-dimensional cones of $\mathcal{F}(f) = \mathcal{F}_{L,L'}(C)$ which are adjacent to ρ. This shows that the maximal monomial in (u, v) which divides $(\psi^\sigma_{\sigma_0})^* f$ is $u^{h_\rho} v^{f_2 \cdot m_0}$. After simplifying by it and setting $u = 0$, one gets the following polynomial equation in the variable v, describing the intersection of the strict transform of C with the v-axis:

$$\sum_{m \in K_\rho \cap M} c_m(f)\, v^{f_2 \cdot (m - m_0)} = 0. \tag{1.38}$$

We recognize here the equation obtained from $f_{K_\rho} = 0$ after the change of variables from (x, y) to (u, v) and the simplification of the highest dividing monomial. This illustrates the importance in our context of the operation of restriction of f to a compact edge of its Newton polygon, introduced in Definition 1.4.2. Using Eq. (1.37), we see that Eq. (1.38) becomes:

$$\sum_{k=0}^{L_\rho} c_{m_0 + k\, \varphi_2}(f)\, v^k = 0. \tag{1.39}$$

The two extreme coefficients $c_{m_0}(f)$ and $c_{m_1}(f)$ of the previous polynomial equation being non-zero, we see that the strict transform of C does not pass through the origin of $\mathbb{C}^2_{u,v}$ and that it intersects the orbit O_ρ in $L_\rho = l_{\mathbb{Z}} K_\rho$ points, counted with multiplicities. The solutions of Eq. (1.39) are the v-coordinates of the intersection points of the strict transform of C with the orbit O_ρ.

By using the same kind of argument for all the cones of the regularization of $\mathcal{F}_{L,L'}(C)$, we may show also that the strict transform of C misses all the singular points of the boundary divisor of $X_{\mathcal{F}_{L,L'}(C)}$. □

Example 1.4.19 Let us give an example of the objects manipulated in the proof of Proposition 1.4.18. Consider the function $f \in \mathbb{C}[[x, y]]$ of Example 1.4.10. Let ρ be the ray of slope $2/1$ of $\mathcal{F}(f)$. Then K_ρ is the side $K_2 := [(3, 4), (7, 2)]$ of $\partial\mathcal{N}(f)$ (see Fig. 1.15). One has $f_1 = e_1 + 2e_2$. A possible choice of the vector φ_2 is $\varphi_2 = -2\epsilon_1 + \epsilon_2$. Therefore $v = x^{-2}y$. Orienting K_ρ by this vector φ_2 one gets $m_0 = (7, 2)$ and $m_1 = (3, 4)$. We saw in Example 1.4.10 that $f_{K_\rho} = x^7y^2 + 2x^5y^3 + x^3y^4 = x^3y^2(x^4 + 2x^2y + y^2)$. As $v = x^{-2}y$, Eq. (1.39) is in this case $1 + 2v + v^2 = 0$. We see that its degree is indeed the integral length L_ρ of the side K_ρ. As it has a double root, the series f is not Newton non-degenerate (see Definition 1.4.21 below). The strict transform of C intersects O_ρ at the single point $v = -1$.

The proof of Proposition 1.4.18 yields easily also a proof of the following proposition :

Proposition 1.4.20 *Let (L, L') be a cross and C a curve singularity on S. Let $f \in \mathbb{C}[[x, y]]$ be a defining function of C relative to any coordinate system (x, y) defining the chosen cross. Then the following conditions are equivalent:*

1. *the curve C is reduced and the Newton modification $\psi^C_{L,L'}$ becomes a toroidal pseudo-resolution of C if one replaces the boundary $\partial S_{\mathcal{F}_{L,L'}(C)}$ by the total transform of the divisor $(\psi^C_{L,L'})^*(C + L + L')$;*
2. *for any ray ρ of the Newton fan $\mathcal{F}_{L,L'}(C)$ which is orthogonal to a compact edge of $\mathcal{N}_{L,L'}(C)$, the polynomial equation (1.39) has only simple roots;*
3. *the defining function f of C has the property that all the restrictions f_K of f to the compact edges K of the Newton polygon $\mathcal{N}(f) = \mathcal{N}_{L,L'}(C)$ define smooth curves in the torus $(\mathbb{C}^*)^2_{x,y}$.*

The plane curve singularities which satisfy the equivalent conditions of Proposition 1.4.20 received a special name:

Definition 1.4.21 Let (L, L') be a cross and C a curve singularity on S. Let $f \in \mathbb{C}[[x, y]]$ be a defining function of C relative to any coordinate system associated to the chosen cross. The function f is called **Newton non-degenerate** and the curve C is called **Newton non-degenerate relative to the cross** (L, L') if the equivalent conditions listed in Proposition 1.4.20 are satisfied.

Usually one speaks about Newton non-degenerate germs of holomorphic functions of several variables. We introduce here the notion of *Newton non-degenerate plane curve singularity relative to a cross* in order to emphasize the underlying geometric phenomena.

Let us come back to Proposition 1.4.18. At each point of intersection o_i of the strict transform $C_{L,L'}$ with the exceptional divisor of $\psi^C_{L,L'}$, one has the following dichotomy:

- Either only one branch of $C_{L,L'}$ passes through o_i, where it is moreover smooth and transversal to the exceptional divisor. The germ A_i at o_i of the exceptional divisor and this branch form a canonical cross on $S_{\mathcal{F}_{L,L'}(C)}$. Then, one reaches locally a toroidal pseudo-resolution of C in the neighborhood of that point.
- Or one does not have a canonical cross, but only a canonical smooth branch: the germ A_i at o_i of the exceptional divisor $(\psi^C_{L,L'})^{-1}(o)$ itself.

In the second case, one may complete A_i into a cross (A_i, L_i) by the choice of a germ L_i of smooth branch transversal to it. Then one is again in the presence of a germ of effective divisor (the germ of the strict transform $C_{L,L'}$ of C by $\psi^C_{L,L'}$) on a germ of smooth surface endowed with a cross (the surface $S_{\mathcal{F}_{L,L'}(C)}$ endowed with the cross (A_i, L_i)). One gets again a Newton polygon, a tropical function, a Newton fan and a Newton modification, and the previous construction may be iterated. This iterative process may be formulated as the following *algorithm of toroidal pseudo-resolution* of the germ C:

Algorithm 1.4.22 Let (S, o) be a smooth germ of surface, L a smooth branch on (S, o) and C a reduced germ of curve on (S, o), which does not contain the branch L in its support.

STEP 1. If (L, C) is a cross, then STOP.
STEP 2. Choose a smooth branch L' on (S, o), possibly included in C, such that (L, L') is a cross.
STEP 3. Let $\mathcal{F}_{L,L'}(C)$ be the Newton fan of C relative to the cross (L, L'). Consider the associated Newton modification $\psi^C_{L,L'} : (S_{\mathcal{F}_{L,L'}(C)}, \partial S_{\mathcal{F}_{L,L'}(C)}) \to (S, L + L')$ and the strict transform $C_{L,L'}$ of C by $\psi^C_{L,L'}$ (see Definition 1.4.14).
STEP 4. For each point $\tilde{o}$ belonging to $C_{L,L'} \cap \partial S_{\mathcal{F}_{L,L'}(C)}$, denote:

- $L :=$ the germ of $\partial S_{\mathcal{F}_{L,L'}(C)}$ at $\tilde{o}$;
- $C :=$ the germ of $C_{L,L'}$ at $\tilde{o}$;
- $o := \tilde{o}$;
- $S := S_{\mathcal{F}_{L,L'}(C)}$.

STEP 5. GO TO STEP 1.

Note that one considers that only the smooth branch L is given at the beginning, and that the second branch L' of the cross (L, L') is chosen when one executes STEP 2 for the first time. Note also that the algorithm is non-deterministic, as it involves choices of supplementary branches.

A variant of this algorithm, obtained by replacing Step 3 by a Step 3^{reg}, will be studied in Sect. 1.4.3. It produces a *toroidal embedded resolution* of C instead of a *pseudo-resolution* (see Definition 1.4.15).

Proposition 1.4.20 means that if C is Newton non-degenerate relative to the cross (L, L') chosen at Step 2 of Algorithm 1.4.22, then this algorithm stops after performing only one Newton modification. More generally, a fundamental property of Algorithm 1.4.22 is:

Theorem 1.4.23 *Algorithm 1.4.22 stops after a finite number of iterations.*

Proof Assume that A is a curve singularity on the smooth germ of surface (S, o), obtained after a finite number of steps of the algorithm, and that $(L \cdot A)_o = 1$. Then (L, A) is a cross and the algorithm stops. Therefore, in order to show that the algorithm stops, it is enough to show that after a finite number of steps all the local intersection numbers of the strict transform $C_{L,L'}$ of C with the exceptional divisor are equal to 1.

By the end statement of Proposition 1.4.18, a sequence of such intersection numbers at *infinitely near points of o* (see Definition 1.4.31) which dominate each other is necessarily decreasing:

$$(C \cdot L)_o \geq (C_1 \cdot E_1)_{o_1} \geq \cdots \geq (C_k \cdot E_k)_{o_k} \geq \cdots . \tag{1.40}$$

At the k-th iteration of the algorithm we are considering the strict transform C_k of C at a point o_k, which belongs to the component E_k of the exceptional divisor.

The sequence (1.40) being composed of positive integers, it necessarily stabilizes. If the stable value is 1 for all choices of sequence $o, o_1, o_2, \ldots$, then the algorithm stops after a finite number of steps.

Let us reason by contradiction, assuming the contrary. Therefore, one may find a sequence as before for which the stable intersection number is $n > 1$. Let us assume without loss of generality, by starting our analysis after the stabilization took place, that:

$$(C \cdot L)_o = (C_1 \cdot E_1)_{o_1} = \cdots = (C_k \cdot E_k)_{o_k} = \cdots = n > 1. \tag{1.41}$$

Therefore, for every $k \geq 1$, (E_k, C_k) is not a cross at o_k. By STEP 2 of the algorithm, a smooth germ L_k was chosen at o_k such that (E_k, L_k) is a cross at o_k.

Let us reformulate the first equality

$$(C_1 \cdot E_1)_{o_1} = (C \cdot L)_o \tag{1.42}$$

of the sequence (1.41) in terms of Newton polygons. By applying again the end statement of Proposition 1.4.18, we see that $(C_1 \cdot E_1)_{o_1}$ is less or equal to the integral length $l_{\mathbb{Z}} K$ of the compact edge K of $\mathcal{N}_{L,L'}(C)$ whose orthogonal ray corresponds to the prime exceptional curve E_1. One has equality if and only if the strict transform of C intersects E_1 at a single point. In turn, the integral length $l_{\mathbb{Z}} K$ is less or equal to the height $(C \cdot L)_o = n$ of $\mathcal{N}_{L,L'}(C)$ (the ordinate of its lowest point on the vertical axis), with equality if and only if K is the only compact edge of $\mathcal{N}_{L,L'}(C)$ and $K = [(0, n), (m_1 n, 0)]$, with $m_1 \in \mathbb{N}^*$.

As a consequence, *one has the equality* (1.42) *if and only if* $\mathcal{N}_{L,L'}(C)$ *has a single compact edge, of the form* $[(0, n), (m_1 n, 0)]$*, with* $m_1 \in \mathbb{N}^*$*, and the associated polynomial in one variable has only one root in* $\mathbb{C}^*$. In terms of local coordinates (x, y) on (S, o) defining the cross (L, L') and a defining unitary polynomial $f \in \mathbb{C}[[x]][y]$ of the plane curve singularity C (see Theorem 1.6.1 below), equality holds in (1.42) if and only if f is of the form $f = (y - c_1 x^{m_1})^n + \cdots$, with $c_1 \in \mathbb{C}^*$, $m_1 \in \mathbb{N}^*$ and where we wrote only the restriction f_K of f to the compact edge K of the Newton polygon $\mathcal{N}_{L,L'}(C)$, in the sense of Definition 1.4.2. Then, STEP 3 is performed simply by considering the morphism:

$$\begin{cases} x = x_1, \\ y = x_1^{m_1} (w_1 + c_1), \end{cases} \tag{1.43}$$

where (x_1, w_1) are local coordinates at o_1 and $Z(x_1) = (E_1, o_1)$. The hypothesis (1.41) implies that (E_1, C_1) is not a cross. Denote by L'_1 the smooth branch at o_1 obtained by applying again STEP 2. Therefore, (E_1, L'_1) is a cross at o_1. By the formal version of the implicit function theorem, we can choose local coordinates (x_1, u_1) defining the cross (E_1, L'_1) in such a way that $u_1 = w_1 - \phi_1(x_1)$, for some $\phi_1 \in \mathbb{C}[[t]]$ with $\phi_1(0) = 0$.

Let us define $y_1 := y - x^{m_1}(c_1 + \phi_1(x))$ and denote $L_1 := Z(y_1)$. Notice that the strict transform of L_1 by the modification (1.43) is equal to L'_1 and that (1.43) can be rewritten

$$\begin{cases} x \ = x_1, \\ y_1 = x_1^{m_1} u_1 \end{cases} \tag{1.44}$$

with respect to the local coordinates (x, y_1) and (x_1, u_1). Let us denote by $f_1 \in \mathbb{C}[[x_1]][u_1]$ the monic polynomial defining C_1 relative to the coordinates (x_1, u_1) (see again Theorem 1.6.1). Reasoning as before, the hypothesis (1.41) implies that the polynomial f_1 is of the form $f_1 = (u_1 - c_2 x_1^{m_2})^n + \cdots$, where $c_2 \in \mathbb{C}^*$, $m_2 \in \mathbb{N}^*$ and the exponents of the monomials $x_1^i u_1^j$ which were omitted verify that $i + m_2 j > m_2 n$ and $0 \leq j < n$. Notice that the order of vanishing of f along E_1 is equal to nm_1. We recover a defining function of C with respect to the coordinates (x, y_1) by expressing, using the relation (1.44), the monomials appearing in the product $x_1^{m_1 n} \cdot f_1(x_1, u_1)$ as monomials in (x, y_1). We get a defining function of C of the form $(y_1 - c_2 x_1^{m_1+m_2})^n + \cdots$, where the exponents of the monomials $x_1^i y_1^j$ which are not written above verify that $i+(m_1+m_2)j > (m_1+m_2)n$ and $0 \leq j < n$.

By induction on $k \geq 1$, one may show similarly that:

- The branch $L'_k = Z(u_k)$ is the strict transform of a smooth branch $L_k = Z(y_k)$ at S, where (x, y_k) is a local coordinate system defining a cross at o and

$$y_k = y_{k-1} - x^{m_1+\cdots+m_k}(c_k + \phi_k(x)), \tag{1.45}$$

where $\phi_k \in \mathbb{C}[[t]]$ satisfies $\phi_k(0) = 0$.

- The composition of the maps in the algorithm expresses as

$$\begin{cases} x &= x_1, \\ y_k &= x_1^{m_1+\cdots+m_k} \, u_k, \end{cases} \tag{1.46}$$

  with respect to the local coordinates (x_k, u_k) at o_k and the coordinates (x, y_k) at o.
- There exists a defining function of C of the form:

$$(y_k - c_k x^{m_1+\cdots+m_k})^n + \cdots$$

  where the exponents of monomials $x^i y_k^j$ which are not written above verify that $i + (m_1 + \cdots + m_k)j > (m_1 + \cdots + m_k)n$ and $0 \leq j < n$.

In particular, we have shown that the Newton polygon $\mathcal{N}_{L,L_k}(C)$ has only one compact edge with vertices $(0, n)$ and $(m_1 + \cdots + m_k, 0)$, where $L_k \cdot C = m_1 + \cdots + m_k$. When we look at the polygons $\mathcal{N}_{L,L_k}(C)$ as subsets of $\mathbb{R}^2$, we get a nested sequence:

$$\mathcal{N}_{L,L'}(C) \supset \mathcal{N}_{L,L_1}(C) \supset \cdots \supset \mathcal{N}_{L,L_{k-1}}(C) \supset \mathcal{N}_{L,L_k}(C). \tag{1.47}$$

By (1.45), one has that $y_k = y - \xi_k(x)$ with $\xi_k(x) \in \mathbb{C}[[x]]$. One may check, using the shape of relation (1.45), that the sequence $(\xi_k(x))_{k\geq 1}$ converges to a series $\xi_\infty(x)$ in the complete ring $\mathbb{C}[[x]]$. Set $y_\infty := y - \xi_\infty(x)$ and $L_\infty := Z(y_\infty)$. Then (L, L_∞) is a cross at o. We deduce that $L_\infty \cdot C = \nu_x f(x, \xi_\infty(x)) = +\infty$ and by (1.47) one gets the inclusion $\mathcal{N}_{L,L_\infty}(C) \subset \mathcal{N}_{L,L_k}(C)$, for every $k \geq 1$. These two facts together imply that the Newton polygon $\mathcal{N}_{L,L_\infty}(C)$ has only one vertex $(0, n)$. Therefore, a local defining series for C is of the form $(y_\infty)^n$. Since $n > 1$, C would not be a reduced germ, contrary to the hypothesis. □

Remark 1.4.24 The argument used in the proof of Theorem 1.4.23 coincides basically with one step of the proof of the Newton-Puiseux theorem (see Theorems 1.2.20 and 1.6.1), as presented in Teissier's survey [123]. Unlike the rest of the proof of this theorem, this particular step holds without making any assumption on the characteristic of the base field.

Algorithm 1.4.22 involves a finite number of choices, those of the smooth branches introduced in order to get crosses each time one executes STEP 2. Let us introduce the following notations:

Notations 1.4.25 Assume that one executes Algorithm 1.4.22 on (S, o), starting from the curve singularity C and the smooth branch L. Then:

1. $\{\boxed{o_i}, \ i \in \boxed{I}\}$ is the set of points at which one applies STEP 1 or STEP 2. One assumes that $\{1\} \subseteq I$ and that $o_1 := o$.
2. $\{\boxed{(A_i, B_i)}, \ i \in I\}$ is the set of crosses considered each time one applies STEP 1 or STEP 2. Therefore $A_1 = L$ and for $i \in I \setminus \{1\}$, the branch A_i is included

in the exceptional divisor of the Newton modification performed at the previous iteration.

3. $\boxed{J} \subseteq I$ consists of those $j \in I$ for which one performs STEP 2 at o_j. Denote by $\boxed{L_j}$ the projection on S of the branch B_j, for every $j \in J$. Therefore, B_i is a strict transform of a branch of C whenever $i \in I \setminus J$ and B_j is the strict transform of L_j whenever $j \in J$.
4. $\boxed{S^{(1)}} := S$. For $k \geq 1$, the surface $S^{(k+1)}$ is obtained from $\boxed{S^{(k)}}$ by performing simultaneously the Newton modification of STEP 3 at all the points o_j of $S^{(k)}$ at which one executes STEP 2. At such a point, denote by $\boxed{\mathcal{F}_{A_j,B_j}(C)}$ the corresponding fan. It is the Newton fan of the germ of strict transform of C at o_j, relative to the cross (A_j, B_j).
5. The previous simultaneous Newton modification is denoted $\boxed{\pi^{(k)}} : S^{(k+1)} \to S^{(k)}$. We call it the **$k$-th level of Newton modifications**.
6. The toroidal boundary $\boxed{\partial S^{(k)}}$ is by definition the total transform on $S^{(k)}$ of all the crosses which appeared in the algorithm until performing STEP 2 at all the points of $S^{(k)}$. In particular, $\partial S = L + L_1$. Each morphism $\pi^{(k)} : (S^{(k+1)}, \partial S^{(k+1)}) \to (S^{(k)}, \partial S^{(k)})$ belongs to the toroidal category, as $(\pi^{(k)})^{-1}(\partial S^{(k)}) \subseteq \partial S^{(k+1)}$.
7. $\boxed{\pi} := \pi^{(1)} \circ \cdots \circ \pi^{(h)}$, where $\boxed{h}$ is the number of modifications $\pi^{(k)}$ produced by the algorithm. We denote by $\boxed{\Sigma}$ the source of π. Therefore, $\pi : \Sigma \to S$ is a modification of the initial germ S.
8. $\boxed{\partial\Sigma}$ denotes $\partial S^{(h)}$. It is the underlying reduced divisor of the total transform $\pi^*(\hat{C}_\pi)$ of the completion $\hat{C}_\pi = C + \sum_{j \in J} L_j$, in the sense of Definition 1.4.15.

There are a lot of notations here! The only way to get used to them, to understand how those objects are related, and why they are important, is to look at examples. That is why we made a detailed one below (see Example 1.4.28). In fact, all the works which deal in a detailed way with processes of resolution of singularities introduce analogously plenty of notations (see for instance Zariski [136], Zariski [137], Lejeune-Jalabert [78], A'Campo and Oka [8], Casas [19], Wall [131] or Greuel, Lossen and Shustin [59]). This is one of the main advantages we see for the notion of *lotus* attached below to such a resolution process (see Definition 1.5.26): it allows to get a simultaneous global view of the previous objects.

We can state in the following way the output of Algorithm 1.4.22 in terms of Definition 1.4.15:

Proposition 1.4.26 *The morphism $\pi : (\Sigma, \partial\Sigma) \to (S, L+L')$ is a toroidal pseudo-resolution of C.*

Remark 1.4.27 We formulated Algorithm 1.4.22 only for *reduced* curve singularities C. It extends readily to an algorithm applicable to any C, by agreeing that one runs it on the reduction of C. One may agree also to define the fan tree of an arbitrary curve singularity C as the fan tree of its reduction (see Definition 1.4.33), each leaf being decorated with the multiplicity of the corresponding branch inside the divisor C. Similar conventions may be chosen in order to associate a lotus to an arbitrary curve singularity C. As we do not use those more general notions in this text, we will not introduce them formally.

Let us give now an example of application of Algorithm 1.4.22. Instead of starting from a particular equation, we will assume that the algorithm involves three levels of toroidal modifications with prescribed Newton fans and we will describe from them the toroidal boundary of the final surface. We will see in Example 1.6.29 below how to write concrete equations for branches C_i and L_j appearing in a toroidal resolution process structured as in Example 1.4.28. The idea is to associate to the Newton polygons of the process a *fan tree* (see Definition 1.4.33), which may be transformed into an *Eggers-Wall tree* (see Definition 1.6.28), which in turn allows to write Newton-Puiseux series defining the branches C_i and L_j. One may take as their defining functions in $\mathbb{C}[[x, y]]$ the minimal polynomials of those Newton-Puiseux series.

Example 1.4.28 We will use Notations 1.4.25, but we will denote in the same way a branch and its various strict transforms by the modifications produced by the algorithm. In particular, we will write L_j instead of B_j, for any $j \in J$.

Assume that, relative to the first cross (L, L_1), which lives on $S^{(1)} = S$, the Newton fan $\mathcal{F}_{L,L_1}(C)$ of the curve singularity C is as represented on the top of Fig. 1.16. Therefore it is the same fan $\mathcal{F}(3/5, 2/1, 5/2)$ as in Fig. 1.8. The associated Newton modification $\pi^{(1)}$ is represented on the bottom of Fig. 1.16.

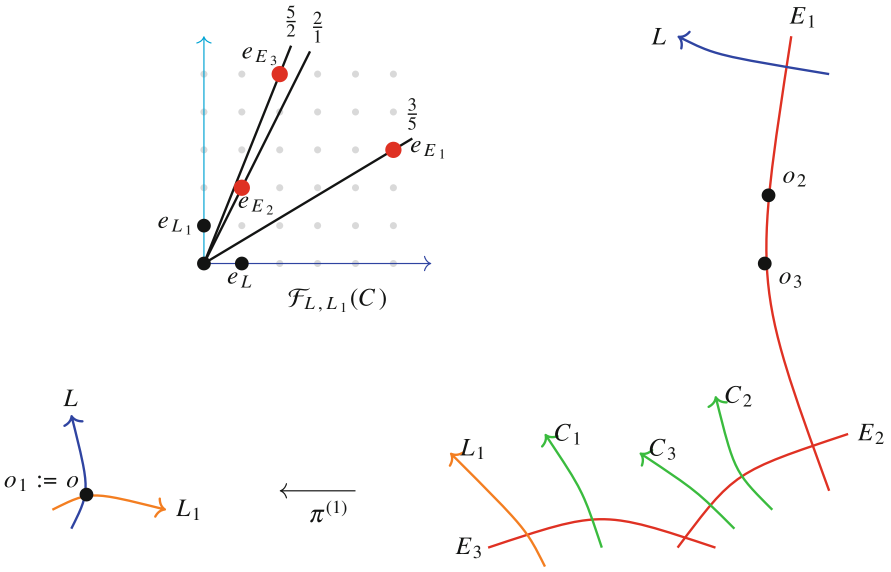


Fig. 1.16 First level of Newton modifications in Example 1.4.28

We have drawn schematically the two boundaries $\partial S^{(1)} = L + L_1$ and $\partial S^{(2)} = L + E_1 + E_2 + E_3 + L_1 + C_1 + C_2 + C_3$. The components E_i of the exceptional divisor of $\pi^{(1)}$ correspond to the rays $\mathbb{R}_+ \, e_{E_i}$ of the Newton fan $\mathcal{F}_{L,L_1}(C)$. We assume that there are three intersection points of the strict transform C_{L,L_1} of C by $\pi^{(1)}$ at which the algorithm stops at STEP 1. The corresponding components of C are denoted C_1, C_2, C_3. By contrast, at the points o_2 and o_3, one has to apply STEP 2 of Algorithm 1.4.22 (which implies that $\{2, 3\} \subseteq J$).

One introduces two new smooth branches L_2 and L_3 passing through o_2 and o_3 respectively, transversally to the exceptional divisor $E_1 + E_2 + E_3$ of $\pi^{(1)}$. Both points o_2 and o_3 belong to the component E_1. Now one may get the second level of Newton modifications, by looking at the Newton fans $\mathcal{F}_{E_1,L_2}(C)$ and $\mathcal{F}_{E_1,L_3}(C)$ (note that we have written (E_1, L_j) instead of (A_j, L_j), because for $j \in \{2, 3\}$, A_j is the germ of E_1 at o_j). We assume that those Newton fans are as represented on the top of Fig. 1.17. The corresponding composition $\pi^{(2)}$ of Newton modifications at o_2 and o_3 is represented on the bottom of the figure, through a schematic drawing

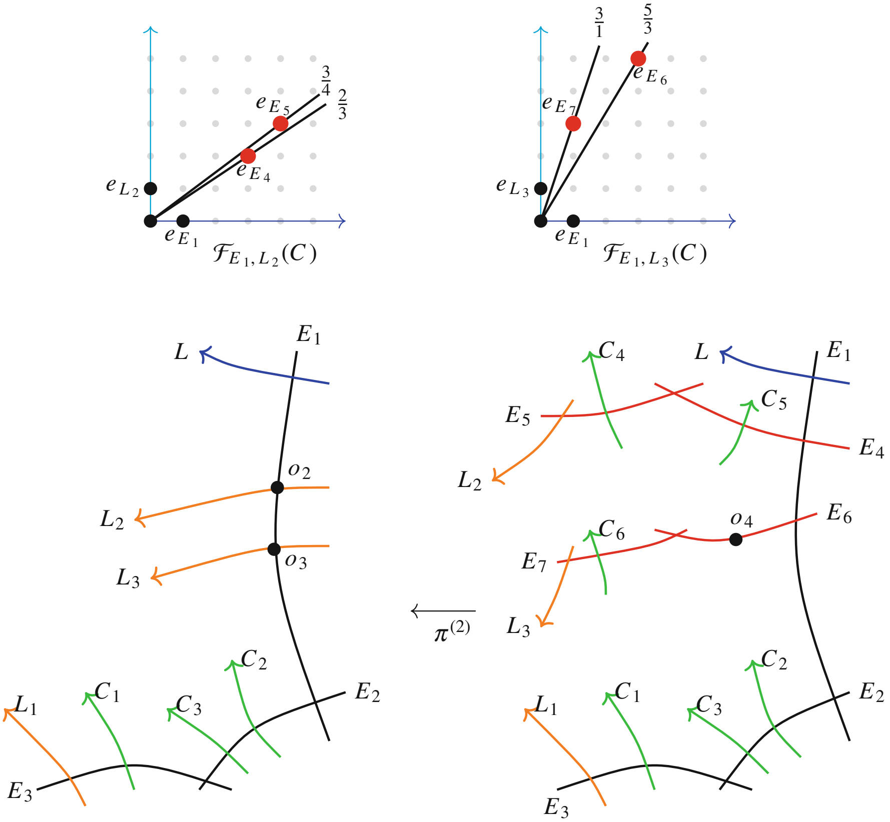


Fig. 1.17 Second level of Newton modifications in Example 1.4.28

of $\partial S^{(2)} + L_2 + L_3$ and of $\partial S^{(3)} = \partial \Sigma$. We assume that the process stops at STEP 1 at three more points, through which pass the strict transforms of the branches C_4, C_5, C_6 of C (see the right bottom part of Fig. 1.17). There remains one point o_4, lying on the component E_6 of the exceptional divisor $E_4 + E_5 + E_6 + E_7$ of $\pi^{(2)}$, at which one has to perform STEP 2.

One completes then the germ A_4 of E_6 at o_4 into a cross (E_6, L_4), represented on the left bottom part of Fig. 1.18. We assume now that the Newton fan $\mathcal{F}_{E_6,L_4}(C)$ is as drawn on the top of the figure. It has only one ray distinct from the edges of the cone $\mathbb{R}_+\langle e_{E_6}, e_{L_4}\rangle$. Therefore, the corresponding Newton modification, which alone gives the third level of Newton modifications $\pi^{(3)}$, introduces only one more irreducible component of exceptional divisor, denoted E_8. It is cut by the strict transform of one more branch of C, denoted C_7 and represented on the bottom right part of Fig. 1.18. The whole curve schematically represented here is the boundary $\partial\Sigma$. On the bottom left is represented the divisor $\partial S^{(3)} + L_4$. The toroidal pseudo-resolution of C produced by the algorithm is the composition $\pi^{(1)} \circ \pi^{(2)} \circ \pi^{(3)} : (\Sigma, \partial\Sigma) \to (S, L + L_1)$. The singular points of the total surface $\Sigma := S^{(3)}$ correspond bijectively to the non-regular 2-dimensional cones of the Newton fans $\mathcal{F}_{L,L_1}(C)$, $\mathcal{F}_{E_1,L_2}(C)$, $\mathcal{F}_{E_1,L_3}(C)$ and $\mathcal{F}_{E_6,L_4}(C)$ produced by the algorithm. We represented them as small blue discs on the bottom right of Fig. 1.18.

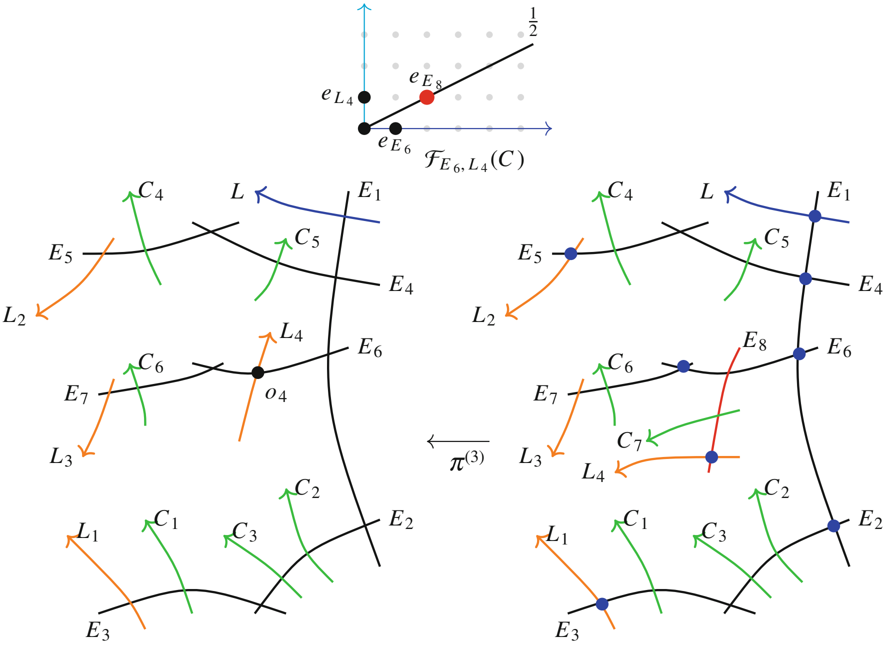


Fig. 1.18 Third level of Newton modifications in Example 1.4.28

1.4.3 From Toroidal Pseudo-Resolutions to Embedded Resolutions

In this subsection, we explain how to get an embedded resolution of $C \hookrightarrow S$ from one of the toroidal pseudo-resolutions produced by Algorithm 1.4.22. Recall first from Definition 1.4.15 the difference between toroidal pseudo-resolutions and embedded ones: in the first ones the source of the modification may have toric singularities, while in the second ones the source is required to be smooth.

Consider a toroidal pseudo-resolution morphism $\pi : (\Sigma, \partial\Sigma) \to (S, L + L')$ of C produced by Algorithm 1.4.22 (we speak about "a morphism" instead of "the morphism", because of the choices of smooth branches $(L_j)_{j\in J}$ involved in its construction, see Definition 1.4.25). The surface Σ has a finite number of singular points. As explained in Example 1.4.36, they correspond to the 2-dimensional non-regular cones of the Newton fans which appeared during the process. Proposition 1.3.28 shows that one may resolve minimally those singular points by taking the regularization of each such cone. In fact, those regularizations glue into the regularizations of the Newton fans.

A way to regularize all the Newton fans produced by Algorithm 1.4.22 is to run a variant of it, obtained by always replacing STEP 3 with the following "regularized" version of it:

STEP 3^{reg}. *Let* $\mathcal{F}^{reg}_{L,L'}(C)$ *be the regularized Newton fan of* C *relative to the cross* (L, L') *and let* $\psi^{C,reg}_{L,L'} : (S_{\mathcal{F}^{reg}_{L,L'}(C)}, \partial S_{\mathcal{F}^{reg}_{L,L'}(C)}) \to (S, L + L')$ *be the associated Newton modification. Consider the strict transform* $C_{L,L'}$ *of* C *by* $\psi^{C,reg}_{L,L'}$.

We did not change the notations for the successive strict transforms of C from STEP 3 to STEP 3^{reg}, because this variant of the algorithm does never modify the surfaces produced by the first algorithm in the neighborhood of those strict transforms. Indeed, the strict transforms never pass through the singular points of the modified surfaces $S^{(k)}$ (see Proposition 1.4.18 and Notations 1.4.25).

One has the following description of the result of running the "regularized" algorithm:

Proposition 1.4.29 *Let* $\pi : (\Sigma, \partial\Sigma) \to (S, L+L')$ *be a toroidal pseudo-resolution obtained by running Algorithm 1.4.22. Assume that one replaces always STEP 3 with STEP* 3^{reg} *above, choosing the same smooth germs* $(L_j)_{j\in J}$ *as in the construction of* π. *Then one gets a morphism in the toroidal category* $\boxed{\pi^{reg}}$: $(\Sigma^{reg}, \partial\Sigma^{reg}) \to (S, L + L')$, *which is moreover an embedded resolution of* C *and which factors as* $\pi^{reg} = \pi \circ \eta$, *where* $\eta : (\Sigma^{reg}, \partial\Sigma^{reg}) \to (\Sigma, \partial\Sigma)$ *is a modification in the toroidal category whose underlying modification of complex surfaces is the minimal resolution of the complex surface* Σ.

Let us look at the underlying morphism of complex surfaces $\pi^{reg} : \Sigma^{reg} \to S$. Both surfaces are smooth, therefore this morphism is a composition of blow ups of points, by the following theorem of Zariski (see [61, Corollary 5.4] or [113, Vol.1, Ch. IV.3.4, Thm.5]):

Theorem 1.4.30 *Let $\psi : S_2 \to S_1$ be a modification of a smooth complex surface S_1, with S_2 also smooth. Then ψ may be written as a composition of blow ups of points.*

In Sect. 1.5 we will describe explicitly the combinatorics of the decomposition of $\pi^{reg} : \Sigma^{reg} \to S$ into blow ups of points.

Let us recall the following classical terminology about objects associated to a process of blow ups of points, starting from $o \in S$ (see [78], [19, Chap. 3], [102] and [96]):

Definition 1.4.31 Let (S, o) be a smooth germ of surface.

- An **infinitely near point** of o is either o or a point of the exceptional divisor of a smooth modification of (S, o). Two such points, on two modifications, are considered to be the same, if the associated bimeromorphic map between the two modifications is an isomorphism in their neighborhoods.
- If o_1 and o_2 are two infinitely near points of o, then one says that o_2 **is proximate to** o_1, written $\boxed{o_2 \to o_1}$, if o_2 belongs to the strict transform of the irreducible rational curve created by blowing up o_1. If moreover there is no point o_3 such that $o_2 \to o_3 \to o_1$, one says that o_1 **is the parent of** o_2.
- A **finite constellation (above** o**)** is a finite set C of infinitely near points of o, closed under the operation of taking the parent.
- The **Enriques diagram** $\boxed{\Gamma(C)}$ of the finite constellation C is the rooted tree with vertex set C, rooted at o, and such that there is an edge joining each point of C with its parent.

Note that the proximity binary relation on the set of all the infinitely near points of o is not a partial order, as it is neither reflexive, nor transitive. For instance, if o_1 belongs to the exceptional divisor E_0 of the blow up of o and o_2 belongs to the exceptional divisor of the blow up of o_1 but not to the strict transform of E_0 by this blow up, then $o_2 \to o_1 \to o$, but $o_2 \not\to o$. Therefore, the Enriques diagram of a finite constellation encodes only part of the proximity binary relation on it. For this reason, Enriques introduced in [35] supplementary rules for the drawing of his diagrams, allowing to reconstruct completely the proximity relation. Namely, the edges of the Enriques diagram are moreover either *straight* or *curved* and there are *breaking points* between some pairs of successive straight edges. As we do not insist on those aspects, we do not give the precise definitions, sending the interested reader to the literature cited above.

1.4.4 The Fan Tree of a Toroidal Pseudo-Resolution Process

In this subsection we explain how to associate a *fan tree* to each process of toroidal pseudo-resolution of a curve singularity C on the smooth germ of surface (S, o) (see Definition 1.4.33). It is a couple formed by a rooted tree and a $[0, \infty]$-valued function constructed from the Newton fans created by the process. It turns out that it is isomorphic to the dual graph of the boundary $\partial\Sigma$ of the source surface Σ of the toroidal pseudo-resolution morphism $\pi : (\Sigma, \partial\Sigma) \to (S, \partial S)$ (see Proposition 1.4.35).

Fan trees are constructed from *trunks* associated with Newton fans. Let us define first those trunks:

Definition 1.4.32 Let N be a 2-dimensional lattice endowed with a basis (e_1, e_2) and let $\mathcal{F}$ be a Newton fan of N relative to this basis, in the sense of Definition 1.4.9. Its **trunk** $\boxed{\theta(\mathcal{F})}$ is the segment $[e_1, e_2] \subseteq \sigma_0$ endowed with the **slope function** $\boxed{\mathbf{S}_{\mathcal{F}}} : [e_1, e_2] \to [0, \infty]$ which associates with each point $w \in [e_1, e_2]$ the slope in the basis (e_1, e_2) of the ray $\mathbb{R}_+ w$ generated by it. Its **marked points** are the points of intersection of $[e_1, e_2]$ with the rays of $\mathcal{F}$. If $\mathcal{E} \subseteq \mathbb{Q}_+ \cup \{\infty\}$, we denote by $\boxed{\theta(\mathcal{E})}$ the trunk of the fan $\mathcal{F}(\mathcal{E})$ introduced in Definition 1.3.4.

Note that the slope function of a trunk is a homeomorphism. Several examples of trunks are represented in Fig. 1.19.

Assume now that we apply Algorithm 1.4.22 to the curve singularity C living on the smooth germ of surface (S, o). Consider the set $\{(A_i, B_i), i \in I\}$ of crosses produced by the algorithm, as explained in Notations 1.4.25. Note that we consider also the crosses at which the algorithm stops at an iteration of STEP 1. Denote by $\boxed{(e_{A_i}, e_{B_i})}$ the basis (e_1, e_2) of the weight lattice N_{A_i, B_i}. The segment $[e_{A_i}, e_{B_i}]$ is the trunk $\theta(\mathcal{F}_{A_i, B_i}(C))$. The following definition uses Notations 1.4.25:

Definition 1.4.33 The **fan tree** $\boxed{(\theta_\pi(C), \mathbf{S}_\pi)}$ **of the toroidal pseudo-resolution** $\pi : (\Sigma, \partial\Sigma) \to (S, L + L')$ **of** C is a pair formed by a rooted tree $\theta_\pi(C)$ and a **slope function** $\mathbf{S}_\pi : \theta_\pi(C) \to [0, \infty]$ obtained by gluing the disjoint union of the trunks $(\theta(\mathcal{F}_{A_i, B_i}(C)), \mathbf{S}_{\mathcal{F}_{A_i, B_i}(C)})_{i \in I}$ in the following way:

1. Label each marked point with the corresponding irreducible component E_k, L_j or C_l of the boundary $\partial\Sigma$ of the toroidal surface $(\Sigma, \partial\Sigma)$.
2. Identify all the points of $\bigsqcup_{i \in I} \theta(\mathcal{F}_{A_i, B_i}(C))$ which have the same label. The result of this identification is the fan tree $\theta_\pi(C)$ and the images inside it of the marked points of $\bigsqcup_{i \in I} \theta(\mathcal{F}_{A_i, B_i}(C))$ are its **marked points**. We keep for each one of them the same label as in the initial trunks.
3. The **root** of $\theta_\pi(C)$ is the point labeled by the initial smooth branch L.
4. For every $i \in I$, the restriction of the function $\mathbf{S}_\pi$ to every half-open trunk $\theta(\mathcal{F}_{A_i, B_i}(C)) \setminus \{e_{A_i}\} = (e_{A_i}, e_{B_i}] \hookrightarrow \theta_\pi(C)$ is equal to $\mathbf{S}_{\mathcal{F}_{A_i, B_i}(C)}$.
5. At the root, $\mathbf{S}_\pi(L) = \mathbf{S}_{\mathcal{F}_{L, L_1}(C)}(L) = 0$.

As in any rooted tree, the root L defines a partial order $\boxed{\preceq_L}$ on the set of vertices of the fan tree $\theta_\pi(C)$ (that is, on its set of marked points), by declaring that $P \preceq_L Q$ if and only if the unique segment $[L, P]$ joining L and P inside the tree is included in the analogous segment $[L, Q]$.

Note that the slope function $\mathbf{S}_\pi$ is discontinuous at all the marked points of $\theta_\pi(C)$ resulting from the identification of points of two different trunks, its directional limits jumping from a positive value to 0 when one passes from one trunk to another one in increasing way relative to the partial order $\preceq_L$. It follows that the fan tree of a toroidal pseudo-resolution determines the trunks $(\theta(\mathcal{F}_{A_i,B_i}(C)), \mathbf{S}_{\mathcal{F}_{A_i,B_i}(C)})_{i\in I}$.

Example 1.4.34 Consider again the toroidal pseudo-resolution process of Example 1.4.28. The construction of the trunks associated to its Newton fans is represented in Fig. 1.19 for all the crosses at which one applies STEP 2 of the algorithm, that is, for the crosses (A_i, B_i) with $i \in J$. The remaining crosses are those at which the algorithm stops while executing STEP 1. The corresponding trunks are represented on the bottom line of Fig. 1.19. Figure 1.20 shows the construction of the fan tree from the previous collection of trunks. In order to make clear the process of gluing of points with the same label, the upper part of the figure shows again the whole collection of trunks, as well as the labels of its marked points.

The following proposition is an easy consequence of Definition 1.4.33 and of Proposition 1.3.24 (recall that the notion of *dual graph* of an abstract simple normal crossings curve was explained in Definition 1.3.22):

Proposition 1.4.35 *The fan tree $\theta_\pi(C)$ is isomorphic to the dual graph of the boundary $\partial\Sigma$ of the source of the toroidal pseudo-resolution $\pi : (\Sigma, \partial\Sigma) \to (S, L + L')$ of the curve singularity C, by an isomorphism which respects the labels.*

Example 1.4.36 Proposition 1.4.35 is illustrated in Fig. 1.21 with the fan tree of the bottom of Fig. 1.20 and the boundary $\partial\Sigma$ of the bottom right of Fig. 1.18. Both of them correspond to the toroidal pseudo-resolution process of Example 1.4.28. The singular points of Σ may be found out from the knowledge of the slope function on the trunks composing the fan tree. Indeed, consider the slopes β/α and δ/γ of two consecutive vertices of the trunk of one of the Newton fans of the pseudo-resolution process. Then the matrix $\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$ is of determinant ± 1 if and only if the intersection point o_i of the irreducible components of $\partial\Sigma$ which corresponds to this edge is non-singular on Σ. Moreover, the surface singularity (Σ, o_i) is analytically isomorphic to the germ at its orbit of dimension 0 of the affine toric surface generated by the cone $\mathbb{R}_+\langle \alpha\, e_1 + \beta\, e_2, \gamma\, e_1 + \delta\, e_2\rangle$ and the lattice $N = \mathbb{Z}\langle e_1, e_2\rangle$. As in Fig. 1.18, the singular points on $\partial\Sigma$ are indicated by small blue discs. The corresponding edges of the fan trees are represented also in blue. Note that in the previous explanation it was important to say that one has to work with the slope function on the individual trunks, instead of the slope function of the fan tree. For instance, if one looks at the intersection point of the components E_1 and E_4, the corresponding slopes are to be read on the trunk $\theta(\mathcal{F}_{E_1,L_2}(C))$ (they are therefore $0/1$ and $2/3$, and the associated

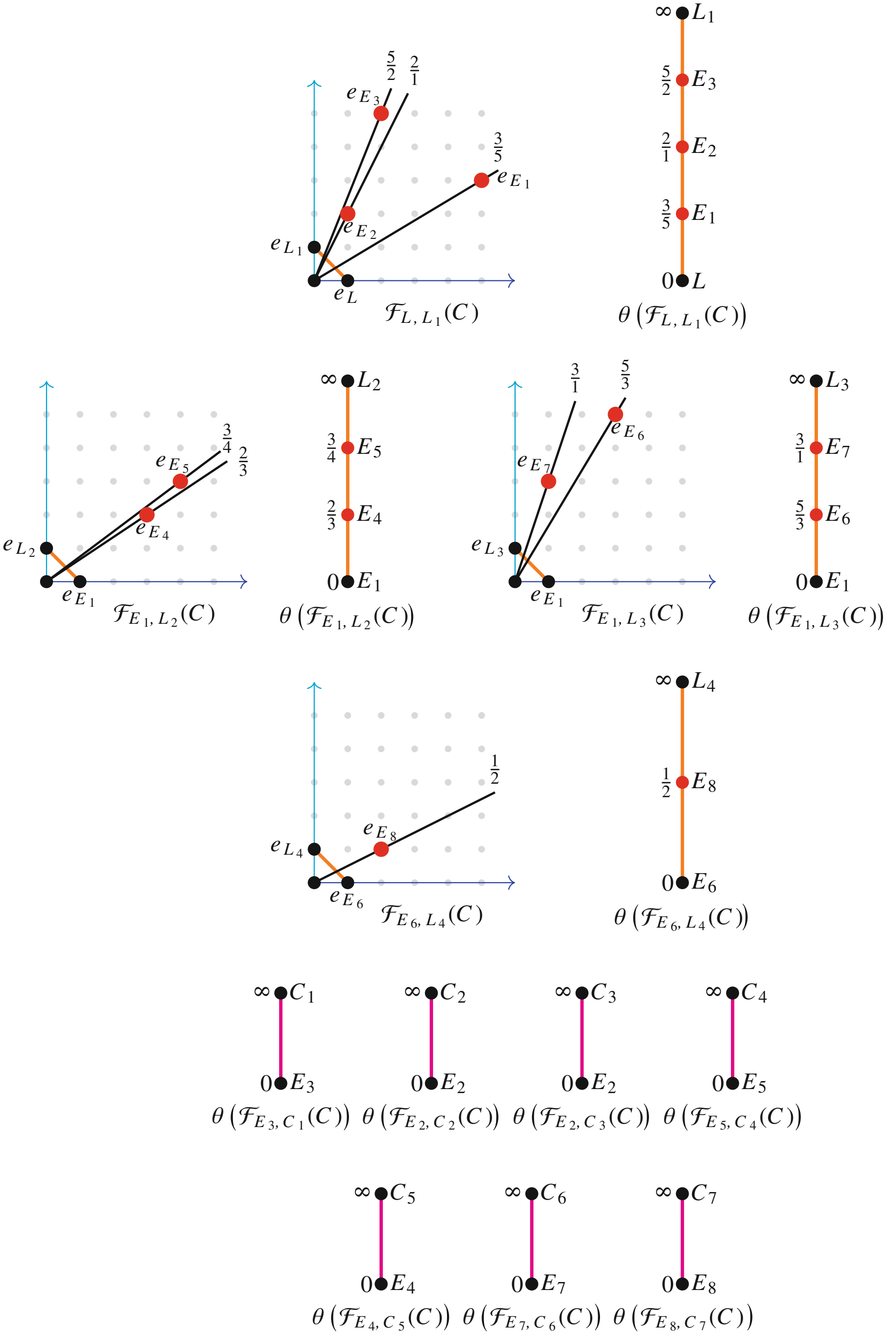


Fig. 1.19 The trunks associated to the toroidal pseudo-resolution of Example 1.4.28

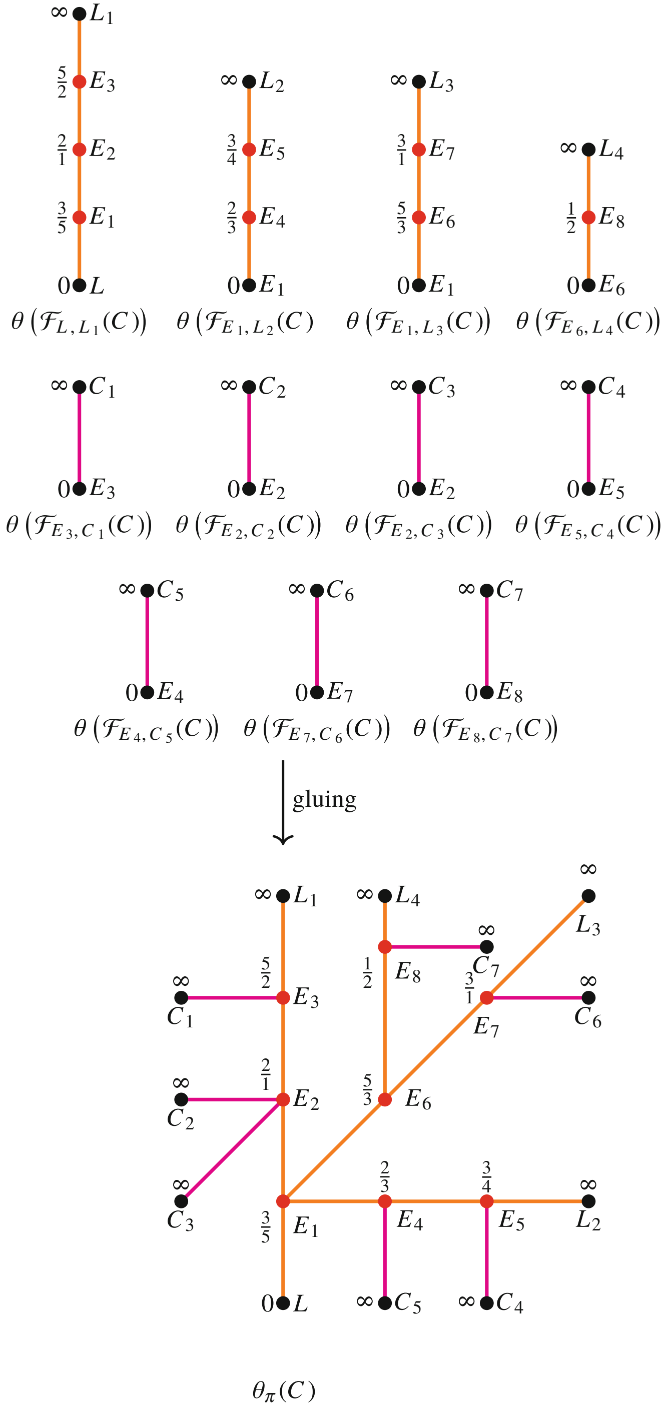


Fig. 1.20 Construction of the fan tree of the toroidal pseudo-resolution of Example 1.4.28

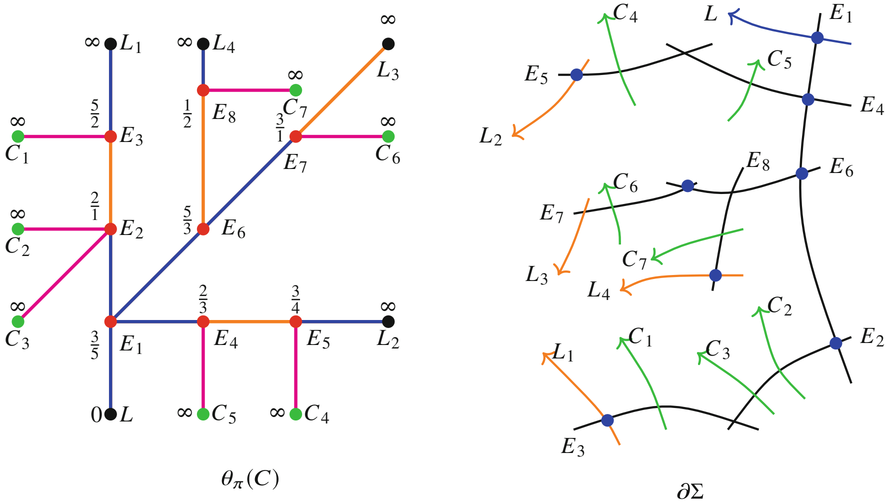


Fig. 1.21 The fan tree $\theta_\pi(C)$ is isomorphic to the dual graph of the toroidal boundary $\partial\Sigma$

matrix $\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$ is not unimodular), not on the fan tree $\theta_\pi(C)$ (which would give the slopes $3/5$ and $2/3$, whose associated matrix $\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$ is unimodular).

1.4.5 Historical Comments

The oldest method to study a plane curve singularity C, imagined by Newton around 1665, but published only in 1736 as [88], is to express it first in local coordinates (x, y) as the vanishing locus of a power series $f(x, y)$ satisfying $f(0, 0) = 0$ and $f(0, y) \neq 0$, then to compute iteratively a formal power series $\eta(x)$ with *rational* positive exponents such that $f(x, \eta(x)) = 0$. Whenever $\frac{\partial f}{\partial y}(0, 0) \neq 0$, there is only one such series $\eta(x)$ which has moreover only integral exponents. This series is simply the Taylor expansion at the origin of the explicit function $y(x)$ whose existence is ensured by the implicit function theorem applied to the function $f(x, y)$ in the neighborhood of $(0, 0)$. But, if $\frac{\partial f}{\partial y}(0, 0) = 0$, then there are at least two such series, their number being equal to the order in y of the series $f(0, y)$.

As explained on the example studied in Sect. 1.2.6, the first step of Newton's iterative method consists in finding the possible leading terms $c\, x^\alpha$ of the series $\eta(x)$. His main insight was that if one substitutes $y := c\, x^\alpha$ in the series $f(x, y)$, getting a formal power series with rational exponents in the variable x, then *there are* *at least two terms of this series with minimal exponent, and the sum of all such terms vanishes*. This fact has two consequences. First, there is a finite number of possible exponents α, which are the slopes of the rays orthogonal to the compact edges of the *Newton polygon* of $f(x, y)$. Secondly, for a fixed exponent α_K corresponding to the compact edge K, there is a finite number of values of the leading coefficient c, given by the roots of the algebraic equation $f_K(x, c\, x^{\alpha_K}) = 0$, where f_K is the restriction of f to K in the sense of Definition 1.4.2.

Newton's explanations were much developed in Cramer's 1750 book [27, Chapter VII], which seems also interesting to us in this context for its interpretation of the weights of the variables x and y as orders of magnitude for infinitely small quantities.

Figures 1.22 and 1.23 are extracted from [88, I, Section XXX] and [27, Section 103] respectively. The first one represents the only drawing of Newton polygon in

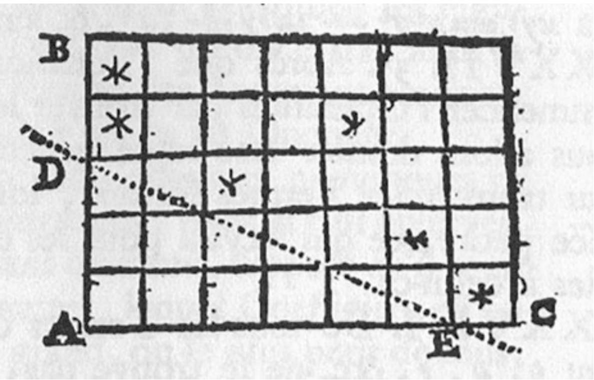


Fig. 1.22 The first Newton polygon

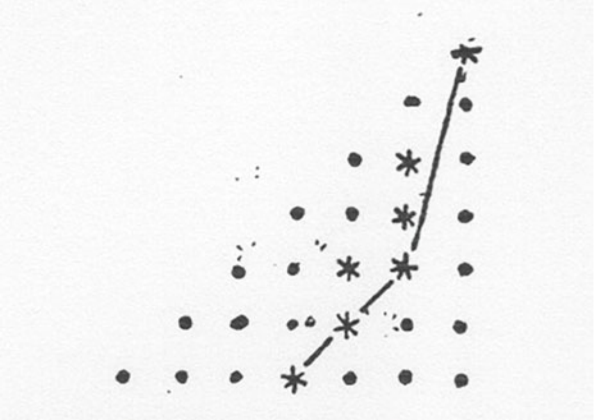
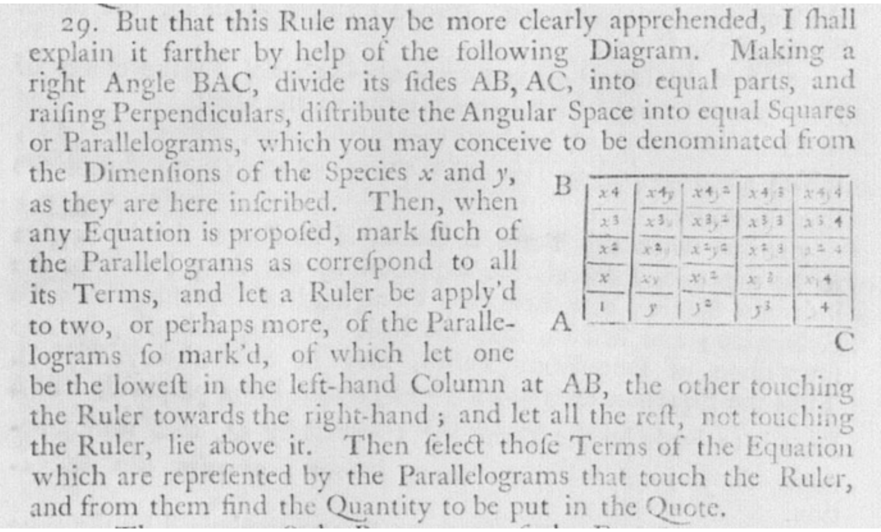


Fig. 1.23 The compact sides of a Newton polygon, as represented by Cramer



29. But that this Rule may be more clearly apprehended, I ſhall explain it farther by help of the following Diagram. Making a right Angle BAC, divide its ſides AB, AC, into equal parts, and raiſing Perpendiculars, diſtribute the Angular Space into equal Squares or Parallelograms, which you may conceive to be denominated from the Dimenſions of the Species x and y, as they are here inſcribed. Then, when any Equation is propoſed, mark ſuch of the Parallelograms as correſpond to all its Terms, and let a Ruler be apply'd to two, or perhaps more, of the Parallelograms ſo mark'd, of which let one be the loweſt in the left-hand Column at AB, the other touching the Ruler towards the right-hand; and let all the reſt, not touching the Ruler, lie above it. Then ſelect thoſe Terms of the Equation which are repreſented by the Parallelograms that touch the Ruler, and from them find the Quantity to be put in the Quote.

x^4	x^4y	x^4y^2	x^4y^3	x^4y^4
x^3	x^3y	x^3y^2	x^3y^3	x^3y^4
x^2	x^2y	x^2y^2	x^2y^3	x^2y^4
x	xy	xy^2	xy^3	xy^4
1	y	y^2	y^3	y^4

Fig. 1.24 Newton's ruler

Newton's book. Strictly speaking, what we call the *Newton polygon* of a series in two variables was not formally introduced in the book. Newton explained only how to move a ruler in order to get a first bounded edge of the polygon (see Fig. 1.24). More details about Newton's and Cramer's ideas on this subject may be found in Ghys' 2017 book [50, Pages 43–68].

Newton wrote that his procedure may be performed iteratively in order to compute as many terms of the series $\eta(x)$ as desired. He also sketched in [88, Ch. I.LII] an explanation of the fact that, whenever $f(x, y)$ converges, the formal series with rational exponents $\eta(x)$ obtained by continuing forever the procedure also converge and satisfy indeed, all of them, the relation $f(x, \eta(x)) = 0$. But it was Puiseux, in his 1850 paper [106], who proved rigorously that one gets indeed as many series as the order in y of $f(0, y)$, that all of them are obtained by substituting some root $x^{1/n}$ of the variable x into formal power series with integral exponents, and that those formal power series are in fact convergent in a neighborhood of the origin. In order to honor his work, the formal or convergent power series in a variable x of the form $\xi(x^{1/n})$, where $\xi(x)$ is a usual power series and $n \in \mathbb{N}^*$ are called nowadays *Puiseux series* or *Newton-Puiseux series*.

Puiseux's approach to the proofs of the existence and the convergence of these series avoided the use of roots $x^{1/n}$, by performing changes of variables of the form $x = x_1^q,\ y = c_1 x_1^p + y_1$ or of the form $x = x_1^q,\ y = x_1^p(c_1 + y_1)$, where c_1 is a non-zero constant and p/q is the irreducible expression of one of the exponents α_K given by the Newton polygon of f. Both changes of variables are compositions of a birational change of variables and of the monomial change of variables $x = u^q,\ y = u^p\, v$. This monomial change of variables is birational only when $q = 1$, that is, when $\alpha_K \in \mathbb{N}^*$. Therefore Puiseux's changes of variables are in general not birational. Nevertheless, by Lemma 1.6.24 below, such a map can be seen as the local analytical expression of a birational map, with respect to a particular choice of local coordinates.

Zariski saw this non-birationality as a drawback, and in his 1939 paper [136] he introduced alternative changes of variables of the form $x = x_1^q(c_1 + y_1)^{q_1}$, $y = x_1^p(c_1 + y_1)^{p_1}$, where $(p_1, q_1) \in \mathbb{N}^* \times \mathbb{N}^*$ and $p_1 q - q_1 p = 1$. This last condition means that Zariski's changes of variables are birational.

Let us discuss now the toric approach to the study of plane curve singularities. Note that the changes of variables used by Puiseux and by Zariski are compositions of affine morphisms and of toric ones. This fact became clear after the development of toric geometry (see Sect. 1.3.5).

The systematic study of plane curve singularities using sequences of toric modifications began with Mutsuo Oka's 1995–96 papers [8, 83, 93], the first one written in collaboration with Lê and the second one with A'Campo (see also Oka's 1997 book [94, Ch. III, Sect. 4]). Oka gave an introduction to this approach in his 2010 paper [95], through the detailed examination of the case of one branch. The second author generalized this approach to *quasi-ordinary* hypersurface singularities of arbitrary dimension in his 2003 paper [52] and applied it to the study of deformations of real plane curve singularities in the 2010 papers [53] and [54], the second one written in collaboration with Risler.

Also during the 1990s, Pierrette Cassou-Noguès started studying plane curve singularities using Puiseux's non-birational toric morphisms, called *Newton maps*. References to her early works on the subject, done partly in collaboration, may be found in her 2011 paper [20] with Płoski, her 2014–15 papers [22, 23] with Veys, her 2014 paper with Libgober [21] and her 2018 paper with Raibaut [24].

In his 1997 paper [129], Veys considered the *log-canonical model* of a plane curve singularity, obtained by contracting certain exceptional divisors on its minimal embedded resolution, in order to study associated zeta functions. The modification from the log-canonical model to the ambient germ of the plane curve singularity may be seen as a morphism associated with a toroidal pseudo-resolution of this singularity. A toroidal pseudo-resolution algorithm for plane curve singularities was described by the second author in [52, Section 3.4]. A more general algorithm was given by Cassou-Noguès and Libgober in [21, Section 3]. Our Algorithm 1.4.22 of toroidal pseudo-resolution generalizes them, since it does not depend on the choice of special kinds of coordinates.

There are several approaches for the search of *optimal choices* of smooth branches in STEP 2 of Algorithm 1.4.22. *Assume first that C is a branch*, that $f \in \mathbb{C}[[x]][y]$ is the monic polynomial of degree n defining C in the local coordinate system (x, y) and that the line $L = Z(x)$ is transversal to C. Let a be a divisor of n. The *a-th approximate root* $h \in \mathbb{C}[[x]][y]$ of f is the unique monic polynomial of degree a such that the degree in y of $f - h^{n/a}$ is smaller than $n - a$. The importance of approximate roots for the study of plane curve singularities and of the algebraic embeddings of $\mathbb{C}$ in $\mathbb{C}^2$ was emphasized by Abhyankar and Moh in their 1973–75 papers [2] and [3]. Certain approximate roots of f, called *characteristic approximate roots*, have the property that their strict transforms can be chosen at

STEP 2 of Algorithm 1.4.22, providing in this way a toroidal pseudo-resolution of C with the minimal number of Newton modifications. This number is precisely the number of *characteristic exponents* of C with respect to x (see Sect. 1.6). This approach was explained by A'Campo and Oka in their 1996 paper [8].

Some properties of the approximate roots may fail when working with a base field of positive characteristic. By contrast, the more general combinatorial notion of *semiroot/maximal contact curve* can be defined over fields of arbitrary characteristic and plays a similar role (see the papers [77] of Lejeune-Jalabert and [49] of the first author and Płoski). For details on applications of approximate roots and semiroots to the study of plane curve singularities, see the paper [60] of Gwoździewicz and Płoski and [99] of the third author. Proposition 1.4.35 above implies that if $\pi : (\Sigma, \partial\Sigma) \to (S, L + L')$ is a toroidal embedded resolution of C which defines its minimal resolution, then the irreducible components of the associated completion $\hat{C}_\pi = \pi(\partial\Sigma)$ may be thought as generalizations of the notion of semiroot to plane curve singularities with an arbitrary number of branches (see also the final comments in Example 1.6.33 below).

Assume now that C is an arbitrary plane curve singularity. The minimal number of Newton modifications involved in the construction of a toroidal pseudo-resolution C was characterized by Lê and Oka in [83] in terms of properties of the dual graph of its minimal embedded resolution.

Another toric approach to the study of plane curve singularities was initiated in Goldin and Teissier's 2000 paper [51], in the case of branches. They first reembedded in a special way the initial germ of surface in a higher dimensional space, then they resolved the branch by just one toric modification of that space. Their approach was done in the spirit of the philosophy of Teissier's 1973 paper [119], in which he saw all equisingular plane branches as deformations of a single branch of higher embedding dimension, the germ at the origin of their common *monomial curve*. A generalization of some of the results in [51] to the case of quasi-ordinary hypersurface singularities was obtained by the second author in [52]. The theoretical possibility of studying analogously singularities of any dimension was established by Tevelev in his 2014 paper [125]. See Teissier's comments in [124, Section 11] for more details about his toric approach to the study of singularities.

The notions of *Newton non-degenerate polynomials* and *series* were introduced by Kouchnirenko in his 1976 paper [74], using the last characterization of Proposition 1.4.20. A version of the first characterization was essential in Varchenko's theorem in [128] about the monodromy of Newton non-degenerate holomorphic series. Then Khovanskii introduced in [73] *Newton non-degenerate complete intersection singularities*, a notion which was much studied by Mutsuo Oka in a series of papers, which were the basis of his 1997 monograph [94]. Characterizations of Newton non-degenerate singularities, analogous to those of Proposition 1.4.20, are in fact true for complete intersection singularities (see Oka's book [94] or Teissier's paper [122, Section 5]). This last paper contains interesting comments about the evolution of the notion of Newton non-degeneracy, and an extension of it to arbitrary singularities, which are not necessarily complete intersections. This extension was further studied in Fuensanta Aroca, Gómez-Morales and Shabbir's paper [9].

Let us discuss now the notion of *tropicalization* trop^f introduced in Definition 1.4.4. The union of the rays of the Newton fan $\mathcal{F}(f)$ which intersect the interior of the regular cone σ_0 is the *tropical zero-locus* of the function trop^f, as defined in tropical geometry, that is, the locus of non-differentiability of the continuous piecewise linear function trop^f. It is also part of the *local tropicalization* of the zero locus $Z(f) \hookrightarrow (\mathbb{C}^2, 0)$ of f, as defined by Stepanov and the third author in [105] for complex analytic singularities of arbitrary dimension embedded in germs of affine toric varieties. The local tropicalization contains also portions at infinity, in a partial compactification of the cone defining the ambient toric variety, in order to keep track of the intersections of the singularity with all the toric orbits.

A precursor of the notion of local tropicalisation was introduced under the name of "*tropism of an ideal*" by Maurer in his 1980 article [85], which was unknown to the authors of [105] when they wrote that paper. In our case, the tropism of the ideal $(f) \subseteq \mathbb{C}[[x, y]]$ is the set of lattice points lying on the rays of $\mathcal{F}(f)$ which are different from the edges of the cone σ_0. The term "*tropism*" had been used before by Lejeune-Jalabert and Teissier in their 1973 paper [79], in the expression "*tropisme critique*". They saw this notion as a measure of anisotropy, as explained by Teissier in [65, Footnote to Sect. 1]:

> As far as I know the term did not exist before. We tried to convey the idea that giving different weights to some variables made the space "anisotropic", and we were intrigued by the structure, for example, of anisotropic projective spaces (which are nowadays called weighted projective spaces). From there to "tropismes critiques" was a quite natural linguistic movement. Of course there was no "tropical" idea around, but as you say, it is an amusing coincidence. The Greek "Tropos" usually designates change, so that "tropisme critique" is well adapted to denote the values where the change of weights becomes critical for the computation of the initial ideal. The term "Isotropic", apparently due to Cauchy, refers to the property of presenting the same (physical) characters in all directions. Anisotropic is, of course, its negation. The name of Tropical geometry originates, as you probably know, from tropical algebra which honours the Brazilian computer scientist Imre Simon living close to the tropics, where the course of the sun changes back to the equator. In a way the tropics of Capricorn and Cancer represent, for the sun, critical tropisms.

1.5 Lotuses

Throughout this section, we will assume that C is *reduced*. We explain the notion of *Newton lotus* (see Definition 1.5.4), its relation with continued fractions (see Sect. 1.5.2) and how to construct a more general *lotus* from the fan tree of a toroidal pseudo-resolution process (see Definition 1.5.26). It is a special type of simplicial complex of dimension 2, built from the Newton lotuses associated with the Newton fans generated by the process, by gluing them in the same way one glued the corresponding trunks into the fan tree. It allows to visualize the combinatorics of the decomposition of the embedded resolution morphism into point blow ups, as well as the associated Enriques diagram and the final dual graphs (see Theorem 1.5.29). We show by two examples that its structure depends on the choice of auxiliary curves introduced each time one executes STEP 2 of Algorithm 1.4.22, that is, on the choice of completion $\hat{C}_\pi$ of C (see Sect. 1.5.4). In Sect. 1.5.5 we define an operation of *truncation* of the lotus of a toroidal pseudo-resolution and we explain some of its uses. In the final Sect. 1.5.6 we give historical information about other works in which appeared objects similar to the notion of lotus.

1.5.1 The Lotus of a Newton Fan

In this subsection, whose content is very similar to that of [102, Section 5], we give a first level of explanation of the subtitle of this article, a second level being described in Sect. 1.5.3. Namely, we introduce the notion of *lotus* $\Lambda(\mathcal{F})$ of a Newton fan $\mathcal{F}$ (see Definition 1.5.4). If the fan originates from a Newton polygon $\mathcal{N}(f)$, that is, if $\mathcal{F} = \mathcal{F}(f)$ (see Definition 1.4.9), we imagine $\Lambda(\mathcal{F})$ as a *blossoming* of $\mathcal{N}(f)$. The lotus of a Newton fan $\mathcal{F}$ allows to understand visually the decomposition into blow ups of the toric modification defined by the regularized fan $\mathcal{F}^{reg}$. For instance, the dual graph of the final exceptional divisor, the Enriques diagram and the graph of the proximity relation of the associated constellation embed naturally in it, as subcomplexes of its 1-skeleton (see Propositions 1.5.11, 1.5.14 and 1.5.16).

Lotuses are built from *petals*, which are triangles with supplementary structure (see Fig. 1.25):

Definition 1.5.1 Let N be a 2-dimensional lattice and let (e_1, e_2) be a basis of it. Denote by $\boxed{\delta(e_1, e_2)}$ the convex and compact triangle with vertices $e_1, e_2, e_1 + e_2$, contained in the real plane $N_{\mathbb{R}}$. It is the **petal associated with the basis** (e_1, e_2). Its **base** is the segment $[e_1, e_2]$, oriented from e_1 to e_2. The points e_1 and e_2 are called the **basic vertices** of the petal. Its **lateral edges** are the segments $[e_i, e_1 + e_2]$, for each $i \in \{1, 2\}$.

Once the petal $\delta(e_1, e_1 + e_2)$ is constructed, the construction may be repeated starting from each one of the bases $(e_1, e_1 + e_2)$ and $(e_1 + e_2, e_2)$ of N, getting two

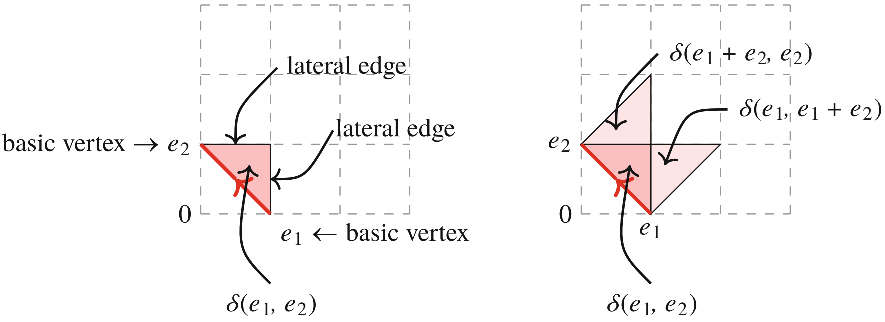


Fig. 1.25 Vocabulary and notations about petals

new petals $\delta(e_1, e_1+e_2)$ and $\delta(e_1+e_2, e_2)$, and so on. Note that the bases produced by this process are ordered such as to define always the same orientation of the real plane $N_{\mathbb{R}}$—we say that they are **positive bases**. In this way, one progressively constructs an infinite simplicial complex embedded in the cone σ_0: at the n-th step, one adds 2^n petals to those already constructed. Each petal, with the exception of the first one $\delta(e_1, e_2)$, has a common edge—its base—with exactly one of the petals constructed at the previous step, called its **parent**.

The pairs of vectors $(f_1, f_2) \in N^2$ which appear as bases of petals $\delta(f_1, f_2)$ during the previous process may be characterized in the following way (see [102, Remarque 5.1]):

Lemma 1.5.2 *A segment $[f_1, f_2]$, oriented from f_1 to f_2, is the base of a petal $\delta(f_1, f_2)$ constructed during the previous process if and only if (f_1, f_2) is a positive basis of the lattice N contained in the cone σ_0. Said differently, if a positive basis (f_1, f_2) of N is contained in the cone σ_0 and is different from (e_1, e_2), then there exists a unique permutation (i, j) of $(1, 2)$ such that $f_j - f_i \in \sigma_0 \cap N$.*

We are ready to define the simplest kinds of lotuses:

Definition 1.5.3 The simplicial complex obtained as the union of all the petals constructed by the previous process starting from the basis (e_1, e_2) of N, is called **the universal lotus** $\boxed{\Lambda(e_1, e_2)}$ **relative to** (e_1, e_2) (see Fig. 1.26). A **lotus** Λ **relative to** (e_1, e_2) is either the segment $[e_1, e_2]$ or the union of a non-empty set of petals of the universal lotus $\Lambda(e_1, e_2)$, stable under the operation of taking the parent of a petal. The segment $[e_1, e_2]$ is called the **base** of Λ. If Λ is of dimension 2, then the petal $\delta(e_1, e_2)$ is called its **base petal**. The point e_1 is called the **first basic vertex** and e_2 the **second basic vertex** of the lotus. The lotus is oriented by restricting to it the orientation of $N_{\mathbb{R}}$ induced by the basis (e_1, e_2).

A lotus may be associated with any set $\mathcal{E} \subseteq [0, \infty]$ or with any Newton fan:

Definition 1.5.4 Let N be a lattice of rank 2, endowed with a basis (e_1, e_2).

- If $\lambda \in (0, \infty)$, then its **lotus**, denoted $\boxed{\Lambda(\lambda)}$, is the union of petals of the universal lotus $\Lambda(e_1, e_2)$ whose interiors intersect the ray of slope λ. If $\lambda \in \{0, \infty\}$, then its lotus $\Lambda(\lambda)$ is just $[e_1, e_2]$.
- If $\mathcal{E} \subseteq [0, \infty]$, then its **lotus** $\boxed{\Lambda(\mathcal{E})}$ is the union $\bigcup_{\lambda \in \mathcal{E}} \Lambda(\lambda)$ of the lotuses of its elements.
- If $\mathcal{F}$ is a Newton fan and $\mathcal{F} = \mathcal{F}(\mathcal{E})$ in the sense of Definition 1.3.4, we say that $\boxed{\Lambda(\mathcal{F})} := \Lambda(\mathcal{E})$ is the **lotus of the fan** $\mathcal{F}$.
- A **Newton lotus** is the lotus of a Newton fan. That is, it is a lotus relative to (e_1, e_2) with a finite number of petals.

We could have called the lotuses relative to (e_1, e_2) *finite lotuses* instead of *Newton lotuses*. We chose the second terminology because in Definition 1.5.26 below we will introduce a more general kind of lotuses with a finite number of

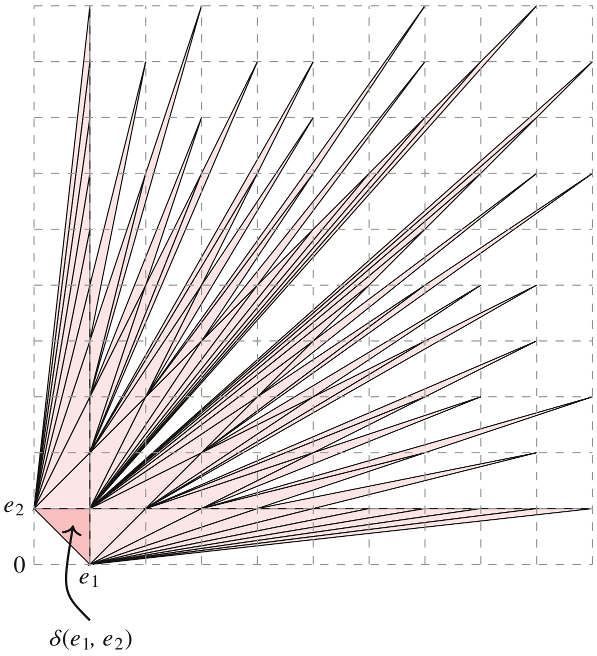


Fig. 1.26 Partial view of the universal lotus $\Lambda(e_1, e_2)$ relative to (e_1, e_2)

petals, and we want to distinguish the class of lotuses of Newton fans inside that more general class of lotuses.

A lotus $\Lambda(\mathcal{E})$, for $\mathcal{E} \subseteq [0, \infty]$, is a Newton lotus if and only $\mathcal{E}$ is a finite set of non-negative rational numbers. Note that, as illustrated for instance by Example 1.5.9 below, the structure of the lotus $\Lambda(\mathcal{E})$ does not allow to reconstruct the initial set $\mathcal{E}$. For this reason, we enrich $\Lambda(\mathcal{E})$ with several *marked* points, whose knowledge allows to reconstruct $\mathcal{E}$ unambiguously:

Definition 1.5.5 Fix a Newton lotus Λ.

- If $\Lambda \neq [e_1, e_2]$, we denote by $\boxed{\partial_+ \Lambda}$ the compact and connected polygonal line defined as the complement of the open segment (e_1, e_2) in the boundary of the lotus Λ. If $\Lambda = [e_1, e_2]$, we set $\partial_+ \Lambda := [e_1, e_2]$. The polygonal line $\partial_+ \Lambda \subseteq \Lambda$ is called the **lateral boundary** of the lotus Λ.
- We denote by $\boxed{p_\Lambda}$ the homeomorphism $p_\Lambda : [0, \infty] \to \partial_+ \Lambda$ which associates with any $\lambda \in [0, \infty]$ the unique point $p_\Lambda(\lambda) \in \partial_+ \Lambda$ of slope λ. If $\Lambda = \Lambda(\mathcal{E})$ where $\mathcal{E} \subseteq \mathbb{Q}_+ \cup \{\infty\}$ is finite and $\lambda \in \mathcal{E}$, then we call $p_{\Lambda(\mathcal{E})}(\lambda)$ the **marked point of** λ (or of the ray of slope λ) in the lotus $\Lambda(\mathcal{E})$. We consider $\Lambda(\mathcal{E})$ as a **marked lotus** using those marked points.

Remark 1.5.6 Notice that if $\lambda \in \mathcal{E}$, then $p_{\Lambda(\mathcal{E})}(\lambda)$ is by construction the unique primitive element $p(\lambda)$ of the lattice N, which has slope λ relative to the basis (e_1, e_2). Therefore, it is independent of the remaining elements of the set $\mathcal{E}$.

We distinguish also by geometric properties several vertices of a Newton lotus:

Definition 1.5.7 Assume that Λ is a Newton lotus. A vertex of Λ different from e_1 and e_2 is called a **pinching point** of the lotus Λ if it belongs to a unique petal of it. If the lotus Λ is two-dimensional, then the lattice point which is connected to e_2 (resp. to e_1) inside the lateral boundary $\partial_+\Lambda$ of Λ is called the **last interior point** (resp. **first interior point**) of the lateral boundary.

Remark 1.5.8 The pinching points of a Newton lotus $\Lambda(\mathcal{E})$ are part of its marked points. Two Newton lotuses $\Lambda(\mathcal{E}_1)$ and $\Lambda(\mathcal{E}_2)$ coincide as unmarked simplicial complexes if and only if their sets of pinching points coincide.

Example 1.5.9 In Fig. 1.27 are represented the lotuses $\Lambda\,(3/5)$ and $\Lambda(\mathcal{E})$, where $\mathcal{E} = \{3/5, 2/1, 5/2\}$ is the set whose fan $\mathcal{F}(\mathcal{E})$ was drawn in Fig. 1.8. The lotus $\Lambda\,(3/5)$ has only one pinching point, which is $p\,(3/5)$. The pinching points of $\Lambda(\mathcal{E})$ are $p\,(3/5)$ and $p\,(5/2)$. Its marked points are $p\,(3/5)$, $p\,(2/1)$ and $p\,(5/2)$. This differentiates it from the lotus $\Lambda(3/5, 5/2) := \Lambda(\{3/5, 5/2\})$, which is the same simplicial complex if one forgets their respective marked points. The first interior point of $\Lambda(\mathcal{E})$ is $p\,(1/2)$ and its last interior point is $p\,(3/1)$.

By comparing Figs. 1.27 and 1.9, which we combined in Fig. 1.28, one sees that the lateral boundary of the lotus $\Lambda(3/5, 2/1, 5/2)$ is exactly the polygonal line constructed when one performed the regularization of the fan $\mathcal{F}(3/5, 2/1, 5/2)$ (see Proposition 1.3.9). This is a general phenomenon, as shown by the following proposition.

Proposition 1.5.10 *Let $\mathcal{F}$ be a fan subdividing the cone σ_0. Then the regularization $\mathcal{F}^{reg}$ of $\mathcal{F}$ is obtained by subdividing σ_0 using the rays generated by all the lattice points lying along the lateral boundary $\partial_+\Lambda(\mathcal{F})$ of the lotus $\Lambda(\mathcal{F})$.*

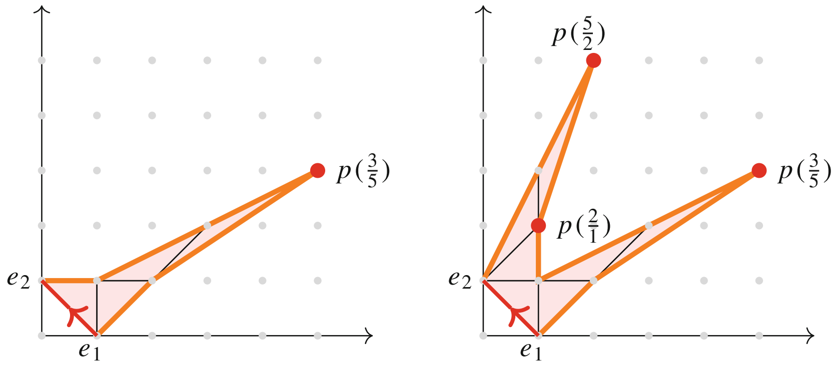


Fig. 1.27 The Newton lotuses $\Lambda\,(3/5)$, $\Lambda\,(3/5, 2/1, 5/2)$ and their marked points

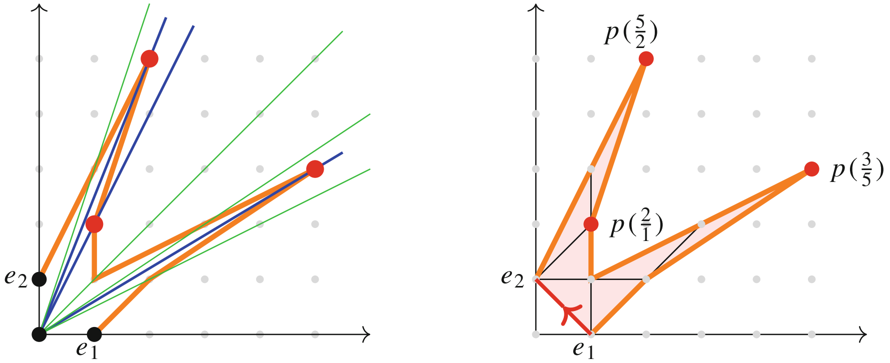


Fig. 1.28 The regularized fan $\mathcal{F}^{reg}$ (3/5, 2/1, 5/2) and the Newton lotus Λ (3/5, 2/1, 5/2)

Proof Consider two successive marked points $p(\lambda)$ and $p(\mu)$ of the lateral boundary $\partial_+\Lambda(\mathcal{F})$. They are primitive elements of the ambient lattice N. Denote by $p(\lambda)+\mathbb{R}_+p(\lambda)$ the closed half line originating from the point $p(\lambda)$ and generated by the vector $p(\lambda)$. Consider analogously the half-line $p(\mu) + \mathbb{R}_+p(\mu)$. Let $P(\lambda, \mu)$ be the polygonal line joining the points $p(\lambda)$ and $p(\mu)$ inside $\partial_+\Lambda(\mathcal{F})$. Consider the union of the three previous polygonal lines: $Q(\lambda, \mu) := (p(\lambda) + \mathbb{R}_+p(\lambda)) \cup P(\lambda, \mu) \cup (p(\mu) + \mathbb{R}_+p(\mu))$.

As the pinching points of $\Lambda(\mathcal{F})$ belong to the marked points, this shows that there are no pinching points in the interior of the polygonal line $P(\lambda, \mu)$. Therefore, $Q(\lambda, \mu)$ is the boundary of a closed convex set $\hat{Q}(\lambda, \mu)$ contained in the cone $\mathbb{R}_+\langle p(\lambda), p(\mu)\rangle$. The complement $\mathbb{R}_+\langle p(\lambda), p(\mu)\rangle \setminus \hat{Q}(\lambda, \mu)$ is contained in the union of the complement $\Lambda(\mathcal{F}) \setminus \partial_+\Lambda(\mathcal{F})$ and the convex hull of the points $0, e_1, e_2$ deprived of the segment $[e_1, e_2]$. Therefore, the origin 0 is the only point of N contained in $\mathbb{R}_+\langle p(\lambda), p(\mu)\rangle \setminus \hat{Q}(\lambda, \mu)$. As all the vertices of $Q(\lambda, \mu)$ belong to N, this shows that $\hat{Q}(\lambda, \mu)$ is the convex hull of the set $\mathbb{R}_+\langle p(\lambda), p(\mu)\rangle \cap (N \setminus \{0\})$. One concludes using Proposition 1.3.9. □

Consider again Fig. 1.28. As shown by Proposition 1.3.24, the polygonal line on the left side gives a concrete embedding of the dual graph of the boundary $\partial X_{\mathcal{F}^{reg}}$. But it does not show the order in which were performed the blow ups into which the associated modification $\psi^{\mathcal{F}}_{\sigma_0} : X_{\mathcal{F}} \to X_{\sigma_0}$ decomposes (see Theorem 1.4.30). It turns out that this order is indicated by the lotus on the right side of Fig. 1.28. To understand this fact, recall first the combinatorial description of the blow up of the orbit of dimension 0 of the smooth affine toric surface X_{σ_0}, explained in Example 1.3.27: one gets it by subdividing the cone σ_0 using the ray generated by $e_1 + e_2$. In terms of the associated bases of N, one replaces the basis (e_1, e_2) by the pair of bases $(e_1, e_1 + e_2)$ and $(e_1 + e_2, e_2)$. Graphically, this may be understood as the passage from the base $[e_1, e_2]$ of the petal $\delta(e_1, e_2)$ seen as the simplest 2-dimensional lotus (see Definition 1.5.1) to its lateral boundary $[e_1, e_1 + e_2] \cup [e_1 + e_2, e_2]$. Again by Proposition 1.3.24, we may see this passage as the replacement of the dual graph of ∂X_{σ_0} by the dual graph of the boundary of the blown up toric surface. Now, each new petal in the lotus $\Lambda(\mathcal{F})$ corresponds to the blow up of an orbit of dimension 0 of the previous toric surface. Its base may be seen as the dual graph of the irreducible components of the boundary meeting at that point. One gets:

Proposition 1.5.11 *Let $\mathcal{F}$ be a Newton fan. Then:*

- *The lateral boundary $\partial_+\Lambda(\mathcal{F})$ of the lotus $\Lambda(\mathcal{F})$ is the dual graph of the boundary $\partial X_{\mathcal{F}^{reg}}$ of the smooth toric surface $X_{\mathcal{F}^{reg}}$. Two vertices of it are joined by an edge of the lotus $\Lambda(\mathcal{F})$ if and only if the corresponding orbits have intersecting closures at some moment of the process of creation of $\partial X_{\mathcal{F}^{reg}}$ by blow ups of orbits of dimension* 0*, which are particular infinitely near points of $O_{\sigma_0} \in X_{\sigma_0}$.*
- *If one associates with each orbit of dimension* 0 *the corresponding petal of $\Lambda(\mathcal{F})$, then the parent map on the set of petals induces on the previous set of* 0*-dimensional orbits the restriction of the parent relation defined on the set of infinitely near points of O_{σ_0} (see Definition 1.4.31).*

Let us set a notation for the constellation created during a toric blow up process (see Definition 1.4.31):

Definition 1.5.12 Let $\mathcal{F}$ be a Newton fan. Denote by $\boxed{C_{\mathcal{F}}}$ the finite constellation above O_{σ_0} consisting of the 0-dimensional orbits O_σ, where σ varies among the regular 2-dimensional cones of the blow up process leading to the smooth toric surface $X_{\mathcal{F}^{reg}}$. It is **the constellation of the fan** $\mathcal{F}$.

Let σ be one of the cones mentioned in Definition 1.5.12. It is of the form $\mathbb{R}_+\langle f_1, f_2\rangle$, where (f_1, f_2) is a positive basis of the lattice N. Proposition 1.5.11 shows that one may represent the 0-dimensional orbit O_σ either by the edge $[f_1, f_2]$ of the lotus $\Lambda(\mathcal{F})$ or by the petal $\delta(f_1, f_2)$. How to understand the Enriques diagram of the constellation $C_{\mathcal{F}}$ using the lotus $\Lambda(\mathcal{F})$? It turns out that this may be done easily using the representing edges $[f_1, f_2]$. In order to explain it, let us introduce first the following definition (see Figs. 1.29 and 1.30):

Definition 1.5.13 Let $\delta(f_1, f_2)$ be a petal of the universal lotus $\Lambda(e_1, e_2)$. Assume that it is different from $\delta(e_1, e_2)$, which means that there exists a unique permutation (i, j) of $(1, 2)$ such that $f_j - f_i \in \sigma_0 \cap N$ (see Lemma 1.5.2). Then its **Enriques edge** is its lateral edge $[f_j, f_1 + f_2]$, that is, its unique lateral edge which extends an edge of its parent petal. The **Enriques tree** of a lotus Λ is:

- the union of the Enriques edges of all its petals different from $\delta(e_1, e_2)$, rooted at its vertex $e_1 + e_2$, whenever Λ is of dimension 2;
- the vertex $e_1 + e_2$ of $\delta(e_1, e_2)$, if $\Lambda = [e_1, e_2]$.

The **extended Enriques tree** of a lotus Λ is:

- the union of the Enriques subtree and of the lateral edge $[e_1, e_1 + e_2]$ of the base petal $\delta(e_1, e_2)$ of Λ, whenever Λ is of dimension 2;
- the lateral edge $[e_1, e_1 + e_2]$ of $\delta(e_1, e_2)$, if $\Lambda = [e_1, e_2]$.

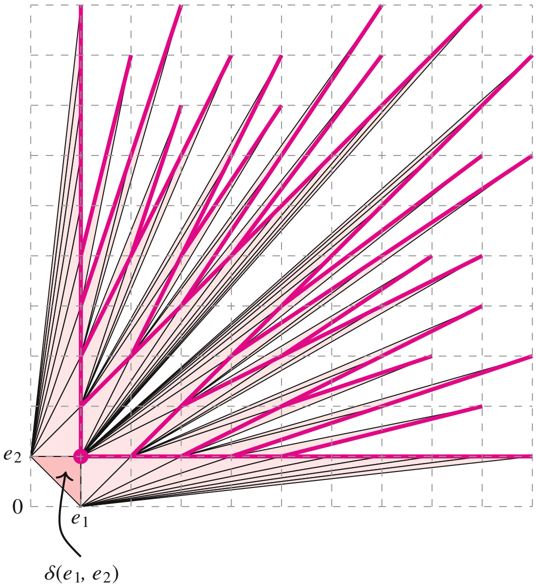


Fig. 1.29 Partial view of the Enriques subtree of the universal lotus $\Lambda(e_1, e_2)$

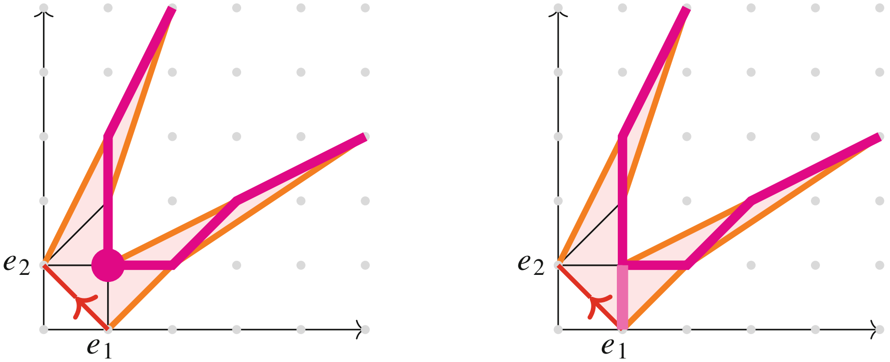


Fig. 1.30 The Enriques tree and the extended Enriques tree of the lotus Λ (3/5, 2/1, 5/2)

One has the following interpretation of the Enriques diagram of the constellation of the fan $\mathcal{F}$ using the lotus $\Lambda(\mathcal{F})$. It allows to understand for which reason we defined the Enriques tree of a lotus reduced to the base $[e_1, e_2]$ in the previous way:

Proposition 1.5.14 *Let $\mathcal{F}$ be a Newton fan. Then the Enriques diagram $\Gamma(C_{\mathcal{F}})$ of the constellation $C_{\mathcal{F}}$ of $\mathcal{F}$ (see Definition 1.5.12) is isomorphic to the Enriques subtree of the lotus $\Lambda(\mathcal{F})$. This isomorphism sends each orbit O_σ belonging to $C_{\mathcal{F}}$ onto the point $f_1 + f_2$, if $\sigma = \mathbb{R}_+\langle f_1, f_2\rangle$.*

Proof The basic idea is that we have a bijection between the set of infinitely near points of O_{σ_0} and the set of prime exceptional divisors created by blowing them up. Therefore, the parent binary relation may be thought as a binary relation on the set of those prime exceptional divisors. In this proposition, we restrict to the divisors which are the orbit closures $\overline{O}_\rho$, where ρ varies among the rays of the regularization $\mathcal{F}^{reg}$ of $\mathcal{F}$ which are distinct from the edges of σ_0. Each such a ray is generated by a lateral vertex of $\Lambda(\mathcal{F})$, therefore the parent binary relation among those orbit closures may be also seen as a binary relation among those lateral vertices. One may prove by induction on this number of rays, that is, on the number of petals of the associated lotus $\Lambda(\mathcal{F})$, that the pairs of related vertices are precisely those which are connected by an edge in the Enriques tree of $\Lambda(\mathcal{F})$.

The case $\mathcal{F} = \sigma_0$ corresponds to a constellation formed by O_{σ_0} alone. In this case one looks at the prime divisor created by blowing it up, that is, at $\overline{O}_{\mathbb{R}_+\langle e_1+e_2\rangle}$. This explains why we defined $\Gamma(C_{\sigma_0})$ as the vertex $e_1 + e_2$ of the petal $\delta(e_1, e_2)$. □

Remark 1.5.15 The reason why we introduced also the notion of *extended Enriques tree* in Definition 1.5.13, in addition to that of *Enriques tree*, will become clear after understanding point (8) of Theorem 1.5.29. Briefly speaking, the constellations associated to the toroidal pseudo-resolution processes have associated lotuses which are glued from lotuses of Newton fans. An analog of Proposition 1.5.14 is also true for them. The corresponding Enriques tree contains the Enriques trees of the Newton fans created by the toroidal process, but also other edges. Those supplementary edges are precisely the edges which have to be added to the Enriques tree of a Newton fan in order to get the corresponding extended Enriques tree (see Definition 1.5.26 below).

The lotus $\Lambda(\mathcal{F})$ contains also the *graph of the proximity binary relation* on the constellation $C_{\mathcal{F}}$, whose set of vertices is the given constellation, two points being joined by an edge if and only if one of them is proximate to the other one (see Definition 1.4.31):

Proposition 1.5.16 *Let $\mathcal{F}$ be a fan refining the regular cone σ_0. Then the graph of the proximity binary relation on the finite constellation $C_{\mathcal{F}}$ is isomorphic to the union of the edges of the lotus $\Lambda(\mathcal{F})$ which do not contain the vertices e_1 and e_2.*

The proof of this proposition is based on the same principles as the proof of Proposition 1.5.14 and is left to the reader.

1.5.2 Lotuses and Continued Fractions

In this subsection we explain a way to build, up to isomorphism, the lotus of a finite set of positive rational numbers in the sense of Definition 1.5.4, starting from the continued fraction expansions of its elements. Namely, given a positive rational number λ, we show how to construct an *abstract lotus* $\Delta(\lambda)$ starting from the continued fraction expansion of λ (see Definition 1.5.18) and we explain that $\Delta(\lambda)$ is isomorphic to the lotus $\Lambda(\lambda)$. Then we show how to glue two abstract lotuses $\Delta(\lambda)$ and $\Delta(\mu)$ in order to get a simplicial complex isomorphic to the lotus $\Lambda(\lambda, \mu)$ (see Proposition 1.5.23). This extends readily to arbitrary finite sets of positive rationals.

Recall first the following classical notion:

Definition 1.5.17 Let $k \in \mathbb{N}^*$ and let $a_1, \dots, a_k$ be natural numbers such that $a_1 \geq 0$ and $a_j > 0$ if $j \in \{2, \dots, k\}$. The **continued fraction** with **terms** $a_1, \dots, a_k$ is the non-negative rational number:

$$\boxed{[a_1, a_2, \dots, a_k]} := a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_k}}}.$$

Any $\lambda \in \mathbb{Q}_+^*$ may be written uniquely as a continued fraction $[a_1, a_2, \dots, a_k]$ if one imposes the constraint that $a_k > 1$ whenever $\lambda \neq 1$. One speaks then of the **continued fraction expansion** of λ. Note that its first term a_1 vanishes if and only if $\lambda \in (0, 1)$.

Definition 1.5.18 Let $\lambda \in \mathbb{Q}_+^*$. Consider its continued fraction expansion $\lambda = [a_1, a_2, \dots, a_k]$. Its **abstract lotus** $\boxed{\Delta(\lambda)}$ is the simplicial complex constructed as follows:

- Start from an affine triangle $[A_1, A_2, V]$, with vertices A_1, A_2, V.
- Draw a polygonal line $P_0 P_1 P_2 \dots P_{k-1}$ whose vertices belong alternatively to the sides $[A_1, V]$, $[A_2, V]$, and such that $P_0 := A_2$ and

$$\begin{cases} P_1 \in [A_1, V), \text{ with } P_1 = A_1 \text{ if and only if } a_1 = 0, \\ P_i \in (P_{i-2}, V) \text{ for any } i \in \{2, \dots, k-1\}. \end{cases}$$

  By convention, we set also $P_{-1} := A_1$, $P_k := V$. The resulting subdivision of the triangle $[A_1, A_2, V]$ into k triangles is the **zigzag decomposition associated with** λ.
- Decompose then each segment $[P_{i-1}, P_{i+1}]$ (for $i \in \{0, \dots, k-1\}$) into a_{i+1} segments, and join the interior points of $[P_{i-1}, P_{i+1}]$ created in this way to P_i. One obtains then a new triangulation of the initial triangle $[A_1, A_2, V]$, which is by definition the abstract lotus $\Delta(\lambda)$.

The **base** of the abstract lotus $\Delta(\lambda)$ is the segment $[A_1, A_2]$, oriented from A_1 to A_2. One orients also the other edges of $\Delta(\lambda)$ in the following way:

- $[P_{i-1}, P_i]$ is oriented from P_i to P_{i-1}, for each $i \in \{1, \dots, k-1\}$.
- An edge joining P_i to a point of the open segment (P_{i-1}, P_{i+1}) is oriented towards P_i.
- An edge contained in a segment $[V, A_j]$ is oriented towards A_j, for each $j \in \{1, 2\}$.

The abstract lotus $\Delta(\lambda)$ of $\lambda \in \mathbb{Q}_+^*$ is a simplicial complex of pure dimension 2, isomorphic to a convex polygon triangulated by diagonals intersecting only at vertices and with a distinguished oriented base. It is well-defined, up to combinatorial isomorphism of polygons triangulated by diagonals intersecting only at vertices, respecting the bases and their orientations. The orientations of its other edges are in fact determined by the orientation of the base. Those orientations will not be important in the sequel, excepted in Proposition 1.5.21 below. For this reason we do not draw them in our examples of abstract lotuses.

Example 1.5.19 Figures 1.31 and 1.32 represent the previous constructions applied to the numbers $\lambda = [4, 2, 5]$ and $\mu = [3, 2, 1, 4]$. On the left are shown the initial zigzag decompositions and on the right the final abstract lotuses $\Delta(\lambda)$ and $\Delta(\mu)$.

The abstract lotus of a positive rational number is isomorphic with its lotus:

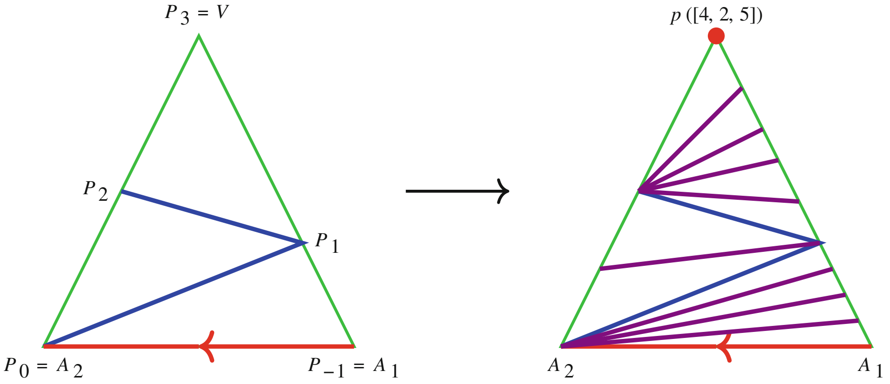


Fig. 1.31 The construction of the abstract lotus $\Delta([4, 2, 5])$

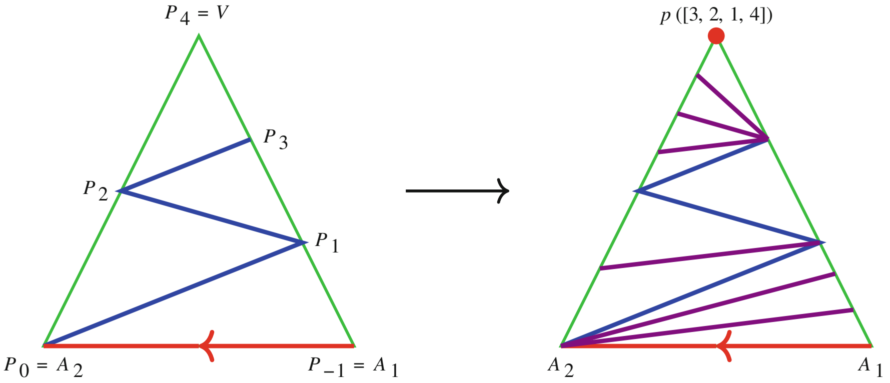


Fig. 1.32 The construction of the abstract lotus $\Delta([3, 2, 1, 4])$

Proposition 1.5.20 *There is a unique isomorphism between the lotus* $\Lambda(\lambda)$ *and the abstract lotus* $\Delta(\lambda)$*, seen as simplicial complexes with a marked point and an oriented base.*

Proof The isomorphism sends A_i to e_i for $i = 1, 2$. The proof may be done by induction on k, the number of terms in the continued fraction expansion of λ. We leave the details to the reader. □

The previous isomorphism does not always send the orientations of the edges of $\Lambda(\lambda)$ as chosen after Definition 1.5.1 onto the orientations of the edges of $\Delta(\lambda)$ as fixed in Definition 1.5.18. The possibility of defining various canonical orientations on the edges of a lotus of the form $\Lambda(\lambda)$ may be useful in applications.

The rational number $\lambda > 0$ may be recovered in the following way from the structure of the corresponding abstract lotus:

Proposition 1.5.21 *Assume that* $\lambda = p_2/p_1$ *with* $p_1, p_2 \in \mathbb{N}^*$ *coprime. Then, for each* $j \in \{1, 2\}$*, the positive integer* p_j *is equal to the number of oriented paths not containing the base* $[A_1, A_2]$ *and going from* V *to* A_j *inside the* 1*-skeleton of* $\Delta(\lambda)$*, oriented as in Definition 1.5.18.*

This proposition may be easily proved by induction on the number of petals of $\Delta(\lambda)$. It shows a way in which the numbers leading to the construction of a Newton lotus may be interpreted as combinatorial invariants of the lotus, seen purely as a marked simplicial complex with oriented base.

Example 1.5.22 In Fig. 1.33 is represented the case $(p_1, p_2) = (2, 3)$ of Proposition 1.5.21. We have drawn twice the lotus $\Delta(3/2) = \Delta([1, 2])$. On the right are drawn the 2 oriented paths starting from V and arriving at A_1. On the left are drawn the 3 oriented paths starting from V and arriving at A_2. We see that the constraint not to contain the base is necessary, otherwise one would obtain 2 more paths from V to A_2 by adding the base to the paths from V to A_1.

Suppose now that one has two numbers $\lambda, \mu \in \mathbb{Q}_+^*$. If $\lambda = [a_1, \ldots, a_k]$ and $\mu = [b_1, \ldots, b_l]$, let $j \in \{0, \ldots, \min\{k, l\}\}$ be maximal such that $a_i = b_i$ for all $i \in \{1, \ldots, j\}$. We may assume, up to permutation of λ and μ, that $k = j$ or $a_{j+1} < b_{j+1}$. Define then:

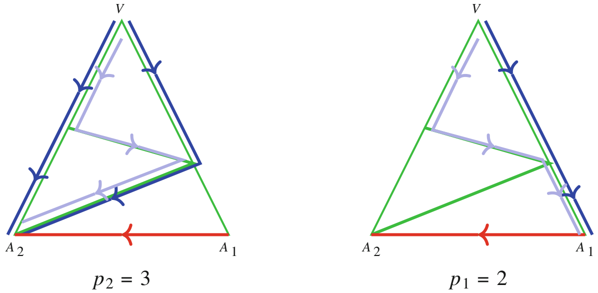


Fig. 1.33 An illustration of Proposition 1.5.21 for $p_2/p_1 = 3/2$

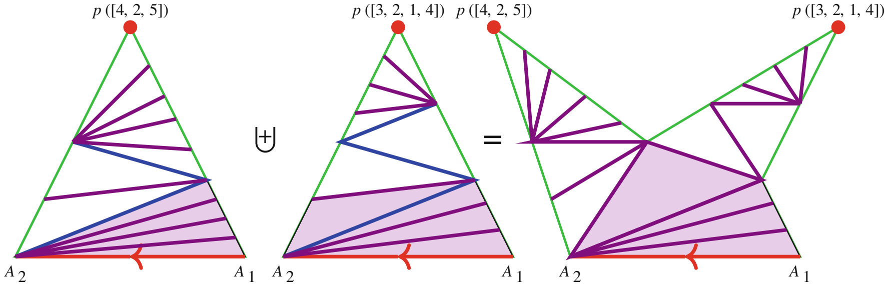


Fig. 1.34 The abstract lotus $\Delta([4, 2, 5], [3, 2, 1, 4])$

$$\boxed{\lambda \wedge \mu} = \boxed{\mu \wedge \lambda} := \begin{cases} [a_1, \dots, a_j], \text{ if } k = j, \\ \left[a_1, \dots, a_j, a_{j+1}\right], \text{ if } k = j + 1, \\ \left[a_1, \dots, a_j, a_{j+1} + 1\right], \text{ if } k > j + 1. \end{cases} \tag{1.48}$$

Next proposition explains that the symmetric binary operation $\wedge$ on $\mathbb{Q}_+^*$ allows to describe the intersection of two lotuses of the form $\Lambda(\lambda)$:

Proposition 1.5.23 *For any $\lambda, \mu \in \mathbb{Q}_+^*$, one has:*

$$\Lambda(\lambda) \cap \Lambda(\mu) = \Lambda(\lambda \wedge \mu).$$

Therefore, the lotus $\Lambda(\lambda, \mu)$ is isomorphic as a simplicial complex with an oriented base to the triangulated polygon obtained by gluing $\Delta(\lambda)$ and $\Delta(\mu)$ along $\Delta(\lambda \wedge \mu)$.

Proof Assume that $\lambda = [a_1, \dots, a_k]$. Proposition 1.5.20 shows in particular that the lotus $\Lambda(\lambda)$ has $n := a_1 + \dots + a_k$ petals. Denote by $(\lambda_i)_{1 \leq i \leq n}$ the sequence of positive rationals such that the successive non-basic vertices of the petals of $\Lambda([a_1, \dots, a_k])$ are the primitive vectors $p(\lambda_1), \dots, p(\lambda_n)$. The sequence of continued fraction expansions of $(\lambda_i)_{1 \leq i \leq n}$ is:

$$[1], [2], \dots, [a_1], [a_1, 1], [a_1, 2], \dots, [a_1, a_2], [a_1, a_2, 1], \dots, [a_1, \dots, a_k]. \tag{1.49}$$

One may prove this fact at the same time as Proposition 1.5.20, by making now an induction on the number n of petals of $\Lambda([a_1, \dots, a_k])$, instead of the number k of terms of the continued fraction.

The proposition results then by combining the previous fact with formula (1.48). □

Example 1.5.24 Let us consider the two rational numbers $\lambda = [4, 2, 5]$ and $\mu = [3, 2, 1, 4]$ of Example 1.5.19. Then $j = 0, k = 3, l = 4$, therefore $j + 1 < \min\{k, l\}$ and $\lambda \wedge \mu = [3 + 1] = 4$. The lotus $\Lambda(\lambda, \mu)$ is therefore isomorphic to the triangulated polygon with an oriented base of the right side of Fig. 1.34.

Iterating the gluing operation, one may construct an **abstract lotus** $\boxed{\Delta(\lambda_1, \dots, \lambda_k)}$ combinatorially equivalent to any given Newton lotus $\Lambda(\lambda_1, \dots, \lambda_k)$, seen as a triangulated polygon with marked points and oriented base. One gets an abelian monoid of (abstract) lotuses, the monoid operation $\boxed{\uplus}$ generalizing the gluing operation of Fig. 1.34. Namely, if $\mathcal{E}_1$ and $\mathcal{E}_2$ are finite subsets of $\mathbb{Q}_+ \cup \{\infty\}$, then:

$$\boxed{\Delta(\mathcal{E}_1) \uplus \Delta(\mathcal{E}_2)} := \Delta(\mathcal{E}_1 \cup \mathcal{E}_2). \tag{1.50}$$

The neutral element of this monoid is the segment $[A_1, A_2] = \Delta(\emptyset) = \Delta(0) = \Delta(\infty) = \Delta(\{0, \infty\})$.

1.5.3 The Lotus of a Toroidal Pseudo-Resolution

In this subsection we reach a second level of explanation of the subtitle of this article, the first level having been reached in Sect. 1.5.1 above. Namely, we define a new kind of lotus by gluing the lotuses associated to the Newton fans produced by Algorithm 1.4.22 (see Definition 1.5.26). We illustrate this definition by our recurrent example (see Example 1.5.28) and by the case of an arbitrary branch (see Example 1.5.30). Finally, we show how this lotus allows to visualize many objects associated to the regularized algorithm and with the decomposition into blow ups of points of the embedded resolution produced by it (see Theorem 1.5.29).

Consider again a reduced curve singularity C on the smooth germ of surface (S, o). Fix a smooth branch L on (S, o), and run Algorithm 1.4.22. Denote as before by $\pi : (\Sigma, \partial\Sigma) \to (S, L + L')$ a resulting toroidal pseudo-resolution of C. We associated to it a fan tree $(\theta_\pi(C), \mathbf{S}_\pi)$, as explained in Definition 1.4.33. One may associate an analogous fan tree $(\theta_{\pi^{reg}}(C), \mathbf{S}_{\pi^{reg}})$ to the toroidal resolution $\pi^{reg} : (\Sigma^{reg}, \partial\Sigma^{reg}) \to (S, L + L')$ defined in Sect. 1.4.3 (see Proposition 1.4.29). One sees that the trunks used in the two constructions are the same, as well as the gluing rules. What changes is that $\theta_{\pi^{reg}}(C)$ has more vertices than $\theta_\pi(C)$, those labeled by the irreducible components of the exceptional divisor of the modification $\eta : \Sigma^{reg} \to \Sigma$ which resolves the singularities of the surface Σ. Therefore:

Proposition 1.5.25 *Seen as rooted trees endowed with* $[0, \infty]$*-valued functions, the fan trees* $(\theta_\pi(C), \mathbf{S}_\pi)$ *and* $(\theta_{\pi^{reg}}(C), \mathbf{S}_{\pi^{reg}})$ *coincide. The second one contains more vertices than the first one, labeled by the irreducible components of the exceptional divisor of the minimal resolution* $\eta : \Sigma^{reg} \to \Sigma$. *The fan tree* $\theta_{\pi^{reg}}(C)$ *of the toroidal resolution* π^{reg} *is isomorphic to the dual graph of the boundary* $\partial\Sigma^{reg}$ *by an isomorphism which respects the labels of the irreducible components.*

The disadvantage of the fan tree $(\theta_{\pi^{reg}}(C), \mathbf{S}_{\pi^{reg}})$ is that one cannot see on it at a glance the partial order of the blow ups leading to the resolution $\pi^{reg} : \Sigma \to S$ of C. We explained in Sect. 1.5.1 that this order may be visualized by using the notion of lotus, for each Newton modification of the regularized algorithm obtained by replacing STEP 3 with STEP 3^{reg}. In order to visualize the blow up structure of the resolution process leading to the modification $\pi^{reg} : (\Sigma^{reg}, \partial\Sigma^{reg}) \to (S, L + L')$, we glue those lotuses using the same rules as those allowing to construct the fan tree from its trunks (see Definition 1.4.33):

Definition 1.5.26 Let C be a reduced curve singularity and (L, L') be a cross on the smooth germ (S, o). The **lotus** $\boxed{\Lambda_\pi(C)}$ **of the toroidal pseudo-resolution** π : $(\Sigma, \partial\Sigma) \to (S, L + L')$ **of** C is a simplicial complex of dimension 2 endowed with a marked oriented edge called its **base**. It is obtained by gluing the disjoint union of the lotuses $(\Lambda(\mathcal{F}_{A_i,B_i}(C)))_{i\in I}$ in the following way:

1. Label each vertex of those lotuses with the corresponding irreducible component E_k, L_j or C_l of the boundary $\partial\Sigma^{reg}$ of the smooth toroidal surface $(\Sigma^{reg}, \partial\Sigma^{reg})$.
2. Identify all the vertices of $\bigsqcup_{i\in I} \Lambda(\mathcal{F}_{A_i,B_i}(C))$ which have the same label. The result of this identification is $\Lambda_\pi(C)$ and the images inside it of the labeled points of $\bigsqcup_{i\in I} \Lambda(\mathcal{F}_{A_i,B_i}(C))$ are its vertices. We keep for each one of them the same label as in the initial lotuses.

Introduce the following terminology for the anatomy of $\Lambda_\pi(C)$:

- The **petals** of $\Lambda_\pi(C)$ are the images by the gluing morphism of the petals of the initial lotuses $(\Lambda(\mathcal{F}_{A_i,B_i}(C)))_{i\in I}$.
- Its **base** is the edge labeled by the initial cross (L, L_1) and its **basic petal** is the petal having it as base.
- Its **basic vertices** are the images inside it of the basic vertices of the 2-dimensional lotuses $(\Lambda(\mathcal{F}_{A_j,B_j}(C)))_{j\in J}$ which were not identified with other vertices.
- Its **lateral boundary** $\boxed{\partial_+\Lambda_\pi(C)}$ is the image by the gluing morphism of the union of the lateral boundaries $(\partial_+\Lambda(\mathcal{F}_{A_i,B_i}(C)))_{i\in I}$ in the sense of Definition 1.5.5.
- Its **lateral vertices** are the vertices of $\Lambda_\pi(C)$ which are not basic.
- Its **membranes** are the images inside it of the lotuses $\Lambda(\mathcal{F}_{A_i,B_i}(C))$ used to construct it.
- Its **Enriques tree** is the union of the Enriques tree of $\Lambda(\mathcal{F}_{A_1,B_1}(C))$ (remember that $(A_1, B_1) = (L, L')$) and of the extended Enriques trees of the other Newton fans $\Lambda(\mathcal{F}_{A_i,B_i}(C))$ (see Definition 1.5.13).

We introduce the notion of *Enriques tree* of a lotus in order to be able to state point (8) of Theorem 1.5.29 below. See also Remark 1.5.15.

Remark 1.5.27 The lateral boundary $\partial_+\Lambda_\pi(C)$ is a *covering subtree* of the 1-skeleton of the lotus $\Lambda_\pi(C)$, that is, a subtree containing all of its vertices. The membranes of $\Lambda_\pi(C)$ may be obtained by removing all the vertices of $\Lambda_\pi(C)$ and by taking the closures inside $\Lambda_\pi(C)$ of the connected components of the resulting topological space. The lotus $\Lambda_\pi(C)$ is a *flag complex*, that is, it may be reconstructed

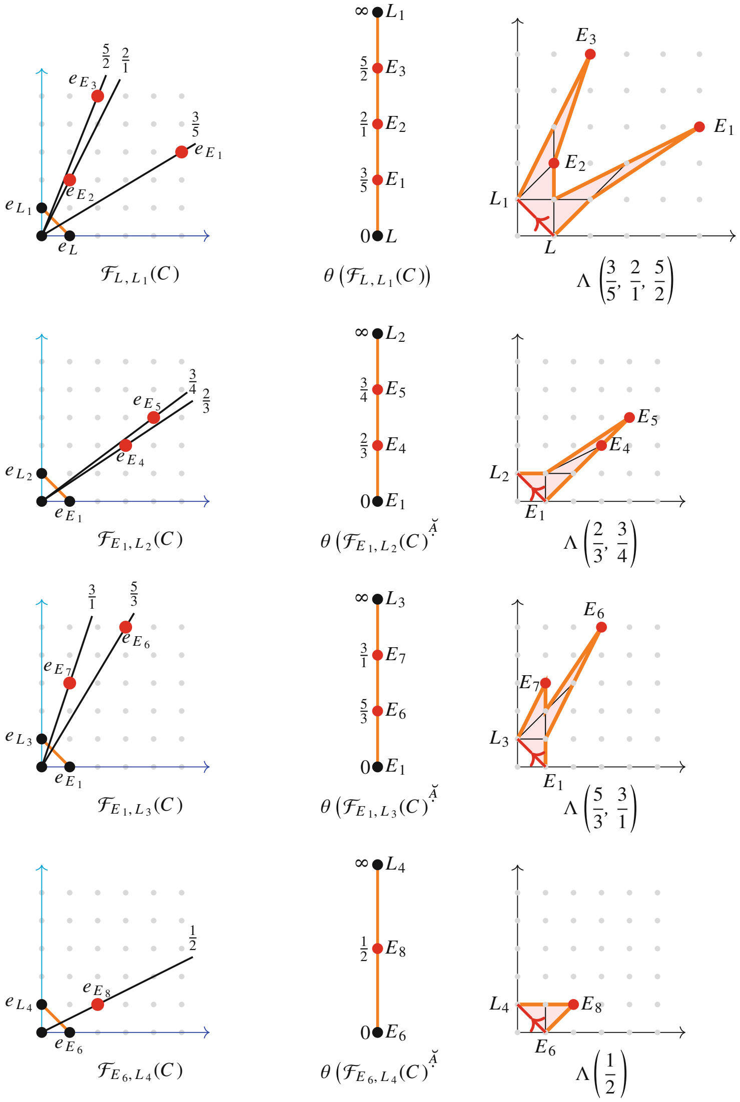


Fig. 1.35 The 2-dimensional Newton lotuses of Example 1.5.28

from its 1-skeleton by filling each complete subgraph with k vertices by a $(k-1)$-dimensional simplex. It turns out that there are such complete subgraphs only for $k \in \{1, 2\}$, for which values of k the filling process adds nothing new, and for $k = 3$, for which one gets all the petals of the lotus.

Example 1.5.28 Consider the toroidal pseudo-resolution process of Example 1.4.28. The construction of the corresponding fan tree was explained in

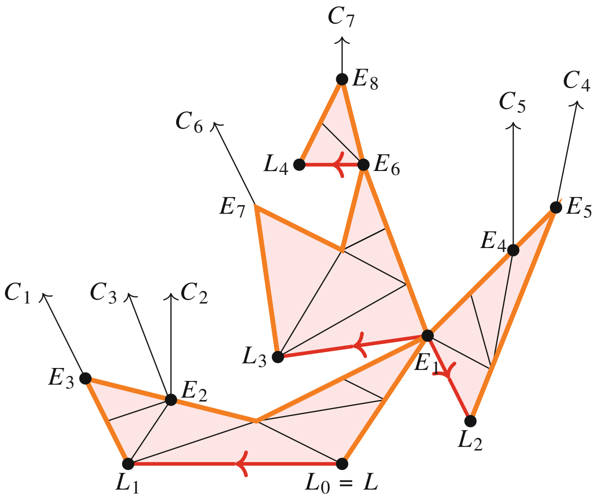
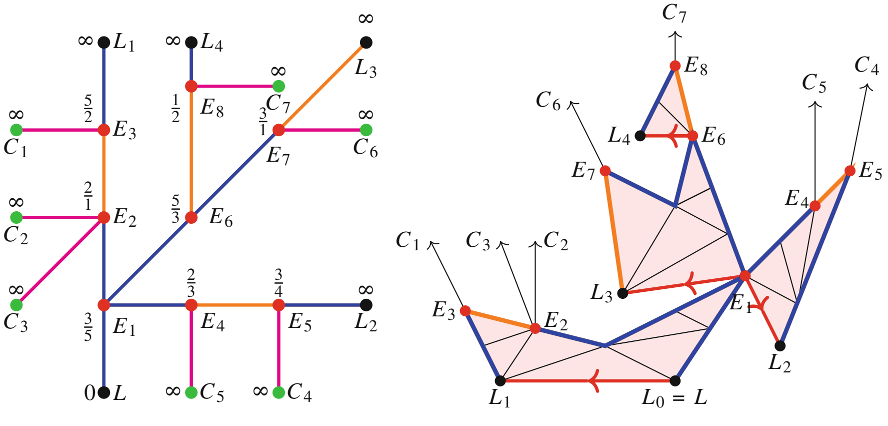


Fig. 1.36 The lotus of the toroidal pseudo-resolution of Example 1.5.28



$\theta_\pi(C)$ $\Lambda_\pi(C)$

Fig. 1.37 Comparison of the fan tree and the lotus of Example 1.5.28

Example 1.4.36 and illustrated in Fig. 1.20. The left column of Fig. 1.35 represents the Newton fans produced each time one runs STEP 2 of Algorithm 1.4.22. The middle column shows the associated trunks and the right column the corresponding lotuses.

The associated lotus $\Lambda_\pi(C)$ is represented in Fig. 1.36. It has 4 membranes of dimension 2 and 7 membranes of dimension 1. The oriented base of each lotus $\Lambda(\mathcal{F}_{A_i,B_i}(C))$ used to construct it is indicated in red. The base of $\Lambda_\pi(C)$ is the oriented edge whose vertices are labeled by L and L_1. The basic vertices of $\Lambda_\pi(C)$

are those labeled by L, L_1, L_2, L_3, L_4. The part of the lateral boundary $\partial_+ \Lambda_\pi(C)$ contained in the 2-dimensional lotuses $(\Lambda(\mathcal{F}_{A_j,B_j}(C)))_{j\in J}$ is represented in orange. In order to get the whole lateral boundary, one has to add the 1-dimensional lotuses of the fans associated to the crosses at which one stops at STEP 1, that is, the segments $[E_3, C_1]$, $[E_2, C_2]$, $[E_2, C_3]$, $[E_5, C_4]$, $[E_4, C_5]$, $[E_7, C_6]$ and $[E_8, C_7]$.

In Fig. 1.37 are represented side by side the fan tree $\theta_\pi(C)$ and the lotus $\Lambda_\pi(C)$. Note that the fan tree is homeomorphic (forgetting the values of the slope function at its vertices) with the lateral boundary $\partial_+ \Lambda_\pi(C)$, by a homeomorphism which preserves the labels. This is a general fact, as formulated in point (4) of Theorem 1.5.29 below. This homeomorphism is not an isomorphism of trees because some of the edges of the fan tree—the blue ones—get subdivided in the lateral boundary of the lotus. Those are precisely the edges which correspond to the singular points of the surface Σ. One may see on the lateral boundary the structure of the exceptional divisor of the minimal resolution of each such point.

For instance, the intersection point of the curves E_1 and E_6 on Σ gets resolved by replacing that point with an exceptional divisor with two components. Their self-intersection numbers in the smooth surface Σ^{reg} are -4 and -3, as results from point (5) of Theorem 1.5.29.

Here comes the announced visualization of the structure of the decomposition of the modification $\pi^{reg} : \Sigma^{reg} \to S$ into blow ups of points in terms of the anatomy of the lotus $\Lambda_\pi(C)$ (see Definition 1.5.26):

Theorem 1.5.29 *Let C be a reduced curve singularity on the smooth germ of surface (S, o). Consider a toroidal pseudo-resolution $\pi : (\Sigma, \partial\Sigma) \to (S, L+L')$ of C produced by Algorithm 1.4.22. Its lotus $\Lambda_\pi(C)$ represents the following aspects of the associated embedded resolution $\pi^{reg} : (\Sigma^{reg}, \partial\Sigma^{reg}) \to (S, L + L')$:*

1. *Its basic edges represent the crosses with respect to which STEP 2 of Algorithm 1.4.22 was applied.*
2. *Its basic vertices represent the branches $(L_j)_{j\in J}$ of the crosses used during the process, which were introduced each time one executed STEP 2.*
3. *Its lateral vertices represent the irreducible components E_k of the exceptional divisor $(\pi^{reg})^{-1}(o)$ of the smooth modification $\pi^{reg} : \Sigma^{reg} \to S$.*
4. *Its lateral boundary $\partial_+ \Lambda_\pi(C)$ is the dual graph of the boundary divisor $\partial\Sigma^{reg}$ and is homeomorphic with the fan tree $\theta_{\pi^{reg}}(C)$, by a homeomorphism which respects the labels.*
5. *The opposite of the number of petals of $\Lambda_\pi(C)$ containing a given lateral vertex is the self-intersection number of the irreducible component of $(\pi^{reg})^{-1}(o)$ represented by that lateral vertex.*
6. *The edges of $\Lambda_\pi(C)$ represent the affine charts used in the decomposition of π into a composition of blow ups of points, and the pairs of irreducible components of $(\pi^{reg})^{-1}(\sum_{j\in J} L_j)$ which are strict transforms of crosses used at some stage of the composition of blow ups.*

7. *The graph of the proximity binary relation on the constellation which is blown up is the full subgraph of the* 1*-skeleton of the lotus* $\Lambda_\pi(C)$ *on its set of non-basic vertices.*
8. *The Enriques tree of* $\Lambda_\pi(C)$ *is the Enriques diagram of the constellation of infinitely near points at which are based the crosses introduced during the blow up process leading to the boundary* $\partial \Sigma^{reg}$.

Proof Points (1) and (2) result from Proposition 1.4.18. Points (3) and (4) result from Propositions 1.4.35, 1.5.10 and 1.5.25. Point (5) results from Corollary 1.2.28 and Proposition 1.2.37. A prototype of this result had been stated in [102, Thm. 6.2]. Points (6) and (7) result from Proposition 1.5.16. Point (8) results from Proposition 1.5.14. □

Example 1.5.30 Assume that C is a branch. Its fan tree $\theta_\pi(C)$ is a segment $[L, C]$. Denote its interior vertices by $P_1 \prec_L \cdots \prec_L P_k = P$, with $k \geq 1$. Here $\preceq_L$ denotes the total order on $\theta_\pi(C)$ induced by the root L. Consider the continued fraction expansions of their slopes $\mathbf{S}_\pi(P_j) = [p_j, q_j, \dots]$, for all $j \in \{1, \dots, k\}$. Then the lotus $\Lambda_\pi(C)$ is represented in Fig. 1.38. We explain in Examples 1.6.32 and 1.6.33 below how to give examples of branches which admit a pseudo-resolution process with such a lotus.

Example 1.5.31 Let us consider again our recurrent example of toroidal pseudo-resolution. Its associated lotus was represented in Fig. 1.36. In Fig. 1.39 are represented the Enriques trees and extended Enriques trees of its membranes of

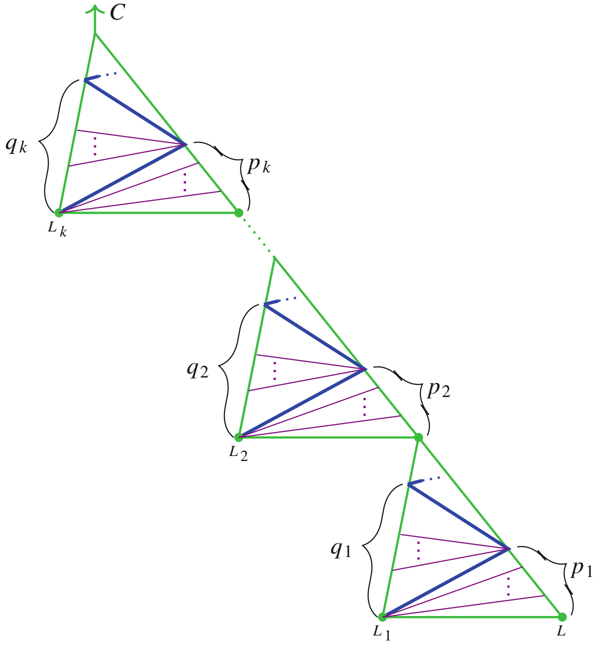


Fig. 1.38 The lotus of toroidal pseudo-resolution for one branch from Example 1.5.30

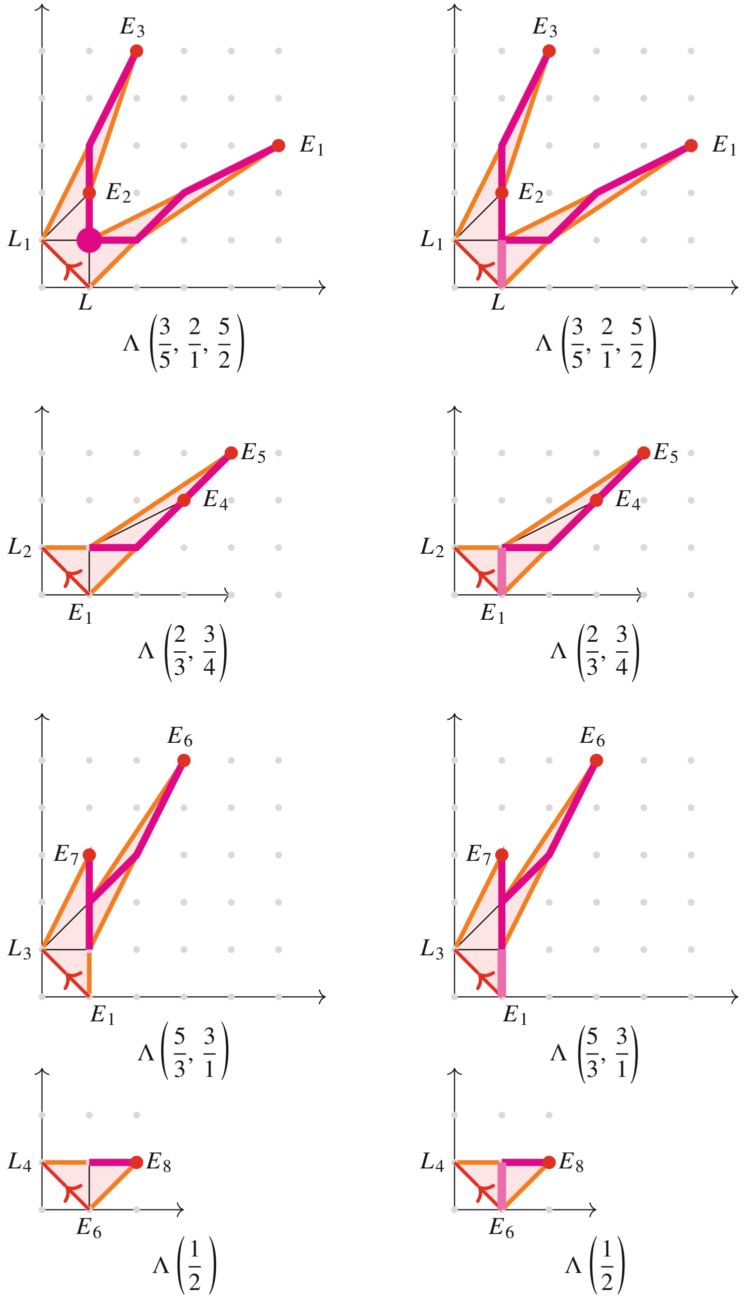


Fig. 1.39 The Enriques trees and the extended Enriques trees in Example 1.5.31

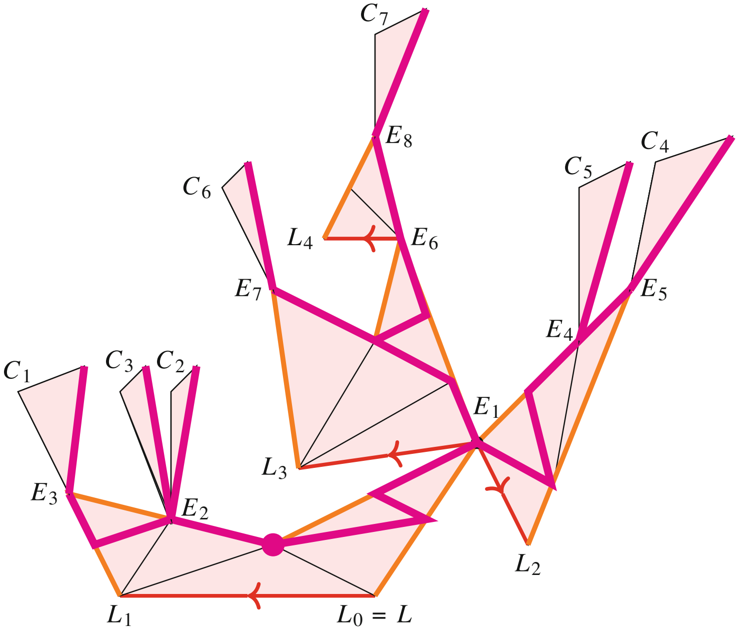


Fig. 1.40 The Enriques tree of the toroidal pseudo-resolution of Example 1.5.31

dimension 2. Finally, in Fig. 1.40 is represented its full Enriques tree. In this figure we have also represented the petals associated to the pairs (E_i, C_j), in order to draw the end edges of the Enriques tree.

1.5.4 The Dependence of the Lotus on the Choice of Completion

In this subsection we show using two examples that the lotus $\Lambda_\pi(C)$ of a toroidal pseudo-resolution process π of a plane curve singularity $C \hookrightarrow S$ depends on the choice of auxiliary curves added each time one executes STEP 2 of Algorithm 1.4.22, that is, on the choice of completion $\hat{C}_\pi$ of C (see Definition 1.4.15).

In the following two Examples 1.5.32 and 1.5.33, we build the lotuses $\Lambda_\pi(C)$ associated with two distinct embedded resolutions $\pi : (\Sigma, \partial\Sigma) \to (S, \partial S)$ of the curve singularity $C = Z(f)$, defined by the power series $f := y^2 - 2xy + x^2 - x^3 \in \mathbb{C}[[x, y]]$, relative to local coordinates (x, y) on the germ (S, o). These examples illustrate the fact that the associated lotus $\Lambda_\pi(C)$ (see Definition 1.5.26), which is based on the toroidal structure of Σ, depends on the choices of auxiliary curves done at STEP 2 of the Algorithm 1.4.22, that is, on the choice of *completion* $\hat{C}_\pi$ of C (see Definition 1.4.15). In both examples we run Algorithm 1.4.22 with $L = Z(x)$,

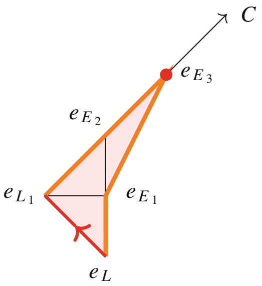


Fig. 1.41 The lotus $\Lambda_\pi(C)$ of Example 1.5.32

replacing STEP 3 by STEP 3^{reg} as we explained in Sect. 1.4.3, and taking different choices of auxiliary curves. The output, which determines the toroidal boundary on Σ, provides two different lotuses. On both of them we recognize the same weighted dual graph of the final total transform of C, thanks to point (4) of Theorem 1.5.29.

Example 1.5.32 We start the algorithm by choosing $L_1 := Z(y - x)$. The cross (L, L_1) at o is defined by the local coordinate system $(x, y_1 := y - x)$. Relative to these coordinates, C has local equation $y_1^2 - x^3 = 0$. The Newton polygon $\mathcal{N}_{L,L_1}(C)$ has only one edge and its orthogonal ray has slope 3/2, hence $\mathcal{F}_{L,L'}(C) \simeq \mathcal{F}(3/2)$. The first trunk is just the segment $[e_L, e_{L_1}]$ with its point of slope 3/2 marked.

The Newton modification $\pi := \psi_{L,L_1}^{C,reg} : (\Sigma, \partial\Sigma) \to (S, \partial S)$ has three exceptional divisors E_1, E_2 and E_3 which correspond to the rays of the regularization $\mathcal{F}^{reg}(3/2) = \mathcal{F}(1, 2, 3/2)$ of the fan $\mathcal{F}(3/2)$ of slopes 1 and 2 and 3/2 respectively. In this case, the strict transform C_{L,L_1} of C is smooth and intersects transversally the component E_3 of the exceptional divisor, that is, the Newton modification π is an embedded resolution of C. Note that when running the Algorithm 1.4.22, we include the cross (E_3, C_{L,L_1}) in the toroidal structure of the boundary of Σ.

The lotus $\Lambda_\pi(C)$ is built by gluing the lotus $\Lambda_{L,L_1}(C) = \Lambda(3/2)$ with the lotus $[e_{E_3}, C]$ associated to the cross (E_3, C_{L,L_1}), identifying the points labeled by E_3 (see Fig. 1.41).

Example 1.5.33 We start the algorithm by choosing $L_1 := Z(y)$ and the cross (L, L_1) on (S, o). The Newton polygon $\mathcal{N}_{L,L_1}(C)$ has only one edge and its orthogonal ray has slope 1, hence $\mathcal{F}_{L,L_1}(C) \simeq \mathcal{F}(1)$. The first trunk is the segment $[e_L, e_{L_1}]$ with its midpoint marked. The first lotus is just the petal $\Lambda_1 := \Lambda(1) = \delta(e_L, e_{L_1})$ with base $[e_L, e_{L_1}]$.

The Newton modification ψ_{L,L_1}^C is the usual blow up of the point o. We restrict it to the chart $\mathbb{C}^2_{v_1,v_2}$, where $x = v_1$, $y = v_1 v_2$. The strict transform $C_1 := C_{L,L_1}$ is defined in this chart by the equation $v_2^2 - 2v_2 + 1 - v_1 = 0$. The exceptional divisor $E_1 := Z(v_1)$ intersects the strict transform C_1 at the point o_1 defined by $v_2 = 1$. When running the algorithm, we have to choose a smooth branch B_2 such that (E_1, B_2) defines a cross at o_1. We set $B_2 := Z(v_2 - 1)$ and $u_1 := v_2 - 1$.

Then, the local coordinates (v_1, u_1) define the cross (E_1, B_2). We denote by L_2 the projection to S of the line $B_2 = Z(u_1)$, which is parametrized by $v_1 = t$ and $v_2 = 1$. One gets that L_2, which is parametrized by $x = t,\ y = t$, has local equation $y - x = 0$.

The strict transform C_1 has local equation $u_1^2 - v_1 = 0$. The Newton polygon $\mathcal{N}_{E_1,B_2}(C_1)$ has only one edge and its orthogonal ray has slope $1/2$, hence its associated fan is $\mathcal{F}_{E_1,B_2}(C_1) \simeq \mathcal{F}(1/2)$. The second trunk is just the segment $[e_{E_1}, e_{L_2}]$ with a marked point of slope $1/2$. The modification $\psi_{E_1,B_2}^{C_1,reg}$ defined by the regularization of this fan has two exceptional divisors E_2 and E_3 corresponding to the rays of the regularization of the fan $\mathcal{F}(1/2)$ of slopes 1 and $1/2$ respectively. When we consider the regularization of the fan $\mathcal{F}_{E_1,B_2}(C_1)$, we have to mark an additional point of slope 1 in the second trunk $[e_{E_1}, e_{L_2}]$. The associated lotus is $\Lambda_2 := \Lambda(1/2)$, with base $[e_{E_1}, e_{L_2}]$.

In this example, the composition $\pi := \psi_{E_1,B_2}^{C_1,reg} \circ \psi_{L,L_1}^{C} : (\Sigma, \partial\Sigma) \to (S, \partial S)$ is an embedded resolution of C, since the strict transform C_2 of C is smooth and intersects transversally the exceptional divisor of π at a point $o_2 \in E_3$. Notice that when running the algorithm, we have to consider also the cross (E_3, C_2) at o_2. Its trunk coincides with its associated lotus. It is just the segment $\Lambda_3 := [e_{E_3}, C]$, with no marked points.

The lotus $\Lambda_\pi(C)$ is represented in Fig. 1.43. It is obtained from $\Lambda_1,\ \Lambda_2$ and Λ_3 (see Fig. 1.42) by identifying the points with the same label.

Remark 1.5.34 The lotus $\Lambda_\pi(C)$ may be embedded canonically into the set of **semivaluations** of the local $\mathbb{C}$-algebra $\hat{\mathcal{O}}_{S,o}$ (semi-valuations are defined similarly to valuations, but dropping the last condition from Definition 1.2.19). Indeed, its base membrane $\Lambda(\mathcal{F}_{L,L_1}(C))$ embeds into the regular cone σ_0^{L,L_1} of Definition 1.3.32, which may be interpreted valuatively by associating to each $w \in \sigma_0^{L,L_1}$ the valuation ν_w defined by Eq. (1.32). Each other membrane may be similarly interpreted valuatively, and one may show that one gets in this way an embedding. Details may be found in [102, Section 7].

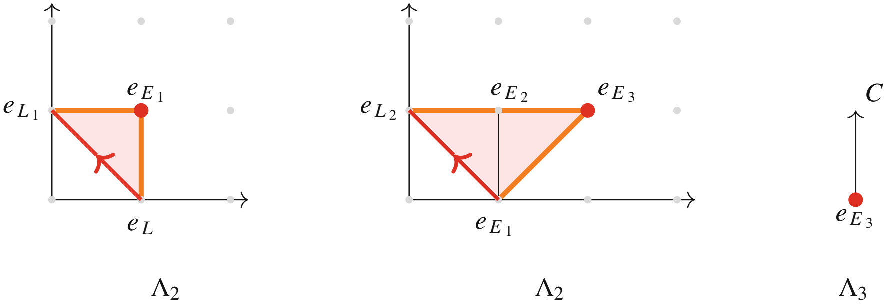


Fig. 1.42 The Newton lotuses $\Lambda_1,\ \Lambda_2$ and Λ_3 of Example 1.5.33

1.5.5 Truncated Lotuses

In this subsection we introduce an operation of *truncation* of the lotus of a toroidal pseudo-resolution of a plane curve singularity C, and we explain how to use it in order to visualize the dual graph of the total transform of C on the associated embedded resolution, as well as the Enriques diagram of the constellation of infinitely near points blown up for creating this resolution, in a way different from that formulated in point (8) of Theorem 1.5.29.

Recall first from Definition 1.5.26 the construction of the lotus $\Lambda_\pi(C)$ of a toroidal pseudo-resolution $\pi : (\Sigma, \partial\Sigma) \to (S, L + L')$ of the curve singularity $C \hookrightarrow S$. As stated in point (4) of Theorem 1.5.29, its lateral boundary $\partial_+\Lambda_\pi(C)$ is isomorphic to the dual graph of the boundary divisor $\partial\Sigma^{reg}$. Here Σ^{reg} denotes the minimal resolution of Σ, and $\partial\Sigma^{reg}$ is the total transform on it of the boundary divisor $\partial\Sigma$ of the toroidal surface $(\Sigma, \partial\Sigma)$. The divisor $\partial\Sigma^{reg}$ is also the total transform of the completion $\hat{C}_\pi$ of C relative to π, that is, the sum of the total transform of C by the smooth modification $\pi^{reg} : \Sigma^{reg} \to S$ and of the strict transforms of the branches L_j introduced while running Algorithm 1.4.22.

How to get the dual graph of the total transform of C on Σ^{reg} from the lateral boundary $\partial_+\Lambda_\pi(C)$? One has simply to remove the ends of $\partial_+\Lambda_\pi(C)$ which are labeled by the branches L_j, as well as the edges which connect them to other vertices of $\partial_+\Lambda_\pi(C)$. This *truncation operation* performed on the tree $\partial_+\Lambda_\pi(C)$ may be seen as the restriction of a similar operation performed on the whole lotus $\Lambda_\pi(C)$. Let us explain this truncation operation on $\Lambda_\pi(C)$, as well as some of its uses.

Consider first a petal $\delta(e_1, e_2)$ associated to a base (e_1, e_2) of a lattice N (see Definition 1.5.1). Its **axis** is the median $[(e_1 + e_2)/2, e_1 + e_2]$ of the petal, joining the vertex $e_1 + e_2$ to the midpoint of the opposite edge. This axis decomposes the petal into two **semipetals**.

The **semipetals** of a lotus are the semipetals of all its petals. Using this vocabulary, as well as that introduced in Definition 1.5.26 about the anatomy of lotuses of toroidal pseudo-resolutions, we may define now the operation of truncation of such a lotus:

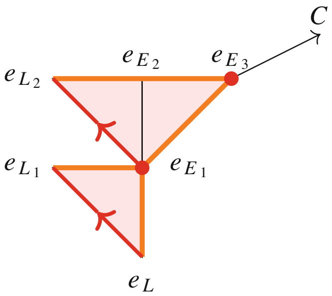


Fig. 1.43 The lotus $\Lambda_\pi(C)$ of Example 1.5.33

Definition 1.5.35 Let $\Lambda_\pi(C)$ be the lotus of a toroidal pseudo-resolution π of the plane curve singularity $C \hookrightarrow (S, o)$. Its **truncation** $\boxed{\Lambda_\pi^{tr}(C)}$ is the union of the axis of its basic petal, of all the semipetals which do not contain basic vertices and of all the membranes which are segments, that is, of the edges of $\Lambda_\pi(C)$ which have an extremity labeled by a branch of C. The **lateral boundary** $\boxed{\partial_+\Lambda_\pi^{tr}(C)}$ of $\Lambda_\pi^{tr}(C)$ is the part of the lateral boundary of $\Lambda_\pi(C)$ which remains in $\Lambda_\pi^{tr}(C)$.

Truncating the lotus $\Lambda_\pi(C)$ corresponds to forgetting its points whose corresponding semivaluations depend on the choice of the branches L_j. One keeps only those semivaluations determined by the given curve singularity C and by the infinitely near points through which pass its strict transforms during the blow up process (see Remark 1.5.34). In fact, the third author had introduced truncated lotuses in [102]—under the name of *sails*—as objects which represent the combinatorial type of a blow up process of a finite constellation, without considering any supplementary branches passing through the points of the constellation.

By construction, the lateral boundary $\partial_+\Lambda_\pi^{tr}(C)$ is isomorphic to the dual graph of the total transform $(\pi^{reg})^*(C)$. One may read again the self-intersection number of each irreducible component of the exceptional divisor of π^{reg} as the opposite of the number of petals, semi-petals and axis containing the vertex representing it.

Note that both lotuses of Figs. 1.41 and 1.43 have the same truncations. The reason is that their associated toroidal pseudo-resolutions lead to the same embedded resolution of C by regularization and that the truncated lotus is a combinatorial object encoding the decomposition of this resolution into blow ups of points (Fig. 1.44).

Example 1.5.36 For instance, in Fig. 1.45 is shown the truncation of the lotus of Fig. 1.36. Its lateral boundary is emphasized using thick orange segments. The component of the exceptional divisor represented by the unique vertex of the lotus contained in the axis has self-intersection number -4, as this vertex is contained in the axis, in two semi-petals and in one petal of $\Lambda_\pi^{tr}(C)$.

Consider now the Enriques tree of the toroidal pseudo-resolution π. Its edges are certain lateral edges of the 2-dimensional petals of $\Lambda_\pi(C)$ and of the 2-

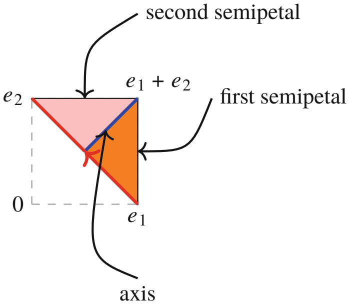


Fig. 1.44 The two semipetals and the axis of the petal $\delta(e_1, e_2)$

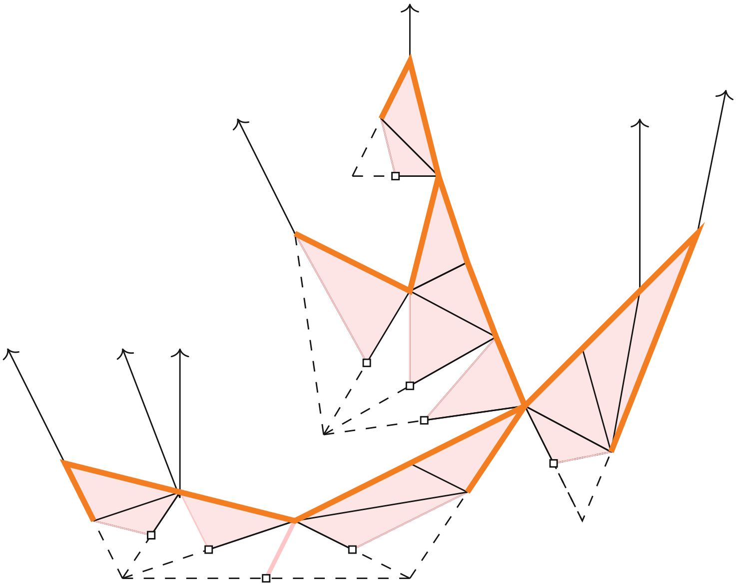


Fig. 1.45 The truncation of the lotus of Fig. 1.36 (see Example 1.5.36)

dimensional petals constructed from the 1-dimensional petals of $\Lambda_\pi(C)$ as bases (see Definition 1.5.26). For each edge $[A, B]$ of the Enriques tree, one may consider instead the homothetic segment $(1/2)[A, B]$, joining the points $(1/2)A$ and $(1/2)B$. This homothety is well-defined if one interprets the elements of the segment $[A, B]$ as valuations (see Remark 1.5.34). If both A and B are vertices of the same petal, then the segment $(1/2)[A, B]$ joins two edge midpoints of this petal. Otherwise, the interior points of the segment $(1/2)[A, B]$ are disjoint from the lotus $\Lambda_\pi(C)$.

The union of such segments $(1/2)[A, B]$—which were called *ropes* by the third author in [102]—is isomorphic to the Enriques tree of π. Therefore it is another representation of the Enriques diagram of the constellation whose blow up creates the resolution π^{reg}.

It is convenient to draw in a same picture both the truncation $\Lambda_\pi^{tr}(C)$ and the union of the ropes. For instance, for the lotus of Fig. 1.36 this union is represented on the right side of Fig. 1.46. For comparison, the Enriques tree is represented on the left side. An advantage of the right-side drawing is that the ropes whose interiors lie outside the truncation are exactly the ropes which were represented by Enriques as curved arcs. One may similarly determine from this drawing which edges go straight in Enriques' convention. For details, one may consult [102, Thm. 6.2]. Note that the *kites* of the title of [102] (in French *cerf-volants*) were the unions of truncated lotuses and of their ropes, as represented on the right side of Fig. 1.46.

Assume now that the combinatorial type of a plane curve singularity is given either using the dual graph of its total transform by an embedded resolution, weighted by the self-intersection numbers of the components of its exceptional

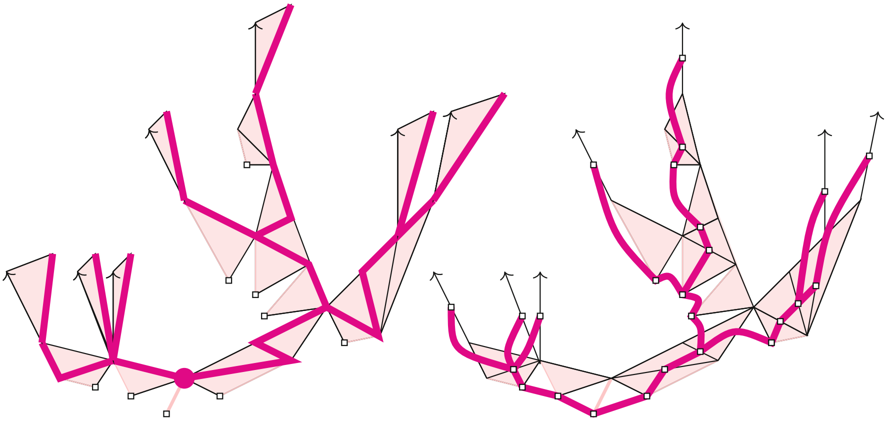


Fig. 1.46 Two ways of visualizing the Enriques tree on a truncated lotus

divisor, or using the Enriques diagram of the decomposition of the resolution morphism into blow ups of infinitely near points of o. How to get a series $f \in \mathbb{C}[[x, y]]$ defining a curve singularity with the given combinatorial type?

One may apply the following steps:

- Pass from the given tree to the associated truncated lotus. If the given tree is an Enriques diagram, it may be more convenient for drawing purposes to think about it as the union of ropes of the truncated lotus which is searched for.
- Complete the truncated lotus into a lotus having it as truncation. This step is not canonical, as shown by the comparison of Figs. 1.41 and 1.43 above.
- Proceed as in Example 1.6.29 below, by constructing the fan tree of the lotus, then the associated Eggers-Wall tree and writing finally a finite set of Newton-Puiseux series whose associated Eggers-Wall tree is isomorphic with this one.

1.5.6 Historical Comments

The study of plane curve singularities by using sequences of blow ups of points was initiated by Max Noether in his 1875 paper [89], and became common in the meantime, as shown by the works [90] of Noether, [35] of Enriques and Chisini, [126] of Du Val and [134, Sections I.2, II.2], [135] of Zariski.

Nowadays, a modification of $\mathbb{C}^2$ obtained as a sequence of blow ups of points is studied most of the time through the structure of its exceptional divisor. One encodes the incidences between its components, as well as their self-intersection numbers in a *weighted dual graph*, which is a tree (see [104] for a description of the development of this idea). When one looks at an *embedded resolution* of the plane curve singularity C, one adds new vertices to this graph, corresponding to the strict transforms of the branches of C.

The dual trees of exceptional divisors were not the first graphs associated with a process of blow ups of points. Another kind of tree, an *Enriques diagram*, encoding the proximity relation between the infinitely near points which are blown up in the process (see Definition 1.4.31), was associated with such a process in the 1917 book [35] of Enriques and Chisini. An example of an Enriques diagram, extracted from [35, Page 383], may be seen in Fig. 1.47. Details about the notion of Enriques diagram may be found in Casas' book [19] or in the third author's papers [96, 102], the second written in collaboration with Pe Pereira. The proximity relation was extended to higher dimensions by Semple in his 1938 paper [112]. Details about this generalization and about other approaches to the study of curve singularities of higher embedding dimension may be found in Campillo and Castellanos' 2005 book [18].

In order to understand the relation between the Enriques diagram of a finite constellation and the dual graph of the blow up of the constellation, the third author introduced the notion of *kite* in his 2011 paper [102]. A kite was defined by gluing *lotuses* into a *sail*, and attaching then *ropes* to this sail. The ropes were lying inside each lotus as the veins in a leaf, and they allowed to visualize the Enriques diagram. In turn, the dual graph could be visualised as the lateral boundary of the sail. A sail was composed not only of *petals*, but also of *axes* and *semi-petals*. The lotuses were also used in Castellini's thesis [25], written under the supervision of the third author. Castellini was able to do everything with petals, eliminating the use of axes,

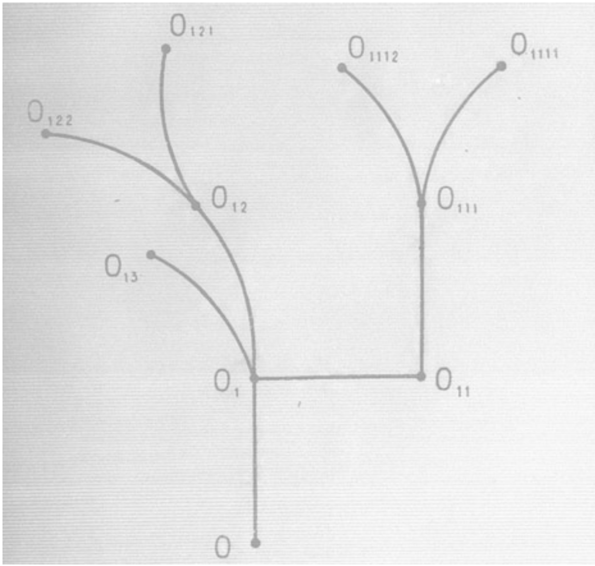
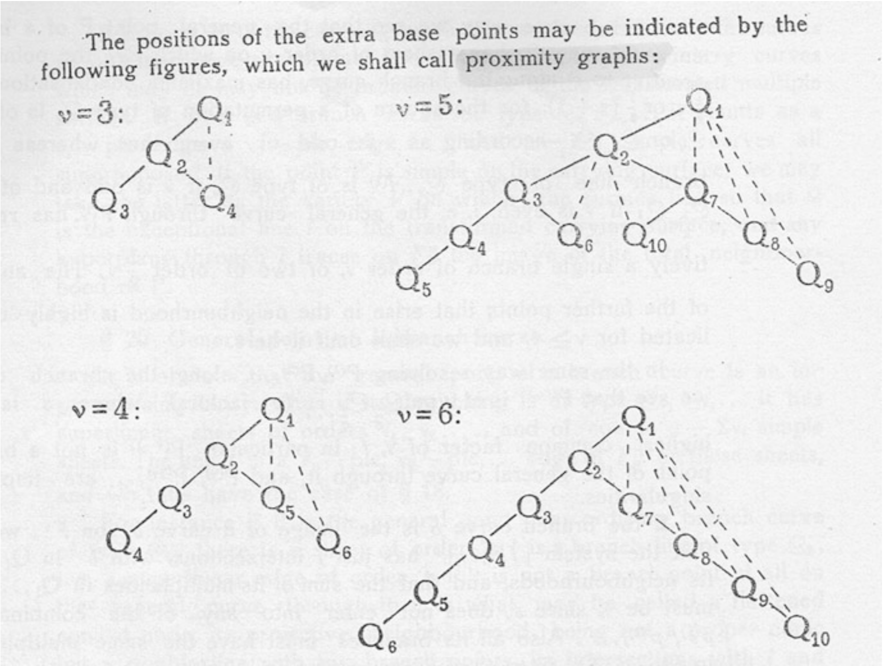


Fig. 1.47 An Enriques diagram

semi-petals and ropes, as what we call here the *Enriques tree* of a lotus proved to be more convenient to visualize the Enriques diagram. Also, the terminology was simplified, the gluing of lotuses resulting again in lotuses, instead of sails, as we do in the present paper.

It turns out that lotuses already appeared in disguise before the paper [102]. Their oldest ancestor is probably the *proximity relation*, defined in Enriques and Chisini's book [35, Page 381]. Indeed (see Theorem 1.5.29 (7)), the graph of the proximity relation among all the points whose blow up composes the embedded resolution produced by the second algorithm described in our paper may be identified with the full subgraph of the 1-skeleton of the associated lotus on the set of vertices which are not basic. The oldest drawings of such proximity graphs seem to be those of Du Val's 1944 paper [127] (see Figs. 1.48 and 1.49, in which one may also recognize what we call the "Enriques tree" of a lotus, drawn with continuous segments). Before, the proximity binary relation was related to the exceptional divisor of the associated blow up process in Barber and Zariski's 1935 paper [12] and Du Val's 1936 paper [126]. Du Val introduced the notion of *proximity matrix*, equivalent to that of proximity binary relation. In his 1939 paper [136], Zariski began a new ideal-theoretical and valuation-theoretical trend in the study of infinitely near points. A geometrical presentation of the previous approaches of study of infinitely near points was given by Lejeune-Jalabert in her 1995 paper [78].



The positions of the extra base points may be indicated by the following figures, which we shall call proximity graphs:

Fig. 1.48 Du Val's "proximity graphs"

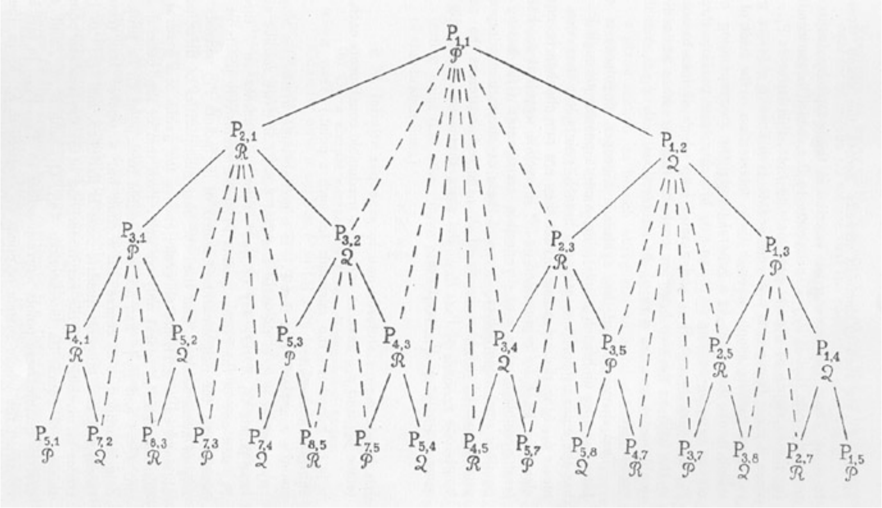


Fig. 1.49 Du Val's version of universal lotus

The graph of the proximity relation was mentioned again by Deligne in his 1973 paper [29], by Morihiko Saito in his 2000 paper [108] and by Wall in his 2004 book [131, Sections 3.5, 3.6]. One may find drawings of simple such graphs only in the first and the third reference.

Another occurrence of lotuses in disguise may be found in Schulze-Röbbecke's 1977 Diplomarbeit [111] written under the supervision of Brieskorn. In that paper are described particular divides (generic immersions of segments in a disc) obtained by applying to *branches* A'Campo's method of constructing δ-constant deformations explained in the 1974–75 papers [6] and [7]. The diagram of Fig. 1.50, extracted from page 57 of [111], indicates the general shape of the divides constructed in that paper. One may recognize inside it part of the lotus associated to a toroidal resolution process of a branch. In his already mentioned 2015 PhD thesis [25], Castellini could extend Schulze-Röbbecke's description to arbitrary plane curve singularities, using in a crucial way the notion of lotus of a blow up process.

Let us discuss now the relation of the universal lotus introduced in Definition 1.5.3 with other objects and constructions. The Enriques tree of the universal lotus $\Lambda(e_1, e_2)$ is an embedding into the cone σ_0 of almost all the *Stern-Brocot tree* defined by Graham, Knuth and Patashnik in [57, Page 116], in reference to the 1858 paper [117] of Stern and the 1860 paper [16] of Brocot. This tree represents the successive generation of the positive rational numbers starting from the sequence $(0/1, 1/0)$. At each step of the generating process, one performs the *Farey addition* $(a/b, c/d) \to (a+c)/(b+d)$ on the pairs of successive terms of the increasing sequence of rationals obtained at the previous steps. The vertices of the Stern-Brocot tree correspond bijectively with the positive rationals. For each Farey

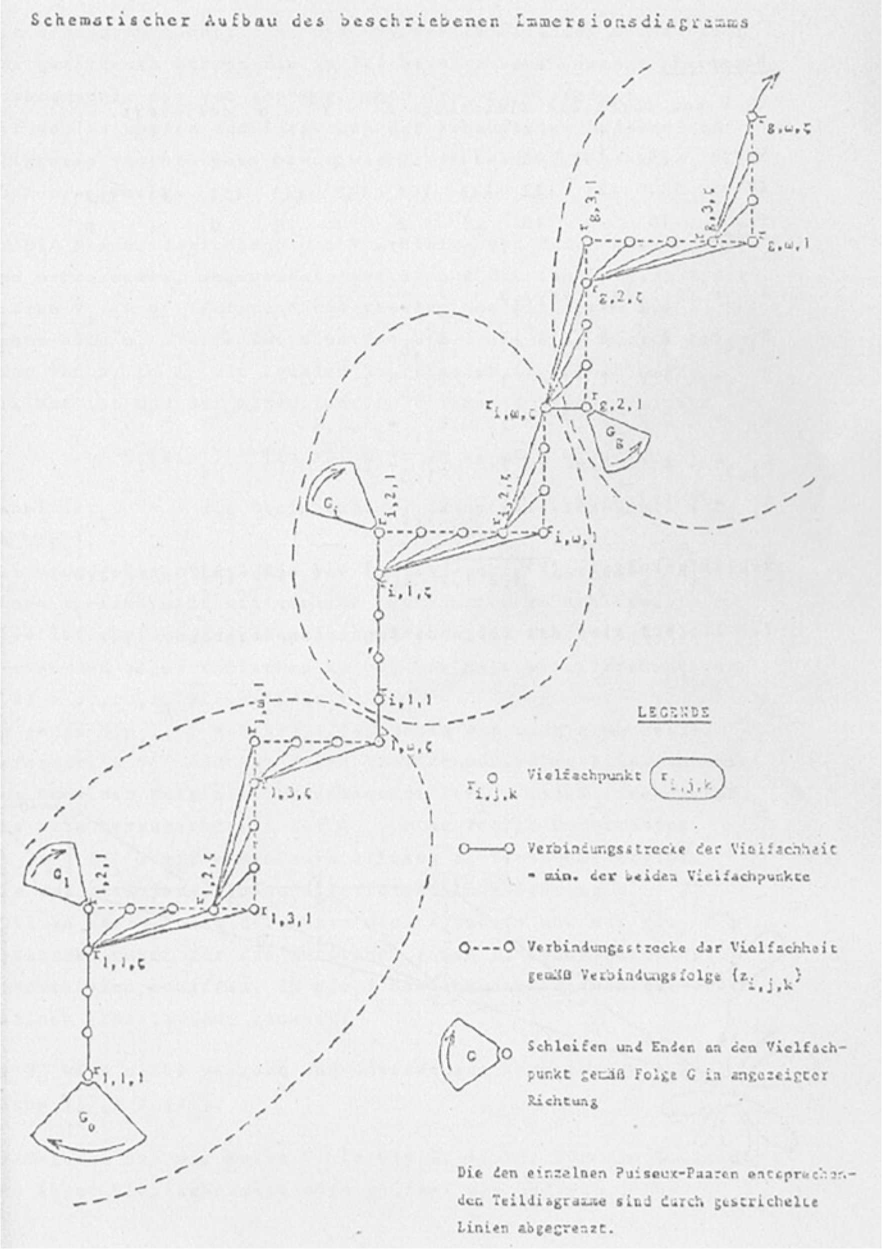


Fig. 1.50 The general shape of Schulze-Röbbecke's divides

addition $(a/b, c/d) \to (a+c)/(b+d)$ in which c/d was created after a/b, one joins the vertices corresponding to c/d and to $(a+c)/(b+d)$. The embedding of the Stern-Brocot tree represented in Fig. 1.29 is obtained by sending each vertex corresponding to $\lambda \in \mathbb{Q} \cap (0, \infty)$ to the primitive vector $p(\lambda) \in N \cap \sigma_0$ (see Notations 1.3.2) and each edge to a Euclidean segment. Another embedding in the cone σ_0 of the same part of the Stern-Brocot tree as above was described in [102, Rem. 5.7]. That embedding may be obtained from the embedding of Fig. 1.29 by applying a homothety of factor $1/2$.

The sequence of continued fractions (1.49) appearing in the proof of Proposition 1.5.23 was called the *slow approximation* (“*approximation lente*”) of $[a_1, \ldots, a_k]$ in Lê, Michel and Weber's paper [82, Appendice]. They used such sequences in order to describe the construction of the dual graph of the minimal embedded resolution of a plane curve singularity starting from the generic characteristic exponents of its branches and the orders of coincidence between such branches.

The *zigzag decompositions* introduced in Definition 1.5.18 are a variant of the *zigzag diagrams* of the third's author 2007 paper [101, Section 5.2]. Those diagrams allow to relate geometrically the usual continued fractions to the so-called *Hirzebruch-Jung continued fractions*. Those Hirzebruch-Jung continued fractions are the traditional tool, going back to Jung's 1908 paper [68] and Hirzebruch's 1953 paper [62], to describe the regularization of a 2-dimensional strictly convex cone. They are also crucial for the understanding of *lens spaces*, which becomes obvious once one sees that those 3-manifolds are exactly the links of toric surface singularities. See Weber's survey [133] for more details and historical explanations about the relations between lens spaces and complex surface singularities.

In [87, Section 9.1], Neumann and Wahl described a method for reconstructing the dual graph of the minimal resolution of a complex normal surface singularity whose link is an integral homology sphere from the so-called *splice diagram* of the link. This method is based on the construction of a finite sequence of rationals interpolating between two given positive rational numbers λ and μ. It may be described in the following way using lotuses of sequences of positive rational numbers:

- Construct by successive additions of petals the lotus $\Lambda(\lambda, \mu)$ as the union of $\Lambda(\lambda)$ and $\Lambda(\mu)$.
- Consider the increasing sequence of slopes of vertices of $\Lambda(\lambda, \mu)$ lying between λ and μ, that is, of vertices of the lateral boundary $\partial_+\Lambda(\lambda, \mu)$ (see Definition 1.5.5) lying on the arc joining the primitive vectors $p(\lambda)$ and $p(\mu)$ of N.

In [40, Section 2.2], Fock and Goncharov described the *tropical boundary hemisphere of the Teichmüller space of the punctured torus* as an infinite simplicial complex with integral vertices embedded in the real affine space associated to a two-dimensional affine lattice. This simplicial complex is a union of universal lotuses (see [40, Fig. 1]).

1.6 Relations of Fan Trees and Lotuses with Eggers-Wall Trees

In Sect. 1.6.1 we explain how to associate an *Eggers-Wall* tree $\Theta_L(C)$ to a plane curve singularity $C \hookrightarrow (S, o)$, relative to a smooth branch L. It is a rooted tree endowed with three structure functions, the *index* $\mathbf{i}_L$, the *exponent* $\mathbf{e}_L$ and the *contact complexity* $\mathbf{c}_L$. In Sect. 1.6.2 we express the Newton polygon of C relative to a cross (L, L') in terms of the Eggers-Wall tree $\Theta_L(C + L')$ of $C + L'$ relative to L (see Corollary 1.6.17). In Sect. 1.6.5 we prove that the fan tree $\theta_\pi(C)$ associated with a toroidal pseudo-resolution process of C is canonically isomorphic with the Eggers-Wall tree $\Theta_L(\hat{C}_\pi)$ of the completion of C relative to this process (see Theorem 1.6.27), and we explain how to compute the triple $(\mathbf{i}_L, \mathbf{e}_L, \mathbf{c}_L)$ of functions starting from the slope function of the fan tree (see Proposition 1.6.28). As a prerequisite, in Sects. 1.6.3 and 1.6.4 we prove *renormalization formulae*, which compare the Eggers-Wall tree of C relative to L and those of its strict transform relative to the exceptional divisor of a Newton modification.

1.6.1 Finite Eggers-Wall Trees and the Universal Eggers-Wall Tree

In this subsection we define the *Eggers-Wall tree* $\Theta_L(C)$ of a *reduced* plane curve singularity $C \hookrightarrow (S, o)$ relative to a smooth branch L (see Notations 1.6.7). It is constructed from the Newton-Puiseux series of C relative to a local coordinate system (x, y) such that $L = Z(x)$ (see Definition 1.6.3), but it is independent of this choice (see Proposition 1.6.6). It is a rooted tree whose root is labeled by L and whose leaves are labeled by the branches of C. It is endowed with three functions, the *index* $\mathbf{i}_L$, the *exponent* $\mathbf{e}_L$ and the *contact complexity* $\mathbf{c}_L$, which allow to compute the characteristic exponents of the Newton-Puiseux series mentioned above and the intersection numbers of the branches of C (see Proposition 1.6.11). Finally, we introduce the *universal Eggers-Wall tree* of (S, o) relative to L (see Definition 1.6.12), as the projective limit of the Eggers-Wall trees of the plane curve singularities contained in S. For more details and proofs one may consult our papers [45, Subsection 4.3] and [46, Section 3].

Let L be a smooth branch on (S, o). Assume in the whole subsection that C is *reduced*. Let (x, y) be a local coordinate system on (S, o), such that $L = Z(x)$, and let $f \in \mathbb{C}[[x, y]]$ be a defining function of C in this coordinate system. As a consequence of the Newton-Puiseux Theorem 1.2.20, one has:

Theorem 1.6.1 *Assume that C does not contain L, that is, that x does not divide $f(x, y)$. Then there exists a finite set $\boxed{\mathcal{Z}_x(f)}$ of Newton-Puiseux series of $\mathbb{C}[[x^{1/\mathbb{N}}]]$ and a unit $u(x, y)$ of the local ring $\mathbb{C}[[x, y]]$, such that:*

$$f(x,y) = u(x,y) \prod_{\eta(x) \in \mathcal{Z}_x(f)} (y - \eta(x)). \tag{1.51}$$

The set $\mathcal{Z}_x(f)$ is obviously independent of the defining function f of C. For this reason, we will denote it instead $\mathcal{Z}_x(C)$. It is the disjoint union of the sets $\mathcal{Z}_x(C_l)$, when C_l varies among the branches of C. It allows to associate to f the following objects:

Definition 1.6.2 Let (x, y) be a local coordinate system of (S, o) such that $L = Z(x)$ and let C be a reduced curve singularity on (S, o) not containing L.

- The finite subset $\boxed{\mathcal{Z}_x(C)} := \mathcal{Z}_x(f)$ from the statement of Theorem 1.6.1 is called the set of **Newton-Puiseux roots** of C relative to x.
- The **order of coincidence** $\boxed{k_x(\xi, \xi')}$ of two Newton-Puiseux series ξ, ξ' is equal to $\nu_x(\xi - \xi')$.
- The **order of coincidence** $\boxed{k_x(C_l, C_m)}$ of two distinct branches C_l and C_m of C is the maximal order of coincidence of Newton-Puiseux roots of the two branches: $\max\{k_x(\xi, \xi'),\ \xi \in \mathcal{Z}_x(C_l),\ \xi' \in \mathcal{Z}_x(C_m)\}$.
- The **set of characteristic exponents** $\boxed{\mathrm{Ch}_x(C_l)}$ of a branch C_l of C relative to the variable x is the set of orders of coincidence of pairs of distinct Newton-Puiseux roots of it: $\{k_x(\xi, \xi'),\ \xi, \xi' \in \mathcal{Z}_x(C_l),\ \xi \neq \xi'\}$.

This shows that for each $\xi \in \mathcal{Z}_x(C_l)$, there exists some $\xi' \in \mathcal{Z}_x(C_m)$ such that $\nu_x(\xi - \xi') = k_x(C_l, C_m)$. Therefore, knowing a Newton-Puiseux root of C_l determines some Newton-Puiseux root of C_m until their order of coincidence $k_x(C_l, C_m)$. This fact motivates the following construction of a rooted tree endowed with two functions:

Definition 1.6.3 Let (x, y) be a local coordinate system such that $L = Z(x)$ and C be a reduced curve singularity on (S, o).

- The **Eggers-Wall tree** $\boxed{\Theta_x(C_l)}$ of a branch $C_l \neq L$ of C relative to x is a compact segment endowed with a homeomorphism $\boxed{\mathbf{e}_x} : \Theta_x(C_l) \to [0, \infty]$ called the **exponent function**, and with **marked points**, which are the preimages by the exponent function of the characteristic exponents of C_l relative to x. The point $(\mathbf{e}_x)^{-1}(0)$ is labeled by L and $(\mathbf{e}_x)^{-1}(\infty)$ is labeled by C_l. The **index function** $\boxed{\mathbf{i}_x} : \Theta_x(C_l) \to \mathbb{N}^*$ whose value $\mathbf{i}_x(P)$ on a point $P \in \Theta_x(C_l)$ is equal to the lowest common multiple of the denominators of the exponents of the marked points belonging to the half-open segment $[L, P)$.
- The **Eggers-Wall tree** $\boxed{\Theta_x(L)}$ is reduced to a point labeled by L, at which $\mathbf{e}_x(L) = 0$ and $\mathbf{i}_x(L) = 1$.
- The **Eggers-Wall tree** $\boxed{\Theta_x(C)}$ of C relative to x is obtained from the disjoint union of the Eggers-Wall trees $\Theta_x(C_l)$ of its branches by identifying, for each pair of distinct branches C_l and C_m of C, their points with equal exponents not greater than the order of coincidence $k_x(C_l, C_m)$. Its **marked points** are

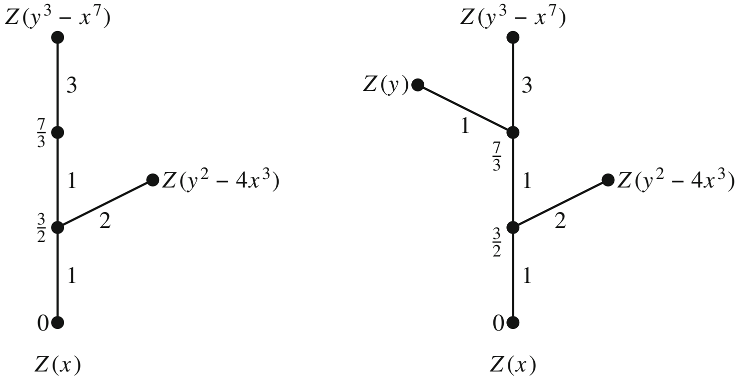


Fig. 1.51 The Eggers-Wall trees of $Z(f(x, y))$ and $Z(xyf(x, y))$ from Example 1.6.4

its ramification points and the images of the marked points of the trees $\Theta_x(C_l)$ by the identification map. Its **labeled points** are analogously the images of the labeled points of the trees $\Theta_x(C_l)$, the identification map being label-preserving. The tree is rooted at the point labeled by L. It is endowed with an **exponent function** $\boxed{\mathbf{e}_x} : \Theta_x(C) \to [0, \infty]$ and an index function $\boxed{\mathbf{i}_x} : \Theta_x(C) \to \mathbb{N}^*$ obtained by gluing the exponent functions and index functions on the trees $\Theta_x(C_l)$ respectively.

Note that, by construction, the exponent function is surjective in restriction to every segment $[L, C_l] = \Theta_x(C_l)$ of $\Theta_x(C)$ such that $C_l \neq L$ and that the ends of $\Theta_x(C)$ are labeled by the branches of C and by the smooth reference branch L. The marked points of $\Theta_x(C)$ which are images of marked points of the subtrees $\Theta_x(C_l)$ may be recovered from the knowledge of the index function, as its set of points of discontinuity. Therefore, the index function is constant on each open edge between two consecutive marked points of $\Theta_x(C)$. Moreover, it is continuous from below relative to the partial order $\preceq_L$ defined by the root L of $\Theta_x(C)$.

The Eggers-Wall tree allows to determine visually the characteristic exponents of each branch C_l. One has simply to follow the segment going from the root to the leaf representing the branch and to read all the vertex weights of the discontinuity points of the index function. In particular, if an internal vertex of such a segment is not a ramification vertex of the tree, then its exponent is necessarily a characteristic exponent of C_l.

Example 1.6.4 Consider again the plane curve singularity $C = Z(f(x, y))$ of Sect. 1.2.6. That is, $f(x, y) = (y^2 - 4x^3)(y^3 - x^7)$. Its Eggers-Wall tree is drawn on the left side of Fig. 1.51. On the right side is drawn the Eggers-Wall tree of the singularity $Z(xy(y^2 - 4x^3)(y^3 - x^7))$, which is the sum of C and of the coordinate axes.

Look at the segment joining the root to the branch $Z(y^3 - x^7)$, on the left side of Fig. 1.51. It contains two internal vertices, with exponents $3/2$ and $7/3$. The vertex of exponent $7/3$ is not a ramification vertex of the tree, therefore $7/3$ is a characteristic exponent of this branch. In turn, $3/2$ is not a characteristic exponent of this branch, as the value of the index function does not increase when crossing the corresponding vertex. Note that, by contrast, it increases when crossing the same vertex on the segment joining the root to the leaf corresponding to the branch $Z(y^2 - 4x^3)$, which shows that $3/2$ is a characteristic exponent of that branch.

We have represented both the Eggers-Wall tree of C and of its union with the coordinate axes in order to show that *the second one is homeomorphic to the dual graph of the total transform of the union by its minimal embedded resolution*, while our example shows that this is not true if one looks at the total transform of C alone (see Fig. 1.7). The previous homeomorphism is a general phenomenon, valid for any plane curve singularity, as seen by combining Proposition 1.4.35 and Theorem 1.6.27 below. Note that in full generality one needs to add to C more branches than simply the coordinate axes, considering a *completion* in the sense of Definition 1.4.15.

Example 1.6.5 Consider a plane curve singularity C whose branches C_i, $1 \leq i \leq 3$, are defined by the Newton-Puiseux series ξ_i, where:

$$\xi_1 = x^{7/2} - x^4 + 2x^{17/4} + x^{14/3}, \quad \xi_2 = x^{5/2} + x^{8/3}, \quad \xi_3 = x^2.$$

The sets of characteristic exponents of the branches are $\mathrm{Ch}_x(C_1) = \{7/2,\ 17/4,\ 14/3\}$, $\mathrm{Ch}_x(C_2) = \{5/2, 8/3\}$, $\mathrm{Ch}_x(C_3) = \emptyset$. One has $k_x(C_1, C_2) = 5/2$, $k_x(C_1, C_3) = k_x(C_2, C_3) = 2$. The Eggers-Wall trees $\Theta_x(C_1)$ and $\Theta_x(C)$ relative to x are drawn in Fig. 1.52. We represented the value of the corresponding exponent near each marked or labeled point, and the value of the corresponding index function near each edge.

In fact, the objects introduced in Definition 1.6.3 depend only on C and L, not on the coordinate system (x, y) such that $L = Z(x)$ (see [45, Proposition 103]):

Proposition 1.6.6 *Let (x, y) be a local coordinate system such that $L = Z(x)$ and C be a reduced curve singularity on (S, o). Then the tree $\Theta_x(C)$ endowed with the pair of functions $(\mathbf{i}_x, \mathbf{e}_x)$ is independent of the choice of local coordinate system such that $L = Z(x)$.*

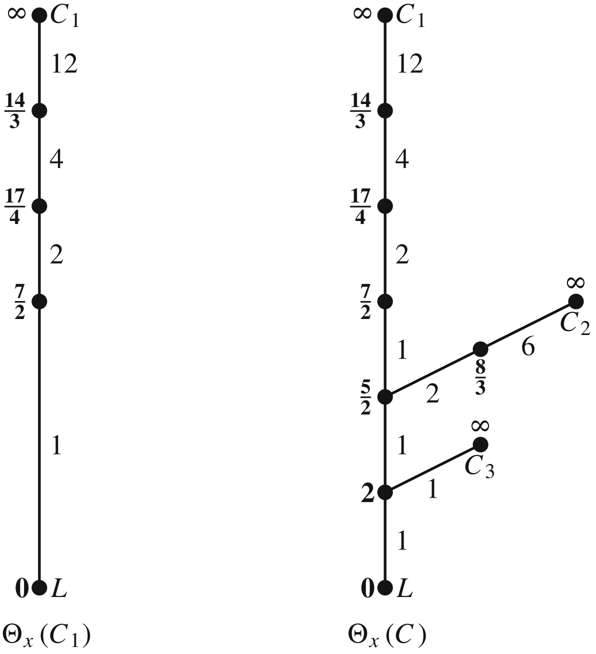


Fig. 1.52 The Eggers-Wall tree of the curve singularities C_1 and C of Example 1.6.5

Proposition 1.6.6 motivates us to introduce the following notations:

Notations 1.6.7 Let L be a smooth branch and C be a reduced curve singularity on (S, o). We denote $\boxed{(\Theta_L(C), \mathbf{i}_L, \mathbf{e}_L)} := (\Theta_x(C), \mathbf{i}_x, \mathbf{e}_x)$, for any local coordinate system (x, y) on (S, o) such that $L = Z(x)$. We say that this rooted tree endowed with two structure functions is the **Eggers-Wall tree of C relative to L**.

Remark 1.6.8 Let L be a smooth branch and C be a reduced curve singularity on (S, o). Then for any point $Q \in \Theta_L(C)$, we have:

$$\mathbf{i}_L(Q) = \min\{\mathbf{i}_L(A)\,,\ A \text{ is a branch on } S \text{ such that } Q \preceq_L A\}, \tag{1.52}$$

where $Q \preceq_L A$ has a meaning in the Eggers-Wall-tree $\Theta_L(C + A) \supseteq \Theta_L(C)$. Indeed, if $Q \preceq_L C_l$ for a branch C_l of C, and if B is a branch on S parametrized by the truncation of a Newton-Puiseux series $\xi \in Z_x(C_l)$, obtained by keeping only the terms of ξ of exponent $< \mathbf{e}_L(Q)$, then $Q \preceq_L B$ and $\mathbf{i}_L(Q) = \mathbf{i}_L(B)$.

The exponent function and the index function determine a third function on the tree $\Theta_L(C)$, the *contact complexity* function (see [46, Def. 3.19]):

Definition 1.6.9 Let C be a reduced curve singularity on (S, o). The **contact complexity function** $\boxed{\mathbf{c}_L} : \Theta_L(C) \to [0, \infty]$ is defined by the formula:

$$\mathbf{c}_L(P) := \int_L^P \frac{d\mathbf{e}_L}{\mathbf{i}_L}.$$

Note that in restriction to a segment $[L, C_l] = \Theta_L(C_l)$ of $\Theta_L(C)$, the contact complexity function is a bijection $[L, C_l] \to [0, \infty]$.

Remark 1.6.10 It follows immediately from Definition 1.6.9 that the contact complexity function together with the index function determine the exponent function by the following formula:

$$\mathbf{e}_L(P) = \int_L^P \mathbf{i}_L d\mathbf{c}_L. \tag{1.53}$$

The importance of the contact complexity function stems from the following property, which in different formulation goes back at least to Smith [115, Section 8], Stolz [118, Section 9] and Max Noether [90]:

Proposition 1.6.11 *Let L be a smooth branch and C be a reduced curve singularity on (S, o), not containing L. Let A and B be two distinct branches of C. Denote by $\boxed{A \wedge_L B}$ the infimum of the points of $\Theta_L(C)$ labeled by A and B, relative to the partial order $\preceq_L$ defined by the root L. Then:*

$$\mathbf{c}_L(A \wedge_L B) = \frac{A \cdot B}{(L \cdot A) \cdot (L \cdot B)}. \tag{1.54}$$

Proof One may find a proof of Proposition 1.6.11 in [131, Thm. 4.1.6]. Let us just sketch the main idea. Fix a local coordinate system (x, y) on (S, o), such that $L = Z(x)$. Start from a normalization of the branch A of the form $u \to (u^n, \zeta(u))$ (see the explanations leading to formula (1.2)). Therefore, $\zeta(x^{1/n})$ is a Newton-Puiseux root of A. By Theorem 1.6.1, one has a defining function of the branch B of the form $\prod_{\eta(x) \in \mathcal{Z}_x(B)} (y - \eta(x))$. Proposition 1.2.8 implies that:

$$A \cdot B = \nu_u \left(\prod_{\eta(x) \in \mathcal{Z}_x(B)} (\zeta(u) - \eta(u^n)) \right) = \sum_{\eta(x) \in \mathcal{Z}_x(B)} \nu_u \left(\zeta(u) - \eta(u^n)\right).$$

The finite multi-set of rational numbers whose elements are summed may be expressed in terms of the characteristic exponents of A and B which are not greater than the order of coincidence of A and B. A little computation finishes the proof.

□

If C and D are two reduced plane curve singularities on (S, o), with $C \subseteq D$, then by construction one has a natural embedding of rooted trees $\Theta_L(C) \subseteq \Theta_L(D)$. The uniqueness of the segment joining two points of a tree allows to define a canonical retraction $\Theta_L(D) \to \Theta_L(C)$. One may consider then either the *direct limit of the previous embeddings*, or the *projective limit of the previous retractions*, for varying C and D. Both limits have natural topologies. The direct limit, which may be thought simply as the union of all Eggers-Wall trees $(\Theta_L(C))_C$, is not compact, but the projective limit is compact. It is in fact a compactification of the direct limit.

For this reason, the projective limit is more suitable in many applications. Let us introduce a special notation for this notion, which will be used in Sect. 1.6.3 below.

Definition 1.6.12 Let L be a smooth branch on (S, o). The **universal Eggers-Wall tree** $\boxed{\Theta_L}$ **of** (S, o) **relative to** L is the projective limit of the Eggers-Wall trees $\Theta_L(C)$ of the various reduced curve singularities C on (S, o), relative to the natural retraction maps $\Theta_L(D) \to \Theta_L(C)$ associated to the inclusions $C \subseteq D$.

1.6.2 From Eggers-Wall Trees to Newton Polygons

In this subsection we explain how the Newton polygon $\mathcal{N}_{L,L'}(C)$ of a plane curve singularity C relative to the cross (L, L') (see Definition 1.4.14) may be determined from the Eggers-Wall tree $\Theta_L(C + L')$ (see Corollary 1.6.17).

The **Minkowski sum** $\boxed{K_1 + K_2}$ of two subsets of a real vector space is the set of sums $v_1 + v_2$, where each v_i varies independently among the elements of K_i. It is a commutative and associative operation. When both subsets are convex, their Minkowski sum is again convex.

The following property is classical and goes back at least to Dumas' 1906 paper [30, Section 3] (where it was formulated in a slightly different, p-adic, context):

Proposition 1.6.13 *If C and D are germs of effective divisors on (S, o), then:*

$$\mathcal{N}_{L,L'}(C + D) = \mathcal{N}_{L,L'}(C) + \mathcal{N}_{L,L'}(D).$$

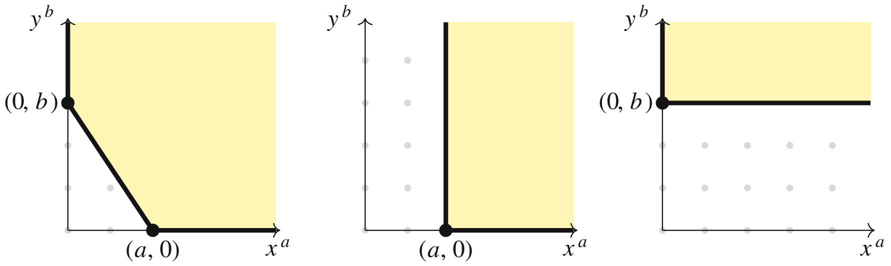
Proof This is a direct consequence of formula (1.35) and Proposition 1.4.12. □

One may extend the notion of Newton polygon to series in two variables with non-negative rational exponents whose denominators are bounded. They have again only a finite number of edges. The simplest Newton polygons are those with at most one compact edge:

Definition 1.6.14 Assume that $a, b \in \mathbb{Q}_+^*$. One associates them the following **elementary Newton polygons** (see Fig. 1.53):

$$\boxed{\left\{\frac{a}{b}\right\}} := \mathcal{N}(x^a + y^b), \quad \boxed{\left\{\frac{a}{\infty}\right\}} := \mathcal{N}(x^a), \quad \boxed{\left\{\frac{\infty}{b}\right\}} := \mathcal{N}(y^b).$$

The quotient a/b is the **inclination** of the elementary Newton polygon $\left\{\frac{a}{b}\right\}$.



$\left\{\frac{a}{b}\right\} := \mathcal{N}(x^a + y^b)$ $\left\{\frac{a}{\infty}\right\} := \mathcal{N}(x^a)$ $\left\{\frac{\infty}{b}\right\} := \mathcal{N}(y^b)$

Fig. 1.53 The elementary Newton polygons $\{\frac{a}{b}\}$, $\{\frac{a}{\infty}\}$, $\{\frac{\infty}{b}\}$

Note that for any $a \in \mathbb{Q}_+^* \cup \{\infty\}$, $b \in \mathbb{Q}_+^*$ and any $d \in \mathbb{N}^*$, one has: $d\left\{\frac{a}{b}\right\} = \left\{\frac{da}{db}\right\}$, where the left-hand side is the Minkowski sum of $\left\{\frac{a}{b}\right\}$ with itself d times. This allows to write:

$$\left\{\frac{a}{b}\right\} = b\left\{\frac{a/b}{1}\right\} \tag{1.55}$$

whenever $b \in \mathbb{N}^*$. The elementary Newton polygons are generators of the semigroup of Newton polygons, with respect to Minkowski sum. In fact one has more:

Proposition 1.6.15 *Each Newton polygon $\mathcal{N}$ may be written in a unique way, up to permutations of the terms, as a Minkowski sum of elementary Newton polygons with pairwise distinct inclinations. Their compact edges are translations of the compact edges of $\mathcal{N}$.*

Proof This is a consequence of the following property, which in turn may be proved by induction on $p \in \mathbb{N}^*$: *If $\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_p$ are elementary Newton polygons with finite non-zero strictly increasing inclinations, then their Minkowski sum $\mathcal{N}$ has exactly p compact edges which are translations of the compact edges of $\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_p$. Moreover, they are met in this order when one lists them starting from the unique vertex of $\mathcal{N}$ lying on the vertical axis.* ☐

The next proposition explains how to compute the Newton polygon of a branch C relative to a cross (L, L'), starting from the Eggers-Wall tree of $C + L'$ relative to L:

Lemma 1.6.16 *Let (L, L') be a cross and let $C \neq L$ be a branch on (S, o). Then the Newton polygon $\mathcal{N}_{L,L'}(C)$ may be expressed as follows in terms of the Eggers-Wall tree $(\Theta_L(C + L'), \mathbf{e}_L, \mathbf{i}_L)$:*

$$\mathcal{N}_{L,L'}(C) = \mathbf{i}_L(C) \left\{ \frac{\mathbf{e}_L(C \wedge_L L')}{1} \right\}.$$

The fan $\mathcal{F}_{L,L'}(C)$ has a unique ray in the interior of the cone σ_0, and its slope is equal to $\mathbf{e}_L(C \wedge_L L')$. That is:

$$\mathcal{F}_{L,L'}(C) = \mathcal{F}\big(\mathbf{e}_L(C \wedge_L L')\big).$$

Proof This is a consequence of Theorem 1.6.1. Indeed, let $f \in \mathbb{C}[[x]][y]$ be a defining function for C relative to a local coordinate system (x, y) defining the cross (L, L'). We know that its set of Newton-Puiseux roots $\mathcal{Z}_x(f)$ has $C \cdot L = \mathbf{i}_L(C)$ elements. All of them have the same support, since C is a branch, which implies that they form a single orbit under the Galois action of multiplication of $x^{1/\mathbf{i}_L(C)}$ by the group of $\mathbf{i}_L(C)$-th roots of 1. The order of any such series is equal to $k_x(L', C) = \mathbf{e}_L(C \wedge_L L')$. We deduce from Proposition 1.6.13 that the Newton polygon $\mathcal{N}_{L,L'}(C)$ is equal to the Minkowski sum of the factors of f in formula (1.51). The first assertion follows since the Newton polygon of $y - \eta(x)$ is equal to $\left\{ \frac{\mathbf{e}_L(C \wedge_L L')}{1} \right\}$, for any series $\eta(x) \in \mathcal{Z}_x(f)$, and then by taking into account formula (1.55). The second assertion is an immediate consequence of the first one. □

As a corollary we get the announced expression of the Newton polygon relative to (L, L') of a reduced curve singularity C in terms of the Eggers-Wall tree $\Theta_L(C+L')$ of $C + L'$ relative to L:

Corollary 1.6.17 *Let (L, L') be a cross and let C be a reduced curve singularity on (S, o) not containing the branch L. The Newton polygon $\mathcal{N}_{L,L'}(C)$ of the germ C with respect to the cross (L, L') is equal to the Minkowski sum:*

$$\sum_l \mathbf{i}_L(C_l) \left\{ \frac{\mathbf{e}_L(C_l \wedge_L L')}{1} \right\}, \tag{1.56}$$

where C_l runs through the branches of C.

Proof By Proposition 1.6.13, the Newton polygon $\mathcal{N}_{L,L'}(C)$ is the Minkowski sum of the Newton polygons of its branches. One uses then Lemma 1.6.16 for each such branch. □

Note that the previous result extends to not necessarily reduced curve singularities C if one defines their Eggers-Wall tree as the Eggers-Wall tree of their reduction, each leaf being endowed with the multiplicity of the corresponding branch in the divisor C. Then, in the right-hand side of Eq. (1.56), each branch C_l has to be counted with its multiplicity.

1.6.3 Renormalization of Eggers-Wall Trees

Let (L, L') be a cross on (S, o). In this subsection we will denote sometimes by $\boxed{\Theta_{o,L}(C)}$ the Eggers-Wall tree denoted before by $\Theta_L(C)$, in order to emphasize the point at which it is based. Indeed, we want to compare the previous tree with the Eggers-Wall tree $\Theta_{o_w,E_w}(C_w)$ of the germ (C_w, o_w) of the strict transform C_w of C at a smooth point o_w of the exceptional divisor E_w of a Newton modification relative to the cross (L, L'), with respect to the germ at o_w of the exceptional divisor E_w itself. Notice that if C is a reduced curve, then the strict transform C_w may consist of several germs of curves, one for each point of intersection of C_w with E_w. We show that the universal Eggers-Wall tree Θ_{o_w,E_w} in the sense of Definition 1.6.12 embeds naturally in the universal Eggers-Wall tree $\Theta_{o,L}$ and we explain how to relate their triples of structure functions (index, exponent and contact complexity). We conceive the passage from $\Theta_{o,L}(C)$ to $\Theta_{o_w,E_w}(C_w)$ as a *renormalization operation*, which explains the title of this subsection. We will give another proof of the renormalization Proposition 1.6.22 in Sect. 1.6.4, in terms of Newton-Puiseux series.

Let us fix a cross (L, L') on (S, o). Fix also a weight vector $\boxed{w} = \boxed{c_w} e_1 + \boxed{d_w} e_2 \in \sigma_0 \cap N_{L,L'}$. Denote by $\boxed{\pi_w : (S_w, \partial S_w) \to (S, L + L')}$ the modification obtained by subdividing σ_0 along the ray $\mathbb{R}_+ w$. If A is a branch on S, we denote by $\boxed{A_w}$ the strict transform of A by π_w. We look at the modification π_w in the toroidal category, relative to the boundaries $\partial S := L + L'$ and $\partial S_w := L_w + E_w + L'_w$, where $\boxed{E_w}$ is the exceptional divisor of the morphism π_w.

Denote by $\boxed{W}$ the point of $\Theta_L(L')$ corresponding to w, that is, the unique point of $\Theta_L(L')$ whose exponent is the slope of the ray $\mathbb{R}_+ w$ in the basis (e_1, e_2):

$$\mathbf{e}_L(W) = \frac{d_w}{c_w}. \tag{1.57}$$

Since (L, L') is a cross on (S, o) and $W \in \Theta_L(L')$, one has that $\mathbf{i}_L(W) = 1$. Therefore, by Definition 1.6.9, the contact complexity of W is:

$$\mathbf{c}_L(W) = \frac{d_w}{c_w}. \tag{1.58}$$

Recall that $A \wedge_L B$ denotes the infimum of the points A and B of the universal Eggers-Wall tree $\Theta_{o,L}$ relative to the partial order $\preceq_L$ induced by the root L. We need the following lemma:

Lemma 1.6.18 *Let A be a branch on (S, o) different from L, L'. The following properties are equivalent:*

1. *The strict transform A_w of A by π_w intersects $E_w \setminus (L_w \cup L'_w)$.*
2. *The fan $\mathcal{F}_{L,L'}(A)$ is the subdivision of σ_0 along the ray $\mathbb{R}_+ w$.*

3. $A \wedge_L L' = W$.

In addition, if these properties hold, then the order of vanishing of A along E_w is equal to $d_w \mathbf{i}_L(A)$ and the intersection number $E_w \cdot A_w$ is $\mathbf{i}_L(A)/c_w$.

Proof The equivalence of these three properties is immediate from Propositions 1.4.18 and 1.6.16. Recall that the order of vanishing $\mathrm{ord}_{E_w}(A)$ is by definition the multiplicity of E_w in the divisor $(\pi_w^* L)$, that is, the value taken by the divisorial valuation ord_{E_w} defined by E_w on a defining function f of A. Thanks to Proposition 1.4.18, this value is equal to $\mathrm{trop}^A_{L,L'}(w)$, which may be written $d_w \mathbf{i}_L(A)$ by Lemma 1.6.16. By Proposition 1.4.18, $E_w \cdot A_w$ is equal to the integral length of the compact edge of the Newton polygon $\mathcal{N}_{L,L'}(A)$. The equality $E_w \cdot A_w = \mathbf{i}_L(A)/c_w$ follows by using Lemma 1.6.16 again. □

Lemma 1.6.19 *Let A and B be two branches on (S, o). Consider the point $W \in \Theta_L(L')$ fixed above, determined by relation (1.57). Assume that $W = A \wedge_L L' = B \wedge_L L'$ inside the universal Eggers-Wall tree Θ_L. Then the following conditions are equivalent:*

1. $A \wedge_L B = W$.
2. $A \cdot B = \dfrac{d_w}{c_w}(L \cdot A)(L \cdot B)$.
3. $A_w \cap E_w \neq B_w \cap E_w$.

Proof

Proof of *1* ⇒ *2* This implication is a consequence of Formulae (1.54) and (1.58).

Proof of *2* ⇒ *1* Let us denote by W' the point $A \wedge_L B$. The assumption $W \preceq_L A$, $W \preceq_L B$ implies that $W \preceq_L W'$. By Formula (1.54), we get $\mathbf{c}_L(W') = d_w/c_w = \mathbf{c}_L(W)$. Since the function $\mathbf{c}_L$ is strictly increasing on $[L, A]$, the inequalities $L \preceq_L W \preceq_L W' \preceq_L A$ imply that $W = W'$.

Proof of *1* ⇔ *3* Let (x, y) be a system of local coordinates defining the cross (L, L'). Denote by f_A a defining function of A with respect to this system and by K_A the compact edge of the Newton polygon $\mathcal{N}_{L,L'}(A)$. By the proof of Lemma 1.6.16, if α_A is the coefficient of x^{d_w/c_w} in a fixed Newton-Puiseux series of A, then the restriction of f_A to the compact edge K_A is equal to:

$$\left(\prod_{\gamma^{c_w}=1} (y - \alpha_A \, \gamma \, x^{d_w/c_w}) \right)^{\mathbf{i}_L(A)/c_w} = (y^{c_w} - \alpha_A^{c_w} x^{d_w})^{\mathbf{i}_L(A)/c_w}.$$

We consider similar notations for the branch B. By Proposition 1.4.18, the point of intersection of the strict transform of A_w with E_w is parametrized by the coefficient $\alpha_A^{c_w}$. The desired equivalence follows since $\alpha_A^{c_w} \neq \alpha_B^{c_w}$ if and only if for every $\gamma \in \mathbb{C}$ with $\gamma^{c_w} = 1$, one has that $\alpha_A \neq \gamma \cdot \alpha_B$, which is also equivalent to the property $k_L(A, B) = d_w/c_w$ by the definition of the order of coincidence (see Definition 1.6.2). □

Proposition 1.6.20 *Let A and B be two branches on S. Consider the point $W \in \Theta_L(L')$ fixed above. Assume that $W = A \wedge_L L' = B \wedge_L L'$. Then:*

1. $L \cdot A = c_w(E_w \cdot A_w)$.
2. $A \cdot B = A_w \cdot B_w + \dfrac{d_w}{c_w}(L \cdot A)(L \cdot B)$.
3. $A_w \cdot B_w > 0$ *if and only if* $W \prec_L A \wedge_L B$.
4. $\mathbf{c}_L(A \wedge_L B) = \dfrac{1}{c_w^2}\mathbf{c}_{E_w}(A_w \wedge_{E_w} B_w) + \dfrac{d_w}{c_w}$.

Proof Notice first that the hypothesis and Lemma 1.6.18 imply that the strict transforms $A_w,\ B_w$ of A and B by π_w intersect $E_w \setminus (L_w \cup L'_w)$. If C is a branch on (S, o), denote by $\boxed{(\pi_w^* C)_{ex}}$ the *exceptional part* of the total transform divisor $(\pi_w^* C) = (\pi_w^* C)_{ex} + C_w$ of C on S_w.

Proof of *1* We have:

$$\begin{aligned} L \cdot A &\overset{(i)}{=\!=} (\pi_w^* L) \cdot (\pi_w^* A) = \\ &\overset{(ii)}{=\!=} (\pi_w^* L) \cdot A_w = \\ &\overset{(iii)}{=\!=} (\pi_w^* L)_{ex} \cdot A_w = \\ &\overset{(iv)}{=\!=} \mathrm{ord}_{E_w}(L)(E_w \cdot A_w) = \\ &\overset{(v)}{=\!=} \nu_w(\chi^{\epsilon_1})(E_w \cdot A_w) = \\ &\overset{(vi)}{=\!=} ((c_w e_1 + d_w e_2) \cdot \epsilon_1)(E_w \cdot A_w) = \\ &\overset{(vii)}{=\!=} c_w(E_w \cdot A_w). \end{aligned}$$

Let us explain each one of the previous equalities:

- Equality (i) results from the birational invariance of the intersection product, if one works with total transforms of divisors.
- Equality (ii) is a consequence of the equality $(\pi_w^* L) \cdot (\pi_w^* A)_{ex} = 0$, which results from the *projection formula* (see [61, Appendix A1]), applied to the divisors L on S, $(\pi_w^* A)_{ex}$ on S_w and to the proper morphism π_w.
- Equality (iii) follows from the hypothesis $L_w \cdot A_w = 0$ and the bilinearity of the intersection product.
- Equality (iv) is a consequence of the equality $(\pi_w^* L)_{ex} = \mathrm{ord}_{E_w}(L) E_w$.
- Equality (v) results from the equalities $\mathrm{ord}_{E_w} = \nu_w$ (see Eq. (1.32)) and $x = \chi^{\epsilon_1}$.
- Equality (vi) results from the fact that $w = c_w e_1 + d_w e_2$.
- Equality (vii) results from the fact that (ϵ_1, ϵ_2) is the dual basis of (e_1, e_2).

Proof of *2*. Let us choose a branch A' on (S, o) such that:

$$\mathbf{i}_L(A) = \mathbf{i}_L(A') \text{ and } W = A \wedge_L L' = A' \wedge_L L'. \tag{1.59}$$

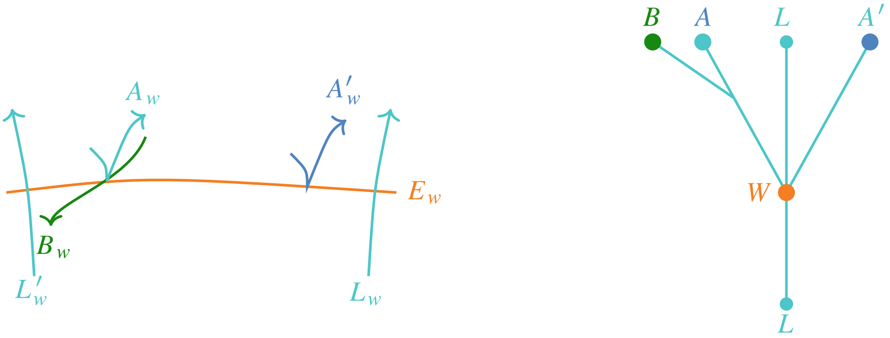


Fig. 1.54 The choice of branch A' in the proof of Proposition 1.6.20 (2)

Using Lemma 1.6.19, we can translate this hypothesis in terms of the total transform of the branches A, A' by π_w. On the left side of Fig. 1.54 is represented the total transform of $L + L' + A + A' + B$ by π_w and on its right side is represented the Eggers-Wall tree $\Theta_L(L + L' + A + A' + B)$, for some branch B. Then:

$$\begin{aligned}
A \cdot B &\overset{(i)}{=\!=} (\pi_w^* A) \cdot (\pi_w^* B) = \\
&\overset{(ii)}{=\!=} (\pi_w^* A) \cdot B_w = \\
&\overset{(iii)}{=\!=} A_w \cdot B_w + (\pi_w^* A)_{ex} \cdot B_w = \\
&\overset{(iv)}{=\!=} A_w \cdot B_w + (\pi_w^* A')_{ex} \cdot B_w = \\
&\overset{(v)}{=\!=} A_w \cdot B_w + (\pi_w^* A') \cdot B_w = \\
&\overset{(vi)}{=\!=} A_w \cdot B_w + A' \cdot B = \\
&\overset{(vii)}{=\!=} A_w \cdot B_w + (L \cdot A')(L \cdot B)\, \mathbf{c}_L(W) = \\
&\overset{(viii)}{=\!=} A_w \cdot B_w + (L \cdot A)(L \cdot B) \frac{d_w}{c_w}.
\end{aligned}$$

Let us explain each one of the previous equalities:

- Equalities (i) and (ii) are analogs of the equalities (i) and (ii) in the proof of point (1) above.
- Equality (iii) results from the bilinearity of the intersection product.
- Equality (iv) results from the hypothesis (1.59) and Lemma 1.6.18, which imply that $\text{ord}_{E_w}(A) = \text{ord}_{E_w}(A')$. Then one concludes using the equality $(\pi_w^* C)_{ex} \cdot B_w = \text{ord}_{E_w}(C)(E_w \cdot B_w)$, for each $C \in \{A, A'\}$.
- Equality (v) results from the fact that, by construction, A'_w and B_w are disjoint.
- Equality (vi) results from the projection formula.
- Equality (vii) results from Lemma 1.6.19.
- Equality $(viii)$ results from Eq. (1.58) and from the equality $L \cdot A = L \cdot A'$, which is a consequence of the hypothesis (1.59) and the equality $L \cdot C = \mathbf{i}_L(C)$ for each $C \in \{A, A'\}$.

Proof of *3* By hypothesis, the strict transforms A_w and B_w intersect the set $E_w \setminus (L_w \cup L'_w)$, which is equal to the torus orbit $O_{\mathbb{R}_+ w}$. By the proof of Proposition 1.4.18, this implies that w is orthogonal to the compact edges of the Newton polygons $\mathcal{N}_{L,L'}(A)$ and $\mathcal{N}_{L,L'}(B)$. Lemma 1.6.16 implies that $\mathbf{e}_L(W) = \mathbf{e}_L(A \wedge_L L') = \mathbf{e}_L(B \wedge_L L')$. As the three points $W, A \wedge_L L', B \wedge_L L'$ belong to the segment $[L, L']$ and that $\mathbf{e}_L$ is strictly increasing on it, we get the equalities $W = A \wedge_L L' = B \wedge_L L'$. This implies that $W \preceq_L A,\ W \preceq_L B$. The claim follows from point (2) by using Lemma 1.6.19.

Proof of *4*. Dividing both sides of the formula of point (2) by the product $(L \cdot A)(L \cdot B)$, we get:

$$\frac{A \cdot B}{(L \cdot A) \cdot (L \cdot B)} = \frac{A_w \cdot B_w}{(L \cdot A) \cdot (L \cdot B)} + \frac{d_w}{c_w}.$$

Using point (1), we get:

$$\frac{A_w \cdot B_w}{(L \cdot A) \cdot (L \cdot B)} = \frac{1}{c_w^2} \frac{A_w \cdot B_w}{(E_w \cdot A_w) \cdot (L_w \cdot B_w)}.$$

By applying formula (1.54) twice we obtain the desired formula:

$$\mathbf{c}_L(A \wedge_L B) = \frac{1}{c_w^2} \mathbf{c}_{E_w}(A_w \wedge_{E_w} B_w) + \frac{d_w}{c_w}. \tag{1.60}$$

□

Let us define in combinatorial terms a natural embedding of the universal Eggers-Wall tree Θ_{o_w, E_w} into the universal Eggers-Wall tree $\Theta_{o,L}$ (see 1.6.12):

Definition 1.6.21 Let A_w be a branch on the germ of surface (S_w, o_w). Denote by A its image by the modification π_w. The **natural embedding** of the universal Eggers-Wall tree Θ_{o_w, E_w} into the universal Eggers-Wall tree $\Theta_{o,L}$ is defined by sending each point Q of the Eggers-Wall segment $\Theta_{o_w, E_w}(A_w)$ to the unique point Q' of $\Theta_{o,L}(A)$ which satisfies:

$$\mathbf{c}_L(Q') = \frac{1}{c_w^2} \mathbf{c}_{E_w}(Q) + \frac{d_w}{c_w}. \tag{1.61}$$

If (C_w, o_w) is a reduced curve on (S_w, o_w), then the embedding of the Eggers-Wall tree $\Theta_{o_w, E_w}(C_w)$ in $\Theta_{o,L}(C)$ is well-defined thanks to Formula (1.60) applied to any pair $A_w,\ B_w$ of branches of (C_w, o_w). That is, the embeddings of the Eggers-Wall segments of its branches glue into an embedding of $\Theta_{o_w, E_w}(C_w)$ in $\Theta_{o,L}(C)$. Notice that the root E_w of $\Theta_{o_w, E_w}(C_w)$ corresponds to the point $W \in \Theta_{o,L}(L')$ defined by relation (1.57) and that the leaf of Θ_{o_w, E_w} labeled by A_w corresponds to the leaf of $\Theta_{o,L}$ labeled by A.

The following proposition describes how to pass from the functions $(\mathbf{i}_{E_w}, \mathbf{e}_{E_w})$ on the tree $\Theta_{o_w,E_w}(C_w)$ to the functions $(\mathbf{i}_L, \mathbf{e}_L)$ on $\Theta_{o,L}(C)$:

Proposition 1.6.22 *Let (C_w, o_w) be a reduced curve singularity on (S_w, o_w). Identify the tree $\Theta_{o_w,E_w}(C_w)$ with the subtree of $\Theta_{o,L}(C)$ defined by the natural embedding of Definition 1.6.21. One has the following relations in restriction to this subtree:*

1. $\mathbf{i}_L = c_w \, \mathbf{i}_{E_w}$.
2. $\mathbf{e}_L = \dfrac{1}{c_w}\mathbf{e}_{E_w} + \dfrac{d_w}{c_w}$.

Proof **Proof of *1*.** We show first the assertion for an end of $\Theta_{o_w,E_w}(C_w)$ corresponding to a branch B_w of C_w. By the definition of the index function, we have the equalities $\mathbf{i}_L(B) = L \cdot B$ and $\mathbf{i}_{E_w}(B_w) = E_w \cdot B_w$. Combining these equalities with point (1) of Proposition 1.6.20, we get:

$$\mathbf{i}_L(B) = c_w \mathbf{i}_{E_w}(B_w). \tag{1.62}$$

Let $Q \neq E_w$ be any rational point of $\Theta_{o_w,E_w}(C_w)$. By the equality (1.52), there exists a branch A_w on the germ of surface (S_w, o_w) such that $\mathbf{i}_{E_w}(A_w) = \mathbf{i}_{E_w}(Q)$. We get:

$$\mathbf{i}_L(Q) \overset{(1.52)}{\leq} \mathbf{i}_L(A) \overset{(1.62)}{=} c_w \mathbf{i}_{E_w}(A_w) = c_w \mathbf{i}_{E_w}(Q).$$

This implies that $\mathbf{i}_L(Q) \leq c_w \mathbf{i}_{E_w}(Q)$. Analogously, using again equality (1.52), there exists a branch B on the germ (S, o) such that $W \prec_L Q \prec_L B$ and $\mathbf{i}_L(B) = \mathbf{i}_L(Q)$. By Definition 1.6.21 of the natural embedding of Θ_{o_w,E_w} in $\Theta_{o,L}$, this implies that $Q \prec_{E_w} B$. Therefore:

$$\mathbf{i}_L(Q) = \mathbf{i}_L(B) \overset{(1.62)}{=} c_w \mathbf{i}_{E_w}(B_w) \overset{(1.52)}{\geq} c_w \mathbf{i}_{E_w}(Q).$$

It follows that $\mathbf{i}_L(Q) = c_w \mathbf{i}_{E_w}(Q)$. We have shown that the equality in point (1) holds in restriction to the rational points of $\Theta_{o_w,E_w}(C_w)$, and by the continuity properties of the index functions, it holds for every point of $\Theta_{o_w,E_w}(C_w)$.

Proof of *2*. Let P be a point of $\Theta_{o_w,E_w}(C_w)$. This implies that $W \preceq_L P$. By the integral formula (1.53), we get:

$$\mathbf{e}_L(P) = \int_L^P \mathbf{i}_L d\mathbf{c}_L = \int_L^W \mathbf{i}_L d\mathbf{c}_L + \int_W^P \mathbf{i}_L d\mathbf{c}_L.$$

Using again Eq. (1.53), we have:

$$\int_L^W \mathbf{i}_L d\mathbf{c}_L = \mathbf{e}_L(W) = \frac{d_w}{c_w}. \tag{1.63}$$

We compute the second integral $\int_W^P \mathbf{i}_L d\mathbf{c}_L$ by making a change of variable. Differentiating formula (1.61), we get $d\mathbf{c}_L = (1/c_w^2) d\mathbf{c}_{E_w}$. Using the expression for $\mathbf{i}_L$ of point (1), we obtain:

$$\int_W^P \mathbf{i}_L d\mathbf{c}_L = \frac{1}{c_w} \int_W^P \mathbf{i}_{E_w}\, d\mathbf{c}_{E_w} = \frac{1}{c_w} \mathbf{e}_{E_w}(P), \tag{1.64}$$

where we have used again the integral formula (1.53). We end the proof by combining the equalities (1.63) and (1.64):

$$\mathbf{e}_L(P) = \frac{d_w}{c_w} + \frac{1}{c_w} \mathbf{e}_{E_w}(P).$$

□

Remark 1.6.23 Identify the tree Θ_{o_w, E_w} with the subtree of the universal Eggers-Wall tree $\Theta_{o,L}$ defined by the embedding of Definition 1.6.21. As a consequence of Proposition 1.6.22, the two formulae stated in it also hold on Θ_{o_w, E_w}.

1.6.4 Renormalization in Terms of Newton-Puiseux Series

We give a different proof of Proposition 1.6.22 by using Newton-Puiseux series. This proof relates the Newton modifications in the toroidal category of Definition 1.4.14 with the *Newton maps*, which appear sometimes in the algorithmic construction of Newton-Puiseux series (see Sect. 1.6.6).

We keep the notations introduced at the beginning of Sect. 1.6.3. Let A be a branch on (S, o) such that A_w intersects E_w at a point $o_w \in E_w \setminus (L_w \cup L'_w)$. Consider local coordinates (x, y) defining the cross (L, L'). Recall from Definition 1.6.2 that $\mathcal{Z}_x(A)$ denotes the set of Newton-Puiseux roots of A relative to x. Let us choose $\eta \in \mathcal{Z}_x(A)$. It may be expressed as:

$$\eta = \sum_{k \geq m} \alpha_k x^{k/n}, \tag{1.65}$$

where $n = A \cdot L,\ m = A \cdot L'$. Hence $\alpha_m \neq 0$. All the series in $\mathcal{Z}_x(A)$ have the same support, since they form a single orbit under the Galois action of multiplication of $x^{1/n}$ by the complex n-th roots of 1 (see Remark 1.2.21).

Let us denote $p := \gcd(n, m)$. Our hypothesis that A_w meets $E_w \setminus (L_w \cup L'_w)$ implies that:

$$n = c_w \cdot p, \quad m = d_w \cdot p. \tag{1.66}$$

The branch A is defined by $f = 0$, where:

$$f = \prod_{\gamma^n=1} (y - (\gamma \cdot \eta)(x)) = (y^{c_w} - \alpha_m^{c_w} x^{d_w})^p + \dots . \tag{1.67}$$

We have only written on the right-hand side of (1.67) the restriction of f to the unique compact edge of the Newton polygon of $f(x, y)$.

Lemma 1.6.24 *There exist local coordinates* (x_1, y_1) *on the germ* (S_w, o_w) *such that* $E_w = Z(x_1)$ *and the map* π_w *is defined by:*

$$\begin{cases} x = x_1^{c_w}, \\ y = x_1^{d_w}(\alpha_m + y_1). \end{cases} \tag{1.68}$$

Proof Consider a vector $w' = a_w e_1 + b_w e_2$ such that:

$$b_w c_w - a_w d_w = 1. \tag{1.69}$$

Therefore the cone $\sigma = \mathbb{R}_+\langle w, w' \rangle$ is regular and included in one cone of the fan $\mathcal{F}_{L,L'}(A)$. As explained in the proof of Proposition 1.4.18, we can look at the intersection of A_w with the orbit $O_{\mathbb{R}_+ w} = E_w \setminus (L_w \cup L'_w)$ in the open subset corresponding to this orbit on the toric surface $X_\sigma = \mathbb{C}^2_{u,v}$. The toric morphism $\psi^{\sigma}_{\sigma_0}$ is the monomial map defined by

$$\begin{cases} x = u^{c_w} v^{a_w} \\ y = u^{d_w} v^{b_w} \end{cases}$$

(see Example 1.3.26). The orbit $O_{\mathbb{R}_+ w}$, seen on the surface $\mathbb{C}^2_{u,v}$, is the pointed axis $\mathbb{C}^*_v$. The maximal monomial in (u, v) which divides $(\psi^{\sigma}_{\sigma_0})^* f$ is equal to $(u^{c_w d_w} v^{a_w d_w})^p$. After factoring out this monomial and setting $u = 0$ we get:

$$(v^{a_w c_w - b_w d_w} - \alpha_m^{c_w})^p \overset{(1.69)}{=} (v - \alpha_m^{c_w})^p. \tag{1.70}$$

This shows that the point o_w has coordinates $(u, v) = (0, \alpha_m^{c_w})$. The formulae

$$\begin{cases} u = x_1 (y_1 + \alpha_m)^{-a_w}, \\ v = (y_1 + \alpha_m)^{c_w}, \end{cases} \tag{1.71}$$

define local coordinates (x_1, y_1) at o_w, since the jacobian determinant of $(u, v - \alpha_m^{c_w})$ with respect to (x_1, y_1) does not vanish at $(0, 0)$. Notice also that $Z(x_1) = Z(u) = E_w$. By (1.69) we get:

$$\begin{cases} x = x_1^{c_w}(y_1 + \alpha_m)^{-a_w c_w}(y_1 + \alpha_m)^{a_w c_w} = x_1^{c_w} \\ y = x_1^{d_w}(y_1 + \alpha_m)^{b_w c_w - d_w a_w} = x_1^{d_w}(\alpha_m + y_1). \end{cases} \qquad \square$$

Proposition 1.6.25 *With respect to the coordinates* (x_1, y_1) *introduced in Lemma 1.6.24, the series*

$$\eta_w := \sum_{k>m} \alpha_m x_1^{(k-m)/p},$$

is a Newton-Puiseux series parametrizing the branch A_w *on* (S_w, o_w).

Proof By formula (1.70), we have that $(A_w \cdot E_w)_{o_w} = p$. It follows that the Newton-Puiseux series in $\mathcal{Z}_{x_1}(A_w)$ must have exponents in $(1/p)\mathbb{N}^*$. By composing (1.68) with the change of variable

$$x_1 = x_2^p, \tag{1.72}$$

we get:

$$\begin{cases} x = x_2^n, \\ y = x_2^{d_w p}(\alpha_m + y_1). \end{cases} \tag{1.73}$$

Apply the substitution (1.73) to the factor $y - (\gamma \cdot \eta)(x^{1/n})$, using that $x_2 = x^{1/n}$ by definition, and factor out the monomial $x_2^{d_w p}$. We get the series

$$(\alpha_m + y_1) - \alpha_m \gamma^m - \sum_{k>m} \alpha_k \gamma^k x_2^{k-m} \in \mathbb{C}[[x_2, y_1]]. \tag{1.74}$$

This series has vanishing constant term if and only if $\gamma^m = 1$. Since $\gamma^n = 1$ and $\gcd(n, m) = p$, one may check that this condition holds if and only if $\gamma^p = 1$, and in this case for any $k > m$ one has $\gamma^k = \gamma^{k-m}$. It follows that the series (1.74) which are non-units are precisely the conjugates of the series $y_1 - \eta_w(x_1^{1/p})$ under the Galois action, since $x_2 = x_1^{1/p}$ by definition (1.72). Therefore, the product of all the conjugates of $y_1 - \eta_w(x_1^{1/p})$ under the Galois action defines a polynomial in $\mathbb{C}[[x_1]][y_1]$ which divides the strict transform of f by the map (1.68). The remaining factor is a series with nonzero constant term, and must belong to the ring $\mathbb{C}[[x_1, y_1]]$ since it is invariant under the Galois action. $\square$

Corollary 1.6.26 *Let A, B be two branches on (S, o) such that $o_w \in A_w \cap B_w \cap E_w$. Then:*

$$k_x(A, B) = \frac{d_w}{c_w} + c_w \cdot k_{x_1}(A_w, B_w).$$

Proof By point (3) of Proposition 1.6.20, the inequality $A_w \cdot B_w > 0$ (which results from the hypothesis that $o_w \in A_w \cap B_w$) implies that $k_x(A, B) > d_w/c_w$. It follows that if we fix a Newton-Puiseux series $\eta \in \mathcal{Z}_x(A)$, then there exists $\xi \in \mathcal{Z}_x(B)$ with the same order and the same leading coefficient. We can apply Lemma 1.6.24, using this leading coefficient, to define suitable local coordinates (x_1, y_1) at the point o_1. The formula results from Proposition 1.6.25 by taking into account the facts that $\eta_w \in \mathcal{Z}_{x_1}(A_w)$ and $\xi_w \in \mathcal{Z}_{x_1}(B_w)$. □

Corollary 1.6.26 implies readily Proposition 1.6.22.

1.6.5 From Fan Trees to Eggers-Wall Trees

In this subsection we assume that C is *reduced*. We explain that there exists a canonical isomorphism from the fan tree $\theta_\pi(C)$ of a toroidal pseudo-resolution π of C produced by running Algorithm 1.4.22, to the Eggers-Wall tree of the completion $\hat{C}_\pi$ of C (see Theorem 1.6.27). We also explain how to compute the index, exponent and contact complexity functions on the Eggers-Wall tree from the slope function on the fan tree (see Proposition 1.6.28).

Let L be a smooth branch on the germ (S, o). Assume that we run Algorithm 1.4.22, arriving at a toroidal pseudo-resolution $\pi : (\Sigma, \partial\Sigma) \to (S, L + L')$. Consider the corresponding completion $\hat{C}_\pi$, in the sense of Definition 1.4.15. There are two trees associated with this setting which have their ends labeled by the branches of $\hat{C}_\pi$, the fan tree $\theta_\pi(C)$ and the Eggers-Wall tree $\Theta_L(\hat{C}_\pi)$. How are they related? It turns out that they are isomorphic:

Theorem 1.6.27 *There is a unique isomorphism from the fan tree $\theta_\pi(C)$ to the Eggers-Wall tree $\Theta_L(\hat{C}_\pi)$, which preserves the labels of the ends of both trees by the branches of $\hat{C}_\pi$.*

Proof At the first step of Algorithm 1.4.22, one chooses a smooth branch L' such that (L, L') is a cross on (S, o). By definition, the branch L' is a component of the completion $\hat{C}_\pi$. Let us consider the segment $[L, L']$ of $\Theta_L(\hat{C}_\pi)$ and the first trunk $\theta_{\mathcal{F}_{L,L'}(C)} = [e_L, e_{L'}]$. We have a homeomorphism

$$\Psi_o : [e_L, e_{L'}] \to [L, L'] = \Theta_L(L')$$

sending a vector $w \in [e_L, e_{L'}]$ to the unique point $W \in [L, L']$ whose exponent $\mathbf{e}_L(W)$ is equal to the slope of w with respect to the basis $(e_L, e_{L'})$ of $N_{L,L'}$. By

Corollary 1.6.17, the map Ψ_o defines also a bijection between the set of marked points of the trunk, according to Definition 1.4.33, and the set of the marked points of the tree $\Theta_L(\hat{C}_\pi)$ which belong to the segment $[L, L']$ according to Definition 1.6.3.

Let o_i be a point of $\partial S_{\mathcal{F}_{L,L'}(C)}$, lying on the strict transform of C. The point o_i is considered at the fourth step of Algorithm 1.4.22. Let A_i denote the germ of $\partial S_{\mathcal{F}_{L,L'}(C)}$ at o_i and let (A_i, B_i) be the cross at o_i chosen when one passes again through the first and second steps of Algorithm 1.4.22. By definition, $L_i := \pi_{L,L'}(B_i)$ is a branch of $\hat{C}_\pi$. We denote by $\boxed{\hat{C}_{\pi,o_i}}$ (resp. $\boxed{C_{o_i}}$) the germ of the strict transform of $\hat{C}_\pi$ (resp. C) at the point o_i. We use the Notations 1.4.25. Let us consider the segment $[A_i, L_i]$ of the Eggers-Wall tree $\Theta_{o_i,A_i}(\hat{C}_{\pi,o_i})$ and the trunk $\theta_{\mathcal{F}_{A_i,B_i}(C_{o_i})} = [e_{A_i}, e_{B_i}]$. Arguing as before, we obtain a homeomorphism $\Psi_{o_i} : [e_{A_i}, e_{B_i}] \to [A_i, L_i]$ which sends $w \in [e_{A_i}, e_{B_i}]$ to the unique point $W \in [A_i, L_i]$ such that $\mathbf{e}_{A_i}(W)$ is equal to the slope of w with respect to the basis (e_{A_i}, e_{B_i}) of the lattice N_{A_i,B_i}. In addition, we get also that the homeomorphism Ψ_{o_i} defines a bijection between the marked points of the trunk $\theta_{\mathcal{F}_{A_i,B_i}(C_{o_i})}$ and the marked points of $\Theta_{o_i,A_i}(\hat{C}_{\pi,o_i})$ on the segment $[A_i, L_i]$. By Proposition 1.6.22, we have an embedding of the Eggers-Wall tree $\Theta_{o_i,A_i}(\hat{C}_{\pi,o_i})$ such that the root A_i of this tree is sent to the marked point $L' \wedge_L L_i$ of $\Theta_L(\hat{C}_\pi)$. By Definition 1.4.33, the point e_{A_i} of the trunk $\theta(\mathcal{F}_{A_i,B_i}(C))$ is identified with the marked point labeled by A_i on $\theta(\mathcal{F}_{L,L'}(C))$, during the construction of the fan tree $\theta_\pi(C)$.

If $\mathcal{T}$ is a tree and $P_1, \dots, P_s \in \mathcal{T}$, we denote by $[P_1, \dots, P_s]$ the smallest subtree of $\mathcal{T}$ containing $P_1, \dots, P_s$. We apply this notation for the subtree $[e_L, e_{L'}, e_{B_j}]$ of $\theta_\pi(C)$ and the subtree $[L, L', L_j]$ of $\Theta_L(\hat{C}_\pi)$. The previous discussion implies that the homeomorphisms Ψ_o and Ψ_{o_i} can be glued into a homeomorphism

$$[e_L, e_{L'}, e_{B_j}] \to [L, L', L_j],$$

which sends the ramification vertex e_{A_i} of the tree $[e_L, e_{L'}, e_{B_j}]$ to the ramification vertex $L' \wedge_L L_j$ of $[L, L', L_j]$. We repeat this construction each time we pass through a cross at the first and second steps during the iterations of Algorithm 1.4.22. By induction, we get a finite number of homeomorphisms Ψ_{o_j}, which glue into a homeomorphism $\Psi : \theta_\pi(C) \to \Theta_L(\hat{C}_\pi)$ which respects the labelings of the ends of both trees by the branches of $\hat{C}_\pi$. □

Identify the two rooted trees $\theta_\pi(C)$ and $\Theta_L(\hat{C}_\pi)$ by the isomorphism of Theorem 1.6.27. For every point $P \in \theta_\pi(C)$, define the set $\boxed{\delta_P} \subset [L, P)$ as the finite subset of discontinuity points of the restriction of the slope function $\mathbf{S}_\pi$ to the segment $[L, P)$. If $\lambda \in \mathbb{Q}^*$, denote by $\boxed{\mathrm{den}(\lambda)}$ the denominator q of λ, when one writes it in the form p/q, with $(p, q) \in \mathbb{Z} \times \mathbb{N}^*$, and p, q coprime. The fan tree $\theta_\pi(C)$ comes endowed with only one function, the *slope function* $\mathbf{S}_\pi$, while the Eggers-Wall tree is endowed with the *index* $\mathbf{i}_L$, the *exponent* $\mathbf{e}_L$ and the *contact complexity* $\mathbf{c}_L$ functions. These functions are related by:

Proposition 1.6.28 *For every* $P \in \theta_\pi(C)$*, one has:*

1. $\mathbf{i}_L(P) = \prod_{Q \in \delta_P} \text{den}(\mathbf{S}_\pi(Q))$.
2. $\mathbf{e}_L(P) = \int_L^P \frac{1}{\mathbf{i}_L}\, d\mathbf{S}_\pi$.
3. $\mathbf{c}_L(P) = \int_L^P \frac{1}{\mathbf{i}_L^2}\, d\mathbf{S}_\pi$.

Proof In order to follow the proof, one has to keep in mind the isomorphism of the fan tree with the Eggers-Wall tree built in Theorem 1.6.27. If the set δ_P is empty, that is, if the slope function $\mathbf{S}_\pi$ is continuous in restriction to $[L, P)$, then P belongs to the first trunk $[L, L']$. By definition, for any $Q \in [L, L']$ we have:

$$\mathbf{i}_L(Q) = 1, \quad \mathbf{e}_L(Q) = \mathbf{S}_\pi(Q). \tag{1.75}$$

Hence the equalities (1), (2) and (3) hold trivially for P.

We prove the assertions (1) and (2) by induction on the number of elements of the set δ_P of discontinuity points. Assume that $\delta_P = \{W = W_1, W_2, \dots, W_k\}$ with $k \geq 1$, and $W \prec_L W_2 \prec_L \cdots \prec_L W_k \prec_L P$. By construction, the point W belongs to the first trunk of $\theta_\pi(C)$. Then, using the notation (1.57), we have $\mathbf{e}_L(W) = d_w/c_w = \mathbf{S}_\pi(W)$, with $c_w = \text{den}(\mathbf{S}_\pi(W))$. We decompose the integral of the second member of equality (2) in the form:

$$\int_L^P \frac{1}{\mathbf{i}_L} d\mathbf{S}_\pi = \int_L^W \frac{1}{\mathbf{i}_L} d\mathbf{S}_\pi + \int_W^P \frac{1}{\mathbf{i}_L} d\mathbf{S}_\pi.$$

By (1.75), one has:

$$\int_L^W \frac{1}{\mathbf{i}_L} d\mathbf{S}_\pi = \mathbf{e}_L(W) = \frac{d_w}{c_w}. \tag{1.76}$$

With the notations of Sect. 1.6.3, we consider the reduced curve C_w at (S_w, o_w), consisting of those branches A_w which are the strict transforms of branches A of C such that $W \prec_L P \prec_L A$ (see point (3) of Proposition 1.6.20). Proposition 1.6.22 implies that:

$$\mathbf{i}_L(Q) = c_w \mathbf{i}_{E_w}(Q), \text{ for } Q \in [W, P] \subset \Theta_{E_w}(C_w). \tag{1.77}$$

Hence:

$$\int_W^P \frac{1}{\mathbf{i}_L} d\mathbf{S}_\pi = \frac{1}{c_w} \int_W^P \frac{1}{\mathbf{i}_{E_w}} d\mathbf{S}_\pi = \frac{1}{c_w} \mathbf{e}_{E_w}(P). \tag{1.78}$$

To understand the last equality of (1.78), apply the induction hypothesis to the integral $\int_W^P (1/\mathbf{i}_{E_w}) d\mathbf{S}_\pi$, with respect to the set $\{W_2, \ldots, W_k\}$ of discontinuity points of the restriction of the slope function $\mathbf{S}_\pi$ to $[W, P)$. The equality (2) follows from (1.76), (1.78) and point (2) of Proposition 1.6.22.

The equality (1) follows similarly by (1.77) and the induction hypothesis applied to $\mathbf{i}_{E_w}(P)$.

Let us prove the equality (3). By point (2) one has $d\mathbf{e}_L = (1/\mathbf{i}_L) d\mathbf{S}_\pi$. Therefore:

$$\mathbf{c}_L(P) = \int_L^P \frac{1}{\mathbf{i}_L} d\mathbf{e}_L = \int_L^P \frac{1}{\mathbf{i}_L^2} d\mathbf{S}_\pi.$$

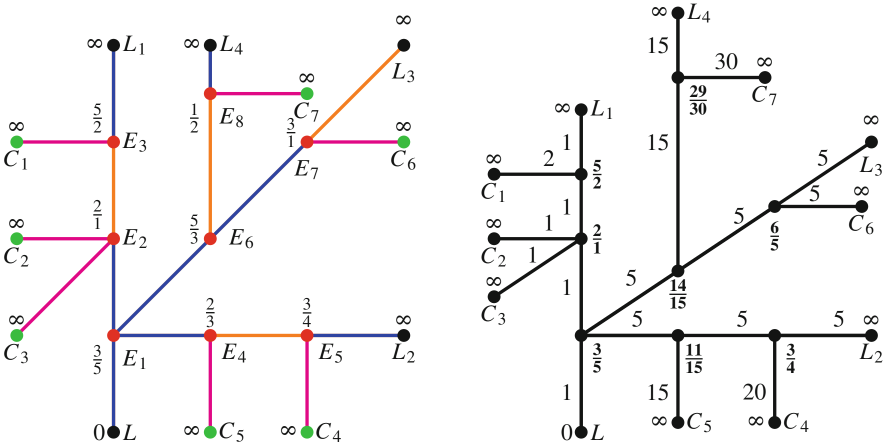
□

Example 1.6.29 Consider the toroidal pseudo-resolution process of Example 1.4.28. Figure 1.55 shows the fan tree $\theta_\pi(C)$ and the corresponding Eggers-Wall tree $\Theta_L(\hat{C}_\pi)$, for which are indicated the values of the exponent and the index functions. We computed them using Proposition 1.6.28. For instance, we have $\mathbf{i}_L(E_6) = 1 \cdot 5 = 5$, $\mathbf{e}_L(E_6) = \frac{3}{5} + \frac{1}{5} \cdot \frac{5}{3} = \frac{14}{15}$, $\mathbf{i}_L(E_8) = 1 \cdot 5 \cdot 3 = 15$ and $\mathbf{e}_L(E_8) = \frac{14}{15} + \frac{1}{15} \cdot \frac{1}{2} = \frac{29}{30}$.

Proposition 1.6.28 allows us to define a concrete reduced curve singularity C which admits the toroidal resolution process described in Example 1.4.28, whose lotus was represented in Fig. 1.36 and whose Enriques tree was represented in Fig. 1.40. Namely, we fix local coordinates (x, y) and we choose Newton-Puiseux series $\eta_1(x), \ldots, \eta_7(x)$ defining branches $C_1, \ldots, C_7$, then we take supplementary series $\lambda_1(x), \ldots, \lambda_4(x)$ defining branches $L_1, \ldots L_4$, such that the Eggers-Wall tree $\Theta_L(C_1 + \cdots + C_7 + L_1 + \cdots + L_4)$ is that on the right side of Fig. 1.55. For instance, one may choose:

$$\eta_1(x) := x^{5/2}, \quad \eta_2(x) := x^2, \quad \eta_3(x) := -x^2, \quad \eta_4(x) := x^{3/5} + x^{3/4},$$
$$\eta_5(x) := x^{3/5} + x^{11/15} \quad \eta_6(x) := 2x^{3/5} + x^{6/5}, \quad \eta_7(x) := 2x^{3/5} + x^{14/15} + x^{29/30},$$
$$\lambda_1(x) := 0, \quad \lambda_2(x) := x^{3/5}, \quad \lambda_3(x) := 2x^{3/5}, \quad \lambda_4(x) := 2x^{3/5} + x^{14/15}.$$

Remark 1.6.30 The right part of Fig. 1.55 shows the Eggers-Wall tree of the completion of a plane curve singularity generated by a toroidal pseudo-resolution process. One may verify that it satisfies the following property which characterizes the Eggers-Wall trees of such completions: *each vertex which is not an end of the tree is contained in the interior of a segment in restriction to which the index function is constant* (in particular, such an Eggers-Wall tree has no vertices of valency 2). When one has such an Eggers-Wall tree, it originates from a fan tree as described in Proposition 1.6.28. But this fan tree is not unique. One has to determine first which segments of the Eggers-Wall tree are trunks of the fan tree, and there may be different choices. For instance, in Fig. 1.55 one could decide that the segment $[L, C_2]$ is a trunk, instead of $[L, L_1]$. Once the trunks are chosen, the sets δ_P are



$\theta_\pi(C)$ $\quad$ $\Theta_L(\hat{C}_\pi)$

Fig. 1.55 The fan tree $\theta_\pi(C)$ and the corresponding Eggers-Wall tree $\Theta_L(\hat{C}_\pi)$ in Example 1.6.29

determined for every point P of the tree. This allows to compute the slope function $\mathbf{S}_\pi$ by integrating the differential relation $d\mathbf{S}_\pi = \mathbf{i}_L d\mathbf{e}_L$, which is a consequence of Proposition 1.6.28 (2).

Proposition 1.6.28 may be written more explicitly as follows:

Corollary 1.6.31 *Let P be a vertex of $\theta_\pi(C) = \Theta_L(\hat{C}_\pi)$, different from the root L. Assume that when one moves on the segment $[L, P]$ from L to P, one meets successively the vertices $P_1, \dots, P_k = P$ of $\delta_P \cup \{P\}$. Denote $\mathbf{S}_\pi(P_j) = d_j/c_j$ with coprime $c_j, d_j \in \mathbb{N}^*$, for all $j \in \{1, \dots, k\}$ (with $c_k = 1$ and $d_k = \infty$ if P is a leaf of the tree). Then:*

1. $\mathbf{i}_L(P) = c_1 \cdots c_{k-1}$.
2. $\mathbf{c}_L(P) = \dfrac{d_1}{c_1} + \dfrac{d_2}{c_1^2 c_2} + \dfrac{d_3}{c_1^2 c_2^2 c_3} + \cdots + \dfrac{d_k}{c_1^2 \cdots c_{k-1}^2 c_k}$.
3. $\mathbf{e}_L(P) = \dfrac{d_1}{c_1} + \dfrac{d_2}{c_1 c_2} + \dfrac{d_3}{c_1 c_2 c_3} + \cdots + \dfrac{d_k}{c_1 \cdots c_k}$.

Example 1.6.32 Let us specialize Corollary 1.6.31 to the case where P is a leaf of $\theta_\pi(C) = \Theta_L(\hat{C}_\pi)$, labeled by a branch C. Therefore the characteristic exponents of a Newton-Puiseux series of C relative to L are:

$$\frac{m_j}{n_1 \cdots n_j} := \frac{d_1}{c_1} + \frac{d_2}{c_1 c_2} + \cdots + \frac{d_j}{c_1 \cdots c_j}, \tag{1.79}$$

for all $j \in \{1, \ldots, k\}$. Here the positive integers $(m_1, \ldots, m_k)$ and $(n_1, \ldots, n_k)$ are chosen such that m_j and n_j are coprime for all $j \in \{1, \ldots, k\}$. The relations (1.79) may be reexpressed in the following way:

$$(c_j, d_j) = (n_j, m_j - n_j \cdot m_{j-1}), \tag{1.80}$$

for all $j \in \{1, \ldots, k\}$ (with the convention $m_0 := 0$). Sometimes the couples (m_j, n_j) are called the *Puiseux pairs* and the couples (d_j, c_j) are called the *Newton pairs* of the given Newton-Puiseux series. The importance of using both sequences of pairs in the topological study of plane curve singularities was emphasized by Eisenbud and Neumann in their book [34, Page 6]. More details may be found in Weber's survey [132, Section 6.1].

Example 1.6.33 This is a continuation of Example 1.6.32. Consider pairs of coprime integers $(n_j, m_j) \in \mathbb{N}^* \times \mathbb{N}^*$ with $n_j > 1$, for $j = 1, \ldots, k$ and the Newton-Puiseux series

$$x^{m_1/n_1} + x^{m_2/(n_1 n_2)} + \cdots + x^{m_k/(n_1 \cdots n_k)},$$

defining a branch C. We can build a toroidal pseudo-resolution π of C with respect to $L = Z(x)$, such that $\hat{C}_\pi = L + C + \sum_{j=1}^{k} L_j$ and the branches $L_1, \ldots, L_k$ are defined by the Newton-Puiseux series:

$$0, \quad x^{m_1/n_1}, \quad x^{m_1/n_1} + x^{m_2/(n_1 n_2)}, \quad \ldots, \quad x^{m_1/n_1} + x^{m_2/(n_1 n_2)} + \cdots + x^{m_{k-1}/(n_1 \cdots n_{k-1})}.$$

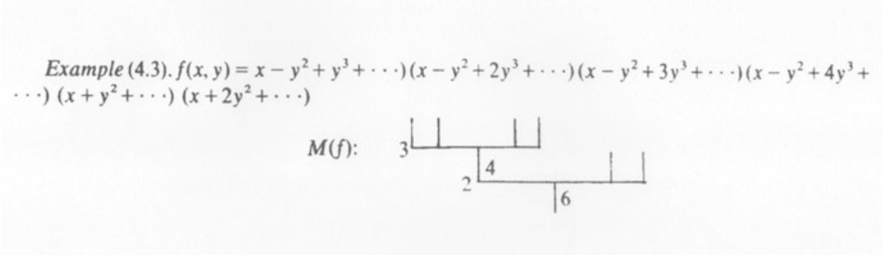
Then the associated lotus is as represented in Fig. 1.38. Using formula (1.80) and the notations introduced in Example 1.5.30, we have:

$$\frac{m_j}{n_j} - m_{j-1} = [p_j, q_j, \ldots],$$

for all $j \in \{1, \ldots, k\}$. In fact, one gets the same lotus whenever C is an arbitrary branch with the previous characteristic exponents relative to L and the branches L_j are *semiroots* of C (see [99, Corollary 5.6]). This shows that our notion of *completion of a reduced curve singularity C relative to a toroidal pseudo-resolution process is a generalization of the operation which adds to a branch a complete system of semiroots relative to L (see [99, Definition 6.4]).*

1.6.6 Historical Comments

Historical information about the notion of *characteristic exponent* may be found in our paper [44, Introduction, Rem. 2.9].



Example (4.3). $f(x, y) = x - y^2 + y^3 + \cdots)(x - y^2 + 2y^3 + \cdots)(x - y^2 + 3y^3 + \cdots)(x - y^2 + 4y^3 + \cdots)\,(x + y^2 + \cdots)\,(x + 2y^2 + \cdots)$

Fig. 1.56 A Kuo-Lu tree

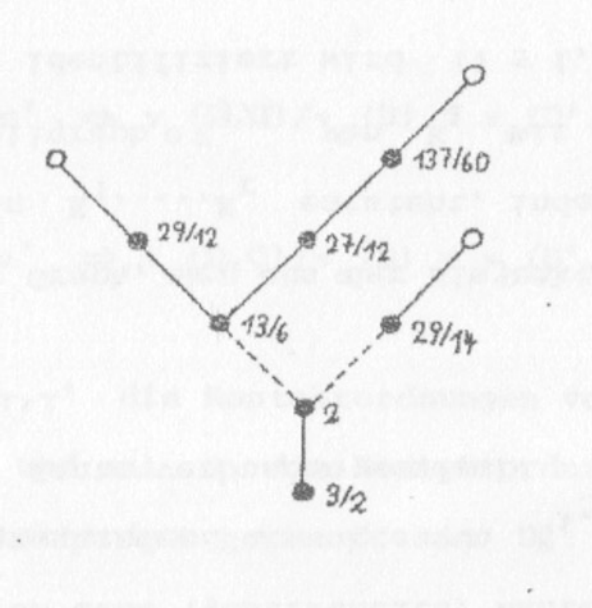


Fig. 1.57 An Eggers tree

In addition to the older Enriques diagrams and dual graphs of exceptional divisors of embedded resolutions, Kuo and Lu associated a third kind of tree to a curve singularity $C = Z(f(x, y))$ in their 1977 paper [75]. An example of such a tree, extracted from their paper, is shown in Fig. 1.57. Their trees were rooted and their sets of leaves were in bijection with the set of Newton-Puiseux series $\eta(x)$ associated with the corresponding plane curve singularity C. They used their trees in order to relate the structure of C to that of its *polar curve* defined by the equation $\dfrac{\partial f}{\partial y} = 0$ (Fig. 1.56).

In his 1983 paper [33], Eggers showed that a kind of Galois quotient of the Kuo-Lu tree of f was more convenient for this purpose. Figure 1.57 shows the first example given in [33]. A variant of the Eggers tree, better suited for computations, was introduced by Wall [130] and presented in more details in his textbook [131, Sections 4.2 and 9.4].

The third author coined in his 2001 thesis [98] the name *Eggers-Wall tree* for Wall's version of Eggers' tree. He proved in [98, Section 4.4] that the Eggers-Wall tree of C relative to generic coordinates could almost always be embedded in the dual graph of the minimal embedded resolution of C as the convex hull of its vertices representing the branches of C. He discovered this fact experimentally, by applying in many examples the first author's algorithm described in her 1996 thesis [42, Section 1.4.6], for the passage from Eggers' tree to the dual graph. Another proof of this embedding result was obtained in terms of certain toroidal-pseudo resolutions introduced by the second author in [52, Section 3.4]. Wall improved the description of this embedding in his 2004 book [131], and Favre and Jonsson explained it differently from their valuative viewpoint in their 2004 book [38, Appendix D2]. Recently, we gave a new viewpoint on this embedding result in [45, Theorem 112], in the framework of Eggers-Wall trees defined relative to arbitrary coordinate systems. It is important to consider the Eggers-Wall tree of C relative to coordinate systems which are not necessarily generic relative to C. Indeed, this freedom is essential when one wants to compare the Eggers-Wall tree of C with that of its strict transform by a blow up or a more complicated toric modification, because after such a modification the natural coordinate x defines the exceptional divisor, and is not necessarily generic with respect to the strict transform. In his paper [100], extracted from his thesis [98], the third author did not consider any genericity hypothesis, in order to extend the definition of this kind of tree to higher dimensional *quasi-ordinary hypersurface singularities*. This generalized notion of Eggers-Wall tree was further developed in connexion with the study of the associated polar hypersurfaces in the 2005 paper [43] of the first and second authors. In turn, the notion of Kuo-Lu tree was extended to quasi-ordinary hypersurface singularities by the first author and Gwoździewicz in their 2015 paper [47] and used again by them in [48], in order to study the structure of higher order polars of such singularities.

The notations for elementary Newton polygons described in Definition 1.6.14 were introduced by Teissier in his 1977 paper [120, Section 3.6], where he restricted them to $a, b \in \mathbb{N}^* \cup \{\infty\}$. Allowing the two numbers in Definition 1.6.14 to be rational is convenient in order to express Newton polygons in terms of Eggers-Wall trees (see Corollary 1.6.17).

Let us consider now the valuative aspects of Eggers-Wall trees. Favre and Jonsson proved in their 2004 book [38] that the set of semivaluations of the local $\mathbb{C}$-algebra $\hat{O}_{S,o}$ which are normalized by the constraint that a defining function x of the smooth germ L has value 1, has a natural structure of rooted real tree, which they called *the valuative tree*. In his 2015 survey [67], Jonsson revisited part of the theory of [38] with a more geometric approach which is valid for algebraically closed fields of arbitrary characteristic. Favre and Jonsson gave several descriptions of its tree structure. In our paper [46, Theorem 8.34] we gave a new description of it, as the universal Eggers-Wall tree of Definition 1.6.12. Namely, we proved that the valuative tree could also be obtained as a projective limit of Eggers-Wall trees. The main point of our proof is that $\Theta_L(C)$ embeds naturally in the valuative tree, for any C. We showed also in [46, Theorem 8.18] that the triple $(\mathbf{i}_L, 1 + \mathbf{e}_L, \mathbf{c}_L)$ is the pullback by this embedding of a triple of three natural functions on the valuative tree: the *multiplicity*, the *log-discrepancy* and the *self-interaction*.

An advantage of the identification of Θ_L with the valuative tree is that it allows to get an interpretation of the points of Θ_L which do not belong to any $\Theta_L(C)$ as special *infinitely singular* semivaluations, in the language of [38] and [67].

Another advantage is obtained when the base algebraically closed field has positive characteristic. Let us *define* the functions $\mathbf{i}_L$, $\mathbf{c}_L$ and $\mathbf{e}_L$ on $\theta_\pi(C)$ by the equalities appearing in Proposition 1.6.28. This provides a *definition* of a notion of Eggers-Wall tree in positive characteristic, where Newton-Puiseux series are not enough for the study of plane curve singularities (see Remark 1.2.17). The approach of Sect. 1.6.3 may be generalized to prove that in restriction to $\theta_\pi(C)$, the multiplicity function relative to L is equal to $\mathbf{i}_L$, the contact complexity function relative to L is equal to $\mathbf{c}_L$ and the log-discrepancy function relative to L is equal to $1 + \mathbf{e}_L$. This abstract Eggers-Wall tree may be associated with the *ultrametric distance* on the branches of C, as described in our paper [45]. It may be seen also as a generalization of the notion of characteristic exponents in positive characteristic introduced in Campillo's book [17], where the author computes these exponents using *Hamburger-Noether expansions* (see [17, Section 3.3]), infinitely near points (see [17, Remark 3.3.8]) or Newton polygons (see [17, Section 3.4]).

Assume now that the germ C is holomorphic. Then the Enriques diagram and the weighted dual graph of the minimal embedded resolution, as well as the Eggers-Wall tree relative to generic coordinates encode the same information, which is equivalent to the embedded topological type of C. Proofs of this fundamental fact may be found in Wall's book [131, Propositions 4.3.8 and 4.3.9].

A basic problem is then to find methods to transform one kind of tree into the two other kinds. Noether described in [90] how to pass from the *characteristic exponents* of an irreducible curve singularity C to the structure of the blow up process leading to an embedded resolution. Enriques and Chisini generalized this approach in [35, Libr. IV, Cap. I] to the case when C is not necessarily irreducible. Namely, they showed how to pass from the characteristic exponents of its branches and the orders of coincidence of pairs of branches in generic coordinates to the associated Enriques diagram.

Zariski and Lejeune-Jalabert proved by different methods in their 1971 paper [137] and 1972 thesis [77] respectively, that the characteristic exponents of the branches of C and the intersection numbers of its pairs of branches determine the embedded topological type of C and the combinatorics of its minimal embedded resolution. This may be seen as a proof of the fact that the weighted dual graph of the minimal embedded resolution is equivalent to the generic Eggers-Wall tree. Methods to pass from the knowledge of the characteristic exponents and intersection numbers to the dual graph were explained by Eisenbud and Neumann [34, Appendix to Ch. 1], Brieskorn and Knörrer [15, Section 8.4], Michel and Weber [86], de Jong and Pfister [66, Section 5.4] and an algorithm was described by the first author in [42, Sect. 1.4.6].

Let us mention now several other trees which were associated to plane curve singularities.

As explained in Sect. 1.4.5, the changes of variables considered by Puiseux (called sometimes *Newton maps*) were compositions of affine and of toric ones, which in general were not birational. Nevertheless, an algorithm of abstract resolution and of computation of Newton-Puiseux series may be developed also using them. A variant of the fan trees, adapted to this context and called *Newton trees*, was used by Cassou-Noguès in her papers mentioned in Sect. 1.4.5, written alone or in collaboration. The Newton trees encode also the toroidal pseudo-resolution processes described in the paper [21] of Cassou-Noguès and Libgober. We refer the reader especially to the papers [20] and [22] for more details about this approach. The changes of coordinates (1.71), which are very similar to Newton maps, were also used in the paper [72] of Kennedy and McEwan to study the monodromy of holomorphic plane curve singularities.

Newton maps and Newton trees have been used to study the singularities of quasi-ordinary hypersurfaces by Artal, Cassou-Noguès, Luengo and Melle Hernández (see for instance [10] and [11]). In their 2014 paper [55], the second author and González Villa compared the Newton maps with the toric morphisms appearing in a toroidal pseudo-resolution of an irreducible germ of quasi-ordinary hypersurface.

Newton trees are algebraic variants of the *splice diagrams* associated by Eisenbud and Neumann in their 1986 book [34] to any oriented graph link in an integral homology sphere, extending a graphical convention introduced by Siebenmann in his 1980 paper [114]. In our recent paper [46, Section 5], we explained how to pass from the Eggers-Wall tree of a holomorphic plane curve singularity C relative to a smooth branch L to the splice diagram of the oriented link of $L + C$ in $\mathbb{S}^3$.

In his 1993 papers [69] and [70], Kapranov associated a version of Kuo and Lu's trees to finite sets of formal power series with complex and real coefficients respectively. He called them *Bruhat-Tits trees*.

A version of Kuo and Lu's trees was used recently by Ghys in his book [50] about the topology of *real* plane curve singularities. He associated two such trees, one for $x > 0$ and another one for $x < 0$ to any germ whose branches are smooth and transversal to the reference branch $x = 0$, and studied their relation, describing all the possible couples of such trees. In a theorem proved with Christopher-Lloyd Simon (see [50, Page 266]), Ghys extended this analysis to all plane curve singularities with only real branches. For this more general problem, it was not any more a variant of Kuo and Lu's tree which was crucial, but a real version of the dual graph of the associated minimal resolution. A different real version of the dual resolution graph was introduced before by Castellini in [25, Chap. 3].

Ghys' version of Kuo and Lu's trees was also used by Sorea in her study [116] of curve singularities defined over $\mathbb{R}$ but without any real branch, that is, singularities of real analytic functions $f(x, y)$ in the neighborhood of a local maximum or minimum. Those trees were related in this work with another kind of tree, defined using Morse theory, the so-called *Poincaré-Reeb tree* of the function f relative to x.

Versions of our fan tree were considered by Weber in his 2008 survey [132] about the embedded topological type of holomorphic plane curve singularities, based on the earlier 1985 preprint [86] of Michel and Weber, which contained also many examples. The reading of Weber's survey [132] should facilitate the interpretations of the objects manipulated in this paper in terms of the embedded topological type of C.

1.7 Overview and Perspectives

We begin this final section by an overview of the content of the paper. Then we formulate a few remarks about perspectives of development of the use of lotuses in the study of singularities. The final Sect. 1.7.3 contains a list of notations used in this paper.

1.7.1 Overview

In this subsection we give an overview of the construction of the fan tree and of the associated lotus from the Newton fans generated by a toroidal pseudo-resolution process of a plane curve singularity. It helps us to understand the relations between Newton polygons, Newton-Puiseux series, iterations of blow ups, final exceptional divisor and the associated Enriques diagrams, dual graphs and Eggers-Wall trees.

We invite the reader to look at Fig. 1.58, which combines Figs. 1.35 and 1.37, but without their labels. Let us recall briefly the names and main properties of the

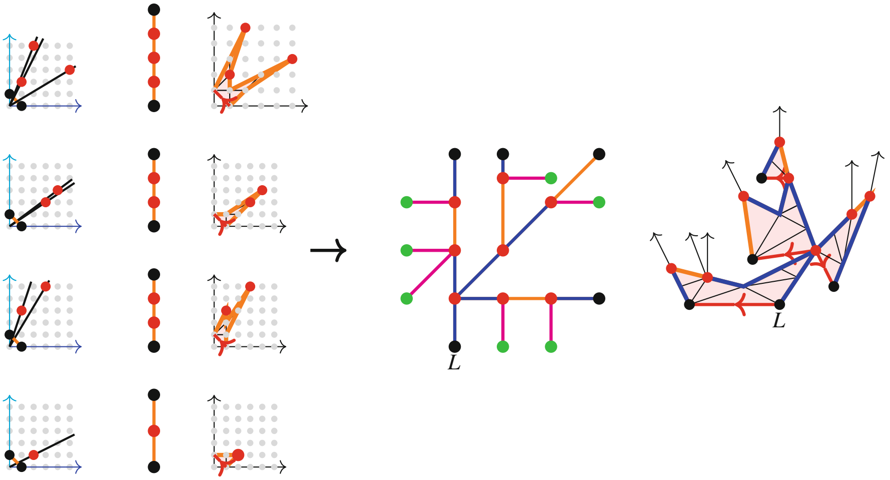


Fig. 1.58 Overview of the constructions of the paper

objects presented in this drawing, which is our way to encode the combinatorics of Algorithm 1.4.22, and how they allow to visualize the relations between Enriques diagrams, dual graphs and Eggers-Wall trees (see Theorem 1.5.29):

1. Given a curve singularity C embedded in a smooth germ of surface S, study it using a *cross* (L, L') (see Definition 1.3.31).
2. Construct the *Newton fan* $\mathcal{F}_{L,L'}(C)$ from the associated *Newton polygon* $\mathcal{N}_{L,L'}(C)$ (see Definitions 1.4.2 and 1.4.14).
3. Draw the *trunk* $\theta_{\mathcal{F}_{L,L'}(C)}$ (see Definition 1.4.32) and the *lotus* $\Lambda(\mathcal{F}_{L,L'}(C))$ (see Definitions 1.5.4 and 1.5.5) of the Newton fan.
4. As a simplicial complex, the lotus of a Newton fan is determined by the continued fraction expansions of the slopes of the fan's rays (see Sect. 1.5.2).
5. Make the *Newton modification* (see Definition 1.4.14) determined by the Newton fan and look at the germs of the strict transform of C at all its intersection points with the exceptional divisor. All those points are smooth on the reduced total transform of $L + L'$. For each such germ of the strict transform of C, complete locally the exceptional divisor into a cross.
6. Each new cross allows to construct again a trunk and a lotus associated to the corresponding germ of the strict transform of C. Combining the corresponding Newton modifications, one gets a new level of Newton modifications.
7. One iterates these constructions until reaching a toroidal surface Σ (see Definition 1.3.29) on which the total transform of C and of all the crosses used during the process is an abstract normal crossings curve, forming the boundary divisor $\partial \Sigma$ of a *toroidal pseudo-resolution* π of C (see Definition 1.4.15). The map π is also a toroidal pseudo-resolution of the *completion* $\hat{C}_\pi = \pi(\partial \Sigma)$ of C relative to π (see Definition 1.4.15), which is a curve singularity containing the branches of C and all the branches whose strict transforms are chosen to define crosses at certain steps of Algorithm 1.4.22.
8. In order to get a global combinatorial view, one constructs the associated *fan tree* $(\theta_\pi(C), \mathbf{S}_\pi)$ (see Definition 1.4.33), by gluing the trunks generated by the toroidal pseudo-resolution process. The function $\mathbf{S}_\pi : \theta_\pi(C) \to [0, \infty]$ is called the *slope function*.
9. The fan tree does not allow to visualize the decomposition of the regularization π^{reg} of π (see Proposition 1.4.29) into blow ups of points. In order to get such a vision, one constructs the *lotus* $\Lambda_\pi(C)$ of the process (see Definition 1.5.26) by gluing the Newton lotuses (see Definition 1.5.4) of the strict transforms of C relative to all the crosses used during the process.
10. The edges of the lotus correspond bijectively to the crosses created during the toroidal embedding resolution process by blow ups of points (see Theorem 1.5.29 (6)). Therefore, one may see the lotus as the space-time of the evolution of the dual graphs of the toroidal surfaces appearing during this process.
11. The graph of the proximity binary relation (see Definition 1.4.31) on the constellation which is blown up is the full subgraph of the 1-skeleton of the lotus $\Lambda_\pi(C)$ on its set of non-basic vertices (see Theorem 1.5.29 (7)).

12. The Enriques diagram (see Definition 1.4.31) of the constellation of infinitely near points blown up in order to decompose π^{reg}, which are the base points of the crosses appearing in the algorithm, is isomorphic with the Enriques tree (see Definition 1.5.26) of the lotus $\Lambda_\pi(C)$.
13. There is a second way of visualizing the Enriques diagram, using a *truncated lotus* $\Lambda_\pi^{tr}(C)$ (see Sect. 1.5.5).
14. The fan tree $\theta_\pi(C)$ is homeomorphic with the lateral boundary $\partial_+\Lambda_\pi(C)$ (see Definition 1.5.26) of the lotus generated by running Algorithm 1.4.22.
15. The lateral boundary $\partial_+\Lambda_\pi(C)$ is isomorphic with the dual graph (see Definition 1.3.22) of the boundary divisor $\partial\Sigma$. There is a simple combinatorial rule for reading on the lotus the self-intersection numbers of the components of the exceptional divisor of the modification π^{reg} (see Theorem 1.5.29 (5)).
16. The fan tree $\theta_\pi(C)$ is also isomorphic with the Eggers-Wall tree $\Theta_L(\hat{C}_\pi)$ (see Definition 1.6.3) of the completion of C relative to the toroidal modification π (see Theorem 1.6.27). The triple of functions (*index* $\mathbf{i}_L$*, exponent* $\mathbf{e}_L$*, contact complexity* $\mathbf{c}_L$) defined on $\Theta_L(\hat{C}_\pi)$ is determined by the *slope function* $\mathbf{S}_\pi$ on the fan tree through explicit formulae (see Proposition 1.6.28).
17. If (L, L') is a cross on S, then the Eggers-Wall tree $\Theta_L(C + L')$ determines the Newton polygon $\mathcal{N}_{L,L'}(C)$ (see Corollary 1.6.17).

1.7.2 Perspectives

In this subsection we give a few perspectives on possible uses of lotuses. We believe that the lotuses of plane curve singularities may be useful in the following research topics:

1. *In the study of the topology of δ-constant deformations of such singularities.* As mentioned in Sect. 1.6.6, Castellini's work [25] gives a first step in this direction. An important advantage of lotuses in this context is that the lotuses of the singularities appearing in the deformations constructed in [25] by A'Campo's method embed in the lotus of the original singularity. This embedding relation is much more difficult to express in terms of classical tree invariants of plane curve singularities. A crucial question is to understand whether this embedding property is specific to A'Campo type deformations, or if it extends to other kinds of δ-constant deformations.
2. *In the analogous study for* real *plane curve singularities.* One should probably describe real variants of the lotuses, embedded canonically up to isotopy in an oriented real plane. Again, Castellini's work [25, Sect. 3.3.2] gives a first step in this direction.
3. *In the extension of the distributive lattice structures described by Pe Pereira and the third author in [96] to arbitrary finite constellations, and in the application of those structures to the problem of adjacency of plane curve singularities.* The natural *operad structure* on the set of finite lotuses associated to toroidal pseudo-

resolution processes (defined by gluing the base of one lotus to an edge of the lateral boundary of another lotus) could be also useful in this direction.

4. *In the study of complex surface singularities through the Hirzebruch-Jung method* (see [103]). This method starts from a finite projection to a germ of smooth surface, and considers then an embedded resolution of the discriminant curve. The lotuses of such discriminant curves could be used as supports for encoding information about the initial finite projection, from which one could read invariants of the surface singularity.

1.7.3 List of Notations

In order to help browsing through the text, we list the notations used for the main objects met in it:

$\left\{ \frac{a}{b} \right\}$	Elementary Newton polygon (see Definition 1.6.14).
$[a_1, \ldots, a_k]$	Continued fraction with terms $a_1, \ldots, a_k$ (see Definition 1.5.17).
$c_m(f)$	Coefficient of the monomial χ^m in the series f (see Definition 1.4.1).
$\mathbf{c}_L$	Contact complexity function (see Definition 1.6.9).
$C_{L,L'}$	Strict transform of C by the Newton modification $\psi^C_{L,L'}$ (see Definition 1.4.14).
$\hat{C}_\pi$	Completion of C relative to the toroidal pseudo-resolution π (see Definition 1.4.15).
$\mathrm{Conv}(Y)$	Convex hull of a subset Y of a real affine space.
χ^m	Monomial with exponent $m \in M$ (see the beginning of Sect. 1.3.2).
∂X	Toric boundary of the toric variety X (see Definition 1.3.18), or toroidal boundary of the toroidal variety X (see Definition 1.3.29).
$\partial_+ \Lambda_\pi(C)$	Lateral boundary of the lotus $\Lambda_\pi(C)$ (see Definition 1.5.5).
$\mathbf{e}_L$	Exponent function (see Definition 1.6.3 and Notations 1.6.7).
f_K	Restriction of f to the compact edge K of its Newton polygon (see Definition 1.4.2).
$\mathcal{F}(f)$	Newton fan of the non-zero series $f \in \mathbb{C}[[x, y]]$ (see Definition 1.4.9).
$\mathcal{F}_{L,L'}(C)$	Newton fan of C relative to the cross (L, L') (see Definition 1.4.14).
$\mathcal{F}^{reg}$	Regularization of the fan $\mathcal{F}$ (see Definition 1.3.8).
$\Gamma(C)$	Enriques diagram of the finite constellation C (see Definition 1.4.31).
$H_{f,\rho}$	Supporting half-plane of the Newton polygon $\mathcal{N}(f)$ determined by the ray $\rho \subset \sigma_0$ (see Proposition 1.4.7).
$\mathbf{i}_L$	Index function (see Definition 1.6.3 and Notations 1.6.7).

$k_x(\xi, \xi')$	Order of coincidence of two Newton-Puiseux series (see Definition 1.6.2).
$k_x(C, C')$	Order of coincidence of two distinct branches, relative to a local coordinate system (x, y) (see Definition 1.6.2).
$l_{\mathbb{Z}}$	Integral length (see Definition 1.3.1).
(L, L')	Cross on a germ of smooth surface (see Definition 1.3.31).
$\Lambda(\mathcal{F})$	Lotus of the Newton fan $\mathcal{F}$ (see Definition 1.5.4).
$\Lambda(\lambda_1, \dots, \lambda_r)$	Lotus associated to the finite set $\{\lambda_1, \dots, \lambda_r\} \subset \mathbb{Q}_+ \cup \{\infty\}$ (see Definition 1.5.4).
$\Lambda_\pi(C)$	Lotus of the toroidal pseudo-resolution π of C (see Definition 1.5.26).
$\Lambda_\pi^{trunc}(C)$	Truncation of the lotus $\Lambda_\pi(C)$ (see Definition 1.5.35).
$m_o(C)$	Multiplicity of the plane curve singularity C at the point o (see Definition 1.2.5).
$M_{L,L'}$	Monomial lattice associated to the cross (L, L'), (see Definition 1.3.32).
$\mathbb{N}$	Set of non-negative integers.
$\mathbb{N}^*$	Set of positive integers.
$N_{L,L'}$	Weight lattice associated to the cross (L, L') (see Definition 1.3.32).
$\mathcal{N}(f)$	Newton polygon of the non-zero series $f \in \mathbb{C}[[x, y]]$ (see Definition 1.4.2).
$\mathcal{N}_{L,L'}(C)$	Newton polygon of C relative to the cross (L, L') (see Definition 1.4.14).
O_ρ	Toric orbit associated to the cone ρ of a fan (see the relation (1.22)).
$\hat{O}_{S,o}$	Completed local ring of the complex surface S at the point o (see Definition 1.2.5).
$\pi^*(C)$	Total transform of a plane curve singularity C by a modification π (see Definition 1.2.31).
$\psi_\sigma^{\mathcal{F}}$	Toric morphism from $X_{\mathcal{F}}$ to X_σ associated to any fan $\mathcal{F}$ which subdivides the cone σ (see relation (1.25)).
$\psi_{L,L'}^C$	Newton modification defined by C relative to the cross (L, L') (see Definition 1.4.14).
$\mathbb{R}_+$	Set of non-negative real numbers.
$\mathcal{S}(f)$	Support of the power series $f \in \mathbb{C}[[x, y]]$ (see Definition 1.4.1).
$\mathbf{S}_\pi$	Slope function of the toroidal pseudo-resolution π of C (see Definition 1.4.33).
σ_0	Regular cone generated by the canonical basis of the lattice $\mathbb{Z}^2$.
$\sigma_0^{L,L'}$	Regular cone generated by the canonical basis of the lattice $N_{L,L'}$ (see Definition 1.3.32).
t^w	One parameter subgroup of the algebraic torus $\mathcal{T}_N$, corresponding to the weight vector $w \in N$ (see the beginning of Sect. 1.3.2).

$\mathcal{T}_N$	Complex algebraic torus with weight lattice N (see formula (1.16)).
trop^f	Tropicalization of the non-zero power series $f \in \mathbb{C}[[x, y]]$ (see Definition 1.4.4).
$\mathrm{trop}^C_{L,L'}$	Tropical function of the curve singularity C relative to the cross (L, L') (see Definition 1.4.14).
$\theta(\mathcal{F})$	Trunk of the fan $\mathcal{F}$ (see Definition 1.4.32).
$\theta_\pi(C)$	Fan tree of the toroidal pseudo-resolution π of C (see Definition 1.4.33).
$\Theta_L(C)$	Eggers-Wall tree of the plane curve singularity C relative to the smooth branch L (see Definition 1.6.3 and Notations 1.6.7).
Θ_L	Universal Eggers-Wall tree (see Definition 1.6.12).
X_σ	Affine toric variety defined by the fan consisting of the faces of the cone σ (see Definition 1.3.14).
$X_{\mathcal{F}}$	Toric variety defined by the fan $\mathcal{F}$ (see Definition 1.3.15).
$\uplus$	Operation of the monoid of abstract lotuses (see formula (1.50)).
$\wedge$	Operation on the set $\mathbb{Q}_+^*$ allowing to describe the intersection of Newton lotuses (see formula (1.48)).
$Z(f)$	Zero-locus of a holomorphic function f or of a formal germ $f \in \hat{\mathcal{O}}_{S,o}$.
$\mathcal{Z}_x(C)$	Set of Newton-Puiseux roots of a plane curve singularity C relative to a local coordinate system (x, y) (see Definition 1.6.2).

Acknowledgments This research was partially supported by the French grants ANR-17-CE40-0023-02 LISA and Labex CEMPI (ANR-11-LABX-0007-01), and also by the Spanish grants MTM2016-80659-P, MTM2016-76868-C2-1-P and SEV-2015-0554. We are grateful to Bernard Teissier for having invited us, when we were his PhD students, to explore the fascinating combinatorial aspects of plane curve singularities, and for everything he taught us about this world. We are grateful to Ana Belén de Felipe and Roberto Castellini for their remarks on lotuses. We thank Patricio Almirón, Pierrette Cassou-Noguès, Manuel González Villa, Carlos Guzmán Durán, Lê Dũng Tráng and Camille Plénat for their comments on preliminary versions of this paper. We are grateful to Dale Cutkosky for several explanations concerning morphisms in the toroidal category. We thank Jose Seade for the invitation to publish this work as a chapter of the "*Handbook of Geometry and Topology of Singularities*", and for his remarks which were very helpful in adapting it to the spirit of this handbook. We thank also the two anonymous referees for their useful recommendations and remarks.

References

1. Abhyankar, S. S. *Concepts of order and rank on a complex space, and a condition for normality.* Math. Ann. 141 (1960), 171–192. 21
2. Abhyankar, S. S., Moh, T. *Newton-Puiseux Expansion and Generalized Tschirnhausen Transformation.* J. Reine Angew. Math. **260** (1973), 47–83; **261** (1973), 29–54. 77
3. Abhyankar, S. S., Moh, T. *Embeddings of the Line in the Plane.* J. Reine Angew. Math. **276** (1975), 148–166. 77

4. Abramovich, D., Karu, K., Matsuki, K., Włodarczyk, J. *Torification and factorization of birational maps.* Journal of the A.M.S. **15** No. 3 (2002), 531–572. 48
5. A'Campo, N. *Sur la monodromie des singularités isolées d'hypersurfaces complexes.* Invent. Math. **20** (1973), 147–169. 3
6. A'Campo, N. *Le groupe de monodromie du déploiement des singularités isolées de courbes planes II.* Proc. of the International Congress of Mathematicians, Vancouver, 1974, 395–404. 108
7. A'Campo, N. *Le groupe de monodromie du déploiement des singularités isolées de courbes planes I.* Math. Annalen **213** No. 1 (1975), 1–32. 108
8. A'Campo, N., Oka, M. *Geometry of plane curves via Tschirnhausen resolution tower.* Osaka J. Math. **33** (1996), 1003–1033. 4, 64, 77, and 78
9. Aroca, F., Gómez-Morales, M., Shabbir, K. *Torical modification of Newton non-degenerate ideals.* Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, **107**, No. 1 (2013), 221–239. 78
10. Artal Bartolo, E., Cassou-Noguès, P., Luengo, I., Melle Hernández, A. *Quasi-ordinary singularities and their zeta functions*, Mem. Amer. Math. Soc., **178**, (2005). 138
11. Artal Bartolo, E., Cassou-Noguès, P., Luengo, I., Melle Hernández, A. *On quasi-ordinary singularities and Newton trees*, Moscow Math. J., **13** No.3 (2013), 365–398. 138
12. Barber, S. F., Zariski, O. *Reducible exceptional curves of the first kind*. American J. of Maths. **57** (1935), 119–141. 107
13. Brauner, K. *Zur Geometrie der funktionen zweier komplexen Veränderlichen III, IV.* Abh. Math. Sem. Hamburg **6** (1928), 8–54. 3
14. Brieskorn, E. *Singularities in the work of Friedrich Hirzebruch.* Surveys in Differential Geom. **VII** (2000), 17–60. 48
15. Brieskorn, E., Knörrer, H. *Plane algebraic curves.* Translated from the German by John Stillwell. Birkhäuser, 1986. 3, 11, 21, and 137
16. Brocot, A. *Calcul des rouages par approximation, nouvelle méthode.* Revue Chrono-métrique **6** (1860), 186–194. 108
17. Campillo, A. *Algebroid Curves in Positive Characteristic.* Lecture Notes in Maths. **813**. Springer, Berlin, 1980. 137
18. Campillo, A., Castellanos, J. *Curve singularities. An algebraic and geometric approach.* Hermann, Paris, 2005. 106
19. Casas-Alvero, E. *Singularities of plane curves.* London Math. Soc. Lecture Note Series **276**. Cambridge Univ. Press, 2000. 21, 64, 69, and 106
20. Cassou-Noguès, P., Płoski, A. *Invariants of plane curve singularities and Newton diagrams.* Univ. Iagel. Acta Math. No. **49** (2011), 9–34. 77 and 138
21. Cassou-Noguès, P., Libgober, A. *Multivariable Hodge theoretical invariants of germs of plane curves. II.* In *Valuation theory in interaction*, A. Campillo, F.-V. Kuhlmann, B. Teissier, eds. EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2014, 82–135. 4, 77, and 138
22. Cassou-Noguès, P., Veys, W. *Newton trees for ideals in two variables and applications.*, Proc. London Math. Society **108**, 4, (2014) 869–910. 77 and 138
23. Cassou-Noguès, P., Veys, W. *The Newton tree: geometric interpretation and applications to the motivic zeta function and the log canonical threshold.* Math. Proc. Cambridge Philos. Soc. **159** (2015), No. 3, 481–515. 77
24. Cassou-Noguès, P., Raibaut, M. *Newton Transformations and the Motivic Milnor Fiber of a Plane Curve.* In *Singularities, Algebraic Geometry, Commutative Algebra and Related Topics. Festschrift for Antonio Campillo on the Occasion of his 65th Birthday.* G.-M. Greuel, L. Narváez and S. Xambó-Descamps eds. Springer, 2018, 145–198. 77
25. Castellini, R. *La topologie des déformations de A'Campo des singularités : une approche par le lotus.* Doctoral Thesis in English, Univ. Lille 1, Sept. 2015. Available at: https://tel.archives-ouvertes.fr/tel-01207005/file/these_castelliniroberto.pdf 106, 108, 138, and 141
26. Cox, D., Little, J., Schenck, H. *Toric varieties.* Springer, 2011. 30, 37, 39, and 48

27. Cramer, G. *Introduction à l'analyse des lignes courbes algébriques.* Frères Cramer et Cl. Philibert, Genève, 1750. 75
28. Cutkosky, S. D. *Resolution of singularities.* Graduate Studies in Maths. **63**. American Math. Society, 2004. 15
29. Deligne, P. *Intersections sur les surfaces régulières*. In *Groupes de monodromie en géométrie algébrique.* SGA 7 II, Lect. Notes in Maths. **340**, Springer, Berlin, 1973, 1–37. 108
30. Dumas, G. *Sur quelques cas d'irréductibilité des polynômes à coefficients rationnels.* Journ. Math. Pures Appl., 6e série, **2** (1906), 191–258. 117
31. Dumas, G. *Sur la résolution des singularités de surfaces.* C.R. Acad. Sci. Paris **152** (1911), 682–684. 48
32. Dumas, G. *Sur les singularités des surfaces.* C.R. Acad. Sci. Paris **154** (1912), 1495–1497. 48
33. Eggers, H. *Polarinvarianten und die Topologie von Kurvensingularitaeten.* Bonner Math. Schriften **147**, 1983. 135 and 136
34. Eisenbud, D., Neumann, W. *Three-dimensional link theory and invariants of plane curve singularities.* Princeton Univ. Press, 1985. 3, 134, 137, and 138
35. Enriques, F., Chisini, O. *Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche II.* Zanichelli, Bologna, 1917. 6, 69, 105, 106, 107, and 137
36. Epple, M. *Branch points of algebraic functions and the beginnings of modern knot theory.* Historia Math. **22** (1995), 371–401. 3
37. Ewald, G. *Combinatorial convexity and algebraic geometry*, Graduate Texts in Maths. **168**, Springer, 1996. 30 and 51
38. Favre, C., Jonsson, M. *The valuative tree.* LNM **1853**. Springer-Verlag, Berlin, 2004. 136 and 137
39. Fischer, G. *Plane algebraic curves.* Student Math. Library **15**. American Math. Soc., 2001. 11 and 15
40. Fock, V. V., Goncharov, A. B. *Cluster Poisson varieties at infinity.* Selecta Math. (N.S.) **22** (2016), no. 4, 2569–2589. 110
41. Fulton, W. *Introduction to toric varieties.* Princeton Univ. Press, 1993. 30, 36, 37, and 39
42. García Barroso, E. R. *Invariants des singularités de courbes planes et courbure des fibres de Milnor.* Tesis, Univ. La Laguna, 1996. See also in *Courbes polaires et courbure des fibres de Milnor des courbes planes.* Thèse, Univ. Paris 7, 2000. Available at http://ergarcia.webs.ull.es/tesis.pdf. 136 and 138
43. García Barroso, E. R., González Pérez, P. D. *Decomposition in bunches of the critical locus of a quasi-ordinary map*, Compositio Math. **141**, (2005), 461–486. 136
44. García Barroso, E. R., González Pérez, P. D., Popescu-Pampu, P. *Variations on inversion theorems for Newton-Puiseux series*, Math. Ann. **368**, (2017), no. 3–4, 1359–1397. 134
45. García Barroso, E. R., González Pérez, P. D., Popescu-Pampu, P. *Ultrametric spaces of branches on arborescent singularities*. In *Singularities, Algebraic Geometry, Commutative Algebra and Related Topics. Festschrift for Antonio Campillo on the Occasion of his 65th Birthday.* G.-M. Greuel, L. Narváez and S. Xambó-Descamps eds. Springer, 2018, 55–106. 111, 114, 136, and 137
46. García Barroso, E. R., González Pérez, P. D., Popescu-Pampu, P. *The valuative tree is the projective limit of Eggers-Wall trees.* Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. **113** (2019), 4051–4105. 3, 111, 115, 136, 137, and 138
47. García Barroso, E. R., Gwoździewicz, J. *Quasi-ordinary singularities: tree model, discriminant, and irreducibility.* Int. Math. Res. Not. IMRN 2015, no. **14**, 5783–5805. 136
48. García Barroso, E. R., Gwoździewicz, J. *Higher order polars of quasi-ordinary singularities.* arXiv:1907.03249. 136
49. García Barroso, E. R., Płoski, A. *An approach to plane algebroid branches*. Rev. Mat. Complut., **28** (1) (2015), 227–252. 78
50. Ghys, É. *A singular mathematical promenade.* ENS Éditions, 2017. Available at http://perso.ens-lyon.fr/ghys/promenade/ 76 and 138

51. Goldin, R., Teissier, B. *Resolving singularities of plane analytic branches with one toric morphism*. Resolution of singularities (Obergurgl, 1997), 315–340, Progr. Math. **181**, Birkhäuser, Basel, 2000. 57 and 78
52. González Pérez, P. D. *Toric embedded resolutions of quasi-ordinary hypersurface singularities.* Ann. Inst. Fourier, Grenoble **53**, 6 (2003), 1819–1881. 4, 77, 78, and 136
53. González Pérez, P. D. *Approximate roots, toric resolutions and deformations of a plane branch.* J. Math. Soc. Japan **62** (2010), no. 3, 975–1004. Erratum in J. Math. Soc. Japan **65** (2013), no. 3, 773–774. 77
54. González Pérez, P. D., Risler, J.-J. *Multi-Harnack smoothings of real plane branches.* Ann. Sci. Éc. Norm. Supér. (4) **43** (2010), no. 1, 143–183. 77
55. González Pérez, P. D., González Villa, M. *Motivic Milnor fiber of a quasi-ordinary hypersurface*, J. Reine Angew. Math. **687** (2014), 159–205. 138
56. González Pérez, P. D., Teissier, B. *Toric geometry and the Semple-Nash modification* Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, **108**, (2014), 1–48. 39
57. Graham, R. L., Knuth, D. E., Patashnik, O. *Concrete Mathematics. A foundation for computer science.* Second Edition. Addison-Wesley Publishing Co., Reading, MA, 1994. 108
58. Greco, S. *Normal varieties.* Academic Press, 1978. 12
59. Greuel, G.-M., Lossen, C., Shustin, E. *Introduction to singularities and deformations.* Springer, 2007. 15 and 64
60. Gwoździewicz, J., Płoski, A. *On the approximate roots of polynomials.* Ann. Pol. Math. **60** (1995) No. 3, 199–210. 78
61. Hartshorne, R. *Algebraic geometry.* Springer, 1977. 19, 20, 21, 22, 69, and 122
62. Hirzebruch, F. *Über vierdimensionale Riemannsche Flächen Mehrdeutiger analytischer Funktionen von zwei komplexen Veränderlichen.* Math. Ann. **126** (1953), 1–22. 48 and 110
63. Hodge, W. V. D. *The isolated singularities of an algebraic surface.* Proc. London Math. Soc. **30** (1930), 133–143. 48
64. Hopf, H. *Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche.* Math. Ann. **104** (1931), 637–665. 17
65. Jensen, A. N., Markwig, H., Markwig, T. *An algorithm for lifting points in a tropical variety.* Collect. Math. **59**, 2 (2008), 129–165. 79
66. de Jong, T., Pfister, G. *Local analytic geometry.* Advanced Lectures in Maths. Friedr. Vieweg & Sohn, 2000. 8, 9, 11, 12, 13, 15, 16, 21, 23, and 138
67. Jonsson, M. *Dynamics on Berkovich spaces in low dimensions.* In *Berkovich spaces and applications*, 205–366. A. Ducros, C. Favre, J. Nicaise eds., Lect. Notes in Maths. **2119**, 2015. 136 and 137
68. Jung, H. W. E. *Darstellung der Funktionen eines algebraischen Körpers zweier unabhängigen Veränderlichen* x, y *in der Umgebung einer stelle* $x = a, y = b$. J. Reine Angew. Math. **133** (1908), 289–314. 48 and 110
69. Kapranov, M. M. *Veronese curves and Grothendieck-Knudsen moduli space* $\overline{M}_{0,n}$. J. Algebraic Geom. **2** (1993), no. 2, 239–262. 138
70. Kapranov, M. M. *The permutoassociahedron, Mac Lane's coherence theorem and asymptotic zones for the KZ equation.* J. Pure Appl. Algebra **85** (1993), no. 2, 119–142. 138
71. Kempf, G., Knudsen, F. F., Mumford, D. and Saint-Donat, B. *Toroidal embeddings. I*, Lecture Notes in Mathematics, Vol. **339**, Springer-Verlag, Berlin, 1973. 48
72. Kennedy, G., McEwan, L. J. *Monodromy of plane curves and quasi-ordinary surfaces*. J. Singul. **1** (2010), 146–168. 138
73. Khovanskii, A. G. *Newton polyhedra, and toroidal varieties.* Funkcional. Anal. i Prilozen. **11** (1977), no. 4, 56–64, 96. English translation in Functional Anal. Appl. **11** (1977), no. 4, 289–296 (1978). 48 and 78
74. Kouchnirenko, A. G. *Polyèdres de Newton et nombres de Milnor.* Inv. Math. **32** (1976), 1–31. 48 and 78
75. Kuo, T. C., Lu, Y. C. *On analytic function germs of two complex variables.* Topology **16** (1977), no. 4, 299–310. 135
76. Laufer, H. B. *Normal two-dimensional singularities.* Princeton Univ. Press, 1971. 19

77. Lejeune-Jalabert, M. *Sur l'équivalence des singularités des courbes algébriques planes. Coefficients de Newton*. In *Introduction à la théorie des singularités I.* Thesis Paris VII (1972). Published in Travaux en Cours **36**. Hermann (1988), 49–124. 78 and 137
78. Lejeune-Jalabert, M. *Linear systems with infinitely near base conditions and complete ideals in dimension two.* In *Singularity theory.* Trieste 1991. D.T. Lê, K. Saito, B. Teissier eds., World Scientific, 1995, 345–369. 64, 69, and 107
79. Lejeune-Jalabert, M., Teissier, B. *Transversalité, polygone de Newton, et installations.* In *Singularités à Cargèse* (Rencontre Singularités Géom. Anal., Inst. Études Sci. de Cargèse, 1972), 75–119. Astérisque **7–8**, Soc. Math. France, Paris, 1973. 79
80. Lê D. T. *Sur les nœuds algébriques.* Compositio Math. **25** (1972), 281–321. 3
81. Lê D. T. *Plane curve singularities and carrousels.* Ann. Inst. Fourier, Grenoble **53** No. 4 (2003), 1117–1139. 3
82. Lê D. T., Michel, F., Weber, C. *Sur le comportement des polaires associées aux germes de courbes planes.* Compos. Math. **72** No. 1 (1989), 87–113. 110
83. Lê D. T., Oka, M. *On resolution complexity of plane curves*, Kodai Math. J., **18**, (1995), no. 1, 1–36. 4, 77, and 78
84. Mac Lagan, D., Sturmfels, B. *Introduction to tropical geometry.* American Math. Soc., 2016. 51
85. Maurer, J. *Puiseux expansion for space curves*. Manuscripta Math. **32** (1980), 91–100. 79
86. Michel, F., Weber, C. *Topologie des germes de courbes planes à plusieurs branches.* Preprint, Univ. de Genève, 1985. 137 and 139
87. Neumann, W., Wahl, J. *Complex surface singularities with integral homology sphere links.* Geom. Topol. **9** (2005), 757–811. 110
88. Newton, I. *The method of fluxions and infinite series.* Printed by H. Woodfall and sold by J. Nourse, London, 1736. Translated into french by M. Buffon, Debure libraire, 1740: *La méthode des fluxions et des suites infinies.* 74, 75, and 76
89. Noether, M. *Ueber die singulären Werthsysteme einer algebraischen Function und die singulären Punkte einer algebraischen Curve.* Math. Annalen **9** (1875), 166–182. 105
90. Noether, M. *Les combinaisons caractéristiques dans la transformation d'un point singulier.* Rend.Circ.Mat.Palermo **IV** (1890), 89–108, 300–301. 105, 116, and 137
91. Oda, T. *Convex bodies and algebraic geometry.* Springer, 1988. 30, 32, 33, 39, 41, and 45
92. Oka, K. *Sur les fonctions analytiques de plusieurs variables. VIII. Lemme fondamental.* J. Math. Soc. Japan **3** (1951), 204–214. 21
93. Oka, M. *Geometry of plane curves via toroidal resolution.* In *Algebraic geometry and singularities.* Progress in Math. **134**, A. Campillo López and L. Narváez Macarro eds., Birkhäuser, 1996, 95–121. 4 and 77
94. Oka, M. *Non-degenerate complete intersection singularity*. Hermann, Paris, 1997. 26, 33, 77, and 78
95. Oka, M. *Introduction to Plane Curve Singularities. Toric Resolution Tower and Puiseux Pairs.* In *Arrangements, local systems and singularities.*, CIMPA Summer School, Galatasaray Univ, Instanbul (2007), F. El Zein et al. eds., Birkhäuser, 2010, 209–245. 77
96. Pe Pereira, M., Popescu-Pampu, P. *Fibonacci numbers and self-dual lattice structures for plane branches.* In *Bridging Algebra, Geometry and Topology*, D. Ibadula et W. Veys Éditeurs, Springer Proceedings in Mathematics and Statistics **96**, 2014, 203–230. 69, 106, and 141
97. Płoski, A. *Introduction to the local theory of plane algebraic curves.* In *Analytic and algebraic geometry*. 115–134, Faculty of Maths. and Computer Science. University of Łódź, 2013. 14
98. Popescu-Pampu, P. *Arbres de contact des singularités quasi-ordinaires et graphes d'adjacence pour les 3-variétés réelles*. Thèse, Univ. Paris 7, 2001. Available at https://tel.archives-ouvertes.fr/tel-00002800v1. 136
99. Popescu-Pampu, P. *Approximate roots.* In *Valuation theory and its applications.* F.V. Kuhlmann et al. eds. Fields Inst. Communications **33**, AMS 2003, 285–321. 78 and 134
100. Popescu-Pampu, P. *Sur le contact d'une hypersurface quasi-ordinaire avec ses hypersurfaces polaires.* Journal of the Inst. of Math. Jussieu **3** (2004), 105–138. 136

101. Popescu-Pampu, P. *The geometry of continued fractions and the topology of surface singularities*. Dans *Singularities in Geometry and Topology 2004*. Advanced Studies in Pure Mathematics **46**, 2007, 119–195. 110
102. Popescu-Pampu, P. *Le cerf-volant d'une constellation.* L'Ens. Math. **57** (2011), 303–347. 69, 80, 81, 97, 101, 103, 104, 106, 107, and 110
103. Popescu-Pampu, P. *Introduction to Jung's method of resolution of singularities.* In *Topology of Algebraic Varieties and Singularities.* Proceedings of the conference in honor of the 60th birthday of Anatoly Libgober. J. I. Cogolludo-Agustin and E. Hironaka eds. Contemporary Mathematics **538**, AMS, 2011, 401–432. 142
104. Popescu-Pampu, P. *From singularities to graphs.* arXiv:1808.00378. 42 and 105
105. Popescu-Pampu, P., Stepanov, D. *Local tropicalization*. In *Algebraic and Combinatorial aspects of Tropical Geometry.* Proceedings Castro Urdiales 2011, E. Brugallé, M.A. Cueto, A. Dickenstein, E.M. Feichtner and I. Itenberg editors, Contemp. Maths. **589**, AMS, 2013, 253–316. 51 and 79
106. Puiseux, V. *Recherches sur les fonctions algébriques.* Journal de maths. pures et appliquées (de Liouville) **15** (1850), 365–480. 48 and 76
107. Reeve, J. *A summary of results in the topological classification of plane algebroid singularities*. Rendiconti Sem. Mat. Torino **14** (1954–55), 159–187. 3
108. Saito, M. *Exponents of an irreducible plane curve singularity.* arXiv:math.AG/0009133. 108
109. Samelson, H. $\pi_3(\mathbb{S}^2)$*, H. Hopf, W. K. Clifford, F. Klein.* In *History of topology*, 575–578, North-Holland, Amsterdam, 1999. 17
110. Schrauwen, R. *Topological series of isolated plane curve singularities.* Enseign. Math. (2) **36** No. 1–2 (1990), 115–141. 3
111. Schulze-Röbbecke, T. *Algorithmen zur Auflösung und Deformation von Singularitäten ebener Kurven.* Bonner Math. Schriften **96**, 1977. 87 pages. 108
112. Semple, J. G. *Singularities of space algebraic curves*. Proc. London Math. Soc. **44** (1938), 149–174. 106
113. Shafarevich, I. *Basic algebraic geometry.* Vol. 1,2. Springer-Verlag, 1994. 11 and 69
114. Siebenmann, L. C. *On vanishing of the Rohlin invariant and nonfinitely amphicheiral homology 3-spheres.* Topology Symposium, Siegen 1979 (Proc. Sympos., Univ. Siegen, Siegen, 1979), pp. 172–222, Lecture Notes in Math. **788**, Springer, Berlin, 1980. 138
115. Smith, H. J. S. *On the higher singularities of plane curves.* Proc. London Math. Soc. **6** (1874), 153–182. 116
116. Sorea, M.-Ş. *The shapes of level curves of real polynomials near strict local minima.* PhD Thesis, Univ. Lille 1, Oct. 2018. Available at: https://tel.archives-ouvertes.fr/tel-01909028 138
117. Stern, M. A. *Über eine zahlentheoretische Funktion.* J. Reine Angew. Math. **55** (1858), 193–220. 108
118. Stolz, O. *Die Multiplicität der Schnittpunkte zweier algebraischer Curven*. Math. Annalen **15** (1879), 122–160. 116
119. Teissier, B. *Appendix* to: Zariski, O. *Le problème des modules pour les branches planes.* École Polytechnique, Paris, 1973. iii+199 pp. Deuxième édition, Hermann, Paris, 1986. An English translation by Ben Lichtin was published in 2006 by the AMS with the title *The moduli problem for plane branches*. 48 and 78
120. Teissier, B. *The hunting of invariants in the geometry of discriminants.* In *Real and complex singularities.* (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), Sijthoff and Noordhoff, Alphen aan den Rijn, 1977, 565–678. 136
121. Teissier, B. *Introduction to curve singularities.* In *Singularity theory.* Trieste 1991. D.T. Lê, K. Saito, B. Teissier eds., World Scientific, 1995, 866–893. 15
122. Teissier, B. *Monomial ideals, binomial ideals, polynomial ideals.* In *Trends in commutative algebra*, 211–246, Math. Sci. Res. Inst. Publ. **51**, Cambridge Univ. Press, Cambridge, 2004. 78
123. Teissier, B. *Complex curve singularities: a biased introduction.* In *Singularities in geometry and topology* , 825–887, World Scientific Publishing, Hackensack, NJ, 2007. 15 and 63

124. Teissier, B. *Overweight deformations of affine toric varieties and local uniformization.* In *Valuation theory in interaction*, 474–565, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2014. 78
125. Tevelev, J. *On a question of B. Teissier.* Collect. Math. **65** (2014), no. 1, 61–66. 78
126. Du Val, P. *Reducible exceptional curves.* Amer. J. Math. **58** (1936) No. 2, 285–289. 105 and 107
127. Du Val, P. *On absolute and non-absolute singularities of algebraic surfaces.* Revue de la Faculté des Sciences de l'Univ. d'Istanbul (A) **91** (1944), 159–215. 107
128. Varchenko, A. *Zeta-function of monodromy and Newton's diagram*. Inv. Math. **37** (1976), 253–262. 48 and 78
129. Veys, W. *Zeta functions for curves and log canonical models.* Proc. London Math. Soc. (3) **74** (1997), no. 2, 360–378. 77
130. Wall, C. T. C. *Chains on the Eggers tree and polar curves.* Proc. of the Int. Conf. on Algebraic Geometry and Singularities (Sevilla, 2001). Rev. Mat. Iberoamericana **19** (2003), no. 2, 745–754. 136
131. Wall, C. T. C. *Singular points of plane curves.* London Math. Society Student Texts **63**. Cambridge Univ. Press, 2004. 3, 6, 11, 19, 21, 22, 64, 108, 116, 136, and 137
132. Weber, C. *On the topology of singularities.* In *Singularities II. Geometric and topological aspects.* J.-P. Brasselet et al. eds., Contemporary Math. **475** (2008), 217–251. 3, 134, and 139
133. Weber, C. *Lens spaces among 3-manifolds and quotient surface singularities.* Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **112** (2018), no. 3, 893–914. 110
134. Zariski, O. *Algebraic surfaces.* Springer-Verlag, 1935. A second supplemented edition appeared in 1971. 6 and 105
135. Zariski, O. *Polynomial ideals defined by infinitely near base points.* Amer. J. Math. **60** No. 1 (1938), 151–204. 105
136. Zariski, O. *The reduction of the singularities of an algebraic surface.* Ann. Math. **40** No. 3 (1939), 639–689. 64, 77, and 107
137. Zariski, O. *General Theory of Saturation and of Saturated Local Rings II: Saturated Local Rings of Dimension 1*. Amer. J. Math. **93** No. 4 (1971), 872–964. 64 and 137

Chapter 2
The Topology of Surface Singularities

Françoise Michel

Contents

Abstract We consider a reduced complex surface germ (X, p). We do not assume that X is normal at p, and so, the singular locus (Σ, p) of (X, p) could be one dimensional. This text is devoted to the description of the topology of (X, p). By the conic structure theorem (see Milnor, *Singular Points of Complex Hypersurfaces*, Annals of Mathematical Studies 61 (1968), Princeton Univ. Press), (X, p) is homeomorphic to the cone on its link L_X. First of all, for any good resolution $\rho : (Y, E_Y) \to (X, 0)$ of (X, p), there exists a factorization through the normalization $\nu : (\bar{X}, \bar{p}) \to (X, 0)$ (see H. Laufer, *Normal two dimensional singularities*, Ann.

F. Michel (✉)
Laboratoire de Mathématiques Emile Picard, Université Paul Sabatier, Toulouse, France
e-mail: fmichel@picard.ups-tlse.fr

J. L. Cisneros Molina et al. (eds.), *Handbook of Geometry and Topology of Singularities I*, https://doi.org/10.1007/978-3-030-53061-7_2

of Math. Studies **71**, (1971), Princeton Univ. Press., Thm. 3.14). This is why we proceed in two steps.

1. When (X, p) a normal germ of surface, p is an isolated singular point and the link L_X of (X, p) is a well defined differentiable three-manifold. Using the good minimal resolution of (X, p), L_X is given as the boundary of a well defined plumbing (see Sect. 2.2) which has a negative definite intersection form (see Hirzebruch et al., *Differentiable manifolds and quadratic forms*, Math. Lecture Notes, vol 4 (1972), Dekker, New-York and Neumann, *A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves*, Trans. Amer. Math. Soc. **268** (1981), p. 299–344).
2. In Sect. 2.3, we use a suitably general morphism, $\pi : (X, p) \to (\mathbb{C}^2, 0)$, to describe the topology of a surface germ (X, p) which has a 1-dimensional singular locus (Σ, p). We give a detailed description of the quotient morphism induced by the normalization ν on the link $L_{\bar{X}}$ of $(\bar{X}, \bar{p})$ (see also Sect. 2.2 in Luengo-Pichon, *Lê 's conjecture for cyclic covers*, Séminaires et congrès 10, (2005), p. 163–190. Publications de la SMF, Ed. J.-P. Brasselet and T. Suwa).

In Sect. 2.4, we give a detailed proof of the existence of a good resolution of a normal surface germ by the Hirzebruch-Jung method (Theorem 2.4.6). With this method a good resolution is obtained via an embedded resolution of the discriminant of π (see Friedrich Hirzebruch, *Über vierdimensionale Riemannsche Flächen mehrdeutiger analytischer Funktionen von zwei komplexen Veränderlichen*, Math. Ann. 126 (1953) p. 1–22). An example is given Sect. 2.6. An appendix (Sect. 2.5) is devoted to the topological study of lens spaces and to the description of the minimal resolution of quasi-ordinary singularities of surfaces. Section 2.5 provides the necessary background material to make the proof of Theorem 2.4.6 self-contained.

2.1 Introduction

Let I be a reduced ideal in $\mathbb{C}\{z_1, \ldots, z_n\}$ such that the quotient algebra $A_X = \mathbb{C}\{z_1, \ldots, z_n\}/I$ is two-dimensional. The zero locus, at the origin 0 of $\mathbb{C}^n$, of a set of generators of I is an analytic surface germ embedded in $(\mathbb{C}^n, 0)$. Let $(X, 0)$ be its intersection with the compact ball B_ϵ^{2n} of radius a sufficiently small ϵ, centered at the origin in $\mathbb{C}^n$, and L_X its intersection with the boundary S_ϵ^{2n-1} of B_ϵ^{2n}. Let Σ be the set of the singular points of $(X, 0)$.

As I is reduced Σ is empty when $(X, 0)$ is smooth, it is equal to the origin when 0 is an isolated singular point, it is a curve when the germ has a non-isolated singular locus (in particular we do not exclude the cases of reducible germs).

If Σ is a curve, $K_\Sigma = \Sigma \cap S_\epsilon^{2n-1}$ is the disjoint union of r one-dimensional circles (r being the number of irreducible components of Σ) embedded in L_X. We say that K_Σ is the link of Σ. By the conic structure theorem (see [18]), for a sufficiently small ϵ, $(X, \Sigma, 0)$ is homeomorphic to the cone on the pair (L_X, K_Σ) and to the cone on L_X when $\Sigma = \{0\}$.

On the other hand, thanks to A. Durfee [7], the homeomorphism class of $(X, \Sigma, 0)$ depends only on the isomorphism class of the algebra A_X (i.e. is independent of a sufficiently small ϵ and of the choice of the embedding in $(\mathbb{C}^n, 0)$). We say that the analytic type of $(X, 0)$ is given by the isomorphism class of A_X and, we say that its topological type is given by the homeomorphism class of the pair $(X, 0)$ if 0 is an isolated singular point, and by the homeomorphism class of the triple $(X, \Sigma, 0)$ if the singular locus Σ is a curve.

Definition 2.1.1 **The link of** $(X, 0)$ is the homeomorphism class of L_X if 0 is an isolated singular point (in particular if $(X, 0)$ is normal at 0), and is the homeomorphic class of the pair (L_X, K_Σ) if the singular locus Σ is a curve.

This paper is devoted to the description of the link of $(X, 0)$.

2.1.1 Good Resolutions

Definition 2.1.2 A morphism $\rho : (Y, E_Y) \to (X, 0)$ where $E_Y = \rho^{-1}(0)$ is the exceptional divisor of ρ, is a **good resolution** of $(X, 0)$ if :

1. Y is a smooth complex surface,
2. the total transform $\rho^{-1}(\Sigma) =: E_Y^+$ is a normal crossing divisor with smooth irreducible components.
3. the restriction of ρ to $Y \setminus E_Y^+$ is an isomorphism.

Definition 2.1.3 Let $\rho : (Y, E_Y) \longrightarrow (X, 0)$ be a good resolution of $(X, 0)$.

The **dual graph associated to** ρ, denoted $G(Y)$, is constructed as follows. The vertices of $G(Y)$ represent the irreducible components of E_Y. When two irreducible components of E_Y intersect, we join their associated vertices by edges whose number is equal to the number of intersection points. A dual graph is a **bamboo** if the graph is homeomorphic to a segment and each vertex represents a rational curve.

If E_i is an irreducible component of E_Y, let us denote by e_i the self-intersection number of E_i in Y and by g_i its genus. To obtain the **weighted dual graph associated to** ρ, denoted $G_w(Y)$, we weight $G(Y)$ as follows. A vertex associated to the irreducible E_i of E_Y is weighted by (e_i) when $g_i = 0$ and by (e_i, g_i) when $g_i > 0$.

For example if $X = \{(x, y, z) \in \mathbb{C}^3, z^m = x^k y^l\}$, where m, k and l are integers greater than two and pairwise relatively prime, Fig. 2.1 describes the shape of the dual graph of the minimal good resolution of $(X, 0)$.

Remark 2.1.4 If $(X, 0)$ is reducible, let $(\cup_{1\leq i\leq r} X_i, 0)$ be its decomposition as a union of irreducible surface germs. Let $\nu_i : (\bar{X}_i, p_i) \to (X_i, 0)$ be the normalization of the irreducible components of $(X, 0)$. The morphisms ν_i induce the normalization morphism on the disjoint union $\coprod_{1\leq i\leq r}(\bar{X}_i, p_i)$.

Fig. 2.1 $G_w(Y)$ when $X = \{(x, y, z) \in \mathbb{C}^3, z^m = x^k y^l\}$. Here $G(Y)$ is a bamboo. The arrows represent the strict transform of $\{xy = 0\}$. In particular if $m = 12$, $k = 5$ and $l = 11$ the graph has three vertices with $e_1 = -3$, $e_2 = -2$, $e_3 = -3$ (see [16, p. 759])

Remark 2.1.5 First of all, for any good resolution $\rho : (Y, E_Y) \to (X, 0)$ there exists a factorization through the normalization $\nu : (\bar{X}, \bar{p}) \to (X, 0)$ (see [11, Thm. 3.14]). In Sect. 2.3, we describe the topology of normalization morphisms. After that it will be sufficient to describe the topology of the links of normal surface germs.

A good resolution is minimal if its exceptional divisor doesn't contain any irreducible component of genus zero, self-intersection -1 and which meets only one or two other irreducible components. Let $\rho : (Y, E_Y) \to (X, 0)$ be a good resolution and $\rho' : (Y', E_{Y'}) \to (X, 0)$ be a good minimal resolution of $(X, 0)$. Then there exists a morphism $\beta : (Y, E_Y) \to (Y', E_{Y'})$ which is a sequence of blowing-downs of irreducible components of genus zero and self-intersection -1 (see [11, Thm 5.9] or [1, p. 86]). It implies the unicity, up to isomorphism, of the minimal good resolution of $(X, 0)$.

As there exists a factorization of ρ' through ν, $(Y', E_{Y'})$ is also the minimal good resolution of $(\bar{X}, \bar{p})$. Let $\bar{\rho} : (Y', E_{Y'}) \to (\bar{X}, \bar{p})$ be the minimal good resolution of $(\bar{X}, \bar{p})$ defined on $(Y', E_{Y'})$. What we said just above implies that $\rho = \nu \circ \bar{\rho} \circ \beta$, i.e. ρ is the composition of the following three morphisms:

$$(Y, E_Y) \xrightarrow{\beta} (Y', E_{Y'}) \xrightarrow{\bar{\rho}} (\bar{X}, \bar{p}) \xrightarrow{\nu} (X, 0)$$

2.1.2 Link of a Complex Surface Germ

In Sect. 2.2, we describe the topology of a plumbing and the topology of its boundary. We explain how the existence of a good resolution describes the link of a normal complex surface germ as the boundary of a plumbing of disc bundles on oriented smooth compact real surfaces with empty boundary. The boundary of a plumbing is, by definition, a plumbed 3-manifold [10, 20] or equivalently a graph manifold in the sense of Waldhausen [23]. The plumbing given by the minimal good resolution of $(X, 0)$ has a normal form in the sense of Neumann [20] and represents its boundary in a unique way.

It implies that the link of a normal complex surface germ $(X, 0)$ determines the weighted dual graph of its good minimal resolution. In particular, if the link is S^3, then the good minimal resolution of $(X, 0)$ is an isomorphism and $(X, 0)$ is smooth at the origin. This is the famous result obtained in 1961 by Mumford [19]. When the singular locus of $(X, 0)$ is an irreducible germ of curve, its link can be S^3. Lê's conjecture, which is still open (see [14] and [2] for partial results), states that it can only happen for an equisingular family of irreducible curves.

In Sect. 2.3, we use a suitably general projection $\pi : (X, 0) \to (\mathbb{C}^2, 0)$ (as told in Sect. 2.3.1) to describe the topology of the restriction $\nu_L : L_{\bar{X}} \to L_X$ of the normalization ν on the link $L_{\bar{X}}$. We will show that ν_L is a homeomorphism if and only if a general hyperplane section of $(X, 0)$ is locally irreducible at z for all points $z \in (\Sigma \setminus \{0\})$. Otherwise, as stated without a proof in Luengo-Pichon [14], ν_L is the composition of two kind of topological quotients: curlings and identifications. Here, we give detailed proofs. Some years ago, John Milnor asked me for a description of the link of a surface germ with non-isolated singular locus. I hope that Sect. 2.3 gives a satisfactory answer.

In Sect. 2.4 we suppose that $(X, 0)$ is **normal**. We use a finite morphism $\pi : (X, 0) \to (\mathbb{C}^2, 0)$ and its discriminant Δ, to obtain a good resolution $\rho : (Y, E_Y) \to (X, 0)$ of $(X, 0)$. We follow Hirzebruch's method (see [9], see also Brieskorn [5] for a presentation of Hirzebruch's work). The scheme to obtain ρ is as in [15], but our redaction here is quite different. In [15], the purpose is to study the behaviour of invariants associated to finite morphisms defined on $(X, 0)$. Here, we explain in detail the topology of each steps of the construction to specify the behaviour of ρ. Hirzebruch's method uses the properties of the topology of the normalization, presented in Sect. 2.3, and the resolution of the quasi-ordinary singularities of surfaces already studied by Jung. This is why one says that this resolution ρ is the Hirzebruch-Jung resolution associated to π. Then L_X is homeomorphic to the boundary of a regular neighborhood of the exceptional divisor E_Y of $\rho : (Y, E_Y) \to (X, 0)$ which is a plumbing as defined in Sect. 2.2.

Section 2.5 is an appendix which can be read independently of the other sections. We suppose again that $(X, 0)$ is **normal**. We give topological proofs of basic results, already used in Sect. 2.4 on finite morphism $\phi : (X, 0) \to (\mathbb{C}^2, 0)$, in the following two cases:

1. The discriminant of ϕ is a smooth germ of curve. Then, in Lemma 2.5.6, we show that $(X, 0)$ is analytically isomorphic to $(\mathbb{C}^2, 0)$ and that ϕ is analytically isomorphic to the map from $(\mathbb{C}^2, 0)$ to $(\mathbb{C}^2, 0)$ defined by $(x, y) \mapsto (x, y^n)$.
2. The discriminant of ϕ is a normal crossing. By definition $(X, 0)$ is then a quasi-ordinary singularity and its link is a lens space. We prove that the minimal resolution of $(X, 0)$ is a bamboo of rational curves (Proposition 2.5.7).

Section 2.6 is an example of Hirzebruch-Jung's resolution.

2.1.3 Conventions

The boundary of a topological manifold W will be denoted by $b(W)$.

A **disc** (resp. an **open disc**) will always be an oriented topological manifold orientation preserving homeomorphic to $\{z \in \mathbb{C}, |z| \leq 1\}$ (resp. to $\{z \in \mathbb{C}, |z| < 1\}$).

A **circle** will always be an oriented topological manifold orientation preserving homeomorphic to $S = \{z \in \mathbb{C}, |z| = 1\}$. Moreover, for $0 < \alpha$, we use the following notation: $D_\alpha = \{z \in \mathbb{C}, |z| \leq \alpha\}$, and $S_\alpha = b(D_\alpha)$.

2.2 The Topology of Plumbings

In this Section $(X, 0)$ is a **normal** complex surface germ.

The name "plumbing" was introduced by David Mumford in [19]. There, he showed that the topology of a resolution of a normal singularity of a complex surface can be described as a "plumbing".

In [9], Hirzebruch constructed good resolutions of normal singularities. Let $\rho : (Y, E_Y) \to (X, 0)$ be a good resolution of the normal germ of surface $(X, 0)$. Each irreducible component E_i of the exceptional divisor is equipped with its normal complex fiber bundle. With their complex structure the fibers have dimension 1. So, a regular compact tubular neighbourhood $N(E_i)$ of E_i in Y, is a disc bundle. As E_i is a smooth compact complex curve, E_i is an oriented differential compact surface with an empty boundary. Then, the isomorphism class, as differential bundle, of the disc bundle $N(E_i)$ is determined by the genus g_i of E_i and its self-intersection number e_i in Y. The complex structure gives an orientation on Y and on E_i, these orientations induce an orientation on $N(E_i)$ and on the fibers of the disc bundle over E_i.

Remark 2.2.1 By definition $(X, 0)$ is a sufficiently small compact representative of the given normal surface germ. Let k be the number of irreducible components of E_Y, $M(Y) = \cup_{1 \leq i \leq k} N(E_i)$ is a compact neighborhood of E_Y. There exists a retraction by deformation $R : Y \to M(Y)$ which induces a homeomorphism from the boundary of Y, $b(Y) = \rho^{-1}(L_X)$, to the boundary $b(M(Y))$. So, the boundary of $M(Y)$ is the link of $(X, 0)$.

Definition 2.2.2 Let $N(E_i), i = 1, 2$, be two oriented disc bundles on oriented smooth compact differentiable surfaces, with empty boundary, $E_i, i = 1, 2$, and let $p_i \in E_i$. **The plumbing of** $N(E_1)$ **and** $N(E_2)$ **at** p_1 **and** p_2 is equal to the quotient of the disjoint union of $N(E_1)$ and $N(E_2)$ by the following equivalence relation. Let D_i be a small disc neighbourhood of p_i in E_i, and $D_i \times \Delta_i$ be a trivialization of $N(E_i)$ over D_i, $i = 1, 2$. Let $f : D_1 \to \Delta_2$ and $g : \Delta_1 \to D_2$ be two orientation preserving diffeomorphisms such that $f(p_1) = 0$ and $g(0) = p_2$.

For all $(v_1, u_1) \in D_1 \times \Delta_1$, the equivalence relation is $(v_1, u_1) \sim (g(u_1), f(v_1))$.

Remark 2.2.3 The diffeomorphism class of the plumbing of $N(E_1)$ and $N(E_2)$ at (p_1, p_2) does not depend upon the choices of the trivializations nor on the choices of f and g. Moreover, in the plumbing of $N(E_1)$ and $N(E_2)$ at p_1 and p_2:

1. The image of E_1 intersects the image of E_2 at the point p_{12} which is the class, in the quotient, of $(p_1 \times 0) \sim (p_2 \times 0)$.

2. The plumbing is a gluing of $N(E_1)$ and $N(E_2)$ around the chosen neighbourhoods of $(p_1 \times 0)$ and $(p_2 \times 0)$.
3. In the plumbing, $D_1 \times 0 \subset E_1$ is identified, via f, with the fiber $0 \times \Delta_2$ of the disc bundle $N(E_2)$ and the fiber $0 \times \Delta_1$ of $N(E_1)$ is identified, via g, with $D_2 \times 0 \subset E_2$.

Definition 2.2.4 More generally we can perform the plumbing of a family $N(E_i), i = 1, \dots, n$, of oriented disc bundles on oriented smooth compact differentiable surfaces E_i with empty boundary, at a finite number of pairs of points $(p_i, p_j) \in E_i \times E_j$. Let g_i be the genus of E_i and e_i be the self-intersection number of E_i in $N(E_i)$. The vertices of the **weighted plumbing graph** associated to such a plumbing represent the basis $E_i, i = 1, \dots, n$, of the bundles. These vertices are weighted by e_i when $g_i = 0$, and by (e_i, g_i) when $0 < g_i$. Each edge which relates (i) to (j), represents an intersection point between the image of E_i and E_j in the plumbing.

In the boundary of the plumbing of the family $N(E_i), i = 1, \dots, n$, the intersections $b(N(E_i)) \cap b(N(E_j)$ are a union of disjoint tori which is the **family of plumbing tori** of the plumbing.

We can perform a plumbing between $N(E_i)$ and $N(E_j)$ at several pairs of points of $E_i \times E_j$ if and only if every two such pairs of points (p_i, p_j) and (p'_i, p'_j) are such that $p_i \neq p'_i$ and $p_j \neq p'_j$. Let $k_{ij} \geq 0$ be the number of these pairs of points. Obviously, k_{ij} is the number of disjoint tori which form the intersection $b(N(E_i)) \cap b(N(E_j)$ and also the number of edges which relate the vertices associated to E_i and E_j in the plumbing graph associated to the plumbing.

An oriented disc bundle $N(E)$ on a differential compact surface E of genus g and empty boundary is determined as differentiable bundle by g and by the self-intersection number of E in $N(E)$. If two plumbings have the same weighted plumbing graph, there exists a diffeomorphism between the two plumbings such that its restriction on the corresponding disc bundles is an isomorphism of differentiable disc bundles.

Proposition 2.2.5 *Let $\rho : (Y, E_Y) \to (X, 0)$ be a good resolution of the normal germ of surface $(X, 0)$. Then a regular neighbourhood, in Y, of the exceptional divisor E_Y, is diffeomorphic to a plumbing of the disc bundles $N(E_i)$. The plumbings are performed around the double points $p_{ij} = E_i \cap E_j$.The associated weighted plumbing graph coincides with the weighted dual graph $G_w(Y)$ of ρ. To each point $p_{ij} \in (E_i \cap E_j)$ we associate a torus $T(p_{ij}) \subset (b(N(E_i)) \cap b(N(E_j)))$.*

Proof We choose trivializations of the disc bundles $N(E_i)$ and $N(E_j)$ in a small closed neighborhood V of p_{ij}. First, we center the trivializations at $(0, 0) = p_{ij}$ and we parametrize V as disc a bundle

1. over E_i by $V_i = \{(v_i, u_i) \in D_i \times \Delta_i\}$, where $D_i \times 0$ is a disc neighborhood of $(0, 0) = p_{ij}$ in E_i and $v_i \times \Delta_i$ is the normal disc fiber at $v_i \in D_i$.
2. over E_j by $V_j = \{(v_j, u_j) \in D_j \times \Delta_j\}$, where $D_j \times 0$ is a disc neighborhood of $(0, 0) = p_{ij}$ in E_j and $v_j \times \Delta_j$ is the normal disc fiber at $v_j \in D_j$.

As E_Y is a normal crossing divisor, we can parametrize V in such a way that $E_Y \cap V = \{uv = 0\}$ where $v = v_i = u_j$ and $u = v_j = u_i$. These equalities provide the plumbing of $N(E_i)$ and $N(E_j)$ around p_{ij}. By construction , the associated weighted plumbing graph is equal to $G_w(Y)$. □

Definition 2.2.6 The union of disc bundles $M(Y) = \cup_{1 \leq i \leq k} N(E_i)$ is **the plumbing** associated to $\rho : (Y, E_Y) \to (X, 0)$.

With the above notation, in a neighborhood of p_{ij}, there is a unique connected component of the intersection $(b(N(E_i)) \cap b(N(E_j)))$ which is parametrized by the torus $b(D_i) \times b(\Delta_i)$ which is glued point by point with $b(D_j) \times b(\Delta_j)$.

Definition 2.2.7 The image of $(b(D_i) \times b(\Delta_i)) \sim (b(D_j) \times b(\Delta_j))$ in the boundary of $M(Y)$ is the **plumbing torus** $T(p_{ij})$ **associated to** p_{ij}.

2.3 The Topology of the Normalization

In this Section $(X, 0)$ is the intersection of a reduced complex surface germ, which can have a 1-dimensional singular locus, with the compact ball B^{2n}_ϵ of radius a small ϵ (i.e. where ϵ is as in Milnor's Theorem 2.10 of [18]), centered at the origin in $\mathbb{C}^n$. As in the Introduction (Sect. 2.1), L_X is the intersection of X with the boundary S^{2n-1}_ϵ of B^{2n}_ϵ.

2.3.1 L_X as Singular Covering over S^3

We choose a general projection $\pi : (X, 0) \to (\mathbb{C}^2, 0)$. We denote by Γ the singular locus of π (in particular $\Sigma \subset \Gamma$) and by Δ its discriminant ($\Delta = \pi(\Gamma)$). In fact it is sufficient to choose new coordinates in $\mathbb{C}^n$, $(x, y, w_1, \ldots, w_{n-2}) \in \mathbb{C}^n$, such that the restriction on $(X, 0)$ of the projection

$$(x, y, w_1, \ldots, w_{n-2}) \mapsto (x, y),$$

denoted by π, is finite and such that, for a sufficiently small α with $\alpha < \epsilon$, and all $a \in \mathbb{C}$ with $|a| \leq \alpha$, the hyperplanes $H_a = \{x = a\}$ meet transversally the singular locus Γ of π. In particular, $H_0 \cap \Gamma = \{0\}$.

Convention and Notation

Let $D_\alpha \times D_\beta \in \mathbb{C}^2$ be a polydisc at the origin in $\mathbb{C}^2$ where $0 < \alpha < \beta < \epsilon$ are chosen sufficiently small such that the following two points are satisfied:

I) $\mathcal{B} = B^{2n}_\epsilon \cap \pi^{-1}(D_\alpha \times D_\beta)$ is a good semi-analytic neighborhood of $(X, 0)$ in the sense of A. Durfee [7]. Then $(X \cap \mathcal{B}, 0)$ is homeomorphic to $(X, 0)$. In this section $(X, 0)$ is given by $(X \cap \mathcal{B}, 0)$. The link $L_X = X \cap b(\mathcal{B})$ is the link of X. The link of Γ is the link $K_\Gamma = \Gamma \cap b(\mathcal{B})$ embedded in L_X.

II) We have the following inclusion:

$$K_\Delta = \Delta \cap ((S_\alpha \times D_\beta) \cup (D_\alpha \times S_\beta)) \subset (S_\alpha \times D_\beta).$$

In this section, we choose such a K_Δ to represent the link of Δ embedded in the 3-sphere (with corners) $((S_\alpha \times D_\beta) \cup (D_\alpha \times S_\beta))$. Let $\delta_j, 1 \leq j \leq r$, be the r branches of the discriminant Δ. Let $N(K_\Delta)$ be a tubular compact neighborhood of K_Δ. So, $N(K_\Delta)$ is a disjoint union of r solid tori. For a sufficiently small $N(K_\Delta)$, the union $N(K_\Gamma)$ of the connected components of $L_X \cap \pi^{-1}(N(K_\Delta))$ which contain a connected component of K_Γ, constitutes a tubular compact neighbourhood of K_Γ in L_X.

Let us denote by $\overset{\circ}{N}(K_\Delta)$ the interior of $N(K_\Delta)$. **The exterior M of the link K_Δ** is defined by:

$$M = ((S_\alpha \times D_\beta) \cup (D_\alpha \times S_\beta)) \setminus \overset{\circ}{N}(K_\Delta).$$

Moreover, let γ be a branch of the singular locus Γ of π. So, $\pi(\gamma) = \delta$ is a branch of Δ. Let $N(K_\delta)$ (resp. $N(K_\gamma)$) be the connected component of $N(K_\Delta)$ (resp. of $N(K_\Gamma)$) which contains the link K_δ (resp. K_γ).

Remark 2.3.1 The restriction $\pi_L : L_X \to ((S_\alpha \times D_\beta) \cup (D_\alpha \times S_\beta))$ of π to L_X is a finite morphism, its restriction on M is a finite regular covering. If γ is not a branch of the singular locus Σ of X, π_L restricted to $N(K_\gamma)$ is a ramified covering with K_γ as ramification locus. If γ is a branch of Σ, $N(K_\gamma)$ is a singular pinched solid torus as defined in Definition 2.3.13 and π_L restricted to $N(K_\gamma)$ is singular all along K_γ.

2.3.2 Waldhausen Graph Manifolds and Plumbing Graphs

Definition 2.3.2 A Seifert fibration on an oriented, compact 3-manifold is an oriented foliation by circles such that every leaf has a tubular neighbourhood (which is a solid torus) saturated by leaves. A Seifert 3-manifold is an oriented, compact 3-manifold equipped with a Seifert fibration.

Remark 2.3.3

1. A Seifert 3-manifold M can have a non-empty boundary. As this boundary is equipped with a foliation by circles, if $B(M)$ is non-empty it is a disjoint union of tori.
2. Let D be a disc and r be a rotation of angle $2\pi q/p$ where (q, p) are two positive integers prime to each other and $0 < q/p < 1$. Let T_r be the solid torus equipped with a Seifert foliation given by the trajectories of r in the following mapping torus:

$$T_r = D \times [0, 1]/(z, 1) \sim (r(z), 0).$$

In particular, $l_0 = (0 \times [0,1])/(0,1) \sim (0,0)$ is a core of T_r. The other leaves are (q,p)-torus knots in T_r. Let T_0 be $D \times S$ equipped with the trivial fibration by circles $l(z) = \{z\} \times S,\ z \in D$. A solid torus $T(l)$ which is a tubular neighbourhood of a leave l of a Seifert 3-manifold M is either

1) orientation and foliation preserving homeomorphic to T_0. In this case, l is a regular Seifert leave.
2) or, is orientation and foliation preserving homeomorphic to T_r. In this case, l is an exceptional leave of M.

3. The compactness of M implies that the set of exceptional leaves is finite.

Definition 2.3.4 Let M be an oriented and compact 3-manifold. The manifold M is a **Waldhausen graph manifold** if there exists a finite family $\mathcal{T}$, of disjoint tori embedded in M, such that if $M_i, i = 1, \ldots, m$, is the family of the closures of the connected components of $M \setminus \mathcal{T}$, then M_i is a Seifert manifold for all $i,\ 1 \leq i \leq m$. We assume that it gives us a finite decomposition $M = \cup_{1 \leq i \leq m} M_i$ into a union of compact connected Seifert manifolds which satisfies the following properties:

1. For each $M_i, i = 1, \ldots, m$, the boundary of M_i is in $\mathcal{T}$ i.e. $b(M_i) \subset \mathcal{T}$.
2. If $i \neq j$ we have the inclusion $(M_i \cap M_j) \subset \mathcal{T}$.
3. The intersection $(M_i \cap M_j)$, between two Seifert manifolds of the given decomposition, is either empty or equal to the union of the common boundary components of M_i and M_j.

Such a decomposition $M = \cup_{1 \leq i \leq m} M_i$, is **the Waldhausen decomposition of M, associated to the family of tori $\mathcal{T}$.**

Remark 2.3.5 One can easily deduce from Definition 2.2.4, that the family of the plumbing tori gives a decomposition of the boundary of a plumbing as a union of Seifert manifolds because the boundary of a disc bundle is a circle bundle. So, the boundary of a plumbing is a Waldhausen graph manifold.

In [20], W. Neumann shows how to construct a plumbing from a given Waldhausen decomposition of a 3-dimensional oriented compact manifold.

As in Sect. 2.3.1, we consider the exterior $M = ((S_\alpha \times D_\beta) \cup (D_\alpha \times S_\beta)) \setminus \mathring{N}(K_\Delta)$ of the link K_Δ. The following proposition is well known (for example see [8, 17]). Moreover, a detailed description of M, as included in the boundary of the plumbing graph given by the minimal resolution of Δ, is given in [12, p. 147–150].

Proposition 2.3.6 *The exterior M of the link of a plane curve germ Δ is a Waldhausen graph manifold. The minimal Waldhausen decomposition of M can be extended to a Waldhausen decomposition of the sphere $((S_\alpha \times D_\beta) \cup (D_\alpha \times S_\beta))$ in which the connected components of K_Δ are Seifert leaves. Moreover, with such a Waldhausen decomposition, the solid tori connected components of $N(K_\Delta)$ are saturated by Seifert leaves which are oriented circles transverse to $(a \times D_\beta)$, $a \in S_\alpha$. The cores K_Δ of $N(K_\Delta)$ are a union of these Seifert leaves.*

2.3.3 The Topology of L_X When L_X Is a Topological Manifold

If $(X, 0)$ is not normal, let $\nu_L : L_{\bar{X}} \to L_X$ be the normalization of $(X, 0)$ restricted to the link of $(\bar{X}, p)$ (if $(X, 0)$ is normal ν_L is the identity).

Remark 2.3.7 The link of a normal complex surface germ is a Waldhausen graph manifold. Indeed, the composition morphism $\pi_L \circ \nu_L$ is a ramified covering with the link K_Δ as set of ramification values:

$$(\pi_L \circ \nu_L) : L_{\bar{X}} \to ((S_\alpha \times D_\beta) \cup (D_\alpha \times S_\beta)).$$

We can take the inverse image under $\pi_L \circ \nu_L$ of the tori and of the Seifert leaves of a Waldhausen decomposition of $((S_\alpha \times D_\beta) \cup (D_\alpha \times S_\beta))$ in which K_Δ is a union of Seifert leaves, to obtain a Waldhausen decomposition of $L_{\bar{X}}$. Then, the plumbing calculus [20] describes $L_{\bar{X}}$ as the boundary of a plumbing without the help of a good resolution of $(\bar{X}, p)$.

If the singular locus $(\Sigma, 0)$ of $(X, 0)$ is one-dimensional, let $(\sigma, 0)$ be a branch of $(\Sigma, 0)$ and s be a point of the intersection $\sigma \cap \{x = a\}$. Let $\delta = \pi(\sigma)$ be the branch of the discriminant Δ which is the image of σ by the morphism π. Then, $\pi_L(s) = (a, y) \in (S_\alpha \times D_\beta)$. Let $N(K_\delta)$ be a solid torus regular neighbourhood of K_δ in $(S_\alpha \times D_\beta)$ and let $N(K_\sigma)$ be the connected component of $(\pi_L)^{-1}(N(K_\delta))$ which contains s (and K_σ).

Let (C, s) be the germ of curve which is the connected component of $N(K_\sigma) \cap \{x = a\}$ which contains s. For a sufficiently small $\alpha = |a|$, (C, s) is reduced and its topological type does not depend upon the choice of s. In particular, **the number of the irreducible components of** (C, s) is well defined, let us denote this number **by** $k(\sigma)$.

Definition 2.3.8

1. By definition (C, s) is the **hyperplane section germ** of σ at s.
   If $k(\sigma) = 1$, σ is a **branch** of Σ **with irreducible hyperplane sections**. Let $\Sigma = \Sigma_1 \cup \Sigma_+$ where Σ_1 is the union of the branches of Σ with irreducible hyperplane sections and Σ_+ is the union of the branches of Σ with reducible hyperplane sections.
2. Let $D_i, 1 \leq i \leq k$ be k oriented discs centered at $0_i \in D_i$. A k**-pinched disc** $k(D)$ is a topological space orientation preserving homeomorphic to the quotient of the disjoint union of the k discs by the identification of their centers in a unique point $\tilde{0}$ i.e. $0_i \sim 0_j$ for all i and j where $1 \leq i \leq k, 1 \leq j \leq k$. **The center of** $k(D)$ is the equivalence class $\tilde{0}$ of the centers 0_i, $1 \leq i \leq k$.
3. If $h : k(D) \to k(D)'$ is a homeomorphism between two k-pinched discs with $k > 1$, $h(\tilde{0})$ is obviously the center of $k(D)'$. We say that h **is orientation preserving** if h preserves the orientation of the punctured k-pinched discs $(k(D) \setminus \{\tilde{0}\})$ and $(k(D)' \setminus \{\tilde{0}\})$.

Lemma 2.3.9 *Let (C, s) be the germ of curve which is the connected component of $N(K_\sigma) \cap \{x = a\}$ which contains s. Then, C is a $k(\sigma)$-pinched disc centered at s and $N(K_\sigma)$ is the mapping torus of C by an orientation preserving homeomorphism h which fixes the point s.*

Proof As (C, s) is a germ of curve with $k(\sigma)$ branches, up to homeomorphism (C, s) is a $k(\sigma)$-pinched disc with $s = \tilde{0}$.

We can saturate the solid torus $N(K_\delta) = \pi(N(K_\sigma))$ with oriented circles such that K_δ is one of these circles and such that the first return homeomorphism defined by these circles on the disc $\pi(C)$ is the identity. Let γ be one circle of the chosen saturation of $N(K_\delta)$. Then $\pi^{-1}(\gamma) \cap N(K_\sigma)$ is a disjoint union of oriented circles because π restricted to $N(K_\sigma) \setminus K_\sigma$ is a regular covering and $(\pi^{-1}(K_\delta) \cap N(K_\sigma)) = K_\sigma$. So, $N(K_\sigma)$ is equipped with a saturation by oriented circles. The first return map on C along the so constructed circles is an orientation preserving homeomorphism h such that $h(s) = s$ because K_σ is one of the given circles. □

Lemma 2.3.10 *As above, let (C, s) be the hyperplane section germ at $s \in \sigma \cap \{x = a\}$. Let $\bar{\sigma}_j$, $1 \leq j \leq n$, be the n irreducible components of $\nu_L^{-1}(\sigma)$ and let d_j be the degree of ν_L restricted to $\bar{\sigma}_j$. Then, we have*

$$k(\sigma) = d_1 + \cdots + d_j + \cdots + d_n.$$

Proof The normalization ν restricted to $\bar{X} \setminus \bar{\Sigma}$, where $\bar{\Sigma} = \pi^{-1}(\Sigma)$, is an isomorphism. The number n of the irreducible components of $\nu_L^{-1}(\sigma)$ is equal to the number of the connected components of $\nu_L^{-1}(N(K_\sigma))$. So, n is the number of the connected components of the boundaries $b(\nu_L^{-1}(N(K_\sigma)))$ which is equal to the number of the connected components of $b(N(K_\sigma))$. Let τ_j, $1 \leq j \leq n$, be the n disjoint tori which are the boundary of $N(K_\sigma)$. The degree d_j of ν restricted to $\bar{\sigma}_j$ is equal to the number of points of $\nu_L^{-1}(s) \cap (\bar{\sigma}_j)$.

Let (γ_j, s) be an irreducible component of (C, s) such that $m_j = b(\gamma_j) \subset \tau_j$. The normalization ν restricted to $(\nu_L^{-1}(\gamma_j \setminus \{s\}))$ is an isomorphism over the punctured disc $(\gamma_j \setminus \{s\})$. So, the intersection $\nu_L^{-1}(\gamma_j) \cap \bar{\sigma}_j$ is a unique point p_j. As $(\bar{X}, p)$ is normal, p_j is a smooth point of $(\bar{X}, p)$ and then, $\nu_L^{-1}(\gamma_j)$ is irreducible and it is the only irreducible component of $\nu_L^{-1}(C)$ at p_j. By symmetry there is exactly one irreducible component of $\nu_L^{-1}(C)$ at every point of $\nu_L^{-1}(s) \cap (\bar{\sigma}_j)$.

So, d_j is the number of the meridian circles of the solid torus $N(K_{\bar{\sigma}_j})$ obtained by the following intersection $(\nu_L^{-1}(C)) \cap (\nu_L^{-1}(\tau_j))$. But ν restricted to $(\nu_L^{-1}(\tau_j))$ is an isomorphism and d_j is also the number of connected components of $C \cap \tau_j$. So, $d_1 + \cdots + d_j + \cdots + d_n$, is equal to the number of connected components of $b(C) = C \cap b(N(K_\sigma))$ which is the number of irreducible components of (C, s).

□

Remark 2.3.11 A well-known result of analytic geometry could be roughly stated as follows: "The normalization separates the irreducible components". Here, $(X, 0)$ has $k(\sigma)$ irreducible components around $s \in \sigma$. Using only basic topology, Lemma 3.3.4 proves that $(\nu_L^{-1}(s))$ has $k(\sigma) = d_1 + \cdots + d_j + \cdots + d_n$ distinct points and that there is exactly one irreducible component of $\nu_L^{-1}(C)$ at every point of $\nu_L^{-1}(s)$. This gives a topological proof that the normalization ν separates the $k(\sigma)$ irreducible components of (C, s) around $s \in \sigma$.

Proposition 2.3.12 *The following three statements are equivalent:*

1. *L_X is a topological manifold equipped with a Waldhausen graph manifold structure.*
2. *The normalization $\nu : (\bar{X}, p) \to (X, 0)$ is a homeomorphism.*
3. *All the branches of Σ have irreducible hyperplane sections.*

Proof The normalization ν restricted to $\bar{X} \setminus \bar{\Sigma}$, where $\bar{\Sigma} = \pi^{-1}(\Sigma)$, is an isomorphism. The normalization is a homeomorphism if and only if ν restricted to $\bar{\Sigma} = \pi^{-1}(\Sigma)$ is a bijection. This is the case if and only if we have $1 = d_1 + \cdots + d_j + \cdots + d_n$ for all the branches σ of Σ. But, by Lemma 2.3.10, $k(\sigma) = d_1 + \cdots + d_j + \cdots + d_n$. This proves the equivalence of the statements 2 and 3.

Let (C, s) be the hyperplane section germ at $s \in \sigma \cap \{x = a\}$. If L_X is a topological manifold, it is a topological manifold at s and $k(\sigma) = 1$ for all branches σ of Σ. If all the branches of Σ have irreducible hyperplane sections, we already know that the normalization $\nu : (\bar{X}, p) \to (X, 0)$ is a homeomorphism. Then, the restriction ν_L of ν to $L_{\bar{X}}$ is also a homeomorphism. By Remark 2.3.7, $L_{\bar{X}}$ is a Waldhausen graph manifold. In particular, we can equip L_X with the Waldhausen graph manifold structure carried by ν_L. This proves the equivalence of the statements 1 and 3. □

2.3.4 Singular L_X, Curlings and Identifications

In Sect. 2.3.3 (Definition 2.3.8), we have considered the union Σ_+ of the branches of the singular locus Σ of $(X, 0)$ which have reducible hyperplane sections. We consider a tubular neighbourhood $N_+ = \cup_{\sigma \subset \Sigma_+} N(K_\sigma)$ of the link K_{Σ_+} of Σ_+ in L_X. As in the proof of Proposition 2.3.12, the exterior $M_1 = L_X \setminus \overset{\circ}{N}_+$, of K_{Σ_+} in L_X, is a topological manifold because ν restricted to $\nu^{-1}(M_1)$ is a homeomorphism. From now on σ is a branch of Σ_+. The definition of Σ_+ implies that L_X is topologically singular at every point of K_σ. In this section, we show that $N(K_\sigma)$ is a singular pinched solid torus. In Lemma 2.3.9, it is shown that $N(K_\sigma)$ is the mapping torus of a $k(\sigma)$-pinched disc by an orientation preserving homeomorphism. But, the homeomorphism class of the mapping torus of a homeomorphism h depends only on the isotopy class of h. Moreover the isotopy class of an orientation preserving homeomorphism h of a k-pinched disc depends only on the permutation induced by h on the k discs. In particular, if $h : D \to D$ is an orientation preserving homeomorphism of a disc D the associated mapping torus

$$T(D,h) = [0,1] \times D/(1,x) \sim (0,h(x))$$

is homeomorphic to the standard torus $S \times D$.

Definition 2.3.13

1. Let $k(D)$ be the k-pinched disc quotient by identification of their centrum of k oriented and ordered discs $D_i, 1 \leq i \leq k$. Let $c = c_1 \circ c_2 \circ \cdots \circ c_n$ be a permutation of the indices $\{1,\dots,k\}$ given as the composition of n disjoint cycles $c_j, 1 \leq j \leq n$, where c_j is a cycle of order d_j. Let $\tilde{h}_c$ be an orientation preserving homeomorphism of the disjoint union of $D_i, 1 \leq i \leq k$ such that $\tilde{h}_c(D_i) = D_{c(i)}$ and $\tilde{h}_c(0_i) = 0_{c(i)}$. Then, $\tilde{h}_c$ induces an orientation preserving homeomorphism h_c on $k(D)$. By construction we have $h_c(\tilde{0}) = \tilde{0}$. A **singular pinched solid torus associated to the permutation** c is a topological space orientation preserving homeomorphic to the mapping torus $T(k(D),c)$ of h_c:

   $$T(k(D),c) = [0,1] \times k(D)/(1,x) \sim (0,h_c(x))$$

   The **core** of $T(k(D),c)$ is the oriented circle $l_0 = [0,1] \times \tilde{0}/(1,\tilde{0}) \sim (0,\tilde{0})$. A homeomorphism between two singular pinched solid tori is orientation preserving if it preserves the orientation of $k(D) \setminus \{\tilde{0}\}$ and the orientation of the trajectories of h_c in its mapping torus $T(k(D),c)$.
2. A d**-curling** C_d is a topological space homeomorphic to the following quotient of a solid torus $S \times D$:

   $$C_d = S \times D/(u,0) \sim (u',0) \Leftrightarrow u^d = u'^d.$$

   Let $q : (S \times D) \to C_d$ be the associated quotient morphism. By definition, $l_0 = q(S \times \{0\})$ is the **core of** C_d.

Example 2.3.14 Let $X = \{(x,y,z) \in \mathbb{C}^3 \; where \; z^d - xy^d = 0\}$. The normalization of $(X,0)$ is smooth i.e. $\nu : (\mathbb{C}^2,0) \to (X,0)$ is given by $(u,v) \mapsto (u^d, v, uv)$. Let $T = \{(u,v) \in (S \times D) \subset \mathbb{C}^2\}$. Let $\pi_x : \nu(T) \to S$ be the projection $(x,y,z) \mapsto x$ restricted to $\nu(T)$. Here the singular locus of $(X,0)$ is the line $\sigma = (x,0,0), x \in \mathbb{C}$. We have $N(K_\sigma) = L_X \cap (\pi_x^{-1}(S)) = \nu(T)$ as a tubular neighbourhood of K_σ. Let $q : T \to C_d$ be the quotient morphism defined above. There exists a well defined homeomorphism $f : C_d \to N(K_\sigma)$ which satisfies $f(q(u,v)) = (u^d, v, uv)$. So, $N(K_\sigma)$ is a d-curling and K_σ is its core. Moreover, f restricted to the core l_0 of C_d is a homeomorphism onto K_σ.

Figure 2.2 shows schematically $\bar{\Gamma} = \nu^{-1}(\Gamma) \subset \bar{X}$ and Δ when Σ is irreducible and $\Gamma \setminus \Sigma$ has two irreducible components.

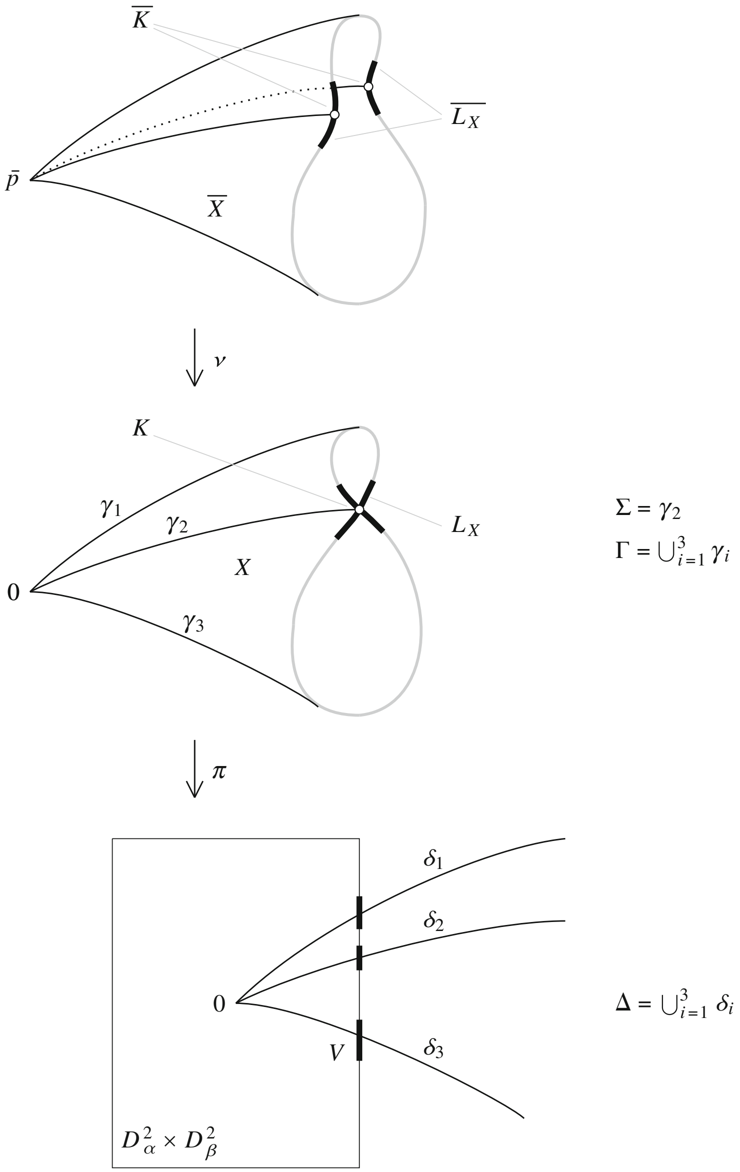


Fig. 2.2 Schematic picture of π and ν when there is a 2-curling on $\Sigma = \gamma_2$

Lemma 2.3.15 *A d-curling is a singular pinched solid torus associated to a d-cycle, i.e. if c is a d-cycle, then* C_d *is homeomorphic to* $T(d(D), c)$.

Proof We use the notation of Example 2.3.14. The model of d-curling obtained in this example is the tubular neighbourhood $N(K_\sigma)$ of the singular knot of the link L_X

of $X = \{(x, y, z) \in \mathbb{C}^3 \; where \; z^d - xy^d = 0\}$. As we work up to homeomorphism, it is sufficient to prove that $N(K_\sigma)$ is a singular pinched solid torus associated to a d-cycle. We can saturate the solid torus T by the oriented circles $l_b = S \times \{b\}$, $b \in D$. The circles $\nu(l_b), b \in D$ also saturate $N(K_\sigma)$ with oriented circles. The fiber $\pi_x^{-1}(a) = (C, (a, 0, 0))$ is a singular fiber of the fibration $\pi_x : \nu(T) \to S_\alpha$. The equation of the curve germ C at $(a, 0, 0)$ is $\{z^d - ay^d = 0\}$, this is a plane curve germ with d branches. So, C is homeomorphic to a d-pinched disc. Moreover, the first return along the circles $\nu(l_b)$ is a monodromy h of π_x which satisfies the conditions given in Definition 3.4.1 to obtain a singular pinched solid torus associated to a d-cycle.

Indeed, $(\pi_x \circ \nu) : T \to S_\alpha$ is a trivial fibration with fiber $\nu^{-1}(C) = \{(\{u_i\} \times D_\beta), u_i^d = a\}$ which is the disjoint union of d ordered meridian discs of T. The first return h_T along the oriented circles l_b is a cyclic permutation of the ordered d meridian discs and $(h_T)^d$ is the identity morphism. Moreover ν restricted to $T \setminus (S \times \{0\})$ is a homeomorphism. As h_T is a lifting of h by ν, the monodromy h determines $N(K_\sigma)$ as a singular pinched solid torus associated to a d-cycle. □

Proposition 2.3.16 *Let σ be a branch of the singular locus of $(X, 0)$ which has a reducible hyperplane section. Let (C, s) be the hyperplane section germ at $s \in \sigma \cap \{x = a\}$. Let $\bar\sigma_j$, $1 \le j \le n$, be the n irreducible components of $\nu_L^{-1}(\sigma)$ and let d_j be the degree of ν_L restricted to $\bar\sigma_j$. Let c_j be a d_j-cycle and let $c = c_1 \circ c_2 \circ \cdots \circ c_n$ be the permutation of $k(\sigma) = d_1 + \cdots + d_j + \cdots + d_n$ elements which is the composition of the n disjoint cycles c_j. A tubular neighbourhood $N(K_\sigma)$ of K_σ is a singular pinched solid torus associated to the permutation c. Moreover, the restriction of ν to $\coprod_{1 \le j \le n} N(K_{\bar\sigma_j})$ is the composition of two quotients: the quotients which define the d_j-curlings followed by the quotient f_σ which identifies their cores.*

Proof Let $N(K_{\bar\sigma_j})$, $1 \le j \le n$ be the n connected components of $\nu^{-1}(N(K_\sigma))$. So, $N(K_\sigma) \setminus K_\sigma$ has also n connected components and $(N(K_\sigma))_j = \nu(N(K_{\bar\sigma_j}))$ is the closure of one of them. Every $N(K_{\bar\sigma_j})$ is a solid torus and the restriction of ν to its core $K_{\bar\sigma_j}$ has degree d_j. The intersection $(\nu^{-1}(C)) \cap N(K_{\bar\sigma_j})$ is a disjoint union of d_j ordered and oriented meridian discs of $N(K_{\bar\sigma_j})$. We can choose a homeomorphism $g_j : (S \times D) \to N(K_{\bar\sigma_j})$ such that $(\nu \circ g_j)^{-1}(C) = \{u\} \times D$, $u^{d_j} = 1$.

The model of a d_j-curling gives the quotient $q_j : (S \times D) \to C_{d_j}$. As in Example 2.3.14, there exists a unique homeomorphism $f_j : C_{d_j} \to (N(K_\sigma))_j$ such that $f_j \circ q_j = \nu \circ g_j$. So, $(N(K_\sigma))_j$ is a d_j-curling. In particular, if ν_j is the restriction of ν to $N(K_{\bar\sigma_j})$, then $\nu_j = f_j \circ q_j \circ (g_j)^{-1}$. Up to homeomorphism ν_j is equivalent to the quotient which defines the d_j-curling.

But for all j, $1 \le j \le n$, we have $\nu(K_{\bar\sigma_j}) = (K_\sigma)$. Up to homeomorphism, $N(K_\sigma)$ is obtained as the quotient of the disjoint union of the d_j-curlings by the identification of their cores. The disjoint union of the f_j induces a homeomorphism f_σ from

$$N = (\coprod_{1 \le j \le n} C_{d_j})/q_j(u, 0) \sim q_i(u, 0) \Leftrightarrow \nu(g_j(u, 0)) = \nu(g_i(u, 0))$$

onto $N(K_\sigma)$. Up to homeomorphism, the restriction of ν to $\coprod_{1\leq j\leq n} N(K_{\bar{\sigma}_j})$ is the composition of two quotients: the quotients which define the d_j-curlings followed by the quotient f_σ which identifies their cores. It is sufficient to prove that $N = T(k(\sigma)(D), c)$ where c is the composition of n disjoint cycles c_j of order d_j. By Lemma 2.3.15, $C_{d_j} = T(d_j(D), c_j)$ and it is obvious that the identifications correspond to the disjoint union of the cycles. □

2.4 Hirzebruch-Jung's Resolution of $(X, 0)$

In this section $(X, 0)$ is a normal surface germ.

Let $\pi : (X, 0) \longrightarrow (\mathbb{C}^2, 0)$ be a finite analytic morphism which is defined on $(X, 0)$. For example π can be the restriction to $(X, 0)$ of a linear projection, as chosen in the beginning of Sect. 2.3.1. But the construction can be performed with any finite morphism π. We denote by Γ the singular locus of π and by $\Delta = \pi(\Gamma)$ its discriminant.

Let $r : (Z, E_Z) \to (\mathbb{C}^2, 0)$ be the minimal embedded resolution of Δ, let $E_Z = r^{-1}(0)$ be the exceptional divisor of r, and let $E_Z^+ = r^{-1}(\Delta)$ be the total transform of Δ. The irreducible components of E_Z are smooth complex curves because the resolution r is obtained by a sequence of blowing up of points in a smooth complex surface. Let us denote by E_Z^0 the set of the smooth points of E_Z^+. So, $E_Z^+ \setminus E_Z^0$ is the set of the double points of E_Z^+.

Here, we give a detailed construction of the Hirzebruch-Jung resolution $\rho : (Y, E_Y) \to (X, 0)$ associated to π. This will prove the existence of a good resolution of $(X, 0)$. As the link L_X is diffeomorphic to the boundary of Y, this will describe L_X as the boundary of a plumbing. In particular, we will explain how to obtain the dual graph $G(Y)$ of E_Y when we have the dual graph $G(Z)$ associated to E_Z. Knowing the Puiseux expansions of all the branches of Δ, there exists an algorithm to compute the dual graph $G_w(Z)$ weighted by the self-intersection numbers of the irreducible components of $E(Z)$ (For example see [6] and Chap. 6 and 7 in [17]). Except in special cases, the determination of the self-intersection numbers of the irreducible components of E_Y is rather delicate.

2.4.1 *First Step: Normalization*

We begin with the minimal resolution r of Δ. The pull-back of π by r is a finite morphism $\pi_r : (Z', E_{Z'}) \to (Z, E_Z)$ which induces an isomorphism from $E_{Z'}$ to E_Z. We denote $r_\pi : (Z', E_{Z'}) \to (X, 0)$, the pull-back of r by π. Figure 2.3 represents the resulting commutative diagram.

In general Z' is not normal. Let $n : (\bar{Z}, E_{\bar{Z}}) \to (Z', E_{Z'})$ be the normalization of Z'.

Fig. 2.3 The diagram of the pull-back of the resolution r by π

$$\begin{tikzcd}
(Z', E_{Z'}) \arrow[r, "r_\pi"] \arrow[d, "\pi_r"'] & (X, 0) \arrow[d, "\pi"] \\
(Z, E_Z) \arrow[r, "r"] & (\mathbb{C}^2, 0)
\end{tikzcd}$$

Remark 2.4.1

1. By construction, the discriminant locus of $\pi_r \circ n$ is included in $E_Z^+ = r^{-1}(\Delta)$ which is the total transform of Δ in Z. As, X is normal at 0, $(X \setminus \{0\})$ has no singular points.
2. As the restriction of r to $Z \setminus E_Z$ is an isomorphism, the restriction of r_π to $Z' \setminus E_{Z'}$ is also an isomorphism. We denote by Γ' (resp. $\bar{\Gamma}$) the closure of $(r_\pi)^{-1}(\Gamma \setminus \{0\})$ in $E_{Z'}$ (resp. the closure of $(r_\pi \circ n)^{-1}(\Gamma \setminus \{0\})$ in $E_{\bar{Z}}$). The restriction of r_π to Γ' (resp. $(r_\pi \circ n)$ on $\bar{\Gamma}$) is an isomorphism onto Γ.
3. The singular locus of Z' is included in $E_{Z'}$. The normalization n restricted to $\bar{Z} \setminus E_{\bar{Z}}$ is an isomorphism.

Notation We use the following notations:

$E^+_{Z'} = E_{Z'} \cup \Gamma'$, and $E^0_{Z'}$ is the set of the points of $E_{Z'}$ which belong to a unique irreducible component of $E^+_{Z'}$. Similarly: $E^+_{\bar{Z}} = E_{\bar{Z}} \cup \bar{\Gamma}$, and $E^0_{\bar{Z}}$ is the set of the points of $E_{\bar{Z}}$ which belong to a unique irreducible component of $E^+_{\bar{Z}}$.

Proposition 2.4.2 *Every singular point of $\bar{Z}$ belongs to at least two irreducible components of $E^+_{\bar{Z}}$. The restriction of the map $(\pi_r \circ n)$ to $E_{\bar{Z}}$ induces a finite morphism from $E_{\bar{Z}}$ to E_Z which is a regular covering from $(\pi_r \circ n)^{-1}(E^0_Z)$ to (E^0_Z).*

Proof As X is normal at 0, $(X \setminus \{0\})$ has no singular points. The pull-back construction implies that:

1. The morphism π_r is finite and its generic degree is equal to the generic degree of π. Indeed, π_r restricted to $E_{Z'}$ is an isomorphism. Moreover, the restriction of π_r to $(Z' \setminus E_{Z'})$ is isomorphic, as a ramified covering, to the restriction of π to $(X \setminus \{0\})$. So, the restriction morphism $(\pi_r)_| : (Z' \setminus E_{Z'}) \to (Z \setminus E_Z)$ is a finite ramified covering with ramification locus Γ'.

2. As the restriction of r to $(Z \setminus E_Z)$ is an isomorphism, then the restriction of r_π to $(Z' \setminus E_{Z'})$ is also an isomorphism. So, the restriction of $(r_\pi \circ n)$ to $(\bar{Z} \setminus E_{\bar{Z}})$ is an analytic isomorphism onto the non-singular analytic set $(X \setminus \{0\})$. It implies that $(\bar{Z} \setminus E_{\bar{Z}})$ is smooth.

If $\bar{P} \in E^0_{\bar{Z}}$, then $P = (\pi_r \circ n)(\bar{P})$ is a smooth point of an irreducible component E_i of E_Z. The normal fiber bundle to E_i in Z can be locally trivialized at P. We can choose a small closed neighborhood N of P in Z such that $N = D \times \Delta$ where D and Δ are two discs , $N \cap E_Z = (D \times 0)$ and for all $z \in D$, $z \times \Delta$ are fibers of the bundle in discs associated to the normal bundle of E_i. We choose $\bar{N} = (\pi_r \circ n)^{-1}(N)$ as closed neighborhood of $\bar{P}$ in $\bar{Z}$. But $\bar{Z}$ is normal and the local discriminant of the restriction $(\pi_r \circ n)_| : (\bar{N}, \bar{P}) \to (N, P)$ is included in $D \times 0$ which is a smooth germ of curve. In that case, the link of $(\bar{N}, \bar{P})$ is S^3 (in Lemma 2.5.6, we give a topological proof of this classical result). As $\bar{Z}$ is normal, by Mumford's Theorem [19], $\bar{P}$ is a smooth point of $\bar{Z}$. This ends the proof of the first statement of the proposition.

Now, we know that the morphism $(\pi_r \circ n)_{|\bar{N}} : (\bar{N}, \bar{P}) \to (N, P)$ is a finite morphism between two smooth germs of surfaces with non-singular discriminant locus. Let d be its generic order. By Lemma 2.5.6, such a morphism is locally isomorphic (as an analytic morphism) to the morphism defined on $(\mathbb{C}^2, 0)$ by $(x, y) \mapsto (x, y^d)$. So, $\bar{D} = (\pi_r \circ n)^{-1}(D \times 0)$ is a smooth disc in $E^0_{\bar{Z}}$ and the restriction of such a morphism to $\{(x, 0), x \in \bar{D}\}$ is a local isomorphism.

By definition of E^0_Z, $P \in (E_i \cap E^0_Z)$ is a smooth point in the total transform of Δ. If we take a smooth germ (γ, P) transverse to E_i at P, then $(r(\gamma), 0)$ is not a branch of Δ. The restriction of π to $\pi^{-1}(r(\gamma) \setminus 0)$ is a regular covering. Let k be the number of irreducible components of $\pi^{-1}(r(\gamma))$. The number k is constant for all $P \in E_i \cap E^0_Z$. Let P' be the only point of $(\pi_r)^{-1}(P)$. Remark 2.3.11, which uses Lemma 2.3.10, shows that the k irreducible components of the germ of curve $((\pi_r)^{-1}(\gamma), P')$ are separated by n. So, the restriction of the map $(\pi_r \circ n)$ to $((\pi_r \circ n)^{-1}(E_i \cap E^0_Z))$ is a regular covering of degree k. □

Definition 2.4.3 A germ $(W, 0)$ of complex surface is **quasi-ordinary** if there exists a finite morphism $\phi : (W, p) \to (\mathbb{C}^2, 0)$ which has a normal-crossing discriminant. A **Hirzebruch-Jung singularity** is a quasi-ordinary singularity of normal surface germ.

Lemma 2.4.4 *Let $\bar{P}$ be a point of $E_{\bar{Z}}$ which belongs to several irreducible components of $E^+_{\bar{Z}}$. Then $\bar{P}$ belongs to two irreducible components of $E^+_{\bar{Z}}$. Moreover, either $\bar{P}$ is a smooth point of $\bar{Z}$ and $E^+_{\bar{Z}}$ is a normal crossing divisor around $\bar{P}$, or $\bar{P}$ is a Hirzebruch-Jung singularity of $\bar{Z}$.*

Proof If $\bar{P}$ be a point of $E_{\bar{Z}}$ which belongs to several irreducible components of $E^+_{\bar{Z}}$ then $P = (\pi_r \circ n)(\bar{P})$ is a double point of E^+_Z. Moreover Z is smooth and E^+_Z is a normal crossing divisor. We can choose a closed neighbourhood N of P isomorphic to a product of discs $(D_1 \times D_2)$, and we take $\bar{N} = (\pi_r \circ n)^{-1}(N)$. For a sufficiently small N, the restriction of $(\pi_r \circ n)$ to the pair $(\bar{N}, \bar{N} \cap E^+_{\bar{Z}})$ is a finite ramified morphism over the pair $(\bar{N}, \bar{N} \cap E^+_{\bar{Z}})$ and the ramification locus is included in the normal crossing divisor $(N \cap E^+_Z)$. The pair $(\bar{N}, \bar{P})$ is normal and the link of the pair $(N, N \cap E^+_Z)$ is the Hopf link in S^3. Then the link of $\bar{N}$ is a lens space, and the link of $(\pi_r \circ n)^{-1}(N \cap E^+_Z)$ has two components (Lemma 2.5.4 gives a topological proof of this classical result). So, $E^+_{\bar{Z}}$ has two irreducible components at $\bar{P}$. We have two possibilities:

1. $\bar{P}$ is a smooth point in $\bar{Z}$. Then the link of the pair $(\bar{N}, \bar{N} \cap E^+_{\bar{Z}})$ is the Hopf link in S^3 and $E^+_{\bar{Z}}$ is a normal crossing divisor at $\bar{P}$.
2. $\bar{P}$ is an isolated singular point of $\bar{Z}$. Then, the link of $\bar{N}$ is a lens space which is not S^3. The point $\bar{P}$ is a Hirzebruch-Jung singularity of $\bar{Z}$ equipped with the finite morphism

$$(\pi_r \circ n)_{|\bar{N}} : (\bar{N}, \bar{N} \cap E^+_{\bar{Z}}) \to (N, N \cap E^+_Z)$$

which has the normal crossing divisor $N \cap E^+_Z$ as discriminant. □

The example given in Sect. 2.6 illustrates the following Corollary.

Corollary 2.4.5 *Let $G(\bar{Z})$ be the dual graph of $E_{\bar{Z}}$. Proposition 2.4.2 and Lemma 2.4.4 imply that $(\pi_r \circ n)$ induces a finite ramified covering of graphs from $G(\bar{Z})$ onto $G(Z)$.*

2.4.2 Second Step: Resolution of the Hirzebruch-Jung Singularities

If $\bar{P}$ is a singular point of $\bar{Z}$, then $P = (\pi_r \circ n)(\bar{P})$ is a double point of E^+_Z. In particular, there are finitely many isolated singular points in $\bar{Z}$. The singularities of $\bar{Z}$ are Hirzebruch-Jung singularities. More precisely, let $\bar{P}_i$, $1 \leq i \leq n$, be the finite set of the singular points of $\bar{Z}$ and let $\bar{U}_i$ be a sufficiently small neighborhood of $\bar{P}_i$ in $\bar{Z}$. We have the following result (see [9] for a proof, see also [11, 22] and [13]) and, to be self-contained, we give a proof in Sect. 2.5.3 (Proposition 2.5.7):

Theorem *The exceptional divisor of the minimal resolution of $(\bar{U}_i, \bar{P}_i)$ is a normal crossings divisor with smooth rational irreducible components and its dual graph is a bamboo (it means is homeomorphic to a segment).*

Let $\bar{\rho}_i : (U'_i, E_{U'_i}) \to (\bar{U}_i, \bar{P}_i)$ be the minimal resolution of the singularity $(\bar{U}_i, \bar{P}_i)$. From [13] (corollary 1.4.3), see also [22] (paragraph 4), the spaces U'_i and the maps $\bar{\rho}_i$ can be glued, for $1 \leq i \leq n$, in a suitable way to give a smooth space Y and a map $\bar{\rho} : (Y, E_Y) \to (\bar{Z}, E_{\bar{Z}})$ satisfying the following property (Fig. 2.4).

$$\begin{tikzcd}
(Y, E_Y) \arrow[rr, "\beta"] \arrow[d, "\bar{\rho}"'] \arrow[rrdd, "\rho"] & & (Y', E_{Y'}) \arrow[dd, "\rho'"] \\
(\bar{Z}, E_{\bar{Z}}) \arrow[rd, "n"] & & \\
& (Z', E_{Z'}) \arrow[r, "r_\pi"] \arrow[d, "\pi_r"'] & (X, 0) \arrow[d, "\pi"] \\
& (Z, E_Z) \arrow[r, "r"] & (\mathbb{C}^2, 0)
\end{tikzcd}$$

Fig. 2.4 The commutative diagram of the morphisms involved in the Hizebruch-Jung resolution ρ of π. By construction $\rho = r_\pi \circ n \circ \bar{\rho}$

Theorem 2.4.6 *Let us denote $\rho = r_\pi \circ n \circ \bar{\rho}$. Then, $\rho : (Y, E_Y) \to (X, p)$ is a good resolution of the singularity (X, p) in which the total transform $\rho^{-1}(\Gamma) = E_Y^+$ of the singular locus Γ of π is a normal crossings divisor.*

Proof The surface Y is smooth because $\bar{\rho}$ is a resolution of all the singular points of $\bar{Z}$. As proved in Proposition 2.4.2 and Lemma 2.4.4, the only possible singular points of the irreducible components of $E_{\bar{Z}}$ are the double points $\bar{P}_i$ of $E_{\bar{Z}}^+$. These points are resolved by the resolutions $\bar{\rho}_i$. So, the strict transform, by $\bar{\rho}$, of the irreducible components of $E_{\bar{Z}}$ are smooth.

The irreducible components of E_Y created during the resolution $\bar{\rho}$ are smooth rational curves. So, all the irreducible components of E_Y are smooth complex curves.

By Lemma 2.4.4, the only possible points of $E_{\bar{Z}}^+$ around which $E_{\bar{Z}}^+$ is not smooth or a normal crossing divisor are the Hirzebruch-Jung singularities $\bar{P}_i$, $1 \leq i \leq n$.

But as the $\bar{\rho}_i$, $1 \le i \le n$, are good resolutions of these singularities, $((\bar{\rho}_i)^{-1}(\bar{U}_i)) \cap (E_Y^+)$, $1 \le i \le n$, are normal crossing divisors. □

As ρ is the composition of three well defined morphisms which depend only on the choice of the morphism π and as we follow the Hirzebruch-Jung method, we have the following definition.

Definition 2.4.7 The morphism $\rho : (Y, E_Y) \to (X, 0)$ is the **Hirzebruch-Jung resolution** associated to π.

Corollary 2.4.8 *The dual graph $G(Y)$ of E_Y is obtained from the dual graph $G(\bar{Z})$ of $E_{\bar{Z}}$ by replacing the edges, which represent the Hirzebruch-Jung singular points of $\bar{Z}$, by a bamboo.*

Let $\rho'' : (Y'', E_{Y''}) \to (X, 0)$ be a good resolution of $(X, 0)$. Let E be an irreducible component of the exceptional divisor $E_{Y''}$ and let E^0 be the set of the smooth points of E in $E_{Y''}$. Let us recall that E is a **rupture component** of $E_{Y''}$ if the Euler characteristic of E^0 is strictly negative. Now we can use the following result (for a proof see [11, Theorem 5.9, p.87]):

Theorem *Let $\rho' : (Y', E_{Y'}) \to (X, 0)$ be the minimal resolution of $(X, 0)$. There exists $\beta : (Y, E_Y) \to (\tilde{Y}', E_{Y'})$ such that $\rho' \circ \beta = \rho$ and the map β consists in a composition of blowing-downs of irreducible components, of the successively obtained exceptional divisors, of self-intersection -1 and genus 0, which are not rupture components.*

2.5 Appendix: The Topology of a Quasi-ordinary Singularity of Surface

2.5.1 Lens Spaces

One can find details on lens spaces and surface singularities in [24]. See also [21].

Definition 2.5.1 A **lens space** L is an oriented compact three-dimensional topological manifold which can be obtained as the union of two solid tori $T_1 \cup T_2$ glued along their boundaries. The torus $\tau = T_1 \cap T_2$ is the Heegaard torus of the given decomposition $L = T_1 \cup T_2$.

Remark 2.5.2 If L is a lens space, there exists an embedded torus τ in L such that $L \setminus \tau$ has two connected components which are open solid tori $\overset{\circ}{T}_i, i = 1, 2$. Let $T_i, i = 1, 2$, be the two compact solid tori closure of $\overset{\circ}{T}_i$ in L. Of course $\tau = T_1 \cap T_2$. In [3], F. Bonahon shows that a lens space has a unique, up to isotopy, Heegaard torus. This implies that the decomposition $L = T_1 \cup T_2$ is unique up to isotopy, it is "the" Heegaard decomposition of L.

A lens space L with a decomposition of Heegaard torus τ can be described as follows. The solid tori $T_i, i = 1, 2$, are oriented by the orientation induced by L.

Let τ_i be the torus τ with the orientation induced by T_i. By definition a meridian m_i of T_i is a closed oriented circle on τ_i which is the boundary of a disc D_i embedded in T_i. A meridian of a solid torus is well defined up to isotopy. A parallel l_i of T_i is a closed oriented curve on τ_i such that the intersection $m_i \cap l_i = +1$ (we also write m_i (resp. l_i) for the homology class of m_i (resp. l_i) in the first homology group of τ_i). The homology classes of two parallels differ by a multiple of the meridian.

We choose on τ_2, an oriented meridian m_2 and a parallel l_2 of the solid torus T_2. As in [24, p. 23], we write a meridian m_1 of T_1 as $m_1 = nl_2 - qm_2$ with $n \in \mathbb{N}$ and $q \in \mathbb{Z}$ where q is well defined modulo n. As m_1 is a closed curve on τ, q is prime to n. Moreover, the class of q modulo n depends on the choice of l_2. So, we can chose l_2 such that $0 \leq q < n$.

Let τ be a boundary component of an oriented compact three-dimensional manifold M. Let T be a solid torus given with a meridian m on its boundary. If γ is a circle embedded in τ there is a unique way to glue T to M by an orientation reversing homeomorphism between the boundary of T and τ which send m to γ. The result of such a gluing is unique up to orientation preserving homeomorphism and it is called the **Dehn filling** of M associated to γ.

Definition 2.5.3 By a Dehn filling argument, it is sufficient to know the homology class $m_1 = nl_2 - qm_2$ to reconstruct L. By definition **the lens space** $L(n, q)$ is the lens space constructed with $m_1 = nl_2 - qm_2$. We have two special cases:

1. $m_1 = m_2$, if and only if L is homeomorphic to $S^1 \times S^2$,
2. $m_1 = l_2$ if and only if L is homeomorphic to S^3.

Lemma 2.5.4 *Let $\phi : (W, p) \to (\mathbb{C}^2, 0)$ be a finite morphism defined on an irreducible surface germ (W, p). If the discriminant Δ of ϕ is included in a normal crossing germ of curve, then the link L_W of (W, p) is a lens space. The link K_Γ of the singular locus Γ of ϕ, has at most two connected components. Moreover, K_Γ is a sub-link of the two cores of the two solid tori of a Heegaard decomposition of L_W as a union of two solid tori.*

Proof After performing a possible analytic isomorphism of $(\mathbb{C}^2, 0)$, Δ is, by hypothesis, included in the two axes i.e. $\Delta \subset \{xy = 0\}$.

Let $D_\alpha \times D_\beta \in \mathbb{C}^2$ be a polydisc at the origin in $\mathbb{C}^2$ where $0 < \alpha < \beta < \epsilon$ are chosen sufficiently small as in Sect. 2.3.1. Then, the restriction ϕ_L of ϕ on the link L_W is a ramified covering of the sphere (with corners)

$$\mathcal{S} = (S_\alpha \times D_\beta) \cup (D_\alpha \times S_\beta)$$

with a set of ramification values included in the Hopf link $K_{xy} = (S_\alpha \times \{0\}) \cup (\{0\} \times S_\beta)$.

Let $N(K_{xy})$ be a small compact tubular neighborhood of K_{xy} in $\mathcal{S}$. Then, $N(K_{xy})$ is the union of two disjoint solid tori $T_y = (S_\alpha \times D_{\beta'}), 0 < \beta' < \beta$, and $T_x = (D_{\alpha'} \times S_\beta), 0 < \alpha' < \alpha$. Then, $\phi_L^{-1}(T_x)$ (resp. $\phi_L^{-1}(T_y)$) is a union of

$r_x > 0$ (resp. $r_y > 0$) disjoint solid tori because the set of the ramification values of ϕ_L is included in the core of T_x (resp. T_y).

Let V be the closure, in $\mathcal{S}$, of $\mathcal{S} \setminus N(K_{xy})$. But, V is a thickened torus which does not meet the ramification values of ϕ_L. Then, $\phi_L^{-1}(V)$ is a union of $r > 0$ disjoint thickened tori. But, L_W is connected because (W, p) is irreducible by hypothesis. The only possibility to obtain a connected space by gluing $\phi_L^{-1}(T_x)$, $\phi_L^{-1}(T_y)$ and $\phi_L^{-1}(V)$ along their boundaries is $1 = r = r_x = r_y$.

So, $\phi_L^{-1}(T_x)$ (resp. $\phi_L^{-1}(T_y)$) which is in L_W a deformation retract of $T_2 = \phi_L^{-1}(S_\alpha \times D_\beta)$ (resp. $T_1 = \phi_L^{-1}(D_\alpha \times S_\beta)$) is a single solid torus. Then $\tau = \phi_L^{-1}(S_\alpha \times S_\beta)$ is a single torus. We have proved that L_W is the lens space obtained as the union of the two solid tori T_1 and T_2 along their common boundary $\tau = \phi_L^{-1}(S_\alpha \times S_\beta)$. So, $T_1 \cup T_2$ is a Heegaard decomposition of L_W as a union of two solid tori.

By hypothesis $K_\Delta \subset (S_\alpha \times \{0\}) \cup (\{0\} \times S_\beta)$. Then, K_Γ is included in the disjoint union of $\phi_L^{-1}(S_\alpha \times \{0\})$ and $\phi_L^{-1}(\{0\} \times S_\beta)$ which are the cores of T_1 and T_2. So, K_Γ has at most two connected components. □

Example 2.5.5 Let n and q be two relatively prime strictly positive integers. We suppose that $q < n$. Let $X = \{(x, y, z) \in \mathbb{C}^3 \ s.t. \ z^n - xy^q = 0\}$. The link L_X of $(X, 0)$ is the lens space $L(n, n - q)$.

Indeed, let $\phi : (X, 0) \to (\mathbb{C}^2, 0)$ be the projection $(x, y, z) \mapsto (x, y)$ restricted to X. The discriminant Δ of ϕ is equal to $\{xy = 0\}$. By Lemma 2.5.4, L_X is a lens space. As in the proof of Lemma 2.5.4, $L_X = \phi^{-1}(\mathcal{S})$ where

$$\mathcal{S} = (S_\alpha \times D_\beta) \cup (D_\alpha \times S_\beta).$$

In the proof of Lemma 2.5.4, it is shown that $T_2 = \phi^{-1}(S_\alpha \times D_\beta)$ and $T_1 = \phi^{-1}(D_\alpha \times S_\beta)$) are two solid tori. Let $(a, b) \in (S_\alpha \times S_\beta)$. As n and q are relatively prime $m_1 = \phi^{-1}(\{a\} \times S_\beta)$ and $m_2 = \phi^{-1}(S_\alpha \times \{b\})$ are connected. So, $m_i, \ i = 1, 2$, is a meridian of T_i.

We choose $c \in \mathbb{C}$ such that $c^n = ab^q$. Let $l_2 = \{z = c\} \cap \phi^{-1}(S_\alpha \times S_\beta)$. On the torus $\tau = \phi^{-1}(S_\alpha \times S_\beta)$, oriented as the boundary of T_2, we have $m_2 \cap l_2 = +1$ and $m_1 = nl_2 - (-q)m_2$. As defined in Definition 2.5.3, we have $L_X = L(n, -q) = L(n, n - q)$.

2.5.2 Finite Morphisms with Smooth Discriminant

Lemma 2.5.6 *Let $\phi : (W, p) \to (\mathbb{C}^2, 0)$ be a finite morphism, of generic degree n, defined on a normal surface germ (W, p). If the discriminant of ϕ is a smooth germ of curve, then $(X, 0)$ is analytically isomorphic to $(\mathbb{C}^2, 0)$ and ϕ is analytically isomorphic to the map from $(\mathbb{C}^2, 0)$ to $(\mathbb{C}^2, 0)$ defined by $(x, y) \mapsto (x, y^n)$.*

Proof After performing an analytic automorphism of $(\mathbb{C}^2, 0)$, we can choose coordinates such that $\Delta = \{y = 0\}$.

Let $D_\alpha \times D_\beta \in \mathbb{C}^2$ be a polydisc at the origin in $\mathbb{C}^2$ where $0 < \alpha < \beta < \epsilon$ are chosen sufficiently small as in Sect. 2.3.1. Then, the restriction ϕ_L of ϕ on the link L_W is a ramified covering of the sphere (with corners)

$$\mathcal{S} = (S_\alpha \times D_\beta) \cup (D_\alpha \times S_\beta)$$

with a set of ramification values included in the trivial link $K_y = (S_\alpha \times \{0\})$.

Here, we satisfy the hypotheses of Lemma 2.5.4. So, $T_2 = \phi_L^{-1}(S_\alpha \times D_\beta)$ and $T_1 = \phi_L^{-1}(D_\alpha \times S_\beta)$ are two solid tori with common boundary $\tau = \phi_L^{-1}(S_\alpha \times S_\beta)$. We take $a \in S_\alpha$ and $b \in S_\beta$.

Let us consider $\mathcal{D}_a = \phi_L^{-1}(\{a\} \times D_\beta) \subset T_2$ and $\mathcal{D}_b = \phi_L^{-1}(D_\alpha \times \{b\}) \subset T_1$.

Here the singular locus of ϕ_L is the core of T_2 and does not meet T_1.

The restriction of ϕ_L to $\phi_L^{-1}(D_\alpha \times \{b\})$ is a regular covering of a disc. Then $\mathcal{D}_b$ is a disjoint union of n discs where n is the general degree of ϕ_L. Let m_1 be the oriented boundary of one of the n discs which are the connected components of $\mathcal{D}_b$. By definition m_1 is a meridian of T_1.

The restriction of ϕ_L to $\mathcal{D}_a$ is a covering of a disc and $(a \times 0)$ is the only ramification value . Then $\mathcal{D}_a$ is a disjoint union of d discs where $d < n$. On τ, the intersection between the circles boundaries of $\mathcal{D}_a$ and $\mathcal{D}_b$ is equal to n because it is given by the (positively counted) n points of $\phi_L^{-1}(a \times b)$. The restriction of ϕ_L to T_1 is a Galois covering of degree n which permutes cyclically the connected components of $\mathcal{D}_a$. So, on the torus $\tau = b(T_1)$, any of the d circles boundaries of the connected components of $\mathcal{D}_a$ intersects any of the n circles boundaries of the connected components of $\mathcal{D}_b$. So computed, the intersection $b(\mathcal{D}_a) \cap b(\mathcal{D}_b)$ is equal to nd. But, $nd = n$ because this intersection is given by the n points of $\phi_L^{-1}(a \times b)$.

So, $d = 1$ and $\mathcal{D}_a$ has a unique connected component. The boundary of $\mathcal{D}_a$ is a meridian m_2 of T_2. As m_1 is the boundary of one of the n connected components of $\mathcal{D}_b$, $m_1 \cap m_2 = +1$ and m_1 can be a parallel l_2 of T_2. This is the case 2) in Definition 2.5.3, so the link L_W of (W, p) is the 3-sphere S^3. As (W, p) is normal, by Mumford [19], (W, p) is a smooth surface germ i.e (W, p) is analytically isomorphic to $(\mathbb{C}^2, 0)$. The first part of Lemma 2.5.6 has been proved.

(*) Moreover $\phi_L^{-1}(S_\alpha \times \{0\}) \cup (\{0\} \times S_\beta)$ is the union of the cores of T_1 and T_2. Then, $(S_\alpha \times \{0\}) \cup (\{0\} \times S_\beta)$ **is a Hopf link in the 3-sphere** L_W.

From now on, $\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is a finite morphism and its discriminant locus is $\{y = 0\}$. Let us write $\phi = (\phi_1, \phi_2)$. The link of the zero locus of the function germ

$$(\phi_1.\phi_2) : (\mathbb{C}^2, 0) \to (\mathbb{C}`0)$$

is the link describe above (see (*)), i.e. it is a Hopf link. The function $(\phi_1.\phi_2)$ reduced is analytically isomorphic to $(x, y) \mapsto (xy)$. But ϕ_1 is reduced because its Milnor fiber is diffeomorphic to $\mathcal{D}_a = \phi_L^{-1}(\{a\} \times D_\beta) \subset T_2$ which is a disc. So, ϕ_1 is isomorphic to x.

The Milnor fiber of ϕ_2 is diffeomorphic to the disjoint union of the n discs $\mathcal{D}_b = \phi_L^{-1}(D_\alpha \times \{b\}) \subset T_1$. When the Milnor fiber of a function germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ has n connected components, n is the $g.c.d.$ of the multiplicities of the irreducible factors of f. Here $\phi_2 = g^n$ where g is an irreducible function germ. We already have seen that ϕ_2 reduced is isomorphic to y. This completes the proof that ϕ_2 is isomorphic to y^n and $\phi = (\phi_1, \phi_2)$ is isomorphic to (x, y^n). □

2.5.3 The Hirzebruch-Jung Singularities

Proposition 2.5.7 *Let (W, p) be a normal surface germ such that there exists a finite morphism $\phi : (W, p) \to (\mathbb{C}^2, 0)$ which has a normal-crossing discriminant $(\Delta, 0)$. Then, (W, p) has a minimal good resolution $\rho : (\tilde{W}, E_{\tilde{W}}) \to (W, p)$ such that:*

I) the exceptional divisor $E_{\tilde{W}}$ of ρ has smooth rational irreducible components and its dual graph is a bamboo. We orient the bamboo from the vertex (1) to the vertex (k). The vertices are indexed by this orientation,

II) the strict transform of $\phi^{-1}(\Delta)$ has two smooth irreducible components which meet $E_{\tilde{W}}$ transversally, one of them at a smooth point of E_1 and the other component at a smooth point of E_k.

Proof After performing an analytic isomorphism of $(\mathbb{C}^2, 0)$, we can choose coordinates such that $\Delta = \{xy = 0\}$. We have to prove that there exists a minimal resolution ρ of (W, p) such that the shape of the dual graph of the total transform of Δ in $\tilde{W}$ looks like the graph drawn in Fig. 2.5 where all vertices represent smooth rational curves.

By Lemma 2.5.4, the link L_W of (W, p) is a lens space. If L_W is homeomorphic to S^3, (W, p) is smooth by Mumford [19], and there is nothing to prove. Otherwise, let n and q be the two positive integers, prime to each other, with $0 < q < n$, such that L_W is the lens space $L(n, n - q)$. By Brieskorn [4] (see also Sect. 2.5 in [24]), the normal quasi-ordinary complex surface germs are taut. It means that any normal quasi-ordinary complex surface germ (W', p') which has a link orientation preserving homeomorphic to $L(n, n - q)$ is analytically isomorphic to (W, p). In

Fig. 2.5 The shape of the dual graph of $G(\tilde{W})$ to which we add an arrow to the vertex (1) to represent the strict transform of $\{x = 0\}$ and another arrow to the vertex (k) to represent the strict transform of $\{y = 0\}$

particular, (W, p) and (W', p') have isomorphic minimal good resolutions. Now, it is sufficient to describe the good minimal resolution of a given normal quasi-ordinary surface germ which has a link homeomorphic to $L(n, n-q)$. As explained below, we can use $(\bar{X}, \bar{p})$ where $\nu : (\bar{X}, \bar{p}) \to (X, 0)$ is the normalization of $X = \{(x, y, z) \in \mathbb{C}^3 \ s.t. \ z^n - xy^q = 0\}$. □

Lemma 2.5.8 *Let n and q be two relatively prime positive integers. We suppose that* $0 < q < n$. *Let* $X = \{(x, y, z) \in \mathbb{C}^3 \ s.t. \ z^n - xy^q = 0\}$. *There exists a good resolution* $\rho_Y : (Y, E_Y) \to (X, 0)$ *of* $(X, 0)$ *such that the dual graph* $G(Y)$ *of* E_Y *is a bamboo and the dual graph of the total transform of* $\{xy = 0\}$ *has the shape of the graph given in Fig. 2.5.*

Lemma 2.5.8 implies Proposition 2.5.7. Indeed:

1) In Example 2.5.5, we show that the link L_X of $(X, 0)$ is the lens space $L(n, n-q)$. Let $\nu : (\bar{X}, \bar{p}) \to (X, 0)$ be the normalization of $(X, 0)$. The singular locus of $(X, 0)$ is the line $\Sigma = \{(x, 0, 0), x \in \mathbb{C}\}$. For $a \in \mathbb{C}$, the hyperplane section of X at $(a, 0, 0)$ is the plane curve germ $\{z^n - ay^q = 0\}$. As n and q are prime to each other $\{z^n - ay^q = 0\}$ is irreducible. Then, by Proposition 2.3.12, ν is a homeomorphism. So, the link $L_{\bar{X}}$ of $(\bar{X}, 0)$ is the lens space $L(n, n-q)$.
2) Let $\rho_Y : (Y, E_Y) \to (X, p)$ be a good resolution of $(X, 0)$ given as in Lemma 2.5.8, in particular the dual graph $G(Y)$ of E_Y is a bamboo. As any good resolution factorizes through the normalization $\nu : (\bar{X}, \bar{p}) \to (X, 0)$ (see [11, Thm. 3.14]), there exists a unique morphism $\bar{\rho_Y} : (Y, E_Y) \to (\bar{X}, \bar{p})$ which is a good resolution of $(\bar{X}, \bar{p})$. Let $\rho' : (Y', E_{Y'}) \to (\bar{X}, \bar{p})$ be the minimal good resolution of $(\bar{X}, \bar{p})$. Then, (for example see [11, Thm 5.9] or [1, p. 86]), there exists a morphism $\beta : (Y, E_Y) \to (Y', E_{Y'})$ which is a sequence of blowing-downs of irreducible components of genus zero and self-intersection -1. By Lemma 2.5.8, the dual graph $G(Y)$ is a bamboo and the dual graph of the total transform of $\{xy = 0\}$ has the shape of the graph given in Fig. 2.5. So, the morphism of graph $\beta* : G(Y) \to G(Y')$ induced by β, is only a contraction of $G(Y)$ in a shorter bamboo.

Proof (of Lemma 2.5.8) In X, we consider the lines $l_x = \{(x, 0, 0), x \in \mathbb{C}\}$ and $l_y = \{(0, y, 0), y \in \mathbb{C}\}$ and the singular locus of $(X, 0)$ is equal to l_x. We prove Lemma 2.5.8 by a finite induction on $q \geq 1$.

1) If $q = 1$, $X = \{(x, y, z) \in \mathbb{C}^3 \ s.t. \ z^n - xy = 0\}$ is the well-known normal singularity A_{n-1}. The minimal resolution is a bamboo of $(n-1)$ irreducible components of genus zero. Indeed, to construct $\rho_Y : (Y, E_Y) \to (X, 0)$, it is sufficient to perform a sequence of blowing-ups of points (we blow up $n/2$ points when n is even and $(n-1)/2$ points when n is odd). We begin to blow up the origin, this separates the strict transform of the lines l_x and l_y. The exceptional divisor, in the strict transform of $(X, 0)$ by the blowing-up of the origin in $\mathbb{C}^3$, has two irreducible rational components when $n > 2$ and only one irreducible rational component when $n = 2$. If $n > 2$, we continue by the

blowing-up of the intersection point of the two irreducible components of the exceptional divisor.

2) If $1 < q < n$, we state the following points I and II which describe how we proceed, we justify them just below.

   I) As n and q are relatively prime , the remainder r of the division $n = mq + r$ is prime to q and $1 < r < q$. Let $R : Z \to \mathbb{C}^3$ be a sequence of m blowing-ups of the line l_x in $\mathbb{C}^3$ and of its strict transforms in a smooth 3-dimensional complex space. Let Y_1 be the strict transform of X by R. Let $\rho : (Y_1, E) \to (X, 0)$ be R restricted to Y_1 and let $E = \rho{-}1(0) \subset Y_1$. The total transform of $l_x \cup l_y$ by ρ, which is equal to $E^+ = \rho^{-1}(l_x \cup l_y)$, has a dual graph which is a bamboo as in Fig. 2.5 with $k = m$ vertices. Let l_x^1 be the strict transform of l_x by ρ. Then, l_x^1 only meets the irreducible component of E obtained by the last blowing-up of a line. The equation of Y_1 along l_x^1 is $\{z^r - xy^q = 0\}$.

   II) If $r = 1$, Y_1 is smooth and Lemma 2.5.8 is proved i.e. $\rho_Y = \rho$. If $r > 2$, after the division $q = m'r + r'$ with remainder r', we have $r' < r$. As r is prime to q, r' is prime to r and $0 < r'$. Moreover, we have $r' < q$ because $r < q$. Let $R' : Z' \to Z$ be a sequence of m' blowing-ups of the line l_x^1 and of its strict transforms. Let Y_2 be the strict transform of Y_1 by R' and let $\rho' : (Y_2, E') \to (Y_1, E)$ be R' restricted to Y_2. As $r < q$, ρ' is bijective, the dual graph of $\rho'^{-1}(E^+)$ is equal to the dual graph of E^+, which is a bamboo as in Fig. 2.5 with $k = m$ vertices. Moreover, the equation of Y_2, along the strict transform of l_x^1 by ρ', is $\{z^r - xy^{r'} = 0\}$. As $1 \leq r' < r$ with relatively prime r and r', Lemma 2.5.8 is proved by induction.

Let us justify the above statements I) and II) by an explicit computation of the blowing-up of l_x. We consider $Z_1 = \{((x, y, z), (v : w)) \in \mathbb{C}^3 \times \mathbb{C}P^1, s.\ t.\ wy - vz = 0\}$. By definition, the blowing-up of l_x in $\mathbb{C}^3$, $R_1 : Z_1 \to \mathbb{C}^3$, is the projection on $\mathbb{C}^3$ restricted to Z_1.

As in statement I), we consider $X = \{(x, y, z) \in \mathbb{C}^3\ s.t.\ z^n - xy^q = 0\}$ with $q < n$. We have to describe the strict transform Y_{11} of $(X, 0)$ by R_1, the restriction $\rho_1 : (Y_{11}, E) \to (X, 0)$ of R_1 to Y_{11}, $E_1 = \rho_1^{-1}(0)$ and $E_1^+ = \rho_1^{-1}(l_x \cup L_y)$.

I) In the chart $v = 1$, we have $(Z_1 \cap \{v = 1\}) = \{((x, y, wy), (1 : w)) \in \mathbb{C}^3 \times \mathbb{C}P^1\}$. The equation of $R_1^{-1}(0) \cap \{v = 1\}$ and of $E_1 \cap \{v = 1\}$ is $y = 0$.
The equation of $(R_1^{-1}(X) \cap \{v = 1\}) = (Y_{11} \cap \{v = 1\})$ is $\{w^n y^{n-q} - x = 0\}$. So, all the points of $(\{v = 1\} \cap Y_{11})$ are non singular and $(\{v = 1\} \cap \{x \neq 0\} \cap Y_{11})$ doesn't meet E_1.
The strict transform of l_x is not in $Y_{11} \cap \{v = 1\}$. If $x = 0$, we have:

$$E_1 \cap \{v = 1\} = \{((0, 0, 0), (1 : w)), w \in \mathbb{C}\} \subset Y_{11}.$$

In Y_{11}, the strict transform $\tilde{l_y} = \{((0, y, 0), (1 : 0)), y \in \mathbb{C}\}$ of l_y meets E_1 at $((0, 0, 0), (1 : 0))$.

II) In the chart $w = 1$, we have $(Z_1 \cap \{w = 1\}) = \{((x, vz, z), (v : 1)) \in \mathbb{C}^3 \times \mathbb{C}P^1\}$. The equation of $R_1^{-1}(0) \cap \{w = 1\}$ and of $E_1 \cap \{w = 1\}$ is $z = 0$.
The equation of $(Y_{11} \cap \{w = 1\})$ is $\{z^{n-q} - xv^q = 0\}$. So, the strict transform of l_x is equal to

$$\tilde{l_x} = (\{w = 1\} \cap Y_{11} \cap R_1^{-1}(l_x)) = \{((x, 0, 0), (0 : 1)) \in \mathbb{C}^3 \times \mathbb{C}P^1\}.$$

The strict transform $\tilde{l_x}$ meets E_1 at the point $p_1 = E_1 \cap \tilde{l_x} = ((0, 0, 0), (0 : 1)$. Then, $E_1 = ((0, 0, 0) \times \mathbb{C}P^1)$ is included in Y_{11}, moreover, $\tilde{l_x}$ and $\tilde{l_y}$ meet E_1 at two distinct points. The total transform $E_1^+ = \rho_1^{-1}(l_x \cup L_y)$ consists of one irreducible component E_1 and two germs of curves which meet E_1 in two distinct points. Moreover the equation of Y_{11} along its singular locus $\tilde{l_x}$ is $\{z^{n-q} - xy^q = 0\}$. By induction we obtain, as stated in I), the germ $(Y_1, 0)$ defined by $\{z^r - xy^q = 0\}$ with $1 \leq r = n - mq < q$.

To justify statement II), we again consider the blowing-up of l_x, $R_1 : Z_1 \to \mathbb{C}^3$. Let Y_{12} be the strict transform of Y_1 by R_1 and let $\rho_1' : Y_{12} \to Y_1$ be R_1 restricted to Y_{12}. Then, Y_{12} has the equation $\{w^r - xy^{q-r} = 0\}$ in the chart $v = 1$. For all $x \in \mathbb{C}$, the intersection of Y_{12} with $y = 0$ is the only point $((x, 0, 0)), (1 : 0))$. In the chart $w = 1$, Y_{12} has the equation $\{1 - xv^q z^{q-r} = 0\}$ and has empty intersection with $z = 0$. This proves that ρ_1' is bijective and by induction the map $\rho' : (Y_2, E') \to (Y_1, E)$ describe above in II) is also bijective. □

Examples

1) Let us consider $X = \{(x, y, z) \in \mathbb{C}^3 \ s.t. \ z^n - xy^{n-1} = 0\}$. The link of $(X, 0)$ is the lens space $L(n, 1)$. Let $R_1 : Z_1 \to \mathbb{C}^3$ be the blowing-up of the line l_x in $\mathbb{C}^3$. Let Y be the strict transform of X by R_1. The equation of Y along the strict transform of l_x is $\{z - xy^{n-1} = 0\}$. So, Y is non singular and we have obtained a resolution of X. Here the dual graph of the total transform of $l_x \cup l_y$ is as in Fig. 2.5 with only one vertex.
2) Let us consider $X = \{(x, y, z) \in \mathbb{C}^3 \ s.t. \ z^n - xy^{n-2} = 0\}$ with n odd and $3 < n$. The link of $(X, 0)$ is the lens space $L(n, 2)$. Let $R_1 : Z_1 \to \mathbb{C}^3$ be the blowing-up of the line l_x in $\mathbb{C}^3$. The equation of the strict transform Y_1, of X by R_1, along the strict transform of l_x is $\{z^2 - xy^{n-2} = 0\}$. Let $\rho : (Y_1, E) \to (X, 0)$ be R_1 restricted to Y_1. We write $n = 2m + 3$. As proved above, after m blowing-ups of lines, we obtain a surface Y_2 and a bijective morphism $\rho' : (Y_2, E') \to (Y_1, E)$ such that the equation of Y_2 along the strict transform of l_x is $\{z^2 - xy = 0\}$. The exceptional divisor E of ρ (resp. $E'$of $(\rho \circ \rho')$) is an irreducible smooth rational curve. The blowing-up ρ'', of the intersection point between E' and the strict transform of l_x, is a resolution of Y_2 and the exceptional divisor of ρ'' is a smooth rational curve. Then, $\rho \circ \rho' \circ \rho''$ is a resolution of $X = \{(x, y, z) \in \mathbb{C}^3 \ s.t. \ z^n - xy^{n-2} = 0\}$, the dual graph of its exceptional divisor is a bamboo with two vertices.

2.6 An Example of Hirzebruch-Jung's Resolution

We give the Hirzebruch-Jung resolution of the germ of surface in $\mathbb{C}^3$ which satisfies the following equation:

$$z^2 = (x - y + y^3)(x - y + y^2)(y^{34} - (x - y)^{13}).$$

where $\pi : (X, 0) \to (\mathbb{C}^2, 0)$ is the projection on the (x, y)-plane. It is a generic projection. In [15] this example is also explored when π is replaced by a non generic projection.

The discriminant locus of $\pi = (f, g)$ is the curve Δ which has three components with Puiseux expansions given by :

$$x = y - y^2$$

$$x = y - y^3$$

$$x = y + y^{34/13}$$

Notice that the three components of Δ have 1 as first Puiseux exponent and respectively 2, 3, 34/13 as second Puiseux exponent.

The coordinate axes are transverse to the discriminant locus of π. The dual graph $G(Z)$ is in Fig. 2.6.

The dual graph $G(\bar{Z})$ of $E_{\bar{Z}}$ admits a cycle created by the normalization. The irreducible component E'_9 of E_Y is obtained by the resolution $\bar{\rho}$. The irreducible components of the exceptional divisor associated to the vertices of $G(\bar{Z})$ and $G(Y)$ have genus equal to zero (Fig. 2.7).

The minimal good resolution ρ is obtained by blowing down E'_6. Its dual graph is in Fig. 2.8.

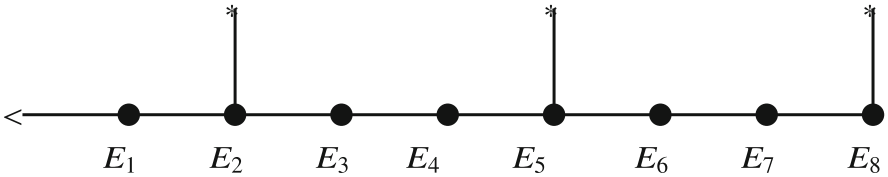


Fig. 2.6 The dual graph of the minimal resolution of Δ. An irreducible component of the strict transform of Δ is represented by an edge with a star. An edge ended by an arrow represents the strict transform of $\{x = 0\}$

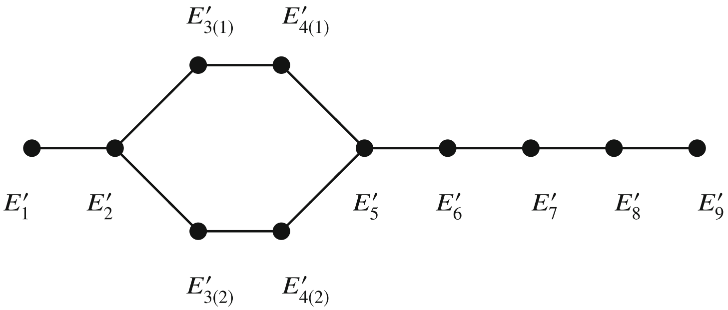


Fig. 2.7 The dual graph $G(Y)$ of the Hirzebruch-Jung resolution associated to π

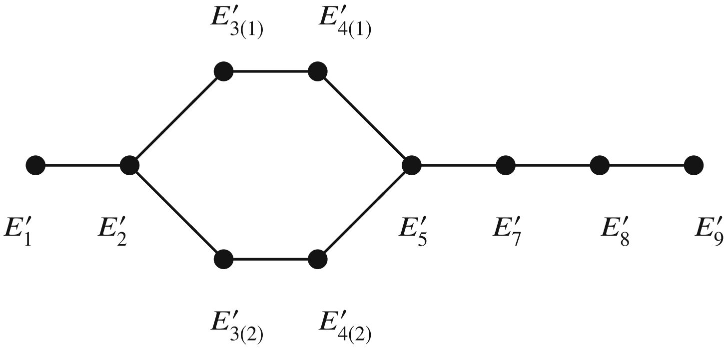


Fig. 2.8 The dual graph $G(Y')$ of the minimal resolution of $(X, 0)$

Acknowledgments I thank Claude Weber for useful discussions.

References

1. W. Barth, C. Peters, A. Van de Ven: *Compact Complex Surfaces,* Ergebnisse der Mathematik, Springer (1984). 154 and 177
2. J. Fernández de Bobadilla: *A reformulation of Lê's conjecture*, Indag. Math.,N.S.,17 (2006), p. 345–352. 155
3. F. Bonahon: *Difféctopies des espaces lenticulaires*, Topology 22 (1983), p. 305–314. 172
4. E. Brieskorn: *Rationale Singularitaten komplexer Flachen,* Invent. Math. 4 (1968), p. 336–358. 176
5. E. Brieskorn: *Singularities in the work of Friedrich Hirzebruch,* Surveys in Differential Geometry 2000, Vol VII (2000), International Press p. 17–60. 155
6. E. Brieskorn and H. Knörrer: *Ebene algebraische Kurven,* Birkhäuser Verlag (1981) 964 p. or *Plane Algebraic Curves,* Birkhäuser Verlag, (1986). 167
7. A. Durfee: *Neighborhoods of algebraic sets,* Trans. Amer. Math. Soc. **276** (1983), 517–530. 153 and 158
8. D. Eisenbud, W: Neumann: *Three-Dimensional Link Theory and Invariants of Plane Curve Singularities,* Annals of Math. Studies 110, Princeton University Press (1985). 160

9. Friedrich Hirzebruch: *Über vierdimensionale Riemannsche Flächen mehrdeutiger analytischer Funktionen von zwei komplexen Veränderlichen*, Math. Ann. 126 (1953) p. 1–22. 155, 156, and 170
10. F. Hirzebruch, W. Neumann and S. Koh: *Differentiable manifolds and quadratic forms*, Math. Lecture Notes, vol 4 (1972), Dekker, New-York. 154
11. H. Laufer: *Normal two dimensional singularities*, Ann. of Math. Studies **71**, (1971), Princeton Univ. Press. 154, 170, 172, and 177
12. D. T. Lê, F. Michel and C. Weber: *Courbes polaires et topologie des courbes planes,* Ann. Scient. Ec. Norm. Sup., série 4, **24** (1991), 141–169. 160
13. D.T. Lê, C. Weber: *Résoudre est un jeu d'enfants*, Sem. Inst. de Estud. con Ibero-america y Portugal, Tordesillas (1998). 170
14. I. Luengo and A. Pichon: *Lê 's conjecture for cyclic covers*, Séminaires et congrès 10, (2005), p. 163–190. Publications de la SMF, Ed. J.-P. Brasselet and T. Suwa. 155
15. H.Maugendre, F.Michel: *On the growth behaviour of Hironaka quotients*, ArXiv Mathematics 2017. Revised version in Journal of Singularities, Vol. 20 (2020), p. 31–53. 155 and 180
16. F. Michel, A. Pichon and C. Weber: *The boundary of the Milnor fiber of Hirzebruch surface singularities*, 745–760 in *Singularity theory*, World Sci. Publ. (2007), Hackensack, NJ. 154
17. F. Michel, C. Weber: *Topologie des germes de courbes planes*, prépublication de l'université de Genève, (1985). 160 and 167
18. J. Milnor: *Singular Points of Complex Hypersurfaces*, Annals of Mathematical Studies 61 (1968), Princeton Univ. Press. 152 and 158
19. D. Mumford: *The topology of normal singularities of an algebraic surface and a criterion for simplicity*, Inst. Hautes Etudes Sci. Publ. Math. *9* (1961), p. 5–22. 154, 156, 169, 175, and 176
20. W. Neumann: *A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves*, Trans. Amer. Math. Soc. **268** (1981), p. 299–344. 154, 160, and 161
21. P. Popescu-Pampu: *Two-dimensional iterated torus knots and quasi-ordinary surface singularities*, C.R.A.S. de Paris 336 (2003), p. 651–656. 172
22. P. Popescu-Pampu: *Introduction to Jung's method of resolution of singularities*, in Topology of Algebraic Varieties and Singularities. Proceedings of the conference in honor of the 60th birthday of Anatoly Libgober, J. I. Cogolludo-Agustin et E. Hironaka eds. Contemporary Mathematics 538, AMS, (2011), p. 401–432. 170
23. F. Waldhausen: *Über eine Klasse von 3-dimensionalen Mannigfaltigkeiten*, Invent. Math. **3** (1967), p. 308–333 and **4** (1967), p. 87–117. 154
24. C. Weber: *Lens spaces among 3-manifolds and quotient surface singularities*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Sci. A Mat. RACSAM 112 (2018), p. 893–914. 172, 173, and 176

Chapter 3
Resolution of Singularities: An Introduction

Mark Spivakovsky

Contents

M. Spivakovsky (✉)
Institut de Mathématiques de Toulouse, UMR CNRS 5219, Toulouse, Cedex, France

Unidad Mixta Internacional 2001 CNRS, Laboratorio Solomon Lefschetz, Cuernavaca, Mexico

Unidad Cuernavaca, Instituto de Matemáticas, UNAM, Cuernavaca, Mexico
e-mail: mark.spivakovsky@math.univ-toulouse.fr

J. L. Cisneros Molina et al. (eds.), *Handbook of Geometry and Topology of Singularities I*, https://doi.org/10.1007/978-3-030-53061-7_3

Abstract The problem of resolution of singularities and its solution in various contexts can be traced back to I. Newton and B. Riemann. This paper is an attempt to give a survey of the subject starting with Newton till the modern times, as well as to discuss some of the main open problems that remain to be solved. The main topics covered are the early days of resolution (fields of characteristic zero and dimension up to three), Zariski's approach via valuations, Hironaka's celebrated result in characteristic zero and all dimensions and its subsequent strengthenings and simplifications, existing results in positive characteristic (mostly up to dimension three), de Jong's approach via semi-stable reduction, Nash and higher Nash blowing up, as well as reduction of singularities of vector fields and foliations. In many places, we have tried to summarize the main ideas of proofs of various results without getting too much into technical details.

3.1 Introduction

Let X be a singular irreducible algebraic variety. A **resolution of singularities** of X is a **birational proper morphism**

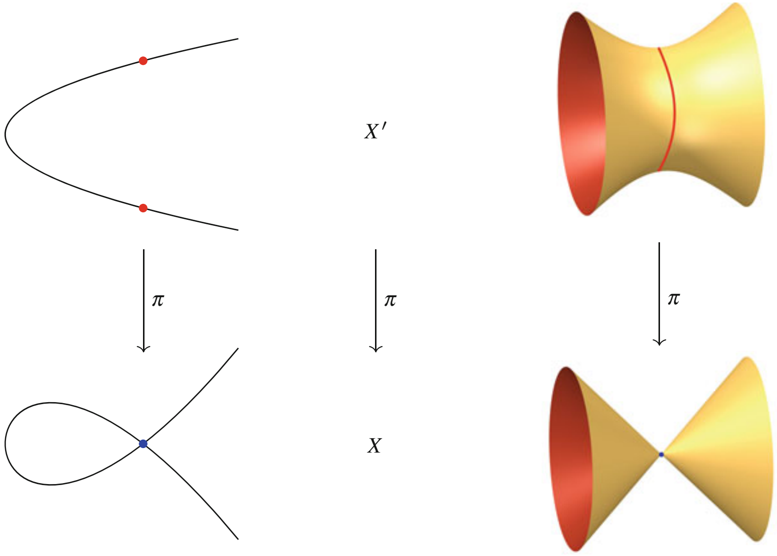
$$\pi : X' \to X \tag{3.1}$$

such that X' is non-singular.

A morphism $\pi : X' \to X$ is said to be **birational** if there exists a proper algebraic subvariety $Y \subsetneqq X$ such that π induces an isomorphism

$$\pi \Big|_{X'\setminus\pi^{-1}(Y)} : X' \setminus \pi^{-1}(Y) \to X \setminus Y \ .$$

The subvariety Y is sometimes called **the center of the blowing up** π and $Y' := \pi^{-1}(Y)$ **the exceptional set of** π.



(a) (b)

Fig. 3.1 Resolution of singularities. The center of the blowing up is in blue and the exceptional set in red. (**a**) Nodal curve $y^2 - x^2 - x^3 = 0$. (**b**) Non-degenerate quadratic cone $z^2 - x^2 - y^2 = 0$

A morphism π is birational if and only if it induces an isomorphism

$$K(X) \cong K(X')$$

between the fields of rational functions of X and X'.

Figure 3.1 depicts resolution of singularities of the nodal curve and of the non-degenerate quadratic cone.

The equivalence relation induced by all the relations of the form $X \sim X'$ where X' admits a birational morphism (3.1) is called **the birational equivalence relation.**

A very closely related question is resolution of singularities of **analytic** varieties. To state it, replace "algebraic" by "analytic" and "birational" by "bimeromorphic" in the definitions above.

Locally (in the sense of valuation theory explained in detail below) resolution of singularities can be understood as parametrizing wedges of the singular variety X by non-singular algebraic varieties. If X is an (analytically) irreducible curve, resolution of singularities of X is the same as a parametrization of X by a non-singular curve.

The goal of this paper is to give a survey of known results about existence and various constructions of resolution of singularities in cases where it has been achieved as well as discuss the status of this problem in cases when it is still open.

3.1.1 Motivation, Significance and Some Applications of Resolution of Singularities

(1) There are many objects and constructions which can only be defined, or at least are much easier to define and study for non-singular varieties. These include Hodge theory, singular and étale cohomology, the canonical divisor, etc.

(2) The classification problem.

> In any branch of mathematics, there are usually guiding problems, which are so difficult that one never expects to solve them completely, yet which provide stimulus for a great amount of work, and which serve as yardsticks for measuring progress in the field. In algebraic geometry such a problem is the classification problem. In its strongest form, the problem is to classify all algebraic varieties up to isomorphism. We can divide the problem into parts. The first part is to classify varieties up to birational equivalence. As we have seen, this is equivalent to the question of classifying all the function fields (finitely generated extension fields over k) up to isomorphism. The second part is to identify a good subset of a birational equivalence class, such as the nonsingular projective varieties, and classify them up to isomorphism. The third part is to study how far an arbitrary variety is from one of the good ones considered above. In particular, we want to know (a) how much do you have to add to a nonprojective variety to get a projective variety, and (b) what is the structure of singularities, and how can they be resolved to give a nonsingular variety?
>
> Robin Hartshorne, *Algebraic Geometry*, §1.8 What is Algebraic Geometry? [88].

From this point of view, resolution of singularities answers a very natural question: does every birational equivalence class contain a non-singular variety (a **non-singular model**) and, more precisely, is every singular variety X birationally dominated by a non-singular one as in (3.1)? Once this question has been answered affirmatively, one may, on the one hand, look for birational invariants, that is, numbers associated to the given birational equivalence class and defined in terms of some non-singular model, and, on the other hand, address the finer questions about the relation between different non-singular models in the given birational equivalence class and what can be said about the relation between the resolution of singularities and the original singular variety which it dominates. This is a very active area of research, known as the Mori program; it has been the stage of some spectacular recent developments.

(3) **Embedded desingularization** is a somewhat stronger form of resolution of singularities, which is particularly useful for applications. Suppose that X is embedded in a regular variety Z. Embedded desingularization asserts that there exists a sequence $\rho : \tilde{Z} \to Z$ of blowings up along non-singular centers (this notion will be defined precisely below), under which the *total transform* $\rho^{-1}(X)$ of X becomes a *divisor with normal crossings*, which means that all

of its irreducible components are smooth hypersurfaces and locally at each point of $\tilde{Z}$, $\rho^{-1}(X)$ is defined by a monomial with respect to some regular system of parameters. Geometrically, this means that at every point of $\tilde{Z}$ there exists a local coordinate system such that $\rho^{-1}(X)$ looks locally like a union of coordinate hyperplanes, counted with certain multiplicities. Thus divisors with normal crossings locally have a very simple structure. There are many situations in which it is useful to know that every closed subvariety can be turned into a divisor with normal crossings by blowing up. For example, this is used for *compactifying* algebraic varieties (problem (b) mentioned in the passage by R. Hartshorne cited above). Let X be a regular algebraic variety over a field k, embedded in some projective space $\mathbb{P}^n_k$. If X is not closed in $\mathbb{P}^n_k$, we can always consider its Zariski closure $\bar{X}$, which is, by definition, projective over k. The problem is that even though we started with a regular X, $\bar{X}$ may well turn out to be singular. Resolution of singularities, together with its embedded version, assures us that, after blowing up closed subvarieties, disjoint from X, we may embed X in a *regular* projective variety X' such that $X' \setminus X$ is a normal crossings divisor.

(4) Finally, resolution of singularities is useful for studying singularities themselves. Namely, let $\xi \in X$ be a singularity and let $\pi : \tilde{X} \to X$ be a desingularization. We may adopt the following philosophy for studying the singularity ξ. All the regular points are locally the same; every singular point is singular in its own way. We may regard resolution of singularities as a way of getting rid of the local complexity of the singularity ξ and turning it into *global* complexity of the regular variety $\tilde{X}$. Thus some global invariant of $\tilde{X}$ may also be regarded as an invariant of the singularity ξ. For example, if X is a surface and the singularity ξ is isolated, then $\pi^{-1}(\xi)$ is a collection of curves on the regular surface $\tilde{X}$. By embedded resolution for curves, we may further achieve the situation where $\pi^{-1}(\xi)$ is a normal crossings divisor (a resolution of singularities having this property is called **a good resolution**). If $\{E_i\}_{1\le i\le n}$, are the irreducible components of $\pi^{-1}(\xi)$, then the intersection matrix $(E_i.E_j)$ (equivalently, the dual graph of the configuration $\bigcup\limits_{i=1}^{n} E_i$) is an important combinatorial invariant associated to the singularity ξ. A good illustration of the usefulness of replacing local difficulties by global is D. Mumford's theorem that asserts that a normal surface singularity which is topologically trivial is regular. More precisely, given a normal surface singularity $\xi \in X$ over $\mathbb{C}$, one may consider its **link**, which is the intersection of X with a small Euclidean sphere centered at ξ. The link is a real 3-dimensional manifold. Mumford's theorem asserts that if the link is simply connected, then ξ is regular. The idea behind Mumford's proof is that the link is nothing but the boundary of a tubular neighbourhood of the collection $\bigcup\limits_{i=1}^{n} E_i$ of non-singular curves on the non-singular surface $\tilde{X}$. This really helps analyze the link.

3.2 A Brief Early History of the Subject: First Constructions of Resolution of Curve Singularities

3.2.1 Newton Polygon and Newton's Rotating Ruler Method for Resolving Plane Curve Singularities

Resolution of singularities of plane curves is due to Newton and Puiseux.

Consider a polynomial or a power series $f(x, y) = \sum\limits_{i,j\in\mathbb{N}} a_{ji}x^i y^j$, where $a_{ij} \in \mathbb{C}$, $f(0, 0) = 0$ and there exists a strictly positive integer n such that

$$a_{0n} \neq 0 \tag{3.2}$$

(that is, the monomial y^n appears in f with a non-zero coefficient). Newton and Puiseux proved that, viewed as an equation in y to be solved in functions of x, $f(x, y) = 0$ has a solution in **Puiseux series** of x (by definition, in a Puiseux series the exponents are rational numbers with bounded denominators).

Theorem 3.2.1 (Newton 1676, Puiseux 1850) *There exists a strictly positive integer m and a Puiseux series $y(x) = \sum\limits_{i=1}^{\infty} c_i x^{\frac{i}{m}}$ such that $f(x, y(x)) \equiv 0$ as a series in $x^{\frac{1}{m}}$.*

Remark 3.2.2 Let $K = \bigcup\limits_{m=1}^{\infty} \mathbb{C}\left(\left(x^{\frac{1}{m}}\right)\right)$. Theorem 3.2.1 says, in particular, that K is algebraically closed. This was the motivation and the point of view adopted by Puiseux.

Newton Polygon

In order to prove Theorem 3.2.1, Newton introduced the notion of **Newton polygon** which, together with its generalization to higher dimensions called **Newton polyhedron** [60, 94–96, 140] has proved to be one of the most fundamental tools in the theory of resolution of singularities.

Let $\mathbb{R}^2_+$ denote the first quadrant of $\mathbb{R}^2$.

Definition 3.2.3 The **Newton polygon** of f, which we will denote by $\Delta(f, y)$, is the convex hull of the set

$$\bigcup_{\substack{(i,j)\in\mathbb{N}^2 \\ a_{ij}\neq 0}} \left((i, j) + \mathbb{R}^2_+)\right) \subset \mathbb{R}^2.$$

Let n be the smallest strictly positive integer satisfying (3.2).

Definition 3.2.4 The vertex $(0, n)$ is called the **pivotal vertex** of $\Delta(f, y)$. The non-vertical edge of $\Delta(f, y)$ containing $(0, n)$ is called the **leading edge** of $\Delta(f, y)$.

As Newton says, to trace the leading edge we put a vertical ruler through $(0, n)$ and rotate it till it hits another point (i, j) with $a_{ij} \neq 0$ (equivalently, another vertex of $\Delta(f, y)$). Let E denote the leading edge of $\Delta(f, y)$. Let $\text{in}_E f := \sum\limits_{(i,j)\in E} a_{ij}x^i y^j$. The polynomial $\text{in}_E f$ is called **the initial form** of f with respect to E. The leading edge, the pivotal point, the initial form of f with respect to an edge and their generalizations to the higher dimensional context of Newton polyhedra play a crucial role in many constructions of resolution of singularities today.

We give a sketch of Newton's proof of Theorem 3.2.1.

Proof If E is horizontal then $y^n \mid f$, so $y = 0$ is a root of f of multiplicity n. Assume that E is not horizontal. Let α be a root of $\text{in}_E f(1, y)$ and s the multiplicity of the root α.

Write the slope of E as $-\frac{q}{r}$, where q and r are two relatively prime strictly positive integers. There are two cases to be considered.

Case 1. We have $\text{in}_E f \neq a_{0n}\left(y - \alpha x^{\frac{r}{q}}\right)^n$. In other words,

$$s < n. \tag{3.3}$$

Put $x_1 = x^{\frac{1}{q}}$ and $y_1 = \frac{y}{x_1^r} - \alpha$. Make the substitution

$$x = x_1^q \tag{3.4}$$

$$y = y_1 x_1^r + \alpha x_1^r. \tag{3.5}$$

Case 2. We have $\text{in}_E f = a_{0n}\left(y - \alpha x^{\frac{r}{q}}\right)^n$. Note that in this case, by Newton's binomial theorem, we have $\left(n - 1, \frac{r}{q}\right) \in E$. This implies that $\frac{r}{q} \in \mathbb{N}$ (in other words, $q = 1$) and $a_{r,n-1} \neq 0$.

Remark 3.2.5 Here we are using in a crucial way the fact that char $\mathbb{C} = 0$. This phenomenon will have important repercussions later when we discuss H. Hironaka's proof of resolution of singularities in characteristic zero and all dimensions, the notions of Tschirnhausen transformation and maximal contact used there and the failure of all them over fields of characteristic $p > 0$.

Put $x_1 = x$ and $y_1 = y - \alpha x_1^r$. Make the substitution

$$x = x_1 \tag{3.6}$$

$$y = y_1 + \alpha x_1^r. \tag{3.7}$$

In both cases, let $f_1(x_1, y_1)$ denote the polynomial or power series, resulting from substituting (3.4)–(3.5) (resp. (3.6)–(3.7)) into f. Let

$$n_1 = n - s \quad \text{in Case 1} \tag{3.8}$$

$$n_1 = n \quad \text{in Case 2.} \tag{3.9}$$

A direct computation shows the following:

(a) the Newton polygon $\Delta(f_1, y_1)$ has $(0, n_1)$ as a vertex
(b) in Case 2, the slope of the leading edge of $\Delta(f_1, y_1)$ is strictly greater than $-\frac{1}{r}$.

Now, iterate the procedure to construct (x_i, y_i) and f_i for $i \in \mathbb{N}$. Since in Case 1 we have $n_1 < n$, Case 1 can occur at most n times. Take $i_0 \in \mathbb{N}$ such that Case 2 occurs for all $i \geq i_0$. For $i > i_0$, let $-\frac{1}{r_i}$ denote the slope of the leading edge of the Newton polyhedron $\Delta(f_i, y_i)$. Our iterative procedure produces $x_i = x_{i_0}$, $y_i = y_{i_0} - \sum\limits_{j=i_0}^{i-1} b_j x_{i_0}^{r_j}$ for suitable $b_j \in \mathbb{C}$. According to statement (b) above, the sequence of integers $(r_i)_i$ is strictly increasing with i, hence goes to ∞ (it may happen that the leading edge of $\Delta(f_i, y_i)$ becomes horizontal for some finite i, in which case we set all the subsequent coefficients b_j to be equal to 0; the procedure will stop here). Let $y_\infty := y_{i_0} - \sum\limits_{j=i_0}^{\infty} b_j x_{i_0}^{r_j}$, substitute $y_{i_0} = y_\infty + \sum\limits_{j=i_0}^{\infty} b_j x_{i_0}^{r_j}$ into f_{i_0} and let f_∞ be the resulting polynomial (resp. power series). The leading edge of $\Delta(f_\infty, y_\infty)$ has slope strictly greater than $-\frac{1}{r_i}$ for all i, hence it is horizontal. Thus $y_\infty^{n_{i_0}} \mid f_\infty$, so $y_{i_0}\left(x_{i_0}\right) := \sum\limits_{j=i_0}^{\infty} b_j x_{i_0}^{r_j}$ is a root of f_{i_0} of multiplicity n_{i_0}.

Let $m := \prod\limits_{j=0}^{i_0} q_i$, $Q := \sum\limits_{j=0}^{i_0} \prod\limits_{\ell=0}^{j} q_\ell$ and $R := \sum\limits_{j=0}^{i_0} \prod\limits_{\ell=0}^{j} r_\ell$. By construction, we have

$$x_{i_0} = x^{\frac{1}{m}} \tag{3.10}$$

and

$$y_{i_0} = yx^{-\frac{R}{Q}} + g\left(x^{\frac{1}{m}}\right), \tag{3.11}$$

where g is a suitable polynomial with complex coefficients. Let $\sum\limits_{i=1}^{\infty} c_i x^{\frac{i}{m}}$ be the Puiseux series $x^{\frac{R}{Q}}\left(\sum\limits_{j=i_0}^{\infty} b_j x_{i_0}^{\frac{r_j}{m}} - g\left(x^{\frac{1}{m}}\right)\right)$. Making the substitution (3.10)–(3.11) back into f_{i_0} and setting $y(x) := \sum\limits_{i=1}^{\infty} c_i x^{\frac{i}{m}}$, we see that $(y - y(x))^{n_{i_0}} \mid f$, that is, $y(x)$ is a root of f of multiplicity n_{i_0}, as desired. □

Remark 3.2.6 Every time Case 1 occurred in Newton's algorithm some choices needed to be made. For example, if Case 1 happens at the first step we had to choose a root α of $\mathrm{in}_E f$. Counted with multiplicity there were $s = n - n_1$ such choices. Starting with the step i_0 we have constructed a root of f of multiplicity n_{i_0}.

Therefore the total number of roots of f obtained by this procedure, counted with multiplicity, is $n_{i_0} + \sum_{j=0}^{i_0-1} \left(n_j - n_{j+1}\right) = n$.

Remark 3.2.7 In the Newton–Puiseux theorem, assume that f is either a polynomial or a convergent power series. It is not hard to show (by estimating the coefficients b_j at each step of the construction) that the Puiseux series produced by Newton's algorithm is also convergent. Assume, in addition, that the plane complex curve $C := \{f(x, y) = 0\}$ is irreducible as an analytic space (in other words, has only one branch near the origin). Then Newton's procedure gives a parametrization of C near the origin by a complex disk with the coordinate x_{i_0}, that is, a resolution of singularities of a suitable neighbourhood of the origin in C. Algebraically, this resolution of singularities is described by the birational, injective ring homomorphism $\mathbb{C}\{x, y\} \hookrightarrow \mathbb{C}\{x_{i_0}\}$, that maps x to $x_{i_0}^m$ and y to $\sum_{i=1}^{\infty} c_i x_{i_0}^i$.

More generally, if the analytic curve C has several branches, parametrizations of each of them are obtained by making suitable choices of roots in Newton's algorithm.

While we are on the subject of resolution of plane curve singularities and Newton polygon, we mention an important work [114] by Monique Lejeune-Jalabert that paved the way to the approach to resolution of singularities and local uniformization via key polynomials (see below).

Global resolution of singularities of analytic curves is due to B. Riemann and was achieved using complex-analytic methods. Indeed, the Riemann surface associated to a complex-analytic curve *is* its resolution of singularities.

Purely algebraic proofs of resolution of curve singularities were given much later by Italian geometers like Albanese [13]. Albanese's proof consists in projecting a singular curve embedded in a projective space of a sufficiently large dimension (more than twice than the degree of the curve) from one of its singular points and showing that this process improves the singularity. Below we will discuss a beautiful one-step procedure defined by O. Zariski [168] that resolves singularities of curves.

3.3 Blowing Up, Multiplicity and the Hilbert–Samuel Function

In this section we introduce one of the main tools for constructing resolution of singularities: blowing up. Blowing up of a variety X along a subvariety Y (more generally, along an ideal sheaf $\mathcal{I}$) is a birational projective morphism $\pi : X' \to X$, defined below, that induces an isomorphism $\pi \big|_{X' \setminus \pi^{-1}(Y)} : X' \setminus \pi^{-1}(Y) \to X \setminus Y$. As we will see, blowing up of a non-singular variety along a non-singular subvariety is again non-singular. Thus a very general idea for constructing a resolution of singularities of a variety X, that we will explain in more detail below, goes as follows.

(1) Embed X in a non-singular variety Z.
(2) Construct a sequence

$$Z \xleftarrow{\rho_1} Z_1 \xleftarrow{\rho_2} \dots \xleftarrow{\rho_i} Z_i \tag{3.12}$$

of blowings up along non-singular centers and study the **strict transform** X_i of X in Z_i (defined below) in the hope of improving and eventually eliminating the singularities of X_i. We now go for precise definitions.

Let X be an affine algebraic variety with coordinate ring A and $I = (f_1, \dots, f_n)$ an ideal of A. As usual, $V(I)$ will denote the zero locus of I.

Definition 3.3.1 The **blowing up** of X along I is the birational projective morphism $\pi : \tilde{X} \to X$, defined as follows. Consider the morphism $\phi : X \backslash V(I) \to X \times_k \mathbb{P}_k^{n-1}$, which sends every $\xi \in X \setminus V(I)$ to $(\xi, (f_1(\xi) : \cdots : f_n(\xi))) \in X \times_k \mathbb{P}_k^{n-1}$. The blowing up $\tilde{X}$ is defined to be the closure $\overline{\phi(X \setminus V(I))} \subset X \times_k \mathbb{P}_k^{n-1}$ in the Zariski topology.

Remark 3.3.2 Since the blowing up $\widetilde{X} = \overline{\phi(X \setminus V(I))} \subset X \times_k \mathbb{P}_k^{n-1}$, the natural projection $X \times_k \mathbb{P}_k^{n-1} \to X$ induces a map $\widetilde{X} \to X$. In particular, $\widetilde{X}$ is projective over X.

The natural map $\pi : \widetilde{X} \to X$ is an isomorphism away from $V(I)$ (the inverse mapping is given by ϕ). This means that the map $\pi : \widetilde{X} \to X$ is birational.

Remark 3.3.3 If X is irreducible (that is, A is an integral domain), then $\widetilde{X}$ is covered by n affine charts U_i, $i \in \{1, \dots, n\}$ with coordinate rings

$$A\left[\frac{f_1}{f_i}, \dots, \frac{f_n}{f_i}\right], 1 \leq i \leq n, \tag{3.13}$$

where the glueing of the charts is implicit in the notation.

Example 3.3.4

1) Blowing up the plane at a point. Let $X = k^2$ be the affine plane, $A = k[x, y]$ its coordinate ring and $I = (x, y)$ the ideal defining the origin. Let (u_1, u_2) be homogeneous coordinates on $\mathbb{P}_k^1$. We have the map $k^2 \backslash \{0\} \to k^2 \times \mathbb{P}_k^1$ that sends the point (x, y) to the point $(u_1 : u_2) \in \mathbb{P}_k^1$.
The blowing up $\widetilde{X}$ is defined in $k^2 \times_k \mathbb{P}_k^1$ by the equation $xu_2 - yu_1 = 0$. For example, if $k = \mathbb{R}$, then $\widetilde{X}$ is nothing but the Möbius band.
Perhaps the most useful way of thinking about the blowing up $\widetilde{X}$ is that it is a variety glued together from two coordinate charts with coordinate rings $k\left[u_1, \frac{u_2}{u_1}\right]$ and $k\left[u_2, \frac{u_1}{u_2}\right]$, where, again, the glueing is implicit in the notation.

2) More generally, we can blow up the affine n-space at the origin. Let

$$A = k[x_1, \dots, x_n], \quad I = (x_1, \dots, x_n).$$

Let $u_1, \dots, u_n$ denote homogeneous coordinates on $\mathbb{P}_k^{n-1}$. Then $\widetilde{X} \subset k^n \times \mathbb{P}_k^{n-1}$ is the subvariety defined by the equations $x_i u_j - x_j u_i$, $1 \le i, j \le n$. Again, $\widetilde{X}$ is covered by n coordinate charts with coordinate rings $k\left[\frac{u_1}{u_i}, \dots, \frac{u_{i-1}}{u_i}, u_i, \frac{u_{i+1}}{u_i}, \dots, \frac{u_n}{u_i}\right]$, $1 \le i \le n$.

3) Even more generally, the blowing up $\widetilde{X}$ of k^n along $(x_1, \dots, x_l)$ for $l < n$ is the subvariety of $k^n \times \mathbb{P}^{l-1}$ defined by the equations $x_i u_j - x_j u_i$, $1 \le i, j \le l$. The blowing up $\widetilde{X}$ is covered by l coordinate charts with coordinate rings $k\left[\frac{u_1}{u_i}, \dots, \frac{u_{i-1}}{u_i}, u_i, \frac{u_{i+1}}{u_i}, \dots, \frac{u_l}{u_i}, u_{l+1}, \dots, u_n\right]$. Intuitively, we may think of this last construction as first blowing up the origin in k^l and then taking the direct product of the whole situation with k^{n-l}.

3.3.1 The Universal Mapping Property of Blowing Up

We now give a characterization of the blowing up of a variety X along an ideal $I \subset A$ by a universal mapping property (in particular, this characterization makes no reference to any particular ideal base $(f_1, \dots, f_n)$ of I).

Let $\pi : \widetilde{X} \to X$ be a morphism of algebraic varieties and $\mathcal{I}$ a coherent ideal sheaf on X. Let $\widetilde{X} = \bigcup\limits_{i,j \in \Phi_i} V_{ij}$ and $X = \bigcup\limits_{1 \le i \le s} U_i$ be the respective coverings by affine charts, where the Φ_i are certain index sets such that $\pi^{-1}(U_i) = \bigcup\limits_{j \in \Phi_i} V_{ij}$, $1 \le i \le s$. Let A_i denote the coordinate ring of U_i and B_{ij} that of V_{ij}. For each i and each $j \in \Phi_i$ we have a homomorphism $A_i \to B_{ij}$. Let $\pi^*\mathcal{I}$ denote the coherent ideal sheaf on $\widetilde{X}$ whose ideal of sections over V_{ij} is $I_i B_{ij}$.

Let X be a scheme and $\mathcal{I}$ a coherent ideal sheaf on X. The idea, which we now explain in detail, is that the blowing up $\pi : \widetilde{X} \to X$ of X along $\mathcal{I}$ is characterized by the universal mapping property with respect to making $\pi^*\mathcal{I}$ invertible (see the Definition below).

Definition 3.3.5 Let I be an ideal in a ring A. The ideal I is said to be **locally principal** if for every maximal ideal $\mathfrak{m}$ of A the ideal $IA_{\mathfrak{m}}$ is principal. The ideal I is said to be **invertible** if for every maximal ideal $\mathfrak{m}$ of A the ideal $IA_{\mathfrak{m}}$ is principal and generated by a non-zero divisor.

Of course, if A is a domain, then invertible and locally principal are the same thing; this case will be our main interest in the present paper.

Definition 3.3.6 An ideal sheaf $\mathcal{I}$ on a variety X is **locally principal** if there exists an affine open cover $X = \bigcup\limits_i U_i$ such that, denoting by A_i the coordinate chart of

U_i, the ideal $\mathcal{I}_{U_i}$ of sections of $\mathcal{I}$ is a principal ideal of A_i for all i. The ideal sheaf $\mathcal{I}$ is said to be **invertible** if each $\mathcal{I}_{U_i}$ is principal and generated by an element which is not a zero divisor.

Again, if X is irreducible then invertible and locally principal are the same thing.

Let the notation be as in (3.13) above. We have

$$IA\left[\frac{f_1}{f_i}, \dots, \frac{f_{i-1}}{f_i}, \frac{f_{i+1}}{f_i}, \dots, \frac{f_n}{f_i}\right] = (f_1, \dots, f_n)A\left[\frac{f_1}{f_i}, \dots, \frac{f_{i-1}}{f_i}, \frac{f_{i+1}}{f_i}, \dots, \frac{f_n}{f_i}\right] =$$
$$= (f_i)A\left[\frac{f_1}{f_i}, \dots, \frac{f_{i-1}}{f_i}, \frac{f_{i+1}}{f_i}, \dots, \frac{f_n}{f_i}\right], \tag{3.14}$$

so that π^*I is invertible on $\widetilde{X}$. Since we are dealing with a *local* property, this statement remains valid even if X is not affine. In other words, if $\pi : \widetilde{X} \to X$ is the blowing up of a coherent ideal sheaf $\mathcal{I}$, then $\pi^*\mathcal{I}$ is invertible.

We now point out that this property is also sufficient to characterize blowing up. Namely, the blowing up π of $\mathcal{I}$ is the smallest (in the sense explained in Theorem 3.3.7 below) projective morphism such that $\pi^*\mathcal{I}$ is invertible. More precisely, we have the following theorem.

Theorem 3.3.7 (The Universal Mapping Property of Blowing Up [88], Proposition II.7.14, p. 164) *Let $\rho : Z \to X$ be a morphism of irreducible algebraic varieties such that $\rho^*\mathcal{I}$ is invertible. Then ρ factors through $\widetilde{X}$ in a unique way.*

Proof We briefly sketch the idea of the proof. Since $\rho^*\mathcal{I}$ is invertible, at each point of Z it must be generated by one of the f_i. Hence Z admits a covering $Z = \bigcup\limits_{i=1}^{n} V_i$ by affine charts with coordinate rings B_i such that $IB_i = (f_i)B_i$. Then $\frac{f_j}{f_i} \in B_i$, so

$$A\left[\frac{f_1}{f_i}, \dots, \frac{f_{i-1}}{f_i}, \frac{f_{i+1}}{f_i}, \dots, \frac{f_n}{f_i}\right] \hookrightarrow B_i. \tag{3.15}$$

The inclusion (3.15) determines a morphism $\lambda_i : V_i \to U_i$ of affine algebraic varieties, where U_i is as in (3.13). Glueing together the morphisms λ_i, $1 \le i \le n$, gives the desired factorization of ρ through $\widetilde{X}$. □

Remark 3.3.8 All of the above definitions, constructions and results can easily be generalized to the case of varieties that may be reducible. We chose to work with irreducible ones to simplify the notation and the exposition.

3.3.2 Strict Transforms

Let Z be an irreducible variety and $\mathcal{I}$ a coherent ideal sheaf on Z. Let $\iota : X \hookrightarrow Z$ be a closed irreducible subvariety of Z with its natural inclusion ι. Let $\pi : \tilde{Z} \to Z$

be the blowing up along $\mathcal{I}$. Let $\widetilde{X} := \overline{\pi^{-1}(X \backslash V(\mathcal{I}))} \subset \tilde{Z}$, where "$\overline{}$" denotes the closure in the Zariski topology.

Definition 3.3.9 The variety $\widetilde{X}$ is called **the strict transform** of X under π.

Of course, $\widetilde{X} \subset \pi^{-1}(X) = \widetilde{X} \cup \pi^{-1}(V(\mathcal{I}))$. To distinguish it from the strict transform, $\pi^{-1}(X)$ is sometimes called **the total transform** of X under π. We state the following useful fact without proof.

Theorem 3.3.10 *The variety $\widetilde{X}$ together with the induced morphism $\rho : \widetilde{X} \to X$ is nothing but the blowing up of the coherent ideal sheaf $\iota^* \mathcal{I}$ on X.*

Example 3.3.11 Let k be a field and u, v—independent variables. Let $Z = k^2$ be the affine plane with coordinate ring $k[u, v]$, $I = (u, v)$ and X—the plane curve $\left\{u^2 - v^3 = 0\right\} \subset Z$.

The blowing up $\widetilde{Z}$ of Z along I is covered by two affine charts with coordinate rings

$$k\left[\frac{u}{v}, v\right] \text{ and } k\left[u, \frac{v}{u}\right].$$

Let us denote the coordinates in the first chart U_1 by u_1, v_1, so that $v = v_1$, $u = u_1 v_1$. Let u_2, v_2 be the coordinates in the second chart U_2, so that $u = u_2$, $v = u_2 v_2$.

To calculate the strict transform $\widetilde{X}$ of U_2, we first find its full inverse image. This inverse image is defined by the equation $u^2 - v^3$, but written in the new coordinates:

$$u^2 - v^3 = u_2^2 - u_2^3 v_2^3 = u_2^2(1 - u_2 v_2^3).$$

Here $u_2 = 0$ is the equation of the exceptional divisor. To obtain the strict transform $\widetilde{X}$, we must factor out the maximal power of u_2 out of the equation. In this case, $\widetilde{X} \cap U_2$ is defined by $1 - u_2 v_2^3$. In U_1, we have

$$u^2 - v^3 = u_1^2 v_1^2 - v_1^3 = v_1^2(u_1^2 - v_1).$$

Here $v_1 = 0$ is the equation of exceptional divisor, so that $\widetilde{X} \cap U_1 = V(u_1^2 - v_1)$. In particular, note that although X had a singularity at the origin, $\widetilde{X}$ is non-singular. Thus, in this example we started with a singular variety X with one singular point, blew up the singularity and found that the strict transform of X became non-singular. That is, we obtained a resolution of singularities of X after one blowing up.

3.3.3 Fundamental Numerical Characters of Singularity: Multiplicity and the Hilbert–Samuel Function

We can now elaborate on the very general description of many constructions of resolution of singularities by sequences of blowings up, given at the beginning of this section.

Typically, we embed the variety X we want to desingularize into an ambient non-singular variety Z. Our goal is to successively construct a sequence (3.12) of blowings up along non-singular centers (that is, blowings up that are isomorphic to 3) of Example 1 locally in the classical or étale topology) and study the strict transform X_i of X in Z_i. We want to choose the center of the blowing up ρ_i at each step so as to "improve" the singularities of X_i. The precise meaning of "improve" is the following. Associate to each singular point ξ of X_i a discrete, **upper-semicontinuous** numerical character $d(\xi)$, that is, an element of a fixed well-ordered set, usually a finite string of non-negative integers or a function $\mathbb{N} \to \mathbb{N}$. Improving the singularities of X_i means ensuring that

$$\max\{d(\xi)\ \mid \xi \in X_{i+1}\} < \max\{d(\xi)\ \mid \xi \in X_i\}. \tag{3.16}$$

Experience shows that the best bet for achieving the strict inequality (3.16) is to blow up the largest possible centers contained in the maximal stratum of $d(\xi)$.

In this subsection we define the most fundamental numerical characters that usually go into the leading place of $d(\xi)$: multiplicity and its generalization—the Hilbert–Samuel function.

Let k be a field, n a strictly positive integer and $X = V(f)$ an $(n-1)$-dimensional hypersurface in k^n. Write $f = \sum\limits_{\alpha} c_\alpha u^\alpha$, where $c_\alpha \in k$, $u = (u_1, \dots, u_n)$, $\alpha = (\alpha_1, \dots, \alpha_n)$ runs over a finite subset of $\mathbb{N}^n$ and $u^\alpha = \prod\limits_{j=1}^{n} u_j^{\alpha_j}$ is the usual multi-index notation. Further, we will use the notation $|\alpha| = \sum\limits_{j=1}^{n} \alpha_j$.

Definition 3.3.12 The **multiplicity** of f at the origin of k^n is the quantity

$$\mathrm{mult}_0 f := \min\{|\alpha| \mid c_\alpha \neq 0\}.$$

The multiplicity at any other point $\xi = (a_1, \dots, a_n)$ of k^n is defined similarly, but using the expansion of f in terms of $u_i - a_i$ instead of the u_i.

Equivalently, the multiplicity of f at ξ is given by $\mathrm{mult}_\xi f = \max\{n \in \mathbb{N} \mid f \in \mathfrak{m}^n\}$, where $\mathfrak{m} = \left\{\frac{g}{h} \;\middle|\; g, h \in k[u], g(\xi) = 0 \neq h(\xi)\right\}$ is the maximal ideal of the local ring of k^n at ξ.

The only problem with this definition is that it is only valid for hypersurfaces whereas we would like to work with varieties of arbitrary codimension. The generalization of multiplicity that is used in many constructions is the Hilbert–Samuel function [21], which we now define.

Definition 3.3.13 Let (A, m, k) be a local Noetherian ring. The **Hilbert–Samuel function** of A is the function $H_{A,m} : \mathbb{N} \to \mathbb{N}$, defined by $H_{A,m}(n) = length\left(\frac{A}{m^{n+1}}\right)$ (considered as an A-module).

By additivity of length,

$$length\left(\frac{A}{m^{n+1}}\right) = \sum_{i=0}^{n} \dim_k \frac{m^i}{m^{i+1}}, \tag{3.17}$$

where the $\frac{m^i}{m^{i+1}}$ are k-modules, that is, k-vector spaces.

Note that since A is Noetherian, each of m^i is finitely generated, so that all the quantities in (3.17) are finite.

Theorem 3.3.14 (Hilbert–Serre) *The function $H_{A,m}(n)$ is a polynomial for $n \gg 0$. In other words, there exists a polynomial $P(n)$ with rational coefficients, such that*

$$P(n) = H_{A,m}(n) \quad for\, n \gg 0.$$

The polynomial $P(n)$ is called **the Hilbert polynomial** of A.

Notation Let $d(A)$ denote the degree of the Hilbert polynomial of A.

Example 3.3.15

1) Let k be a field and $A = k[x_1, \ldots, x_d]$ the polynomial ring in d variables. Let $\mathfrak{m} = (x_1, \ldots, x_d)$ be the maximal ideal corresponding to the origin in k^d. Consider the localization $A_\mathfrak{m}$. The Hilbert-Samuel function of $A_\mathfrak{m}$ is $H_{A_\mathfrak{m},\mathfrak{m}}(n) = length\left(\frac{A}{\mathfrak{m}^{n+1}}\right) = \binom{n+d}{d}$, which is a polynomial in n of degree d. In this case, $H_{A_\mathfrak{m},\mathfrak{m}}(d)$ is a polynomial for *all* n, not merely for n sufficiently large.
2) Let $B := \frac{A_\mathfrak{m}}{(f)}$, where f is a polynomial of multiplicity μ at the origin and let $\mathfrak{n}$ denote the maximal ideal of B. It is not hard to show that

$$\begin{aligned} H_{B,\mathfrak{n}} &= \binom{n+d}{d} && \text{if } n < \mu \\ &= \binom{n+d}{d} - \binom{n+d-\mu}{d} && \text{if } n \geq \mu. \end{aligned} \tag{3.18}$$

Now, $\binom{n+d}{d} - \binom{n+d-\mu}{d}$ is a polynomial of degree $d-1$, whose leading coefficient is $\frac{\mu}{(d-1)!}$. This shows that in the case of hypersurface singularities multiplicity can be recovered from the Hilbert–Samuel function. In fact, in this case multiplicity and the Hilbert–Samuel function are equivalent sets of data.

An important property of multiplicity, the Hilbert–Samuel function and the Hilbert polynomial is that they are **upper semicontinuous**. This means that the stratum of points on an algebraic variety X where the multiplicity (resp. Hilbert–Samuel function, resp. the Hilbert polynomial) is greater than or equal to a given value is a closed algebraic subvariety of X.

3.3.4 Normal Flatness and the Stability of the Hilbert–Samuel Function Under Blowing-Up

In this subsection we provide further details on the above program of resolving the singularities of any algebraic variety by constructing a sequence (3.12) of blowings up that strictly decreases a certain upper semicontinuous numerical invariant $d(\xi)$, $\xi \in X$.

For a point $\xi \in X$, we denote by $\mathcal{O}_{X,\xi}$ the **local ring** of X at ξ, that is, the ring formed by all the rational functions $\frac{g}{h}$ on X whose denominator h does not vanish at ξ. Let $\mathfrak{m}_{X,\xi}$ denote the maximal ideal of $\mathcal{O}_{X,\xi}$; it is the ideal formed by all the $\frac{g}{h}$ such that $g(\xi) = 0$. Write $H_{X,\xi}$ for $H_{\mathcal{O}_{X,\xi},\mathfrak{m}_{X,\xi}}$.

We define the leading component of our numerical invariant $d(\xi)$ to be the Hilbert–Samuel function $H_{X,\xi}$ (resp. $\operatorname{mult}_\xi X$ if X is a hypersurface, where $\operatorname{mult}_\xi X$ denotes the multiplicity at ξ of a local defining equation of X in an ambient non-singular variety Z near ξ).

Let X be an algebraic variety, Y a subvariety of X and ξ a point of Y. Let $\mathcal{I}_Y$ denote the ideal sheaf, defining Y in X. The **normal cone** of Y in X is defined to be the algebraic variety with coordinate ring

$$\bigoplus_{n=0}^{\infty} \frac{\mathcal{I}_Y^n}{\mathcal{I}_Y^{n+1}}.$$

Assume that Y is non-singular.

Definition 3.3.16 (H. Hironaka 1964) We say that X is **normally flat** along Y at ξ if $\bigoplus_{n=0}^{\infty} \frac{\mathcal{I}_{Y,\xi}^n}{\mathcal{I}_{Y,\xi}^{n+1}}$ is a free $\mathcal{O}_{Y,\xi}$-module. We say that X is normally flat along Y if it is normally flat at every point $\xi \in Y$ (equivalently, if $C_{X,Y}$ is flat over Y).

Theorem 3.3.17 (B. Bennett, H. Hironaka) *The variety X is normally flat along Y at ξ if and only if $H_{X,\eta} = H_{X,\xi}$ for all $\eta \in Y$ near ξ (in other words, the Hilbert–Samuel function of X is locally constant on Y near ξ).*

The next theorem (valid over fields of arbitrary characteristic) constitutes the first step of the above program of constructing a resolution of singularities of any algebraic variety by lowering a suitable numerical character $d(\xi)$. Namely, it says that a blowing up along a center Y over which X is normally flat *does not increase* the Hilbert–Samuel function (resp. multiplicity).

Theorem 3.3.18 (H. Hironaka 1964) *Let $Y \subset X$ be a non-singular algebraic subvariety of X over which X is normally flat. Let H denote the common Hilbert–Samuel function $H_{X,\xi}$ for all $\xi \in Y$. Let $\pi : \widetilde{X} \to X$ be the blowing up along Y and $\tilde{\xi} \in \pi^{-1}(Y)$. Then*

$$H_{\tilde{X},\tilde{\xi}} \leq H \tag{3.19}$$

(we compare Hilbert–Samuel functions in the lexicographical order, but in fact all the inequalities we write such as (3.19) hold componentwise, that is, separately for each n).

A subvariety Y as in the Theorem is sometimes referred to as a **permissible center of blowing up** and the blowing π itself as a **permissible blowing up**.

If we can achieve strict inequality in (3.19), our proof of resolution of singularities will be finished by induction. The difficult question is: what to do if equality holds in (3.19)?

3.4 Resolution of Surface Singularities over Fields of Characteristic Zero

Resolution of singularities of surfaces was constructed in late nineteenth—early twentieth century by the Italian school (P. del Pezzo 1892, Beppo Levi 1897 [115, 116], O. Chisini 1921 [51], G. Albanese 1924 [13]) as well as by H.W.E. Jung 1908 [106], followed by the first completely rigorous algebraic proof by R. Walker 1935 [162] and another one by O. Zariski 1939 [169, 171].

Let k be an algebraically closed field of characteristic zero. Below we briefly summarize Beppo Levi's, Jung's and O. Zariski's constructions of resolution of singularities of surface over k, with Beppo Levi's proof valid only for *hypersurfaces*.

3.4.1 Beppo Levi's Method

Let X be an algebraic surface over k, embedded in a smooth threefold Z. For a point $\xi \in X$ let $\mathrm{mult}_\xi X$ denote the multiplicity at ξ of a local defining equation of X in Z near ξ. Beppo Levi's algorithm goes as follows.

1) Let $\mu = \max \left\{ \mathrm{mult}_\xi X \;\middle|\; \xi \in X \right\}$.
2) Let $S_\mu = \left\{ \xi \in X \;\middle|\; \mathrm{mult}_\xi X = \mu \right\}$. By upper semicontinuity of multiplicity, S_μ is an algebraic subvariety of X, that is, a union of algebraic curves and points.

3) First assume that S_μ is *not* a union of a normal crossings divisor with a finite set of points.
4) The set of points of S_μ where it fails to be a normal crossings divisor is finite. Blow up each of these points, and keep doing so until S_μ becomes a union of a normal crossings divisor with a finite set of points.
5) If S_μ is a union of a normal crossings divisor with a finite set of points, let $\pi : \widetilde{X} \to X$ be a blowing up of an irreducible component of S_μ.
6) By Theorem 3.3.18 (which Beppo Levi proved in the special case of two-dimensional hypersurfaces over fields of characteristic zero), we have

$$\mu \geq \max \left\{ \operatorname{mult}_{\tilde{\xi}} \widetilde{X} \;\middle|\; \tilde{\xi} \in \widetilde{X} \right\}. \tag{3.20}$$

7) If equality holds in (3.20), let $\tilde{S}_\mu = \left\{ \tilde{\xi} \in \widetilde{X} \;\middle|\; \operatorname{mult}_{\tilde{\xi}} \widetilde{X} = \mu \right\}$. Again by Theorem 3.3.18 we have $\tilde{S}_\mu \subset \pi^{-1}(S_\mu)$. Observe that $\tilde{S}_\mu$ is again a union of a normal crossings divisor with a finite set of points (or the empty set).
8) Keep repeating the procedure of 5) until the locus of points of multiplicity μ becomes the empty set. This completes the proof by induction on μ.

Remark 3.4.1 Predictably, Beppo Levi's method of resolution of singularities fails starting with dimension three. Reference [144] gives an example of a threefold X in k^4 all of whose singular points have multiplicity 2. The locus of multiplicity 2 is a normal crossings subvariety consisting of two lines that meet each other at the origin. Blowing up any one of the two lines produces a new threefold whose multiplicity 2 locus is a union of three lines. Blowing up one of those three lines yields a threefold containing a singularity, isomorphic to the origin in X. Thus there exists an infinite sequence of blowings up along non-singular components of the locus of multiplicity 2 which does not resolve the singularities of X.

It was later pointed out by Zariski that none of the proofs of resolution of surfaces by the Italian geometers was complete and some were outright wrong. The first completely rigorous algebraic proof was given by R. Walker in 1935 [162].

3.4.2 *Normalization*

Before discussing the proofs by Jung and O. Zariski of 1939, we need to introduce the notion of normalization.

Let A be an integral domain with field of fractions K. We may consider the integral closure $\bar{A}$ of A in K (sometimes it is also called **the normalization of** A). If A if of finite type over k, it is the coordinate ring of an irreducible affine algebraic variety X. The inclusion $A \hookrightarrow \bar{A}$ gives rise to the natural birational finite (hence projective) morphism $\pi : \bar{X} \to X$ of irreducible algebraic varieties. The canonical morphism π is called the **normalization** of the variety X. Because of the uniqueness of normalization, even if X is not affine, the separate normalizations of the various affine charts of X glue together in a natural way to yield the normalization of X.

Definition 3.4.2 An integral domain is said to be **normal** if it coincides with its normalization. An algebraic variety is said to be normal if the coordinate rings of all of its affine charts are normal.

The notion of normalization was defined (surprisingly late—in 1939) by Oscar Zariski [168]. This is a great example of the usefulness of the algebraic language in geometry: this notion, extremely important as it turned out to be, did not occur to anyone until the algebraic language was developed. The importance of normalization for resolution of singularities is explained by the following result.

Theorem 3.4.3 (Zariski) *Let A be a one-dimensional Noetherian local ring. Then A is regular if and only if A is normal.*

Corollary 3.4.4 (Zariski) *If X is a normal algebraic variety,*

$$\dim \ Sing(X) \leq \dim X - 2.$$

Geometrically, Theorem 3.4.3 says that normalization resolves the singularities of curves. More generally, it says that for an arbitrary reduced variety normalization resolves the singularities in codimension 1. When normalization was defined, the theorem of resolution of singularities of curves was known for almost a century, yet it was quite a surprise that it had such a simple and elegant proof and that the procedure for desingularization had such a simple description.

We now summarize Jung's and Zariski's methods for the resolution of surfaces.

3.4.3 Jung's Method

1) Fix a projection $\sigma : X \to \mathbb{C}^2$ from our affine singular surface X to a plane and consider the branch locus C of the σ.
2) Apply embedded resolution of plane curve singularities to the curve C, that is, construct a sequence $\rho : W' \to \mathbb{C}^2$ of point blowings up such that the total transform of C under ρ is a normal crossings divisor.
3) Let $X' := X \times_{\mathbb{C}^2} W'$. We obtain a cartesian square

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "\sigma'"'] & X \arrow[d, "\sigma"] \\ W' \arrow[r, "\rho"] & \mathbb{C}^2 \end{tikzcd} \tag{3.21}$$

4) Let $\bar{X} \to X'$ be the normalization of X'. The branch locus of $\bar{X}$ over W' is still a normal crossings divisor.

5) Observe that the fact that the branch locus of the normal surface $\bar{X}$ has normal crossings implies that the singularities of $\bar{X}$ are of a very special type, namely, cyclic quotient singularities (that is, singularities obtained from $\mathbb{C}^2$ by taking a quotient by a cyclic group; these are precisely the toric ones among the normal surface singularities).
6) Resolve the cyclic quotient singularities by hand.

Remark 3.4.5 Even though normalization was officially defined by Zariski in 1939, Jung constructs it by hand in this special case. Items 4) and 6) in Jung's proof use complex-analytic and topological methods (namely, the theory of ramified coverings of analytic varieties).

3.4.4 Zariski's Method

Let k be an algebraically closed field of characteristic zero and X an algebraic surface over k. Zariski's method for desingularizing X goes as follows.

1) Let $\bar{X} \to X$ be the normalization of X. According to Corollary 3.4.4, $Sing\left(\bar{X}\right)$ has codimension 2 in $\bar{X}$, that is, is a finite union of isolated points.
2) Let $X' \to X$ be the blowing up of all the singular points of $\bar{X}$.
3) Replace X by X' and go back to step 1). Keep iterating steps 1) and 2) until the singularities are resolved.

Remark 3.4.6 Zariski's algorithm has the virtue of being extremely easy to state. However, proving that it works is technically quite difficult (an improved version of this result was given later by J. Lipman). An intermediate step in the proof is to show that after finitely many iterations the resulting surface $X^{(i)}$ has only **sandwiched singularities** (see the definition below).

Definition 3.4.7 A surface singularity (X, ξ) is said to be **sandwiched** if a neighbourhood of ξ in X admits a birational map to a non-singular surface.

Being sandwiched is quite a strong restriction; in particular, sandwiched singularities are rational.

3.5 Oscar Zariski

The appearance on the scene of O. Zariski and his school marks a completely new era in the study of resolution of singularities. In the earlier section we mentioned the introduction of normalization which gives a one-step procedure for desingularizing curves in all characteristics, as well as Zariski's proof of resolution for surfaces. In the late nineteen thirties and early forties Zariski proposed a completely new approach to the problem using valuation theory (building on some earlier ideas of

Krull). In a nutshell this approach can be summarized as saying that valuation theory provides a natural notion of "local" in birational geometry and allows to state a local version of the resolution problem called Local Uniformization.

3.5.1 Valuations

For a detailed treatment of the basics of valuation theory, we refer the reader to [174] and [155].

Definition 3.5.1 **An ordered group** is an abelian group Γ together with a subset $P \subset \Gamma$ (here P stands for "positive elements") which is closed under addition and such that

$$\Gamma = P \coprod \{0\} \coprod (-P).$$

Remark 3.5.2 The above decomposition induces a total ordering on Γ:

$$a < b \iff b - a \in P.$$

Thus an equivalent way to define an ordered group would be "a group with a total ordering which respects addition, that is, $a > 0, b > 0 \implies a + b > 0$".

Note that an ordered group is necessarily torsion-free.

Example 3.5.3 The additive groups $\mathbb{Z}$, $\mathbb{R}$ with the usual ordering are ordered groups. Any subgroup $\Gamma \subset \mathbb{R}$ is an ordered group with the induced ordering (more generally, any subgroup of an ordered group is an ordered group). The group $\mathbb{Z}^n$ with the lexicographical ordering is an ordered group.

All the ordered groups that appear in algebraic geometry are subgroups of groups of the form $\bigoplus\limits_{i=1}^{r} \Gamma_i$, where $\Gamma_i \subset \mathbb{R}$ for all i and the total order is lexicographic.

We are now ready to define valuations. Let K be a field, Γ an ordered group. Let K^* denote the multiplicative group of K.

Definition 3.5.4 **A valuation** of K with value group Γ is a surjective group homomorphism $\nu : K^* \to \Gamma$ such that for all $x, y \in K^*$

$$\nu(x + y) \geq \min\{\nu(x), \nu(y)\}. \tag{3.22}$$

Remark 3.5.5 Let K be a field, ν a valuation of K and x, y non-zero elements of K such that $\nu(x) \neq \nu(y)$. It is a consequence of Definition 3.5.4 that in this case equality must hold in (3.22), that is,

$$\nu(x + y) = \min\{\nu(x), \nu(y)\}. \tag{3.23}$$

Example 3.5.6 Let X be an irreducible algebraic variety, $K = K(X)$ its field of rational functions, $\xi \in X$ such that $\mathcal{O}_{X,\xi}$ is a regular local ring. Let $\mathfrak{m}_{X,\xi}$ be the maximal ideal of $\mathcal{O}_{X,\xi}$. Define $\nu_\xi : K^* \to \mathbb{Z}$ by

$$\nu_\xi(f) = \text{mult}_\xi f = \max\left\{n \;\middle|\; f \in m^n_{X,\xi}\right\}, \; f \in \mathcal{O}_{X,\xi}.$$

The map ν_ξ extends from $\mathcal{O}_{X,\xi}$ to all of K in the obvious way by additivity:

$$\nu_\xi\left(\frac{f}{g}\right) = \nu_\xi(f) - \nu_\xi(g).$$

The map ν_ξ induces a group homomorphism because $\bigoplus \frac{m^n_{X,\xi}}{m^{n+1}_{X,\xi}}$ is an integral domain.

In the above example, note that ξ could be any *scheme-theoretic* point; for example, it could stand for the generic point of an irreducible codimension 1 subvariety. In that case, the condition that $\mathcal{O}_{X,\xi}$ be non-singular holds automatically whenever X is normal (Theorem 3.4.3).

Remark 3.5.7 Let X be an irreducible algebraic variety, A its coordinate ring,

$$K = K(X)$$

the field of fractions of A, $I \subset A$ an ideal. We can generalize the above example as follows. Define

$$\nu_I(f) = \max\{n \mid f \in I^n\}, \quad \text{for } f \in A.$$

In general, ν_I is a pseudo-valuation, which means that the condition of additivity in the definition of valuation is replaced by the *inequality* $\nu_I(xy) \geq \nu_I(x) + \nu_I(y)$. The map ν_I is a valuation if and only if $\oplus \frac{I^n}{I^{n+1}}$ is an integral domain (a condition which always holds if I is maximal and A_I is regular).

Valuations of the form ν_I are called **divisorial**. The reason for this name is that if A is the coordinate ring of an affine algebraic variety X, even if dim $A_I > 1$, we can always blow up X along I. Let $\pi : \widetilde{X} \to X$ be the blowing up along I. Then $K(X) = K\left(\widetilde{X}\right)$.

The property that $\oplus \frac{I^n}{I^{n+1}}$ is a domain means that the exceptional divisor

$$\tilde{D} := V(\pi^* I) = \pi^{-1}(V(I))$$

is irreducible. Then $\mathcal{O}_{\widetilde{X},\tilde{D}}$ is a regular local ring of dimension 1 and $\nu_I = \nu_{\tilde{D}}$ measures the order of zero or pole of a rational function at the generic point of $\tilde{D}$. This example illustrates an important philosophical point about valuations: a valuation is an object associated to the field K, that is, to an entire birational equivalence class, not to a particular model in that birational equivalence class. Thus to study a given valuation, one is free to perform blowings up until one arrives at a model which is particularly convenient for understanding this valuation.

Valuation Rings

Let K be a field, Γ an ordered group, $\nu : K^* \twoheadrightarrow \Gamma$ a valuation of K. Associated to ν is a local subring $(R_\nu, \mathfrak{m}_\nu)$ of K, having K as its field of fractions:

$$\begin{aligned} R_\nu &= \{x \in K^* \mid \nu(x) \geq 0\} \cup \{0\} \\ \mathfrak{m}_\nu &= \{x \in K^* \mid \nu(x) > 0\} \cup \{0\}. \end{aligned} \tag{3.24}$$

Example 3.5.8 (Divisorial Valuations) Let X be an irreducible algebraic variety, $D \subset X$ a closed irreducible subvariety, ξ the generic point of D.

Assume that $\mathcal{O}_{X,\xi}$ is a regular local ring of dimension 1. Let t be a generator of $\mathfrak{m}_{X,\xi}$. Then $K = (\mathcal{O}_{X,\xi})_t$. Indeed, any element $f \in \mathcal{O}_{X,\xi}$ can be written as $f = t^n u$, where $n \in \mathbb{N}$ and u is invertible. For each $f = t^n u$ as above, we have $\nu_D(f) = n$. Then $R_\nu = \mathcal{O}_{X,D}$.

Definition 3.5.9 Let $(R_1, \mathfrak{m}_1)$, $(R_2, \mathfrak{m}_2)$ be two local domains with the same field of fractions K. We say that R_2 **birationally dominates** R_1, denoted $R_1 < R_2$, if

$$R_1 \subset R_2 \quad \text{and} \tag{3.25}$$

$$\mathfrak{m}_1 = \mathfrak{m}_2 \cap R_1. \tag{3.26}$$

Remark 3.5.10 One of the main examples of birational domination encountered in algebraic geometry is the following. Let X be an irreducible algebraic variety and $\pi : X' \to X$ a blowing up of X. Let $\xi \in X$, $\xi' \in X'$ be such that $\xi = \pi(\xi')$. Then $\mathcal{O}_{X,\xi} < \mathcal{O}_{X',\xi'}$.

Theorem 3.5.11 *Let* $(R, \mathfrak{m})$ *be a local domain with field of fractions* K*. The following conditions are equivalent:*

(1) $R = R_\nu$ *for some valuation* $\nu : K^* \twoheadrightarrow \Gamma$

(2) for any $x \in K^*$*, either* $x \in R$ *or* $\frac{1}{x} \in R$ *(or both)*

(3) the ideals of R are totally ordered by inclusion

(4) $(R, \mathfrak{m})$ *is maximal (among all the local subrings of K) with respect to birational domination.*

Remark 3.5.12 Although we omit the proof of Theorem 3.5.11, we note that the proof of the implication (3) $\Longrightarrow$ (1) involves reconstructing the valuation ν (in a unique way, modulo the obvious equivalence relation) from the valuation ring. Hence the valuation ring R_ν determines ν up to equivalence.

For future reference, we define two important numerical characters of valuations: rank and rational rank.

Definition 3.5.13 An subgroup Δ of an ordered group Γ is said to be **isolated** if Δ is a segment with respect to the given ordering: if $a \in \Delta$, $b \in \Gamma$ and $-a \leq b \leq a$ then $b \in \Delta$.

The set of isolated subgroups of an ordered group Γ is totally ordered by inclusion.

Definition 3.5.14 Let ν be a valuation with value group Γ. The **rank** of ν, denoted rk ν, is the number of distinct isolated subgroups of Γ. We have rk $\nu = \dim\ R_\nu$.

Definition 3.5.15 The **rational rank** of ν is, by definition, rat.rk $\nu := \dim_{\mathbb{Q}} \Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$.

Theorem 3.5.11 (in particular, its part (4)) paves the way for a geometric interpretation of valuations. This is due to Zariski in the nineteen forties, when he developed valuation theory with the express purpose of applying it to the problem of resolution of singularities. To explain how valuations provide a natural notion of "local" in birational geometry and to give a precise statement of the Local Uniformization Theorem we need the notion of **center of a valuation** and also that of **local blowing up** with respect to a valuation, which we now define.

Definition 3.5.16 Let $(R, \mathfrak{m}, k)$ be a local domain with field of fractions K and ν a valuation of K. We say that ν is **centered** in R if $R < R_\nu$ (this is equivalent to saying that $\nu(R) \geq 0$ and $\nu(\mathfrak{m}) > 0$).

If X is an irreducible algebraic variety with $K = K(X)$ and ξ a point of X, we say that ν is **centered** in ξ (or that ξ is the center of ν on X) if it is centered in the local ring $\mathcal{O}_{X,\xi}$, that is, if $\mathcal{O}_{X,\xi} < R_\nu$.

The center of a given valuation ν on a variety X is uniquely determined by ν.

Let X be an irreducible algebraic variety, ξ a point of X and $\mathcal{I}$ a coherent ideal sheaf on X. Let $\pi : X_1 \to X$ be the blowing up of X along $\mathcal{I}$. Take a point $\xi_1 \in \pi^{-1}(\xi)$. The map π induces a local homomorphism $\sigma : \mathcal{O}_{X,\xi} \to \mathcal{O}_{X_1,\xi_1}$ of local rings.

Definition 3.5.17 A homomorphism of the form $\sigma : \mathcal{O}_{X,\xi} \to \mathcal{O}_{X_1,\xi_1}$, where ξ_1 is a point of $\pi^{-1}(\xi)$, is called a **local blowing up** of $\mathcal{O}_{X,\xi}$ along $\mathcal{I}_{X,\xi}$. Let ν be a valuation, centered at $\mathcal{O}_{X,\xi}$. We say that σ is a **local blowing up with respect to** ν if ν is centered at $\mathcal{O}_{X_1,\xi_1}$, that is,

$$\nu(\mathcal{O}_{X_1,\xi_1}) \geq 0 \left(\mathcal{O}_{X_1,\xi_1}\right) \geq 0$$

and $\nu\left(\mathfrak{m}_{X_1,\xi_1}\right) > 0$.

Let X be an irreducible algebraic variety, ξ a point of X and ν a valuation of $K = K(X)$, centered at ξ. Let $\pi : X' \to X$ be a birational projective morphism. The following theorem is a version of the **valuative criterion of properness**:

Theorem 3.5.18 *There exists a unique point $\xi' \in \pi^{-1}(\xi)$ such that ν is centered in ξ'.*

The notion of center of a valuation together with Theorem 3.5.18 allows us to divide the problem of resolution of singularities into two parts: local and global. The local version of resolution of singularities is called Local Uniformization.

Let X, K and ν be as above and assume that ν is centered at a point ξ of X.

Definition 3.5.19 A **local uniformization of** X with respect to ν is a birational projective morphism $\pi : X' \to X$ such that the center ξ' of ν in $\pi^{-1}(\xi)$ is a regular point of X'.

Zariski proved in 1940 that if X is an algebraic variety over a field of characteristic zero then X admits a local uniformization with respect to any valuation, centered at a point of X [170]. The same question is still open for fields of characteristic $p > 0$ (the papers [130, 131] and [132] show that to prove Local Uniformization in its full generality, it is sufficient to prove it for valuations of rank 1).

Clearly, a resolution of singularities $\pi : X' \to X$ also constitutes a local uniformization simultaneously with respect to every valuation ν, centered at a point of X. The converse, however, is not so clear: assume that local uniformization is known with respect to every valuation. Does this imply the existence of resolution of singularities of X?

To study this question, Zariski introduced what is known today as the Zariski–Riemann space. Let X be an irreducible algebraic variety. Consider the totality of all the birational projective morphisms $X_\alpha \to X$. This set naturally forms a projective system, whose arrows are birational projective morphisms. Indeed, given two such morphisms

$$X_\alpha \to X \quad \text{and} \tag{3.27}$$

$$X_\beta \to X, \tag{3.28}$$

one can construct a new variety $X_{\alpha\beta}$ together with birational projective morphisms

$$\lambda_\alpha : X_{\alpha\beta} \to X_\alpha, \tag{3.29}$$

$$\lambda_\beta : X_{\alpha\beta} \to X_\beta \tag{3.30}$$

which make the diagram

$$\begin{tikzcd} X_{\alpha\beta} \arrow[r, "\lambda_\beta"] \arrow[d, "\lambda_\alpha"'] & X_\beta \arrow[d, "\pi_\beta"] \\ X_\alpha \arrow[r, "\pi_\alpha"] & X \end{tikzcd} \tag{3.31}$$

commute. The variety $X_{\alpha\beta}$ is the unique irreducible component of the cartesian product $X_\alpha \times_X X_\beta$ which maps dominantly onto X, X_α and X_β. More explicitly, since π_α and π_β are birational, there exist non-empty open subvarieties $U \subset X$, $U_\alpha \subset X_\alpha$ and $U_\beta \subset X_\beta$ such that $\pi_\alpha|_{U_\alpha} : U_\alpha \cong U$ and $\pi_\beta|_{U_\beta} : U_\beta \cong U$. Then

$U \cong U_\alpha \times_U U_\beta$ embeds naturally into $X_\alpha \times_X X_\beta$ as an irreducible open set. The variety $X_{\alpha\beta}$ is nothing but the Zariski closure of U in $X_\alpha \times_X X_\beta$. Geometrically, one should think of $X_{\alpha\beta}$ as the graph of the birational correspondence between X_α and X_β.

Let $S := \underset{\overleftarrow{\alpha}}{\lim} X_\alpha$.

Theorem 3.5.20 (Zariski) *There is a natural bijection between S and the set*

$$M := \{\textit{valuations } \nu \textit{ of } K, \ \textit{centered at points of } X\}.$$

We briefly sketch the proof.

Proof First, fix a valuation ν of K, centered at a point $\xi \in X$. By Theorem 3.5.18, for each $\pi_\alpha : X_\alpha \to X$ in our projective system, there exists a unique $\xi_\alpha \in \pi^{-1}(\xi)$ such that ν is centered at ξ_α. Therefore we can associate to ν a collection $\{\xi_\alpha \in X_\alpha\}_\alpha$, compatible with the morphisms in our projective system, that is, an element of S. This defines a natural map $f : M \to S$.

Conversely, take an element $\{\xi_\alpha \in X_\alpha\}_\alpha \in S$. The local rings $\mathcal{O}_{X_\alpha,\xi_\alpha}$ form a *direct* system, whose arrows are relations of birational domination. It should therefore not come as a surprise that the direct limit $R := \underset{\overrightarrow{\to\alpha}}{\lim} \mathcal{O}_{X_\alpha,\xi_\alpha}$ of this system is a local subring of K, maximal with respect to $<$, that is, a valuation ring. To prove this rigorously, a short argument using the equivalence (1)$\Longleftrightarrow$(2) of Theorem 3.5.11 is required. We omit the details.

This defines the map $g : S \to M$. It is routine to check that the maps f and g are inverse to each other. ☐

Definition 3.5.21 The set S is called the **Zariski–Riemann space** associated to X.

Zariski's original name for this object (in the special case when X was a *projective* variety over k) was **the abstract Riemann surface of the field** K. The thinking was that in the special case when $k = \mathbb{C}$ and $\dim\ X = 1$, the projective system defining S is finite and its inverse limit is nothing but the resolution of singularities of X, that is, a smooth complex projective curve, or a Riemann surface. However, when $\dim\ X \geq 2$, S does not even have a structure of a variety or a scheme, only one of a ringed space. It resembles more John Nash's space of arcs than it does anything like a Riemann surface. This is why the name "Zariski–Riemann space" seems more appropriate.

In order to address the problem of "glueing" the local uniformizations with respect to various valuations, it is useful to introduce a topology on S. Namely, S is naturally endowed with the inverse limit topology (which is usually referred to as the **Zariski topology** on S). By definition of inverse limit, for each X_α in our projective system we have a natural map $\rho_\alpha : S \to X_\alpha$; this map assigns to each valuation ν centered at a point $\xi \in X$ the center of ν in X_α, lying over ξ. A base for the Zariski topology is given by all the sets of the form $\rho_\alpha^{-1}(U)$ where X_α runs over the entire projective system and U over all the Zariski open sets of X_α. In other words, the Zariski topology is the coarsest topology which makes all the maps ρ_α continuous.

Theorem 3.5.22 (Zariski [173], Chevalley) *The topological space S is compact.*

We spell out the main idea of the proof. By definition, S comes with a natural embedding ι into the direct product $\prod\limits_\alpha X_\alpha$. Each X_α is compact with respect to its Zariski topology, hence so is $\prod\limits_\alpha X_\alpha$ by Tychonoff's theorem. If all the topologies in sight were Hausdorff, ι would be a *closed* embedding, and the compactness of S would follow immediately. Indeed, this is how one proves a standard theorem from general topology: an inverse limit of compact Hausdorff spaces is again compact.

Unfortunately, none of the spaces we are working with here are Hausdorff. The next idea is to replace the Zariski topology on the X_α by a finer, Hausdorff topology, pass to the inverse limit and conclude compactness as above, and then observe that the compactness property is preserved by passing from a finer topology to a coarser one. This is, indeed, what Zariski did in the special case of projective varieties over $\mathbb{C}$. He replaced the Zariski topology by the classical Euclidean topology and the proof was completed as above. Finally, Chevalley came up with a proof, which follows roughly the same plan, but is applicable to varieties over fields of any characteristic and even to arbitrary noetherian schemes.

Once Zariski proved the Local Uniformization Theorem in characteristic zero, his plan went as follows. For each valuation $\nu \in S$, let $\pi : X' \to X$ be the local uniformization with respect to ν and let ξ' be the center of ν on X'. Let U denote the preimage in S of the set $Reg(X')$. By definition, U is an open set, containing ν. Furthermore, for every $\nu' \in U$ the map π constitutes local uniformization also with respect to ν'. Conclusion: once we achieve local uniformization with respect to some $\nu \in S$, we automatically achieve it for all the valuations in some open neighbourhood U of ν. Since this can be done for every $\nu \in S$, we obtain an open covering of S by sets U, for each of which there exists a simultaneous local uniformization of all the elements of U. By compactness, this open covering admits a finite subcovering. Finally, we obtain: there exist finitely many birational projective morphisms $\pi_i : X_i \to X$, $1 \le i \le n$, having the following property. Let $\rho_i : S \to X_i$ denote the natural map, given by the definition of projective limit. Then $\bigcup\limits_{i=1}^{n} \rho^{-1}(Reg(X_i)) = S$.

At this point, the problem of resolution of singularities in characteristic zero was reduced to one of "glueing" the n partial desingularizations X_i together to produce a global resolution of singularities. More precisely by induction on n it is sufficient to prove the following:

There exists an algebraic variety X_{12} together with birational projective morphisms

$$\lambda_1 : X_{12} \to X_1 \tag{3.32}$$

$$\lambda_2 : X_{12} \to X_2, \tag{3.33}$$

having the following properties:

1) the diagram

$$\begin{tikzcd}
X_{12} \arrow[r, "\lambda_2"] \arrow[d, "\lambda_1"'] & X_2 \arrow[d, "\pi_2"] \\
X_1 \arrow[r, "\pi_1"] & X
\end{tikzcd} \tag{3.34}$$

commutes

2) we have $Reg(X_{12}) \supset \lambda_1^{-1}(Reg(X_1)) \cup \lambda_2^{-1}(Reg(X_2))$.

The glueing problem is highly non-trivial because the local uniformization algorithms used to construct the partial resolutions X_i depend on the respective valuations. A priori absolutely nothing is known about the nature of the birational correspondences among the various X_i.

Zariski was able to solve this problem in dimension 2 by proving his famous factorization theorem: a birational morphism between non-singular surfaces is a composition of point blowings up (see [10] for a much more difficult version of this result in higher dimensions). It is also worth mentioning that Zariksi's factorization theorem together with Castelnuovo's criterion for contractibility of rational curves on non-singular surfaces implies the existence of *minimal resolution* for surfaces, that is, a resolution such that every other resolution of singularities factors through it.

With much greater difficulty Zariski advanced to dimension three [172]. This work of Zariski was recently generalized and systematized by O. Piltant [136]. Thanks to this, we now have a general procedure for glueing local uniformizations in dimension three in a much more general context and for much more general objects than just algebraic varieties or schemes.

3.6 Resolution of Singularities of Algebraic Varieties over a Ground Field of Characteristic Zero

Almost twenty-five years have passed after Zariski's proof of his Local Uniformization Theorem until H. Hironaka proved the existence of resolution of singularities in characteristic zero without using valuations or the Zariski–Riemann space. This (next) revolution in the field of resolution of singularities is the subject of the present section.

Theorem 3.6.1 (H. Hironaka [93]) *Every variety X over a ground field of characteristic zero admits a resolution of singularities.*

Hironaka's original proof of this was over 200 pages long. It is one of the most technically difficult and one of the most often quoted results of the twentieth century mathematics. We give a very brief sketch of the main ideas of the proof, as seen from 55 years into the future.

Proof

Step 1. The definition of **normally flat** (see Definition 3.3.16 and Theorem 3.3.17).

Step 2:

Proposition 3.6.2 *Let X be an algebraic variety and Y a smooth subvariety of X. Assume that X is normally flat along Y. Let $\pi : \widetilde{X} \to X$ be the blowing up of X along Y. Take a point $\tilde{\xi} \in \pi^{-1}(Y)$. Then the Hilbert–Samuel function $H_{\widetilde{X},\tilde{\xi}}$ is smaller than or equal to the common Hilbert–Samuel function $H_{X,\xi}$ of all the points $\xi \in X$. In particular, the blowing up π does not increase the maximal value of the Hilbert–Samuel function $H_{X,\xi}$ of all the points $\xi \in X$.* □

To complete the proof of the Theorem, it is sufficient to construct a sequence of blowings up of X that decreases the Hilbert–Samuel function *strictly*.

Step 3. Reduce the problem to the case when X is an n-dimensional hypersurface embedded into k^{n+1}:

$$X = V(f), \qquad \text{where } f \in k[x, y], \qquad y \text{ is a single variable and } x = (x_1, \dots, x_n). \tag{3.35}$$

This amounts to choosing a Gröbner basis (or a standard basis in Hironaka's terminology) $(f_1, \dots, f_r)$ of the defining ideal I of X having the following properties.

(a) The maximal locus of the Hilbert–Samuel function of X is equal to the intersection of the loci of maximal multiplicity of the polynomials f_i. In particular, a blowing up center Y is permissible for X if and only if it is simultaneously permissible for each of the hypersurfaces $V(f_i)$. This property holds after any permissible sequence of blowings up under which the maximal value of the Hilbert–Samuel function does not decrease.

(b) Let

$$\pi : \widetilde{X} \to X \tag{3.36}$$

be a permissible sequence of blowings up. The sequence π strictly decreases the maximal value of the Hilbert–Samuel function of X if and only if it strictly decreases the maximal multiplicity of a singularity of at least one of the hypersurfaces $V(f_i)$.

Remark 3.6.3 In 1977 H. Hironaka proved that, regardless of the characteristic of the ground field there exists a basis $(f_1, \dots, f_r)$ of I such that (a) and (b) hold [97].

Step 4. From now on, assume that X is a hypersurface as in (3.35). Let $\mu := \operatorname{mult}_0 f$; assume that μ is the greatest multiplicity of a singular point of X. Using the Henselian Weierstrass Preparation Theorem, further reduce the problem to the case when f has the form $f(x, y) = y^{\mu} + \sum_{i=1}^{\mu} \phi_i(x) y^{\mu-i}$, where $\mathrm{mult}_0 \phi_i \geq i$. This requires replacing X by a suitable étale covering, but we will not dwell on this point here.

Step 5. Make the **Tschirnhausen transformation**, that is, the change of coordinates $y \to y + \frac{1}{\mu}\phi_1(x)$. This amounts to ensuring that in the new coordinates we have

$$\phi_1(x) = 0. \tag{3.37}$$

We will assume that (3.37) holds from now on. In this situation we say that y is a **maximal contact coordinate for** X.

Step 6. The following Proposition is proved by an easy direct calculation.

Proposition 3.6.4

(1) The maximal contact hyperplane $W := \{y = 0\}$ contains all the points of X of multiplicity μ sufficiently close to the origin. In particular, every permissible center Y is contained in W.

(2) Let (3.36) be a permissible blowing up with center Y. Take a point $\tilde{\xi} \in \pi^{-1}(Y)$ and let $\tilde{f}$ be a local defining equation of $\tilde{X}$ near $\tilde{\xi}$. If $\mathrm{mult}_{\tilde{\xi}} \tilde{f} = \mu$ then $\tilde{\xi}$ lies in the strict transform of W. ☐

This looks like a good setup for induction on dim X. Indeed, on the one hand, we are only interested in blowing up centers Y that are contained in the hyperplane W. On the other hand, the only points we are interested in studying *after* blowing up belong to the strict transform of W. Thus the next idea is to try to define a variety V *strictly contained* in W and relate the problem of desingularizing V to that of desingularizing our original variety X.

Step 7. In fact, instead of a variety V we need to consider a more general object: a scheme, defined by the *idealistic exponent*, associated to f. Precisely, consider the ideal $H := \left(\phi_i^{\frac{\mu!}{i}}\right)_{2 \leq i \leq \mu} \subset k[x]$. After defining the notion of a permissible blowing up center for $V(H)$ and showing that a center Y is permissible for $V(H)$ if and only if it is permissible for X, one can use the induction assumption to construct a sequence (3.36) of permissible blowings up that **monomializes** the ideal H (by this we mean that $\pi^* H$ is principal and generated locally near every point of $\widetilde{X}$ by a single monomial in suitable coordinates; this should be thought of as an embedded resolution of $V(H)$). This is an important feature of Hironaka's construction: in order to construct a resolution of singularities of n-dimensional varieties, we need embedded resolution in dimension $n-1$. Because of this, both resolution and embedded resolution are proved by two simultaneous inductions: embedded resolution in dimension $n-1 \implies$ resolution in dimension $n \implies$ embedded resolution in dimension n.

Step 8. By Step 7, assume that H is generated by a monomial ω. To monomialize f it remains to construct a sequence of blowings up along permissible coordinate subvarieties (3.36) such that at each point of $\widetilde{X}$ one of the monomials ω and $y^{\mu!}$ divides the other. This is a special case of a purely combinatorial problem that has been variously called Hironaka's game, Perron's algorithm and resolution of (not necessarily normal) toric varieties by permissible blowings up. It is the combinatorial skeleton of resolution of singularities that appears, implicitly or explicitly in every desingularization algorithm that consists of a sequence of blowings up along non-singular subvarieties of the ambient regular variety. We refer the reader to [71, 142, 170] and [125] for various solutions of this problem (see [143] for a counterexample to a harder version of Hironaka's game, needed for resolution in characteristic $p > 0$). □

Remark 3.6.5 The assumption char $k = 0$ is used crucially in Step 5. Naively, one sees that $\frac{1}{\mu}$ makes no sense when char $k = p > 0$ and $p \mid \mu$. More seriously, R. Narashimhan [128] gave the following example showing that in positive characteristic there might not exist a non-singular subvariety satisfying (1) of Proposition 3.6.4, that is, containing all the points of multiplicity μ sufficiently near the origin.

Example 3.6.6 Let k be a perfect field of characteristic 2 and consider the hypersurface X defined by $f(x, y) = y^2 + x_1x_2^3 + x_2x_3^3 + x_3x_1^7 = 0$ in k^4. This threefold has multiplicity 2 at the origin and all of its points are either non-singular or have multiplicity 2, so its multiplicity 2 locus coincides with the singular locus. The singular locus $Sing(X)$ is defined by $\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3} = 0$, that is, $x_2^3 + x_3x_1^6 = x_1x_2^2 + x_3^3 = x_2x_3^2 + x_1^7 = 0$. We find that $Sing(X)$ is the parametrized curve $t \to \left(t^7, t^{19}, t^{15}, t^{32}\right)$ and that this curve has embedding dimension 4 at the origin. Thus it is not contained in any proper non-singular subvariety of k^4 passing through the origin. This shows that in this case there does not exist a non-singular variety W satisfying (1) of Proposition 3.6.4.

Much work has been done since 1964 to simplify and better understand resolution of singularities in characteristic zero. We mention [18, 20, 22–35, 41–43, 65, 76, 77, 90, 112, 150–152, 160, 163, 166, 167].

Many of the later proofs (starting with Bierstone–Milman and Villamayor) have the following transparent structure. One defines a discrete, upper semi-continuous numerical character of singularity $d(\xi)$, consisting of the Hilbert–Samuel function followed by a finite string of non-negative integers. We regard the set of possible values of $d(\xi)$ as being totally ordered by the lexicographical ordering. One stratifies the singular variety X according to $d(\xi)$. By upper semi-continuity, the maximal stratum S_{max} of $d(\xi)$ is a closed subvariety of X. One shows that S_{max} is a normal crossings subvariety and chooses one of its coordinate subvarieties Y in a canonical way (discussed and explained below by example). One lets $\pi : \widetilde{X} \to X$ be the blowing up along Y and one shows that for every $\tilde{\xi}$ not belonging to the strict transforms of components of S_{max} other than Y we have $d\left(\tilde{\xi}\right) < d(\xi)$. Repeating this procedure for the other components of S_{max} we strictly lower the maximum value of $d(\xi)$. This completes the proof by induction on $d(\xi)$.

3.6.1 Functorial Properties of Resolution in Characteristic Zero

The later proofs cited above are functorial with respect to smooth morphisms (flat morphisms with non-singular fibers). This means that they produce a functor from the category of varieties and smooth morphisms to the category of non-singular varieties and smooth morphisms that assigns to each variety X its resolution of singularities $\widetilde{X}$. Being a functor simply means that each smooth morphism of varieties $\phi : X \to V$ lifts (necessarily uniquely) to a smooth morphism $\tilde{\phi} : \widetilde{X} \to \tilde{V}$ of their resolutions. In particular, if ϕ is an open embedding (resp. an isomorphism), so is $\tilde{\phi}$. In this way we obtain that our procedure of resolution of singularities is equivariant with respect to automorphisms of X, any group actions on X, etc.

Choosing a Unique Coordinate Subvariety of S_{max} in a Canonical Way

We illustrate the situation by example.

Example 3.6.7 Consider the surface X defined by the equation $z^2 - x^3y^3 = 0$. Its singular locus coincides with its locus of multiplicity 2 and is the union of the x-axis and the y-axis. These two axes play a symmetric role (in fact, they can be carried into each other by an automorphism X). From a naive point of view, blowing up the origin does not seem to improve the singularity, so one is tempted to blow up one of the one-dimensional components of $Sing(X)$. However, there is no way to do this and respect the functoriality described above. Even if one did not care for functoriality in its own right, a desingularization algorithm that involves an arbitrary choice of a branch would present serious problems: after all, there could be a singularity that locally looks like X but such that the two branches of $Sing(X)$ are in fact two branches of the same irreducible curve.

The modern solution to this problem goes as follows. Start by blowing up the origin since it is the only canonical choice that can be made. The multiplicity 2 locus $Sing\left(\widetilde{X}\right)$ of $\widetilde{X}$ now consists of three lines: the respective strict transforms $\tilde{L}_x$ and $\tilde{L}_y$ of the x- and the y-axes and the exceptional divisor E. At first glance the singular points $E \cap \tilde{L}_x$ and $E \cap \tilde{L}_y$ look worse than the singularity at the origin that we started with, and $Sing\left(\widetilde{X}\right)$ is again a union of two lines near each of those points. However, they have one important new advantage: there is a natural ordering on the set of irreducible components of the equimultiple locus, namely, the order of appearance of those components in the history of the resolution process until this point. This settles the difficult issue of which component should be blown up first.

This points to another important feature of all the known resolution procedures by permissible blowings up: the choice of the blowing up center at each step depends not only on our singular variety itself but also on the history of the resolution process up to the given point.

Two recent preprints, [11] and [122], get around this problem by working in the 2-category of excellent Deligne–Mumford stacks instead of varieties or schemes (stacks are beyond the scope of this survey, but a definition of excellent and quasi-excellent schemes is given in the Appendix).

Finally, we mention a construction of resolution of singularities of analytic varieties due to J.M. Aroca, H. Hironaka and J.L. Vicente Cordoba [14–16] as well as the paper [98] by H. Hironaka.

3.7 Resolution of Singularities of Algebraic Varieties over a Ground Field of Positive Characteristic

3.7.1 Resolution in Dimensions 1, 2 and 3

As mentioned above, resolution of curve singularities in arbitrary characteristic was settled in 1939 when Zariski defined normalization: this one-step procedure works equally well in characteristic zero and characteristic $p > 0$.

The first proof for surfaces is due to S. Abhyankar in 1956 [1] with subsequent strengthenings by H. Hironaka [100] and J. Lipman [117] to the case of more general 2-dimensional schemes, with Lipman giving necessary and sufficient condition for a 2-dimensional scheme to admit a resolution of singularities. See also [64].

The next breakthrough came in 1966, again due to S. Abhyankar, who proved resolution of singularities for threefolds except in characteristics 2, 3 and 5. The idea of Abhyankar's proof is the following. The starting point of the proof is an Auxiliary Theorem which says that any d-dimensional variety over an algebraically closed ground field can be birationally transformed to a variety having no e-fold point for any $e > d!$. The proof of this Auxiliary Theorem generalizes an argument used by Albanese [13] in the surface case combined with the Veronese embedding. Since $3! = 6$, in the special case $d = 3$ we obtain that our variety has singularities of multiplicity at most 6. If $p > 6$ then all the singularities have multiplicity strictly smaller than the characteristic of the ground field (this is precisely the reason for the restriction on the characteristic of the ground field in Abhyankar's proof). Roughly speaking, in this situation one can imitate characteristic zero methods to finish the proof. Still, Abhyankar's proof is extremely technical and difficult and comprises a total of 508 pages [2–6]. For a more recent and more palatable proof we refer the reader to [66].

Resolution of singularities for surfaces was reproved by J. Giraud in 1983 [80], using a novel idea that has proved to be very influential for subsequent work (see also [52, 57–59]). Namely, let k be a perfect field of characteristic $p > 0$ and consider the (typical and significant) special case of a surface in k^3, defined by an equation of the form

$$f(x_1, x_2, y) = y^p - g(x_1, x_2) = 0, \qquad (3.38)$$

where g is some polynomial in two variables of multiplicity strictly greater than p. If we wanted to imitate characteristic zero methods, we would naturally study the transformation law for g under blowing up and try to relate the resolution problem for f to the problem of monomialization of g. We already mentioned in the previous section that the main obstruction to imitating characteristic zero proof in the case of characteristic $p > 0$ is the non-existence of maximal contact coordinates in the situation when p divides the multiplicity of a defining equation. A natural idea for a replacement of maximal contact coordinates in the case of Eq. (3.38) is to use a transformation of the form

$$y \to y + \phi(x_1, x_2) \qquad (3.39)$$

to make sure that no monomials which are p-th powers appear in g. However, unlike maximal contact coordinates in characteristic zero which are stable under coordinate changes in the x variables and under blowings up that do not lower the Hilbert–Samuel function, the above "maximal contact" coordinates in positive characteristic can be destroyed even by the simplest of linear homogeneous coordinate changes, as the following example shows.

Example 3.7.1 Take $g = x_1 x_2^{2p-1}$ in (3.39). Then g is a single monomial that is not a p-th power. However, after a coordinate change $(x_1, x_2) \to (x_1 + x_2, x_2)$ the new equation involves the monomial x_2^p.

Giraud's idea for dealing with this difficulty was to study the behaviour of the differential dg (instead of that of g itself) under permissible blowings up. The point is that the differential dg is stable under coordinate changes of the form (3.39). The drawback of this approach is that the transformation rules of dg under blowing up are much more complicated than those for g itself. In spite of this, Giraud was able to give a new proof of resolution of surface singularities using this idea.

The method of Giraud was systematically exploited by his Ph.D student V. Cossart to give, in his Ph.D thesis [53], a proof of resolution of singularities of threefolds defined by equations of the form $y^p - f(x_1, x_2, x_3) = 0$, which for a long time had been considered to be the basic and significant special case, exhibiting most of the phenomena and difficulties of the general problem.

The same result was obtained independently and by different methods by T.T. Moh [124]. Both works are of a formidable technical difficulty and comprise hundreds of pages.

It was not until much later that V. Cossart and O. Piltant settled the problem of resolution of threefolds in complete generality (their theorem holds for arbitrary quasi-excellent noetherian schemes of dimension three, including the arithmetic case) in a series of three long papers spanning the years 2008 to 2019 [61–63], building on the earlier works [54–56]. The overall method is based on the idea of Giraud mentioned above. The main point is to prove the Local Uniformization Theorem. After that global resolution of singularities becomes an immediate consequence of Piltant's work [136] that axiomatizes Zariski globalization in three dimensions.

3.7.2 Resolution and Local Uniformization in Dimension Four and Higher

In this subsection we briefly mention and discuss known recent partial results, programs and attempts at proofs in arbitrary dimension.

In the paper [153] Michael Temkin proves a version of the Local Uniformization Theorem in which the required desingularization map $\pi : \widetilde{X} \to X$ is *generically finite* instead of being birational (in other words, it induces a *finite extension*

$$\iota : K(X) \hookrightarrow K\left(\widetilde{X}\right)$$

of function fields instead of an isomorphism). In Temkin's proof the extension ι can be taken to be purely inseparable. Among other things, he gives a rigorous proof of a fact that until then was a mere philosophical belief: to prove Local Uniformization (for varieties over fields of characteristic $p > 0$) it is sufficient to prove it for hypersurfaces defined by equations of the form $y^p + g(x_1, \dots, x_n) = 0$.

A similar, though in some sense complementary result was obtained by H. Knaf and F.-V. Kuhlmann [111]: they also prove Local Uniformization after a finite extension ι of function fields, but in their case the extension ι is Galois (combined with a purely inseparable extension of the residue field of the valuation ring in the case of non-perfect residue fields). In the paper [110] the same authors prove Local Uniformization with respect to Abhyankar valuations. A valuation ν is said to be **Abhyankar** if equality holds in Abhyankar's inequality:

$$\text{rat.rk}\ \nu + \text{tr.deg}(k_\nu/k) = \text{tr.deg}(K/k),$$

where k denotes the ground field, $K = K(X)$ is the field of rational functions of the variety X we want to desingularize and k_ν is the residue field of the valuation ring.

It is well known that to prove the Local Uniformization Theorem it is sufficient to prove it for the case of hypersurfaces (since in the case of general varieties one can handle the defining equations one by one). Let X be a hypersurface in k^n defined by an equation $f(u_1, \dots, u_n) = 0$. We would like to construct a local uniformization with respect to a given valuation ν. Consider the extension

$$\theta : k(u_1, \dots, u_{n-1}) \hookrightarrow \frac{k(u_1, \dots, u_{n-1})[u_n]}{(f)} \tag{3.40}$$

of valued fields.

One way of thinking of the main difficulty of constructing a local uniformization of X with respect to ν is in terms of the **defect** δ of the extension θ (this point of view has been promoted by F.-V. Kuhlmann among others, see [113]). Defining defect is beyond the scope of this survey, but we briefly mention some of its properties relevant to us.

Let

$$p = 1 \qquad \text{if char } k_\nu = 0 \tag{3.41}$$

$$= \text{char } k \qquad \text{if char } k_\nu > 0. \tag{3.42}$$

The defect δ is always a power of p, hence is equal to 1 if char $k = 0$. We have $\delta = 1$ as well in the case of Abhyankar valuations (this explains why the characteristic zero case as well as the case of Abhyankar valuations is easier to handle than the case of arbitrary valuations in characteristic $p > 0$). The philosophy that "all the difficulty of local uniformization lies in the defect" has been understood for some time, but we would like to mention two recent works that make the above statement precise: J.-C. San Saturnino [139, Theorem 6.5] and S. D. Cutkosky–H. Mourtada [67].

We mention two papers by B. Teissier, [148] and [149], that propose another possible approach to constructing Local Uniformization using the graded algebra associated to the given valuation ν and trying to interpret this graded algebra as the coordinate ring of an (infinite-dimensional) toric variety that is a deformation of the variety X we want to desingularize, inspired by the case of plane curve singularities [12, 81].

Finally, for the approach to local uniformization via key polynomials we refer the reader to [7, 8, 68, 69, 91, 92, 118–120, 133, 138, 139, 147, 156–159]. J. Decaup's Ph.D. thesis carries out the program of proving a strengthening of the Local Uniformization Theorem over fields of characteristic zero, but with a view to generalizing the result to fields of positive characteristic.

There has also been recent work whose goal is to construct (or at least make progress toward constructing) global resolution of singularities directly, without going through valuation theory and local uniformization, but the jury is still out on how close to or far from a complete proof we are: [19, 36–39, 44, 89, 101–104, 107, 108].

3.8 An Alternative Approach by J. de Jong et al. via Semi-stable Reduction

In 1996 a major event occurred in the field of resolution of singularities: J. de Jong [70] proved the existence of resolution of singularities for varieties over fields of arbitrary characteristic by alterations:

Definition 3.8.1 An **alteration** is a proper surjective morphism

$$\pi : \widetilde{X} \to X \tag{3.43}$$

such that the induced homomorphism $K(X) \hookrightarrow K\left(\widetilde{X}\right)$ of function fields is finite.

Theorem 3.8.2 *Let X be a variety over a ground field k. There exists an alteration (3.43) such that $\widetilde{X}$ is non-singular. In fact, we can choose $\widetilde{X}$ to be a complement of a normal crossings divisor in some regular projective variety X'.*

We briefly summarize his proof which uses the compactification of moduli stacks of curves of genus g by stable curves (in the special case when X is projective).

Proof

Step 1. Take a sufficiently general projection $\rho : X \to Y$ to a variety Y of dimension dim $X - 1$, so that the fibers of ρ are curves.

Step 2. Normalizing X, we may assume, in addition, that X is normal. After further modifying X by a birational transformation, we may choose the fibration morphism to Y to be generically smooth along any component of any fiber.

Step 3. Choose a sufficiently general and sufficiently ample relative divisor H on X over Y. After taking a base change with an alteration $Y' \to Y$, we may assume that H is a union of sections $\sigma_i : Y \to X$:

$$H = \bigcup_{i=1}^{n} \sigma_i(Y)$$

(this is one of the places in the proof where we actually need to use an alteration rather than a birational map).

Step 4. Since H was chosen sufficiently general and sufficiently ample, for every component of every fiber of ρ there are at least three sections σ_i, intersecting it in distinct points of the smooth locus of ρ. Therefore there exists a Zariski open subset $U \subset Y$ such that for each $\eta \in U$ the fiber $\rho^{-1}(\eta)$, together with the points determined by the σ_i, is a stable n-pointed curve of certain genus g. By definition of the moduli stack $\overline{\mathcal{M}}_{g,n}$ of stable curves of genus g with n marked points, we obtain a unique morphism $U \to \overline{\mathcal{M}}_{g,n}$ such that the family

$$\rho\Big|_{\rho^{-1}(U)} : \rho^{-1}(U) \to U \tag{3.44}$$

is the pullback of the universal family of stable n-pointed curves of genus g over $\overline{\mathcal{M}}_{g,n}$. Now, $\overline{\mathcal{M}}_{g,n}$ admits a finite étale covering $\overline{M} \to \overline{\mathcal{M}}_{g,n}$ by a projective scheme $\overline{M}$; the universal family of stable n-pointed curves of genus g can be lifted to $\overline{M}$. Putting $U' := U \times_{\overline{\mathcal{M}}_{g,n}} \overline{M}$, we obtain a cartesian diagram

$$\begin{tikzcd} U' \arrow[r, "\theta"] \arrow[d, "\lambda"] & \overline{M} \arrow[d] \\ U \arrow[r] & \overline{\mathcal{M}}_{g,n} \end{tikzcd} \tag{3.45}$$

where λ is an alteration and the pullback of the family of curves (3.44) under λ coincides with the pullback of the universal family by θ (this is the second place in the proof where we genuinely need to use alterations rather than birational morphisms).

Step 5. Let X_U denote the preimage of U in X. Let Y' be the closure of

$$Im(U' \to Y \times \bar{M}) \subset Y \times \bar{M}.$$

Then Y' is a projective variety over k and $Y' \to Y$ is an alteration which is generically étale. The smooth stable n-pointed curve $(X_U, \sigma_1|_U, \dots, \sigma_n|_U) \times U'$ extends to a stable n-pointed curve X' over Y'.

Step 6. Replacing Y by Y' and X by X' we reduce the problem to the case in which there exist a stable n-pointed curve $(C, \tau_1, \dots, \tau_n)$ over Y, a nonempty open subvariety $U \subset Y$ and an isomorphism $\beta : C_U \to X_U$ mapping the section $\tau_i|_U$ to the section $\sigma_i|_U$ (where C_U denotes the preimage of U in C). It can be proved that the rational map β can me be made into a morphism, possibly after base change by a birational projective transformation of Y.

To Summarize the Result of Steps 4–6 We started out with a morphism ρ whose *generic* fiber was a stable n-pointed curve of genus g. We ended up with a morphism ψ, *all* of whose fibers are stable n-pointed curves. In other words, we have reduced the problem to the case where all the fibers of ρ are stable pointed curves (and the generic fiber is non-singular).

Step 7. By induction on dim X, resolve the singularities of Y. Furthermore, by the induction hypothesis in the non-projective case we may assume that the non-smooth locus of the morphism ρ is a normal crossings divisor (note that we are using the induction hypothesis in the non-projective case even to prove the result for projective X).

Step 8. At this point the only singularities of C are given by equations of the form

$$xy = t_1^{n_1} \dots t_d^{n_d}.$$

These are resolved explicitly by hand. □

Now assume that char $k = 0$. Shortly after the appearance of de Jong's theorem on alterations D. Abramovich and J. de Jong [9] took it as a starting point to give a new proof of resolution of singularities by birational morphisms in characteristic zero.

Their proof goes as follows. Fix an alteration $X' \to X$ such that X' is non-singular. We may assume that the corresponding finite extension $K(X) \hookrightarrow K(X')$ of function fields is Galois. Let G denote the Galois group $Gal(K(X')/K(X))$. Then G acts on X' and the quotient of this action birationally dominates X. By induction on dim X we may assume that the subvariety $\left\{\xi \in X' \;\middle|\; g(\xi) = \xi \text{ for some } g \in G\right\}$ of points of X' fixed by at least one $g \in G$ is a normal crossings divisor. A few auxiliary blowups make the quotient X'/G toroidal. Finally, the authors apply the well known result on resolution of toroidal singularities [109, Theorem 11*] to finish the argument.

Another proof of resolution of singularities in characteristic zero based on the same idea but quite different in detail from the Abramovich–de Jong one was given independently by F. Bogomolov and T. Pantev [40].

3.9 Resolving Singularities in Characteristic Zero by Nash and Higher Nash Blowing Up: Results and Conjectures

The goal of this section is define Nash and higher Nash blowing up and to give an overview of both known results and conjectures involving their desingularization properties.

H. Hironaka's proof that every algebraic variety over a field of characteristic zero admits a resolution of singularities provided an inspiration to John Nash for several extremely fruitful ideas, one of the most important being the introduction of Nash blowing up as a conjectural method for constructing a canonical resolution of singularities of varieties in characteristic zero.

Let k be a field and X an affine irreducible algebraic variety of dimension n embedded in k^N.

Definition 3.9.1 The **Gauss map** $\phi : X \setminus Sing(X) \to G := Grass(N, n)$ is the map that sends every non-singular point $\xi \in X$ to its tangent space, viewed as a point of G.

Definition 3.9.2 The **Nash blowing up** NX of X is the closure $\overline{graph(\phi)}$ of $graph(\phi)$ in $X \times G$.

We have a canonical map $\mu : NX \longrightarrow X$ induced by the canonical projection of $X \times G$ onto the first factor. Over $X \setminus Sing(X)$ the variety $\mu^{-1}(X \setminus Sing(X))$ is the graph of the Gauss map, hence isomorphic to $X \setminus Sing(X)$. Thus μ is birational. Since G is a projective variety, the morphism μ is projective.

If X is a complete intersection defined by equations

$$f_1(x_1, \ldots, x_N) = \cdots = f_\ell(x_1, \ldots, x_N) = 0$$

then μ coincides with the blowing up of the Jacobian ideal, that is, the ideal generated by all the $(\ell \times \ell)$-minors of the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}\right)_{\substack{1\le i\le \ell \\ 1\le j\le N}}$. Even if X is not a complete intersection, there is a similar description of Nash blowing up in terms of the Jacobian matrix, though it took mankind much longer to come up with it. Namely, let $r = N - n = codim(X, k^N)$. Let M be a submatrix of the Jacobian matrix formed by r rows that are linearly independent as K-vectors (where, as usual, $K = K(X)$ denotes the field of rational functions of X). Then μ coincides with the blowing up of the ideal generated by all the $(r \times r)$-minors of the matrix M.

The above constructions seem, *a priori*, to depend on the chosen embedding $\iota : X \hookrightarrow \mathbb{C}^N$. We now give two other characterization of Nash blowing up, both of them independent of ι.

This construction of an ideal whose blowing up coincides with the Nash blowing up is a special case of a more general construction of the **determinant** of a module (in this case, the module of Kahler differentials of X) due to Rossi in the analytic case and to O. Villamayor [161] in the general setting. Namely, let R be a domain, K its field of fractions and M and R-module. Let $r := \dim_K(M \otimes_R K)$ denote the generic rank of M.

Definition 3.9.3 The **determinant** of M is $Im\left(\bigwedge^r M \longrightarrow \bigwedge^r M \otimes_R K \cong K\right)$.

We think of the determinant as a fractional ideal, that is, an R-submodule of K. Clearing denominators, we can construct a non-canonical isomorphism of R-modules between a fractional ideal and an honest ideal of R. To obtain an ideal whose blowing up coincides with Nash blowing up, we take the determinant of the module $\Omega^1_{X/\mathbb{C}}$ of Kahler differentials whose generic rank is n.

Finally, Nash blowing up can be characterized by a universal mapping property. Namely, we have the following

Proposition 3.9.4 *Let $\mu : X' \to X$ be the Nash blowing up of X. The following statements hold.*

(1) The $O_{X'}$-module $\frac{\mu^*\Omega^n_{X/\mathbb{C}}}{torsion}$ is locally principal (that is, generated by a single element).

(2) the Nash blowing up μ has the universal mapping property with respect to (1). This means, by definition, that every birational morphism $\lambda : V \to X$ such that $\frac{\lambda^*\Omega^n_{X/\mathbb{C}}}{torsion}$ is locally principal factors through X' in a unique way.

With a view of constructing a resolution of singularities of X, consider the sequence

$$X \xleftarrow{\mu_1} X_1 \xleftarrow{\mu_2} \dots \xleftarrow{\mu_i} X_i \xleftarrow{\mu_{i+1}} \dots \tag{3.46}$$

where each μ_i is either a Nash blowing up or a normalized Nash blowing up (that is, a Nash blowing up followed by normalization). The question posed to Hironaka by Nash was: does X_i become non-singular for $i \gg 0$?

An affirmative answer to this question would provide a very simple and natural algorithm for resolving singularities over fields of characteristic zero.

Unfortunately, very little is known about Nash's question, despite considerable effort by many mathematicians. Let us briefly summarize the existing results.

In order to have any hope for the answer to be affirmative, we must at least ensure that no singular variety remains unchanged after Nash blowing up. This is the content of Nobile's Theorem:

Theorem 3.9.5 (Nobile [129]) *The Nash blowing up $\mu : X' \to X$ is an isomorphism if and only if X is non-singular.*

The "if" part of the Theorem is trivial, so its main content is "only if".

Corollary 3.9.6 *If* $\dim X = 1$ *iterating Nash blowing up produces a resolution of singularities.*

Proof Let $\tilde{X} \to X$ be the resolution of singularities of X. As we saw earlier, $\tilde{X}$ is nothing but the normalization of X. In particular, $\mathcal{O}_{\tilde{X}}$ is a finite (hence a noetherian) $\mathcal{O}_X$-module. Now, the sequence of morphisms (3.46) induces a sequence

$$\mathcal{O}_X \xrightarrow{\mu_1^*} \mathcal{O}_{X_1} \xrightarrow{\mu_2^*} \dots \xrightarrow{\mu_i^*} \mathcal{O}_{X_i} \xrightarrow{\mu_{i+1}^*} \dots \tag{3.47}$$

of homomorphisms of rings, with all the $\mathcal{O}_{X_i}$ contained in $\mathcal{O}_{\tilde{X}}$. Since $\mathcal{O}_{\tilde{X}}$ is a noetherian $\mathcal{O}_X$-module, the sequence (3.47) must stabilize after $\mathcal{O}_{X_i}$ for some $i \in \mathbb{N}$. By Nobile's theorem, X_i is non-singular. □

Remark 3.9.7 Assume that $\operatorname{char} k = p > 0$, fix a prime number $q \neq 2$ and consider the plane curve $X = \{f(x, y) = y^p + x^q = 0\}$. This is a complete intersection variety whose Jacobian ideal J is principal (since $\frac{\partial f}{\partial y} = 0$). Hence the Nash blowing up $\mu : X' \to X$ is an isomorphism. Thus Nobile's theorem does not hold over fields of positive characteristic. There seems to be little hope to devise a plausible approach to resolution over fields of characteristic $p > 0$ along the lines of Nash blowing up.

Theorem 3.9.8 (Rebassoo [137]) *Iterating Nash blowings up gives resolution of singularities of any surface X defined in $\mathbb{C}^3$ by an equation of the form*

$$z^a - x^b y^c = 0. \tag{3.48}$$

The proof is quite long and technical. One of the difficulties is that after Nash blowing up X stops being a hypersurface, though, as we will see below, it remains a toric variety.

Theorem 3.9.9 (Hironaka [99]) *Starting with a surface X, consider a sequence (3.46) of morphisms such that each μ_i dominates the Nash blowing up of X_{i-1} (that is, μ_i is a composition of Nash blowing up with another birational projective morphism). There exists $i \in \mathbb{N}$ such that the normalization $\bar{X}_i$ of X_i*

dominates a non-singular surface (in other words, $\bar{X}_i$ *has at most sandwiched singularities).*

Using this result as a starting point, M. Spivakovsky proved in 1985 that iterating *normalized* Nash blowings up resolves the singularities of any surface over a field of characteristic zero:

Theorem 3.9.10 ([146]) *Assume that* dim $X = 2$ *and each* μ_i *in (3.46) is a normalized Nash blowing up. Then* X_i *is non-singular for* $i \gg 0$.

By Hironaka's result, it is enough to prove this Theorem in the case when X has at most sandwiched singularities. Again, the proof is long and technical. The first step is a classification of sandwiched surface singularities, accomplished in [146], building on a classification of valuations in function fields of surfaces [145].

Another important ingredient in the proof is a geometric characterization of Nash blowing up in terms of **polar curves**, inspired by [83, 84].

3.9.1 Nash Blowing Up and the Base Locus of the Polar Curve

Consider a variety X of dimension n embedded in $\mathbb{C}^N$.

Definition 3.9.11 (Lê–Teissier) The **first polar variety** of X is the closure of the critical locus of a generic projection $X \to \mathbb{C}^n$, restricted to $X \setminus Sing(X)$. If X is a surface, the first polar variety is referred to as the **polar curve** of X; it is the critical locus of a generic projection $X \to \mathbb{C}^2$.

One should think of the polar curve as a linear system: as we vary the generic projection, we obtain a family of polar curves, all of them linearly equivalent to each other. In this way, we may talk about the **base locus** of the polar curve. Another way of thinking of polar curves is as zeroes of sections of the sheaf $\Omega^2_{X/\mathbb{C}}$ of Kahler differentials. This is why making this sheaf (modulo torsion) locally principal is equivalent to removing the base locus of the strict transform of the polar curve.

Proposition 3.9.12 ([146]) *Let* X *be a variety of dimension* n.

(1) Consider a birational transformation $\mu : X' \to X$*, dominating the Nash blowing up of* X*. The linear system formed by the strict transforms of the first polar variety has no base points (we say that Nash blowing up resolves the base points of the first polar variety).*

(2) Conversely, assume that μ *resolves the base points of the first polar variety and that* X' *is normal. Then* X' *dominates the Nash blowing up of* X*.*

This leads to the following method of computing the normalized Nash blowing up of any given normal surface singularity (this method is essentially due to G. Gonzalez-Sprinberg [83, 84]). Consider the commutative diagram

$$\begin{tikzcd} Y' \arrow[r, "\sigma"] \arrow[d, "\pi'"'] & Y \arrow[d, "\pi"] \\ X' \arrow[r, "\mu"] & X \end{tikzcd} \tag{3.49}$$

where π and π' are the respective minimal resolutions of singularities of X and X', μ is the normalized Nash blowing up and σ the factorization of $\mu \circ \pi'$ through Y given by definition of the minimal resolution Y.

By Zariski's factorization theorem, σ is a sequence of blowings up of points. Now, μ resolves the base points of the polar curve, hence so does $\mu \circ \pi'$. Since, by Theorem 3.9.12, μ is the "smallest" birational transformation with this property, σ is the smallest sequence of point blowings up that resolves the base points of the strict transform of the polar curve of X in Y. The method for studying the desingularization properties of Nash blowing up, inspired by [83–85], consists of computing directly the strict transform of the polar curve in Y, particularly, its base points, and thus deducing information about σ and Y'.

Once we classify sandwiched singularities, we consider a subclass of them called **minimal singularities** (rational singularities of surfaces with reduced fundamental cycle; this includes all the toric surface singularities). In the case of minimal singularities the polar curve, and thus σ and Y', can be computed explicitly. We show that the number of irreducible exceptional curves of π' is at most one half of the number of irreducible exceptional curves of π. Thus, if we let E be the number of irreducible exceptional curves in the minimal resolution of the surface X, the singularities of X are resolved after at most $\log_2 E$ normalized Nash blowings up.

In the case of non-minimal sandwiched singularities our results are much less explicit, but we are able to get enough information about the polar curve to give an indirect proof that if X has at most sandwiched singularities then after finitely many normalized Nash blowings up the resulting surface X_i has at most minimal singularities. This completes the proof.

3.9.2 Nash Blowing Up of Toric Varieties

Recently, there has been a resurgence of interest in resolution of singularities by iterating Nash blowing up, particularly, in the case of (not necessarily normal) toric varieties. We summarize some of the main advances here.

Let n be a strictly positive integer. Consider a semigroup $\Phi \subset \mathbb{Z}^n$ having the following properties:

(1) Φ generates $\mathbb{Z}^n$ as an additive group
(2) the cone C generated by Φ in $\mathbb{R}^n \supset \mathbb{Z}^n$ is strictly convex (this means that C contains no straight lines). Let $\gamma_1, \dots, \gamma_s$ be a set of generators of Φ (not necessarily minimal).

Definition 3.9.13 The **affine toric variety** X determined by Φ is the image of the map

$$\mathbb{C}^n \to \mathbb{C}^s$$

defined by $t \to (t^{\gamma_1}, \dots, t^{\gamma_s})$ (here we are using the multi-index notation: $t = (t_1, \dots, t_n)$, each γ_i is an n-vector and $t^{\gamma_i} = \prod\limits_{j=1}^{n} t_j^{\gamma_{ij}}$).

As everything else related to toric varieties, the Nash blowing up of such a variety can be described combinatorially. More precisely, we can compute the logarithmic Jacobian ideal explicitly in terms of the elements $\gamma_1, \dots, \gamma_s$. This task was accomplished, independently, in [82] and [86] (the latter paper includes the case of reducible toric varieties). Namely, the module $\Omega^n_{X,\mathbb{C}}$ is generated by elements of the form $dt^{\gamma_{i_1}} \bigwedge \cdots \bigwedge dt^{\gamma_{i_n}}$, where $(i_1, \dots, i_n)$ runs over all the n-tuples of distinct elements of $\{1, \dots, s\}$. We have

$$dt^{\gamma_{i_1}} \bigwedge \cdots \bigwedge dt^{\gamma_{i_n}} = \det\left(\gamma_{i_1}, \dots \gamma_{i_n}\right) t^{\sum\limits_{j=1}^{n} \gamma_{i_j} - n} dt_1 \bigwedge \cdots \bigwedge dt_n.$$

Thus the logarithmic Jacobian ideal we must blow up to compute the Nash blowing up is the ideal generated by all the monomials $t^{\sum\limits_{j=1}^{n} \gamma_{i_j} - n}$ as $(i_1, \dots, i_n)$ runs over all the n-tuples of distinct elements of $\{1, \dots, s\}$ satisfying

$$\det\left(\gamma_{i_1}, \dots \gamma_{i_n}\right) \neq 0. \tag{3.50}$$

Picking one of these monomials specifies a coordinate chart of the Nash blowing up. For example, assume that $\det(\gamma_1, \dots, \gamma_n) \neq 0$ and consider the coordinate chart determined by the monomial $t^{\sum\limits_{j=1}^{n} \gamma_j - n}$. The semigroup Φ_1 that determines the corresponding affine toric variety is generated by $\gamma_1, \dots, \gamma_s$ and all the vectors of the form

$$\sum_{j=1}^{n} \gamma_{i_j} - \sum_{j=1}^{n} \gamma_j, \tag{3.51}$$

where $(i_1, \dots, i_n)$ runs over all the n-tuples of distinct elements of $\{1, \dots, s\}$ satisfying (3.50). Now, an important special case to be considered is one when there exists $j \in \{1, \dots, n\}$ such that $i_{j'} = j'$ for all $j' \in \{1, \dots, n\} \setminus \{j\}$ and $i_j \neq j$. Then the condition (3.50) amounts to saying that

$$\det\left(\gamma_1, \dots, \gamma_{j-1}, \gamma_{i_j}, \gamma_{j+1}, \dots, \gamma_n\right) \neq 0. \tag{3.52}$$

One can show that after a permutation of the n-tuple $(i_1, \dots, i_n)$ we can achieve the situation where condition (3.52) holds for all $j \in \{1, \dots, n\}$ simultaneously. This shows that Φ_1 is generated by $\gamma_1, \dots, \gamma_s$ and all the differences of the form

$$\gamma_i - \gamma_j,\ j \in \{1, \dots, n\},\ i \in \{n+1, \dots, s\}$$
$$\text{such that } \det\left(\gamma_1, \dots, \gamma_{j-1}, \gamma_i, \gamma_{j+1}, \dots, \gamma_n\right) \neq 0 \tag{3.53}$$

A complete list of coordinate charts on the Nash blowing up of the toric variety X is obtained in this way, after imposing the additional condition that the resulting semigroup determines a strictly convex cone.

One way of thinking of the choice of an affine coordinate chart on the Nash blowing up is in terms of valuations. We saw earlier that by a theorem of Zariski fixing a valuation ν of the rational function field $K(X)$ of X is equivalent to specifying a (scheme-theoretic) point called the center of ν on every blowing up of X. Here we are interested in a less precise version of this statement: specifying the values $\nu(t_1),\dots,\nu(t_n)$ of the torus variables $t_1,\dots,t_n$ limits the choice of a coordinate chart to those charts that contain the center of ν. Namely, a coordinate chart as above contains the center of ν if and only if for every pair i, j as in (3.53) we have

$$\nu\left(t^{\gamma_i}\right) \geq \nu\left(t^{\gamma_j}\right). \tag{3.54}$$

In general, even under this restriction the choice of the coordinate chart is not uniquely determined, unless the inequality in (3.54) is strict for all the choices of i, j as in (3.53). This last statement holds whenever the values $\nu(t_1),\dots,\nu(t_n)$ are $\mathbb{Q}$-linearly independent.

In [72] and [73] it is shown that if $\dim X = 2$ and the rank of the group generated by $\nu(t_1)$ and $\nu(t_2)$ coincides with its rational rank then iterating Nash blowing up resolves the singularities of X in all the coordinate charts compatible with the valuation ν. In [82] the same result is proved for X of arbitrary dimension.

The simplest case of a group whose rank differs from its rational rank is that of rank 1 and rational rank 2. Thus the simplest case in which resolution of singularities of toric varieties by iterating Nash blowing up is not known is the following.

An Open Problem

Let $\Phi = (\gamma_1, \dots, \gamma_s) \subset \mathbb{Z}^2$ be a semigroup which generates $\mathbb{Z}^2$ as a group, such that the cone generated by it is strictly convex. Let α be an irrational number. Let $L : \mathbb{Z}^2 \to \mathbb{R}$ be the map given by $L(x, y) = x + \alpha y$. Assume that $L(\Phi \setminus \{0\}) > 0$

and that $L(\gamma_1) < L(\gamma_2) < L(\gamma_j)$ for $j > 2$. Let Φ_1 be the semigroup generated by γ_1, γ_2 and all the differences of the form $\gamma_i - \gamma_1$ and $\gamma_j - \gamma_2$ where $\det(\gamma_i, \gamma_2) \neq 0$ and $\det(\gamma_j, \gamma_1) \neq 0$. Replace Φ by Φ_1 (as we explained above, this corresponds to performing a Nash blowing up of our toric surface and picking the unique coordinate chart prescribed by the valuation such that $\nu(t_1) = 1$ and $\nu(t_2) = \alpha$). Question: is it true that after finitely many iterations of this procedure the resulting semigroup Φ_i is generated by two elements?

There is overwhelming computer evidence that the answer to this question is affirmative. Rebassoo's theorem is a special case of this, providing further evidence. On this subject we also mention the paper [17].

3.9.3 Higher Nash Blowing Up

Let $X \subset \mathbb{C}^N$ be an irreducible affine algebraic variety of dimension n and R its coordinate ring. Consider the map $\lambda : R \otimes_k R \to R$ which sends $a \otimes b$ to ab. Let $I = Ker(\lambda)$. We view I as an R-module via the map $R \to R \otimes_k R, r \to r \otimes 1$.

For $i \in \mathbb{N}, i \geq 2$, the higher Nash blowing up $N_i X$ of X was defined by Oneto and Zatini [134] in terms of the Grassmanian of the i-jet module $\left(\frac{I}{I^{i+1}}\right)^*$ and by Takehiko Yasuda [164] using Hilbert schemes of points of length $\binom{n+i}{n}$, with an alternative, explicit characterization by E. Chavez, D. Duarte and A. Giles in terms of the generalized Jacobian matrix [50]. We summarize the first two constructions here.

For a point $x \in X$. Let $(R_x, \mathfrak{m}_x)$ be the localization of R at the point x and I_x the localization of I. Consider the following $\mathbb{C} = \frac{R_x}{\mathfrak{m}_x}$-vector space:

$$T_x^i X := \left(\frac{I_x}{I_x^{i+1}} \otimes_R \mathbb{C}\right)^*$$

This is a vector space of dimension $L = \binom{i+n}{n} - 1$ whenever x is a non-singular point. Since $X \subset \mathbb{C}^N$, we have $T_x^i X \subset T_x^i \mathbb{C}^N = \mathbb{C}^M$ where $M = \binom{N+i}{N} - 1$, that is, we may view $T_x^i X$ as an element of the Grassmanian $G(M, L)$. Consider the Gauss map:

$$G_i : X \setminus Sing(X) \to G(M, L) \tag{3.55}$$

$$x \to T_x^i X. \tag{3.56}$$

Denote by X_i the Zariski closure of the graph of G_i. Call μ_i the restriction to X_i of the projection of $X \times G(M, L)$ to X.

Definition 3.9.14 ([134, Definition 1.1]) The pair (X_i, μ_i) is called the **Nash blowing up** of X **relative to** $\frac{I}{I^{i+1}}$.

Similarly to the usual Nash blowing up, the Nash blowing up relative to $\frac{I}{I^{i+1}}$ coincides with the blowing up of the determinant of the module $\frac{I}{I^{i+1}}$ [134].

Next, we summarize Yasuda's construction. Consider a $\mathbb{C}$-rational point $x \in X$ and let $\mathfrak{m}$ be the corresponding maximal ideal of R. Let $n = \dim X$. Let $x(i) := Spec \frac{R}{\mathfrak{m}^{i+1}}$ be the i-th infinitesimal neighborhood of x. If X is smooth at x, then $x(i)$ is a closed subscheme of X of length $L + 1 = \binom{i+n}{n}$ (that is, $\frac{R}{\mathfrak{m}^{i+1}}$ has length $L + 1$ as an R-module). Therefore, it corresponds to a point $[x(i)] \in Hilb_{L+1}(X)$, where $Hilb_{L+1}(X)$ is the Hilbert scheme of $(L+1)$-points of X (see [127, Definition 1.2]). If $Reg(X)$ denotes the smooth locus of X, we have a map

$$\delta_i : Reg(X) \to Hilb_{L+1}(X) \tag{3.57}$$

$$x \to [x(i)] \tag{3.58}$$

Definition 3.9.15 ([164, Definition 1.2]) The higher Nash blowup of X of order i, denoted by $N_i X$, is the closure of the graph of δ_n in $X \times_k Hilb_{L+1}(X)$ with reduced scheme structure. By restricting the projection $X \times_k Hilb_{L+1}(X) \to X$ to $N_i X$ we obtain a map $\pi_n : N_i X \to X$.

This map is projective, birational, and is an isomorphism over $Reg(X)$.

Proposition 3.9.16 ([164, Proposition 1.8]) *For every variety X and every strictly positive integer i, we have a canonical isomorphism* $(N_i(X), \pi_n) \cong (X_i, \mu_i)$. *In particular,* $N_1(X)$ *is canonically isomorphic to the classical Nash blowup of X.*

Yasuda conjectured that for i large enough, the i-th Nash blowup of X is non-singular [164, Conjecture 0.2]. If the conjecture were true, this construction would give a one-step resolution of singularities. In the same paper, the author proves that the conjecture is true for curves (here we give the statement only for irreducible varieties whereas Yasuda's result is stated and is proved for varieties that may be reducible.):

Theorem 3.9.17 ([164, Corollary 3.7]) *Let X be an irreducible variety of dimension 1. For i large enough the variety* $N_i X$ *is non-singular.*

The proof of this is not trivial and consists of two parts. First, the author shows that for $i \gg 0$ the transformation N_i separates the (analytic) branches of X, that is, X becomes analytically irreducible at every point. Yasuda goes on to show that each branch gets desingularized by N_i for $i \gg 0$. Precisely, he shows the following. Assume that X is analytically irreducible at a certain point ξ. The resolution of singularities of X gives an injection of $\mathcal{O}_{X,\xi}$ into a regular local ring and thus induces a discrete rank 1 valuation ν on $\mathcal{O}_{X,\xi}$. Consider the semigroup $\Phi := \nu(\mathcal{O}_{X,\xi} \setminus \{0\}) \subset \mathbb{N}$ and let $0 = s_0, s_1, s_2, s_3, \ldots$ be the complete list of elements of Φ arranged in an increasing order.

Theorem 3.9.18 ([164, Theorem 3.3]) *For an integer* $i \in \mathbb{N}$ *the curve* $N_i X$ *is non-singular if and only if* $s_{i+1} - 1 \in \Phi$.

Since Φ coincides with $\mathbb{N}$ for $i \gg 0$, Theorem 3.9.18 immediately implies Theorem 3.9.17 in the case of analytically irreducible curves.

Yasuda has stated that the A_3 singularity (that is, the singularity defined by the equation $z^4 - xy = 0$) is probably a counterexample to his conjecture (see [166, Remark 1.5]). Recently Rin Toyama [154] has shown that this is, indeed, the case, building on earlier work by D. Duarte.

Incredibly, the analogue of Nobile's theorem (that is, the statement that a higher Nash blowing up of X is an isomorphism if and only if X is non-singular) is not known for higher Nash blowing up. The best partial results on this subject are due to D. Duarte, who proved it for normal toric varieties [74] and for normal hypersurfaces [75]. It has recently been proved for toric curves [50].

Finally, we mention another conjecture of T. Yasuda about higher Nash blowing up of (analytically) irreducible curves. Let X be an analytically irreducible curve, Φ its associated semigroup and the s_i elements of Φ listed in increasing order, as above.

Conjecture 3.9.19 (Yasuda [165]) Let Φ_i denote the semigroup associated to the analytically irreducible curve $N_i X$. We have $\Phi_i = \{s_\ell - s_j \mid \ell > i, j \leq i\}$.

The paper [50] contains the following results:

(1) a definition of the higher-order Jacobian matrix J of an affine algebraic variety, so that the i-th higher Nash blowing up coincides with the blowing up of an ideal generated by suitable minors of J in a way completely analogous to that of usual Nash blowing up described above
(2) a proof that the higher Nash blowings up of a toric variety are themselves toric varieties
(3) a proof of Conjecture 3.9.19 in the case of toric curves
(4) as an immediate corollary of (3), a proof of the analogue of Nobile's theorem for toric curves
(5) a family of counterexamples to Conjecture 3.9.19 in the general case (namely, the parametrized curves $t \to \left(t^4, t^{4i+2} + t^{4i+3}\right)$ giving a counterexample for each positive integer i).

3.10 Reduction of Singularities of Vector Fields, Foliations by Lines and Codimension One Foliations

Let K be the field of rational functions of a projective algebraic variety M_0 of dimension n over an algebraically closed field k of characteristic zero.

Consider the n-dimensional K-vector space $Der_k K$ of k-derivations from K to itself.

Definition 3.10.1 A **foliation by lines** is a 1-dimensional K-vector subspace

$$\mathcal{L} \subset Der_k K.$$

Take a regular point P on a projective model M of the field K. We know that

$$Der_k \mathcal{O}_{M,P} \subset Der_k K$$

is a free $\mathcal{O}_{M,P}$-module of rank n generated by the partial derivatives $\frac{\partial}{\partial z_i}$, $i \in \{1, 2, \ldots, n\}$, for a regular system of parameters $z_1, z_2, \ldots, z_n$ of the local ring $\mathcal{O}_{M,P}$.

Definition 3.10.2 The free rank one submodule $\mathcal{L}_{M,P} := \mathcal{L} \cap Der_k \mathcal{O}_{M,P}$ of $Der_k \mathcal{O}_{M,P}$ is called the **local foliation induced by** $\mathcal{L}$ at M, P.

Let $\mathfrak{m}_{M,P}$ denote the maximal ideal of $\mathcal{O}_{M,P}$.

Definition 3.10.3 A germ of a vector field $\xi \in Der_k \mathcal{O}_{M,P}$ is said to be **non-singular** if $\xi \notin \mathfrak{m}_{M,P} Der_k \mathcal{O}_{M,P}$. The germ ξ is **elementary** if it is singular and the k-linear endomorphism

$$\xi : \frac{\mathfrak{m}_{M,P}}{\mathfrak{m}^2_{M,P}} \to \frac{\mathfrak{m}_{M,P}}{\mathfrak{m}^2_{M,P}} \tag{3.59}$$

is not nilpotent.

We say that $\mathcal{L}$ is non-singular (resp. elementary) at P if there is a germ $\xi \in \mathcal{L}_{M,P}$ that is non-singular (resp. elementary). If $Y \subset M$ is an irreducible subvariety, we say that $\mathcal{L}$ is non-singular (resp. elementary) at Y if it is so at a generic point of Y. Note that this definition makes sense only if M itself is non-singular at the generic point of Y.

A plane vector field $D = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$, with a, b two relatively prime polynomials in x and y, defines a one-dimensional saturated foliation $\mathcal{F}$ having singularities at the zeroes of D (that is, the common zeroes of a and b). It was proved by Seidenberg in 1968 [141] that after a finite number of point blowings up of the ambient plane we obtain a foliation $\tilde{F}$ which is given locally at each singular point by a vector field $\tilde{D}$ whose linear part has eigenvalues 1 and λ, with $\lambda \notin \mathbb{Q}_+$ (= strictly positive rational numbers); see also [45]. The above singularities may be thought of as final forms in the sense that they are preserved under all subsequent point blowings up. Note also that these singularities are characterized by the fact that they are elementary in the sense of Definition 3.10.3 and remain elementary after the subsequent blowings up. On the other hand, if the eigenvalues are $1, \lambda \in \mathbb{Q}_+$, the linear part of the vector field (cf. (3.59)) will become nilpotent after finitely many blowings up.

This points to a new feature of the desingularization problem for vector fields and foliations: in general, it is not possible to make them non-singular by blowings up, so one must start by defining the final form of the singularity that one is trying to achieve. This is why in this subject we usually talk about *reduction* rather than resolution of singularities. A counterexample by F. Sanz and F. Sancho shows that starting with dimension three it is not possible to arrive at elementary singularities by a sequence of blowings up along non-singular centers (see the Introduction to [135]). Therefore a new notion of a final form of singularities is needed. In higher dimensions a useful and natural notion seems to be that of log-elementary singularities, motivated by the results of [46].

Let the notation be as in the beginning of this section.

Definition 3.10.4 A germ of a vector field $\xi \in Der_k \mathcal{O}_{M,P}$ is said to be **log-elementary** if there is a regular system of parameters $z_1, z_2, \ldots, z_n$ of $\mathcal{O}_{M,P}$, and an integer e, $0 \le e \le n$ such that ξ has the form $\xi = \sum\limits_{i=1}^{e} a_i z_i \frac{\partial}{\partial z_i} + \sum\limits_{i=e+1}^{n} a_i \frac{\partial}{\partial z_i}$, where $a_i \in \mathcal{O}_{M,P}$ for $i \in \{1, 2, \ldots, n\}$ and $a_j \notin \mathfrak{m}_{M,P}^2$ for at least one index j. We say that $\mathcal{L}$ is log-elementary at P if there is a germ $\xi \in \mathcal{L}_{M,P}$ that is log-elementary. If $Y \subset M$ is an irreducible subvariety, we say that $\mathcal{L}$ is log-elementary at Y if it is so at a generic point of Y.

The following result is the main theorem of [49]:

Theorem 3.10.5 *Assume that* $n = 3$. *Consider a foliation by lines* $\mathcal{L} \subset Der_k K$. *There is a birational projective morphism* $M \to M_0$ *such that* $\mathcal{L}$ *is log-elementary at all the points of* M.

The general structure of the proof is à la Zariski. First, a local uniformization along any valuation ν of K vanishing on k is established: a sequence of blowings up $M \to M_0$ along non-singular centers is constructed such that $\mathcal{L}$ is log-elementary at the center Y of ν on M. After that Theorem 3.10.5 is deduced from the Piltant–Zariski general globalization procedure in dimension three [136]: one just has to check that Piltant's axioms I–VI hold in this special case. The proof of local uniformization of three-dimensional vector fields is inspired by [46] and [47].

We mention, without giving the details, the following related results on reduction of singularities of foliations and vector fields.

(1) The paper [48] constructs a reduction of singularities of codimension 1 foliations in ambient dimension 3.
(2) The paper [135] accomplishes reduction of singularities of real-analytic vector fields; the real setting is used in an essential way in the proof.
(3) The paper [123] proves reduction of singularities of foliations by curves in ambient dimension 3 to canonical ones (the condition of being canonical is somewhat stronger than being log-elementary), but in the 2-category of Deligne–Mumford stacks.
(4) The papers [78, 79] prove the Local Uniformization theorem for codimension one foliations in all dimensions, under two restrictions on the given valuation ν: $rk\ \nu = 1$ and $k_\nu = \mathbb{C}$.

3.11 Appendix

It is natural to pose the problem of resolution of singularities in the more general context of noetherian schemes.

Definition 3.11.1 Let X be a reduced noetherian scheme. A **resolution of singularities** of X is a blowing up $X' \to X$ along a subscheme of X, not containing any irreducible components of X, such that X' is non-singular.

In this Appendix we address the question of the hypotheses that must be imposed on X in order for resolution of singularities to exist. Let $Reg(X)$ denote the set of regular points of X. It is obvious that the following condition is necessary for the existence of a resolution of singularities of X:

(1) $Reg(X)$ must contain a non-empty Zariski open set.
Furthermore, suppose X admits a resolution of singularities $\pi : X' \to X$ and let

$$\bar{\pi} : \bar{X} \to X$$

denote the normalization of X. Then π must factor through $\bar{X}$. We have $\bar{X} = Spec\ \pi_* \mathcal{O}_{X'}$ and $\pi_* \mathcal{O}_{X'}$ is a coherent sheaf of $\mathcal{O}_X$-modules. This gives another necessary condition for the existence of resolution:

(2) $\bar{X}$ must be finite over X.

Moreover, since the usual methods involve blowing up and induction on $\dim X$, we are led to assume that (1) and (2) hold for every reduced scheme of finite type over X. By Nagata's criterion, (1) then implies that X is a J-2 scheme, that is, for every scheme $\tilde{X}$, reduced and of finite type over X, $Reg(\tilde{X})$ is open.

Grothendieck [87, IV.7.9] proved that if all of the irreducible closed subschemes of X and all of their finite purely inseparable covers admit resolution of singularities, then X must satisfy a somewhat stronger condition than (1)∧(2) above, called **quasi-excellence**, which we now define. For a point ξ on a scheme we will denote by $\kappa(\xi)$ the residue field of the local ring of that point.

Definition 3.11.2 ([121, Chapter 13, (33.A), p. 249]) Let $\sigma : X \to Y$ be a morphism of noetherian schemes. We say that σ is **regular** if it is flat, and for every $\xi \in Y$ the fiber $X \times_Y Spec\ \kappa(\xi)$ is geometrically regular over $\kappa(\xi)$ (this means that for every finite field extension $\kappa(\xi) \to k'$, the scheme $X \times_Y Spec\ k'$ is regular).

Remark 3.11.3 If $\kappa(\xi)$ is perfect, the fiber $X \times_Y Spec\ \kappa(\xi)$ is geometrically regular over $\kappa(\xi)$ if and only if it is regular.

Remark 3.11.4 It is known that a morphism of finite type is regular in the above sense if and only if it is smooth (that is, flat with smooth fibers).

3.11.1 Quasi-excellent Schemes

Regular morphisms come up in a natural way when one wishes to pass to the formal completion of a local ring at a singularity:

Definition 3.11.5 ([121, (33.A) and (34.A)]) Let R be a noetherian ring. For a maximal ideal $\mathfrak{m}$ of R, let $\hat{R}_{\mathfrak{m}}$ denote the $\mathfrak{m}$-adic completion of R. We say that R is a **G-ring** if for every maximal ideal $\mathfrak{m}$ of R, the natural map $Spec\ \hat{R}_{\mathfrak{m}} \to Spec\ R$ is a regular morphism.

Definition 3.11.6 ([121, (34.A), p. 259]) Let X be a noetherian scheme. We say that X is **quasi-excellent** if the following two conditions hold:

(1) X is J-2, that is, for every scheme $\tilde{X}$, reduced and of finite type over X, $Reg(\tilde{X})$ is open in the Zariski topology.
(2) For every closed point $\xi \in X$, $\mathcal{O}_{X,\xi}$ is a G-ring.

Remark 3.11.7 If $X = Spec\ R$ with R a **local** noetherian ring then (2) $\Longrightarrow$ (1) in the above definition [121].

A scheme is said to be **excellent** if it is quasi-excellent and universally catenary. In general, rings that arise from natural constructions in algebra and geometry are excellent. Complete and complex-analytic local rings are excellent (see [121, Theorem 30.D] for a proof that every complete local ring is excellent and [121, (33.H), Theorem 78, p. 257] for a proof of finiteness of normalization for quasi-excellent schemes). Both excellence and quasi-excellence are preserved by localization and passing to schemes of finite type over X [121, Chapter 13, (33.G), Theorem 77, p. 254]. In particular, every scheme that is essentially of finite type over a field, $\mathbb{Z}$, $\mathbb{Z}_{(p)}$, $\mathbb{Z}_p$, the Witt vectors or any other excellent Dedekind domain, or over a complete or complex-analytic local ring is excellent. See [126, Appendix A.1, p. 203], for some examples of non-excellent rings.

If X is a quasi-excellent scheme then for every $\xi \in X$ the natural map

$$Spec\ \hat{\mathcal{O}}_{X,\xi} \to X$$

is a regular homomorphism (by Definition 3.11.6 (2)). Thus, the passage to the formal completion is a natural operation in the category of quasi-excellent schemes; in particular, it does not change the nature of singularity.

Once local uniformization is proved in a given context, in order to globalize it and to make it canonical (that is, functorial in the category whose objects are quasi-excellent noetherian schemes and whose morphisms are *regular* morphisms), one is interested in local uniformization algorithms determined, locally at every point ξ, by the formal completion $\hat{\mathcal{O}}_{X,\xi}$ of $\mathcal{O}_{X,\xi}$.

Grothendieck's result means that the largest *subcategory* of the category of noetherian schemes, closed under passing to closed subschemes and finite purely inseparable covers, for which resolution of singularities could possibly exist, is that of *quasi-excellent* schemes. In [87, IV.7.9], Grothendieck conjectures that resolution of singularities exists in this most general possible context.

We take this opportunity to mention a recent paper [105] by L. Illusie, Y. Laszlo and F. Orgogozo, based on the ideas of Ofer Gabber.

References

1. S. Abhyankar, *Local uniformization on algebraic surfaces over ground fields of characteristic* $p \neq 0$, Ann. of Math. 63, 491–526 (1956) 215
2. S. Abhyankar, *Reduction to multiplicity less than p in a p-cyclic extension of a two dimensional regular local ring (*p *= characteristic of the reside field)*, Math. Annalen 154 28–55 (1964) 215
3. S. Abhyankar, *Resolution of Singularities of Embedded Algebraic Surfaces*, Academic Press, New York and London (1966)
4. S. Abhyankar, *An algorithm on polynomials in one indeterminate with coefficients in a two dimensional regular local domain,* Annali di Matematica Pura ed Applicata 71 25–60 (1966)
5. S. Abhyankar, *Uniformization in a p-cyclic extension of a two dimensional regular local domain of residue field of characteristic p,* Festschrift zur Gedächtnisfeier für Karl Weierstrass 1815–1965, Wissenschaftliche Abhandlungen des Landes Nordrhein-Westfalen 33 Westdeutscher Verlag, Köln und Opladen 243–317 (1966)
6. S. Abhyankar, *Nonsplitting of valuations in extensions of two dimensional regular local domains,* Math. Annalen 170 87–144 (1967) 215
7. S. Abhyankar and T.T. Moh, *Newton–Puiseux expansion and generalized Tschirnhausen transformation I,* Reine Agew. Math. 260, 47–83 (1973) 218
8. S. Abhyankar and T.T. Moh, *Newton–Puiseux expansion and generalized Tschirnhausen transformation II,* Reine Agew. Math. 261, 29–54 (1973) 218
9. D. Abramovich and J. de Jong, *Smoothness, Semistability and Toroidal Geometry,* J. Alg. Geom. 6, 789–801 (1997) 220
10. D. Abramovich, K. Karu, K. Matsuki and J. Włodarczyk, *Torification and factorization of birational maps*, J. Amer. Math. Soc. 15 531–572 (2002) 210
11. D. Abramovich, M. Temkin, J. Włodarczyk, *Functorial embedded resolution via weighted blowings up*, arXiv:1906.07106 215
12. N. A'Campo and M. Oka, *Geometry of plane curves via Tschirnhausen resolution tower,* Osaka J. Math. 33 1003–1033 (1996) 218
13. G. Albanese, *Transformazione birazionale di una superficie algebrica qualunque in un'altra priva di punti multipli,* Rend. Circ. mat. Palermo 48 (1924) 191, 199, and 215
14. J.-M. Aroca, H. Hironaka and J.-L. Vicente, *The theory of maximal contact,* Memo. Mat. del Inst. Jorge Juan, Madrid 29 (1975) 215
15. J.-M. Aroca, H. Hironaka and J.-L. Vicente, *Desingularization Theorems*, Memo. Mat. del Inst. Jorge Juan, Madrid 30 (1977)
16. J.-M. Aroca, H. Hironaka, J.-L. Vicente, *Complex Analytic Desingularization*, ISBN 978-4-431-49822-3, Springer (2018) 215
17. A. Atanasov, C. Lopez, A. Perry, N. Proudfoot and M. Thaddeus, *Resolving Toric Varieties with Nash Blowups,* Experiment. Math. Vol 20, 3 288–303 (2011) 228
18. A. Belotto da Silva, E. Bierstone, V. Grandjean and P. Milman, *Resolution of singularities of the cotangent sheaf of a singular variety*, Adv. Math. 307 780–832 (2017) 213
19. A. Benito and O. Villamayor, Monoidal transforms and invariants of singularities in positive characteristic, Compositio Math. 149, no. 8 1267–1311 (2013) 218
20. A. Benito, S. Encinas and O. Villamayor, *Some natural properties of constructive resolution of singularities*, Asian J. Math 15, no 2 141–192 (2011) 213

21. B. Bennett, *On the characteristic functions of a local ring,* Ann. of Math. 91 25–87 (1970) 197
22. E. Bierstone, S. Da Silva, P. Milman F. Vera Pacheco, *Desingularization by blowings-up avoiding simple normal crossings*, Proc. Amer. Math. Soc. 142, no. 12 4099–4111 (2014) 213
23. E. Bierstone, D. Grigoriev, P. Milman and J. Włodarczyk, *Effective Hironaka resolution and its complexity*, Asian J. Math. 15, no. 2 193–228 (2011)
24. E. Bierstone, P. Lairez and P. Milman, *Resolution except for minimal singularities II: The case of four variables*, Adv. Math. 231, no. 5 3003–3021 (2012)
25. E. Bierstone and P. Milman, *Uniformization of analytic spaces*, Journal of the AMS 2, 4 801–836 (1989)
26. E. Bierstone and P. Milman, *A simple constructive proof of canonical resolution of singularities,* Effective Methods in Algebraic Geometry, Progress in Math. Birkhäuser Boston 94 11–30 (1991)
27. E. Bierstone and P. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. 128, 2 207–302 (1997)
28. E. Bierstone and P. Milman, *Standard basis along a Samuel stratum and implicit differentiation*, Fields Inst. Communications, Arnold Volume 24 81–113 (1999)
29. E. Bierstone and P. Milman, *Desingularization of toric and binomial varieties*, J. Algebraic Geom. 15, 3 443–486 (2006)
30. E. Bierstone and P. Milman, *Functoriality in resolution of singularities*, Publ. Res. Inst. Math. Sci 44, no. 2 609–639 (2008)
31. E. Bierstone and P. Milman, *Resolution except for minimal singularities I*, Adv. Math. 231, no. 5 3022–3053 (2012)
32. E. Bierstone, P. Milman and M. Temkin, $\mathbb{Q}$-*universal desingularization*, Asian J. Math. 15, no. 2 229–249 (2011)
33. E. Bierstone and F. Vera Pacheco, *Resolution of singularities of pairs preserving semi-simple normal crossings*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM 107, no. 1 159–188 (2013)
34. E. Bierstone and F. Vera Pacheco, *Desingularization preserving stable simple normal crossings*, Israel J. Math. 206, no. 1 233–280 (2015)
35. R. Blanco, *Complexity of Villamayor's algorithm in the non-exceptional monomial case*, International Journal of Mathematics 20 (06), 659–678 (2009) 213
36. R. Blanco, *Desingularization of binomial varieties in arbitrary characteristic. Part I. A new resolution function and their properties* Mathematische Nachrichten 285 (11–12), 1316–1342 (2012) 218
37. R. Blanco, *Desingularization of binomial varieties in arbitrary characteristic. Part II: Combinatorial desingularization algorithm,* Quarterly journal of mathematics 63 (4), 771–794 (2012)
38. R. Blanco, S. Encinas *Coefficient and elimination algebras in resolution of singularities,* Asian J. Math. 15 (2) 251–272 (2011)
39. R. Blanco, S. Encinas, *Embedded desingularization of toric varieties,* arXiv:0901.2211 218
40. F. Bogomolov and T. Pantev, *Weak Hironaka Theorem*, Math. Res. Lett. 3, no. 3 299–307 (1996) 221
41. A. Bravo, S. Encinas and O.E. Villamayor, *A simplified proof of desingularization and applications*, Rev. Mat. Iberoamericana 21, 2 349–458 (2005) 213
42. A. Bravo and O.E. Villamayor, *Strengthening the Theorem of Embedded desingularization*, Math. Res. Letters 8 79–90 (2001)
43. A. Bravo and O. Villamayor, *A strengthening of resolution of singularities in characteristic zero*, Proc. London. Math. Soc. (3), 86 (2) 327–357 (2003) 213
44. A. Bravo, O.E. Villamayor, *Singularities in positive characteristic, stratification and simplification of the singular locus*, Adv. in Math. 224 1349–1418 (2010) 218
45. F. Cano, *Desingularization of plane vector fields*, 37 Transactions of the AMS 296, Number 1 (1986) 231

46. F. Cano, *Desingularization strategies for three-dimensional vector fields*, Lecture Notes in Mathematics, 1259 Springer–Verlag Berlin(1987) 232
47. F. Cano, *Final forms for a three-dimensional vector field under blowing-up,* Annales de l'Iinstitut Fourier, tome 37 no 2, 151–193 (1987) 232
48. F. Cano, *Reduction of the singularities of codimension one singular foliations in dimension three*, Annals of Mathematics 160 (3) 907–1011 (2004) 232
49. F. Cano, C. Roche and M. Spivakovsky, *Reduction of singularities of three-dimensional line foliations,* Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matemáticas 108, Issue 1 221–258 (2014) 232
50. E. Chávez Martínez A. Giles Flores, D. Duarte, *On higher Nash blowups of toric varieties and two conjectures on toric curves,* arXiv:1803.04595v2 (2019) 228 and 230
51. O. Chisini, *La risoluzione delle singolaritá di una superficie mediante transformazioni birazionali dello spazio*, Mem. Accad. Sci. Bologna VII. s.8 (1921) 199
52. V. Cossart, *Desingularization of embedded excellent surfaces*, Tohoku Math. J. 33 25–33 (1981) 215
53. V. Cossart, *Forme normale pour une fonction sur une variété de dimension trois en caractéristique positive*, Thèse d'Etat, Orsay (1988) 216
54. V. Cossart, *Contact maximal en caractéristique positive et petite multiplicité*, Duke Math. J. 63 57–64 (1991) 217
55. V. Cossart, *Modèle projectif et désingularisation*, Math. Ann. 293 115–121 (1992)
56. V. Cossart, *Désingularisation en dimension 3 et caractéristique p*, Proceedings de La Rabida, Progress in Mathematics 134 1–7 Birkhauser (1996) 217
57. V. Cossart *Uniformisation et désingularisation des surfaces d'après Zariski*, Resolution of Singularities: A research textbook in tribute to Oscar Zariski edited by H. Hauser, J. Lipman, F. Oort, A. Quirós, Progress in Mathematics 181 Birkhauser Verlag, Basel/Switzerland 239–258 (2000) 215
58. V. Cossart, J. Giraud and U. Orbanz, *Resolution of surface singularities (with an appendix by H. Hironaka)*, Lecture Notes in Mathematics 1101 Springer-Verlag, Berlin (1984)
59. V. Cossart, U. Jannsen and S. Saito, *Canonical embedded and non-embedded resolution of singularities for excellent two-dimensional schemes,* preprint arXiv:0905.2191 (2009) 215
60. V. Cossart, U. Jannsen, B. Schober, *Invariance of Hironaka's characteristic polyhedron*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A, Matemáticas, DOI: 10.1007/s13398-019-00672-x 1–25 (2019) 188
61. V. Cossart and O. Piltant, *Resolution of singularities of threefolds in positive characteristic I. Reduction to local uniformization on Artin-Schreier and purely inseparable coverings*, J. Algebra 320, no. 3 1051–1082 (2008) 216
62. V. Cossart and O. Piltant, *Resolution of singularities of threefolds in positive characteristic II*, J. Algebra 321, no. 7 1836–1976 (2009)
63. V. Cossart and O. Piltant, *Resolution of singularities of arithmetical threefolds,* J. of Algebra 529 268–535 (2019) 216
64. V. Cossart, B. Schober, *A strictly decreasing invariant for resolution of singularities in dimension two,* arXiv:1411.4452v2 (2014) 215
65. S. D. Cutkosky, *Resolution of singularities*, AMS, Providence, RI, Graduate Studies in Mathematics 63 (2004) 213
66. S. D. Cutkosky, *Resolution of singularities for 3-folds in positive characteristic*, American Journal of Mathematics 131, no 1 59–127 (2009) 215
67. S.D. Cutkosky and H. Mourtada, *Defect and Local Uniformization,* arXiv:1711.02726 218
68. J. Decaup, *Uniformisation locale simultanée par monomialisation d'éléments clefs,* Thèse de Doctorat, Institut de Mathématiques de Toulouse (2018) 218
69. J. Decaup, W. Mahboub and M. Spivakovsky *Abstract key polynomials and comparison theorems with the key polynomials of Mac Lane – Vaquié*, arXiv:1611.06392 218
70. J. de Jong, *Smoothness, semi-stability and alterations*, Publ. IHES 83 51–93 (1996) 218
71. M. de Moraes, J. Novacoski, *Perron transforms and Hironaka's game,* arXiv:1907.02094 213

72. D. Duarte, *Nash modification on toric surfaces,* Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales, Serie A Matemáticas 108 (1) 153–171 (2012) 227
73. D. Duarte, *Nash modification on toric surfaces and higher Nash blowup on normal toric varieties,* Ph.D. thesis, Université Paul Sabatier, Toulouse III (2013) 227
74. D. Duarte, *Higher Nash blowup on normal toric varieties,* Journal of Algebra 418 110–128 (2014) 230
75. D. Duarte, *Computational aspects of the higher Nash blowup of hypersurfaces,* Journal of Algebra 477 211–230 (2017) 230
76. S. Encinas and H. Hauser, *Strong resolution of singularities in characteristic zero*, Comment. Math. Helv. 77 821–845 (2002) 213
77. S. Encinas and O. Villamayor, *Good points and constructive resolution of singularities,* Acta Mathematica 181 109–158 (1998) 213
78. M. Fernandez Duque, *Elimination of resonances in codimension one foliations,* Publ. Mat. 59 75–97 (2015) 232
79. M. Fernandez Duque, *Local Uniformization of Foliations for Rational Archimedean Valuations,* arXiv:1611.08730 232
80. J. Giraud, *Forme normale pour une fonction sur une surface de caractéristique positive*, Bull. Soc. Math. France 111 109–124 (1983) 215
81. R. Goldin and B. Teissier, *Resolving singularities of plane analytic branches with one toric morphism*, Resolution of singularities Obergurgl, 1997, Progr. Math.181 Birkhauseraddr Basel 315–340 (2000) 218
82. P. González Perez, B. Teissier, *Toric Geometry and the Semple-Nash modification,* Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales, Serie A Matemáticas, DOI 10.1007/s13398-012-0096-0 (2012) 226 and 227
83. G. Gonzalez-Sprinberg, *Éventails en Dimension 2 et Transformé de Nash,* Publ. de l'E.N.S, Paris (1977) 224 and 225
84. G. Gonzalez-Sprinberg, *Résolution de Nash des points doubles rationnels,* Ann. Inst. Fourier, Grenoble 32, 2 111–178 (1982) 224 and 225
85. G. Gonzalez-Sprinberg, *Désingularisation des Surfaces par des Modifications de Nash Normalisées,* Séminaire N. Bourbaki, Fevrier 1986, Expose 661 225
86. D. Grigoriev, P. Milman, *Nash desingularization for binomial varieties as Euclidean division, a priori termination bound, polynomial complexity in dim 2,* Adv. Math. 231, no. 6 3389–3428 (2012) 226
87. A. Grothendieck, *Éléments de Géométrie Algébrique IV,* Publ. IHES 24 (1965) 233 and 235
88. R. Hartshorne, *Algebraic Geometry,* Graduate Texts in Mathematics, Springer 1st ed. (1977) 186 and 194
89. H. Hauser, *Seventeen Obstacles for Resolution of Singularities*, Progress in Mathematics, The Brieskorn Anniversary Volume, Arnold, V. I., Greuel, G.-M., Steenbrink, J., editors Birkhäuseraddr Boston 289–313 (1998) 218
90. H. Hauser and J. Schicho, *A game for the resolution of singularities,* Proceedings of the London Mathematical Society, Bd. 105, S. 1149–1182. (2012) 213
91. F. J. Herrera Govantes, M. A. Olalla Acosta, M. Spivakovsky, *Valuations in algebraic field extensions*, Journal of Algebra 312, Issue 2 1033–1074 (2007) 218
92. F. J. Herrera Govantes, W. Mahboub, M. A. Olalla Acosta, M. Spivakovsky, *Key polynomials for simple extensions of valued fields*, arXiv:1406.0657 218
93. H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. Math. 79 109–326 (1964) 210
94. H. Hironaka, *Characteristic polyhedra of singularities,* J. Math. Kyoto Univ. 7 251–293 (1968) 188
95. H. Hironaka, *Certain numerical characters of singularities,* J. Math. Kyoto Univ. 10 327–334 (1970)
96. H. Hironaka, *Schemes, etc.* Proc. 5th Nordic Summer School in Math., Oslo 291–313 (1970) 188

97. H. Hironaka, *Idealistic exponents of singularity,* Algebraic geometry (J. J. Sylvester Sympos., Johns Hopkins Univ., Baltimore, Md., 1976), Johns Hopkins Univ. Press, Baltimore, Md., 52–125 (1977) 211
98. H. Hironaka, *Bimeromorphic smoothing of a complex analytic space,* Acta Math. Vietnam. 2 (2) 103–168 (1977) 215
99. H. Hironaka, *On Nash blowing-up,* Arithmetic and Geometry II, Progr. Math., vol 36, Birkhauser Boston, Mass., 103–111 (1983) 223
100. Hironaka, H, *Desingularization of excellent surfaces,* Notes by B. Bennett at the Conference on Algebraic Geometry, Bowdoin 1967. Reprinted in: Cossart, V., Giraud, J., Orbanz, U.: Resolution of surface singularities. Lecture Notes in Math. 1101, Springer (1984) 215
101. H. Hironaka, *Theory of infinitely near singular points,* J. Korean Math. Soc. 40, 5 901–920 (2003) 218
102. H. Hironaka, *Three key theorems on infinitely near singularities,* Singularités Franco-Japonaises, Sémin. Congr. 10 87–126 (2005)
103. H. Hironaka, *A program for resolution of singularities, in all characteristics* $p > 0$ *and in all dimensions ,* preprint for series of lectures in "Summer School on Resolution of Singularities" at International Center for Theoretical Physics, Trieste, June 12–30 (2006)
104. H. Hironaka, *Resolution of singularities in positive characteristic,* Available at http://people.math.harvard.edu/~hironaka/pRes.pdf 218
105. L. Illusie, Y. Laszlo, F. Orgogozo, *Travaux de Gabber sur l'uniformisation locale et la cohomologie etale des schemas quasi-excellents,* Seminaire a l'Ecole polytechnique 2006–2008, arXiv:1207.3648 235
106. H.W.E. Jung, *Darstellung der Funktionen eines algebraischen Körpers zweier unabhängigen Veränderlichen* x, y *in der Umgebung einer Stelle* $x = a$, $y = b$, J. Reine Angew. Math. 133 289–314 (1908) 199
107. H. Kawanoue, *Toward resolution of singularities over a field of positive characteristic Part I. Foundation: the language of the idealistic filtration,* Publ. RIMS, Kyoto Univ. 43 819–909 (2007) 218
108. H. Kawanoue and K. Matsuki, *Toward resolution of singularities over a field of positive characteristic (The idealistic filtration program) Part II. Basic invariants associated to the idealistic filtration and their properties,* Publ. RIMS Kyoto Univ. 46 359–422 (2010) 218
109. G. Kempf, F. Knudsen, D. Mumford and B. Saint-Donat, *Toroidal Embeddings I,* Springer, LNM 339 (1973) 221
110. H. Knaf and F.-V. Kuhlmann, *Abhyankar places admit local uniformisation in any characteristic,* Annales de l'ENS Série 4: 38, no. 6 833–846 (2005) 217
111. H. Knaf and F.-V. Kuhlmann, *Every place admits local uniformization in a finite extension of the function field,* Advances in Math. 221 428–453 (2009) 217
112. J. Kollár, *Lectures on resolution of singularities,* Annals of Mathetmatics Studies, 166, Princeton University Press (2007) 213
113. F.-V. Kuhlmann, *Valuation theoretic and model theoretic aspects of local uniformization,* Resolution of Singularities - A Research Textbook in Tribute to Oscar Zariski, H. Hauser, J. Lipman, F. Oort, A. Quiros (editors.), Progress in Mathematics 181 Birkhäuser Verlag Basel 381–456 (2000) 218
114. M. Lejeune-Jalabert, *Contributions à l'étude des singularités du point de vue du polygone de Newton,* Thése d'Etat, Université Paris 7 (1973) 191
115. Beppo Levi, *Sulla risoluzione delle singolarità puntuali delle superficie algebriche dello spazio ordinario per transformazioni quadratiche,* Ann. Mat. pura appl. II. s. 26 (1897) 199
116. Beppo Levi, *Risoluzione delle singolarità puntuali delle superficie algebriche,* Atti Accad. Sci. Torino 33 66–86 (1897) 199
117. J. Lipman, *Desingularization of two-dimensional schemes,* Ann. Math.107 151–207 (1978) 215
118. S. MacLane, *A construction for prime ideals as absolute values of an algebraic field,* Duke Math. J. 2 492–510 (1936) 218

119. S. MacLane, *A construction for absolute values in polynomial rings,* Transactions of the AMS 40 363–395 (1936)
120. S. MacLane and O.F.G Schilling, *Zero-dimensional branches of rank one on algebraic varieties,* Ann. of Math. 40, 3 (1939) 218
121. H. Matsumura, *Commutative Algebra,* Benjamin/Cummings Publishing Co. Reading, Mass. (1970) 233 and 234
122. M. McQuillan and G. Marzo, *Very fast, very functorial, and very easy resolution of singularities*, arXiv:1906.06745 215
123. M. McQuillan and D. Panazzolo, *Almost étale resolution of foliations*, J. Differential Geometry 95 279–319 (2013) 232
124. T.T. Moh, *On a Newton polygon approach to the uniformization of singularities in characteristic p*, in Algebraic geometry and singularities (La Rábida, 1991) , Progr. Math. 134, Birkhäuser, Basel, 49–93 (1996) 216
125. B. Molina-Samper, *Combinatorial Aspects of Classical Resolution of Singularities,* Preprint, arXiv:1711.08258 213
126. M. Nagata, *Local Rings,* Krieger Publishing Co. Huntington, NY (1975) 234
127. H. Nakajima, *Lectures on Hilbert Schemes of Points on Surfaces,* University Lecture Series, Vol. 18, American Mathematical Society, Providence (1991) 229
128. R. Narasimhan, *Hyperplanarity of the equimultiple locus,* Proc. Amer. Math. Soc. 87, 3 403–408 (1983) 213
129. A. Nobile, *Some properties of the Nash blowing-up*, Pacific Journal of Mathematics 60, 297–305 (1975) 223
130. J. Novacoski and M. Spivakovsky, *Reduction of local uniformization to the rank one case,* Valuation Theory in Interaction EMS Series of Congress Reports European Mathematical Society 404–431 (2014) 207
131. J. Novacoski and M. Spivakovsky, *On the local uniformization problem,* Algebra, Logic and Number Theory Banach Center Publ. 108 231–238 (2016) 207
132. J. Novacoski and M. Spivakovsky, *Reduction of local uniformization to the case of rank one valuations for rings with zero divisors,* Michigan Math. J. 66, Issue 2 277–293 (2017) 207
133. J. Novacoski and M. Spivakovsky, *Key polynomials and pseudo-convergent sequences,* Journal of Algebra 495 199–219 arXiv:1611.05679 (2018) 218
134. A. Oneto, E. Zatini, *Remarks on Nash blowing-up,* Commutative algebra and algebraic geometry, II (Italian) (Turin 1990), Rend. Sem. Mat. Univ. Politec. Torino 49, no. 1, 71–82 (1991) 228 and 229
135. D. Panazzolo, *Resolution of singularities of real-analytic vector fields in dimension three,* Acta Math. 197 167–289 (2006) 232
136. O. Piltant, *An axiomatic version of Zariski's patching theorem,* Rev. T. Acad. Cienc. Exactas Fis. Nat. Ser. A. Math. RACSAM 107 91–121 (2013) 210, 217, and 232
137. V. Rebassoo, *Desingularisation properties of the Nash blowing-up process,* Ph.D Thesis, University of Washington (1977). 223
138. J.-C. San Saturnino, *Théorème de Kaplansky effectif et uniformisation locale des schémas quasi-excellents,* Thèse de Doctorat, Institut de Mathématiques de Toulouse, July 2013 218
139. J.-C. San Saturnino, *Defect of an extension, key polynomials and local uniformization,* J. of Algebra 481, 91–119 (2017) 218
140. B. Schober, *Characteristic polyhedra of idealistic exponents with history,* PhD thesis, University of Regensburg, http://epub.uni-regensburg.de/28877/ (2013) 188
141. A. Seidenberg, *Reduction of the singularities of the differential equation $Ady = Bdx$*, Am. J. of Math. 90 248–269 (1968) 231
142. M. Spivakovsky, *A solution to Hironaka's polyhedra game,* Arithmetic and Geometry II, Papers dedicated to I. R. Shafarevich on the occasion of his sixtieth birthday, M. Artin and J. Tate, editors Birkhäuser, 419–432 (1983) 213
143. M. Spivakovsky, *A counterexample to Hironaka's "hard" polyhedra game,* Publ. RIMS Kyoto University 18, 3 1009–1012 (1982) 213

144. M. Spivakovsky, *A counterexample to the theorem of Beppo Levi in three dimensions,* Invent. Math 96 181–183 (1989) 200
145. M. Spivakovsky, *Valuations in function fields of surfaces,* Amer. J. Math 112, 1 107–156 (1990) 224
146. M. Spivakovsky, *Sandwiched Singularities and Desingularization of Surfaces by Normalized Nash Transformations,* Annals of Mathematics, Second Series, Vol. 131, No. 3, 411–491 (1990) 224
147. M. Spivakovsky, *Resolución de singularidades y raices aproximadas de Tschirnhausen*, Seminarios temáticos — Instituto de Estudios con Iberoamérica y Portugal, Seminario Iberoamericano de Matemáticas IV, notes by Fernando Sanz 3–17 (1997) 218
148. B. Teissier, *Valuations, deformations and toric geometry,* Proceedings of the Saskatoon Conference and Workshop on valuation theory, Vol II, F-V. Kuhlmann, S. Kuhlmann, M. Marshall, editors, Fields Institute Communications 33 361–459 (2003) 218
149. B. Teissier, *Overweight deformations of affine toric varieties and local uniformization,* Valuation Theory in Interaction, EMS Series of Congress Reports European Mathematical Society 474–565 (2014) 218
150. M. Temkin, *Desingularization of quasi-excellent schemes of characteristic zero,* Adv. Math. 219 488–522 (2008) 213
151. M. Temkin, *Absolute desingularization in characteristic zero,* Motivic integration and its interactions with model theory and non-archimedean geometry, Volume II London Math. Soc. Lecture Note Ser. 384 213–250 (2011)
152. M. Temkin, *Functorial desingularization of quasi-excellent schemes in characteristic zero: the non-embedded case,* Duke Journal of Mathematics 161 2208–2254 (2012) 213
153. M. Temkin, *Inseparable local uniformization,* J. Algebra 373 65–119 (2013) 217
154. R. Toh-Yama, *Higher Nash blowups of* A_3*-singularity,* arXiv:1804.11050 (2018) 230
155. M. Vaquié, *Valuations*, Proceedings of the Working Week on Resolution of Singularities, Tirol Progr. in Math.181 539–590 (2000) 203
156. M. Vaquié, *Famille admise associée à une valuation de* $K[x]$, Singularités Franco-Japonaises Séminaires et Congrès 10, Actes du colloque franco-japonais, juillet 2002, édité par J.-P. Brasselet et T. Suwa, Société Mathématique de France Paris 391–428 (2005) 218
157. M. Vaquié, *Algèbre graduée associée à une valuation de* $K[x]$, Singularities in Geometry and Topology 2004 Advanced Studies in Pure Mathematics, Actes du Troisième Congrès Franco-Japonais sur les Singularités en Géométrie et Topologie, Sapporo, Japon, septembre 2004, édités par J.-P. Brasselet et T. Suwa 46 259–271 (2007)
158. M. Vaquié, *Extension d'une valuation,* Trans. Amer. Math. Soc 359, no. 7 3439–3481 (2007)
159. M. Vaquié, *Famille admissible de valuations et défaut d'une extension,* J. Algebra 311, no. 2 859–876 (2007) 218
160. O. Villamayor, *Constructiveness of Hironaka resolution,* Ann. Sci. Ec. Norm. Sup. 4, t. 2 1–32 (1989) 213
161. O. Villamayor, *On flattening of coherent sheaves and of projective morphisms*, Journal of Algebra 295 119–140 (2006) 222
162. R. J. Walker, *Reduction of singularities of an algebraic surface,* Ann. of Math. 36 336–365 (1935) 199 and 200
163. J. Włodarczyk, *Simple Hironaka resolution in characteristic zero,* J. Amer. Math. Soc. 18 (4) 779–822 (2005) 213
164. T. Yasuda, *Higher Nash blowups,* Compositio Math. 143, no. 6 1493–1510 (2007) 228 and 229
165. T. Yasuda, *Flag higher Nash blowups,* Communications in Algebra, 37, 1001–1015 (2009) 230
166. B. Youssin, *Newton polyhedra without coordinates,* Mem. AMS 87, 433 (1990) 213 and 230
167. B. Youssin, *Newton polyhedra of ideals,* Mem. AMS 87, 433 (1990) 213
168. O. Zariski, *Some Results in the Arithmetic Theory of Algebraic Varieties,* American J. of Math. 61 (2) 249–294 (1939) 191 and 201
169. O. Zariski, *The reduction of singularities of an algebraic surface,* Ann. of Math 40 639–689 (1939) 199

170. O. Zariski, *Local uniformization theorem on algebraic varieties,* Ann. of Math. 41 852–896 (1940) 207 and 213
171. O. Zariski, *A simplified proof for resolution of singularities of an algebraic surface,* Ann. of Math. 43 583–593 (1942) 199
172. O. Zariski, *Reduction of singularities of algebraic three dimensional varieties,* Ann. of Math. 45 472–542 (1944) 210
173. O. Zariski, *The compactness of Riemann manifold of an abstract field of algebraic functions,* Bull. Amer. Math. Soc.45 683–691 (1944) 209
174. O. Zariski and P. Samuel *Commutative Algebra, Vol. II* Springer-Verlag (1960). 203

Chapter 4
Stratification Theory

David Trotman

Contents

Abstract This is a survey of stratification theory in the differentiable category from its beginnings with Whitney, Thom and Mather until the present day. We concentrate mainly on the properties of C^∞ stratified sets and of stratifications of subanalytic or definable sets, with some reference to stratifications of complex analytic sets. Brief mention is made of the theory of stratified mappings.

4.1 Stratifications

The idea behind the notion of stratification in differential topology and algebraic geometry is to partition a (possibly singular) space into smooth manifolds with some control on how these manifolds fit together.

D. Trotman (✉)
Institut Mathématique de Marseille, Aix Marseille Univ., CNRS, Marseille, France
e-mail: david.trotman@univ-amu.fr

J. L. Cisneros Molina et al. (eds.), *Handbook of Geometry and Topology of Singularities I*, https://doi.org/10.1007/978-3-030-53061-7_4

In 1957 Whitney [146] showed that every real algebraic variety V in $\mathbf{R}^n$ can be partitioned into finitely many connected smooth submanifolds of $\mathbf{R}^n$. This he called a *manifold collection*. Such a partition is obtained by showing that the singular part of V is again algebraic and of dimension strictly less than that of V. One obtains thus a filtration of V by algebraic subvarieties,

$$V \supset SingV \supset Sing(SingV) \supset \dots$$

In 1960, Thom [120] replaced the term "manifold collection" of Whitney, by "stratified set", introduced the notions of "stratum" and "stratification", and initiated a theory of stratified sets and stratified maps. Later, in 1964, Thom proposed that a stratification should have the property that transversality of a map $g : \mathbf{R}^m \to \mathbf{R}^n$ to the strata of a stratified set in $\mathbf{R}^n$ be an open condition on maps in $C^\infty(\mathbf{R}^m, \mathbf{R}^n)$, and that there should be some "local triviality" in a neighbourhood of each stratum.

As a result Whitney refined his definition in 2 papers [147, 148] which appeared in 1965, concerning stratifications of real and complex analytic varieties, introducing his conditions (a) and (b). He proved the existence of stratifications satisfying these conditions for any real or complex analytic variety, remarked that Thom's openness of transversality follows from condition (a) and conjectured a local fibration property (known as Whitney's holomorphic fibering conjecture, this was finally proved by Parusinski and Paunescu in 2017 [102] after a partial version obtained by Hardt and Sullivan in 1988 [49]). Thom then developed a theory of C^∞ stratified sets, described in detail in his 1969 paper entitled "*Ensembles et morphismes stratifiés*" [123]. The following year Mather gave a series of lectures at Harvard giving a revised account of Thom's theory of stratified sets and maps, and it is Mather's definitions that have been generally used since then. Mather's 1970 notes of his lectures, which circulated widely via photocopies, were finally published in 2012 [81]. The reader may profitably consult also [36] and [105] for detailed presentations of the theory of stratified sets.

I will now describe what has become the accepted notion of Whitney stratification.

Definition 4.1.1 Let Z be a closed subset of a differentiable manifold M of class C^k. A C^k *stratification* of Z is a filtration by closed subsets

$$Z = Z_d \supset Z_{d-1} \supseteq \cdots \supseteq Z_1 \supseteq Z_0$$

such that each difference $Z_i - Z_{i-1}$ is a differentiable submanifold of M of class C^k and dimension i, or is empty. Each connected component of $Z_i - Z_{i-1}$ is called a *stratum* of dimension i. Thus Z is a disjoint union of the strata, denoted $\{X_\alpha\}_{\alpha \in A}$, and Z is a *stratified set*.

Example 4.1.2 The filtration of a realisation of a simplicial complex defined by skeleta is a C^∞ stratification, where the strata are the open simplices.

We would like our stratifications to "look the same" at different points on the same stratum. This turns out to be possible if "looking the same" is interpreted as having neighbourhoods which are homeomorphic, a kind of equisingularity. Various stronger equisingularity conditions, also called regularity conditions, have been introduced ensuring this. An obvious necessary condition is as follows:

Definition 4.1.3 A stratification $Z = \bigcup_{\alpha \in A} X_\alpha$ satisfies the *frontier condition* if $\forall(\alpha, \beta) \in A \times A$ such that $X_\alpha \cap \overline{X_\beta} \neq \emptyset$, one has $X_\alpha \subseteq \overline{X_\beta}$. As the strata are disjoint this means that either $X_\alpha = X_\beta$ or $X_\alpha \subset \overline{X_\beta} \setminus X_\beta$.

Definition 4.1.4 One says that a stratification is *locally finite* if the number of strata is locally finite.

4.2 Whitney's Conditions (*a*) and (*b*)

The most widely used of the different regularity conditions proposed so as to provide adequate "equisingularity" of a stratification are the conditions (a) and (b) of Whitney.

Definition 4.2.1 Take two adjacent strata X and Y, i.e. two C^1 submanifolds of M such that $Y \subset \overline{X} \setminus X$, so that X is adjacent to Y. The pair (X, Y) is said to satisfy Whitney's condition (a) at $y \in Y$, or to be (a)-regular at y if : for all sequences $\{x_i\} \in X$ with limit y such that, in a local chart at y, $\{T_{x_i}X\}$ tends to τ in the grassmannian G^{dimM}_{dimX}, one has $T_yY \subseteq \tau$.

When every pair of adjacent strata of a stratification is (a)-regular (at each point) then we say that the stratification is (a)-regular.

Definition 4.2.2 The pair (X, Y) is said to satisfy Whitney's condition (b) at $y \in Y$, or to be (b)-regular at y if : for all sequences $\{x_i\} \in X$ and $\{y_i\} \in Y$ with limit y such that, in a local chart at y, $\{T_{x_i}X\}$ tends to τ and the lines $\overline{x_i y_i}$ tend to λ, one has $\lambda \in \tau$.

When every pair of adjacent strata of a stratification is (b)-regular (at each point) then we say that the stratification is (b)-regular.

Definition 4.2.3 Let Z be a closed subset of a differentiable manifold M of class C^1. When $Z = \bigcup_{\alpha \in A} X_\alpha$ is a locally finite (b)-regular stratification satisfying the frontier condition, we say we have a *Whitney stratification* of Z.

Remark 4.2.4 It will be a nontrivial consequence of the theory that the frontier condition is automatically satisfied by pairs of adjacent strata of a locally finite (b)-regular stratification.

Definition 4.2.5 Let $\pi : T_Y \to Y$ be the retraction of a C^1 tubular neighbourhood of Y in M. A pair of adjacent strata (X, Y) is said to be (b^π)-regular if for all sequences $\{x_i\}$ in X such that x_i tends to y and the lines $\overline{x_i \pi(x_i)}$ tend to λ and the tangent planes $T_{x_i}X$ tend to τ, then $\lambda \in \tau$.

When every pair of adjacent strata of a stratification is (b^π)-regular (at each point) then we say that the stratification is (b^π)-regular.

Exercises

1. $(b) \Rightarrow (a)$.
2. $(b) \Leftrightarrow (b^{\pi}) \quad \forall \pi$.
3. (b) holds if both (a) and (b^{π}) hold for some π.
4. If (X, Y) is (b)-regular at $y \in Y$, then $\dim Y < \dim X$.

   The following standard example due to Whitney shows that (a) does not imply (b).

Example 4.2.6 Let $Z = Z_2 = \{y^2 = t^2x^2 + x^3\} \subset \mathbf{R}^3$. Set $Z_1 = \{(0, 0, t)|t \in \mathbf{R}\}$ and $Z_0 = \emptyset$. Then $Z_2 \supset Z_1 \supset Z_0 = \emptyset$ is a filtration defining a C^{∞} stratification with 4 strata of dimension 2 and one stratum of dimension 1. The strata are defined as follows : $X_1 = (Z_2 - Z_1) \cap \{t > 0\} \cap \{x < 0\}$, $X_2 = (Z_2 - Z_1) \cap \{t < 0\} \cap \{x < 0\}$, $X_3 = (Z_2 - Z_1) \cap \{y < 0\} \cap \{x > 0\}$, $X_4 = (Z_2 - Z_1) \cap \{y > 0\} \cap \{x > 0\}$, $Y = Z_1$. One can check that the pairs of strata (X_3, Y) and (X_4, Y) are (b)-regular, and in fact they form C^{∞} manifolds with boundary, while (X_1, Y) and (X_2, Y) are not (b)-regular at $(0, 0, 0)$, although they are (a)-regular. Note that the frontier condition does not hold for (X_1, Y) and (X_2, Y). It is possible to unite X_1 and X_2 into one connected stratum by turning Y into a circle, so that the frontier condition holds. But (b) will still fail.

Next we give examples showing that (b^{π}) does not imply (a).

Example 4.2.7 Let $Z = \{y^2 = tx^2\} \subset \mathbf{R}^3$; with filtration $Z = Z_2 \supset Z_1 = (Ot) \supset Z_0 = \emptyset$. The stratification is (b^{π})-regular if π is the canonical projection onto the t-axis, but it is not (a)-regular, and does not satisfy the frontier condition.

Example 4.2.8 Let $Z = \{x^3 + 3xy^5 + ty^6 = 0\} \subset \mathbf{R}^3$, with filtration $Z = Z_2 \supset Z_1 = (Ot) \supset Z_0 = \emptyset$. Here the stratification is not (a)-regular, but is $(b^{\pi})-$regular where π is projection to the t-axis, and satisfies the frontier condition.

Wall [144] conjectured geometric versions of conditions (a) and (b), and these conjectures were proved in [131]. Different proofs were given later by Hajto [43] and by Perkal [104]. Recall that each tubular neighbourhood of a submanifold Y of a manifold M is given by a diffeomorphism ϕ defined on a neighbourhood U of Y. We denote by $\pi_{\phi} : U \to Y$ the associated retraction and by $\rho_{\phi} : U \to [0, 1)$ the associated tubular function.

Theorem 4.2.9 (Trotman [131]) *Let X, Y be disjoint C^1 submanifolds in a C^1 manifold M, with $Y \subset \overline{X} \setminus X$. Then X is (b)-regular (resp. (a)-regular) over Y if and only if for every C^1 diffeomorphism ϕ defining a tubular neighbourhood of Y the map $(\pi_{\phi}, \rho_{\phi})|_X$ (resp. $\pi_{\phi}|_X$) is a submersion.*

The theorem implies that conditions (a) and (b) are C^1 invariants. Examples exist showing that it is not sufficient to take C^2 diffeomorphisms ϕ [62].

One of the main reasons that Whitney stratifications are of interest is because analytic varieties can be Whitney stratified.

Theorem 4.2.10 (Whitney [147, 148]) *Every analytic variety (in* $\mathbf{R}^n$ *or* $\mathbf{C}^n$*) admits a Whitney stratification whose strata are analytic (hence* C^∞*) manifolds.*

The main point in the proof of this theorem is to show that the Whitney conditions are stratifying, i.e. true on an analytic open dense set of a given subspace. This is proved by contradiction using a wing lemma. Also in 1965, Łojasiewicz proved the existence of Whitney stratifications of semi-analytic sets [79]. Hironaka and Hardt proved that the same is true of every subanalytic set [46, 55]. Hironaka's proof uses resolution of singularities. More accessible existence proofs for semialgebraic sets using Whitney's wing lemma method are due to Thom [122] and to Wall [144]. A more elementary proof for subanalytic sets is due to Denkowska, Wachta and Stasica [22, 24]. More generally, existence theorems for Whitney stratifications of definable sets in o-minimal structures [25] have been given by Loi [78], by van den Dries and Miller [26], and by Nguyen, Trivedi and Trotman [95]. Another proof of the o-minimal case follows from the work of Halupczok and Yin [44].

Whitney's theorem above is a pure existence statement, proved by contradiction using Whitney's wing lemma. Teissier in [118] obtained a much more precise result for complex analytic sets: a complex analytic stratification is Whitney (b)-regular if and only if the multiplicities of its polar varieties are constant along strata. So Whitney regularity is equivalent to the constance of a finite set of numerical invariants. The existence theorem follows. Teissier's theorem also implies the existence, for a complex analytic set, of a canonical minimal Whitney stratification of which all others are refinements (see [118]). It also gives rise to the most general Plücker formula, expressing the degree of the dual variety of a projective variety in terms of topological characters of its canonical Whitney stratification and its general plane sections [37]. Another way of characterising Whitney conditions for complex analytic varieties was developed by Gaffney [32] using the integral closure of modules. Gaffney also gives real analogues characterising Whitney (b)-regularity using the real integral closure.

There are other situations where Whitney stratifications arise naturally. The stratification of a smooth manifold by the orbit types of a proper Lie group action is Whitney regular (this was known to Bierstone in the 1970s [10] and was reproved several times, cf. [105]), but in fact a much stronger result holds: it is smoothly locally trivial [35, 145]! Another situation where a natural Whitney stratification turns out to be smoothly local trivial is the partition of a compact smooth manifold into unstable manifolds of a generic Morse function. That this is a Whitney stratification was proved by Nicolaescu [96], while Laudenbach proved the stronger smooth local triviality [74].

One can ask why one should study Whitney's condition (a), as it is strictly weaker than condition (b). One reason is that it is both simple to understand and easy to check. A second reason is that it is a necessary and sufficient condition for transversality to the strata of a stratification to be an *open* property, as we shall see in the next theorem, often cited in the literature.

Definition 4.2.11 We say that a map $f : N \to M$ between C^1 manifolds is *transverse* to a C^1 stratification of a closed set $Z \subset M$, if $\forall x \in N$ such that $f(x) \in Z$, then

$$(df)_x T_x N + T_{f(x)} X = T_{f(x)} M$$

where X is the stratum containing $f(x)$, i.e. the map f is transverse to each stratum of the stratification of Z.

Theorem 4.2.12 (Trotman [130]) *A locally finite C^1 stratification of a closed subset Z of a C^1 manifold M is (a)-regular if and only if for every C^1 manifold N, $\{f \in C^1(N, M) | f$ is transverse to the strata of $Z\}$ is an open set in the Whitney C^1 topology.*

The sufficiency of (a)-regularity here is due to Feldman in 1965 [28]. A simple proof of necessity in Theorem 4.2.12 can be extracted from the proof of a recent relative version of the theorem given by Trivedi and Trotman [125].

A partial version of this theorem in the holomorphic case is due to Trivedi [124]. Let $H(M, N)$ denote the space of holomorphic mappings between two complex analytic manifolds M and N.

Theorem 4.2.13 (Trivedi) *Let M be a Stein manifold and N be an Oka manifold. Let Σ be a stratification of a complex analytic subvariety in N. Let r be the minimum of the dimensions of strata in Σ. If dim $M =$ dim $N - r$ and there exists a compact set K in M such that the set of maps $T_K = \{f \in H(M, N) : f \pitchfork_K \Sigma\}$ is open in $H(M, N)$, then Σ is an a-regular stratification.*

Another application of Whitney (a)-regularity is the following.

Theorem 4.2.14 (Kuo-Li-Trotman [70]) *Let X be a stratum of an (a)-regular stratification of a subset Z of $\mathbf{R}^n$. For each $x \in X$ and for every pair of Lipschitz transversals M_1, M_2 to X at x (a Lipschitz transversal is defined to be the graph of a Lipschitz map $N_x X \to T_x X$), there is a homeomorphism*

$$(M_1, Z \cap M_1, x) \to (M_2, Z \cap M_2, x).$$

Such results justify the separate study and verification of (a)-regularity.

A natural geometric operation is to take transverse intersections of geometric objects. Suppose Z and Z' are two closed stratified sets of a manifold M. Denote the set of strata by Σ and Σ' respectively. If Σ and Σ' are transverse, i.e. if for all $X \in \Sigma$, and for all $X' \in \Sigma'$, X and X' are transverse as submanifolds of M, then we can stratify $Z \cap Z'$ by $\Sigma \cap \Sigma' = \{X \cap X' | X \in \Sigma, X' \in \Sigma'\}$. Moreover $Z \cup Z'$ is naturally stratified by adding the complements in Z (resp. Z') $\{(X \setminus X \cap Z') | X \in \Sigma\}$ (resp. $\{(X' \setminus X' \cap Z) | X' \in \Sigma'\}$).

Theorem 4.2.15 *If (Z, Σ) and (Z', Σ') are Whitney (b)-regular (resp. (a)-regular), and have transverse intersections in M, then $(Z \cap Z', \Sigma \cap \Sigma')$ is (b)-regularly (resp. (a)-regularly) stratified, as is also $Z \cup Z'$.*

This can often be useful. The case of (b)-regularity was treated by Gibson in 1976 [36]. A general theorem of this kind was proved by Orro and Trotman in 2002 [98, 99] for a large class of regularity conditions including the (w)-regularity of the next section.

Other useful properties of Whitney stratified sets include stability under taking products, and triangulability.

Products If Z and Z' are Whitney stratified then so is $Z \times Z'$. A similar result is true for the (w)-regular stratified sets of the next section.

Triangulation It is known that compact Whitney stratified sets are triangulable (Goresky [38], Verona [143], Shiota [112]). The non-compact case follows from another result of Shiota [111] stating that every Whitney stratified set is homeomorphic to a subanalytic set, from which triangulability follows by citing the triangulability of subanalytic sets due to Hironaka [56] and Hardt [47].

However there remains an outstanding open question [39]: does a Whitney stratified set (Z, Σ) have a triangulation whose open simplexes are the strata of a Whitney stratification refining Σ ? In other words, does every Whitney stratified set admit a Whitney triangulation ? The existence of Whitney (b)-regular triangulations was proved by Shiota for semialgebraic sets [114], and by Czapla [19] for definable sets in o-minimal structures.

As we want our stratifications to "look the same" at different points of a given stratum one might hope that there is a C^1 diffeomorphism mapping neighbourhoods of a point y_1 on Y to neighbourhoods of another point y_2 on Y. This is not true in general, as illustrated by the following celebrated example.

Example 4.2.16 (Whitney) Let $Z = \{(x, y, t)|xy(x-y)(x-ty) = 0, t \neq 1\} \subset \mathbf{R}^3$, stratified by $Z = Z_2 \supset Z_1 = (Ot)$. This is a family of 4 lines parametrised by t. The stratification is (b)-regular, but there is no C^1 diffeomorphism mapping Z_{t_1} to Z_{t_2} where $Z_t = Z \cap (\mathbf{R}^2 \times \{t\})$, because of the cross-ratio obstruction. (A linear isomorphism of the plane preserving three distinct lines through a point preserves also any fourth line through that point.)

One may observe that Z in the previous example is a union of eight C^1 manifolds-with-boundary, with $(0t)$ the common boundary. Pawłucki [103] proved a general theorem showing this property: if X and Y are subanalytic adjacent strata such that X is (b)-regular over Y, and $\dim X = \dim Y + 1$, then $X \cup Y$ is a finite union of C^1 manifolds-with-boundary with common boundary Y. A generalisation to definable sets in polynomially bounded o-minimal structures was given by Trotman and Valette [135], who show also that this property fails for definable sets in non polynomially bounded o-minimal structures.

Although Example 4.2.16 means we cannot expect to have in general local C^1 triviality of Whitney stratified sets even in the real algebraic case, we can obtain what is known as local topological triviality. The following, together with Whitney's existence Theorem 4.2.10, constitutes the fundamental theorem of stratification theory.

Theorem 4.2.17 (Thom-Mather [81, 123]) *Let* (Z, Σ) *be a Whitney stratified subset of a* C^2 *manifold* M*. Then for each stratum* $Y \in \Sigma$ *and each point* $y_0 \in Y$ *there is a neighbourhood* U *of* y_0 *in* M*, a stratified set* $L \subset S^{k-1}$ *and a homeomorphism*

$$h : (U, U \cap Z, U \cap Y) \to (U \cap Y) \times (B^k, c(L) \cap B^k, 0)$$

such that $p_1 \circ h = \pi_Y$*, where* $c(L)$ *is the cone on the link* L *with vertex* 0*,* B^k *is the* k*-ball centred at* 0*, and* π_Y *is the projection onto* $U \cap Y$ *of a tubular neighbourhood.*

A typical application of this theorem is Fukuda's proof that the number of topological types of polynomial functions $p : \mathbf{R}^n \to \mathbf{R}$ of given degree d is finite [30].

Theorem 4.2.17 applies without any hypothesis of analyticity or subanalyticity. If the strata are assumed to be semialgebraic, Coste and Shiota [18] have shown that the trivialising homeomorphism h may be chosen to be semialgebraic, using real spectrum methods in their proof. See Shiota's book [112] for a more general result applying to definable sets and providing a definable trivialisation. The proof of Mather [81] of Theorem 4.2.17 uses the notion of controlled vector field, and the homeomorphism h resulting from Mather's proof is obtained by integrating such controlled vector fields, so that the resulting homeomorphism h will not in general be semialgebraic even if the strata are semialgebraic.

A (stratified) vector field v on a stratified set (Z, Σ) is defined by a collection of vector fields $\{v_X | X \in \Sigma\}$. It is *controlled* when $(\pi_Y)_* v_X(x) = v_Y(\pi_Y(x))$ and $(\rho_Y)_* v_X(x) = 0$ on a tubular neighbourhood T_Y of Y, where T_Y is part of a set of compatible tubular neighbourhoods called control data. See Mather's notes [81] for details of the theory of control data and of controlled vector fields. It was not until 1996 that a proof was published that these stratified controlled vector fields could be assumed to be *continuous* : given a vector field v_Y on a stratum Y of a Whitney stratified set, or indeed a Bekka stratified set (see Sect. 4.8 below), there exists a continuous controlled stratified vector field $\{v_X\}$ on M extending v_Y (Shiota [112] for Whitney stratified sets, du Plessis [106] for the more general Bekka stratified sets). This result has been used for example by Hamm [45] to simplify some statements in stratified Morse theory [41], and by S. Simon to prove a stratified version of the Poincaré-Hopf theorem [116].

The proof of local topological triviality and conicality of Whitney stratified sets as stated in Theorem 4.2.17 is in fact an easy consequence of the following more general first isotopy lemma of Thom [81, 123]:

Theorem 4.2.18 *Let Z be a Whitney stratified subset of a C^2 manifold M, and let $f : M \to \mathbf{R}^k$ be a C^2 map such that $f|Z$ is proper and the restriction of f to each stratum of Z is a submersion. Then there is a stratum-preserving homeomorphism $h : Z \to \mathbf{R}^k \times (f^{-1}(0) \cap Z)$ which commutes with the projection to $\mathbf{R}^k$, so that the fibres of $f|_Z$ are homeomorphic by a stratum-preserving homeomorphism.*

There is a second isotopy lemma for stratified maps satisfying Thom's (a_f) condition (see Definition 4.4.1 below), a relative version of condition (a) [81, 82, 123]. These two isotopy lemmas were first used in the proof of the difficult topological stability theorem : the space of topologically stable mappings is dense in the space of proper smooth mappings between two smooth manifolds [36, 82, 83, 107, 121, 123]. A recent strengthening of Theorem 4.2.17, obtaining continuity of the tangent spaces to the leaves defined by fixing points in the normal slice, implies the density of strongly topologically stable mappings in the space of proper mappings [89]. Strong topological stability refers to imposing continuity of the commuting homeomorphisms as functions of a varying map.

4.3 The Kuo-Verdier Condition (*w*)

Condition (a) for (X, Y) says that the distance between the tangent space to X at x and the tangent space to Y at y tends to zero as x tends to y. Kuo and Verdier studied what happens when the rate of vanishing of this distance is $O(|x - \pi_Y(x)|)$ [67, 142].

Definition 4.3.1 Two adjacent strata (X, Y) in a C^1 manifold M are said to be (w)-regular at $y_0 \in Y$, or to satisfy the Kuo-Verdier condition (w) , if there exist a constant $C > 0$ and a neighbourhood U of y_0 in M such that

$$d(T_y Y, T_x X) < C||x - y||$$

$\forall x \in U \cap X, \forall y \in U \cap Y.$

Here, for vector subspaces V and W of an inner product space E,

$$d(V, W) = \sup\{\inf\{\sin\theta(v, w)|w \in W^*\}|v \in V^*\}$$

where $\theta(v, w)$ is the angle between v and w.

Note that $d(V, W) = 0 \Leftrightarrow V \subset W$, and that $d(V, W) = 1 \Leftrightarrow \exists v \in V^*, v \perp W$.

Proposition 4.3.2 (Kuo [66]) *For semi-analytic X and Y, $(w) \Rightarrow (b)$.*

Verdier observed that Kuo's proof that (w) implies (b) in [66] (where Kuo takes as hypothesis a weaker condition, that he called the ratio test) works too for subanalytic sets [142], and Loi [78] extended this result to the case of definable sets in o-minimal structures.

So (w)-regularity is a stronger regularity condition than (b) for definable stratified sets (it no longer implies (b) in general for C^∞ stratified sets as shown by Example 4.3.7 below). Moreover it turns out to be generic too, as the following theorem shows.

Theorem 4.3.3 (Verdier [142]) *Every subanalytic set admits a locally finite (w)-regular stratification.*

This is also true for definable sets in arbitrary o-minimal structures as shown by Loi [78]. Other proofs in the subanalytic case are due to Denkowska and Wachta [23], and to Łojasiewicz, Stasica and Wachta [80], both of these proofs avoiding resolution of singularities. Another proof, due to Kashiwara and Schapira [63], follows from the equivalence of (w) and their microlocal condition μ [134].

For complex analytic sets a major result proved in 1982 by Teissier, with a contribution by Henry and Merle, implies the equivalence of (b) and (w) [50, 118]. Real algebraic examples showing that (b) does not imply (w) are common because (b) is a C^1 invariant [131] while (w) is not a C^1 invariant (although it is a C^2 invariant), as shown by the following example.

Example 4.3.4 (Brodersen-Trotman [15]) Let $Z = \{y^4 = t^4x + x^3\} \subset \mathbf{R}^3$. Then the stratification of Z defined by $Z = Z_2 \supset Z_1 = (Ot)$ is (b)-regular but not (w)-regular. Z is actually the graph of the C^1 function $f(x,t) = (t^4x + x^3)^{1/4}$.

Infinitely many real algebraic examples with (b) holding but not (w) may be found in the combined classifications of Juniati, Noirel and Trotman [59, 60, 97, 133]. The first such semialgebraic example was given in [128].

One can characterise (w)-regularity using stratified vector fields as follows.

Proposition 4.3.5 (Brodersen-Trotman [15]) *A stratification is (w)-regular $\Leftrightarrow$ every vector field on a stratum Y extends to a rugose stratified vector field in a neighbourhood of Y.*

Here a stratified vector field $\{v_X : X \in \Sigma\}$ is called *rugose* near y_0, in a stratum Y, when there exists a neighbourhood U of y_0 and a constant $C > 0$, such that for all adjacent strata X, $\forall x \in U \cap X$, $\forall y \in U \cap Y$,

$$\| v_X(x) - v_Y(y) \| \leq C \| x - y \| .$$

This resembles an asymmetric Lipschitz condition, and poses the question of when the extension of a Lipschitz vector field can be chosen to be Lipschitz. This we will discuss in Sect. 4.5.

There is a somewhat weaker version of the Thom-Mather isotopy theorem, due to Verdier [142] in 1976, for his (w)-regular stratified sets. He obtains local topological triviality but not the local conicality of Theorem 4.2.17.

Theorem 4.3.6 (Verdier) *Let (Z, Σ) be a (w)-regular C^2 stratified subset of a C^2 manifold M. Then for each stratum $Y \in \Sigma$ of codimension k in M, and each point $y_0 \in Y$ there is a neighbourhood U of y_0 in M, a stratified set $N \subset B^k$ and a rugose* *homeomorphism*

$$h : (U, U \cap Z, U \cap Y) \to (U \cap Y) \times (B^k, N, 0)$$

such that $p_1 \circ h = \pi_Y$*, where* B^k *is the* k*-ball centred at* 0*, and* π_Y *is the projection onto* $U \cap Y$ *of a tubular neighbourhood.*

The proof is by integration of rugose vector fields [142]. Another approach to this isotopy theorem was given by Fukui and Paunescu [31].

Example 4.3.7 The topologist's sine curve in $\mathbf{R}^2$, with Z the closure of $\{y = \sin(1/x) : x > 0\}$, provides an example of a (w)-regular stratified set Z which is not Whitney stratified : (b^π) fails at every point of the 1-dimensional stratum on the y-axis. Clearly the conical conclusion of the Thom-Mather isotopy Theorem 4.2.17 fails to hold.

Remark 4.3.8 The homeomorphism obtained in the Thom-Mather isotopy Theorem 4.2.17 is also rugose, because it is controlled, given by integrating controlled vector fields (see [81]).

4.4 Stratified Maps

Knowing that subanalytic sets may be stratified with regularity conditions ensuring local topological control one can ask whether similar structure theorems can be proved for mappings. Hardt [46] proved that every proper real analytic mapping between real analytic manifolds may be stratified, in the sense that one may find Whitney stratifications of source and target such that restricted to each stratum of the source the map is a submersion onto the stratum in the target. However Thom[120] had already observed in 1962 that in a family of polynomial maps the topological type can vary continuously. He proposed a type of regularity condition on maps to avoid this phenomenon [123] as follows.

Definition 4.4.1 A map f defined on a stratified set in a manifold M with Y a stratum is said to satisfy the Thom condition (a_f) at $y \in Y$ when f is of constant rank on each stratum and

$$T_y(Y \cap f^{-1}(f(y))) \subset \lim_{x \to y} T_x(X \cap f^{-1}(f(x))),$$

where, for X a stratum and $x \in X$ tending to y, the limit is taken in the appropriate grassmannian G^{dimM}_{dimX-k} if f restricted to X has rank k.

When the map f is constant on X and Y this is just (a)-regularity.

Thom conjectured in the 1960s that proper stratified maps satisfying (a_f) should be triangulable. This was proved by Shiota in 2000 [113] after an earlier partial result by Verona in his book [143]. See also [115] for the non-proper semialgebraic case.

A striking result [76] by Lê and Saito in complex equisingularity, showing the naturality of Thom's condition, is that constancy of the Milnor numbers of a family of isolated hypersurface singularities defined by

$$F : (\mathbf{C}^n \times \mathbf{C}, 0 \times \mathbf{C}) \to (\mathbf{C}, 0)$$

is equivalent to the map F satisfying (a_F) with respect to the stratification with 3 strata,

$$\mathbf{C}^{n+1} \supset F^{-1}(0) \supset (0 \times \mathbf{C}).$$

If instead of inclusion in Definition 4.4.1 which says the distance between the two tangent spaces goes to zero, one requires that this distance is bounded above by a constant times the distance of x to Y, one then obtains a condition (w_f) generalising the Kuo-Verdier condition (w) of Definition 4.3.1. A study of (w_f) in the complex analytic case with geometric characterisations, analogous to Teissier's study of (w) and (b) in [118] was carried out by Henry, Merle and Sabbah [53]. Gaffney and Kleiman give algebraic versions of (w_f) in the complex case see [33]. In the real C^∞ case (w_f) is one of a family of regularity conditions on maps and spaces studied by Trotman and Wilson in [137].

For subanalytic functions one can always stratify a map so that (w_f) holds (Hironaka [57] for (a_f) and Parusiński [100] for (w_f)). The blowup of a point in the plane provides a counterexample to the existence of a stratification (a_f) when the target space has dimension at least 2.

Associated to Thom maps (stratified maps satisfying (a_f)) there is a second isotopy lemma for which we refer to [81] and [82]. This is important in the study of topological stability of mappings [83], Mather using it to complete the proof of the density of the set of topologically stable mappings between smooth manifolds.

Having seen that conditions (a) and (w) have relative versions (a_f) and (w_f) one may wonder about a possible relative version (b_f) of condition (b). So far there have been 3 different conditions called (b_f) in the literature, introduced and used respectively by Thom [123], by Henry and Merle [51] and by Nakai [90]. There is also a condition (D) due to Goresky [39]. No comparative study of these conditions has been undertaken. However Murolo has recently worked out properties of Goresky's condition (D) [88].

4.5 Lipschitz Stratifications

Mostowski in 1985 [85] introduced conditions (L) on a stratification, which further strengthen the Kuo-Verdier condition (w) and these imply the possibility of extending Lipschitz vector fields and can indeed be characterised by the existence of certain Lipschitz extensions of Lipschitz vector fields (see Theorem 4.5.3 below) [101].

Definition 4.5.1 (Mostowski) Let $Z = Z_d \supset \cdots \supset Z_\ell \neq \emptyset$ be a closed stratified set in $\mathbf{R}^n$. Write $\overset{\circ}{Z}_j = Z_j - Z_{j-1}$. Let $\gamma > 1$ be a fixed constant. A *chain* for a point $q \in \overset{\circ}{Z}_j$ is a strictly decreasing sequence of indices $j = j_1, j_2, \ldots, j_r = \ell$ such that each $j_s (s \geq 2)$ is the greatest integer less than j_{s-1} for which

$$\mathrm{dist}(q, Z_{j_s - 1}) \geq 2\gamma^2 \mathrm{dist}(q, Z_{j_s}).$$

For each $j_s, 1 \leq s \leq r$, choose $q_{j_s} \in \overset{\circ}{Z}_{j_s}$ such that $q_{j_1} = q$ and $|q - q_{j_s}| \leq \gamma \mathrm{dist}(q, Z_{j_s})$.

If there is no confusion we call $\{q_{j_s}\}_{s=1}^r$ a chain of q.

For $q \in \overset{\circ}{Z}_j$, let $P_q : \mathbf{R}^n \to T_q(\overset{\circ}{Z}_j)$ be the orthogonal projection to the tangent space and let $P_q^\perp = I - P_q$ be the orthogonal projection to he normal space $(T_q(\overset{\circ}{Z}_j))^\perp$.

Definition 4.5.2 (Mostowski) A stratification $\Sigma = \{Z_j\}_{j=\ell}^d$ of Z is said to be a *Lipschitz stratification*, or to satisfy the (L)-conditions, if for some constant $C > 0$ and for every chain $\{q = q_{j_1}, \ldots, q_{j_r}\}$ with $q \in \overset{\circ}{Z}_{j_1}$ and each k, $2 \leq k \leq r$,

$$| P_q^\perp P_{q_{j_2}} \cdots P_{q_{j_k}} | \leq C \mid q - q_{j_2} \mid / d_{j_k - 1}(q) \qquad (L1)$$

and for each $q' \in \overset{\circ}{Z}_{j_1}$ such that $| q - q' | \leq (1/2\gamma)\, d_{j_1 - 1}(q)$,

$$| (P_q - P_{q'}) P_{q_{j_2}} \cdots P_{q_{j_k}} | \leq C \mid q - q' \mid / d_{j_k - 1}(q) \qquad (L2)$$

and

$$| P_q - P_{q'} | \leq C \mid q - q' \mid / d_{j_1 - 1}(q) \qquad (L3).$$

Here $\mathrm{dist}(-, Z_{\ell-1}) \equiv 1$, by convention.

It is not hard to show that for a given Lipschitz stratification $\exists\, C > 0$ such that $\forall x \in \overset{\circ}{Z}_j, \forall y \in \overset{\circ}{Z}_k, k < j$ then

$$|P_x^\perp P_y| \leq \frac{C|x - y|}{\mathrm{dist}(y, Z_{k-1})},$$

and because $|P_x^\perp P_y| = d(T_y \overset{\circ}{Z}_k, \overset{\circ}{Z}_j)$, (w)-regularity follows with a precise estimation for the constant (which can tend to infinity as y approaches Z_{k-1}).

Parusiński has given the following characterisation of Mostowski's Lipschitz conditions in terms of extensions of vector fields.

Theorem 4.5.3 (Parusiński [101]) *A stratification $Z = Z_d \supset Z_{d-1} \supset \cdots \supset Z_0 \supset Z_{-1} = \emptyset$ is Lipschitz if and only if there exists a constant $K > 0$ such that for every subset $W \subset Z$ such that*

$$Z_{j-1} \subset W \subset Z_j$$

for some $j = \ell, \ldots, d$ where ℓ is the lowest dimension of a stratum of Z, each Lipschitz Σ-compatible vector field on W with Lipschitz constant L which is bounded on $W \cap Z_\ell$ by a constant $C > 0$, can be extended to a Lipschitz Σ-compatible vector field on Z with Lipschitz constant $K(L + C)$.

He also proved an existence theorem for subanalytic sets.

Theorem 4.5.4 (Parusiński [101]) *Every subanalytic set admits a Lipschitz stratification. Moreover such Lipschitz stratifications are locally bilipschitz trivial.*

The initial existence theorem for Lipschitz stratifications was for complex analytic sets, due to Mostowski in 1985 [85]. It is not true that definable sets in arbitrary o-minimal structures admit Lipschitz stratifications.

Example 4.5.5 (Parusinski) Let $X(t)$ be the union of the x-axis and the graph $y = x^t (x > 0)$ in $\mathbf{R}^3 = (x, y, t)$. Then the Lipschitz types of $X(t)$ are distinct for all $t > 1$. By Miller's dichotomy every non polynomially bounded o-minimal structure contains this as a definable set.

However we do have an existence theorem in the polynomially bounded case.

Theorem 4.5.6 (Nguyen-Valette [93]) *Every definable set in a polynomially bounded o-minimal structure admits a definable Lipschitz stratification.*

Halupczok and Yin have given another proof of this result [44].

It is clear that the (L)-conditions are much more of a constraint than is (w). Here are some simple examples showing that the two conditions are distinct.

Example 4.5.7 (Mostowski) In $\mathbf{C}^4$ or $\mathbf{R}^4$ let $Z = \{y = z = 0\} \cup \{y = x^3, z = tx\}$. Then (w) holds along the t-axis, but (L) fails.

Example 4.5.8 (Koike-Juniati [61]) In $\mathbf{R}^3$ let $Z = \{y^2 = t^2x^2 + x^3, x \geq 0\}$ and stratify by $Z = Z_2 \supset Z_1 = \langle Ot \rangle$. It is easy to check that (w) holds for this semialgebraic example, while $(L2)$ fails : let $q = q_{j_1} = q_2 = (t^2, \sqrt{2}t^3, t), q' = (t^2, -\sqrt{2}t^3, t), q_{j_2} = q_1 = (0, 0, t)$, as $t \to 0$.

4.5.1 Teissier's Criteria for a Good Equisingularity Condition

In his 1974 Arcata lectures [117] Teissier gave a list of criteria for a good equisingularity condition E on a stratification of a complex analytic set; E-regularity should in particular:

1) be as strong as possible;
2) be generic, i.e. every complex analytic set should possess an E-regular stratification;
3) imply local topological triviality along strata;
4) imply equimultiplicity;
5) be preserved after intersection with generic linear spaces containing a given stratum, locally linearised ($E \Rightarrow E^*$, see below for a precise definition).

Criteria 2) to 5) hold for Whitney (b)-regularity (see Teissier [118]), which turns out to be equivalent to (w) in the complex case as noted above. Criterion 5) is an essential part of the proof of this result via the equimultiplicity of polar varieties. (Recall that (b) does not imply (w) for real algebraic varieties by Example 4.3.4.)

Criterion 4) is a theorem of Hironaka from 1969 [54].

Theorem 4.5.9 (Hironaka) *For a complex analytic Whitney stratified variety V the pointwise multiplicity $m(V, p)$ is constant on each stratum.*

Definition 4.5.10 (E^*-regularity) Let M be a C^2 manifold. Let Y be a C^2 submanifold of M and let $y \in Y$. Let X be a C^2 submanifold of M such that $y \in \overline{X}$ and $Y \cap X = \emptyset$. Let E denote an equisingularity condition (e.g. (b), (w), (L)). Then (X, Y) is said to be $E_{\mathrm{cod}k}$*-regular* at y $(0 \leq k \leq \mathrm{cod}Y)$ if there exists an open dense subset U^k of the grassmannian of codimension k subspaces of T_yM containing T_yY, such that if W is a C^2 submanifold of M with $Y \subset W$ near y, and $T_yW \in U^k$, then W is transverse to X near y, and $(X \cap W, Y)$ is E-regular at y.

One says finally that (X, Y) is E^**-regular* at y if (X, Y) is $E_{\mathrm{cod}k}$-regular for all $k, 0 \leq k < \mathrm{cod}Y$.

Theorem 4.5.11 (Navarro Aznar-Trotman [92]) *For subanalytic stratifications, $(w) \Rightarrow (w^*)$, and if $\dim Y = 1$, $(b) \Rightarrow (b^*)$.*

The fact that (b^*)-regular stratifications exist for subanalytic sets allows one to prove that stratified Morse functions (in the sense of Goresky and MacPherson [41]) exist and are generic, using (b_{cod1}). The rapid spiral is an example of a Whitney stratified set for which no (stratified) Morse functions exist [41].

Question: Is it true that $(b) \Rightarrow (b^*)$ for subanalytic stratifications in general, i.e. when $\dim Y \geq 2$?

Theorem 4.5.12 (Teissier [118]) *For complex analytic stratifications, $(b) \Rightarrow (b^*)$.*

Theorem 4.5.13 (Juniati-Trotman-Valette [61]) *For subanalytic stratifications, $(L) \Rightarrow (L^*)$.*

According to the 1974 criteria of Teissier [117], Whitney regularity is a good equisingularity condition. Because Mostowski's Lipschitz condition (L) is stronger it may be considered better as it also satisfies Teissier's criteria.

Many results concerning E^*-regularity for different equisingularity conditions E in the complex analytic context are described in [77], including a kind of converse to the Thom-Mather local triviality Theorem 4.2.17, namely that (TT^*) implies (b), where (TT) means local topological triviality along strata.

4.6 Definable Trivialisations

We have seen that Whitney (b)-regularity ensures local topological triviality. Mostowski and Parusiński proved that an (L)-regular stratification of a subanalytic set is locally bilipschitz trivial (Theorem 4.5.4). It is natural to ask if such trivialisations can be chosen to be definable. Or specifically if Z is a semialgebraic set, is there some stratification which is locally semialgebraically trivial ? This was proved by Hardt in 1980 [48]. His method was improved by G. Valette who obtained local semialgebraic bilipschitz triviality [139, 140].

Theorem 4.6.1 (Hardt) *Semialgebraic sets admit locally semialgebraically trivial stratifications.*

Theorem 4.6.2 (Valette) *Semialgebraic sets admit locally semialgebraically bilipschitz trivial stratifications.*

There are also subanalytic versions of these results. For semialgebraic (b)-regular stratifications Coste and Shiota [18] proved a semialgebraic isotopy theorem using real spectrum methods. See the book of Shiota [112] for further details and references.

A recent (2017) very powerful theorem by Parusiński and Paunescu [102], proving the Whitney fibering conjecture of 1965 [147], produces a subanalytic trivialisation of a given stratified analytic variety (real or complex) which is moreover arc-analytic, as is its inverse. The hypothesis on the stratified set is a type of Zariski equisingularity, stronger than (w)-regularity, hence implying Whitney (b)-regularity by Proposition 4.3.2. The relation between this notion of Zariski equisingularity and Mostowski's Lipschitz condition of Definition 4.5.2 is currently being studied in the case of complex analytic varieties by Parusiński and Paunescu. See Parusiński's contribution to this handbook for details of their work.

4.7 Abstract Stratified Sets

One may begin the study of differentiable manifolds in two ways, either by starting with the abstract definition and eventually proving the existence of an embedding into euclidean space [58], or by starting with submanifolds of euclidean space

[42] so that the abstract concept is obtained by taking an equivalence class by diffeomorphisms. In a similar way there is a definition of abstract stratified set, due to Mather [81]. He developed this definition by adapting ideas of Thom [123], who gave a different definition of an abstract stratified set, so that the resulting spaces are called Thom-Mather stratified sets.

Definition 4.7.1 An *abstract stratified set* is a triple $(Z, \Sigma, \mathcal{T})$ satisfying 9 axioms:

A1) Z is a locally compact second countable Hausdorff space, hence metrisable.
A2) Σ is a partition of Z into locally closed subsets, called the strata.
A3) Each stratum is a topological manifold with a differentiable structure of class C^k.
A4) Σ is locally finite.
A5) Σ satisfies the frontier property.
A6) $\mathcal{T}$ is a triple $(\{T_X\}, \{\pi_X\}, \{\rho_X\})$ where for each $X \in \Sigma$, T_X is an open neighbourhood of X in V, $\pi_X : T_X \longrightarrow X$ is a continuous retraction of T_X onto X, and $\rho_X : T_X \longrightarrow [0, \infty)$ is a continuous function. We call T_X the *tubular neighbourhood* of X, π_X the *local retraction* of T_X onto X, and ρ_X the *tubular function* of X.
A7) $X = \{v \in T_X : \rho_X(v) = 0\}$.

Notation For strata X, Y, let $T_{X,Y} = T_X \cap Y$, let $\pi_{X,Y} = \pi_X|_{T_{X,Y}} : T_{X,Y} \longrightarrow X$ and let $\rho_{X,Y} = \rho_X|_{T_{X,Y}} : T_{X,Y} \longrightarrow (0, \infty)$.

A8) For each pair of strata X, Y, $(\pi_{X,Y}, \rho_{X,Y}) : T_{X,Y} \longrightarrow X \times (0, \infty)$ is a C^k submersion (hence $\dim X < \dim Y$ if $T_{X,Y} \neq \emptyset$).
A9) For strata $W < X < Y$ we have $\pi_{W,X} \circ \pi_{X,Y}(z) = \pi_{W,Y}(z)$, and $\rho_{W,X} \circ \pi_{X,Y}(z) = \rho_{W,Y}(z)$. These are called *control conditions* .

Such abstract stratified sets are triangulable, as shown by Goresky [38] and by Verona [143].

Mather's proof in [81] of the first isotopy lemma of Thom for stratified submersions on Whitney stratified sets (Theorem 4.2.18 above) uses Mather's result that every Whitney stratified subset of a manifold admits the structure of an abstract stratified set. He then proves the isotopy lemma in the abstract context.

It is then natural to ask about an embedding theorem for abstract stratified sets, similar to the embedding theorem for smooth manifolds. Teufel [119] and Natsume [91] proved that every abstract stratified set of dimension n can be embedded in $\mathbf{R}^{2n+1}$ as a Whitney stratified set. Noirel [97] improved their statements by showing that the resulting Whitney stratified set may be made subanalytic as may the induced local retractions and tubular functions. Also he showed that the embedded stratification may be made (w)-regular (hence also (b)-regular by the subanalytic version of Proposition 4.3.2). Moreover the embedded set and the induced control data can be made semialgebraic if the set is compact [97].

Note that in the C^∞ category (w)-regular stratified sets do not in general admit the structure of a Thom-Mather abstract stratified set because they are not always locally conical as shown by Example 4.3.7. However they are locally topologically trivial as shown directly by Verdier using integration of rugose stratified vector fields [142].

Much work has been done generalising the differential properties of smooth manifolds to abstract stratified sets in the above sense. See Sect. 4.11 below for some references.

4.8 K. Bekka's (*c*)-Regularity

It can be important to be more precise as to when a stratification is locally topologically trivial in the sense of Theorem 4.2.17, for example when classifying topologically or when studying topological stability (cf. work of Damon, Looijenga, Wirthmüller and the book of du Plessis and Wall [107]). Then one needs the weakest regularity condition on a stratification ensuring local topological triviality. This principle led to the introduction of the following condition.

Definition 4.8.1 (K. Bekka) A stratified set (Z, Σ) in a manifold M is (c)-regular if for every stratum Y of Σ there exists an open neighbourhood U_Y of Y in M and a C^1 function $\rho_Y : U_Y \to [0, \infty)$ such that ${\rho_Y}^{-1}(0) = Y$ and the restriction $\rho_Y|_{U_Y \cap \mathrm{Star}(Y)}$ is a Thom map, where $\mathrm{Star}(Y) = \bigcup\{X \in \Sigma | X \geq Y\}$, i.e. $\forall X \in \mathrm{Star}(Y)$, with $\rho_{XY} = \rho_Y|_X$ and $x \in X$,

$$\lim_{x\to y} T_x({\rho_{XY}}^{-1}(\rho_Y(x))) \supseteq T_yY \quad \forall y \in Y.$$

Note that $\rho_Y : U_Y \to [0, \infty)$ is defined globally on a neighbourhood of Y. So this is not a local condition. Local (c)-regularity is developed and used by Schürmann [109].

Theorem 4.8.2 (Bekka [4]) *(c)-regular stratifications are locally topologically trivial along strata.*

The proof is by proving the existence of an abstract stratified structure of Thom-Mather which allows the use of Mather's theory of controlled stratified vector fields [81] and implies that the conclusions of Theorems 4.2.17 and 4.2.18 are satisfied. If one only requires constance of homological or cohomological data then one can weaken (c) even further—see chapter 4 of the book of Schürmann [109].

Characterisations of condition (c) are given by Bekka and Koike in [5].

We saw how (w) and (L) are characterised by the existence of appropriate lifts of vector fields. Here is the corresponding result for (c)-regularity.

Theorem 4.8.3 (du Plessis-Bekka [106]) *A stratification is (c)-regular $\Leftrightarrow$ every C^1 vector field on a stratum Y admits a continuous controlled stratified extension to a neighbourhood of Y.*

This means that there exists a family of vector fields $\{v_X | X \in Star(Y)\}$ such that $v = \bigcup v_X$ is continuous (in TM), while being controlled as defined above.

How do (b) and (c) compare ?

I proved [131] that (b) over a stratum Y is equivalent to the property that for every C^1 tubular neighbourhood T_Y of Y the restriction to neighbouring strata of the associated map (π_Y, ρ_Y) is a submersion, where $\pi_Y : T_Y \to Y$ is the canonical retraction and $\rho_Y : T_Y \to [0, 1)$ the canonical distance function (see Theorem 4.2.9).

In comparison, (c) says that there exists *some* C^1 function ρ vanishing on Y (not necessarily associated to a tubular neighbourhood: ρ can be degenerate, e.g. weighted homogeneous, or even flat on Y) such that for every C^1 tubular neighbourhood T_Y of Y the restriction to neighbouring strata of the map (π_Y, ρ) is a submersion [4].

One can prove easily that (b) implies (c) while there are examples showing that the converse is false [4]. See [6] for real algebraic examples. There are complex algebraic examples due to Briançon and Speder [14]: these consist of 1-parameter families of complex hypersurfaces with isolated singularities defined by $F : \mathbf{C}^3 \times \mathbf{C}, 0 \times \mathbf{C} \to \mathbf{C}, 0$ such that $(F^{-1}(0), 0 \times \mathbf{C})$ is (c)-regular (because weighted homogeneous) but not (b)-regular. It is unknown whether topologically trivial complex analytic stratifications are always (c)-regular, or even whether they are (a)-regular (a question of Thom).

Several authors have used (c)-regularity as a means of providing sufficient conditions for the existence of a real Milnor fibration associated to a real analytic map [108, for example].

A recent theorem of Murolo, du Plessis and Trotman [89] states that for Whitney (b)-regular or Bekka (c)-regular stratified sets the Thom-Mather isotopy theorem can be improved so as to provide a smooth form of the Whitney fibering conjecture. One can ensure that the fibres of the trivialising homeomorphism h in the Thom-Mather isotopy Theorem 4.2.17 (or Theorem 4.5.4) for fixed points of $c(L)$ have continuously varying tangent spaces as one goes to the base stratum X, or changes stratum in the star of X. Moreover the associated wings obtained by fixing a point on the link L can be made (c)-regular.

Sandwiched between Whitney (b)-regularity and Bekka's (c)-regularity there is a condition known as weak Whitney regularity. For a pair of adjacent strata (X, Y) we assume that for some choice of local coordinates at a point $y_0 \in Y$ the angle θ between secant lines and the tangent space to X is bounded away from $\pi/2 : \exists \delta > 0$ such that

$$\theta(\overline{xy}, T_x X) < \pi/2 - \delta$$

for all $x \in X$ in some neighbourhood U of y_0.

We call this condition (δ) and the combined condition $(a + \delta)$ (when both (a) and (δ) hold) is known as weak Whitney regularity . The proof that weak Whitney regularity implies (c) (for a standard tubular function ρ_Y associated to a tubular neighbourhood) is in [6]. Real algebraic examples exist showing that the converse is not true. No complex examples are known.

It is a curious fact that weak Whitney regularity for a family of complex hypersurfaces with isolated singularities implies equimultiplicity [136], generalising Hironaka's theorem in this case [54]. It is unknown whether topologically trivial families are equimultiple (a parametrised version of the famous Zariski problem [150] concerning topological invariance of the multiplicity of an isolated complex hypersurface singularity). The examples of Briançon and Speder [14] of μ-constant families of hypersurfaces which are not Whitney regular turn out to be weakly Whitney regular—see [8] and the correction [9]. One can then ask whether μ-constant families of hypersurfaces are always weakly Whitney regular. This would imply topological triviality via (c)-regularity and Bekka's Theorem 4.8.2 [4] that (c) implies topological triviality, and thus extend the Lê-Ramanujam theorem (which uses the h-cobordism theorem) to the missing surface case [75].

We note that weakly Whitney stratified sets in general have similar metric properties to Whitney stratified sets - they are of finite geodesic diameter if compact for example [7]. Also weakly Whitney stratified sets with a smooth singular set of codimension 1 have finite volume. This is not true in general if the singular set has codimension 2 or if the depth is at least 2 [29].

4.9 Condition (t^k)

We return to the first example of Whitney, $Z = \{y^2 = t^2x^2 + x^3\}$. Slice the surface by a plane S transverse to the t-axis at 0. Then the topological type of the germ at 0 of the intersection $Z \cap S$ is constant, i.e. independent of S. Remember that Whitney (a) holds. Thom noticed this and mentioned it to Kuo, who proved the following theorem [68].

Theorem 4.9.1 (Kuo 1978) *If (X, Y) is (a)-regular at $y \in Y$ then (h^∞) holds, i.e. the germs at y of intersections $S \cap X$, where S is a C^∞ submanifold transverse to Y at $y \in S \cap Y$ and $dim S + dim Y = dim M$, are homeomorphic.*

It later turned out [132] that one can replace (h^∞) by (h^1), meaning one considers all C^1 transversals S, and weaken (a) to (t^1), defined as follows.

Definition 4.9.2 A pair of strata (X, Y) is (t^k)-regular at $y \in Y$ if for every C^k submanifold S transverse to Y at $y \in Y \cap S$, there is a neighbourhood U of y such that S is transverse to X on $U \cap X$ $(1 \leq k \leq \infty)$.

Clearly (a) implies (t^1). The converse does not hold as first shown in [127, 129]. The converse does hold in the subanalytic case if we allow transversals of arbitrary dimension [132]. In the case of transversals of complementary dimension there are semialgebraic examples with (t^1) but not (a) [132], and there are even complex algebraic examples [34].

Theorem 4.9.3 (Trotman [132]) *If we restrict to transversals of complementary dimension to* Y, (t^1) *is equivalent to* (h^1).

Theorem 4.9.4 (Trotman-Wilson [137]) *For subanalytic strata,* (t^k) *is equivalent to the finiteness of the number of topological types of germs at* y *of* $S \cap X$ *for* S *a* C^k *transversal to* Y $(k \geq 2)$ *of complementary dimension.*

The proofs that I developed with Kuo and with Wilson use the "Grassmann blowup" introduced by Kuo and myself [71]. Let

$$E^{n,d} = \{(L, x) | x \in L\} \subset G^{n,d} \times \mathbf{R}^n$$

for $d < n$, with projection to $G^{n,d}$, denote the canonical d-plane bundle. Let $\beta = \beta_{n,d}$ denote projection to $\mathbf{R}^n$. When $d = 1$ this is the usual blowup of $\mathbf{R}^n$ with centre 0.

Suppose $X, Y \subset \mathbf{R}^n$ and $0 \in Y$ with $d = \operatorname{codim} Y$.

Let $\tilde{X} = \beta^{-1}(X)$ and let $\tilde{Y} = \{(L, 0) | L$ is transverse to Y at $0\}$. The following striking theorem results from work by Kuo and myself [71], completed by work with Wilson [137].

Theorem 4.9.5 (X, Y) *is* (t^k)*-regular at* $0 \in Y$ *if and only if* $(\tilde{X}, \tilde{Y})$ *is* (t^{k-1})*-regular at every point of* $\tilde{Y}$ $(k \geq 1)$.

When $k = 1$, (t^0) is equated with (w), the Kuo-Verdier condition of Definition 4.3.1. So in particular, (w)-regularity is the first in an infinite sequence of (t^k)-regularity conditions !

Now we can see how to prove that (t^1) implies (h^1) by using the Verdier isotopy Theorem 4.3.6 for (w)-regular stratifications in the Grassmann blowup, although this was not the original proof.

The (t^k) conditions were used to characterise jet sufficiency by Trotman and Wilson, generalising theorems of Bochnak, Kuo, Lu and others, and realising part of the early programme of Thom (1964). See [137] for details. Work with Gaffney and Wilson [34] developed an algebraic approach to the (t^k) conditions, using integral closure of modules.

To illustrate the difference between (t^2) and (t^1), and the previous theorem, look at the Koike-Kucharz example [65] given by $Z = \{x^3 - 3xy^5 + ty^6 = 0\} \subset \mathbf{R}^3$ stratified as usual by (X, Y) with Y the t-axis and X its complement $Z - Y$. Then (X, Y) is (t^2) but not (t^1) at 0. It is easy to check that there are 2 topological types of germs at 0 of intersections $S \cap X$ where S is a C^2 submanifold transverse to Y at 0. However the number of topological types of such germs for S of class C^1 is infinite, even uncountable. It is easy to construct similar examples showing (t^k) does not imply (t^{k-1}).

This example arose from the discovery independently by S. Koike and W. Kucharz that the 6-jet $x^3 - 3xy^5$ has infinitely many topological types among its representatives of class C^7, but only finitely many (in fact two) among its representatives of class C^8. Such an example contradicts a conjecture of Thom from [123]. The relation of these properties of jets with stratification theory and the conditions (t^1) and (t^2) was pointed out by Kuo and Lu [69].

On a historical note, condition (t) with no specification on the differentiability of the transversals was first introduced by Thom in 1964 [121], before the appearance of Whitney's conditions (a) and (b). Thom claimed that (t) implies the openness of the set of maps transverse to a stratification [121, 123]. This is true in the semialgebraic case because then (t) implies (a) and one can use Theorem 4.2.12, but is false for C^∞ stratified sets, again using Theorem 4.2.12 and examples with (t) but not (a) [127].

4.10 Density and Normal Cones

We saw above Hironaka's Theorem 4.5.9 (from [54]) that complex analytic Whitney stratifications are equimultiple along strata. What is a real version of this statement ?

The multiplicity $m(V, p)$ at a point p of a complex analytic variety V is the number of points near p in the intersection of V with a generic plane L missing p, of complementary dimension to that of V. This positive integer is equal to the Lelong number, or density $\theta(V, p)$ of V at p defined as the limit as ϵ tends to 0 of the quotient $\frac{\mathrm{vol}(V \cap B_\epsilon(p))}{\mathrm{vol}(P \cap B_\epsilon(p))}$ where P is a plane containing p of the same dimension as V. Kurdyka and Raby showed that the density is well-defined for subanalytic sets, as a positive real number [72]. It is thus natural to conjecture (I did so in 1988) that the density of a subanalytic set is continuous along strata of a subanalytic Whitney stratification, as a generalisation of Hironaka's theorem to the real case. This was partially proved by G. Comte in his thesis (1998) for subanalytic (w)-regular stratifications [17], and more generally for subanalytic (b^*)-regular stratifications. The general conjecture was proved for subanalytic (b)-regular stratifications by G. Valette in 2008 [141]. Valette also showed that the density is a Lipschitz function along strata of a subanalytic (w)-regular stratification. Analogous theorems for the continuity (resp. Lipschitz variation) of Lipschitz-Killing invariants along strata of a definable Whitney (resp. (w)-regular) stratification were proved by Nguyen and Valette in 2018 [94].

For a long time it was thought that Whitney regularity might impose restrictions on the space of limits of tangents to a stratified set. In the case of an isolated singularity this was shown not to be the case by a construction of Kwiecinski and Trotman proving that any continuum (compact connected set) can be realised as the tangent cone or Nash fibre of a Whitney (b)-regular stratified set at an isolated singular point [73].

In the paper [54] about equimultiplicity, Hironaka proved results about the *normal cones* of complex analytic Whitney stratifications that one can generalise to the subanalytic case as follows. Suppose Z is a stratified subset of $\mathbf{R}^n$ and let Y be a stratum. Let π_Y be the projection of a tubular neighbourhood of Y and let

$\mu(v) = \frac{v}{\|v\|}$. The normal cone is defined to be:

$$C_Y Z = \overline{\{(x, \mu(x\pi_Y(x)))|x \in Z - Y\}|_Y} \subset \mathbf{R^n} \times S^{n-1}.$$

Let $p : C_Y Z \to Y$ be the canonical projection.

Theorem 4.10.1 *A* (b)*-regular subanalytic stratification of a subanalytic set is*

(npf) normally pseudo-flat, i.e. p is an open map, and

(n) for each stratum Y and each point y of Y, the fibre $(C_Y Z)_y$ *of the normal cone at y is equal to the tangent cone* $C_y(Z_y)$ *at y to the special fibre* ${\pi_Y}^{-1}(y)$*.*

The proofs are by integration of vector fields [52, 54, 98].

The result is not true for definable sets in non-polynomially bounded o-minimal structures, as shown by the following examples, together with Miller's dichotomy that an o-minimal structure is polynomially bounded if and only if it does not contain the exponential function as a definable function [84].

Example 4.10.2 Take Z in $\mathbf{R^3}$ to be the graph of the function $f : [0, \infty) \times \mathbf{R} \to \mathbf{R}$ defined by

$$z = f(x, y) = x - \frac{x}{ln(x)} ln(y + (x^2 + y^2)^{\frac{1}{2}}).$$

Stratify Z by $Z_1 = \{0y\} \subset Z$. One checks easily that $(C_Y Z)_0$ is an arc, while $C_0(Z_0)$ is a point so that the criterion (n) above fails. Moreover the example is not normally pseudoflat, nor (b^*)-regular, but it is Whitney (b)-regular (see [138] and [135]).

Example 4.10.3 Consider the closure of the graph in $\mathbf{R}^3$ of the function

$$g(x, y) = y^{x^2+1}$$

defined on $\mathbf{R} \times (0, \infty]$. This is an example of Pawłucki of a definable stratified set which is (b)-regular but not a C^1 manifold with boundary [103]. It is not normally pseudoflat. Also the three dimensional stratified set defined by the span of this graph and the plane $\{z = 0\}$ provides the first example of a definable Whitney stratified set for which the density is not continuous along a stratum. For details see [135].

In [98] real algebraic (a)-regular examples are given showing that (n) does not imply (npf) and conversely.

Example 4.10.4 First let $(0z) = Z_1 \subset Z = \{x(x^2+y^2)z^2-(x^2+y^2)^2+xy^2 = 0\}$. Then (a) and (n) hold but (npf) fails.

Example 4.10.5 Finally look again at $\{y^2 = t^2x^2 + x^3\}$, stratified by the t-axis and its complement. Here (n) fails, because $(C_Y Z)_0$ consists of 2 points while $C_0(Z_0)$ consists of 1 point, but it is normally pseudoflat.

4.11 Algebraic Topology of Stratified Spaces

Because stratified sets are a generalisation of smooth manifolds to singular spaces it is natural to study the analogues of the highly developed theories concerning the algebraic and differential topology of manifolds.

For example Morse theory has been generalised to stratified Morse theory by Goresky and MacPherson [41]. Not all Whitney stratified sets admit Morse functions in their sense, however Morse functions (exist and) are dense on subanalytic sets. See the contribution of Mark Goresky to this handbook for an account of the current state of stratified Morse theory. Also Poincaré duality is a fundamental property of compact smooth manifolds. To provide a suitable generalisation of this duality Goresky and MacPherson developed intersection homology for stratified spaces in 1980 [40].

In his 1976 thesis Goresky developed a geometric theory for homology and cohomology carried by Whitney stratified chains and cochains [39]. He proved that the homology of a compact smooth manifold can be represented by Whitney stratified cycles, and that the cohomology of a compact Whitney stratified set can be represented by Whitney stratified cocycles. Murolo [87] showed how to obtain an isomorphism between the homologies and cohomologies.

A basic theorem in smooth manifold theory is the Poincaré-Hopf theorem equating the Euler characteristic of a compact manifold, possibly with boundary, with the total index of a vector field with isolated zeros. For stratified vector fields on a Whitney stratified set one has to impose restrictions on the vector field, for example to be radial, i.e. exiting from a family of tubes around each stratum, as first defined by M.-H. Schwartz [110]. She used in fact the stronger (w)-regular stratifications in the case of real analytic manifolds with boundary. More general theorems are due to Simon [116] for radial vector fields on (c)-regular stratified sets and to King and Trotman [64] who allow more general stratified sets (including closure orderable subanalytic partitions of a given subanalytic set) and more general vector fields : semi-radial vector fields (which never point orthogonally into a tube) and even arbitrary (generic) vector fields by introducing a notion of virtual index. The very large quantity of results concerning index theorems and Chern classes for singular real and complex analytic varieties up to 2009, almost always using Whitney stratifications, is described by Brasselet, Seade and Suwa in their book [13].

There are versions of the De Rham theorem for stratified spaces and intersection cohomology due to several authors, including Brasselet, Hector and Saralegui [11], also Brasselet and Legrand [12]. Extensive work on the signature of compact stratified pseudomanifolds is due to Albin, Leichtnam, Mazzeo and Piazza [1], related to Melrose's iterated fibration construction. These references are mere examples in a large body of literature.

The study of the topology of Whitney stratified sets is very much alive. Recent work includes a study of their combinatorial properties by Ehrenborg, Goresky and Readdy [27], and a stratum-sensitive approach to homotopy theory in Woolf's transversal homotopy theory [149]. The precise relation of Whitney stratified sets and Thom maps to the deep work of Ayala, Francis, Tanaka and Rozenblyum on local properties of a new class of conically smooth stratified spaces is currently conjectural [2, 3].

4.12 Real World Applications

In so far as (a)-regular stratification is essential in the proof that the space of smooth functions corresponding to the elementary catastrophes is an open set (by Theorem 4.2.12 [126, 130]), so that the properties of the functions are stable, there are hundreds of very varied applications of Whitney stratifications in papers on applications of catastrophe theory to physics (e.g. gravitational lensing), engineering, ship design, economics, urban geography, paleontology, psychology, biology, etc.

Canny used Whitney stratifications to define roadmaps (curves connecting two points in a semialgebraic set) in his prize-winning work on finding simple exponential algorithms for the generalised piano-mover's problem [16] in theoretical robotics. He uses a general position trick to avoid using the doubly exponential algorithm constructing a Whitney stratification of a given algebraic set [86].

More recently Damon, Giblin and Haslinger and Damon with Gasparovic have used extensively Whitney stratifications in their work on the mathematics of natural images and on skeletal structures [20, 21].

References

1. Albin, P., Leichtnam, E., Mazzeo, R., Piazza, P.: The signature package on Witt spaces, Ann. Sci. École. Norm. Supér. **(4) 45** (2012), no. 2, 241–310. 266
2. Ayala, D., Francis, J., Rozenblyum, N.: A stratified homotopy hypothesis, J. Eur. Math. Soc. **21** (2019), no. 4, 1071–1178. 267
3. Ayala, D., Francis, J., Tanaka, H.L.: Local structures on stratified spaces, Adv. Math. **307** (2017), 903–1028. 267
4. Bekka, K.: C-régularité et trivialité topologique, in *Singularity theory and its applications, Warwick 1989, Part I*, Lecture Notes in Math. **1462**, Springer, Berlin, 1991, 42–62. 260, 261, and 262
5. Bekka, K., Koike, S.: The Kuo condition, an inequality of Thom's type and (C)-regularity, Topology **37** (1998), no. 1, 45–62. 260
6. Bekka, K., Trotman, D.: Weakly Whitney stratified sets, in *Real and complex singularities* (Proceedings, Sao Carlos 1998, edited by J.W. Bruce and F. Tari), Chapman and Hall/CRC (2000), 1–15. 261 and 262
7. Bekka, K., Trotman, D.: On metric properties of stratified sets, Manuscripta Math. **111** (2003), no. 1, 71–95. 262
8. Bekka, K., Trotman, D.: Briançon-Speder examples and the failure of weak Whitney regularity. J. Singul. **7** (2013), 88–107. 262

9. Bekka, K., Trotman, D.: Corrigendum: Briançon-Speder examples and the failure of weak Whitney regularity, J. Singul. **17** (2018), 214–215. 262
10. Bierstone, E.: *The structure of orbit spaces and the singularities of equivariant mappings*, Monografías de Matemática **35**, Instituto de Matemática Pura e Aplicada, Rio de Janeiro, 1980. 247
11. Brasselet, J.-P., Hector, G., Saralegi, M.: Théorème de de Rham pour les variétés stratifiées, Ann. Global Anal. Geom. **9** (1991), no. 3, 211–243. 266
12. Brasselet, J.-P., Legrand, A.: Un complexe de formes différentielles à croissance bornée sur une variété stratifiée, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **21** (1994), no. 2, 213–234. 266
13. Brasselet, J.-P., Seade, J., Suwa, T.: *Vector fields on singular varieties*, Lecture Notes in Mathematics **1987**, Springer-Verlag, Berlin, 2009. 266
14. Briançon, J., Speder, J.-P.: La trivialité topologique n'implique pas les conditions de Whitney, C. R. Acad. Sci. Sér. A-B **280** (1975), no. 6, 1365–1367. 261 and 262
15. Brodersen, H., Trotman, D.: Whitney (b)-regularity is strictly weaker than Kuo's ratio test for real algebraic stratifications, Math. Scand., **45** (1979), 27–34. 252
16. Canny, J. F.: *The complexity of robot motion planning*. ACM Doctoral Dissertation Awards 1987, MIT Press, Cambridge, MA,1988. 267
17. Comte, G.: Équisingularité réelle : nombres de Lelong et images polaires, Ann. Sci. École Norm. Sup., Paris **(4) 33** (2000), 757–788. 264
18. Coste, M., Shiota, M.: Thom's first isotopy lemma: a semialgebraic version, with uniform bound, in *Real analytic and algebraic geometry (Trento, 1992)*, de Gruyter, Berlin, 1995, 83–101. 250 and 258
19. Czapla, M.: Definable triangualtions with regularity conditions, Geometry and Topology **16** (2012), no. 4, 2067–2095. 249
20. Damon, J., Gasparovic, E.: *Medial/skeletal linking structures for multi-region configurations*, Mem. Amer. Math. Soc. **250** (2017), no. 1193. 267
21. Damon, J., Giblin P., Haslinger, G.: *Local Features in Natural Images via Singularity Theory*, Lecture Notes in Mathematics **2165**, Springer, 2016. 267
22. Denkowska, Z., Stasica, J.: *Ensembles sous-analytiques à la polonaise*, Hermann, Paris, 2007. 247
23. Denkowska, Z., Wachta, K.: Une construction de la stratification sousanalytique avec la condition (w), Bull. Polish Acad. Sci. Math. **35** (1987), no. 7–8, 401–405. 252
24. Denkowska, Z., Wachta, K., Stasica, J.: Stratification des ensembles sous-analytiques avec les propriétés (A) et (B) de Whitney, Univ. Iagel. Acta Math. **25** (1985), 183–188. 247
25. van den Dries, L.: *Tame Topology and o-minimal structures*, Cambridge University Press, London Mathematical Society Lecture Note Series **248**, 1998. 247
26. van den Dries, L., Miller, C.: Geometric categories and o-minimal structures, Duke Math. Journal **84** (1996), 497–540. 247
27. Ehrenborg, R., Goresky, M., Readdy, M.: Euler flag enumeration of Whitney stratified spaces. Adv. Math. **268** (2015), 85–128. 266
28. Feldman, E.: Geometry of immersions I, Trans. Amer. Math. Soc. **120** (1965), 185–224. 248
29. Ferrarotti, M.: Volume on stratified sets, Ann. Mat. Pura Appl. (4) **144** (1986), 183–201. 262
30. Fukuda, T.: Types topologiques des polynômes, Inst. Hautes Études Sci. Publ. Math. **46** (1976), 87–106. 250
31. Fukui, T., Paunescu, L.: Stratification theory from the weighted point of view, Canadian J. Math. **53** (2001), 73–97. 253
32. Gaffney, T.: Integral closure of modules and Whitney equisingularity, Invent. Math. **107** (1992), no. 2, 301–322. 247
33. Gaffney, T., Kleiman, S.: W_f and integral dependence, in *Real and complex singularities* (SÂÑo Carlos, 1998), Chapman and Hall/CRC Res. Notes Math., **412**, Boca Raton, FL, (2000), 33–45. 254

34. Gaffney, T., Trotman, D., Wilson, L.: Equisingularity of sections, (t^r) condition, and the integral closure of modules, J. Alg. Geom. **18** (2009), no. 4, 651–689. 262 and 263
35. Giacomoni, J.: On the stratification by orbit types II, arXiv:1704.06121, 2017. 247
36. Gibson, C. G., Wirthmüller, K., du Plessis, A. A., Looijenga, E. J. N.: *Topological stability of smooth mappings*, Lecture Notes in Math. **552**, Springer-Verlag, 1976. 244, 249, and 251
37. Giles Flores, A., Teissier, B.: Local polar varieties in the geometric study of singularities, Annales de la Faculté des Sciences de Toulouse **27**, no. 4 (2018), 679–775. 247
38. Goresky, M.: Triangulation of stratified objects, Proc. A.M.S. **72** (1978), no. 1, 193–200. 249 and 259
39. Goresky, M.: Whitney stratified chains and cochains,. Trans. Amer. Math. Soc. **267** (1981), no. 1, 175–196. 249, 254, and 266
40. Goresky, M., MacPherson, R.: Intersection homology theory, Topology **19** (1980), 135–162. 266
41. Goresky, M., MacPherson, R.: *Stratified Morse theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), **14**, Springer-Verlag, Berlin, 1988. 250, 257, and 266
42. Guillemin, V., Pollack, A.: *Differential topology*, Prentice Hall, 1974. 259
43. Hajto, Z.: On the equivalence of Whitney (a)-regularity and (a_s)-regularity, Univ. Iagel. Acta Math. **25** (1985), 299–303. 246
44. Halupczok, I., Yin, Y.: Lipschitz stratifications in power-bounded o-minimal fields, J. Eur. Math. Soc. **20** (2018), no. 11, 2717–2767. 247 and 256
45. Hamm, H.: On stratified Morse theory, Topology **38** (1999), no. 2, 427–438. 250
46. Hardt, R.: Stratification of real analytic mappings and images, Inventiones Math. **28** (1975), 193–208. 247 and 253
47. Hardt, R.: Triangulation of subanalytic sets and proper light subanalytic maps, Inventiones Math. **38** (1976/77), no. 3, 207–217. 249
48. Hardt, R.: Semi-algebraic local-triviality in semi-algebraic mappings, Amer. J. Math. **102** (1980), no. 2, 291–302. 258
49. Hardt, R., Sullivan, D.: Variation of the Green function on Riemann surfaces and Whitney's holomorphic stratification conjecture, I.H.E.S. Publ. Math. **68** (1988), 115–137. 244
50. Henry, J.-P., Merle, M.: Limites de normales, conditions de Whitney et éclatement d'Hironaka, in Singularities, Proc. Sympos. Pure Math. **40** (Amer. Math. Soc, Providence, R.I., 1983), Part 1, 575–584. 252
51. Henry, J.-P., Merle, M.: Conditions de régularité et éclatements, Ann. Inst. Fourier (Grenoble) **37** (1987), no. 3, 159–190. 254
52. Henry, J.-P., Merle, M.: Stratifications de Whitney d'un ensemble sous-analytique, C. R. Acad. Sci. Paris, série I, **t. 308** (1989), 357–360. 265
53. Henry, J.-P., Merle, M., Sabbah, C.: Sur la ciondition de Thom stricte pour un morphisme analytique complexe, Ann. Ecole Norm. Sup. **(4) 17** (1984), no. 2, 227–268. 254
54. Hironaka, H.: Normal cones in analytic Whitney stratifications, Inst. Hautes Études Sci. Publ. Math. **36** (1969), 127–138. 257, 262, 264, and 265
55. Hironaka, H.: Subanalytic sets, in *Number theory, algebraic geometry and commutative algebra, in honor of Yasuo Akizuki*, Kinokuniya, Tokyo (1973), 453–493. 247
56. Hironaka, H.: Triangulations of algebraic sets, in *Algebraic geometry* (Proc. Sympos. Pure Math. Vol **29**, Humboldt State Univ., Arcata, Calif., 1974), Amer. Math. Soc., Providence (1975), 165–185. 249
57. Hironaka, H.: Stratifications and flatness, in *Real and complex singularities*,(Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), Sijthoff and Noordhoff, Alphen aan den Rijn (1977), 199–265. 254
58. Hirsch, M.: *Differential topology*, Springer, 1976. 258
59. Juniati, D.: *De la régularité Lipschitz des espaces stratifiés*, University of Provence thesis, 2002. 252
60. Juniati, D., Noirel, L., Trotman, D.: Whitney, Kuo-Verdier and Lipschitz stratifications for the surfaces $y^a = z^b x^c + x^d$, Topology Appl. **234** (2018), 335–347. 252

61. Juniati, D., Trotman, D., Valette, G.: Lipschitz stratifications and generic wings, Journal of the London Math. Soc. **(2) 68** (2003), 133–147. 256 and 257
62. Kambouchner, A., Trotman, D.: Whitney (a)-faults which are hard to detect, Ann. Sci. École Norm. Sup. (4) **12** (1979), no. 4, 465–471. 246
63. Kashiwara, M., Schapira, P.: *Sheaves on manifolds*, Springer-Verlag, Berlin, 1990. 252
64. King, H., Trotman, D.: Poincaré-Hopf theorems on singular spaces, Proc. Lond. Math. Soc. **(3) 108** (2014), no. 3, 682–703. 266
65. Koike, S., Kucharz, W.: Sur les réalisations de jets non suffisants, C. R. Acad. Sci. Paris Sér. A-B **288** (1979), no. 8, A457-A459. 263
66. Kuo, T.-C.: The ratio test for analytic Whitney stratifications, in *Proceedings of Liverpool Singularities - Symposium I* (C.T.C. Wall, ed.), Springer Lecture Notes in Math. **192** (1971), 141–149. 251
67. Kuo, T.-C.: Characterizations of v-sufficiency of jets, Topology **11** (1972), 115–131. 251
68. Kuo, T.-C.: On Thom-Whitney stratification theory, Math. Ann. **234** (1978), no. 2, 97–107. 262
69. Kuo, T.-C., Lu, Y.-C.: Sufficiency of jets via stratification theory, Inventiones Math. **57** (1980), no. 3, 219–226. 264
70. Kuo, T.-C., Li, P.-X., Trotman, D. J. A.: Blowing-up and Whitney (a)-regularity, Canad. Math. Bull. **32** (1989), no. 4, 482–485. 248
71. Kuo, T.-C., Trotman, D. J. A.: On (w) and (t^s)-regular stratifications, Inventiones Mathematicae **92** (1988), 633–643. 263
72. Kurdyka, K., Raby, G.: Densité des ensembles sous-analytiques, Ann. Inst. Fourier **39** (1989), 735–771. 264
73. Kwieciński, M., Trotman, D.: Scribbling continua in $\mathbf{R}^n$ and constructing singularities with prescribed Nash fibre and tangent cone, Topology and its Applications **64** (1995), 177–189. 264
74. Laudenbach, F.: *Transversalité, courants et théorie de Morse*, Éditions de l'École Polytechnique, Palaiseau, 2012. 247
75. Lê D. T., Ramanujam, C. P.: The invariance of Milnor's number implies the invariance of the topological type, Amer. J. Math. **98** (1976), no. 1, 67–78. 262
76. Lê D. T., Saito, K.: La constance du nombre de Milnor donne des bonnes stratifications, C. R. Acad. Sci. Paris **277** (1973), 793–795. 254
77. Lê D.T., Teissier, B.: Cycles évanescents, sections planes et conditions de Whitney II, in *Singularities*, Part 2 (Arcata, Calif., 1981), Proc. Sympos. Pure Math. **40**, Amer. Math. Soc., Providence, RI. (1983), 65–103. 258
78. Loi, T. L.: Verdier and strict Thom stratifications in o-minimal structures, Illinois J. Math. **42** (1998), 347–356. 247, 251, and 252
79. Łojasiewicz, S.: *Ensembles semi-analytiques*, I.H.E.S. notes, 1965. 247
80. Łojasiewicz, S., Stasica, J., Wachta, K.: Stratifications sous-analytiques. Condition de Verdier, Bull. Polish Acad. Sci. Math. **34** (1986), no. 9–10, 531–539. 252
81. Mather, J.: Notes on topological stability, Harvard University, 1970 and Bull. Amer. Math. Soc. N.S. **49** (2012), no. 4, 475–506. 244, 250, 251, 253, 254, 259, and 260
82. Mather, J. N.: Stratifications and mappings, in *Dynamical Systems* (M. M. Peixoto, ed.), Academic Press, N. Y. (1973), 195–232. 251 and 254
83. Mather, J. N.: How to stratify mappings and jet spaces, in *Singularités d'Applications Différentiables, Plans-sur-Bex 1975*, Lecture Notes in Mathematics **535**, Sprenger-Verlag, Berlin (1976), 128–176. 251 and 254
84. Miller, C.: Exponentiation is hard to avoid, Proc. Amer. Math. Soc. **122** (1994), 257–259. 265
85. Mostowski, T.: *Lipschitz equisingularity*, Dissertationes Math. (Rozprawy Mat.) **243** (1985), 46 pp. 254 and 256
86. Mostowski, T., Rannou, E.: Complexity of the computation of the canonical Whitney stratification of an algebraic set in $\mathbf{C}^n$, in *Applied algebra, algebraic algorithms and error-correcting codes (New Orleans, LA, 1991)*, Lecture Notes in Comput. Sci., **539**, Springer, Berlin, (1991), 281–291, 267

87. Murolo, C.: Whitney homology, cohomology and Steenrod squares, Ricerche di Matematica **43** (1994), 175–204. 266
88. Murolo, C.: Stratified submersions and condition (D), J. Singul. **13** (2015), 179–204. 254
89. Murolo, C., du Plessis, A. A., Trotman, D. J. A.: *On the smooth Whitney fibering conjecture*, preprint, 2017. 251 and 261
90. Nakai, I.: Elementary topology of stratified mappings, in *Singularities - Sapporo 1998*, Advanced Studies in Pure Mathematics **29** (2000), 221–243. 254
91. Natsume, H.: The realization of abstract stratified sets, Kodai Math. J. **3** (1980), no. 1, 1–7. 259
92. Navarro Aznar, V., Trotman, D. J. A.: Whitney regularity and generic wings, Annales Inst. Fourier, Grenoble **31** (1981), 87–111. 257
93. Nguyen, N., Valette, G.: Lipschitz stratifications in o-minimal structures, Ann. Sci. Ecole Norm. Sup. **(4) 49** (2016), no. 2, 399–421. 256
94. Nguyen, N., Valette, G.: Whitney stratifications and the continuity of local Lipschitz-Killing curvatures, Ann. Inst. Fourier Grenoble **68** (2018), no. 5, 2253–2276. 264
95. Nguyen, N., Trivedi, S., Trotman, D.: A geometric proof of the existence of definable Whitney stratifications, Illinois J. of Math. **58** (2014), 381–389. 247
96. Nicolaescu, L.: *An Invitation to Morse Theory*, Universitext, Springer, 2011. 247
97. Noirel, L.: *Plongements sous-analytiques d'espaces stratifiés de Thom-Mather*, University of Provence thesis, 1996. 252 and 259
98. Orro, P., Trotman, D.: Cône normal et régularités de Kuo-Verdier, Bull. Soc. Math. France **130** (2002), 71–85. 249 and 265
99. Orro, P., Trotman, D.: Regularity of the transverse intersection of two regular stratifications, in *Real and complex singularities*, London Math. Soc. Lecture Note Ser., **380**, Cambridge Univ. Press, Cambridge (2010), 298-304. 249
100. Parusiński, A.: Limits of tangent spaces to fibres and the w_f condition, Duke Math. J. **72** (1993), no. 1, 99–108. 254
101. Parusiński, A.: *Lipschitz stratification of subanalytic sets*, Ann. Sci. École Norm. Sup. **(4) 27** (1994), no. 6, 661–696. 255 and 256
102. Parusiński, A., Paunescu, L.: Arc-wise analytic stratification, Whitney fibering conjecture and Zariski equisingularity, Advances in Math. **309** (2017), 254–305. 244 and 258
103. Pawłucki, W.: Quasi-regular Boundary and Stokes' formula for a sub-analytic leaf, in *Seminar on Deformations, Łodz-Warsaw 1981–83*, Springer Lecture Notes in Math. **1165** (1985), 235–252. 249 and 265
104. Perkal, N.: On proving the geometric versions of Whitney regularity, J. London Math. Soc. (2) **29** (1984), no. 2, 343–351. 246
105. Pflaum, M.: *Analytic and Geometric Study of Stratified Spaces*, Lecture Notes in Mathematics **1768**, Springer, 2001. 244 and 247
106. du Plessis, A.: Continuous controlled vector fields , in *Singularity theory* (Liverpool, 1996, edited by J. W. Bruce and D. M. Q. Mond), London Math. Soc. Lecture Notes **263**, Cambridge Univ. Press, Cambridge, (1999), 189–197. 250 and 260
107. du Plessis, A., Wall, T.: *The geometry of topological stability*, Oxford Science Publications, Clarendon Press, 1995. 251 and 260
108. Ruas, M. A. S., Araújo dos Santos, R. N.: Real Milnor fibrations and (c)-regularity, Manuscripta Math. **117** (2005), no. 2, 207–218. 261
109. Schürmann, J.: *Topology of singular spaces and constructible sheaves*, Mathematics Institute of the Polish Academy of Sciences. Mathematical Monographs (New Series)], **63**, Birkhäuser Verlag, Basel, 2003. 260
110. Schwartz, M.-H.: *Champs radiaux sur une stratification analytique*, Travaux en Cours **39**, Hermann, Paris, 1991. 266
111. Shiota, M.: Triangulation of subanalytic sets, in *Singularities*, Banach Center Publications, vol. **20**, PWN Polish Scientific Publishers, Warsaw (1988), 385–395. 249
112. Shiota, M.: *Geometry of semialgebraic and subanalytic sets*, Progress in Mathematics **150**, Birkhäuser Boston, Inc., Boston, MA (1997). 249, 250, and 258

113. Shiota, M.: Thom's conjecture on triangulations of maps, Topology **39** (2000), no. 2, 383–399. 253
114. Shiota, M.: Whitney triangulations of semialgebraic sets, Ann. Polon. Math. **87** (2005), 237–246. 249
115. Shiota, M.: Triangulations of non-proper semialgebraic Thom maps, in *The Japanese-Australian Workshop on Real and Complex singularities - JARCS III*, Proc. Centre Math. Appl. Austral. Nat. Univ. **43**, Austral. Nat. Univ., Canberra (2010), 127–140. 254
116. Simon, S.: Champs totalement radiaux sur une structure de Thom-Mather, Ann. Inst. Fourier (Grenoble) **45** (1995), no. 5, 1423–1447. 250 and 266
117. Teissier, B.: Introduction to equisingularity problems, in *A. M. S. Algebraic Geometry Symposium, Arcata 1974*, Providence; Rhode Island (1975), 593–632. 257 and 258
118. Teissier, B.: Variétés polaires II: multiplicités polaires, sections planes et conditions de Whitney, in *Algebraic Geometry, La Rabida 1981*, Lecture Notes in Math. **961** (Springer, Berlin, 1982), 314–491. 247, 252, 254, and 257
119. Teufel, M.: Abstract prestratified sets are (b)-regular, J. Differential Geom. **16** (1981), no. 3, 529–536. 259
120. Thom, R.: La stabilité topologique des applications polynomiales, L'Enseignement Math. **8** (1962), 24–33. 244 and 253
121. Thom, R.: Local topological properties of differentiable mappings, in *Differential analysis*, Bombay Colloquium 1964, Oxford Univ. Press, 191–202. 251 and 264
122. Thom, R.: Propriétés différentielles locales des ensembles analytiques, in *Séminaire Bourbaki 1964–65*, exposé no. **281**, Astérisque **9** (1964–66), 69–80. 247
123. Thom, R.: Ensembles et morphismes stratifiés, Bull. Amer. Math. Soc. **75** (1969), 240-284. 244, 250, 251, 253, 254, 259, and 264
124. Trivedi, S.: Stratified transversality of holomorphic maps, International J. of Math. **24** (2013), no. 13, 1350106. 248
125. Trivedi, S., Trotman, D.: Detecting Thom faukts in stratified mappings, Kodai Math. J. **37** (2014), no. 2, 341–354. 248
126. Trotman, D. J. A.: *The classification of elementary catastrophes of codimension* ≤ 5, M. Sc. thesis, Warwick University 1973. Reprinted in Zeeman, E. C.: *Catastrophe theory. Selected papers, 1972–1977*. Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1977. 267
127. Trotman, D. J. A.: A transversality property weaker than Whitney (a)-regularity, Bull. London Math. Soc. **8** (1976), 225–228. 262 and 264
128. Trotman, D. J. A.: Counterexamples in stratification theory: two discordant horns, in *Real and complex singularities* (ed. P. Holm), Oslo 1976, Nordic Summer School, Sijthoff and Noordhoff, 1977, 679–686. 252
129. Trotman, D. J. A.: *Whitney stratifications: faults and detectors*, Warwick University thesis, 1977. 262
130. Trotman, D. J. A.: Stability of transversality to a stratification impliues Whitney (a)-regularity, Inventiones Math. **50** (1979), 273–277. 248 and 267
131. Trotman, D. J. A.: Geometric versions of Whitney regularity for smooth stratifications, Ann. Scient. École Normale Sup. (4) 12 (1979), 453–463. 246, 252, and 261
132. Trotman, D.: Transverse transversals and homeomorphic transversals, Topology **24** (1985), 25–39. 262 and 263
133. Trotman, D.: On the canonical Whitney stratifications of algebraic hypersurfaces, in *Séminaire sur la Géométrie Algébrique Réelle* (dir. J.-J. Risler), Publ. Math. Univ. Paris VII, **24** (1987), vol. 1, 123–152. 252
134. Trotman, D.: Une version microlocale de la condition (w) de Verdier, Ann. Inst. Fourier (Grenoble) **39** (1989), 482–485. 252
135. Trotman, D., Valette, G.: On the local geometry of definably stratified sets, in *Ordered algebraic structures and related topics*, Contemporary Math. 697, AMS, Providence, RI (2017), 349–366. 249 and 265

136. Trotman, D., van Straten, D.: Weak Whitney regularity implies equimultiplicity for families of complex hypersurfaces, Ann. Fac. Sci. Toulouse Math. **(6) 25** (2016), no. 1, 161–170. 262
137. Trotman, D. J. A., Wilson, L. C.: Stratifications and finite determinacy, Proc. London Math. Soc. **(3) 78** (1999), no. 2, 334–368. 254 and 263
138. Trotman, D., Wilson, L.: (r) does not imply (n) or (npf) for definable sets in non polynomially bounded o-minimal structures, Advanced Studies in Pure Mathematics **43** (2006), 463–475. 265
139. Valette, G.: A bilipschitz version of Hardt's theorem, C. R. Acad. Sci. Paris **340** (2005), no. 12, 895–900. 258
140. Valette, G.: Lipschitz triangulations, Illinois J. Math. **49** (2005), no. 3, 953–979. 258
141. Valette, G.: Volume, Whitney conditions and Lelong number, Ann. Polon. Math. **93** (2008), 1–16. 264
142. Verdier, J.-L.: Stratifications de Whitney et théorème de Bertini-Sard, Inventiones Math. **36** (1976), 295–312. 251, 252, 253, and 260
143. Verona, A.: *Stratified mappings - structure and triangulability*. Lecture Notes in Mathematics, **1102**. Springer-Verlag, Berlin, 1984. 249, 254, and 259
144. Wall, C. T. C.: Regular stratifications, in *Dynamical systems — Warwick 1974* (Proc. Sympos. Appl. Topology and Dynamical Systems, Univ. Warwick, Coventry, 1973/1974; presented to E. C. Zeeman on his fiftieth birthday), Lecture Notes in Math. **468**, Springer, Berlin, (1975), 332–344. 246 and 247
145. Wall, C. T. C.: *Differential topology*, Cambridge studies in advanced mathematics **156**, Cambridge University Press, 2016. 247
146. Whitney, H.: Elementary structure of real algebraic varieties, Annals of Math. (2) **66** (1957), 545–556. 244
147. Whitney, H.: Local properties of analytic varieties, in *Differential and Combinatorial Topology* (ed. S. S. Cairns), Princeton Univ. Press (1965), 205–244. 244, 247, and 258
148. Whitney, H.: Tangents to an analytic variety, Annals of Math. (2) **81** (1965), 496–549. 244 and 247
149. Woolf, J.: Transversal homotopy theory, Theory and Applications of Categories **24(7)** (2010), 148–178. 267
150. Zariski, O.: Open questions in the theory of singularities, Bull. Amer. Math. Soc. **77 (4)** (1971), 481–491. 262

Chapter 5
Morse Theory, Stratifications and Sheaves

Mark Goresky

Contents

M. Goresky (✉)
School of Mathematics, Institute for Advanced Study, Princeton, NJ, USA
e-mail: goresky@ias.edu

J. L. Cisneros Molina et al. (eds.), *Handbook of Geometry and Topology of Singularities I*, https://doi.org/10.1007/978-3-030-53061-7_5

Abstract After the local topological structure of stratified spaces was determined by R. Thom (Bull. Amer. Math. Soc., **75** (1969), 240–284) and J. Mather (**Notes on topological stability**, lecture notes, Harvard University, 1970) it became possible (see Kashiwara and Schapira, **Sheaves on Manifolds**, Grundlehren der math. Wiss. **292**, Springer Verlag Berlin, Heidelberg, 1990; Goresky and MacPherson, **Stratified Morse Theory**, Ergebnisse Math. **14**, Springer Verlag, Berlin, Heidelberg, 1988; Schürmann, **Topology of Singular Spaces and Constructible Sheaves**, Monografie Matematyczne **63**, Birkhäuser Verlag, Basel, 2003) to analyze constructible sheaves on a stratified space using Morse theory. Although the detailed proofs are formidable, the statements and main ideas are simple and intuitive. This article is a survey of the constructions and results surrounding this circle of ideas.

5.1 Introduction

The stratified Morse theory of [29, 31] and the theory of constructible sheaves in [44] are two sides of the same coin. These books contain many parallel and overlapping results of a body of material that was developed in the 1980s. A brief outline applying Morse theory to constructible sheaves appears in Appendix 6.A of [31] and a complete and parallel development of the two theories is presented in [70]. In this article we provide a rapid and hopefully intuitive view of this circle of ideas.

In many situations the "nondegeneracy" conditions of Morse theory may be relaxed, which leads to a rich theory involving the topology of singular spaces, sheaves and maps, some of which we describe in Sects. 5.3.2, 5.11.1, 5.11.2, 5.12 below.

The author is grateful to Vidit Nanda and to an anonymous referee for many helpful corrections and suggestions on an earlier version of this paper. Some of the diagrams in this paper were typeset using Paul Taylor's diagrams.sty package.

5.2 Preliminaries

A *pair* of topological spaces (A, B) means that $B \subset A$. A product of pairs $(A, B) \times (X, Y)$ is the pair $(A \times X, A \times Y \cup B \times X)$. If $Z = X \cup_B A$ is obtained by attaching a space A along a subspace B using two inclusions $B \to A$ and $B \to X$, by abuse of notation we write simply $Z = X \cup (A, B)$. If $f : X \to \mathbb{R}$ is a continuous mapping and $a \in \mathbb{R}$ define

$$X_{\leq a} = \{x \in X |\ f(x) \leq a\}$$

and similarly for $X_{<a}, X_{>a}$, etc. For $a < b \in \mathbb{R}$, Morse theory addresses the question of how to obtain $X_{\leq b}$ from $X_{\leq a}$ by attaching a topological space A along a subspace $B \subset A$ using an embedding $B \to X_{\leq a}$. In this case the pair (A, B) is said to be Morse data for f over the interval $[a, b]$ and the excision isomorphism implies that $H_i(X_{\leq b}, X_{\leq a}) \cong H_i(A, B)$. One possible answer, of course, is the pair

$$(X_{[ab]}, X_b) = (f^{-1}([a, b]), f^{-1}(b))$$

which we refer to as *coarse Morse data*. One objective in Morse theory is to find explicit Morse data (A, B) that is as simple as possible.

5.3 Review of Smooth Morse theory

5.3.1 Manifolds

Let M be a smooth n-dimensional manifold and $f : M \to \mathbb{R}$ a smooth proper Morse function, that is, a function with isolated critical points (meaning $df(p) = 0$) and nondegenerate Hessian matrix (in local coordinates)

$$H(f)(p) = \left(\frac{\partial^2 f(p)}{\partial x_i \partial x_j} \right)$$

at each critical point p. The *Morse index* λ at such a critical point p is the dimension of the greatest subspace on which $H(f)(p)$ is negative definite. The zeroth theorem in Morse theory says: if $[a,b] \subset \mathbb{R}$ contains no critical values then $M_{\le a}$ is diffeomorphic (as a manifold with boundary) to $M_{\le b}$. The first fundamental theorem of Morse theory says that $(D^\lambda, \partial D^\lambda) \times D^{n-\lambda}$ is Morse data for f at p:

Theorem 5.3.1 *If $p \in M$ is a nondegenerate critical point with isolated critical value $v = f(p)$ (meaning that no other critical points have critical value v) then for any $\epsilon > 0$ sufficiently small, the smooth manifold with boundary $M_{\le v+\epsilon}$ is homeomorphic to the adjunction space*

$$M_{\le v-\epsilon} \bigcup (D^\lambda \times D^{n-\lambda}, \partial D^\lambda \times D^{n-\lambda}) \tag{5.1}$$

where λ is the Morse index of f at the critical point p, and where D^λ denotes the (unit) disk of dimension λ and boundary sphere ∂D^λ.

An immediate consequence is that for any local coefficient system $E \to M$ (see Sect. 5.6.1) of finitely generated abelian groups,

$$H_i(M_{\le v+\epsilon}, M_{\le v-\epsilon}; E) \cong \begin{cases} E_p & i = \lambda \\ 0 & \text{otherwise} \end{cases}$$

where E_p denotes the stalk of E at the critical point p. There are two additional facts:

- The adjunction is local near the critical point p. Thus, if there are several critical points with the same critical value v and various but possibly different Morse indices $\lambda_1, \lambda_2, \cdots$ then by choosing ϵ sufficiently small the various embeddings ∂D^{λ_i} can be chosen disjoint and the pairs $(D^{\lambda_i}, \partial D^{\lambda_i})$ may be adjoined independently.
- By "straightening the angle" [59] p. 34, [71] where the attaching occurs, the homeomorphism (5.1) may be realized as a diffeomorphism of manifolds with boundary.

5.3.2 *Perfect Morse-Bott Functions*

Morse theory may also be applied to smooth functions $f : M \to \mathbb{R}$ with minimally degenerate critical points. A *nondegenerate critical submanifold* $V \subset M$ is a submanifold such that $df(x)|T_xV = 0$ for all $x \in V$, and the Hessian$H(f)(x)$ is nondegenerate on the normal space $T_{V,x}M = T_xM/T_xV$. A choice of Riemannian metric yields a decomposition $T_VM = E^+ \bigoplus E^-$ into positive/negative eigenbundles. The rank of E^- is the Morse index of the restriction $f|W$ to a normal slice W through V so it is referred to as the Morse index of f on V.

A Morse-Bott function is one whose critical points consist of nondegenerate critical submanifolds. Bott's extension of Morse's theorem is:

Theorem 5.3.2 *With $f : M \to \mathbb{R}$ smooth and proper as above, if the critical value v is isolated and corresponds to a single connected critical submanifold V then for sufficiently small $\epsilon > 0$ the space $M_{\leq v+\epsilon}$ is homotopy equivalent to the space obtained from $M_{\leq v-\epsilon}$ by attaching the disk bundle $D(E^-)$ of E^- along its boundary sphere bundle $S(E^-) = \partial D(E^-)$.*

If $S^1 = \{e^{i\theta}\}$ acts by Hamiltonian diffeomorphisms on a symplectic manifold (M, ω) with resulting vector field Y corresponding to $\partial/\partial\theta$, then the *moment map*[1] $\mu : M \to \mathbb{R}$ is a Morse-Bott function [4].

Let R be a commutative (coefficient) ring. A Morse-Bott function $f : M \to \mathbb{R}$ is *perfect* (for R) if the connecting homomorphisms for the long exact sequences involving $H_i(M_{\leq v+\epsilon}, M_{\leq v-\epsilon}; R)$ vanish. If f is R-perfect and if the negative bundles E^- of the critical submanifolds are also R-orientable then the Thom isomorphism gives a non-canonical decomposition

$$H^i(M; R) \cong \bigoplus\nolimits_V H^{i-\lambda_V}(V; R) \tag{5.2}$$

where the sum is taken over all critical submanifolds V and where λ_V is the corresponding Morse index of f on V.

This exceptional situation of a perfect Morse function, as described in [10], arises when $R = \mathbb{Q}$ and $M \subset \mathbb{CP}^N$ is a nonsingular complex projective variety that is preserved by an algebraic action of $\mathbb{C}^*$, in which case the action of $S^1 \subset \mathbb{C}^*$ is Hamiltonian with respect to the canonical symplectic form[2] on M. Then the vector bundles $E^\pm \to V$ arise geometrically: let $V \subset M$ be a connected component of the fixed point set and let

$$\begin{aligned} V^- &= \left\{ x \in M | \lim_{t\to\infty} t.x \in V \right\} \\ V^+ &= \left\{ x \in M | \lim_{t\to 0} t.x \in V \right\}. \end{aligned} \tag{5.3}$$

Theorem 5.3.3 ([10]) *The projection $V^+ \to V$ (resp. $V^- \to V$) has the natural structure of an algebraic bundle of affine spaces that is diffeomorphic to the bundle $E^+ \to V$ (resp. $E^- \to V$) and the moment map $\mu : M \to \mathbb{R}$ is a perfect Morse-Bott function.*

Equation (5.2) (equivalent to the perfection of the Morse function) then follows from the *Bialynicki-Birula decomposition* $M = \coprod_V V^-$ and the Weil conjectures (proved by Grothendieck and Deligne) which describe cohomology by counting points of these varieties mod p. See also Sect. 5.11.1 below.

---

[1] Characterized up to an additive constant by the condition that $d\mu = \iota_Y \omega$ (interior product).

[2] The imaginary part of the Fubini-Study metric.

5.4 Stratified Spaces

A stratification of a closed subset $W \subset M$ is a locally finite decomposition into (disjoint) smooth locally closed submanifolds, called strata, $W = \coprod X_i$ such that the closure of each stratum is a union of strata of smaller dimension. A stratified mapping $W \to W'$ between stratified spaces is a continuous mapping that takes strata to strata and is smooth on each stratum. It is a *stratified homeomorphism* if it has an inverse that is also a stratified mapping.

Write $X < Y$ if $X \neq Y$ are strata of W with $X \subset \overline{Y}$. The pair $X < Y$ is said to satisfy Whitney's conditions at a point $x \in X$ if the following holds:

> suppose $y_i \in Y$ and $x_i \in X$ are sequences that converge to the same point $x \in X$; suppose the secant lines $\overline{x_i, y_i}$ converge to some limiting line $\ell \subset T_xM$ and suppose the tangent planes $T_{y_i}Y$ converge to some limiting plane $\tau \subset T_xM$. Then **(A)** $T_xX \subset \tau$ and **(B)** $\ell \subset \tau$.

Convergence of these lines and planes may be taken in the appropriate bundle of Grassmannians over M or equivalently, they may be taken with respect to some, and hence any, local coordinate system on M containing x. Condition (B) implies condition (A). A stratification of a space W is said to be a Whitney stratification if Whitney's conditions hold at every point x with respect to every pair of strata $X < Y$.

If M is a (real) analytic manifold and $W_1, \cdots, W_r$ are analytic, semi-analytic or sub-analytic subsets then there exists a Whitney stratification of M so that each W_j and each multi-intersection of the $\{W_j\}$ are unions of strata, cf. [20, 25, 35–37, 51, 57].

The Whitney conditions are a sort of "no-wiggle" condition as points in Y approach points in X but they imply the fundamental structure theorem of the Thom-Mather theory of Whitney stratified spaces: the space W is topologically locally trivial along each stratum X of W and each point in X has a basis of *basic neighborhoods*, all of which are stratified-homeomorphic. We make this statement precise in Sects. 5.4.1, 5.4.2.

5.4.1 Normal Slice and Link

Recall that submanifolds $S, N \subset M$ of a smooth manifold M are *transverse* if $T_pS + T_pN = T_pM$ for each point $p \in S \cap N$, in which case the intersection is a smooth submanifold $P = S \cap N$ and we write $P = S \pitchfork N$. If $W, W' \subset M$ are Whitney stratified subsets, and if each stratum of W is transverse to each stratum of W' then the intersection $W \pitchfork W'$ is Whitney stratified with strata of the form $S \pitchfork S'$ where S (resp. S') run through the strata of W (resp. W').

Let S be a stratum of dimension s in a Whitney stratified (closed) subset $W \subset M$ of some smooth n dimensional manifold M. Fix $p \in S$ and let $T \subset M$ be a smooth submanifold (or germ of a submanifold) such that $S \pitchfork T = \{p\}$. This implies, by Whitney's condition B that for every stratum $R > S$, the transversality condition $R \pitchfork T$ also holds at points in R that are sufficiently close to $p \in S$.

Let $B_\epsilon(p)$ be the (closed) ball of radius $\epsilon > 0$ (with respect to some Riemannian metric on M) centered at $p \in S$. Whitney's condition B implies that for sufficiently small $\epsilon > 0$,

(*) the boundary sphere $\partial B_\epsilon(p)$ is transverse to every stratum of W and of $T \cap W$ and the same holds for all $0 < \epsilon' \leq \epsilon$.

Define the *normal slice*,

$$(N_\epsilon(p), \partial N_\epsilon(p)) = T \cap W \cap (B_\epsilon(p), \partial B_\epsilon(p)) \tag{5.4}$$

with its induced stratification. Its boundary

$$L_S(p) = \partial N_\epsilon(p) = T \cap W \cap \partial B_\epsilon(p) \tag{5.5}$$

is called the *link* of the stratum S at p. These objects are related by a local statement (Theorem 5.4.1) and a global statement (Sect. 5.4.4) below.

Theorem 5.4.1 ([31, §7]) *The normal slice is stratified-homeomorphic to the cone over the link, that is, there exists a stratum preserving homeomorphism, smooth on each stratum,*

$$N_\epsilon(p) \cong c\,(\partial N_\epsilon(p))$$

which takes the cone point to the point p. Suppose $p, p' \in S$ lie in the same connected component of the stratum $S \subset W$. Suppose $N_\epsilon(p)$ and $N'_{\epsilon'}(p')$ are normal slices at p, p' taken with respect to different choices of submanifold T, T', different Riemannian metrics on M and different values ϵ, ϵ'. If $\epsilon, \epsilon' > 0$ satisfy (*) above then there is a stratified homeomorphism

$$(N_\epsilon(p), \partial N_\epsilon(p)) \cong (N'_{\epsilon'}(p'), \partial N'_{\epsilon'}(p')).$$

Moveover, there is a stratum preserving homeomorphism of pairs

$$\begin{aligned}(U_p, \partial U_p) &= (B_\epsilon(p), \partial B_\epsilon(p)) \cap W \\ &\cong (B_\epsilon(p) \cap S, \partial B_\epsilon(p) \cap S) \times (N_\epsilon(p), \partial N_\epsilon(p)) \\ &\cong (D^s, \partial D^s) \times (c(\partial N_\epsilon(p)), \partial N_\epsilon(p)).\end{aligned} \tag{5.6}$$

where $s = \dim(S)$ and D^s denotes the closed unit disk.

5.4.2 Basic Neighborhood

The homeomorphism (5.6) implies that the intersection $L_p = \partial B_\epsilon(p) \cap W$ (the *link* of p in W) is homeomorphic to the s-fold suspension of $L_S(p) = \partial N_\epsilon(p)$ and that the whole closed neighborhood $\overline{U}_p = B_\epsilon(p) \cap W$ is homeomorphic to the product $\overline{U}_p \cong D^s \times N_\epsilon(p) \cong D^s \times c(L_S(p))$.

5.4.3 Deformation Arguments

Theorems 5.4.1, 5.5.1, 5.5.2, 5.5.3, 5.8.1, 5.9.1 (below) and many others like these are proven in [31]. The proofs involve deforming, in a smooth stratum preserving way, one subset, for example $\mathcal{S}_0 = N_\epsilon(p)$, into another subset, say, $\mathcal{S}_1 = N'_{\epsilon'}(p')$. Although these sets are complicated, they arise from very simple pictures Y_0, Y_1 respectively, usually in $\mathbb{R}^2$. Theorem 5.4.2 [31, Theorems 4.3, 4.4] below says that a deformation $\{Y_t\}$ of the simple "picture" gives rise to a corresponding deformation $\{\mathcal{S}_t\}$ of the (more complicated) set. For example, Theorem 5.5.2 below corresponds to moving the point (ϵ, δ) into the point (ϵ', δ') within the set of allowable possible choices.

Theorem 5.4.2 (Moving the wall) *Let $W \subset M$ be a Whitney stratified set and let $\phi : M \to \mathbb{R}^2$ be a smooth mapping so that $\phi|W$ is proper. Let $Y \subset \mathbb{R}^2 \times \mathbb{R}$ be a closed Whitney stratified subset so that the projection $\pi : Y \to \mathbb{R}$ to the second factor is a submersion (everywhere surjective differential) on each stratum. Considering the second factor $\mathbb{R}$ to be a parameter space, let $Y_t = \pi^{-1}(t)$.*

$$\begin{tikzcd} W \arrow[r, phantom, "\subset"] & M \arrow[d, "\phi"'] & \\ & \mathbb{R}^2 \arrow[r, phantom, "\supset"] & Y_t \end{tikzcd}$$

Suppose the restriction $f|S$ to each stratum S of W is transverse to each stratum of Y_t, for all $t \in \mathbb{R}$. Then there is a stratified homeomorphism

$$W \cap \phi^{-1}(Y_0) \cong W \cap \phi^{-1}(Y_1)$$

This is little more than a restatement of Thom's first isotopy lemma (see Theorem 5.5.1 below) and in fact the target space $\mathbb{R}^2$ may be replaced by an arbitrary smooth manifold. A similar result holds for pairs of spaces [31, §4.4]. A illustrative example in [31, §4.5] shows that Morse data for a Morse function on a smooth manifold is a product of cells. Cohomological (rather than homeomorphism) deformation arguments, applicable to a wide range of sheaves and spaces, are proven in [44, Theorem 2.7.2], [43, Theorem 1.4.3].

5.4.4 Control Data and Canonical Retraction

Let $W \subset M$ be a closed Whitney stratified subset. The local structure of W near a point in a stratum S, as described in Theorem 5.4.1, is actually a consequence of the existence (due to R. Thom [77] and J. Mather [56]) of a global system of *control data*: a collection $\{(\pi_S, \rho_S) : T_S(\epsilon) \to S \times [0, \epsilon)\}$ of data for each stratum S, where $T_S(\epsilon)$ is a tubular neighborhood of S in M, ρ_S is a tubular "distance" function vanishing exactly on S, where (π_S, ρ_S) is a submersion when restricted to each stratum $R > S$ and where $\pi_S \pi_R = \pi_S$ and $\rho_S \pi_R = \rho_S$ whenever both sides of these equations are defined.

It follows (after much work, see [56, 77]) that each fiber $\pi_S^{-1}(x)$ is stratified-homeomorphic to the normal slice $N_\epsilon(p) \cong c(\partial N_\epsilon(p))$ and the whole tubular neighborhood is stratified-homeomorphic to the mapping cylinder,

$$T_S(\epsilon) \cong \mathrm{cyl}\left(\partial T_S(\epsilon) \xrightarrow{\pi_S} S\right)$$

by homeomorphisms which (by a choice of a *family of lines* [26]) may be chosen to be compatible among strata.

Suppose $Z \subset W$ is a closed union of strata and let $T_Z(2\epsilon) = \bigcup_S T_S(2\epsilon)$ be the union of the tubular neighborhoods of strata $S \subset Z$ and similarly for $T_Z(\epsilon)$. Then the projections and mapping cylinder structures may be assembled into a stratum preserving deformation retraction [27, §7], unique up to stratum preserving isotopy, $T_Z(2\epsilon) \to T_Z(2\epsilon)$ whose restriction

$$r_Z : T_Z(\epsilon) \to Z \tag{5.7}$$

retracts $T_Z(\epsilon)$ to Z and agrees with the tubular projections: if $x \in T_S(\epsilon) - \cup_{R<S} T_R(2\epsilon)$ then $r_Z(x) = \pi_S(x) \in S$. If a point x is in the region

$$T_S(\epsilon) \cap (T_R(2\epsilon) - T_R(\epsilon))$$

then $r_Z(x) = r_Z(\pi_S(x))$ and $r_Z|S$ shrinks towards R along the mapping cylinder lines (Fig. 5.1). So, for all $y \in S$ there exists $y' \in S$ with $r_Z(y') = y$ and

$$r_Z^{-1}(y) \cong \pi_S^{-1}(y') \cong \pi_S^{-1}(y)$$

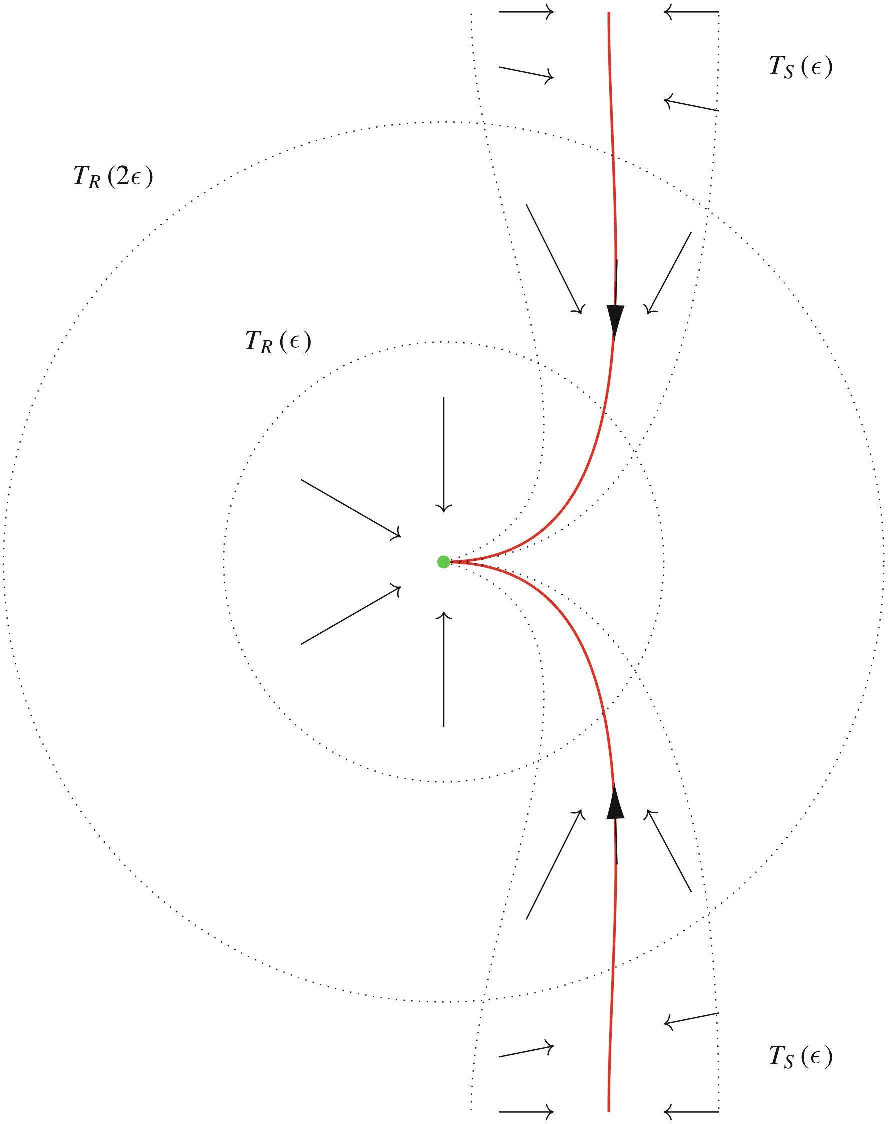


Fig. 5.1 Tubular neighborhoods and retraction for $R < S$

5.5 Stratified Morse Theory

5.5.1 Conormal Vectors

Let M be a smooth manifold and let $W \subset M$ be a Whitney stratified (closed) subset. Let X be a stratum of W and let $p \in X$.

A cotangent vector $\xi \in T_p^*M$ is said to be *conormal* to X if its restriction vanishes: $\xi|T_pX = 0$. The collection of all conormal vectors to X in M is denoted T_X^*M. It is a smooth conical Lagrangian locally closed submanifold of T^*M.

If $f : M \to \mathbb{R}$ is smooth, its restriction $f|X$ to X is a Morse function if and only if the graph of df is transverse to T_X^*M in T^*M (see, for example, [44, p. 311], [70, p. 286]).

A subspace $\tau \subset T_pM$ will be said to be a *limit of tangent spaces from W* if there is a stratum $Y > X$ ($Y \neq X$) and a sequence of points $y_i \in Y$, $y_i \to p$ such that the tangent spaces $T_{y_i}Y$ converge to τ. A conormal vector $\xi \in T_X^*M$ at p is *nondegenerate* if $\xi(\tau) \neq 0$ for every limit $\tau \subset T_pM$ of tangent spaces from larger strata $Y > X$. The set of nondegenerate conormal vectors is denoted Λ_X. Evidently,

$$\Lambda_X = T_X^*M - \bigcup_{Y>X} \overline{T_Y^*M}$$

where the union is over all strata $Y > X$ (including the case $Y = M - W$ because T_M^*M is the zero section, and elements of Λ_X are necessarily nonzero).

5.5.2 Morse Functions

Let $f : M \to \mathbb{R}$ be a smooth function and let $\lambda = df(p) \in T_p^*M$. A *critical point of $f|W$* is a point $p \in X$ in some stratum X such that $df(p)|T_pX = 0$, that is, $\lambda \in T_X^*M$. (In particular, every zero dimensional stratum is a critical point.) The value $v = f(p)$ is said to be an isolated critical value of $f|W$ if no other critical point $q \in W$ of $f|W$ has $v = f(q)$. We say that f is a Morse function for W (cf. [50]) if

- its restriction to W is proper
- $f|X$ has isolated nondegenerate critical points for each stratum X,
- at each critical point $p \in X$ the covector $\lambda = df(p) \in \Lambda_X$ is nondegenerate, that is, $df(p)(\tau) \neq 0$ for every limit of tangent spaces $\tau \subset T_pM$ from larger strata $Y > X$.

In the case of a 1-dimensional target, Thom's First Isotopy Lemma [56, 77], becomes the zeroth theorem of SMT, which says:

Theorem 5.5.1 *Let $f : M \to \mathbb{R}$ be a smooth proper function, let $W \subset M$ be a Whitney stratified closed subset and suppose that $[a, b] \subset \mathbb{R}$ contains no critical* *values of the restriction of f to any stratum of W. Then $W_{\leq a}$ is homeomorphic to $W_{\leq b}$ by a stratum preserving homeomorphism that is smooth on each stratum.*

5.5.3 Normal Morse Data

Suppose that $f : M \to \mathbb{R}$ is a smooth proper mapping that is Morse on $W \subset M$ as above. Suppose S is a stratum of W of dimension s and p is a (nondegenerate) critical point of $f|S$. Let $(N_\epsilon(p), \partial N_\epsilon(p))$ be a normal slice (5.4) to the stratum at p with $\epsilon > 0$ chosen sufficiently small so as to satisfy (*) in Sect. 5.4.1. Set $v = f(p)$. The nondegeneracy of the conormal vector $\xi = df(p)$ implies there exists $\delta > 0$ so that

(**) $f|N_\epsilon(p)$ has no critical points on any stratum of $N_\epsilon(p) \cap f^{-1}[v-\delta, v+\delta]$ other than $\{p\}$, and the same holds for all $\delta' \leq \delta$.

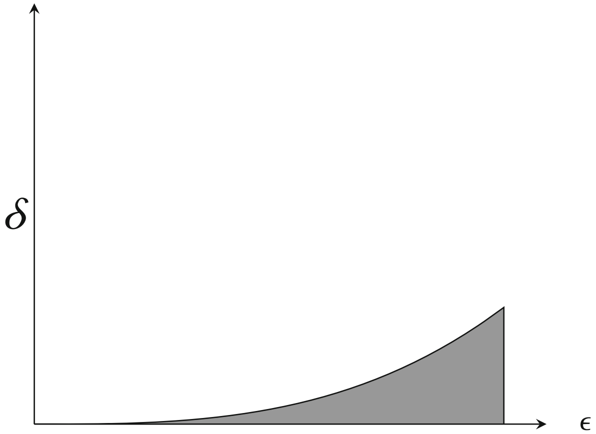
In this case we write $0 < \delta \ll \epsilon$. The set of possible choices for ϵ, δ will be an open region in the (ϵ, δ) plane as in Fig. 5.2:

The *normal Morse data* for f at p is defined to be the coarse Morse data of the normal slice, that is, the pair

$$\left(N_\epsilon(p)_{[v-\delta,v+\delta]}, N_\epsilon(p)_{v-\delta}\right) = N_\epsilon(p) \cap \left(f^{-1}[v-\delta, v+\delta],\ f^{-1}(v-\delta)\right)$$

Theorem 5.5.2 ([31, Theorem 3.6.2]) *Suppose the stratum S is connected, $p' \in S$ is a nondegenerate critical point of a second function $f' : M \to \mathbb{R}$ with $\xi' = df'(p') \in \Lambda_S$ and suppose that ξ, ξ' are in the same connected component of Λ_S. Then there is a stratified homeomorphism between the normal Morse data for f at p and normal Morse data for f' at p'.*

Fig. 5.2 $\delta \ll \epsilon$ region



5.5.4 Main Theorem

With $p \in S \subset W \subset M \to \mathbb{R}$ as in Sect. 5.5.2 above, let λ denote the Morse index of $f|S$ at the critical point p. Define the *tangential Morse data* to be the pair $(D^\lambda, \partial D^\lambda) \times D^{s-\lambda}$, see Eq. (5.1). The main theorem [31, Theorem 3.7] in stratified Morse theory says that Morse data at an isolated critical point is the product of the tangential Morse data with the normal Morse data. The proof involves repeated application of Theorem 5.4.2 to provide a sequence of stratum preserving deformations.

Theorem 5.5.3 *Suppose $[a, b] \subset \mathbb{R}$ contains a single (isolated) critical value $v \in (a, b)$ of $f|W$ corresponding to a nondegenerate critical point $p \in S$. Suppose $0 \leq \delta \ll \epsilon$ are chosen as in (*) Sect. 5.4.1 and (**) Sect. 5.5.3 above so that the normal Morse data is well defined. Then there is a homeomorphism between the space $W_{\leq b}$ and the space obtained from $W_{\leq a}$ by attaching the pair*

$$\left(D^\lambda, \partial D^\lambda\right) \times D^{s-\lambda} \times \left(N_\epsilon(p)_{[v-\delta, v+\delta]}, N_\epsilon(p)_{v-\delta}\right).$$

Hence $H_i(W_{\leq b}, W_{\leq a}) \cong H_{i-\lambda}(N_\epsilon(p)_{[v-\delta,v+\delta]}, N_\epsilon(p)_{\leq v-\delta})$ for all i.

5.5.5 Illustration

Figure 5.3 illustrates Theorem 5.5.3. The stratified space W is like three pages of a book glued along the spine with two "pages" going up and one "page" going down. The critical value is $v = 0$. The Morse function is the height function and the height $-\delta$ cuts the stratified space W along the red horizontal slice. The normal slice is a "Y". The tangential and normal Morse data and their product is shown in the second row of the diagram with the red region marking the subspace.

To obtain $W_{\leq \delta}$ from $W_{\leq -\delta}$, attach the Morse data long the red subspace. In order to do this, it is necessary to deform the Morse data, stretching it so that the red vertical "ends" of the Morse data become horizontal, so they can be lined up with $f^{-1}(-\delta)$. This simple example illustrates the complexity of the deformation arguments, which take about 50 pages in [31].

5.5.6 Existence of Morse Functions

If M is an analytic manifold and $W \subset M$ is a subanalytically Whitney stratified subanalytic set then the collection of Morse functions is open and dense in the space of smooth mappings $M \to \mathbb{R}$ that are proper on W, in the Whitney C^∞ topology, cf. [64, 67].

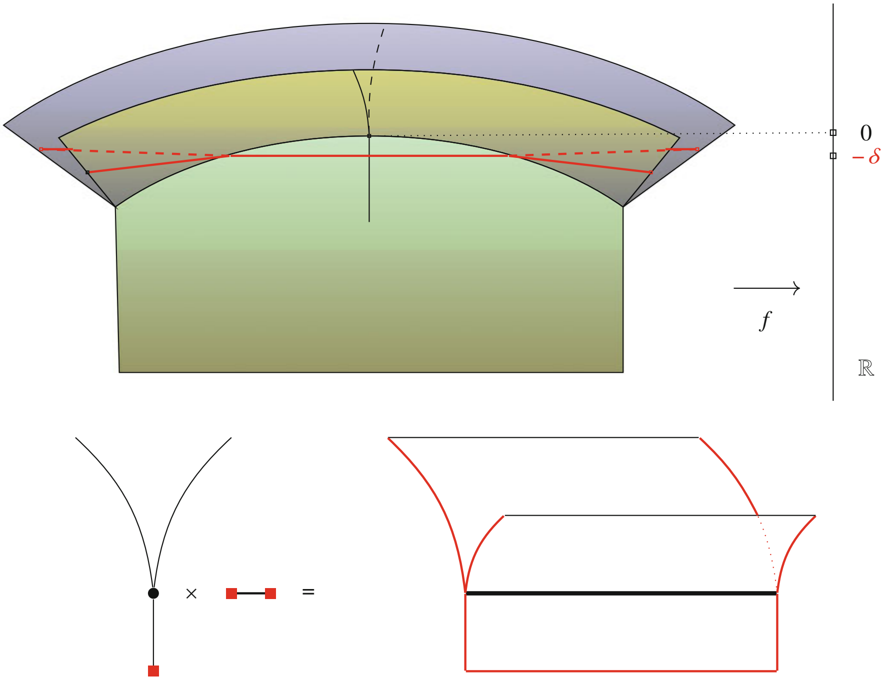


Fig. 5.3 Normal Morse data × Tangential Morse data = Total Morse data. Glue along the red subspace

5.6 Recollections on Sheaves

5.6.1 Presheaves and Sheaves

A presheaf S of abelian groups on a topological space X is a contravariant functor from the category of open subsets of X (and inclusions) to the category of abelian groups (and homomorphisms). Elements of $S(U)$ are called "sections" over the open set $U \subset X$.

A presheaf S is a sheaf if it is "locally defined", that is, if the following condition holds. Let $\{U_\alpha\}_{\alpha\in I}$ be a (possibly infinite) collection of open subsets of X and set $U = \cup_{\alpha\in I} U_\alpha$. Suppose that $s_\alpha \in S(U_\alpha)$ are sections such that $s_\alpha|U_\alpha \cap U_\beta = s_\beta|U_\alpha \cap U_\beta$ for all $\alpha, \beta \in I$. Then there is a unique section $s \in S(U)$ so that $s|U_\alpha = s_\alpha$ for all $\alpha \in I$.

For any presheaf S the stalk at $x \in X$ is the direct limit $S_x = \lim\limits_{x\in U} S(U)$. There is a unique topology on the *leaf space* $LS = \bigcup_{x\in U} S_x$ so that each stalk has the discrete topology and so that the projection $\pi : LS \to X$ is locally (near each point in LS) a homeomorphism. If $U \subset X$ is open define the group of sections

$$\Gamma(U, LS) = \{s : U \to LS |\ s \text{ is continuous and } \pi \circ s = Id\}.$$

The restriction homomorphisms $S(U) \to S_x$ $(x \in U)$ determine a canonical homomorphism

$$\phi_U : S(U) \to \Gamma(U, LS).$$

The presheaf S is a sheaf if and only if ϕ_U is an isomorphism for all open sets $U \subset X$, in which case the group of sections is commonly denoted

$$\Gamma(U, S) = S(U) = \Gamma(U, LS)$$

If S is a presheaf then the functor $U \mapsto \Gamma(U, LS)$ is a sheaf and so this gives a canonical *sheafification* operation which identifies the category of sheaves as a full subcategory of the category of presheaves. A *local coefficient system* is a locally trivial sheaf $LS \to X$.

5.6.2 Čech Cohomology

Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be a collection of open sets that cover X. A Čech q-cochain σ assigns to each ordered collection $\{U_0, U_1, \cdots, U_q\}$ of elements of $\mathcal{U}$ with nonempty intersection, an element

$$\sigma(U_0, U_1, \cdots, U_q) \in \Gamma(U_0 \cap U_1 \cap \cdots \cap U_q, S)$$

that is antisymmetric: for any permutation π,

$$\sigma(U_{\pi(0)}, U_{\pi(1)}, \cdots, U_{\pi(q)}) = \text{sign}(\pi)\sigma(U_0, \cdots, U_q).$$

(The sign corresponds to a choice of orientation of the associated simplex in the nerve of the covering $\mathcal{U}$). The group of Čech q-cochains for the cover $\mathcal{U}$ is denoted $\check{C}^q_{\mathcal{U}}(X; S)$. The coboundary operator $d^q : \check{C}^q_{\mathcal{U}} \to \check{C}^{q+1}_{\mathcal{U}}$ is defined as follows. Let $(U_0, U_1, \cdots, U_{q+1})$ be an ordered collection of elements of $\mathcal{U}$ and set $V = U_0 \cap U_1 \cap \cdots \cap U_{q+1}$. Then

$$(d^q\sigma)(U_0, U_1, \cdots, U_{q+1}) = \sum_{j=0}^{q+1}(-1)^j \sigma(U_0, \cdots, \widehat{U}_j, \cdots, U_{q+1})|V$$

(where $|V$ denotes the restriction to V and $\widehat{U}_j$ means "omit U_j"). Then $d^{q+1} \circ d^q = 0$. Define

$$\check{H}^q_{\mathcal{U}}(X; S) = \ker(d^q)/\mathrm{Im}(d^{q-1}).$$

Note that $\check{H}^0_{\mathcal{U}}(X; S) = \Gamma(X, S)$. The Čech cohomology $\check{H}^q(X; S)$ is defined to be the limit over all open coverings of $\check{H}^q_{\mathcal{U}}(X; S)$ but typically fewer open sets suffice:

Theorem 5.6.1 *Suppose the open cover $\mathcal{U}$ has the property that $H^q(U_J; S) = 0$ for all $q > 0$ and for every $J \subset I$, where $U_J = \cap_{j \in J} U_j$. Then for all $q \geq 0$ the natural homomorphism is an isomorphism:*

$$\check{H}^q_{\mathcal{U}}(X; S) \xrightarrow{\cong} \check{H}^q(X; S).$$

5.6.3 Resolutions

The second way to construct the cohomology of a sheaf is with a resolution. Recall that a sheaf I on X is flabby if $\Gamma(X, I) \to \Gamma(U, I)$ is surjective for every open set $U \subset X$. It is soft if $\Gamma(X, I) \to \Gamma(K, I)$ is surjective for every closed set $K \subset X$. It is injective if: for every morphism $f : S \to I$ and for every injection $g : S \to T$ there exists a morphism $h : T \to I$ so that $f = h \circ g$. An injective (resp. flabby, resp. soft) resolution of S is an exact sequence

$$0 \to S \to I^0 \to I^1 \to I^2 \to \cdots \tag{5.8}$$

where I^j are injective (resp. flabby, resp. soft) sheaves.[3] (A sequence of sheaves is exact if and only if it is exact on the stalks.) For the following see, for example, [7, §4].

Theorem 5.6.2 *Suppose S is a sheaf on a locally compact, paracompact Hausdorff topological space X. Let (5.8) be an injective (resp. flabby, resp. soft) resolution of S. Then the Čech cohomology $\check{H}^*(X; S)$ coincides with the cohomology of the complex of global sections,*

$$\Gamma(X, I^0) \to \Gamma(X, I^1) \to \Gamma(X, I^2) \to \cdots$$

and is therefore independent of the choice of resolution $I^\bullet$.

---

[3] Injectivity is an algebraic as well as a topological condition. The constant sheaf $\mathbb{Z}$ on a point is flabby and soft but not injective. It has an injective resolution $\mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$.

5.6.4 Chains and Cochains

In many cases the central object of study is the resolution itself. For example, if M is a smooth manifold then the complex of differential forms is a fine (hence flabby) resolution of the constant sheaf $\mathbb{R}$:

$$0 \to \mathbb{R} \to \mathbf{\Omega}^0 \to \mathbf{\Omega}^1 \to \mathbf{\Omega}^2 \to \cdots$$

because the Poincaré lemma says that it is exact on stalks. Therefore $H^i(X; \mathbb{R})$ is isomorphic to the cohomology of the complex $\Gamma(X, \mathbf{\Omega}^\bullet) = \Gamma(X, \mathbf{\Omega}^\bullet)$ of global sections, a theorem of G. deRham. More generally let X be a topological space and let R be a commutative ring. If $U \subset X$ is an open set then the complex of singular chains $(C_r(U; R), \partial_r)$ consists of *finite* formal sums (with coefficients in R) of singular simplices whose image is contained in U. Its dual is the complex of singular cochains $(C^r(U; R) = \mathrm{Hom}(C_r(U; R), R), d^r = \partial_r^*)$ which evidently form a complex of sheaves $\mathbf{C}^\bullet$ on X by allowing U to vary over all open subsets of X. It is a flabby resolution of the constant sheaf and the cohomology of the complex of global sections is the singular cohomology of X.

Unfortunately the singular chains $C_r(U; R)$ do not form a sheaf because restriction maps are not defined for $V \subset U$. Borel and Moore [11] solved this problem by dualizing again. If R is a field define

$$\omega_{BM}^{-r} = \omega_r^{BM}(U; R) = \mathrm{Hom}_R(\Gamma_c(U, \mathbf{C}^r), R)$$

where Γ_c denotes sections with compact support.[4]

Theorem 5.6.3 ([11]) *The Borel-Moore complex of chains $\omega_{BM}^{-r}(U)$ form an injective complex (see Sect. 5.6.6) of sheaves $\omega_{BM}^\bullet$whose stalk cohomology (see Sect. 5.6.6) is the local homology: $H_x^{-r}(\omega_{BM}^\bullet) \cong H_r(X, X - x; R)$.*

5.6.5 PL Chains

A more concrete description (see [30]) of the Borel-Moore sheaf of chains exists if the space X has a piecewise linear (or subanalytic or o-minimal) structure. Let R be a commutative ring. Suppose X is a simplicial complex, $U \subset X$ is an open subset and T is a locally finite triangulation of U. Define $C_i^T(U; R)$ to be the group of T-simplicial chains in U, that is, finite sums $\xi = \sum a_j \sigma_j$ where $a_j \in R$ and $\sigma_j \subset U$ is a (closed) i dimensional simplex, with the usual boundary operator. If T' refines T there is a canonical inclusion $C_i^T(U; R) \to C_i^{T'}(U; R)$. Set $C_i^{PL}(U; R) =$

---

[4] For more general rings it is necessary to replace R by an injective resolution, in which case the Hom above becomes a double complex and ω_{BM}^{-r} is defined to be the associated single complex.

$\varinjlim C_i^T(U;\, R)$. If $V \subset U$ are open and if T is a locally finite triangulation of U there exists a locally finite triangulation T' of V so that every simplex of T' is contained in a single simplex of T so we obtain restriction maps $C_i^{PL}(U;\, R) \to C_i^{PL}(V : R)$ and therefore a soft sheaf $\mathbf{C}_i^{PL}$ on X of "locally finite chains" or "infinite chains" on X. The complex of soft sheaves $\omega_{PL}^{\bullet}$ with $\omega_{PL}^{-i} = \mathbf{C}_i^{PL}$ and $d = \partial$ (the differential increases degree) is quasi-isomorphic to the Borel-Moore complex.

5.6.6 Complexes of Sheaves

A (bounded below) complex of sheaves (of abelian groups)

$$\cdots \longrightarrow S^0 \xrightarrow{d^0} S^1 \xrightarrow{d^1} S^2 \xrightarrow{d^2} \cdots$$

on a topological space X is a collection $\{S^i\}$ ($i \in \mathbb{Z}$) which vanish for i sufficiently small, and satisfy $d \circ d = 0$. For each $x \in X$ there is a resulting complex of stalks, $\cdots \to S_x^0 \to S_x^1 \to \cdots$ whose cohomology $H^i(S_x^{\bullet})$ is called the *stalk cohomology* of the complex $S^{\bullet}$. Since sheaves form an abelian category we may form the cohomology of the sequence $S^{\bullet}$ in the category of sheaves. Thus, the i-th cohomology sheaf of $S^{\bullet}$ is

$$\mathbf{H^i}(S^{\bullet}) = \mathbf{ker}(d^i)/\mathbf{Im}(d^{i-1})$$

and its stalk coincides with the stalk cohomology, that is, $\mathbf{H_x^i}(S^{\bullet}) = H^i(S_x^{\bullet})$. A *morphism* $S^{\bullet} \to T^{\bullet}$ of complexes of sheaves is a collection of sheaf morphisms $S^r toT^r$ that commute with the differentials. It is said to be a *quasi-isomorphism* if it induces isomorphisms $\mathbf{H}^r(S^{\bullet}) \to \mathbf{H}^r(T^{\bullet})$ for all r, which is the same as saying that it induces an isomorphism on stalk cohomology $\mathbf{H}_x^r(S^{\bullet}) \cong \mathbf{H}_x^r(T^{\bullet})$ for all r and for all $x \in X$. If each T^r is injective (resp. flabby, resp. soft) then such a quasi-isomorphism is said to be an injective (resp. flabby, resp. soft) *resolution* of $S^{\bullet}$.

To find such a resolution, first choose injective (resp. flabby, resp. soft) resolutions of each S^j so that these fit together into a commuting double complex with horizontal and vertical differentials d_h, d_v respectively,

$$\begin{tikzcd}
S^2 \arrow[r, hook] & I^{02} \arrow[r] & I^{12} \arrow[r] & I^{22} \arrow[r] & \cdots \\
S^1 \arrow[r, hook] \arrow[u, "d_S"'] & I^{01} \arrow[r] \arrow[u] & I^{11} \arrow[r] \arrow[u] & I^{21} \arrow[r] \arrow[u] & \cdots \\
S^0 \arrow[r, hook] \arrow[u, "d_S"'] & I^{00} \arrow[r, "d_h"'] \arrow[u, "d_v"'] & I^{10} \arrow[r] \arrow[u] & I^{20} \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

Define the associated *single complex* $J^\bullet$ by adding along diagonals,

$$J^r = \bigoplus_{p+q=r} I^{pq} \quad \text{with } d(c_{pq}) = (d_h + (-1)^p d_v)c_{pq}$$

for $c_{pq} \in I^{pq}$. Then $d \circ d = 0$ and the homomorphism $S^\bullet \to J^\bullet$ is a quasi-isomorphism, hence an injective (resp. flabby, resp. soft) resolution of $S^\bullet$.

5.6.7 Cohomology

Let $S^\bullet$ be a complex of sheaves on a topological space X. The total cohomology

$$H^r(X; S^\bullet) = H^r(\Gamma(X, J^\bullet))$$

is defined to be the cohomology of the complex of global sections of any injective, flabby or soft resolution $J^\bullet$ of $S^\bullet$. It is independent of the resolution and, more generally, the following fact from homological algebra is messy but straight forward. It is the main technical tool for establishing cohomology isomorphisms because it reduces such questions to isomorphisms on stalk cohomology.

Theorem 5.6.4 *Let $S^\bullet$ be a complex of sheaves and let I^{pq} be a double complex of injective resolutions as above. Then this double complex determines a spectral sequence with*

$$E_2^{pq} = H^p(X; \mathbf{H}^q(S^\bullet)) \implies H^{p+q}(X; S^\bullet).$$

Consequently a quasi-isomorphism $S^\bullet \to T^\bullet$ induces an isomorphism

$$H^r(U; S^\bullet) \to H^r(U; T^\bullet)$$

for any open subset $U \subset X$ and for all r.

5.7 Derived Category and Constructible Sheaves

5.7.1 Construction of the Derived Category

Good general references for derived categories are [23, 24]; a quick summary is in [30]. Grothendieck recognized (see Theorem 5.6.4) that for most purposes, quasi-isomorphic (complexes of) sheaves behave alike, so there should be a category in which such sheaves become isomorphic. This dream was realized by Jean-Louis Verdier ([79, 80], who added enough morphisms to the category of sheaves so that every quasi-isomorphism acquired an inverse. An object in the (bounded) derived category

$$D^b(X) = D^b(Sh_X)$$

of sheaves on X is a complex of sheaves $A^\bullet$, bounded from below ($A^j = 0$ for $j \ll 0$), whose cohomology sheaves are also bounded from above $\mathbf{H}^j(A^\bullet) = 0$ for $j \gg 0$). A morphism $A^\bullet \to B^\bullet$ is an equivalence class of diagrams

$$\begin{tikzcd} & C^\bullet \arrow[dl, "qi"'] \arrow[dr] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$

where $C^\bullet \to A^\bullet$ is a quasi-isomorphism, and where two such morphisms $A^\bullet \leftarrow C_1^\bullet \to B^\bullet$ and $A^\bullet \leftarrow C_2^\bullet \to B^\bullet$ are considered to be equivalent if there exists a diagram that is commutative up to chain homotopy:

$$\begin{tikzcd} & C_1^\bullet \arrow[dl] \arrow[dr] & \\ A^\bullet & C_3^\bullet \arrow[l, "qi"'] \arrow[u] \arrow[d] \arrow[r] & B^\bullet \\ & C_2^\bullet \arrow[ul] \arrow[ur] & \end{tikzcd}$$

5.7.2 *Derived Functor*

If $A^\bullet \to B^\bullet$ is a quasi-isomorphism of complexes of sheaves and if each B^j is injective then there exists an inverse up to homotopy, $B^\bullet \to A^\bullet$. In fact, the homotopy category of (bounded below) complexes of injective sheaves is equivalent to the derived category (see [24, §III.5]). Moreover, there is a canonical functorial construction of an injective resolution of any complex of sheaves, due to Godement. Accordingly, if T is a *left exact* functor from the category of sheaves $\mathcal{S}h_X$ to some abelian category $\mathcal{A}$, Verdier defines its right derived functor $RT(S^\bullet) = T(J^\bullet)$ where $S^\bullet \to J^\bullet$ is the Godement injective resolution. This procedure passes to the derived category producing a *right derived functor*

$$RT : D^b(\mathcal{S}h_X) \to D^b(\mathcal{A}).$$

It is possible to replace the injective resolution $J^\bullet$ with any T-acyclic or T-adapted resolution, see [23, §4.3] or [24, §III.6.3]. For the functor $T = \Gamma$ of global sections, and for the functors $T = f_*, f_!$ (push forward, push forward with proper support, see below), fine sheaves and soft sheaves are T-acyclic.

5.7.3 Derived Push Forward

If $f : X \to Y$ is a continuous map and S is a sheaf on X then its push forward is denoted $f_*(S)$. If $A^\bullet$ is a complex of sheaves on X the derived functor is $Rf_*(A^\bullet) = f_*(J^\bullet)$ where $A^\bullet \to J^\bullet$ is an injective, flabby or soft resolution of $A^\bullet$ and there is a canonical isomorphism

$$H^i(X; A^\bullet) \cong H^i(Y; Rf_*(A^\bullet)) \tag{5.9}$$

for all i, which is to say that the cohomology of X may be computed locally on Y. In many applications this isomorphism replaces arguments involving the Leray-Serre spectral sequence, illustrating the power and convenience of the derived category.

If f is proper, the stalk cohomology of the derived push forward is $H^i_y(Rf_*(A^\bullet)) = H^i(f^{-1}(y); A^\bullet)$ and the cohomology sheaf $\mathbf{H}^i(Rf_*(A^\bullet))$ is classically denoted $R^i f_*(A^\bullet)$. The push forward with proper support and its derived functor are denoted $f_!$ and $Rf_!$ respectively.

5.7.4 Mapping Cone

The following construction works in any Abelian category $\mathcal{A}$ but we are mostly concerned with the category of sheaves on some space. The *mapping!cone* $C^\bullet = C^\bullet(\phi)$ of a morphism $\phi : A^\bullet \to B^\bullet$ of complexes is the complex $C^r = A^{r+1} \bigoplus B^r$ with differential $d_C(a, b) = (d_A(a), (-1)^{\deg(a)}\phi(a)+d_B(b))$. It is the total complex of the double complex

$$\begin{tikzcd}
{} & {} \\
A^2 \arrow[u, "d"] \arrow[r, "\phi"] & B^2 \arrow[u, "d"'] \\
A^1 \arrow[u, "d"] \arrow[r, "\phi"] & B^1 \arrow[u, "d"'] \\
A^0 \arrow[u, "d"] \arrow[r, "\phi"] & B^0 \arrow[u, "d"']
\end{tikzcd}$$

with morphisms $\beta : B^\bullet \to C^\bullet$ and $\gamma : C^\bullet \to A[1]^\bullet$ where $A[1]^j = A^{j+1}$. Denote this by:

$$\begin{tikzcd}
A^\bullet \arrow[rr, "\phi"] & & B^\bullet \arrow[dl] \\
& C^\bullet(\phi) \arrow[ul, "{[1]}"] &
\end{tikzcd}$$

Lemma 5.7.1 *If ϕ is injective then there is a natural quasi-isomorphism*

$$\mathrm{coker}(\phi) \cong C^\bullet(\phi).$$

If ϕ is surjective then there is a natural quasi-isomorphism

$$C^\bullet(\phi) \cong \ker(\phi)[1].$$

There are natural quasi-isomorphisms $A^\bullet[1] \cong C^\bullet(\beta)$ and $B^\bullet[1] \cong C^\bullet(\gamma)$ so that any side of this triangle determines the third element up to quasi-isomorphism. This triangle determines a long exact sequence on cohomology

$$\cdots \to \mathbf{H}^{r-1}(B^\bullet) \to \mathbf{H}^{r-1}(C^\bullet) \to \mathbf{H}^r(A^\bullet) \to \mathbf{H}^r(B^\bullet) \to \mathbf{H}^r(C^\bullet) \to \cdots .$$

A triangle of morphisms of complexes

$$\begin{tikzcd}
X^\bullet \arrow[rr] & & Y^\bullet \arrow[dl] \\
& Z^\bullet \arrow[ul, "{[1]}"] &
\end{tikzcd}$$

is said to be a *distinguished triangle* if it is homotopy equivalent to a triangle

$$\begin{tikzcd}
A^\bullet \arrow[rr, "\phi"] & & B^\bullet \arrow[dl] \\
& C^\bullet(\phi) \arrow[ul, "{[1]}"] &
\end{tikzcd}$$

5.7.5 Restriction to Subspaces

There are two ways to restrict a sheaf S on a topological space X to a closed subspace $i_Z : Z \to X$. The ordinary restriction $S|Z = i_Z^* S$ is the sheaf whose leaf space (Sect. 5.6.1) is $\pi^{-1}(Z)$ where $\pi : LS \to X$ is the leaf space of S. Let $j : U = X - Z \to X$ be the inclusion. Then there is a distinguished triangle:

$$\begin{tikzcd}
Rj_!j^*(A^\bullet) \arrow[rr] & & A^\bullet \arrow[dl] \\
& Ri_{Z*}i_Z^*(A^\bullet) \arrow[ul, "{[1]}"] &
\end{tikzcd}$$

The long exact cohomology sequence is that of the pair

$$H^r(X, Z; A^\bullet) = H^r(X; Rj_!j^*A^\bullet).$$

The second type of restriction, denoted $i_Z^! S$, is the restriction to Z of the presheaf with sections supported in Z, that is

$$i_Z^!(S) = i_Z^*(S_Z) \text{ where } \Gamma(V, S_Z) = \{s \in \Gamma(V, S) | \operatorname{supp}(s) \subset Z\}.$$

The group of global sections is

$$\Gamma(Z, i_Z^!(S)) = \varinjlim_{V \supset Z} \Gamma_Z(V, S)$$

(the limit is over open sets $V \subset X$ containing Z). The functor $i_Z^!$ is a right adjoint to the pushforward with compact support $(i_Z)_!$. For any $A^\bullet \in D^b(X)$ there is a distinguished triangle,

$$\begin{tikzcd}
Ri_{Z*}i_Z^!(A^\bullet) \arrow[rr] & & A^\bullet \arrow[dl] \\
& Rj_*j^*(A^\bullet) \arrow[ul, "{[1]}"] &
\end{tikzcd} \qquad (5.10)$$

which gives the long exact sequence for the pair:

$$H^r(X, X - Z; A^\bullet) \cong H^r(X; Ri_{Z*}i_Z^!A^\bullet).$$

The object $i_Z^!A^\bullet$ is denoted $R\Gamma_Z A^\bullet$ in [44].

5.7.6 Constructible Sheaves

Let X be a Whitney stratified space. A complex of sheaves $A^\bullet$ on X is said to be (cohomologically) constructible with respect to this stratification if each of the cohomology sheaves $\mathbf{H}^r(A^\bullet)$ is locally constant on each stratum of X and its stalk is finite dimensional at each point. The constant sheaf, the sheaf of singular cochains, and the Borel-Moore sheaf of chains are constructible. If we do not specify a stratification, a complex of sheaves on X is said to be *constructible* if it is (cohomologically) constructible with respect to *some* Whitney stratification. If X is real or complex algebraic, analytic, or subanalytic then the relevant stratification is assumed to be algebraic, analytic, etc. The *constructible derived category* $D^b_c(X)$ is the corresponding full subcategory of the derived category.

Lemma 5.7.2 *Suppose $A^\bullet$ is a complex of sheaves, constructible with respect to some Whitney stratification of X. Let $x \in X$ and let U_x be a basic neighborhood as described in Sect. 5.4.2. Then there is a canonical isomorphism*

$$H^r(U_x; A^\bullet) \cong \mathbf{H}^r_x(A^\bullet)$$

between the sheaf cohomology of U_x and the stalk cohomology of $A^\bullet$.

The stratified homeomorphism $\overline{U}_x \cong D^s \times N_\epsilon(x)$ of Eq. (5.6) and the constructibility hypothesis imply that $H^r(U_x; A^\bullet) \cong H^r(N_\epsilon(x); A^\bullet)$. But the normal slice is a cone and the cohomology sheaves $\mathbf{H}^r(A^\bullet)$ are locally constant along the cone lines. So there is a one parameter family of shrinking maps $\theta_t : N_\epsilon(x) \to N_\epsilon(x)$ (with θ_1 the identity and θ_0 the map to the cone point) inducing quasi-isomorphisms $\theta_t^*(A^\bullet) \to A^\bullet$ for all $t > 0$. Therefore its cohomology $H^r(N_\epsilon(x))$ coincides with $\mathbf{H}^r_x(A^\bullet)$. The local topological triviality of a stratification gives the following fact, crucial for many arguments involving constructible sheaves because it produces a constructible sheaf on a larger set than it started with:

Lemma 5.7.3 *Suppose X is Whitney stratified and $\Sigma \subset X$ is a closed union of strata with complement $U = X - \Sigma$ and inclusion $j : U \to X$. Let $A^\bullet$ be a complexes of sheaves on U that is constructible with respect to the (induced) stratification of U. Then the complexes $Rj_*(A^\bullet)$ and $Rj_!(A^\bullet)$ on X are constructible with respect to the given stratification of X.*

5.7.7 Verdier Duality

Let X be a Whitney stratified space as above. The sheaf $\omega^\bullet_X$ of Borel-Moore chains is a *dualizing complex* and the dual of a complex of sheaves $A^\bullet \in D^b_c(X)$ is defined to be the sheaf

$$\mathbb{D}_X(A^\bullet) = \mathbf{RHom}^\bullet(A^\bullet, \omega^\bullet_X).$$

(See [11, 23, 79, 80, §5.16], [39, §6], [44, §3], [21, §3].) The operation **RHom** may be replaced by **Hom** if an injective model[5] is used for $\omega^\bullet_X$. The dual of the constant sheaf is $\omega^\bullet_X$. There is a canonical double duality isomorphism $\mathbb{D}_X(\mathbb{D}_X(A^\bullet)) \cong A^\bullet$

---

[5] If the coefficient ring is a field then a flabby or soft model of ω_X suffices, see footnote 3.

in $D^b_c(X)$. If $f : X \to Y$ then duality switches Rf_* and $Rf_!$. It also switches f^* with $f^!$ which may be taken to give a definition of $f^!$, that is,

$$f^!(A^\bullet) = \mathbb{D}_X(f^*(\mathbb{D}_Y(A^\bullet))$$

which agrees with the operation $i^!$ of Sect. 5.7.5 for closed embeddings $i : Z \to W$ and agrees with $j^* = j^!$ for open embeddings $j : U \to W$.

5.8 Morse Theory of Constructible Sheaves

5.8.1 Basic Result

Throughout this section we fix a Whitney stratified closed subset $W \subset M$ and a complex of sheaves $A^\bullet$ on W that is (cohomologically) constructible with respect to this stratification. Since the homeomorphism in Theorem 5.5.3 is stratum preserving, the same deformation argument as in Lemma 5.7.2 gives the following.

Theorem 5.8.1 *Let $f : M \to \mathbb{R}$ be a smooth function and suppose $f|W$ is proper. Suppose $X \subset W$ is a stratum and that $x_0 \in X$ is a nondegenerate critical point of f with isolated critical value $v = f(x_0) \in (a, b)$ and Morse index λ. Suppose there are no more critical values of $f|W$ in the interval $[a, b]$. Choose $0 < \delta \ll \epsilon$ as in (*) and (**) and let $N = N_\epsilon(x_0)$ be the normal slice. Then there is a natural isomorphism of* Morse groups

$$H^r(W_{\leq b}, W_{\leq a}; A^\bullet) \cong H^{r-\lambda}\left(N_{[v-\delta,v+\delta]}, N_{v-\delta}; A^\bullet|N\right).$$

5.8.2 Sheaf Theoretic Expression

Kashiwara and Schapira [44, §5.1, §5.4] and Schürmann [70] prefer a sheaf-theoretic expression for the Morse group. Let

$$x_0 \in X \subset W \subset M \xrightarrow{f} \mathbb{R}$$

as in Theorem 5.8.1 above and suppose $f(x_0) = 0$ is an isolated critical value of $f|W$. Let $A^\bullet \in D^b_c(W)$ be a constructible complex of sheaves. Set

$$Z = \{x \in W | f(x) \geq 0\}$$

with inclusion $i : Z \to W$ and let $S_Z^\bullet = i_Z^! A^\bullet = R\Gamma_Z A^\bullet$ denote the sheaf obtained from $A^\bullet$ with sections supported in Z, cf. Sect. 5.7.5. Let $U = B_\epsilon(x_0) \cap W$ be a basic neighborhood of the critical point x_0. If $a < 0 < b$ and $[a, b]$ contains no critical values other than 0 then for $0 < \delta \ll \epsilon$ Thom's first isotopy lemma (Theorem 5.5.1 above) gives isomorphisms of the Morse groups:

$$\begin{aligned} H^r(W_{\leq b}, W_{\leq a}; A^\bullet) &\cong H^r(U_{\leq \delta}, U_{\leq -\delta}; A^\bullet) \\ &\cong H^r(U_{\leq \delta}, U_{<0}; A^\bullet) \\ &\cong H^r(U_{\leq \delta}; i_Z^! A^\bullet) \\ &\cong \mathbf{H}^{\mathbf{r}}_{x_0}(i_Z^! A^\bullet) = \mathbf{H}^{\mathbf{r}}_{x_0}(R\Gamma_Z A^\bullet) \end{aligned}$$

since the stalk cohomology is the limit as $\epsilon, \delta \to 0$, but changing ϵ, δ does not change the cohomology provided $0 < \delta \ll \epsilon$ remain in the region shown in Fig. 5.2, that is, they satisfy (*) Sect. 5.4.1 and (**) of Sect. 5.5.3. If we apply the main theorem in stratified Morse theory, this Morse group is identified with

$$\mathbf{H}^{\mathbf{r}-\lambda}_{x_0}(i_{Z\cap N}^!(A^\bullet|N))$$

where $N = T \cap W \cap B_\epsilon(p)$ denotes the normal slice to the stratum X. Except for the shift λ (which comes from the tangential Morse data), this expression depends only on the (nondegenerate) covector $\xi = df(x_0) \in \Lambda_X$, cf. Theorem 5.5.2. In Sect. 5.8.3 we arrange that $\lambda = 0$.

5.8.3 *Characteristic Cycle*

Let $A^\bullet$ be a constructible complex of sheaves on a Whitney stratified subset $W \subset M$. From the preceding paragraph, for each stratum X of W, for each point $x_0 \in X$ and for each nondegenerate conormal vector $\xi \in \Lambda_X$ at x_0 there is a collection of *Morse groups*

$$H^r(\xi; A^\bullet) := H^r(N_{\leq \delta}, N_{<0}; A^\bullet|N) \cong H^r_{x_0}(i_{Z\cap N}^!(A^\bullet|N)) \cong H^r_{x_0}(i_Z^!(A^\bullet))$$

that measures the local change in cohomology for any smooth function $\phi : M \to \mathbb{R}$ chosen so that

- $\phi(x_0) = 0$
- $d\phi(x_0)|T_{x_0}X = 0,$
- $d\phi(x_0) = \xi \in \Lambda_X$ is nondegenerate
- $\phi|X$ has a local nondegenerate minimum at x_0

If ξ varies within a single connected component Λ_α of Λ_X the Morse group $H^r(\xi, A^\bullet)$ does not change, nor does the Euler characteristic

$$\chi(\xi; A^\bullet) = \sum_{r \geq 0} (-1)^r \mathrm{rank}(H^r(\xi; A^\bullet) \otimes \mathbb{Q}). \tag{5.11}$$

Kashiwara's idea [42] is to use these coefficients to create a Lagrangian cycle. (cf. [44, §IX], [70, §5.2].)

Each $T^*_X M$ is a smooth Langrangian submanifold of T^*M and the union $\bigcup_X T^*_X M$ is closed by Whitney's condition A. However, the closure $\overline{T^*_X M}$ could be wild unless we assume, as we do for the rest of this article, that W is a subanalytic subset of an orientable real analytic manifold M Then an orientation of M induces an orientation on $T^*_X M$ (cf. [69, §2]) and the set of nondegenerate conormal vectors $\Lambda_X \subset T^*_X M$ breaks into finitely many connected components Λ_α. Define the characteristic cycle

$$CC(A^\bullet) = \sum_X \sum_\alpha m_\alpha \left[\overline{\Lambda_\alpha}\right] \tag{5.12}$$

where the first sum is over strata X, the second sum is over connected components Λ_α of Λ_X, where $m_\alpha = \chi(\xi; A^\bullet) \in \mathbb{Z}$ for any $\xi \in \Lambda_\alpha$, and where Λ_α is oriented as above.

Theorem 5.8.2 ([42]) *If $A^\bullet$ is cohomologically constructible then $CC(A^\bullet)$ is well defined and is a Borel-Moore Lagrangian cycle in $H_n^{BM}(T^*M)$ supported on $T^*_W M = \bigcup_X T^*_X M$.*

For any triangulation of $CC(A^\bullet)$ the interior of each n-dimensional simplex will be contained in a connected component Λ_α of the nondegenerate covectors of some stratum X, and so the above prescription will define a simplicial chain (with infinite support) in T^*M. Kashiwara's theorem is that its homological boundary vanishes, cf. the discussion [70, §5.0.1].

The characteristic cycle construction is natural with respect to push forward, pullback and Verdier duality [44, §9.4]. The characteristic cycle has many applications in the theory of $\mathcal{D}$-modules [55], the Gauss-Manin connection [46] and representation theory [69, 75].

5.8.4 Euler Characteristic

Let $A^\bullet$ be a complex of sheaves of k-vector spaces (where k is a field) that is constructible with respect to a Whitney stratification (with connected strata) of a closed subanalytic subset $W \subset M$. The Euler characteristic of the stalk cohomology at a point $x \in W$ is

$$\chi_x(A^\bullet) = \sum_{j \geq 0} (-1)^j \dim H_x^j(A^\bullet). \tag{5.13}$$

It is independent of the point x as it varies within a single connected component of a single stratum, which is to say that it is a *constructible function*.

The Euler characteristic with compact support is additive. Therefore, if the cohomology with compact support $H_c^r(W; A^\bullet)$ is finite dimensional for all r then the Euler characteristic with compact support

$$\chi_c(W; A^\bullet) = \sum_{r \geq 0} (-1)^r \dim H_c^r(W; A^\bullet) = \sum_X \chi_c(X) \chi_x(A^\bullet) \tag{5.14}$$

is defined and finite, where the sum is over the strata of W and $x \in X$. *Kashiwara's index theorem* [41] says that if W is compact then the Euler characteristic is the intersection product of the zero section with the characteristic cycle:

$$\chi(W; A^\bullet) = \chi_c(W; A^\bullet) = T_M^* M \cap CC(A^\bullet).$$

5.9 Complex Stratified Morse Theory

5.9.1 Levi Form

Let M be a complex n dimensional manifold and let $f : M \to \mathbb{R}$ be a smooth function. The *E. E. Levi form* at $x \in M$ is the Hermitian form

$$L_f(x) = \partial \bar{\partial} f(x) = \left(\frac{\partial^2 f(x)}{\partial z_i \partial \bar{z}_j} \right)$$

defined on the tangent space $T_x M$. The associated quadratic form satisfies

$$L_f(x)(v) = \sum_{i,j} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} v_i \bar{v}_j = \frac{1}{4}(H_f(x)(v) + H_f(x)(iv))$$

where

$$H_f(x)(v) = \sum_{i,j} \frac{\partial^2 f(x)}{\partial x_i \partial x_j} v_i v_j$$

is the quadratic form associated to the Hessian of f at x by forgetting the complex structure on M. If $N \subset M$ is a complex submanifold containing x then $L_f(x)|N = L_{f|N}(x)$ but the same does not hold for H_f unless $df(x) = 0$.

Now suppose that $df(x) = 0$ and that $H_f(x)$ is nondegenerate. Let $\lambda_x(f)$ be the Morse index of f at x, that is, the (real) dimension of the largest (real) subspace of T_xM on which H_f is negative definite. Let $\sigma_x(f)$ the complex dimension of the largest complex subspace on which L_f is negative definite and let $\nu_x(f)$ be the nullity of L_f. An exercise (see [31, §4.A.2], [5, p. 311]) gives:

$$\lambda_x(f) \geq \sigma_x(f) + \nu_x(f)$$
$$\lambda_x(-f) \geq \sigma_x(-f) + \nu_x(f).$$

If $f : \mathbb{C}^N \to \mathbb{R}$ is the distance from a codimension r linear subspace of $\mathbb{C}^N$ and $M \subset \mathbb{C}^N$ is a submanifold of complex dimension n and if $f|M$ has a nondegenerate critical point at $x \in M$ then the Morse index λ of $f|M$ at x satisfies $n \geq \lambda \geq n - r$.

5.9.2 Local Structure of Complex Varieties

Throughout this chapter, $W \subset M$ denotes a complex analytic subvariety of a complex analytic variety, together with a complex analytic stratification *with connected strata* of W. If X is a stratum of W and $p \in X$ then there is a canonical isomorphism of real vector spaces

$$T_p^*(X) = \mathrm{Hom}_{\mathbb{R}}(T_pX, \mathbb{R}) \cong \mathrm{Hom}_{\mathbb{C}}(T_pX, \mathbb{C}). \tag{5.15}$$

In this case, a theorem of B. Teissier [76] states that the set of degenerate covectors (that is, $\xi \in T_X^*M$ such that $\xi(\tau) = 0$ for some limit τ of tangent spaces from some larger stratum $Y > X$) form a proper complex analytic (conical) subvariety of the conormal space $T_{X,p}^*M$ of complex codimension ≥ 1. So its complement $\Lambda_{X,p}$ is connected and the normal Morse data (Sect. 5.5.3) is independent (up to stratum preserving homeomorphism) of the choice of covector $\xi \in \Lambda_X$.

By choosing local coordinates near $p \in M$, replacing M by $\mathbb{C}^m$, any $\xi \in T_p^*(M)$ may be realized as the differential of a complex linear function $\pi : \mathbb{C}^m \to \mathbb{C}$, or equivalently, using (5.15) as the differential of a real linear function $\phi = Re(\pi) : \mathbb{C}^m \to \mathbb{R}$. By choosing an analytic submanifold T transversal to X with $T \pitchfork X = \{p\}$ we may also arrange (locally) that the normal slice $N = T \pitchfork W$ to the stratum X is a closed complex analytic subvariety of $\mathbb{C}^m$, Whitney stratified with strata $T \cap Y$ where $Y \geq X$ runs through strata of W. It has a zero dimensional stratum $T \cap X = \{p\} = \{0\}$, and we may assume that $\pi(0) = 0$.

5.9.3 Complex Link

Assume as above that $N \subset \mathbb{C}^m$, $\{p\} = \{0\} \subset N$ is a zero dimensional stratum, that $\pi : \mathbb{C}^m \to \mathbb{C}$ is linear and $\xi = d\pi(0) \in \Lambda_X$ is a nondegenerate covector. Let $r(z)$ denote the square of the distance in $\mathbb{C}^m$ from the origin. As in Sect. 5.5.3 there is an open region $0 < \delta \ll \epsilon \subset \mathbb{R}^2$ such that for any pair (δ, ϵ) in this region the following holds:

- $\partial B_\epsilon(0)$ is transverse to each stratum of N
- for each stratum $Y \cap T$ of N (where $Y > X$) the restriction $\pi|(Y \cap T)$ has no critial points with critical values in the disk $D_\delta(0)$ except for the case $\{0\} = X \cap T$
- for each stratum $Y \cap T$ of N and for any point $z \in Y \cap T \cap \partial B_\epsilon(0)$ such that $|\pi(z)| \leq \delta$, the complex linear map

$$(dr(z), d\pi(z)) : T_z(Y \cap T) \to \mathbb{C}^2$$

has rank 2. (Such points z do not exist if $\dim(Y \cap T) < 2$.)

With this data, identify $\delta = \delta + 0i \in \mathbb{C}$ and define the complex link

$$\mathcal{L} = \pi^{-1}(\delta) \cap N \cap B_\epsilon(0),\ \partial\mathcal{L} = \pi^{-1}(\delta) \cap N \cap \partial B_\epsilon(0). \tag{5.16}$$

It is a single fiber of the (stratified) fiber bundle over the circle $S^1 = \partial D_\delta$,

$$\mathcal{E} = \pi^{-1}(\delta e^{i\theta}) \cap N \cap B_\epsilon(0),\ \partial\mathcal{E} = \pi^{-1}(\delta e^{i\theta}) \cap N \cap \partial B_\epsilon(0)$$

$$(\mathcal{E}, \partial\mathcal{E}) \to S^1 = \partial D_\delta = \{\delta e^{i\theta} |\, 0 \leq \theta \leq 2\pi\}$$

and the boundary $\partial\mathcal{E}$ is a trivial bundle over $S^1 = \partial D_\delta$. See Fig. 5.4.

Theorem 5.9.1 ([31]) *The bundle $\mathcal{E}$ is stratified-homeomorphic to the mapping cylinder of a (stratified) monodromy homeomorphism*

$$\mu : (\mathcal{L}, \partial\mathcal{L}) \to (\mathcal{L}, \partial\mathcal{L})$$

that is the identity on $\partial\mathcal{L}$ and is well defined up to stratum preserving isotopy. The link $L_X(p)$ (Eq. (5.5)) is homeomorphic to the "cylinder with caps",

$$L_X(p) \cong \mathcal{E} \cup_{\partial\mathcal{E}} (\partial\mathcal{L} \times D_\delta)\,. \tag{5.17}$$

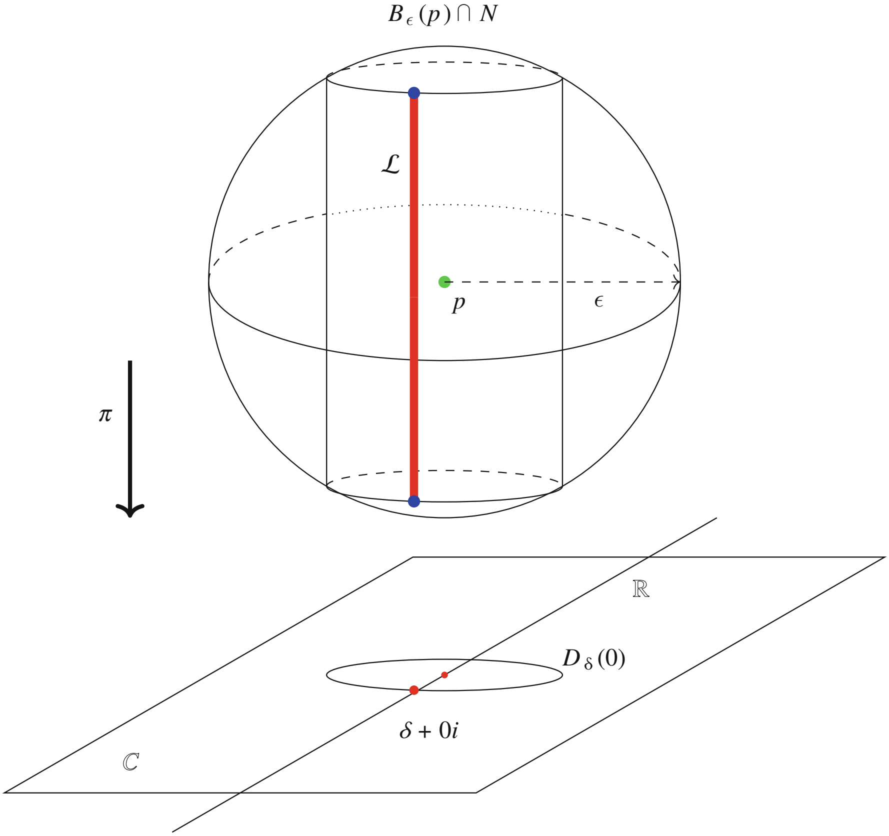


Fig. 5.4 Normal slice with complex link, ϵ-ball and δ-disk

5.9.4 *First Consequences*

Let $f : W \to \mathbb{R}$ be a Morse function with a nondegenerate critical point $p \in X \subset W$. Let $\mathcal{L}$ be the complex link of the stratum X. Since the set Λ_X of nondegenerate covectors is connected, the complex link is independent (up to stratum preserving homeomorphism) of the covector $\xi \in \Lambda_X$ that is used in its definition and we may take $\xi = df(p)$. Let $A^\bullet \in D^b_c(W)$ be a constructible complex of sheaves.

Theorem 5.9.2 *The normal Morse data for f at the critical point $p \in X \subset W$ has the homotopy type of the pair*

$$(cone(\mathcal{L}), \mathcal{L}) .$$

The Morse group $H^r(\xi, A^\bullet)$ *at* p *is:*

$$H^r(N, N_{<0}; A^\bullet) \cong H^r(N_\epsilon(p), \mathcal{L}; A^\bullet) = \mathbf{H}^r_p(R\Gamma_Z A^\bullet)$$

where $Z = \{z \in N | \ f(z) \geq 0\}$*, cf. Sect. 5.8.2 and* $N_\epsilon(p) = N \cap B_\epsilon(p)$*.*

5.10 Complex Morse Theory of Sheaves

5.10.1 The Braid Diagram

Throughout this section we fix a constructible complex of sheaves $A^\bullet \in D^b_c(W)$ and $x \in X$. The homeomorphisms described in the preceding section are stratum preserving so they induce isomorphisms on cohomology with coefficients in $A^\bullet$ and they allow us to interpret these cohomology groups,

$$\begin{aligned} H^r(N - x; A^\bullet) &\cong H^r(L_X(x); A^\bullet) & \qquad H^r(N, N - x; A^\bullet) &\cong H^r(i^!_x A^\bullet) \\ H^r(N; A^\bullet) &\cong H^r(i^*_x A^\bullet) & H^r(N, N_{<0}; A^\bullet) &\cong H^r(\xi; A^\bullet) \\ H^r(N_{<0}; A^\bullet) &\cong H^r(\mathcal{L}; A^\bullet) & H^{r+1}(N - x, N_{<0}; A^\bullet) &\cong H^r(\mathcal{L}, \partial\mathcal{L}; A^\bullet) \end{aligned}$$

By (5.17) the "variation" map $I - \mu : H^r(\mathcal{L}; A^\bullet) \to H^r(\mathcal{L}, \partial\mathcal{L}; A^\bullet)$ may be identified with the connecting homomorphism in the third row of this display, that is, the long exact sequence for the pair $(N - x, N_{<0})$, cf. [28]. As in [31, p. 215] the long exact sequences for the triple of spaces

$$N_{<0} \subset N - \{x\} \subset N$$

may be assembled into a braid diagram with exact sinusoidal rows (Fig. 5.5). (cf. [70, §6.1] where the same sequences are considered separately):

$$\begin{tikzcd}
H^r(i^*_x A^\bullet) \arrow[rr, bend left] \arrow[dr] & & H^r(\mathcal{L}, A^\bullet) \arrow[rr, bend left, "var"] \arrow[dr] & & H^r(\mathcal{L}, \partial\mathcal{L}; A^\bullet) \arrow[rr, bend left] \arrow[dr] & & H^{r+2}(i^!_x A^\bullet) \\
& H^r(L; A^\bullet) \arrow[ur] \arrow[dr] & & H^{r+1}(\xi; A^\bullet) \arrow[ur] \arrow[dr] & & H^{r+1}(L; A^\bullet) \arrow[ur] \arrow[dr] & \\
H^{r-1}(\mathcal{L}, \partial\mathcal{L}; A^\bullet) \arrow[ur] \arrow[rr, bend right] & & H^{r+1}(i^!_x A^\bullet) \arrow[ur] \arrow[rr, bend right] & & H^{r+1}(i^*_x A^\bullet) \arrow[ur] \arrow[rr, bend right] & & H^{r+1}(\mathcal{L}; A^\bullet)
\end{tikzcd}$$

Fig. 5.5 Braid diagram

5.10.2 *Euler Characteristics and the Characteristic Cycle*

The results in Sects. 5.8.3 and 5.8.4 become simpler in the presence of a complex structure. Let W be a complex analytic variety with a fixed analytic Whitney stratification with connected strata. Let $A^\bullet \in D^b_c(W)$. The Euler characteristics of the normal Morse data (5.11), of the complex link (5.16), and of the stalk cohomology (5.13) are constructible functions (that is, constant on strata). For a stratum X of W we denote these values by

$$m_X(A^\bullet),\ \chi(\mathcal{L}, A^\bullet),\ \chi_x(A^\bullet) = m_X(A^\bullet) + \chi(\mathcal{L}, A^\bullet)$$

respectively (where $x \in X$). Equation (5.14) becomes[6] the sum over strata,

$$\chi(W; A^\bullet) = \sum_X \chi(X)\chi_x(A^\bullet) = \chi_c(W; A^\bullet)$$

(where $x \in X$) assuming the cohomology $H^*(W; A^\bullet)$ is finite dimensional.

The characteristic cycle (5.12) of $A^\bullet$ is the sum over all strata $X \subset W$:

$$CC(A^\bullet) = \sum_X m_X(A^\bullet)\left[\overline{T^*_X M}\right].$$

Euler characteristics of the normal Morse data and of the stalk cohomology at a point $p \in W$ are related by a formula of [15], the sum over strata X such that $\overline{X} \ni p$:

$$\chi_x(A^\bullet) = \sum_X (-1)^{\dim_{\mathbb{C}} X} m_X(A^\bullet)\mathbf{Eu}_p(\overline{X}).$$

Here, $\mathbf{Eu}_p(\overline{X})$ is MacPherson's *local Euler obstruction* [53] and the sign $(-1)^{\dim_{\mathbb{C}} X}$ arises due to choice of orientations, cf. [70, §5.0.3].

When $A^\bullet = \mathbb{Q}$ this formula can be inverted, so as to express $\mathbf{Eu}_p(W)$ as a linear combination of the various $m_X(\mathbb{Q})$ by considering $\mathbf{Eu}_Y(\overline{X}) = \mathbf{Eu}_y(\overline{X})$ (where $y \in Y$) to be a square matrix of integers indexed by strata $Y < X$ (say, with respect to a total ordering of the strata that respects the natural ordering). It is lower triangular, of determinant 1, so it has an inverse that is also a lower triangular matrix of integers with 1s on the diagonal.

---

[6] By the universal coefficient theorem, the Euler characteristic may be computed with coefficients in any field. If X is a smooth n-dimensional manifold then $H^i_c(X; \mathbb{Z}/(2)) \cong H_{n-i}(X; \mathbb{Z}/(2))$ by Poincaré duality hence $\chi_c(X) = (-1)^n \chi(X)$.

5.10.3 Vanishing Conditions

The braid diagram, together with induction and the estimates in Sect. 5.9.1, may be used to prove many vanishing theorems and Lefschetz-type theorems in sheaf cohomology, see [28, 31, 34, 42, 70]. The following serve as illustrations. To avoid issues of torsion and injective resolutions of R-modules, from now on we assume that all sheaves are sheaves of vector spaces over a field k (usually $k = \mathbb{Q}$). Using the convex function distance^2 from the point $\{x\}$ below, and induction, one finds the following two results (cf. [3]) which may be proven together (since the statement for one becomes the inductive step for the other):

Theorem 5.10.1 *Suppose $A^\bullet \in D^b_c(W)$ is a complex of sheaves of k-vector spaces on a complex analytic set W such that for each stratum X and for each point $x \in X$, with $i_x : \{x\} \to W$, the stalk cohomology vanishes:*

$$H^r(i_x^* A^\bullet) = 0 \textit{ whenever } r > \operatorname{codim}_{\mathbb{C}}(X). \tag{5.18}$$

Then $H^r(\mathcal{L}; A^\bullet) = 0$ for all $r > \ell = \dim_{\mathbb{C}}(\mathcal{L})$. If the Verdier dual $\mathbb{D}_W(A^\bullet)$ satisfies (5.18), or equivalently, (if W has pure (complex) dimension n and)

$$H^r(i_x^! A^\bullet) = 0 \textit{ whenever } r < n + \dim_{\mathbb{C}}(X), \tag{5.19}$$

then $H^r(\mathcal{L}, \partial\mathcal{L}; A^\bullet) = 0$ for all $r < \ell$.

Theorem 5.10.2 *Let W be a Stein space or an affine complex algebraic variety of dimension n and let $A^\bullet \in D^b_c(W)$ be a complex of sheaves on W that satisfies (5.18). Then $H^r(W; A^\bullet) = 0$ for all $r > n$. Let W be a projective variety and let H be a hyperplane that is transverse to each stratum of some Whitney stratification of W. Let $A^\bullet$ be a complex of sheaves on W that satisfies (5.19). Then $H^r(W, W \cap H; A^\bullet) = 0$ for all $r < n$.*

5.10.4 Homotopy Version

Homotopy versions of these statements follow from the same induction, by replacing Theorem 5.10.1 with Theorem 5.10.3 below, see [31].

Theorem 5.10.3 *The complex link $\mathcal{L}$ of a point x in a stratum X of a complex analytically stratified complex analytic set $W \subset M$ has the homotopy type of a CW complex of dimension $\leq \ell = \dim_{\mathbb{C}}(\mathcal{L}) = \operatorname{codim}_W(X) - 1$. Moreover $\mathcal{L}$ may be obtained from $\partial\mathcal{L}$ by attaching cells of dimension $\geq \ell$.*

Consequently, a Stein space or affine complex algebraic variety of dimension n has the homotopy type of a CW complex with cells of dimension $\leq n$. Partially weakening the hypotheses in Theorems 5.10.1, 5.10.2, or 5.10.3 will result in a partial weakening of the conclusions, so Grothendieck's conjectures [33] on rectifiable homotopical depth and their homological analogues may be proven this way, see [34] and [70, §6.0].

5.10.5 Perverse Sheaves

The standard reference for this section is [8] but a great survey is [16]. See also [44, 48]. Suppose $W \subset M$ is an algebraic variety of pure dimension n. A complex of sheaves $A^\bullet$ on W is said to be (middle) *perverse* if it satisfies both (5.18) and (5.19) with respect to some[7] algebraic stratification of W. It is an *intersection complex* (IC) if it satisfies the stronger conditions, obtained by replacing $>$ with $\geq$ and $<$ with $\leq$ in (5.18) and (5.19). An IC sheaf on W is determined by its restriction E to the top stratum, which is (isomorphic to) a local coefficient system [30] so we may denote it unambiguously by $IC_W(E)$.

The category $\mathcal{P}(W)$ of perverse sheaves is the full subcategory of $D_c^b(W)$ whose objects are perverse. It is an Abelian, Artinian and Noetherian subcategory and is preserved under Verdier duality. The simple perverse sheaves are the shifted IC sheaves, $IC_V(E_V)[\dim(V)]$ of irreducible subvarieties $V \subset W$ and irreducible local systems E_V defined on the nonsingular part of V. Using the braid diagram and induction it is easy to show [44, Thm. 10.3.12] that:

Theorem 5.10.4 *A constructible complex $A^\bullet$ on W is perverse if and only if for every stratum X of W, for every $x \in W$ and for every nondegenerate covector $\xi \in T_X^*M$ at x, the Morse groups $H^r(\xi, A^\bullet) = 0$ vanish unless $r = \operatorname{codim}(X)$.*

Consequently, Morse theory applied to perverse sheaves reduces to the familiar situation in which the nonzero Morse groups live in a single degree.

The Abelian category of perverse sheaves was first discovered in conjunction with the Kazhdan-Lusztig conjecture [47, Conj. 1.5] whose proof (cf. [9, 14]) involved the Riemann-Hilbert correspondence which we state here without explaining the terms, cf. [12]. Let M be a complex analytic manifold and $\mathcal{D}_M$ its sheaf of differential operators. Let $D_{rh}^b(\mathcal{D}_M)$ be the derived category of (coherent sheaves of) modules over $\mathcal{D}_M$ whose cohomology sheaves are holonomic with regular singularities. Then the de Rham functor defines an equivalence [40, 58] of derived categories $D_{rh}^b(\mathcal{D}_M) \to D_c^b(M)$ which commutes with direct images, inverse images and duality, and it restricts to an equivalence between the abelian category of holonomic modules with regular singularities and the abelian category of perverse sheaves on M.

---

[7] In some situations, such as when a variety is stratified by the orbits of an algebraic group action, it is convenient to consider the category of perverse sheaves constructible with respect to a fixed stratification.

5.10.6 Further Properties

In this section W is a complex projective algebraic variety and "sheaf" means sheaf of $\mathbb{Q}$-vector spaces or $\mathbb{C}$-vector spaces, but the theory extends to $\mathbb{Q}_\ell$-sheaves on schemes over any field. Properties of perverse sheaves are most conveniently expressed by introducing Deligne's *degree shift* which is assumed throughout [8] and which we shall use in this paragraph, replacing IC_W with $IC_W[\dim(W)]$ so that Verdier duality is symmetric about degree 0. Hence, if $i : V \subset W$ is a (closed) subvariety and $\mathcal{L}$ is a local system on the nonsingular part of V then $Ri_* IC_V(\mathcal{L})$ is perverse on W and more generally (using this degree shift), i^*, $i^!$ and $Ri_* = Ri_!$ take perverse sheaves to perverse sheaves. The conditions (5.18) and (5.19) that $A^\bullet$ should be a perverse sheaf are that for all $x \in W$ (with $i_x : \{x\} \to W$) and for all $r \in \mathbb{Z}$ the following holds:

$$\dim\{x \in W |\ H^r(i_x^* A^\bullet) \neq 0\} \leq -r$$
$$\dim\{x \in W |\ H^r(i_x^! A^\bullet) \neq 0\} \leq r.$$

The *perverse cohomology* functors ${}^p\mathcal{H}^j : D_c^b(W) \to \mathcal{P}(W)$ take distinguished triangles to long exact sequences (of perverse sheaves), commute with Verdier duality, and identify perverse sheaves[8] as those complexes $A^\bullet \in D_c^b(W)$ such that ${}^p\mathcal{H}^r(A^\bullet) = 0$ for all $r \neq 0$.

If $f : W \to Y$ is a proper algebraic map, the *decomposition theorem* with coefficients in $\mathbb{Q}$ [8, §5.4] says:

Theorem 5.10.5 *There is a decomposition in* $D_c^b(Y)$*,*

$$Rf_*(IC_W) \cong \bigoplus_{i\in\mathbb{Z}} {}^p\mathcal{H}^i(Rf_*(IC_W)[-i], \tag{5.20}$$

and a hard Lefschetz morphism $\eta : {}^p\mathcal{H}^i(Rf_*(IC_W)) \to {}^p\mathcal{H}^{i+2}(Rf_*(IC_W))$ *which induces isomorphisms of perverse sheaves,*

$$\eta^r : {}^p\mathcal{H}^{-r}(Rf_*(IC_W)) \to {}^p\mathcal{H}^r(Rf_*(IC_W))$$

for all $r \geq 1$. *There is a stratification* $Y = \coprod_\beta Y_\beta$ *with local systems* E_β *on* Y_β *and a further decomposition*

$${}^p\mathcal{H}^i(Rf_*(IC_W)) \cong \bigoplus\nolimits_\beta IC(\overline{Y_\beta}; E_\beta)$$

of each factor in (5.20) into a direct sum of IC sheaves of subvarieties.

---

[8] Similarly the Abelian category of sheaves is equivalent to the full subcategory of $D_c^b(W)$ whose objects $A^\bullet$ satisfy: $\mathbf{H}^r(A^\bullet) = 0$ for $r \neq 0$.

This is one of the deepest and most useful results in mathematics, with many applications to algebraic geometry, representation theory, combinatorics, number theory, automorphic forms and other areas of mathematics. See, for example, [16, 38, 52, 60, 61, 63, 72].

5.11 Perfect Morse Functions and Fixed Point Theorems

5.11.1 Torus Actions

The Morse-Bott theory (Sect. 5.3.2) of critical points for smooth manifolds also has various extensions to singular spaces. Suppose the torus $T = \mathbb{C}^*$ acts algebraically on a (possibly singular) normal projective algebraic variety W with resulting moment map $\mu : W \to \mathbb{R}$ as in Sect. 5.3.2. Let $W^T = \coprod_r V_r$ denote the fixed point components of the torus action and define $V_r^{\pm}$ as in (5.3) with inclusions

$$V_r \xrightarrow{j_r^{\pm}} V_r^{\pm} \xrightarrow{h_r^{\pm}} W \tag{5.21}$$

On each V_r there is a complex of sheaves,

$$IC_r^{!*} = (j_r^+)^!(h_r^+)^*(IC_W) \cong (j_r^-)^*(h_r^-)^!(IC_W)$$

representing cohomology with closed supports in the directions flowing into V_r and with compact supports in the directions flowing away from V_r. (The isomorphism is proven in [6] and [32]). In [49] F. Kirwan uses the decomposition theorem (Theorem 5.10.5) to prove the following.

Theorem 5.11.1 ([49]) *The moment map μ is a perfect Morse Bott function and it induces a decomposition for all i, expressing the intersection cohomology of W as a sum of locally defined cohomology groups of the fixed point components:*

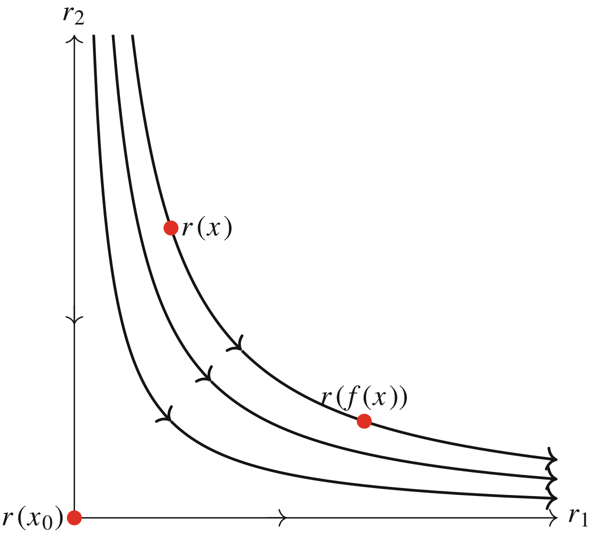
$$IH^i(W) \cong \bigoplus_r H^i(V_r; IC_r^{!*}). \tag{5.22}$$

The result also generalizes to actions of a torus $T = (\mathbb{C}^*)^m$. In [6] the sheaf $IC_r^{!*}$ is shown to be a direct sum of IC sheaves of subvarieties of V_r.

5.11.2 Hyperbolic Lefschetz Numbers

The time $t = 1 \in \mathbb{C}^*$ map of a $\mathbb{C}^*$ action (see Sects. 5.11.1, 5.3.2 above) is an example of a map with *hyperbolic* fixed points. For general hyperbolic maps the full

Fig. 5.6 Behavior near a hyperbolic fixed point



decomposition (5.22) may fail but the Lefschetz number can still be expressed as a sum of explicit local contributions.

Let $f : W \to W$ be a subanalytic self map defined on a subanalytically stratified subanalytic set W. A connected component V of the fixed point set of f is said to be hyperbolic[9] if there is a neighborhood $U \subset W$ of V and a subanalytic mapping $r = (r_1, r_2) : U \to \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ so that $r^{-1}(0) = V$ and so that $r_1(f(x)) \geq r_1(x)$ and $r_2(f(x)) \leq r_2(x)$ for all $x \in U$. Hyperbolic behavior of $f : W \to W$ is illustrated in Fig. 5.6. (Flow lines connecting $r(x)$ and $r(f(x))$ do not exist in general).

Let $V^+ = r^{-1}(Y)$ and $V^- = r^{-1}(X)$ where X, Y denote the X and Y axes in $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ with inclusions $j^{\pm}, h^{\pm}$ as in (5.21). If $A^{\bullet} \in D^b_c(W)$ define

$$A^{!*} = (j^+)^!(h^+)^*A^{\bullet} \text{ and } A^{*!} = (j^-)^*(h^-)^!A^{\bullet}.$$

A morphism $\Phi : f^*(A^{\bullet}) \to A^{\bullet}$ is called a *lift* of f to $A^{\bullet} \in D^b_c(W)$. Such a lift induces a homomorphism $\Phi_* : H^i(W; A^{\bullet}) \to H^i(W; A^{\bullet})$ and defines the Lefschetz number

$$\text{Lef}(f, A^{\bullet}) = \sum_{i\in\mathbb{Z}} (-1)^i \text{ Trace}\left(\Phi_* : H^i \to H^i\right).$$

---

[9] Examples of non-hyperbolic fixed points include the point at infinity of the extension to $\mathbb{CP}^1$ of the map $z \mapsto z + 1$, for $z \in \mathbb{C}$.

If $V \subset W$ is a hyperbolic component of the fixed point set then Φ also induces self maps $\Phi_V^{!*}$ on $H^i(V; A^{!*})$ and $\Phi_V^{*!}$ on $H^i(V; A^{*!})$ and in [32] it is proven that associated *local Lefschetz numbers* $\mathrm{Lef}(\Phi_V^{!*}; A^{!*})$ and $\mathrm{Lef}(\Phi_V^{*!}; A^{*!})$ are equal.

Theorem 5.11.2 ([32]) *Given* $f : W \to W$, $A^\bullet \in D_c^b(W)$ *and* $\Phi : f^*(A^\bullet) \to A^\bullet$ *as above. Suppose that* W *is compact and that all connected components of the fixed point set are hyperbolic. Then the global* $\mathrm{Lef}(f, A^\bullet)$ *is the sum over connected components of the fixed point set of the local Lefschetz numbers:*

$$\mathrm{Lef}(f, A^\bullet) = \sum_V \mathrm{Lef}(\Phi_V^{!*}; A^{!*}) = \sum_V \mathrm{Lef}(\Phi_V^{*!}; A^{*!}).$$

Moreover, each local Lefschetz number $\mathrm{Lef}(\Phi_V^{!*})$ is the Euler characteristic of a constructible function $\mathrm{Lef}(\Phi_x, A^{!*})$ for $x \in V$ (see Sect. 5.8.4 above). Let $V = \coprod V_r$ be a stratification of the fixed point component V so that the pointwise Lefschetz number $\mathrm{Lef}(\Phi_x, A^{!*})$ is constant on each stratum V_r, and call it $L_r(\Phi; A^{!*})$. If V is compact then (cf. [32, §11.1]),

$$\mathrm{Lef}(\Phi_V^{!*}; A^{!*}) = \sum_r \chi_c(V_r) L_r(\Phi; A^{!*}).$$

5.12 Specialization

5.12.1 Specialization by Retraction

The geometry described in Sect. 5.9.3 and Fig. 5.4, associated to a nondegenerate covector $\xi = d\pi(0)$ extends with very few modifications to much more general situations. Let $X \subset M$ be a complex $(n+1)$ dimensional analytic subvariety of some complex analytic manifold M and let $f : X \to \mathbb{C}$ be a proper complex analytic mapping. Such a mapping can be stratified with complex analytic strata, which in the target space $\mathbb{C}$ consists of discrete points. We wish to understand the local behavior of f near one such stratum which we may take to be $0 \in \mathbb{C}$. The *central fiber* $X_0 = f^{-1}(0)$ is a closed union of strata and $X_t = f^{-1}(t)$ is called "the" nearby fiber if $t \neq 0$ is sufficiently small (see below). Let

$$r_0 : T_{X_0}(\epsilon) \to X_0$$

denote the canonical retraction (Sect. 5.4.4) of a neighborhood T_{X_0} of X_0, whose fiber at $x \in X_0$ is stratified-homeomorphic to the normal slice $N_\epsilon(x)$ at x through the stratum $S \subset X_0$ containing x.

As in Sect. 5.9.3 there is an open region, $0 < \delta \ll \epsilon$ in the (δ, ϵ) plane so that if (δ, ϵ) lies in this region and if $t \in \mathbb{C}^*$, $0 < |t| \leq \delta$ then the pre-image $X_t = f^{-1}(t)$ is contained in $T_{X_0}(\epsilon)$ and is transverse to $\partial T_S(\epsilon)$ for each stratum $S \subset X_0$. The *specialization map* $\psi : X_t \to X_0$ is the restriction $\psi = r_0|X_t$. The fiber $\psi^{-1}(x)$ of the specialization map is therefore the *Milnor fiber* of f, that is, the intersection of the normal slice $N(x)$ with a ball $B_\epsilon(x)$ and with the nearby fiber X_t. Its (real) dimension is $2c$ where c denotes the complex codimension in X_0 of the stratum S containing x. (So the codimension of S in X is $c + 1$. If the differential $df(x)$ is a nondegenerate covector then the fiber $\psi^{-1}(x)$ is the complex link $\mathcal{L}$ in X of the stratum S, cf. Sect. 5.9.3). The monodromy $\mu : X_t \to X_t$ is a stratum preserving homeomorphism and $\psi \circ \mu = \psi$.

5.12.2 Nearby Cycles

With $f : X \to \mathbb{C}$ and $t \in \mathbb{C}^*$ as above, and $i_t : X_t \to X$, let $A^\bullet \in D^b_c(X)$. The sheaf

$$\psi_f(A^\bullet) = R\psi_* i_t^* A^\bullet = R\psi_*(A^\bullet|X_t) \in D_b(X_0)$$

is called the sheaf of *nearby cycles* on X_0. Its isomorphism class in $D^b_c(X_0)$ is independent of the choice of t (provided $|t| < \delta$ as above). Its cohomology is $H^r(X_0; \psi_f(A^\bullet)) \cong H^r(X_t; A^\bullet|X_t)$, cf. Eq. (5.9). The stalk cohomology of $\psi_f(A^\bullet)$ at a point $x \in X_0$ is the cohomology of the Milnor fiber, as described above. The monodromy passes to a morphism $\mu : \psi_f(A^\bullet) \to \psi_f(A^\bullet)$. This sheaf may also be constructed [19] without choosing $t \in \mathbb{C}^*$: let

$$\tilde{U} = U \times_{f,e} \mathbb{C} \to U = X - X_0 = f^{-1}(\mathbb{C} - \{0\})$$

be the infinite cyclic cover obtained by pulling back $U \to \mathbb{C}^*$ under the map $e : \mathbb{C} \to \mathbb{C}^*$, $e(z) = \exp(2\pi i z)$ as in the following diagram,

$$\begin{tikzcd}
\widetilde{U} \arrow[r, "\pi"] \arrow[d] & U \arrow[r, hook, "i"] \arrow[d, "f"'] & X \arrow[d] & X_0 \arrow[l, "j"'] \arrow[d] \\
\mathbb{C} \arrow[r, "e"'] & \mathbb{C}^* \arrow[r, hook] & \mathbb{C} & 0 \arrow[l]
\end{tikzcd}$$

Then $\psi_f(A^\bullet) = j^* Ri_* R\pi_* \pi^*(A^\bullet|U)$.

As in Sect. 5.9.1 the index estimates for (a Morse perturbation of the) distance from x and induction show that $\psi^{-1}(x)$ has the homotopy type of a CW complex of dimension $\leq c$, where c denotes the codimension (in X_0) of the stratum containing x. Moreover, if $A^\bullet$ is a complex of sheaves on X that satisfies (5.18) of Sect. 5.10.3 then the same argument implies that $\psi_f(A^\bullet)$ also satisfies (5.18). Since the pushforward under a proper mapping commutes with Verdier duality, this proves (see [28, 31, §6.A], [44, §10], [8, §4.4], [13, Thm. 1.2]) that specialization takes constructible sheaves to constructible sheaves and preserves perverse sheaves (using Deligne's degree shift):

Theorem 5.12.1 *Suppose $f : X \to C$ is a proper complex algebraic map to a curve C. Let $p \in C$ be a point and $X_p = f^{-1}(p)$ be the "central fiber". Let $\psi : X_t \to X_p$ be the specialization map, for $t \in C$ sufficiently close to p. Let $A^\bullet$ be a perverse sheaf on X. Then $A^\bullet|X_t$ is perverse on X_t and $R\psi_*(A^\bullet|X_t) = \psi_f(A^\bullet)$ is perverse on X_p.*

If $D_\delta \subset \mathbb{C}$ is a sufficiently small disk about $0 \in \mathbb{C}$ then for all i,

$$H^i(f^{-1}(D_\delta); A^\bullet) \cong H^i(X_0; A^\bullet)$$

for any $A^\bullet \in D^b_c(X)$. The *local invariant cycle theorem* [8, §6.2.9], a corollary of Theorem 5.10.5, says that every monodromy-invariant class in $IH^i(X_t)$ extends to a class in $IH^i(f^{-1}(D_\delta))$:

Theorem 5.12.2 *The natural homomorphism to the invariant classes*

$$H^i(X_0; IC_X|X_0) \to IH^i(X_t)^{\pi_1} = H^i(X_0; \psi_f(IC_X))^{\pi_1}$$

is surjective, where $\pi_1 \cong \mathbb{Z}$ is the monodromy action.

5.12.3 Vanishing Cycles

There is a canonical morphism $A^\bullet|X_0 \to \psi_*(A^\bullet)$ that arises from the restriction of sheaves for the inclusion $X_t \subset T_{X_0}(\epsilon)$. The sheaf of *vanishing cycles* $\phi_f(A^\bullet)$ is the third term in the resulting distinguished triangle:

$$\begin{tikzcd} A^\bullet|X_0 \arrow[rr] & & \psi_f(A^\bullet) \arrow[dl] \\ & \phi_f(A^\bullet) \arrow[ul, "{[1]}"] & \end{tikzcd} \qquad (5.23)$$

Let $Z = \{x \in X |\ Re(f(x)) \geq 0\}$ with inclusion $i_Z : Z \to X$ and $j : X_0 \to Z$. The sheaf of vanishing cycles is $\phi_f(A^\bullet) \cong j^* i^!_Z A^\bullet$ and its stalk cohomology $\mathbf{H^r}_x(i^!_Z A^\bullet)$ at $x \in X_0$ is the local Morse group (with degree shift of 1) for the function $Re(f) : X \to \mathbb{R}$, even though the covector $\xi = df(x)$ may be degenerate. (If $x_0 \in X_0$ is a 0-dimensional stratum and if $df(x_0)$ is a nondegenerate covector then this is exactly the Morse group for $Re(f)$ at x_0 and the exact sequence on cohomology from (5.23) may be found in the braid diagram Sect. 5.10.1.) The action of the monodromy $\mu : X_t \to X_t$ extends to a morphism $\mu : \phi_f(A^\bullet) \to \phi_f(A^\bullet)$ and the variation map $I - \mu$ extends naturally to a morphism

$$\mathrm{var} : \phi_f(A^\bullet) \to \psi_f(A^\bullet).$$

If $A^\bullet$ is a perverse sheaf then so are $\psi_f(A^\bullet)$ and $\phi_f(A^\bullet)$.

In particular if $X = \mathbb{C}$ is stratified with a single stratum at $0 \in \mathbb{C}$ and if $A^\bullet$ is constructible and perverse with respect to this stratification then $V = \psi_f(A^\bullet)$ and $W = \phi_f(A^\bullet)$ are (quasi-isomorphic to) vector spaces in degree zero with the following result (cf. [54, §6], [22, 81, §4]):

Theorem 5.12.3 *The category of perverse sheaves on* $(\mathbb{C}, \{0\})$ *is equivalent to the category of diagrams*

$$\begin{tikzcd} V \arrow[r, shift left, "\alpha"] & W \arrow[l, shift left, "\beta"] \end{tikzcd}$$

where $I + \alpha\beta$ *and* $I + \beta\alpha$ *are invertible.*

Vanishing cycles may be used to give quiver-like descriptions of the category of perverse sheaves in many other situations [54]. Oscillatory integrals and exponential sums may be estimated using vanishing cycles [1, 2, 17, 18, 45, 66, 68, 78]. The Fourier transform has a sheaf theoretic analog, the *geometric Fourier transform* [13, 44, 48] that is constructed using vanishing cycles and has many applications to representation theory and symplectic geometry (see for example [61, 62]). Mixed Hodge structures are constructed on the cohomology of vanishing cycles [65, 73, 74]. Morse theory and structure of the singularities plays a key role in the analysis of each of these fascinating applications.

References

1. Arnold, V. I.: Remarks about the stationary phase method and Coxeter numbers, Usp. Mat. Nauk **28**, no. 5 (1973), 17–44. (Russian Mathematical Surveys (1973), **28**(5), 19–48.) 316
2. Arnold, V. I., Gusein-Zade, S. M., Varchenko, A. N.: **Singularities of Differentiable Maps Volume II**, Birkäuser, 1988. 316
3. Artin, M.: Théorème de finitude pour un morphisme propre; dimension cohomologique des schémas algébriques affines, in SGA 4, Lecture Notes in Math **305**, Springer-Verlag, New York, 1973, pp. 145–167. 308
4. Atiyah, M. F.: Convexity and commuting Hamiltonians, Bull. London Math. Soc. **14** (1982), 1–15. 279
5. Barth, W: Larsen's theorem on the homotopy groups of projective manifolds of small embedding codimension. Proc. Symp. Pure Math. **29**, Amer. Math. Soc. 1975, 307–313. 303
6. Braden, T.: Hyperbolic localization of intersection cohomology, Trans. Groups, **8** (2003), 209–216. 311
7. Bredon, G.: **Sheaf Theory**, Graduate Texts in Mathematics **170**, Springer Verlag, N.Y., 1997. 290

8. Beilinson, A., Bernstein, J., Deligne, P., Gabber, O.: **Faisceux Pervers**, Astérisque **100**, Soc. Mat. France, Paris, 1982. 309, 310, 314, and 315
9. Beilinson, A., Bernstein, J.: Localisation de $\mathfrak{g}$-modules, C. R. Acad. Sci. Paris **292** (1981), 15–18. 309
10. Bialynicki-Birula, A.: On fixed points of torus actions on projective varieties. Bull. de l'Acad. Polon. des Sciences **32** (1974), 1097–1101. 279
11. Borel, A., Moore, J. C.: Homology theory for locally compact spaces, Mich. Math. J., **7** (1960), 137–169. 291 and 298
12. Borel, A. et al: **Algebraic D-Modules**, Perspectives in Mathematics **2**, Academic Press, N.Y., 1987. 309
13. Brylinski, J. L.: Transformations canonique, dualité projective, théorie de Lefschetz, transformation de Fourier et sommes trigonométriques, **Géometrie et Analyse Microlocales**, Astérisque **140–141**, Soc. Mat. de France, Paris (1986), 3–134. 314 and 316
14. Brylinski, J. L., Kashiwara, M.: Kazhdan-Lusztig conjecture and holonomic systems, Inv. Math. **64** (1981), 387–410. 309
15. Brylinski, J. L., Dubson, A., Kashiwara, M.: Formule d'indice pour les modules holonomes et obstruction d'Euler locale. C. R. Acad. Sci. Paris **293** (1981), 573–576. 307
16. deCataldo, M., Migliorini, L.: The decomposition theorem, perverse sheaves and the topology of algebraic maps, Bull. Amer. Math. Soc. **46** (2009), 535–633. 309 and 311
17. Delabaere, E., Howls, C. J.: Global asymptotics for multiple integrals with boundaries, Duke Math. J. **112** (2) (2002), 199–266. 316
18. Denef, J., Loeser, F.: Weights of exponential sums, intersection cohomology, and Newton polyhedra, Inv. Math., **106** (1991), 275–294. 316
19. Deligne, P.: Le formalisme des cycles évanescents, Exp. XIII, in **Groupes de monodromie en géométrie algébrique**, SGA7 part II, pp. 82–115. Lecture Notes in Mathematics, **340** Springer-Verlag, Berlin-New York, 1973. 314
20. Denkowska, Z., Stasica, J., Denkowski, M., Teissier, B.: **Ensembles Sous-analytiques à la Polonaise**, Travaux en cours **69**, Hermann, Paris, 2008. 280
21. Dimca, A.: **Sheaves in Topology**, Universitext, Springer Verlag, N.Y., 2004. 298
22. Galligo, A., Granger, M., Maisonobe, P.: D-modules et faisceaux pervers dont le support singulier et un croisement normal, Ann. Inst. Fourier **35** I (1985), 1–48. 316
23. Gelfand, S. I., Manin, Y.: **Homological Algebra**, Algebra V, Encyclopedia of mathematical sciences **38**, Springer Verlag, N.Y., 1994. 293, 294, and 298
24. Gelfand, S. I., Manin, Y.: **Methods of Homological Algebra**, Springer monographs in mathematics, Springer Verlag, New York, 2002. 293 and 294
25. Gibson, C. G., Wirtmüller, K., du Plessis, A. A., Looijenga, E.: **Topological Stability of Smooth Mappings**, Lecture Notes in Mathematics **552**, Springer Verlag, NY, 1976. 280
26. Goresky, M.: Triangulations of stratified objects, Proc. Amer. Math. Soc. **72** (1978), 193–200, 283
27. Goresky, M.: Whitney stratified chains and cochains, Trans. Amer. Math. Soc. **267** (1981), 175–196. 283
28. Goresky, M., MacPherson, R.: Morse theory and intersection homology, in **Analyse et topologie sur les espaces singuliers**, Astérisque **101–102**, Soc. Math. France, Paris (1981), 135–192. 306, 308, and 314
29. Goresky, M., MacPherson, R.: Stratified Morse theory, in **Singularities**, Proc. Symp. Pure Math. **40** Part I, 517–533. Amer. Math. Soc., Providence R. I., 1983. 276
30. Goresky, M., and MacPherson, R.: Intersection homology II, Invent. Math. **71** (1983), 77–129. 291, 293, and 309
31. Goresky, M., MacPherson, R.: **Stratified Morse Theory**, Ergebnisse Math. **14**, Springer Verlag, Berlin, Heidelberg, 1988. 276, 281, 282, 286, 287, 303, 304, 306, 308, and 314
32. Goresky, M., MacPherson, R.: Local contribution to the Lefschetz fixed point formula, Inv. Math. **111** (1993), 1–33. 311 and 313
33. Grothendieck, A.: Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA2), Masson & North-Holland, Paris, 1968. 309

34. Hamm, H., Lê, D. T.: Rectified homotopical depth and Grothendieck conjectures, in **The Grothendieck Festchrift Volume II**, Progress in Mathematics **87**, Birkhäuser Boston, 1990. 308 and 309
35. Hardt, R.: Stratification of real analytic mappings and images, Invent. Math. **28** (1975), 193–208. 280
36. Hironaka, H.: Subanalytic sets, in **Number Theory, Algebraic Geometry, and Commutative Algebra (Dedicated to Akizuki)**, Kinokunia, Tokyo Japan, 1973, 453–493.
37. Hironaka, H.: Stratification and flatness, in **Real and Complex Singularities**, Nordic Sumer School (Oslo, 1976), Sijthoff-Noordhoff, Groningen, 1977. 280
38. Huh, J., Wang, B.: Enumeration of points, lines, planes, etc., Acta Math., **218** (2017), 297–317 311
39. Iverson, B.: **Sheaf Theory**, Universitext, Springer Verlag N.Y., 1986. 298
40. Kashiwara, M.: Faisceaux constructibles et systèmes holonomes d'équations aux dérivées partielles linéaires à points singuliers réguliers, Sem. Goulaouic-Schwartz 1979–80, Exp. 19. École Polytechnique, 1980. 309
41. Kashiwara, M.: Index theorem for constructible sheaves. In **Systèmes differentiels et singularités**, Astérisque **130**, Soc. Mat. France, Paris (1985), 193–205. 302
42. Kashiwara, M.: Character, character cycle, fixed point theorem and group representation. Adv. Stud. Pure Math **14** (1988), 369–378. 301 and 308
43. Kashiwara, M., Schapira, P.: **Microlocal study of sheaves** Astérisque **128**, Soc. Math. France, Paris, 1985. 282
44. Kashiwara, M., Schapira, P.: **Sheaves on Manifolds**, Grundlehren der math. Wiss. **292**, Springer Verlag Berlin, Heidelberg, 1990. 276, 282, 285, 297, 298, 299, 301, 309, 314, and 316
45. Katz, N.: **Gauss Sums, Kloosterman Sums, and Monodromy Groups**, Annals of Mathematics Studies **124**, Princeton University Press, Princeton N.J., 1988. 316
46. Katz, N., Oda, T.: On the differentiation of de Rham cohomology classes with respect to a parameter, J. Math. Kyoto Univ. **1** (1968), 199–213. 301
47. Kazhdan, D., Lusztig, G.: Representations of Coxeter Groups and Hecke Algebras, Invent. Math. **53** (1979), 165–184. 309
48. Kiehl, R., Weissauer, R.: **Weil Conjectures, Perverse Sheaves and l'adic Fourier Transform**, Ergeb. Math. **42**, Springer Verlag, 2001. 309 and 316
49. Kirwan, F.: Intersection homology and torus actions, Jour. Amer. Math. Soc., **1** (no. 2) (1988), 388–400. 311
50. Lazzeri, F.: Morse theory on singular spaces, **Singularités à Cargèse**, Astérisque **7–8**, Soc. Mat. France, Paris (1973), 263–268. 285
51. Lojasiewicz, S.: Ensembles semi-analytiques. Preprint, I.H.E.S. 1972. 280
52. Looijenga, E.: L^2 cohomology of locally symmetric varieties, Comp. Math. **67** (1988), 3–20. 311
53. MacPherson, R.: Chern classes for singular algebraic varieties, Ann. Math. **100** (1974), 423–432. 307
54. MacPherson, R., Vilonen, K.: Elementary construction of perverse sheaves, Inv. Math., **84** (1986), 403–435, 316
55. Maisonobe, P., Sabbah, C. (eds.): **D-modules cohérents et holonomes**, Les cours du CIMPA, Travaux en cours, **45**, Paris, Hermann, 1993. 301
56. Mather, J.: **Notes on topological stability**, lecture notes, Harvard University, 1970. Reprinted in Bull. Amer. Math. Soc. **49** (2012), 475–506. 283 and 285
57. Mather, J.: Stratifications and mappings, in **Dynamical Systems** (M. M. Peixoto, ed.), Academic Press, N.Y. 1973. 280
58. Mebkhout, Z.: Une équivalence de catégories et une autre équivalence de catégories, Comp. Math. **51** (1984), 55–69. 309
59. Milnor, J.: Differentiable manifolds which are homotopy spheres, mimeographed notes, Princeton University, 1959. 278
60. Mirkoviĉ, I., Vilonen, K.: Geometric Langlands duality and representations of algebraic groups over commutative rings, Ann. Math. **166** (2007), 94–143. 311

61. Nadler, D.: Perverse sheaves on real loop Grassmannians, Invent. Math. **159** (2005), 1–73. 311 and 316
62. Nadler, D.: Springer theory via the Hitchin fibration, Comp. Math. **147** (2011) no. 5, 1635–1670. 316
63. Ngo, B. C.: Le lemme fondamental pour les algèbres de Lie, Publ. Math. IHES, **111** (2010), 1–169. 311
64. Orro, P.: Espaces conormaux et densité des fonctions de Morse, C.R.Acad.Sci.Paris, **305** (1987), 269–272. 287
65. Peters, C., Steenbrink, J.: **Mixed Hodge Structures**, Ergeb. Math. **52**, Springer Verlag, N.Y., 2008. 316
66. Pham, F.: Vanishing homologies and the n variable saddlepoint method, Proc. Symp. Pure Math. **40** (Part II), **Singularities**, 319–333, Amer. Math. Soc., Providence R.I., 1983. 316
67. Pignoni, R.: Density and stability of Morse functions on a stratified space, Ann. Scuola Norm. Sup. Pisa (4) **4** (1979), 592–608. 287
68. Sabbah, C.: **Vanishing cycles of polynomial maps**, Lecture notes, Nice, Nov 2008. 316
69. Schmidt, W., Vilonen, K.: Characteristic cycles of constructible sheaves, Inv. Math. **124** (1996), 451–502. 301
70. Schürmann, J.: **Topology of Singular Spaces and Constructible Sheaves**, Monografie Matematyczne **63**, Birkhäuser Verlag, Basel, 2003. 276, 285, 299, 301, 306, 307, 308, and 309
71. Smale, S.: Generalized Poincaré's conjecture in dimensionsl greater than four, Ann. Math. **74** (1961), 391–406. 278
72. Springer, T. A. : Quelques applications de la cohomologie d'intersection, Sem. Bourbaki **589**, Astérisque **92–93**, Soc. Mat. France, Paris (1982), 249–273. 311
73. Steenbrink, J.: Limits of Hodge structures, Invent. Math. **31** (1976), 229–257. 316
74. Steenbrink,J., Zucker, S.: Variations of mixed Hodge structure I, Invent. Math **80** (3) (1983), 489–542. 316
75. Tanisaki, T.: Characteristic varieties of highest weight modules and primitive quotients, Adv. Studies Pure Math. **14** (1988), **Representations of Lie Groups, Kyoto, Hiroshima 1986**, Academic Press, N.Y., 1988, pp. 1–30. 301
76. Teissier, B.: Variétés polaries II: Multiplicités polaries, sections planes et conditions de Whitney. in **Algebraic Geometry**, Proceedings, La Rabida 1981. Lecture Notes in Mathematics **961**, Springer Verlag, 1982. 303
77. Thom, R.: Ensembles et morphismes stratifiés, Bull. Amer. Math. Soc., **75** (1969), 240–284. 283 and 285
78. Varchenko, A.: Newton polyhedra and estimation of oscillating integrals, Funkt. Anal. i Ego Pril **10**(3) (1976) , 13–38. (Funct. Anal. Appl. **10**(3) (1976), 175–196. 316
79. Verdier, J. L.: Catégories dérive'es état 0, unpublished preprint, I.H.E.S., 1963; reprinted in SGA $4\frac{1}{2}$, Lecture Notes in Mathematics **569**, Springer Verlag, 1977. 293 and 298
80. Verdier, J. L.: **Des catégories dérivees des catégories abéliennes**, thesis, 1967, reprinted in Astérisque, **239**, Soc. Math. France., Paris (1996). 293 and 298
81. Verdier, L. L.: Extension of a perverse sheaf over a closed subspace, in **Systèmes Différentiels et Singularités**, Astérisque **130**, Soc. Mat. France. (1985), 210–217. 316

Chapter 6
The Topology of the Milnor Fibration

Dũng Tráng Lê, Juan José Nuño-Ballesteros, and José Seade

Contents

D. T. Lê (✉)
Centre de Mathématiques et Informatique, Université d'Aix-Marseille, Marseille, France
e-mail: ledt@ictp.it

J. J. Nuño-Ballesteros
University of Valencia, Valencia, Spain
e-mail: juan.nuno@uv.es

J. Seade
Instituto de Matemáticas, Universidad Nacional Autónoma de México, Mexico City, Mexico

Unidad Mixta Internacional 2001 CNRS, Laboratorio Solomon Lefschetz, Cuernavaca, Mexico
e-mail: jseade@im.unam.mx

J. L. Cisneros Molina et al. (eds.), *Handbook of Geometry and Topology of Singularities I*, https://doi.org/10.1007/978-3-030-53061-7_6

Abstract The fibration theorem for analytic maps near a critical point published by John Milnor in 1968 is a cornerstone in singularity theory. It has opened several research fields and given rise to a vast literature. We review in this work some of the foundational results about this subject, and give proofs of several basic "folklore theorems" which either are not in the literature, or are difficult to find. Examples of these are that if two holomorphic map-germs are isomorphic, then their Milnor fibrations are equivalent, or that the Milnor number of a complex isolated hypersurface or complete intersection singularity $(X, \underline{0})$ does not depend on the choice of functions that define it. We glance at the use of polar varieties to studying the topology of singularities, which springs from ideas by René Thom. We give an elementary proof of a fundamental "attaching-handles" theorem, which is key for describing the topology of the Milnor fibers. This is also related to the so-called "carousel", that allows a deeper understanding of the topology of plane curves and has several applications in various settings. Finally we speak about Lê's conjecture concerning map-germs $\mathbb{C}^2 \to \mathbb{C}^3$, and about the Lê-Ramanujam theorem, which still is open in dimension 2.

6.1 Introduction

Milnor's fibration theorem in [65] concerns the geometry and topology of analytic maps near their critical points. Consider a holomorphic map $(\mathbb{C}^{n+1}, \underline{0}) \xrightarrow{f} (\mathbb{C}, 0)$ taking the origin to 0, and for simplicity assume for a moment that f has an isolated critical point at $\underline{0}$. Since f is analytic, there exists $r > 0$ sufficiently small so that $0 \in \mathbb{C}$ is the only critical value of the restriction $f|_{\mathring{\mathbb{B}}_r}$, where $\mathring{\mathbb{B}}_r$ is the open ball of radius r and center at $\underline{0}$. Set

$$V := f^{-1}(0) \cap \mathring{\mathbb{B}}_r \quad \text{and} \quad V^* := V - \{\underline{0}\} .$$

So V^* is an n-dimensional complex manifold. We know from [65] that V^* meets transversally every sufficiently small sphere $\mathbb{S}_\varepsilon$ in $\mathbb{C}^{n+1}$ centered at $\underline{0}$ and contained in $\mathring{\mathbb{B}}_r$. The variety $L_V := V \cap \mathbb{S}_\varepsilon$ is called the link of the singularity $\underline{0}$ of $f^{-1}(0)$ and its homeomorphism type does not depend on the choice of the sphere (by [65, Theorem 2.10] or [11]). Then Milnor's theorem in [65] (Corollary 4.5) says that for every such sphere $\mathbb{S}_\varepsilon$ we have a smooth fiber bundle

$$\varphi := \frac{f}{|f|} : \mathbb{S}_\varepsilon - L_V \longrightarrow \mathbb{S}^1 . \tag{6.1}$$

This is known as the first version of Milnor fibration theorem, or also as the spherical Milnor fibration. There is a second version of this theorem, the "tube fibration", in which the fibers F_f are diffeomorphic to the complex manifolds obtained by considering a regular value t sufficiently near $0 \in \mathbb{C}$ and looking at the piece of $f^{-1}(t)$ contained within the open ball $\mathring{\mathbb{B}}_\varepsilon$ bounded by $\mathbb{S}_\varepsilon$. Each of these two versions of the fibration theorem has its own importance and interest, and this has led to a vast literature (see for instance [71] for a survey on the topic).

The purpose of this work is to discuss several results which concern the foundations of these fibrations. These are all well-known results, but difficult to find in the literature and splitted into many papers. Here we present these foundational results in a unified, coherent way.

We begin this article by discussing the existence of local "Milnor balls" and "Milnor spheres". This holds for arbitrary real analytic map-germs $f\colon (X,\underline{0}) \to (\mathbb{R}^k, 0)$, where X is a real analytic variety of dimension $> k$, and it means a sufficiently small sphere $\mathbb{S}_\varepsilon$ (or a ball $\mathbb{B}_\varepsilon$ bounded by $\mathbb{S}_\varepsilon$) centered at the critical point $\underline{0} \in X \subset \mathbb{R}^N$ such that every sphere in $\mathbb{R}^N$ centered at $\underline{0}$ and contained in $\mathbb{B}_\varepsilon$ intersects the zero locus $V = f^{-1}(0) \cap X$ transversally in the stratified sense. That means that the germ $(X,\underline{0})$ is equipped with a Whitney stratification (see [17] or Sect. 6.11 below) for which V is a union of strata, and every stratum that has $\underline{0}$ in its closure intersects transversally all the spheres as above. This proves that V has a local conical structure. The set $L_V := V \cap \mathbb{S}_\varepsilon$ is called the link of the singularity.

For simplicity we restrict the discussion in Sect. 6.2 to the case where X is complex analytic with an isolated singularity at $\underline{0}$ and f is holomorphic. We show in an elementary way the existence of Milnor spheres and balls whenever f has an isolated critical point. In that section we also show the existence of a tube fibration as mentioned above. This is called *the local Milnor fibration* of f at $\underline{0}$. The fibers $F_t = f^{-1}(t) \cap \mathring{\mathbb{B}}_\varepsilon$ are *the Milnor fibers* of f at 0. This is based on a Bertini-Sard type observation by René Thom for complex analytic maps, and a version of Ehresmann's fibration lemma for manifolds with boundary. We give here a proof of that lemma. Next we define the geometric monodromy of the tube fibration and show that two locally smooth fibrations $E_1 \to \mathbb{S}^1$ and $E_2 \to \mathbb{S}^1$ are isomorphic if they have diffeomorphic fibers and isomorphic geometric monodromies. The rest of Sect. 6.2 is devoted to proving (Proposition 6.2.12) that if $f_1 : (X_1, \underline{0}_1) \to (\mathbb{C}, 0)$ and $f_2 : (X_2, \underline{0}_2) \to (\mathbb{C}, 0)$ are two isomorphic germs of holomorphic functions, then the local Milnor fibrations are diffeomorphic.

The next Section discusses explicit examples where it is easy to describe the topology of the Milnor fiber and see a difference between the isolated and the non-isolated singularity case. We also introduce the Pham join J for the polynomials:

$$(z_0, \ldots, z_n) \xrightarrow{f} z_0^{a_0} + \ldots + z_n^{a_n}, \qquad a_i \geq 2\,.$$

One has that J is a deformation retract of the Milnor fiber, and J has the homotopy type of a bouquet of spheres of middle dimension. This is the birth of Milnor's theorem discussed later, that, in the isolated singularity case, the Milnor fiber has the homotopy type of a bouquet of spheres of middle dimension.

In the case the singularities of f are not isolated, there is a similar Milnor fibration; we envisage this in Sect. 6.4. We still assume that X is a complex analytic space with an isolated singularity at $\underline{0}$, but now $(X,\underline{0}) \xrightarrow{f} (\mathbb{C}, 0)$ may have a non-isolated singularity at $\underline{0}$. We show (Theorem 6.4.1) that one has in this case a Milnor tube fibration. This uses Thom's A_f condition, that we explain in the text, and a remarkable result of Hironaka stating that every complex analytic map $f : X \to Y$ to a complex analytic space of $\dim Y = 1$ can be stratified so that f satisfies the A_f condition. Furthermore, the A_f stratification of f can be obtained from refining a given stratification of f. One has, even in this general setting, that the Milnor fiber is a Stein manifold, and therefore it has the homotopy type of a CW-complex of middle dimension, by Andreotti-Frankel's theorem in [2]. This theorem uses elementary complex analysis to show that up to a small perturbation, the function distance to the origin restricted to a Stein n-manifold is a plurisubharmonic Morse function (see for instance [19, Def. 1 Chapter IX, Section C]) and the Morse index at each critical point is $\leq n$.

In Sect. 6.5 we return to Milnor's original formulation of the spherical fibration theorem, similar to (6.1) above, for functions $(X, \underline{0}) \xrightarrow{f} (\mathbb{C}, 0)$ where X has an isolated singularity at $\underline{0}$, and we sketch the proof of this fibration property using the previous tube fibration.

Section 6.6 focuses on the case where f defines an isolated complete intersection singularity germ (an ICIS). We recall in this section Hamm's fibration theorem for ICIS, and the definition of the Milnor number of f as the rank of the middle homology of the Milnor fiber. We prove that if $f, g \colon (\mathbb{C}^{n+k}, \underline{0}) \to (\mathbb{C}^k, \underline{0})$ define the same ICIS of dimension n, $V = f^{-1}(\underline{0}) = g^{-1}(\underline{0})$ (as complex spaces), then the Milnor fibres of f and g are diffeomorphic. Hence, if $f \colon (\mathbb{C}^{n+k}, \underline{0}) \to (\mathbb{C}^k, \underline{0})$ defines an ICIS $V = f^{-1}(0)$ of dimension n, then we may define the *Milnor number* of $(V, \underline{0})$ as $\mu(V, \underline{0}) := \mu(f)$ and this is independent of the choice of the representative f.

We also recall in Sect. 6.6 a general fibration theorem where we consider a complex analytic space X of pure dimension $n + k$, a holomorphic map germ $f := (f_1, \ldots, f_k) \colon (X, \underline{0}) \to (\mathbb{C}^k, \underline{0})$ and let $V = f^{-1}(\underline{0})$. We assume that f defines an *isolated singularity* in the sense that at each point of $V - \{\underline{0}\}$ sufficiently near to $\underline{0}$ the space X is not singular and the mapping f is a submersion. The case $k = 1$ is specially interesting and that is the subject of Sect. 6.11, where we prove a general fibration theorem for map-germs $(X, \underline{0}) \xrightarrow{f} (\mathbb{C}, 0)$ where X may have arbitrary singular locus.

Section 6.7 is a turning point in this presentation, since we begin here the use of polar varieties to study the topology of singularities. This springs from ideas of René Thom [77] and was introduced by B. Teissier and D. T. Lê (see for instance [32, 36, 37, 74, 75]). The idea is to study the topology of the germ in question by looking at the slices by a linear form; the points of non-transversality determine the corresponding polar variety. This is somehow a reminiscent of Morse theory where one studies the topology of manifolds by slicing them with the level surfaces of a Morse function. Here we replace Morse functions by linear forms.

We conclude Sect. 6.7 with Lê's attaching theorem, stating that if $(\mathbb{C}^{n+1}, \underline{0}) \xrightarrow{f} (\mathbb{C}, 0)$ is the germ of a holomorphic function with a possibly non-isolated critical point at $\underline{0}$, F_t is a local Milnor fiber at $\underline{0}$ and H is a sufficiently general hyperplane near $\underline{0}$, then F_t is obtained from the Milnor fiber of the slice $F_t \cap H$ by attaching a certain number of n-handles. The number of such handles is the intersection number $\big(\Gamma_{f,\ell} \cdot V(f)\big)$, where $\Gamma_{f,\ell}$ is the polar curve of f relative to a general linear form ℓ. Furthermore, if f has an isolated critical point, then this number equals the sum of Milnor numbers $\mu(f) + \mu(f, \ell)$.

An immediate consequence of Lê's attaching theorem is that in the isolated singularity case, the Milnor fiber F_f is diffeomorphic to a $2n$-ball to which one attaches μ handles of middle dimension, where μ is the Milnor number. This was conjectured by Milnor and proved in [50]. When the map f has non-isolated critical points, the Milnor fiber F_f has more complicated topology. In fact the ideas described in Sect. 6.7 naturally lead to the concept of Lê cycles and Lê numbers, introduced by D. Massey in [56, 57, 59]. The Lê cycles are local analytic cycles associated to the germ of f at $\underline{0}$, defined by means of polar varieties; the Lê numbers are the multiplicities of those cycles. There is a Lê number in each (complex) dimension, from 0 to that of the singular set of f. Then Lê-Perron's theorem generalizes to saying that the Milnor fiber of a map germ $(\mathbb{C}^{n+1}, \underline{0}) \overset{f}{\rightarrow} (\mathbb{C}, 0)$ is diffeomorphic to a $2n$-ball to which one attaches handles of various Morse indices, and this is dictated by the corresponding Lê numbers. This is the content of Sect. 6.10 where we briefly discuss the topology of the Milnor fiber of map-germs $\mathbb{C}^n \to \mathbb{C}$.

In Sect. 6.8 we briefly describe the *carousel* associated to a map-germ $\mathbb{C}^2 \to \mathbb{C}$. This was introduced by D. T. Lê (see for instance [38, 41]) and it springs also from the ideas described in Sect. 6.7. Given a map-germ $f : (\mathbb{C}^2, \underline{0}) \to (\mathbb{C}, 0)$, say irreducible, then we know that its link is an iterated torus knot determined by the Puiseux pairs of f. The point is that the Puiseux expansions actually give an additional structure near the singular point and this gives rise to what Lê called the carousel associated to the singularity. This is obtained by considering an auxiliary linear form ℓ, general enough for f, and looking at the distribution of points $\{z_j\}$ in the intersection $\{\ell = t\} \cap \{g(x, y) = 0\}$. Then the carousel arises by a careful study of how the Puiseux pairs describing the points where the line $\{\ell = t\}$ meets the Milnor fiber, which are distributed regularly around each point $\{z_j\}$ and their distribution is determined iteratively by the Puiseux pairs.

In the following Sect. 6.9 we describe some results where carousels are used. The first of these is a theorem from [38] stating that if $f : \mathcal{U} \to \mathbb{C}$ is a non-constant reduced analytic function defined in a neighborhood of $\underline{0}$ in $\mathbb{C}^{n+1}$ with a critical point at $\underline{0}$ (not necessarily isolated), then there is a geometric monodromy of the Milnor fibration of f at $\underline{0}$ which does not have a fixed point. As a corollary one gets A'Campo's theorem in [1] that the Lefschetz number of the monodromy of f at $\underline{0}$ is zero.

Finally, in Sect. 6.12 we look at two open problems. The first of these is a long standing open problem that comes from the Problems Section in [81], and according to the first named author, was originally stated by M. Oka as follows: let $(X, \underline{0})$ be a surface in $(\mathbb{C}^3, \underline{0})$; if its link L_X is homeomorphic to a sphere, then $(X, \underline{0})$ is the total space of a Whitney equisingular family of irreducible plane curves. This question was reformulated by the first named author in terms of the injectivity of a holomorphic map germ from $(\mathbb{C}^2, \underline{0})$ to $(\mathbb{C}^3, \underline{0})$ and it is known as Lê's Conjecture: if $f : (\mathbb{C}^2, \underline{0}) \to (\mathbb{C}^3, \underline{0})$ is holomorphic and injective, then f has rank ≥ 1 at $\underline{0}$. We prove in Sect. 6.12 that these two conjectures are actually equivalent.

6.2 Background on Milnor Fibrations

Let $(X, \underline{0})$ be the germ at the origin $\underline{0}$ in some complex space $\mathbb{C}^N$ of an irreducible complex analytic variety X of dimension $n + 1$.

First we consider the case where the analytic space X has an isolated singular point at $\underline{0}$. Let $(X, \underline{0}) \overset{f}{\to} (\mathbb{C}, 0)$ be a non-constant holomorphic function. We first assume that the function f has no critical points in $\mathcal{U} - \underline{0}$, where $\mathcal{U}$ is a neighborhood of $\underline{0}$ in X. If a function f of X satisfies this property, we shall say that the holomorphic function f has an isolated singularity at the point $\underline{0}$.

Let $\varepsilon > 0$ be small enough so that the ball $\mathring{\mathbb{B}}_\varepsilon \subset \mathbb{C}^N$ with center at $\underline{0}$ and radius ε is a Milnor ball for the germs of X at $\underline{0}$ with respect to the given stratification. That is, every sphere contained in $\mathring{\mathbb{B}}_\varepsilon$ centered at $\underline{0}$ meets transversally X. The existence of Milnor balls is given by the following lemma attributed to Whitney (see [83, Theorem 22.1]):

Lemma 6.2.1 *Let (z_n) be a sequence of non-singular points of X which tends to $\underline{0}$. Consider $(T_{z_n}(X))$ the sequence of tangent spaces of X at z_n. Assume that this sequence of tangent spaces has a limit T and the sequence of lines $(l_{\underline{0},z_n})$ has a limit ℓ, then we have:*

$$\ell \subset T.$$

Proof Since the space $\mathbb{P}^{N-1}$ of lines through $\underline{0}$ and the Grassmann space $G(\dim X, N)$ of $\dim X$-linear subspaces of $\mathbb{C}^N$ are compact, we may choose a sub-sequence of (z_n) such the sequences of lines $(\underline{0}z_n)$ and tangent spaces $T_{z_n}(X)$ have limits ℓ and T.

In fact, let $\mathcal{U}$ be an open neighborhood of $\underline{0}$ in X and consider the mapping:

$$\mathcal{U} \cap X - \underline{0} \xrightarrow{\Phi} \mathbb{P}^{n-1} \times G(\dim X, N)$$

defined by $\Phi(z) = (\underline{0}z, T_z(X))$. The graph of Φ in $\mathcal{U} \cap X \times \mathbb{P}^{n-1} \times G(\dim X, N)$ has a closure which is an analytic subspace (see *e.g.* Theorem 16.4 of [83]). Therefore, by the Curve Selection Lemma [65, Lemma 3.1], the limit (ℓ, T) is also the limit along a complex analytic arc $p : (\mathbb{C}, 0) \to (X, \underline{0})$ where $p(0) = \underline{0}$ and $p(t) \in X - \underline{0}$ for $t \neq 0$. Now, for $t \neq 0$ the tangent vector:

$$\frac{d\,p(t)}{dt} = \dot{p}(t)$$

of the analytic arc at the point $p(t)$ is also a tangent vector in $T_{p(t)}(X)$. We are going to prove that when t tends to zero the line $0p(t)$ and the line defined by $\dot{p}(t)$ have the same limit. We have:

$$p(t) = a_i t^i + a_{i+1} t^{i+1} + \dots$$

where $i \geq 1$ and $a_i \neq 0$. Let $[c]$ be the class in $\mathbb{P}^{N-1}$ of $c \in \mathbb{C}^N - \{0\}$. Then,

$$[p(t)] = [a_i t^i + a_{i+1} t^{i+1} + \dots] = [a_i + a_{i+1} t + \dots] \to [a_i]$$

when $t \to 0$. Analogously,

$$[\dot{p}(t)] = [i a_i t^{i-1} + (i+1) a_{i+1} t^i + \dots] = [i a_i + (i+1) a_{i+1} t + \dots] \to [i a_i] = [a_i]$$

when $t \to 0$.

Since the tangent vector $\dot{p}(t)$ is contained in $T_{p(t)}(X)$, this proves Lemma 6.2.1 in the case X is irreducible at the point $\underline{0}$. When X is not irreducible at $\underline{0}$, we may assume that the sequence (z_n) belongs to an irreducible component of X at $\underline{0}$ by eventually extracting a subsequence. This completes the proof of Lemma 6.2.1. □

Remark 6.2.2 We notice that Lemma 6.2.1 is proving that if we equip a small neighborhood U of $\underline{0}$ with a stratification $(X \cap U \setminus \{\underline{0}\}), \{\underline{0}\})$, then this stratification satisfies Whitney's b-condition and therefore it is Whitney regular, see Sect. 6.11.

A consequence of Lemma 6.2.1 is:

Lemma 6.2.3 *Let X be an equidimensional complex analytic space with an isolated singularity at $\underline{0}$. There is ε_1 such that the ball $\mathring{\mathbb{B}}_{\varepsilon_1}$ is a Milnor ball of X at $\underline{0}$. In this case, the differentiable structure of $X \cap \mathbb{S}_\varepsilon$ does not depend on ε, for $\varepsilon_1 > \varepsilon > 0$, where $\mathbb{S}_\varepsilon$ is the sphere bounding the Milnor ball $\mathring{\mathbb{B}}_\varepsilon$.*

Proof Let us suppose that for all $\varepsilon > 0$ the ball $\mathring{\mathbb{B}}_\varepsilon$ is not a Milnor ball. Let us choose a corresponding sequence ε_n of positive numbers which tends to 0. For each n there is a point $z_n \in X$ where the sphere $\mathbb{S}_{\|z_n\|}$ is not transverse to X at z_n. By choosing a subsequence we may suppose that the sequence (z_n) is chosen such $(\underline{0}z_n)$ has a limit ℓ, the sequence of tangent spaces $(T_{z_n}X)$ has a limit T and the limit of $(T_{z_n}\mathbb{S}_{\|z_n\|})$ is $\mathcal{T}$. Since at the point z_n the space X is not transverse to $\mathbb{S}_{\|z_n\|}$, we have:

$$T_{z_n}X \subset T_{z_n}\mathbb{S}_{\|z_n\|} .$$

Then, at the limit $\lim_n \underline{0}z_n = \ell$ is orthogonal to $\mathcal{T} \supset T = \lim_n T_{z_n}X$, which contradicts Lemma 6.2.1. Therefore, there is an $\varepsilon_1 > 0$ such that $\mathring{\mathbb{B}}_{\varepsilon_1}$ is a Milnor ball of X at $\underline{0}$.

Since we assume that the function f has an isolated singularity at $\underline{0}$, the subspace $V(f) = \{f = 0\}$ of X has an isolated singularity at $\underline{0}$ and we can choose ε_1 such that the ball $\mathring{\mathbb{B}}_{\varepsilon_1}$ of $\mathbb{C}^N$ is also a Milnor ball of $V(f)$ at $\underline{0}$.

Therefore, for $\varepsilon_1 > \varepsilon > 0$, the intersection $X \cap \mathbb{S}_\varepsilon$, where $\mathbb{S}_\varepsilon$ is the sphere boundary of $\mathbb{B}_\varepsilon$, is a real analytic manifold. Similarly since the holomorphic function $f : (X, \underline{0}) \to (\mathbb{C}, 0)$ has an isolated singularity at the point $\underline{0}$, for $\varepsilon_1 > \varepsilon > 0$, the intersection $V(f) \cap \mathbb{S}_\varepsilon$ is a real analytic manifold.

We claim that the differentiable structures of $X \cap \mathbb{S}_\varepsilon$ and $V(f) \cap \mathbb{S}_\varepsilon$ do not depend on ε, for $\varepsilon_1 > \varepsilon > 0$. We prove this assertion below for X; the case of $V(f)$ is proved in the same way.

Consider a Milnor ball $\mathbb{B}_{\varepsilon_1}$ for X at $\underline{0}$. Let φ be the restriction to $X \cap \mathring{\mathbb{B}}_{\varepsilon_1} - \underline{0}$ of the distance function to the point $\underline{0} \in \mathbb{C}^N$. It defines a smooth function:

$$\varphi : X \cap \mathring{\mathbb{B}}_{\varepsilon_1} - \underline{0} \longrightarrow (0, \varepsilon_1).$$

The function φ is clearly proper and, by definition of a Milnor ball, it is submersive and surjective. Ehresmann Lemma, whose statement is given below in Lemma 6.2.10, (in this case a basic Lemma of Morse theory is enough) implies that $X \cap \mathbb{S}_{\varepsilon'}$ and $X \cap \mathbb{S}_{\varepsilon''}$ are diffeomorphic for any ε' and ε'' in $(0, \varepsilon_1)$. □

This is why we may define:

Definition 6.2.4 When ε is small enough, one calls $X \cap \mathbb{S}_\varepsilon$ *the link* of X at $\underline{0}$. One defines similarly the link of $V(f)$ at the point $\underline{0}$.

Example 6.2.5

1. Let $X = \mathbb{C}^2$. Assume that the holomorphic function $f : (\mathbb{C}^2, \underline{0}) \to (\mathbb{C}, 0)$ has an isolated critical point at $\underline{0}$. Then $V(f)$ defines the germ of a reduced plane curve $(V(f), \underline{0})$ at $\underline{0}$. The link of $\mathbb{C}^2$ at $\underline{0}$ is a 3-sphere $\mathbb{S}^3_\varepsilon$ and the link $V(f) \cap \mathbb{S}^3_\varepsilon$ of $V(f)$ at $\underline{0}$ defines a link in the 3-sphere $\mathbb{S}^3_\varepsilon$. These links, when they have one component and then are called knots, have been studied by K. Brauner in [3] (see also [33]) and in the case of several components have been studied in [49].
2. Let $X = \mathbb{C}^5$. Consider the polynomials $f = z_1^3 + z_2^{6r-1} + z_3^2 + z_4^2 + z_5^2$. If one varies r from 1 to 28, the link of $V(f)$ gives the 28 classes of oriented manifolds which are homeomorphic to $\mathbb{S}^7$ [5]. The smooth manifolds homeomorphic to a sphere, but not diffeomorphic, are called exotic spheres (see e.g. [65] Chap. 8 and 9).

Let X be a complex analytic space with an isolated singularity at $\underline{0}$. Assume that $(X, \underline{0})$ is embedded in $(\mathbb{C}^N, \underline{0})$. Then, a general linear form of $\mathbb{C}^N$ restricted to X defines a holomorphic function on X with an isolated singularity at $\underline{0}$. Now, we have a local Bertini-Sard type result due to R. Thom:

Lemma 6.2.6 *Let $f : (X, \underline{0}) \to (\mathbb{C}, 0)$ be a germ of holomorphic function on an equidimensional complex analytic space X with an isolated singularity at $\underline{0}$. Then, on a sufficiently small representative X of $(X, \underline{0})$, there exists $\delta > 0$ small enough such that for all t, $\delta > |t| > 0$, the space $\{f = t\} \cap X$ is non-singular.*

Proof Let $\Sigma(f)$ be the set of critical points of f. It is defined by the points where the matrices $(df_1, \ldots, df_k, df)$ have rank $\leq N - \dim X$, where the functions $f_1, \ldots, f_k$ define X at $\underline{0}$ in $\mathbb{C}^N$. Hence, $\Sigma(f)$ is an analytic subspace of X. We can find locally finite connected complex analytic manifolds (Σ_k) such that $\Sigma(f)$ is the disjoint union of the Σ_k:

$$\Sigma(f) = \coprod_k \Sigma_k.$$

Since the partition (Σ_k) is locally finite, in a neighborhood X_0 of $\underline{0}$ in X, there is only a finite number $\Sigma_1, \ldots, \Sigma_n$ which contain $\underline{0}$ in their closures.

The restriction of f to $\Sigma_i - \{\underline{0}\}$ is critical, so it is constant for $i = 1, \ldots, n$. Its value is the same value as $f(\underline{0}) = 0$ because $\Sigma_1, \ldots, \Sigma_n$ are connected and $\underline{0}$ is in the closures of $\Sigma_1, \ldots, \Sigma_n$. Therefore for $\delta > 0$ small enough, $X \cap f^{-1}(t)$ is non-singular for all t such that $\delta > |t| > 0$. □

Remark 6.2.7 If X is not equidimensional, but with an isolated singularity at $\underline{0}$, the space $\{f = t\}$ for t small and $\neq 0$, splits locally at $\underline{0}$ into non-singular pieces in each analytic component of $(X, \underline{0})$.

Remark 6.2.8 Notice that the proof of Lemma 6.2.6 actually proves more than is stated: it shows that taking the representative of X small enough, we can assume that $0 \in \mathbb{C}$ is the only critical value of f.

We have a local fibration theorem. This is the tube fibration mentioned in the introduction:

Proposition 6.2.9 *Let X be an analytic space with an isolated singularity at $\underline{0}$ and let $f : (X, \underline{0}) \to (\mathbb{C}, 0)$ be a holomorphic function with an isolated singularity at the point $\underline{0}$. Let $\mathring{\mathbb{B}}_{\varepsilon_1}$ be a Milnor ball of X and $V(f)$ at $\underline{0}$. Let ε, $0 < \varepsilon < \varepsilon_1$, then there is $\delta > 0$ small enough such that, for all t with $\delta \geq |t|$, the fiber $\{f = t\}$ intersects $\mathbb{S}_\varepsilon \cap X$ transversally in X. We have a local differentiable fibration:*

$$\mathbb{B}_\varepsilon \cap X \cap f^{-1}(\mathring{\mathbb{D}}_\delta) - V(f) \longrightarrow \mathring{\mathbb{D}}_\delta - \{0\}$$

induced by f on $\mathbb{B}_\varepsilon \cap X - V(f)$ where $\mathbb{B}_\varepsilon$ is now the closed ball.

The proof of Proposition 6.2.9 uses Ehresmann Lemma in the case where the source manifold has a boundary. When M has no boundary, which is the original statement of Ehresmann, there are several proofs in the literature, see *e.g.* [7, 8.12]. For completeness let us recall and prove Ehresmann Lemma in the more general setting we need here.

Lemma 6.2.10 *Let $\varphi : M \to N$ be a differentiable map from a smooth manifold M with boundary ∂M to a smooth manifold N. We assume:*

1. *the map φ is proper;*
2. *the restriction of φ to $M - \partial M$ is surjective and submersive;*
3. *if the boundary ∂M is not empty, the restriction of φ to the boundary ∂M is also surjective and submersive.*

Then φ is a locally trivial smooth fibration on N.

Proof To show that φ is a locally trivial fibration we have to find that, for any point $p \in N$, there exists a neighborhood U such that we have the following commutative diagram:

$$\begin{tikzcd}
U \times \varphi^{-1}(p) \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & \varphi^{-1}(U) \arrow[dl, "\varphi_{|}"] \\
& U &
\end{tikzcd}$$

with a diffeomorphism ψ, where π is the projection on the first factor and $\varphi_{|}$ is induced by φ.

Let us construct a vector field whose flow will give us the isomorphism ψ. We may consider an open neighborhood U_0 of the point p in N which is diffeomorphic to an open set of $\mathbb{R}^n$, where n is the dimension of N at the point p and we have local coordinates $x_1, \dots, x_n$.

Consider a point $q \in \varphi^{-1}(p)$. If q is a point of the interior $M - \partial M$, since φ is submersive there are an open neighborhood $V(q)$ of q in $M - \partial M$, an open neighborhood $U(p)$ of p in N and an open neighbourhood $U'(q)$ of q in $\varphi^{-1}(p)$, such that $U(p) \times U'(q)$ is diffeomorphic to $V(q)$ and the first projection of $U(p) \times U'(q)$ onto $U(p)$ is isomorphic to the map induced by φ on $V(q)$ into $U(p)$. We can always assume $U(p) \subset U_0$.

If $q \in \varphi^{-1}(p)$ is a point on the boundary ∂M of M, there is an open neighborhood $V_0(q)$ of q in M which has a boundary $\partial V_0(q)$. Since the restriction $\varphi_{|\partial M}$ of φ to ∂M is submersive at q, we can choose the neighborhood $V_0(q)$ such that there exists an open neighborhood $U(p)$ of p in N and an open neighborhood $U'_0(q)$ of q in $\varphi^{-1}(p)$ for which the product $U(p) \times U'_0(q)$ is diffeomorphic to $V_0(q)$ and the projection of $U(p) \times U'_0(q)$ onto $U(p)$ is isomorphic to the map induced by φ on $V_0(q)$ into $U(p)$. Again we may assume $U(p) \subset U_0$.

Since φ is proper, the fiber $\varphi^{-1}(p)$ is compact. We can choose a finite number of points $q_1, \dots, q_k$ such that the corresponding neighborhoods $V(q_i)$ or $V_0(q_j)$ cover the fiber $\varphi^{-1}(p)$. Therefore we have a closed ball $\mathbb{B}$ centered at p in the local coordinates of N fixed above, such that:

$$\mathbb{B} \subset \cap_i \varphi(V(q_i)) \cap_j \varphi(V_0(q_j)).$$

We denote by $\overset{\circ}{\mathbb{B}}$ the interior of $\mathbb{B}$. We are going to construct a vector field in $\varphi^{-1}(\mathbb{B})$.

Let us call $\partial/\partial x_i$ the vector field associated to the coordinate x_i of N. So if $q \in M - \partial M$, the vector field $\partial/\partial x_i$ defines a vector field in $U(p)$ which extends trivially to $U(p) \times U'(q)$ and, using the diffeomorphism onto $V(q)$, it defines a vector field v_q^i in $V(q)$. The image by φ of this vector field v_q^i is $\partial/\partial x_i$.

Similarly if $q \in \partial M$, by trivial extension of $\partial/\partial x_i$ we have a vector field on $U(p) \times U'_0(q)$ which is tangent to the boundary $U(p) \times \partial U'_0(q)$ and its image by the diffeomorphism obtained above is a vector field v_q^i of $V_0(q)$ which is tangent to the boundary $V_0(q) \cap \partial M$. The image of v_q^i by φ is $\partial/\partial x_i$ in $U(p)$.

Consider as above $q_1, \dots, q_k$ such that the associated open neighborhoods $V(q)$ or $V_0(q)$ cover $\varphi^{-1}(p)$. Let $(\rho_1, \dots, \rho_k)$ be a partition of unity associated to these open neighborhoods. The sums $\sum_1^k \rho_\ell v_\ell^i$ define vector fields W_i, for $i = 1, \dots, n$, on this union of open neighborhoods. Since:

$$\mathbb{B} \subset \cap_i \varphi(V(q_i)) \cap_j \varphi(V_0(q_j)),$$

the vector fields W_i define vector fields $\theta_1, \dots, \theta_n$ on $\varphi^{-1}(\mathbb{B})$. By construction the vector fields $\theta_1, \dots, \theta_n$ are tangent to the boundary of M.

The image of θ_i by φ at a point z of $\varphi^{-1}(\mathring{\mathbb{B}})$ is equal to:

$$d_z\varphi(\sum_{\ell=1}^{k} \rho_\ell v_\ell^i) = (\sum_{\ell=1}^{k} \rho_\ell(z))\partial/\partial x_i = \partial/\partial x_i$$

because $(\rho_1, \dots, \rho_k)$ is a partition of unity. Therefore the vector field image of θ_i by φ is the vector field $\partial/\partial x_i$ of $\mathring{\mathbb{B}}$. Let us choose the open set U to be the open ball $\mathring{\mathbb{B}}$. Now, we define the diffeomorphism $\psi : \mathring{\mathbb{B}} \times \varphi^{-1}(p) \to \varphi^{-1}(\mathring{\mathbb{B}})$.

Let Φ_i be the flow defined by θ_i, *i.e.* the mapping:

$$\Phi_i : (-\varepsilon, +\varepsilon) \times \varphi^{-1}(\mathring{\mathbb{B}}) \to \varphi^{-1}(\mathring{\mathbb{B}})$$

such that $\Phi_i(0, q) = q$ and $\Phi_i(t, q)$ defines the integral path γ tangent to θ_i for which $\gamma(0) = q$. Then define $\psi : \mathring{\mathbb{B}} \times \varphi^{-1}(p) \to \varphi^{-1}(\mathring{\mathbb{B}})$ by:

$$\psi(u_1, \dots, u_n, q) = \Phi_1(u_1, \Phi_2(\dots, \Phi_n(u_n, q)))$$

where $(u_1, \dots, u_n) \in \mathring{\mathbb{B}}$ and $q \in \varphi^{-1}(p)$. The flow $\Phi_i(u_i, q)$ is defined as long as the point it defines lies in $\varphi^{-1}(\mathring{\mathbb{B}})$. In fact the space $\varphi^{-1}(\mathbb{B})$ is compact because φ is proper and the flow curves lie in $\varphi^{-1}(\mathring{\mathbb{B}})$, therefore all the flows are defined in $\varphi^{-1}(\mathring{\mathbb{B}})$.

We have:

$$u = f(\psi(u, q))$$

because, since the image of θ_i by φ is $\partial/\partial x_i$, we have

$$f(\Phi_j(u_j, q)) = f(\Phi_j(0, q)) + u_j e_j = p + u_j e_j$$

as the flow of $\partial/\partial x_j$ is $\phi(u_j, p) = p + u_j e_j$, where e_j is the unit vector field of the j-th coordinate of $\mathring{B}$.

It remains to prove that ψ is a diffeomorphism. The inverse diffeomorphism:

$$\psi^{-1} : \varphi^{-1}(\mathring{\mathbb{B}}) \to \mathring{\mathbb{B}} \times \varphi^{-1}(p)$$

is given by:

$$\psi^{-1}(z) = \Phi_1(-u_1, \Phi_2(\ldots, \Phi(-u_n, y)))$$

where $\varphi(z) = (u_1, \ldots, u_n)$. This proves Ehresmann Lemma. ☐

Remark 6.2.11 We notice that in the statement of Ehresmann Lemma, the manifolds might not be connected.

Let us apply Ehresmann Lemma to prove Proposition 6.2.9. Let $\mathbb{B}_{\varepsilon_0}$ be a ball centered at $\underline{0}$ with radius ε_0 small enough such that it is a Milnor ball. Consider ε, $0 < \varepsilon < \varepsilon_0$, and set $M_0 := \mathbb{B}_\varepsilon \cap X$. One notices that for $\delta > 0$ small enough, by Remark 6.2.8, any $t \neq 0$, $|t| < \delta$, is not a critical value of f restricted to a neighborhood of $\underline{0}$ in X. If X has one analytic component at $\underline{0}$, we just apply Lemma 6.2.6 to conclude that for X and $|t|$ small enough, the fibers $f^{-1}(t)$ with $t \neq 0$ are smooth manifolds. If X has more than one analytic component, since X has an isolated singularity at $\underline{0}$, these components at $\underline{0}$ only meet at $\underline{0}$, so one can apply Lemma 6.2.6 to each analytic component of X at $\underline{0}$ (see Remark 6.2.7).

On the other hand, the sphere $\mathbb{S}_\varepsilon$ is transverse to $V(f) = f^{-1}(0)$ in $\mathbb{C}^N$, therefore for $\delta > 0$ small enough, the fibers $f^{-1}(t)$, for $|t| < \delta$, are transverse to $\mathbb{S}_\varepsilon$ in $\mathbb{C}^N$, by continuity of the transversality.

Let us choose $\delta > 0$ such that, for any $t \neq 0$, $|t| < \delta$, the fiber $f^{-1}(t) \cap \mathbb{B}_\varepsilon$ does not contain critical points of f and $f^{-1}(t)$ intersects $\mathbb{S}_\varepsilon$ transversally in $\mathbb{C}^N$. Now consider:

$$M = \mathbb{B}_\varepsilon \cap M \cap (f^{-1}(\mathring{\mathbb{D}}_\delta) - V(f)) \xrightarrow{f_|} \mathring{\mathbb{D}}^*_\delta$$

where $\mathbb{D}^*_\delta = \mathbb{D}_\delta - \{0\}$ and $f_|$ is induced by f. This map is obviously proper because the closed ball $\mathbb{B}_\varepsilon$ is compact. The second condition of Ehresmann Lemma means that $f_|$ has no critical point in the intersection $\mathring{\mathbb{B}}_\varepsilon \cap X \cap f^{-1}(\mathring{\mathbb{D}}_\delta)$. The third condition means that the restriction of f to the boundary $\mathbb{S}_\varepsilon \cap X \cap f^{-1}(\mathring{\mathbb{D}}_\delta)$ of M has maximal rank. So we can apply Ehresmann Lemma which proves Proposition 6.2.9.

The fibration of Proposition 6.2.9 only depends on the germ $f : (X, \underline{0}) \to (\mathbb{C}, 0)$ as stated in the following Proposition:

Proposition 6.2.12 *Let* $f_1 : (X_1, \underline{0}_1) \to (\mathbb{C}, 0)$ *and* $f_2 : (X_2, \underline{0}_2) \to (\mathbb{C}, 0)$ *be two isomorphic germs of holomorphic functions, then the local fibrations as in Proposition 6.2.9 are diffeomorphic.*

Recall that f_1 and f_2 are isomorphic if we have a commutative diagram:

$$\begin{tikzcd}
(X_1, \underline{0}_1) \arrow[r, "h"] \arrow[d, "f_1"'] & (X_2, \underline{0}_2) \arrow[d, "f_2"] \\
(\mathbb{C}, 0) \arrow[r, "k"] & (\mathbb{C}, 0)
\end{tikzcd} \tag{1}$$

where h and k are isomorphism of complex analytic germs.

Proof Our proof is based on an observation of J. Milnor in [65], at the end of the proof of Lemma 8.3, on the isomorphism class of a fibration on the circle $\mathbb{S}^1$: □

Definition 6.2.13 Let $\varphi : E \to \mathbb{S}^1$ be a locally trivial smooth fibration over the circle $\mathbb{S}^1$. Suppose that φ is proper. The integration of a vector field obtained as a lifting by φ of the unit vector field of $\mathbb{S}^1$ defines, by the first return map, a diffeomorphism of a fiber to itself, called a *geometric monodromy* of the fibration φ.

Since the lifting of the unit vector field of $\mathbb{S}^1$ is not unique, the geometric monodromy is not unique. However its isotopy class is uniquely determined by φ.

Let us set $F_\theta := \varphi^{-1}(e^{i\theta})$ where $i = \sqrt{-1}$. The fibration $\varphi : E \to \mathbb{S}^1$ defines a commutative diagram:

$$\begin{tikzcd}
F_0 \times \mathbb{R} \arrow[r, "h"] \arrow[d, "p"'] & E \arrow[d, "\varphi"] \\
\mathbb{R} \arrow[r, "\pi"] & \mathbb{S}^1
\end{tikzcd}$$

where p is the projection on the second factor and π is the universal covering $\pi(\theta) = e^{i\theta}$ of $\mathbb{S}^1$.

By using the lifting of the unit vector field on $\mathbb{S}^1$, we obtain a smooth diffeomorphism from F_0 into F_θ for $0 \leq \theta \leq 2\pi$. This gives a one parameter family of smooth diffeomorphism from F_0 into F_θ.

One has a trivial fibration $F_0 \times [0, 2\pi]$ onto $[0, 2\pi]$ which gives our fibration $E \to \mathbb{S}^1$ by glueing $F_0 \times \{0\}$ with $F_0 \times \{2\pi\}$ using a geometrical monodromy, *i.e.* the point $(z, 0)$ is identified with $(H(z), 2\pi)$ where H is a geometric monodromy.

Exercise 6.2.14

1. Prove that the preceding construction gives a locally trivial smooth fibration isomorphic to $E \to \mathbb{S}^1$.
2. Prove that all geometric monodromies belong to the same isotopy class of diffeomorphism of F_0 into itself.

The preceding description of a locally smooth fibration $E \to \mathbb{S}^1$ shows that:

Lemma 6.2.15 *Two locally smooth fibrations $E_1 \to \mathbb{S}^1$ and $E_2 \to \mathbb{S}^1$ are isomorphic if they have diffeomorphic fibers and isomorphic geometric monodromies.*

The locally trivial fibrations that we have associated to an isolated singularity of a function on a space with isolated singularity are fibrations $\varphi : \mathcal{E} \to \mathring{\mathbb{D}}^*$ over an open punctured disc $\mathring{\mathbb{D}}^*$.

Considering the diffeomorphism $\delta : \mathring{\mathbb{D}}^* \to \mathbb{S}^1 \times (0, 1)$ of $\mathring{\mathbb{D}}^*$ with a product $\mathbb{S}^1 \times (0, 1)$ given by $\delta(re^{i\theta}) = (e^{i\theta}, r/R)$ where R is the radius of $\mathbb{D}$, the fiber $F_\theta := (\delta \circ \varphi)^{-1}(e^{i\theta}, 1/2)$ and the morphism $\delta \circ \varphi$ induces a trivial smooth fibration on $\mathcal{E}_\theta \simeq (\delta \circ \varphi)^{-1}(\{e^{i\theta}\} \times (0, 1))$ over the interval $(0, 1)$ with fiber F_θ, because any smooth fibration over an interval is trivial.

In fact the projection of $\mathcal{E}$ onto the interval $(0, 1)$ obtained as the composition of $\delta \circ \varphi$ with the second projection p_2 of $\mathbb{S}^1 \times (0, 1)$ is a trivial fibration:

$$p_2 \circ \delta \circ \varphi : \mathcal{E} \to (0, 1).$$

The fibers of this trivial fibration are $\mathcal{E}_r := \varphi^{-1}(\partial \mathbb{D}_r)$ where $\partial \mathbb{D}_r$ is the circle boundary of the disc $\mathbb{D}_r$ centered at 0 of radius r, *i.e.* the fibration is:

$$p_2 \circ \delta \circ \varphi : \mathcal{E} = \coprod_{0<r<R} \mathcal{E}_r \to (0, 1)$$

where $p_2 \circ \delta \circ \varphi(z) = r/R$ if $z \in \mathcal{E}_r$.

We shall show a stronger result: the maps $\mathcal{E}_r \to \partial \mathbb{D}_r$ are isomorphic locally trivial fibrations when $0 < r < R$. To prove this assertion consider the radial vector field w in the punctured disc $\mathring{\mathbb{D}}^*_R$ which is tangent to the radii, aiming to the origin 0 and with norm at a point $x \in \mathring{\mathbb{D}}^*_R$ equal to the distance of x to 0. Since φ is proper, we do as above, in the proof of Ehresmann Lemma (Lemma 6.2.10), and one can lift this vector field as a vector field W on $\mathcal{E}$ which is tangent to the boundary of $\mathcal{E}$ and so that its projection on $\mathring{\mathbb{D}}_R$ gives the flow whose integration give radii of $\mathring{\mathbb{D}}_R$.

For any $z \in \mathcal{E}_r$, let $\Phi(t, z)$ be the flow associated to W such that $\Phi(0, z) = z$ and the projection by φ of $\Phi(t, z)$ in $\mathring{\mathbb{D}}^*$ is a point on the circle of radius passing by $\varphi(z)$ at a distance re^{-t} from $\varphi(z)$.

Therefore the flow $\Phi(t, z)$ for a fixed t defines a diffeomorphism from $\mathcal{E}_{1/2R}$ to $\mathcal{E}_{1/2Re^{-t}}$. The projection of this flow gives a mapping along the radii which sends $\partial \mathbb{D}_{1/2R}$ to $\partial \mathbb{D}_{1/2Re^{-t}}$. Actually we have a diffeomorphism:

$$\Phi : (-\ln 2, +\infty) \times \mathcal{E}_{1/2R} \to \mathcal{E}.$$

We can do this reasoning for any r, $R > r > 0$, in which case the flow is Φ_r so that $\Phi_r(0, z) = z$ for any point $z \in \mathcal{E}_r$. In this case the flow gives a diffeomorphism

$$\Phi_r : (-\ln R + \ln r, +\infty) \times \mathcal{E}_r \to \mathcal{E}.$$

By integration of the vector fields W and w, on one the hand we have a diffeomorphism between the $\mathcal{E}_r$, and on the other hand a diffeomorphism along the radii between the circles $\partial \mathbb{D}_r$. This gives that the fibrations $\mathcal{E}_r \to \partial \mathbb{D}_r$ are isomorphic fibrations and also that the integration of the vector fields gives an isomorphism between $\mathcal{E} \to \mathring{\mathbb{D}}_R$ and $\mathcal{E}_{1/2R} \times (0, R) \to \partial \mathbb{D}_{1/2R} \times (0, R)$.

We have proved:

Lemma 6.2.16 *A proper locally trivial fibration* $\varphi : \mathcal{E} \to \mathring{\mathbb{D}}_R$ *is isomorphic to the local trivial fibration*

$$\varphi_| \times Id : \mathcal{E}_r \times (0, R) \to \partial \mathbb{D}_r \times (0, R)$$

where r *is a real number such that* $R > r > 0$, $\varphi_|$ *is induced by* φ *and* Id *is the identity of the interval* $(0, R)$.

That is, the fibration φ is determined by the locally trivial fibration $\varphi_| : \mathcal{E}_r \to \partial\mathbb{D}_r$.

Now consider a non-negative real analytic function θ defined in an open neighborhood of an isolated singularity $\underline{0}$ in an analytic set X. We suppose that $\theta^{-1}(0) = \{0\}$. Then:

Lemma 6.2.17 *The value* 0 *is not an accumulation point of critical values of* θ.

Proof We give a proof analogous to the one by E. Looijenga in [54, Lemma (2.2)].

In a small neighborhood of $\underline{0}$, we may assume that X is defined by complex equations $f_1, \dots, f_k$ in $\mathbb{C}^N$ and since $X - \{\underline{0}\}$ is non singular, the matrix of the complex differentials of $f_1, \dots, f_k$ has rank $N - \dim X$ for any point of $X - \{\underline{0}\}$ in a small neighborhood of $\underline{0}$. The set of critical points of the real function θ on $X - \{\underline{0}\}$ is contained in the set Y where the matrix of the differentials of the real equations which define X and the differential of θ is $\leq 2(N - \dim X)$. It is clear that Y is real analytic. It suffices to prove that the point $\underline{0}$ is isolated in Y. If it was not, the Curve Selection Lemma (see [54] (2.10) or in the semi-algebraic case [65] Lemma 3.1) would give us a curve $\gamma : (-\varepsilon, \varepsilon) \to X$ such that $\gamma(0) = \underline{0}$ and $\gamma(t) \in Y - \{\underline{0}\}$ for $t \neq 0$. Then:

$$\frac{d\theta \circ \gamma}{dt}(t) = 0$$

by definition of $Y - \{\underline{0}\}$. Then $\theta \circ \gamma$ would be constant and equal to $\underline{0}$ contradicting the hypothesis $\gamma(t) \in Y - \{\underline{0}\}$ for $t \neq 0$. □

Remark 6.2.18 In fact Lemma 6.2.17 has a more general statement:

Lemma 6.2.19 *Let* $\underline{0}$ *be a point in the closure* $\overline{Z}$ *of a non-singular semi-analytic set* Z *and let* $\varphi : \overline{Z} \to \mathbb{R}$ *be a real analytic function. There is a neighborhood* $\mathcal{U}$ *of* $\underline{0}$ *in* $\overline{Z}$ *and* $\delta > 0$ *such that for any* t, $0 < |t| < \delta$, *the space* $\varphi^{-1}(t) \cap Z \cap \mathcal{U}$ *is non-singular.*

In the case we consider here, with $\overline{Z} = X$, $Z = X - \{0\}$ and $\varphi = \theta$, this lemma gives Lemma 6.2.17. The proof is similar to the one of Lemma 6.2.6 and is left to the reader. In the case of semi-algebraic sets, one can find this type of results in [65], Corollary 2.8.

Suppose that we have another non-negative real analytic function r defined on a neighborhood of $\underline{0}$ in X such that $r^{-1}(0) = \{0\}$. Then, we have the following Lemma (cf. [65, Corollary 3.4]):

Lemma 6.2.20 *There is a neighborhood of* $\underline{0}$ *in* X *such that in no point of this neighborhood the differentials of* θ *and* r *are collinear, i.e.* $d\theta = \lambda dr$, *with a factor of collinearity* < 0.

Proof The proof uses again the Curve Selection Lemma. Consider the real analytic set of X where $d\theta$ and dr are collinear. In a neighborhood in X of $\underline{0}$ the space X has local complex analytic equations $f_1 \dots, f_k$ in $\mathbb{C}^N$. The set of points $z \in X - \{\underline{0}\}$

where $d\theta$ and dr are collinear is the set where the differentials of real analytic equations of X, $d\theta$ and dr make a matrix of rank $\leq 2N - 2\dim X + 1$. This defines a real analytic set Γ.

The point $\underline{0}$ is in the closure of Γ. Assume that arbitrarily near to $\underline{0}$, there are points where $d\theta = \lambda dr$ with $\lambda < 0$. This subset of Γ is semi-analytic. By the Curve Selection Lemma there is a curve $\gamma : [0 + \varepsilon) \to \Gamma$ such that $\gamma(0) = \underline{0}$ and $\gamma(t) \in X - \{\underline{0}\}$ for $t > 0$. For any $t > 0$, we have:

$$d\theta(\gamma(t)) = \lambda(t) dr(\gamma(t)) \neq 0$$

with $\lambda(t) < 0$, which means that one of the two functions is decreasing and the other one is increasing along γ, which is obviously impossible. This proves the Lemma.

□

A consequence of the preceding theorem is the following:

Proposition 6.2.21 *Let* r_1 *and* r_2 *be two non-negative real analytic functions defined on a neighborhood of* $\underline{0}$ *in a complex analytic set* X. *We assume that* $\underline{0}$ *is an isolated singularity of* X *and that* $r_1^{-1}(0) = r_2^{-1}(0) = \{\underline{0}\}$. *For* $\varepsilon_i > 0$ *small enough, the set* $r_i \leq \varepsilon_i$ *is compact for* $i = 1, 2$. *If* $\varepsilon_1 > 0$ *and* $\varepsilon_2 > 0$ *are small enough and they are such that* $\{r_1 \leq \varepsilon_1\} \subset \{r_2 \leq \varepsilon_2\}$, *then there is a diffeomorphism from* $r_1 = \varepsilon_1$ *onto* $r_2 = \varepsilon_2$.

Proof Because of Lemma 6.2.17 we can suppose that there is a neighborhood $\mathcal{U}$ where both functions r_1 and r_2 have no critical point in $\mathcal{U} - \{\underline{0}\}$. Then consider the function $\phi(\lambda) := \lambda(r_1 - \varepsilon_1) + (1 - \lambda)(r_2 - \varepsilon_2)$, where $\lambda \in \mathbb{R}$. Consider the unit vector field v_0 on an open neighborhood U of the interval [0, 1].

Since r_1 and r_2 have no critical point in $\mathcal{U} - \{0\}$ and one can choose $\mathcal{U}$ so that the differentials dr_1 and dr_2, when collinear, have the same direction, because of Lemma 6.2.20, the function ϕ has maximal rank on a neighborhood of $\{r_2 \leq \varepsilon_2\} - \{r_1 < \varepsilon_1\}$. One can lift by ϕ the restriction of v_0 to [0, 1] as a non-zero vector field V_0. One does as usually: the lifting is done locally, because ϕ has maximal rank, and by using a partition of unity one obtains a vector field on $\{r_2 \leq \varepsilon_2\} - \{r_1 < \varepsilon_1\}$. Since $\{r_2 \leq \varepsilon_2\} - \{r_1 < \varepsilon_1\}$ is compact, one can use a finite partition of unity.

Then integrating the vector field V_0, one obtains the desired diffeomorphism. If Φ is the flow $\Phi(t, z)$ of V_0 such that $\Phi(0, z) = z \in \{r_1 = \varepsilon_1\}$, then $\Phi(1, z)$ is in $\{r_2 = \varepsilon_2\}$. □

Now, let us come back to the proof of Proposition 6.2.12. To prove that the Milnor fibration of isomorphic germs are isomorphic, according to Lemma 6.2.16, we only need to show that over a small circle centered at 0 the local fibrations of Propositions 6.2.9 associated to f_1 and f_2 are isomorphic.

If f_1 and f_2 are isomorphic germs we have a commutative diagram as in (6.1). Since we have an isomorphism of analytic germs $h : (X_1, \underline{0}_1) \to (X_2, \underline{0}_2)$, there exist open neighborhoods $\mathcal{U}_1$ of $\underline{0}_1$ and $\mathcal{U}_2$ of $\underline{0}_2$ in X_2 such that h induces an isomorphism from $\mathcal{U}_1$ onto $\mathcal{U}_2$ that we shall still denote h. We can assume that $\mathcal{U}_i \subset \mathbb{C}^{N_i}$, for $i = 1$ or 2.

Let $r_i := \sum_1^{N_i} |z_j|^2$ be the square of the distance to $\underline{0}_i$ in $\mathbb{C}^{N_i}$. Then, the ball $\mathbb{B}_\varepsilon^{(i)}$ is the subset $\{r_i \le \varepsilon^2\}$. Let ρ_i be the restriction of r_i to $\mathcal{U}_i \cap X_i$. The pre-image of $\mathring{\mathbb{B}}_\varepsilon^{(2)} \cap X_2$ by the isomorphism $h : (X_1, 0) \cap \mathcal{U}_1 \to (X_2, 0) \cap \mathcal{U}_2$ is the set $\{\rho_2 \circ h < \varepsilon^2\}$.

The functions ρ_i are real analytic respectively in the neighborhood $\mathcal{U}_i$ of $\underline{0}_i$ in X_i.

Notice we can assume that in an open set $\mathcal{U}$ small enough, the functions ρ_1 and $\rho_2 \circ h$ have no critical points in $\mathcal{U} \cap X_1 - \underline{0}_1$, because of Lemma 6.2.17.

Lemma 6.2.22 *Let $\varepsilon_2 > 0$ be such that the set $\{\rho_2 \circ h \le \varepsilon_2^2\}$ is compact and lies inside $\mathcal{U}$. Set $\varepsilon_1 > 0$ such that the compact neighborhood $\mathbb{B}_{\varepsilon_1}^{(1)} \cap X_1$ of $\underline{0}$ in X_1 is contained in $\{\rho_2 \circ h < \varepsilon_2^2\}$. Then there is a smooth vector field v defined on a neighborhood of $\{\rho_2 \circ h \le \varepsilon_2^2\} - \mathring{\mathbb{B}}_{\varepsilon_1}^{(1)} \cap X_1$ whose integration defines an isomorphism of the fibration of $f_1 = k^{-1} \circ f_2 \circ h$ induced on $\{\rho_2 \circ h \le \varepsilon_2^2\}$ with the fibration of f_1 induced on $\mathbb{B}_{\varepsilon_1}^{(1)} \cap X_1$.*

Proof Since ρ_1 and $\rho_2 \circ h$ are real analytic functions which define $\underline{0}_1$ in the sense that $\rho_i^{-1}(0) = \underline{0}_1$ and $(\rho_2 \circ h)^{-1}(0) = \underline{0}_1$, we can apply Lemma 6.2.21 or more precisely adapt its proof to prove Lemma 6.2.22. In fact, we will consider a vector field which is going to give us more than the conclusions of Lemma 6.2.22.

For any point z of $\{\rho_2 \circ h \le \varepsilon_2\} - \mathring{\mathbb{B}}_{\varepsilon_1}^{(1)} \cap X_1$, there is a smooth vector field v_z defined in an open neighborhood U_z of z which satisfies the following conditions:

1. for any $z' \in U_z$, $d_{z'}(\rho_1)(v_{z'}) = -1$ and $d_{z'}(\rho_2 \circ h)(v_{z'}) < 0$;
2. there is τ small enough such that, for any $z' \in U_z \cap \{f_1 = t\}$ with $\tau > |t|$, the vector field $v_{z'}$ is tangent to $U_z \cap \{f_1 = t\}$.

If at z the differentials $d_z(\rho_1)$ and $d_z(\rho_2 \circ h)$ are linearly independent, then the condition (1) is obvious in a small open neighborhood U_z of z in X_1, since, for every $z' \in U_z$ we have that $d_{z'}(\rho_1)$ and $d_{z'}(\rho_2 \circ h)$ are also linearly independent. If at z the differentials $d_z\rho_1$ and $d_z(\rho_2 \circ h)$ are not linearly independent, we have just proved in Lemma 6.2.20 that they necessarily point in the same direction, so we can find v_z so that $d_z\rho_1(v_z) = -1$ and $d_z(v_z) < 0$ and it extends into a smooth vector field which satisfies (1).

To obtain condition (2) it is enough to notice that in the compact set:

$$A_{\varepsilon_1,\varepsilon_2} := \{\rho_2 \circ h \le \varepsilon_2^2\} - \mathring{\mathbb{B}}_{\varepsilon_1}^{(1)} \cap X_1,$$

if t is small enough, the levels $\rho_2 \circ h = \varepsilon^2$ and $\rho_1 = \varepsilon^2$ are transverse to the fibers $f_1 = t$. For any point of $A_{\varepsilon_1,\varepsilon_2} \cap \{f_1 = 0\}$ we can repeat what we have done above on $A_{\varepsilon_1,\varepsilon_2} \cap X_1$. For each point $z \in A_{\varepsilon_1,\varepsilon_2} \cap \{f_1 = 0\}$ there is a neighborhood of z in $A_{\varepsilon_1,\varepsilon_2} \cap \{f_1 = 0\}$ in which there is a smooth vector field $v_{z'}$ such that $d_{z'}(\rho_1)(v_{z'}) = -1$ and $d_{z'}(\rho_2 \circ h)(v_{z'}) < 0$. Since in a neighborhood of each point z of $A_{\varepsilon_1,\varepsilon_2} \cap \{f_1 = 0\}$ the fibers $\{f = t\}$ for small t are fibers of a locally trivial fibration, one can extend that vector field as a smooth vector field on a small neighborhood V_z of $z \in A_{\varepsilon_1,\varepsilon_2} \cap \{f_1 = 0\}$.

Let us consider the open set V of X_1 defined by:

$$V := \cup_{z\in A_{\varepsilon_1,\varepsilon_2}\cap\{f_1=0\}} V_z.$$

Since $A_{\varepsilon_1,\varepsilon_2} \cap \{f_1 = 0\}$ is compact, there is $\delta > 0$ small enough such that:

$$A_{\varepsilon_1,\varepsilon_2} \cap \{|f_1| \le \delta\} \subset V.$$

Assume that, for $z \notin A_{\varepsilon_1,\varepsilon_2} \cap \{|f_1| \le \delta\}$ the open subsets U_z defined above can be chosen to be open subset of in $A_{\varepsilon_1,\varepsilon_2} - \{|f_1| \le \delta\}$. Then:

$$(\cup_{z\in A_{\varepsilon_1,\varepsilon_2}-\{|f_1|\le\delta\}} U_z) \cup (\cup_{z\in A_{\varepsilon_1,\varepsilon_2}\cap\{f_1=0\}} V_z)$$

is an open covering of $A_{\varepsilon_1,\varepsilon_2}$. Consider a partition of unity associated to this covering and glue the local vector fields v_z and V_z with this partition of unity to get a smooth vector field W on an open neighborhood of $A_{\varepsilon_1,\varepsilon_2}$. By integrating this vector field W we retract $\rho_2 \circ h \le \varepsilon_2^2$ into $\rho_1 \le \varepsilon_1^2$ (see [54, p. 23 and 24]) and we retract $\{\rho_2 \circ h \le \varepsilon_2^2\} \cap \{f_1 < \delta\}$ into $\{\rho_1 \le \varepsilon_1^2\} \cap \{f_1 < \delta\}$ along the fibers of f_1. □

Actually the vector field W that we just constructed gives a isomorphism between the fibration:

$$\mathbb{B}^{(1)}_{\varepsilon_1} \cap f_1^{-1}(\mathring{\mathbb{D}}^*_\delta) \to \mathring{\mathbb{D}}^*_\delta$$

and the fibration $\{\rho_2 \circ h \le \varepsilon_2\} \cap f_1^{-1}(\mathring{\mathbb{D}}^*_\delta) \to \mathring{\mathbb{D}}^*_\delta$.

According to Lemma 6.2.16, the fibration $\mathbb{B}^{(1)}_{\varepsilon_1} \cap f_1^{-1}(\mathring{\mathbb{D}}^*_\delta) \to \mathring{\mathbb{D}}^*_\delta$ is given by the fibration $\mathbb{B}^{(1)}_{\varepsilon_1} \cap f_1^{-1}(\partial\mathbb{D}_{\delta/2}) \to \partial\mathbb{D}_{\delta/2}$.

Similarly the fibration $\{\rho_2 \circ h \le \varepsilon_2\} \cap f_1^{-1}(\mathring{\mathbb{D}}^*_\delta) \to \mathring{\mathbb{D}}^*_\delta$ is given by:

$$\{\rho_2 \circ h \le \varepsilon_2\} \cap f_1^{-1}(\partial\mathbb{D}_{\delta/2}) \to \partial\mathbb{D}_{\delta/2}.$$

Therefore the fibrations $\mathbb{B}^{(1)}_{\varepsilon_1} \cap f_1^{-1}(\partial\mathbb{D}_{\delta/2}) \to \partial\mathbb{D}_{\delta/2}$ and $\{\rho_2 \circ h \le \varepsilon_2\} \cap f_1^{-1}(\partial\mathbb{D}_{\delta/2}) \to \partial\mathbb{D}_{\delta/2}$. are isomorphic.

To prove Proposition 6.2.12 we are left to prove that the fibration:

$$\{\rho_2 \circ h \le \varepsilon_2\} \cap f_1^{-1}(\partial\mathbb{D}_{\delta/2}) \to \partial\mathbb{D}_{\delta/2}$$

is isomorphic to $\mathbb{B}^{(2)}_{\varepsilon_2} \cap f_2^{-1}(\partial\mathbb{D}_{\delta/2}) \to \partial\mathbb{D}_{\delta/2}$.

However, the inverse image by h of $\mathbb{B}^{(2)}_{\varepsilon_2} \cap f_2^{-1}(\partial\mathbb{D}_{\delta/2})$ is $\{\rho_2 \circ h \le \varepsilon_2\} \cap f_1^{-1}(\partial\mathbb{D}_{\delta/2})$ and the inverse image of $\mathbb{D}_{\delta/2}$ by the isomorphism k is $\{r \circ k = (\delta/2)^2\}$, where $r(z) = |z|^2$.

We can choose $\beta > 0$ small enough such that:

$$\mathbb{D}_\beta = \{r \leq \beta^2\} \subset \{r \circ k \leq (\delta/2)^2\}.$$

By applying Proposition 6.2.21 we have that $\{r = \beta^2\}$ is diffeomorphic to $\{r \circ k = (\delta/2)^2\}$. Therefore the fibration $\{\rho_2 \circ h \leq \varepsilon_2\} \cap f_1^{-1}(\partial \mathbb{D}_\beta) \to \partial \mathbb{D}_\beta$ is isomorphic to the fibration $\{\rho_2 \circ h \leq \varepsilon_2\} \cap f_1^{-1}(\partial\{r \circ k \leq (\delta/2)^2\}) \to \partial\{r \circ k \leq (\delta/2)^2\}$.

This shows that the following fibrations are isomorphic:

1. $\mathbb{B}_{\varepsilon_1}^{(1)} \cap f_1^{-1}(\partial \mathbb{D}_{\delta/2}) \to \partial \mathbb{D}_{\delta/2}$;
2. $\{\rho_2 \circ h \leq \varepsilon_2\} \cap f_1^{-1}(\partial \mathbb{D}_{\delta/2}) \to \partial \mathbb{D}_{\delta/2}$;
3. $\{\rho_2 \circ h \leq \varepsilon_2\} \cap f_1^{-1}(\partial \mathbb{D}_\beta) \to \partial \mathbb{D}_\beta$;
4. $\{\rho_2 \circ h \leq \varepsilon_2\} \cap f_1^{-1}(\partial\{r \circ k \leq (\delta/2)^2\}) \to \partial\{r \circ k \leq (\delta/2)^2\}$.

In fact, the fibrations (1) and (2) are isomorphic because of the integration of the vector field V constructed above. The fibrations (2) and (3) are isomorphic because of Lemma 6.2.16. The fibrations (3) and (4) are isomorphic because of the reasoning above using Proposition 6.2.21.

The fibration (1) gives the fibration of Proposition 6.2.9 for f_1 and the fibration (4) gives the fibration of Proposition 6.2.9 for f_2, thus proving Proposition 6.2.12.

Definition 6.2.23 The fibration obtained in Proposition 6.2.9 is called *the local Milnor fibration* of f at $\underline{0}$. The fibers $F_t = f^{-1}(t) \cap \mathring{\mathbb{B}}_\varepsilon$ are *the Milnor fibers* of f at 0; these are the interior of the fibers in the compact fibration of Proposition 6.2.9.

Remark 6.2.24 Since we have assumed that the singularity of X and f at $\underline{0}$ is isolated, the fibers $\bar{F}_t := f^{-1}(t) \cap \mathbb{B}_\varepsilon$, $t \in \mathbb{D}_\delta - \{0\}$, are compact manifolds with boundary, embedded in X with trivial normal bundle (since the gradient of f trivializes the normal bundle). It follows from [30] that if the tangent bundle of $X - \{\underline{0}\}$ is trivial (as a C^∞ real vector bundle), then the tangent bundle of the fiber $\bar{F}_t$ is trivial too.

6.3 Examples

Consider the homogeneous map $f : \mathbb{C}^2 \to \mathbb{C}$ defined by $(x, y) \mapsto xy$; this has a unique critical point at $x = 0 = y$. Its zero locus $V(f)$ consists of the two axes $\{x = 0\} \cup \{y = 0\}$ with the origin as an isolated singularity. So its link $L_f := V(f) \cap \mathbb{S}^3$ is the Hopf link. By [65, Lemma 9.4], the Milnor fiber F_f is diffeomorphic to the whole fiber $f^{-1}(1)$, which consists of the points where $x \neq 0$ and $y = 1/x$. Hence F_f is diffeomorphic to a copy of $\mathbb{C}^*$; in fact it is an open cylinder $\mathbb{S}^1 \times \mathbb{R}$, and it can be regarded as being the tangent bundle of the circle. In particular F_f has the homotopy type of $\mathbb{S}^1$.

We extend these considerations to higher dimensions in two different ways. First notice that we can make the change of coordinates $(x, y) \mapsto (z_1, z_2)$ with $z_1 = (x+iy)$ and $z_2 = (x-iy)$, where (x, y) denote here complex variables. In these new coordinates the above map becomes $z_1^2 + z_2^2$ and we may consider, more generally, the homogeneous polynomial:

$$f(z_0, \cdots, z_n) = z_0^2 + \cdots + z_n^2 .$$

The link L_f consists of the points where one has:

$$\mathrm{Re}(z_0^2 + \cdots + z_n^2) = 0 \quad , \quad \mathrm{Im}(z_0^2 + \cdots + z_n^2) = 0 \quad \text{and} \quad |z_0|^2 + \cdots + |z_n|^2 = 1 .$$

Hence L_f is diffeomorphic to the unit sphere bundle of the n-sphere $\mathbb{S}^n$. By Milnor's Lemma 9.4 in [65], the Milnor fiber F_f is diffeomorphic to the set of points where $z_0^2 + \cdots + z_n^2 = 1$, *i.e.*,

$$\mathrm{Re}(z_0^2 + \cdots + z_n^2) = 1 \quad \text{and} \quad \mathrm{Im}(z_0^2 + \cdots + z_n^2) = 0 .$$

This describes the tangent bundle of the n-sphere $\mathbb{S}^n$, and F_f actually is the corresponding open unit disc bundle. In particular F_f has the sphere $\mathbb{S}^n$ as a deformation retract and therefore F_f has non-trivial homology only in dimensions 0 and n; in these dimensions its integral homology is isomorphic to $\mathbb{Z}$.

Starting again with the initial example, consider now the map $f : \mathbb{C}^3 \to \mathbb{C}$ defined by $(x, y, z) \mapsto xyz$. Its zero set $V(f)$ consists of the coordinate planes $\{x = 0\} \cup \{y = 0\} \cup \{z = 0\}$, with the three axes as singular set. The singularities of $V(f) := f^{-1}(0)$ are non-isolated. However we shall see in the next section that one can define a Milnor fibration and a Milnor fiber.

The Milnor fiber F_f is diffeomorphic to $\{xyz = 1\}$, *i.e.*, $x \neq 0$, $y \neq 0$ and $z = 1/xy$. Therefore F_f is diffeomorphic to $\mathbb{C}^* \times \mathbb{C}^*$ and it has the torus $\mathbb{S}^1 \times \mathbb{S}^1$ as a deformation retract. So F_f now has non-trivial homology in dimensions 0, 1, 2.

We know from [65] that the Milnor fiber F_f of an arbitrary holomorphic map-germ $(\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ has the homotopy type of a finite CW-complex of middle-dimension n. This follows too from [2] since F_f is a Stein manifold and, perhaps moving the origin $\underline{0}$ slightly if necessary, the square of the function distance to $\underline{0}$ is a strictly plurisubharmonic Morse function on F_f, so one has severe restrictions on the possible Morse indices.

If we further assume that f has an isolated critical point at $\underline{0}$, then Milnor in [65, Lemma 6.4] used Morse theory to show that F_f is $(n-1)$-connected. Lefschetz duality, together with the above observations about the homology of F, implies that in this setting the fiber F_f has the homotopy type of a bouquet of n-spheres. The number $\mu = \mu(f)$ of such spheres is now called the Milnor number of f.

This will be discussed in the sequel, and it can be seen explicitly for the Pham-Brieskorn singularities, as noticed by Pham in [69] and explained below.

Consider a Pham-Brieskorn polynomial $f : \mathbb{C}^{n+1} \to \mathbb{C}$,

$$(z_0, \ldots, z_n) \overset{f}{\mapsto} z_0^{a_0} + \ldots + z_n^{a_n}, \qquad a_i \geq 2 .$$

The origin $\underline{0} \in \mathbb{C}^{n+1}$ is its only critical point, so $V := f^{-1}(0)$ is a complex hypersurface with an isolated singularity at $\underline{0}$. Let d be the lowest common multiple of the a_i and for each $i = 0, \cdots, n$ set $d_i = d/a_i$. Then for every non-zero complex number $\lambda \in \mathbb{C}^*$ one has a $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$ determined by

$$\lambda \cdot (z_0, \cdots, z_n) \mapsto (\lambda^{d_0} z_0, \cdots, \lambda^{d_n} z_n) .$$

Observe that one has:

$$f(\lambda^{d_0} z_0, \cdots, \lambda^{d_n} z_n) = \lambda^d f(z_0, \cdots, z_n) ,$$

so f is *weighted homogeneous*.

In his pioneer article [69], F. Pham studied the topology of the complex manifold $V_{(a_0,\cdots,a_n)} \subset \mathbb{C}^{n+1}$ defined by:

$$z_0^{a_0} + \cdots + z_n^{a_n} = 1 ,$$

where $n > 0$ and the a_i are integers ≥ 2. It is easy to see that this manifold is diffeomorphic to the Milnor fibre of the complex singularity defined by $f(z) = 0$, where f is the Pham-Brieskorn polynomial $f(z) = z_0^{a_0} + \cdots + z_n^{a_n}$. To explain Pham's results, let G_a denote the finite cyclic group of a^{th} roots of unity. Given the integers $\{a_0, \cdots, a_n\}$, denote by $J = J_{(a_0,\cdots,a_n)}$ the join:

$$J = G_{a_0} * G_{a_1} * \cdots * G_{a_n} \subset \mathbb{C}^{n+1} ,$$

which consists of all linear combinations

$$(t_0\, \omega_0, \cdots, t_n\, \omega_n)$$

with real numbers $t_i \geq 0$ such that $t_0 + \cdots + t_n = 1$ and $\omega_j \in G_{a_j}$. Note that J can be identified with the subset $\mathcal{P} = \mathcal{P}_{(a_0,\cdots,a_n)}$ defined by:

$$\mathcal{P} = \{z \in V_{(a_0,\cdots,a_n)} \mid z_j^{a_j} \in \mathbb{R} \text{ and } z_j^{a_j} \geq 0 , \text{ for all } j = 0, \cdots n\} .$$

To see this notice that $\mathcal{P}$ can also be described by the conditions:

$$t_0 + \cdots + t_n = 1 \text{ , and } z_j = u_j |z_j| \, , \ u_j \in G_{a_j} \, , \ t_j = |z_j|^{a_j}, \text{ for all } j = 0, \cdots, n$$

Hence $\mathcal{P}$ is contained in the manifold $V_{(a_0,\cdots,a_n)}$. The set $\mathcal{P}$ is known as *the Pham join* of the polynomial f. It is not hard to see that $V_{(a_0,\cdots,a_n)}$ has $\mathcal{P}$ as *a deformation* *retract* and therefore its homotopy type is that of $\mathcal{P}$. In fact, given a point $z \in V_{(a_0,\cdots,a_n)}$, first deform each coordinate z_j along a path in $\mathbb{C}$ chosen so that the trajectory described by $z_j^{a_j}$ is the straight line to the nearest point on the real axis, that we denote by $\hat{z}_j$. This carries z into a vector $\hat{z} = (\hat{z}_0, \cdots, \hat{z}_n)$ satisfying $\hat{z}_j^{a_j} \in \mathbb{R}$ for each j, and it is clear that this deformation leaves $V_{(a_0,\cdots,a_n)}$ invariant. Now, whenever one has that $\hat{z}_j^{a_j} < 0$, move $\hat{z}_j$ along a straight line to $0 \in \mathbb{C}$ as follows, and we deform simultaneously the other coordinates, keeping them ≥ 0, so that we remain in $V_{(a_0,\cdots,a_n)}$. Hence the point $\hat{z} = (\hat{z}_0, \cdots, \hat{z}_n)$ moves along a straight line towards a point $\check{z} = (\check{z}_0, \cdots, \check{z}_n) \in V_{(a_0,\cdots,a_n)}$ whose coordinates are all ≥ 0 and one has that each coordinate $\check{z}_j$ is necessarily of the form $t_j\,\omega_j$ for some $t_j \geq 0$ and some $\omega_j \in G_{a_j}$. This gives a deformation of $V_{(a_0,\cdots,a_n)}$ into $\mathcal{P}$ that leaves this set invariant, so the join $\mathcal{P}$ *is a deformation retract* of $V_{(a_0,\cdots,a_n)}$. It is now an exercise to show that $\mathcal{P}$ *has the homotopy type of a wedge (or bouquet) of spheres* of real dimension n. Moreover, the number of spheres in this wedge is $(a_0 - 1) \cdot (a_1 - 1) \cdots (a_n - 1)$. Thus we have obtained:

Theorem 6.3.1 (Pham) *The variety*

$$V_{(a_0,\cdots,a_n)} := \{z \in \mathbb{C}^{n+1} \mid z_0^{a_0} + \cdots + z_n^{a_n} = 1\},$$

has the set $\mathcal{P}$ as a deformation retract. Since $\mathcal{P}$ is homeomorphic to the join $J = G_{a_0} * G_{a_1} * \cdots * G_{a_n} \subset \mathbb{C}^{n+1}$, where G_a is the finite cyclic group of a^{th}-roots of unity, then $V_{(a_0,\cdots,a_n)}$ has the homotopy type of a bouquet $\bigvee \mathbb{S}^n$ of spheres of dimension n, the number of spheres in this wedge being $[(a_0 - 1) \cdot (a_1 - 1) \cdots (a_n - 1)]$.

6.4 Non-isolated Critical Points

In the case where the singularities of f are not isolated, there is a fibration theorem similar to Proposition 6.2.9. We still assume that X is a complex analytic space with an isolated singularity at $\underline{0}$ and that $f : (X, \underline{0}) \to (\mathbb{C}, 0)$ is a holomorphic function with a possibly non-isolated singularity at $\underline{0}$. Notice that $X - f^{-1}(0)$ is non-singular (see Lemma 6.2.6). We have the following fibration theorem:

Theorem 6.4.1 *Let X be a complex analytic space in $\mathbb{C}^N$ with an isolated singularity at $\underline{0}$ and let $f : (X, \underline{0}) \to (\mathbb{C}, 0)$ be a holomorphic function. Then there exist $\varepsilon > 0$ small enough and $\delta > 0$ sufficiently small with respect to ε, such that $(X \cap \mathbb{B}_\varepsilon) - f^{-1}(0)$ is non-singular and*

$$X \cap \mathbb{B}_\varepsilon \cap f^{-1}(\mathbb{D}_\delta^*) \to \mathbb{D}_\delta^*$$

is a locally trivial smooth fibration, where $\mathbb{D}_\delta^*$ is the punctured disc in $\mathbb{C}$ centered at $\underline{0}$ and with radius δ.

Proof That with the above conditions $(X \cap \mathbb{B}_\varepsilon) - f^{-1}(0)$ is non-singular follows immediately from Remark 6.2.8. The proof that we have a fibration as stated is essentially like the proof of Proposition 6.2.9. We are going to apply Ehresmann Lemma (Lemma 6.2.10).

We are choosing ε such that the sphere $\mathbb{S}_\varepsilon$ of radius ε in $\mathbb{C}^N \supset X$ is a Milnor ball for X. This can be done because we have assumed that X has an isolated singularity at $\underline{0}$. However we have to choose $\mathbb{S}_\varepsilon$ such that it is a Milnor sphere for $f^{-1}(0)$. To explain this, we stratify the set $|f^{-1}(0)|$ by a Whitney stratification $\mathcal{S} = (S_\alpha)_{\alpha \in A}$ (see Sect. 6.11 below or [83, Theorem 19.2, p. 540]). Since a Whitney stratification is locally finite, only a finite number of strata S_α contain $\underline{0}$ in their closure. □

Definition 6.4.2 A *Milnor sphere* for $f^{-1}(0)$ is a sphere $\mathbb{S}_\varepsilon$ transverse to all the strata of a Whitney stratification $\mathcal{S} = (S_\alpha)_{\alpha \in A}$ and such that every other sphere of smaller radius and centered at $\underline{0}$ also meets transversally all the strata.

The following is a special case of the general Theorem 6.11.1, see [6]:

Lemma 6.4.3 *There exist Milnor spheres for $f^{-1}(0)$ and if $\mathbb{S}_\varepsilon$ is a Milnor sphere of $f^{-1}(0)$, the topology of $\mathbb{S}_\varepsilon \cap f^{-1}(0)$ does not depend on ε.*

The main result is given by a theorem of H. Hironaka which states that every holomorphic function f can be stratified with a stratification satisfying A_f-condition. Let us explain what the A_f condition is. This notion is due to R. Thom (see [77], Morphismes sans éclatement, p. 257).

Definition 6.4.4

1. An analytic map $F : X \to Y$ between analytic sets X and Y stratified by Whitney stratifications $\mathcal{S} = (X_i)_{i \in I}$ and $\mathcal{T} = (Y_j)_{j \in J}$ is called a *stratified map*, if for any i, there exists j such that $F(X_i) = Y_j$ and $F_| : X_i \to Y_j$ induced by F is a submersion.
2. We say that the stratified map F *satisfies A_f condition* if for any $y \in Y$, for any $x \in X_i \cap f^{-1}(y)$ and any sequence (x_n) of X, which belongs to X_k and converges to x whose sequence of tangent spaces $T_{x_n}(X_k \cap F^{-1}(F(x_n))$ converges to T, we have $T \supset T_x(X_i \cap F^{-1}(y))$.

The (remarkable) result of Hironaka [24] is:

Theorem 6.4.5 *A complex analytic map $f : X \to Y$ into a complex analytic space of* $\dim Y = 1$ *can be stratified such that f satisfies the A_f condition. The A_f stratification of f can be obtained from refining a given stratification of f.*

Let us apply Hironaka Theorem to the case where f is a germ of holomorphic function:

$$f : (X, \underline{0}) \to (\mathbb{C}, 0)$$

such that $X - f^{-1}(0)$ is non-singular.

We may assume that a representative of f is $X \to \mathring{\mathbb{D}}$ where $\mathring{\mathbb{D}}$ is a sufficiently small open disc of $\mathbb{C}$ centered at 0, so a Whitney stratification of this representative is given by $X - f^{-1}(0)$ (which is non-singular) and a Whitney stratification of $f^{-1}(0)$ for X and $\mathbb{D} - \{0\}$ and $\{0\}$ for $\mathbb{D}$.

By Hironaka Theorem, we may refine the Whitney stratification of f such that it satisfies the A_f condition. Let ε small enough such that the sphere $\mathbb{S}_\varepsilon$ is transverse in $\mathbb{C}^N$ to the strata of the A_f stratification contained in $f^{-1}(0)$, and choose ε such that for all $0 < \varepsilon_1 \leq \varepsilon$ the sphere $\mathbb{S}_{\varepsilon_1}$ is transverse to the strata of $f^{-1}(0)$. Then, there is $\delta > 0$, such that for all $|t|$, $0 < |t| < \delta$, the non-singular space $\{f = t\}$ (see Remark 6.2.7) is transverse to the sphere $\mathbb{S}_\varepsilon$ in $\mathbb{C}^N$. If not, there is a sequence (x_n) of points of $(X - f^{-1}(0)) \cap \mathbb{S}_\varepsilon$ which tends to a point x_0 of $f^{-1}(0) \cap \mathbb{S}_\varepsilon$ such that the fiber $f^{-1}(f(x_n))$ is not transversal in $\mathbb{C}^N$ to $\mathbb{S}_\varepsilon$ at x_n. This would imply:

$$T_{x_n} f^{-1}(f(x_n)) \subset T_{x_n} \mathbb{S}_\varepsilon.$$

We can assume that x_n is in the stratum $X_k \subset X - f^{-1}(f(\underline{0}))$ of the A_f stratification of f and $x_0 \in X_\ell \subset f^{-1}(0)$. Therefore:

$$T_{x_n}(X_k \cap f^{-1}(f(x_n))) \subset T_{x_n} f^{-1}(f(x_n)) \subset T_{x_n} \mathbb{S}_\varepsilon.$$

At the limit, since the Grassmannians are compact, we can suppose that the sequence of tangent spaces $T_{x_n}(X_k \cap f^{-1}(f(x_n))$ has a limit T we have:

$$T_{x_0} X_\ell \subset lim_n T_{x_n}(X_k \cap f^{-1}(f(x_n)) = T \subset T_{x_0}(\mathbb{S}_\varepsilon).$$

This would contradict that $\mathbb{S}_\varepsilon$ and the strata of $f^{-1}(f(x_n))$ are transverse in $\mathbb{C}^N$.

Therefore, there is $\delta > 0$ such that for t, $0 < |t| < \delta$, $\{f = t\}$ is transverse to the sphere $\mathbb{S}_\varepsilon$ in $\mathbb{C}^N$.

Remark 6.4.6 In Théorème 4.2.1 of [4], it is shown that stratifications with the Whitney condition (see definition in Sect. 6.11) for f are Thom A_f stratifications. This gives another proof of the existence of stratifications with A_f conditions for f.

Now, we can apply Ehresmann Lemma (Lemma 6.2.10) to:

$$f_| : \mathbb{B}_\varepsilon \cap f^{-1}(\mathring{\mathbb{D}}_\delta - \{0\}) \to \mathring{\mathbb{D}}_\delta - \{0\}$$

when $1 \gg \varepsilon \gg \delta > 0$ as we did in the proof of Proposition 6.2.9. This proves Theorem 6.4.1.

Recall that a *Stein manifold* is a complex manifold M which is holomorphically convex and holomorphically separable, *i.e.*, given any two distinct points there exists a holomorphic function on M which takes distinct values on those points. For instance, the standard complex space $\mathbb{C}^N$ is a Stein manifold, and so is every domain of holomorphy in $\mathbb{C}^N$ and every closed complex submanifold of a Stein manifold. It follows that Milnor fibers F_t are Stein manifolds. One has:

Theorem 6.4.7 *Let X be an equidimensional complex analytic space of dimension n, let $(X, \underline{0}) \xrightarrow{f} (\mathbb{C}, 0)$ be a non-constant holomorphic map and assume X is non-singular away from $f^{-1}(0)$. Let F_t be its Milnor fiber, associated to the fibration in Theorem 6.4.1 (defined as in Definition 6.2.23). Then:*

1. *F_t is a compact manifold with boundary $f^{-1}(t) \cap \mathbb{S}_\varepsilon$, and its interior is a complex manifold of (complex) dimension n.*
2. *If the tangent bundle of $X - \{\underline{0}\}$ is (topologically) trivial as a real or complex bundle, then the tangent bundle of F_t is trivial as a real or complex bundle, respectively.*
3. *The fiber F_t has the homotopy type of a CW-complex of real dimension at most n.*

The third statement above is a special case of a deep theorem in [2] for Stein manifolds in general, largely due R. Thom (*cf.* [65, Chapter 5]). The proof consists in using complex geometry and analysis to show that up to a small perturbation, the function distance to the origin restricted to the fiber F_t, which is Stein, is a plurisubharmonic Morse function and the Morse index at each critical point is $\leq n$.

Milnor in [65] also studies the topology of the link L_V when the ambient space X is $\mathbb{C}^{n+1}$. He proves [65, Theorem 5.2]:

Theorem 6.4.8 *Let $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ be a non-constant holomorphic map-germ. Let $V := f^{-1}(0)$ and let $L_V := V \cap \mathbb{S}_\varepsilon$ be its link. Then the space L_V is $(n-2)$-connected.*

Notice that $L_V := V \cap \mathbb{S}_\varepsilon$ is a real analytic variety of real dimension $(2n-1)$ if f has an isolated singularity at $\underline{0}$ and it is singular if the critical point at $\underline{0}$ is non-isolated.

As Milnor says in [65, p. 45–46], the proof depends on a study of the Morse theory associated with the smooth real valued function $|f|$ on the complement of L_V in the sphere $\mathbb{S}_\varepsilon$, showing that the Morse index of the restriction of $|f|$ to $\mathbb{S}_\varepsilon - U(L_V)$, where $U(L_V)$ is a small open neighborhood of L_V in S_ε, at any critical point is $\geq n$.

6.5 Fibration on the Sphere

We have seen in the introduction that, for a germ of holomorphic function

$$f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0),$$

J. Milnor has proved that, for $\varepsilon > 0$ small enough, the map:

$$\frac{f}{|f|} : \mathbb{S}_\varepsilon^{2n+1} - L_V \longrightarrow \mathbb{S}^1,$$

where $L_V = f^{-1}(0) \cap \mathbb{S}_\varepsilon$ is the link of f at $\underline{0}$, is a smooth fibration (see [65]). In fact in [65] J. Milnor proves that the fiber in this fibration is diffeomorphic to the one in the fibration given in Theorem 6.4.1 when $X = \mathbb{C}^{n+1}$ (see Theorem 5.11 p. 53 of [65]) and we restrict Theorem 6.4.1 to the interior of the ball. This suggests that those two fibrations are isomorphic, and indeed they are, even in a more general setting that we shall discuss in Sect. 6.11.

In this section we shall prove that the result is true under the hypotheses of Theorem 6.4.1.

Theorem 6.5.1 *Let* $f : (X, \underline{0}) \to (\mathbb{C}, 0)$ *be a germ of holomorphic function, set* $V = f^{-1}(0)$ *and assume* $X - V$ *is non-singular. We may assume that* $(X, \underline{0}) \subset (\mathbb{C}^N, \underline{0})$*. Let* $\varepsilon > 0$ *be small enough so that if* $\mathbb{S}_\varepsilon$ *is the sphere centered at* $\underline{0}$ *in* $\mathbb{C}^N$ *with radius* ε*, then* $L_X := \mathbb{S}_\varepsilon \cap X$ *is the link of* X *at* $\underline{0}$ *and* $L_V := f^{-1}0 \cap \mathbb{S}_\varepsilon$ *is the link of* V*. Then*

$$\phi := \frac{f}{|f|} : L_X - L_V \longrightarrow \mathbb{S}^1,$$

is a smooth locally trivial fibration. Furthermore, this fibration is isomorphic to the fibration of Theorem 6.4.1 followed by the projection $\pi : \mathbb{D}^*_\delta \to \mathbb{S}^1$ *given by* $\pi(z) = z/|z|$*.*

Proof The original proof by Milnor in [65] is easily adapted to this more general setting. The first step uses the Curve Selection Lemma to show that the map ϕ has no critical points at all. This implies that the fibers $\phi^{-1}(e^{i\theta})$ are all smooth submanifolds of L_X. The next and harder step makes a sharper use of the Curve Selection Lemma to control the behavior of ϕ as the fibers approach the link L_V, and show that every fiber of ϕ has a neighborhood which is a product.

Nowadays the simplest way to prove Milnor's fibration theorem on the spheres, and more generally Theorem 6.5.1, is by proving first Theorem 6.4.1 and then carrying that fibration by a smooth flow, into the fibration in Theorem 6.5.1. This proves at once both statements in Theorem 6.5.1: that ϕ is the projection map of a C^∞ fiber bundle, and that this bundle is isomorphic to the one in Theorem 6.4.1. This is done by constructing an appropriate vector field.

For this, choose a local C^∞ vector field in a neighborhood of 0 in $\mathbb{C}$ which has an isolated singularity at 0 and everywhere else it is pointing outwards. Then the Curve Selection Lemma yields that for $\varepsilon > 0$ small enough, one can lift this to a C^∞ vector field ξ in $(X - V) \cap \mathbb{B}_\varepsilon$, where $\mathbb{B}_\varepsilon$ is the ball bounded by the Milnor sphere $\mathbb{S}_\varepsilon$, such that:

1. ξ is everywhere transversal to the intersection of X with every sphere in $\mathbb{C}^N$ centered at $\underline{0}$ and contained in $\mathbb{B}_\varepsilon$; and
2. ξ is everywhere transversal to every tube $f^{-1}(C_\delta)$ contained in $X \cap \mathbb{B}_\varepsilon$, where C_δ is the circle in $\mathbb{C}$ centered at 0 and with radius δ.

The existence of the above vector field ξ is an immediate application of Milnor's Corollary 3.4 in [65], stating that if we have two real analytic non-negative functions which vanish at a given point x_o, then their differentials cannot point in exactly opposite directions at any point in a neighborhood of x_o (see Lemma 6.2.20).

Once we have such a vector field, consider a Milnor sphere $\mathbb{S}_{\varepsilon'}$ for some $\varepsilon > \varepsilon' > 0$ and consider the locally trivial fibration in Theorem 6.4.1:

$$X \cap \mathbb{B}_{\varepsilon'} \cap f^{-1}(\mathbb{D}^*_\delta) \to \mathbb{D}^*_\delta\,.$$

where we assume that $\delta > 0$ is small enough also with respect to ε'. Restrict this fibration to the tube $N_f(\varepsilon', \delta) := X \cap \mathbb{B}_{\varepsilon'} \cap f^{-1}(\partial(\mathbb{D}^*_\delta))$.

Now, for each $x \in N_f(\varepsilon', \delta)$, take the integral line $\gamma_\xi(x)$ of ξ passing through x and follow it until γ_ξ hits the sphere $\mathbb{S}_{\varepsilon'}$. This gives a diffeomorphism

$$h_\xi : N_f(\varepsilon', \delta) \longrightarrow X \cap (\mathbb{S}_{\varepsilon'} - f^{-1}(\mathring{\mathbb{D}}_\delta))$$

which is the identity over $N_f(\varepsilon', \delta) \cap \mathbb{S}_{\varepsilon'}$. We may now define a projection:

$$\psi : L_X - L_V \longrightarrow \mathbb{S}^1$$

by $\psi = f/|f|$ in $L_X \cap f^{-1}(\mathbb{D}^*_\delta)$ and by $\psi = \pi \circ f \circ h_\xi^{-1}$ elsewhere, where π is the radial projection $\mathbb{D}^*_\delta \to \mathbb{S}^1$ defined by $x \mapsto x/|x|$. This gives a fiber bundle

$$\psi : L_X - L_V \longrightarrow \mathbb{S}^1,$$

and by construction this is isomorphic to the bundle in Theorem 6.4.1.

So far, the discussion actually holds in the real analytic category, *i.e.* for X real analytic and f a real analytic map into $\mathbb{R}^2$. We refer to [9, 71] for a thorough discussion of that setting.

Yet, we are still missing something important to reach Theorem 6.5.1. So far, we can assure that ψ can be taken to be $\phi := f/|f|$ only near L_V. Away from a neighborhood of L_V we have no control of the projection ψ, which depends upon the choice of the vector field ξ. To finish the proof of Theorem 6.5.1 we need to show that there exists a vector field ζ which has the same properties (1) and (2) as ξ, and satisfies also a third property:

(3) For each integral line $\gamma(t)$ of ζ in $(X - V) \cap \mathbb{B}_\varepsilon$, one has that the argument of the complex number $f(x)$ is constant for all $x \in \gamma(t)$. That is, if $x_1, x_2 \in (X - V) \cap \mathbb{B}_\varepsilon$ are contained in the same integral curve of ζ, then:

$$\frac{f(x_1)}{|f(x_1)|} = \frac{f(x_2)}{|f(x_2)|}\,.$$

This last property is always satisfied for holomorphic mappings $X \to \mathbb{C}$, but it may not be so for real analytic functions. That is the d-regularity condition discussed in [9, 71]. The proof that it is satisfied for holomorphic functions uses elementary complex linear algebra, as in the proof of [65, Lemma 4.6]. □

Exercise 6.5.2 Prove the existence of the vector field ζ satisfying the condition (3) using the proof of Lemma 4.6 of [65].

6.6 Milnor Open-Books and the Milnor Number

Let us consider again an equidimensional complex analytic germ $(X, \underline{0})$ in $\mathbb{C}^N$ of dimension n. Now we assume $X - \{\underline{0}\}$ is non-singular, and consider a holomorphic map $f : (X \cap \mathring{\mathbb{B}}_r, \underline{0}) \to (\mathbb{C}, 0)$, where $\mathring{\mathbb{B}}_r$ is a sufficiently small open ball in $\mathbb{C}^N$ centered at the origin $\underline{0}$, with a unique singular point at $\underline{0} \in X$. We set $V = f^{-1}(0)$ and let $L_X = X \cap \mathbb{S}_\varepsilon$ and $L_V = V \cap \mathbb{S}_\varepsilon$ be the links of X and V. Then we know from the previous section that the spherical Milnor fibration (Theorem 4.8 of [65]) extends to this setting and we have a smooth locally trivial fibration:

$$\phi := \frac{f}{|f|} : L_X - L_V \longrightarrow \mathbb{S}^1, \tag{6.2}$$

This fibration gives rise to what is known today as the Milnor open-book of the map-germ f.

Recall that the concept of *open-books* was introduced by E. Winkelnkemper and we refer to his appendix in [70] for a clear account of the subject. An open-book decomposition of a smooth n-manifold M consists of a codimension 2 submanifold N, called *the binding*, embedded in M with trivial normal bundle, together with a fiber bundle decomposition of its complement:

$$\theta : M - N \to \mathbb{S}^1\,,$$

satisfying that on a tubular neighborhood of N, diffeomorphic to $N \times \mathbb{D}^2$, the restriction of θ to the punctured tubular neighborhood $N \times (\mathbb{D}^2 - \{0\})$ is the map $(x, y) \mapsto y/\|y\|$. The fibers of θ are called *the pages* of the open-book. These are all diffeomorphic and each page F can be compactified by attaching the binding N as its boundary, thus getting a compact manifold with boundary.

Milnor's fibration theorem grants that in the isolated singularity case, we get open-book decompositions where the manifold M is L_X, the binding N is the link L_V, the pages of the open-book are the Milnor fibers and the projection map is $\phi := \frac{f}{|f|}$.

We now turn our attention to the topology of the fibers. One has the following theorem due to H. Hamm in [20], which generalizes Milnor's Theorem 6.5 in [65]:

Theorem 6.6.1 *Let the isolated singularity germ* $(X, \underline{0})$ *be a complete intersection and consider the locally trivial fibration*

$$\phi := \frac{f}{|f|} : L_X - L_V \longrightarrow \mathbb{S}^1\,.$$

Then the fibers have the homotopy type of a bouquet of spheres of middle dimension n.

Definition 6.6.2 The number $\mu(f)$ of spheres in this bouquet is known as *the Milnor number* of the singularity.

Theorem 6.6.3 *If* $X = \mathbb{C}^{n+1}$ *and* f *has an isolated critical point at* $\underline{0}$*, then* $\mu(f)$ *equals the intersection number*

$$\mu(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+1,\underline{0}}}{\mathrm{Jac}(f)}\,, \tag{6.3}$$

where Jac(f) *is the Jacobian ideal, generated by the partial derivatives of* f *in* $\mathcal{O}_{n+1,\underline{0}}$.

A consequence of this formula is that the Milnor number is in fact an invariant of $(V, \underline{0})$, where $V = f^{-1}(0)$. That is, it is independent of the choice of the holomorphic function f which defines $(V, \underline{0})$:

Lemma 6.6.4 *Let* $f, g\colon (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ *be holomorphic functions with isolated singularity such that* $f^{-1}(0) = g^{-1}(0)$*. Then* $\mu(f) = \mu(g)$*.*

Proof Since f, g have isolated singularity, they are both reduced and $f^{-1}(0) = g^{-1}(0)$ implies that $g = uf$ for some unit u in $\mathcal{O}_{n+1,\underline{0}}$, by the analytic Nullstellensatz. We have $u(\underline{0}) \neq 0$ and hence we can assume, without loss of generality, that $u(\underline{0}) = 1$. Let $F : (\mathbb{C}^{n+1}, \underline{0}) \times \mathbb{C} \to (\mathbb{C}, 0)$ be the family of function germs given by

$$F(x,t) := F_t(x) := (1 + t(u(x) - 1))f(x).$$

For each $t_0 \in \mathbb{C}$, there exist open neighbourhoods U of $\underline{0}$ in $\mathbb{C}^{n+1}$ and T of t_0 in $\mathbb{C}$ such that $\underline{0}$ is the only singular point of F_t in U, for all $t \in T$. By the principle of conservation of the intersection number (see for instance [28, 6.4.5]),

$$\mu(F_t) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+1,\underline{0}}}{\mathrm{Jac}(F_t)} = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+1,\underline{0}}}{\mathrm{Jac}(F_{t_0})} = \mu(F_{t_0}),$$

for all $t \in T$. But $\mathbb{C}$ is connected and thus, $\mu(F_t)$ is not only locally constant, but globally constant on $t \in \mathbb{C}$. In particular, $\mu(f) = \mu(F_0) = \mu(F_1) = \mu(g)$. □

Definition 6.6.5 Let $(V, \underline{0})$ be a hypersurface with isolated singularity in $(\mathbb{C}^{n+1}, \underline{0})$. The *Milnor number* of $(V, \underline{0})$ is defined as $\mu(V, \underline{0}) := \mu(f)$, where $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ is any reduced holomorphic function such that $f^{-1}(0) = V$.

Remark 6.6.6 Recall that a knot consists of a pair (M, N) of connected, oriented manifolds, where N is a codimension 2 submanifold of M; if N is not connected, then it is called a link in M. Notice that if we are given a holomorphic map-germ

$f\colon (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ with an isolated critical point at $\underline{0}$, and $L_f := f^{-1}(0) \cap \mathbb{S}_\varepsilon$ is the link of f, then the pair $(\mathbb{S}_\varepsilon, L_f)$ is a knot when $n \geq 2$ because of Theorem 6.4.8 (or a link when $n = 1$ and f has several branches). These are called *algebraic knots (or links)*, a concept introduced in [33]. We remark that Lemma 6.6.4 is obviously false in general if we relax the condition $f^{-1}(0) = g^{-1}(0)$ asking just for a homeomorphism $L_f \cong L_g$. Yet, B. Teissier proved in [74] that if the equivalence is as knots, *i.e.*, the knots $(\mathbb{S}_\varepsilon, L_f)$ and $(\mathbb{S}_\varepsilon, L_g)$ are homeomorphic, then f and g have the same Milnor number. More generally if the functions f and g have the same topology at $\underline{0}$, in the sense that there are open sets U and V of $\underline{0}$ and a homeomorphism of U onto V such that $f^{-1}(0) \cap U$ is homeomorphic to $g^{-1}(0) \cap V$, the Milnor fibrations of f and g at $\underline{0}$ are isomorphic. In particular if the critical point of f and g at $\underline{0}$ is also isolated, then their Milnor number at $\underline{0}$ is equal (see [32, Proposition in the introduction]).

Notice that if V has an isolated complete intersection singularity (an ICIS) at $\underline{0}$, say defined by a germ,

$$g := (g_1, \cdots, g_k) : (\mathbb{C}^{n+k}, \underline{0}) \to (\mathbb{C}^k, 0) ,$$

then H. Hamm in [20] observed that one has a fibration in the vein of Theorem 6.4.7, but now this is over the complement of the discriminant of g. In fact one has the following lemma by Greuel (see [18] or [54, Lemma 5.9]).

Lemma 6.6.7 *Let $g = (g_1, \ldots, g_k)\colon (\mathbb{C}^{n+k}, \underline{0}) \to (\mathbb{C}^k, 0)$ be a holomorphic map germ which defines an ICIS $(V, 0)$ of dimension n. Then after a linear change of coordinates in $\mathbb{C}^k$, we can assume that $g' = (g_1, \ldots, g_{k-1})\colon (\mathbb{C}^{n+k}, \underline{0}) \to (\mathbb{C}^{k-1}, 0)$ defines an ICIS $(V', 0)$ of dimension $n+1$, and the zero set of g_k exhibits V as a hypersurface in V' with an isolated singularity $\underline{0} \in V'$.*

Hence the above fibration for ICIS is a special case of Theorem 6.4.7. The remarkable point proved by Hamm is Theorem 6.6.1, concerning the topology of the fibers. Recall that the number of such spheres is called the Milnor number of the ICIS.

One has the Lê-Greuel formula for the Milnor number of an ICIS, which generalizes Theorem 6.6.3:

Theorem 6.6.8 *If $g_1, \cdots, g_k$ and f are holomorphic map germs $(\mathbb{C}^{n+k+1}, \underline{0}) \to (\mathbb{C}, 0)$ such that $h = (g_1, \cdots, g_k)$ and (h, f) define isolated complete intersection germs, then their Milnor numbers are related by:*

$$\mu(h) + \mu(h, f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+k,\underline{0}}}{(h, \mathrm{Jac}_{k+1}(h, f))} , \tag{6.4}$$

where $\mathrm{Jac}_{k+1}(h, f)$ denotes the ideal generated by the determinants of all $(k+1)$ minors of the Jacobian matrix whose rows are the partial derivatives of $g_1, \ldots, g_k, f$.

This formula was proved by Lê [35] and G.-M. Greuel [18]. At about the same time Teissier proved [74, Proposition II.1.2] a *"formule de restriction"*, which is the same theorem in the case where one of the two functions is linear; this is known as "Teissier's Lemma".

Example 6.6.9 Let $g = (g_1, g_2)\colon (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)$ be the map germ given by:

$$g_1(x, y, z) = x^2 + y^2 \text{ and } g_2(x, y, z) = z^2 + xy.$$

Then $V = g^{-1}(0)$ is a 1-dimensional ICIS, but g_1 has not isolated singularity. The coordinate change $(y_1, y_2) \mapsto (y_1 + y_2, y_2)$ transforms g_1 into $g_1 + g_2$ which has now isolated singularity. This illustrates how a linear change of coordinates yields to a set of good representatives of an ICIS.

The fact that the Milnor number of an ICIS $(V, \underline{0})$ is also independent of the choice of the equations is not so easy to prove as in the hypersurface case. In order to prove it, we need another construction of the fibration similar to the Milnor fibration of Proposition 6.2.9. We follow here the formulation of Looijenga's book [54].

Consider a complex analytic space X of pure dimension $n+k$ and a holomorphic map germ $f := (f_1, \dots, f_k)\colon (X, \underline{0}) \to (\mathbb{C}^k, \underline{0})$ and let $V = f^{-1}(\underline{0})$. We assume that f defines an *isolated singularity* in the sense that at each point of $V - \{\underline{0}\}$ sufficiently near to $\underline{0}$ the space X is not singular and the mapping f is a submersion. We choose $\epsilon > 0$ a Milnor radius for V (see Lemma 6.2.3), which means that V is transverse to $\mathbb{S}_{\epsilon'}$ for all $0 < \epsilon' \leq \epsilon$. This implies that $V \cap \mathbb{B}_\varepsilon$ is homeomorphic to the cone over its boundary $V \cap \mathbb{S}_\varepsilon$. Since $V \cap \mathbb{S}_\varepsilon$ is compact, there is a contractible open neighborhood U of $\underline{0}$ in $\mathbb{C}^k$ such that $f|_{X \cap \mathbb{S}_\varepsilon}$ is a submersion along $f^{-1}(U) \cap \mathbb{S}_\varepsilon$. We denote:

$$\overline{X} := f^{-1}(U) \cap \mathbb{B}_\varepsilon, \quad X := f^{-1}(U) \cap \mathring{\mathbb{B}}_\varepsilon, \quad \partial X := f^{-1}(U) \cap \mathbb{S}_\varepsilon,$$

When ε and U are small enough, we call the restriction $f\colon X \to U$ (resp. $f\colon \overline{X} \to U$) a *good representative* of f (resp. a *proper good representative* of f).

When $k = 1$ and $U = \mathring{\mathbb{D}}_\delta$ we obtain the Milnor tubes of Proposition 6.2.9; we study this case in detail in Sect. 6.11. We set the following additional notation:

- X_{sing} is the singular locus of X and $X_{reg} = X - X_{sing}$;
- C is the set of points of X which are in X_{sing} or f is not a submersion.
- For each $A \subset U$, $\overline{X}_A := f^{-1}(A) \cap \overline{X}$ and $X_A := f^{-1}(A) \cap X$.

Theorem 6.6.10 *With the above notations, when $\varepsilon > 0$ and U are sufficiently small we have:*

1. *$f\colon \overline{X} \to U$ is proper and $f\colon \partial X \to U$ is a C^∞-trivial fibration.*
2. *C is analytic in X and closed in $\overline{X}$. Moreover, $f|_C$ is finite (i.e., proper with finite fibres).*
3. *$C - X_{sing}$ is of pure dimension $k - 1$.*

4. *the image D of C by f is a hypersurface in U if $C - X_{sing}$ is dense in C. In this case we call D the discriminant of f.*
5. *Let us suppose that $C - X_{sing}$ is dense in C, the mapping:*

$$f : (\overline{X}_{U-D}, \partial\overline{X}_{U-D}) \to U - D$$

is a C^{∞}-trivial fibration of which each fibre pair $(\overline{X}_s, \partial X_s)$ is a complex analytic manifold with boundary.
6. *f defines an ICIS at every point of $\overline{X}_{reg}$.*
7. *For $k = 1$, if the point $\underline{0}$ is isolated in X_{sing}, the fibration of item (5) is isomorphic to the fibration of Theorem 6.6.1.*
8. *If $X = \mathbb{C}^{n+k}$, then every fibre $\overline{X}_s$ has the homotopy type of a bouquet of n-spheres.*

Proof The reader can look at [54, Theorem 2.8] for items (1) to (6), The statement (7) can be found in [20, Theorem 2.6] and (8) is from [54, Corollary 5.10]. For the reader's convenience we give here a short and sketchy proof of Theorem 6.6.10.

The condition (1) is the consequence of the hypotheses by applying Ehresmann Lemma (Lemma 6.2.10), if $\varepsilon > 0$ and U are small enough, since we have assumed $\underline{0}$ to be an isolated singular point of $V = f^{-1}(0)$.

Since $V = f^{-1}(0)$ has an isolated singularity at $\underline{0}$, the intersection $f^{-1}(0) \cap C$ is the point $\underline{0}$, if $\varepsilon > 0$ is sufficiently small. Then, the restriction of f to $(C, \underline{0})$ is finite, *i.e.* a representative of f with ε and U small enough is proper with finite fibers, by a geometric version of Weierstrass Theorem (see [27]). A Theorem of Remmert shows that, since the restriction of f to C is finite, the image $D = f(C)$ is analytic and it is called the *discriminant* of f. This shows (2) and the fact that D is analytic.

Since the restriction of f to C is finite, the dimensions of C and D are equal. In particular $\dim C \leq k$. Now let $z \in C$ a point of X which is not singular. In a small open neighborhood U_z of z in X the image of $U_z \cap C$ cannot be of dimension k because of Sard's Theorem, Therefore $\dim_z C \leq k-1$. To prove that $\dim_z C \geq k-1$, choose U_z small enough so that we have local coordinates to define f and consider the Jacobian matrix defined on U_z. It gives a map:

$$\Phi_z : U_z \to Hom(\mathbb{C}^{n+k}, \mathbb{C}^k).$$

In U_z the analytic subset C is the inverse image of the subset C of matrices of rank $< k$. We have a general result (see e.g. [26, Theorem 1]):

Lemma 6.6.11 *The matrices of rank $\leq r$ in $Hom(\mathbb{C}^N, \mathbb{C}^k)$ are an algebraic subvariety of codimension $(N - r)(k - r)$.*

Applying this lemma to our situation we find that the space of matrices of rank $< k$ in $Hom(\mathbb{C}^{n+k}, \mathbb{C}^k)$ is a subvariety of codimension $n+1$, therefore the codimension of C at z is $\leq n-1$ which gives $\dim_z C \geq k-1$, so:

$$\dim_z C = k - 1.$$

Moreover, at a point z of X_{reg}, C is defined by the minors of order k of a matrix of size $(n+k) \times k$. So X is determinantal and hence Cohen-Macaulay at z (again by [26]). In particular, C has pure dimension at z. This proves (3).

To prove (4) it is enough to prove that D has pure dimension $k-1$. By (3) we have obtained that $C - X_{sing}$ has pure dimension $k-1$. If $C - X_{sing}$ is dense in C, the space C is also of pure dimension $k-1$. Since D is the image of C by a finite map, it has also pure dimension $k-1$, so D is a hypersurface.

Assertion (5) is a straightforward application of Ehresmann Lemma (Lemma 6.2.10).

If z is a point of $C - X_{sing}$ the germ of (X, z) is non-singular so up to a local choice of coordinates that germ is isomorphic to $(\mathbb{C}^{n+k}, 0)$ and the germ of f at z is isomorphic to a germ $(\mathbb{C}^{n+k}, z) \to (\mathbb{C}, f(z))$. Therefore the fiber $f^{-1}(f(z))$ has dimension n and the singularity z is isolated, since f restricted to $C \cap U_z$ is finite and $f(z)$ is a point of $f(C \cap U_z)$, so the point $z \in C \cap U_z$ is locally isolated in $f^{-1}(f(z) \cap C) \cap U_z$ which means that z is an isolated singularity of the complete intersection $f^{-1}(f(z)) \cap U_z$. This answers to (6).

As stated above, we refer to [54, Corollary 5.10] for a proof of (8). □

Proposition 6.6.12 *Assume that* $f, g \colon (\mathbb{C}^{n+k}, \underline{0}) \to (\mathbb{C}^k, \underline{0})$ *define the same ICIS of dimension* n, $V = f^{-1}(\underline{0}) = g^{-1}(\underline{0})$ *(as complex spaces). Then the Milnor fibres of* f *and* g *are diffeomorphic.*

Proof Since $f^{-1}(\underline{0}) = g^{-1}(\underline{0})$ as complex spaces, the ideals generated by the components of f and g are the same, so $g_i = \sum_{j=1}^{k} a_{ij} f_j$, for some $k \times k$-matrix $A = (a_{ij})$ with entries in $\mathcal{O}_{n+k,\underline{0}}$ such that $\det A(\underline{0}) \neq 0$. After a linear change of coordinates in $\mathbb{C}^k$, we can assume that $A(\underline{0}) = I$, the identity matrix. Let $F \colon (\mathbb{C}^{n+k}, \underline{0}) \times \mathbb{C} \to (\mathbb{C}^k, \underline{0}) \times \mathbb{C}$ be the map germ given by $F(x,t) := (f_t(x), t)$ where

$$f_t := (I + t(A - I))f.$$

Let $f : \overline{X} \to U$ be a proper good representative of f, where $\overline{X} = f^{-1}(U) \cap \mathbb{B}_\epsilon$. For each $t_0 \in \mathbb{C}$, there exists a contractible open neighbourhood T of t_0 in $\mathbb{C}$ such that for each $t \in T$, with $\overline{X}_t = f_t^{-1}(U) \cap \mathbb{B}_\epsilon$, $f_t : \overline{X}_t \to U$ is also a proper good representative of the germ f_t. Consider now $F : \overline{\mathcal{X}} \to U \times T$, where $\overline{\mathcal{X}} := F^{-1}(U \times T) \cap (\mathbb{B}_\varepsilon \times T)$. By using the arguments of Theorem 6.6.10, it follows that

$$F_{|U \times T - D_F} : \overline{\mathcal{X}}_{U \times T - D_F} \to U \times T - D_F$$

is a C^∞-trivial fibration, where D_F is the discriminant of F. If $s \in U - D_{f_t}$, then $(s,t) \in U \times T - D_F$ and $(\overline{X}_t)_s \times \{t\} = \overline{\mathcal{X}}_{(s,t)}$ where $(\overline{X}_t)_s$ is the fiber of f_t over s.

It follows that all the germs f_t with $t \in T$, have diffeomorphic Milnor fibres $(\overline{X}_t)_s$. Since $\mathbb{C}$ is connected, all the germs f_t, with $t \in \mathbb{C}$, have also diffeomorphic Milnor fibres. In particular, $f = f_0$ and $g = f_1$ have diffeomorphic Milnor fibres. □

Definition 6.6.13 Let $f\colon (\mathbb{C}^{n+k}, \underline{0}) \to (\mathbb{C}^k, \underline{0})$ be a map germ which defines an ICIS $V = f^{-1}(0)$ of dimension n. The *Milnor number* of $(V, \underline{0})$ is defined as $\mu(V, \underline{0}) := \mu(f)$.

6.7 Polar Curves and Attaching Handles

The study of the topology of complex analytic functions using polar curves relative to a linear form ℓ springs from ideas of René Thom [77] and was introduced by B. Teissier in [74] and D. T. Lê in [32]. This was extended in [36, 37], to germs of holomorphic functions $f : (X, x) \to (\mathbb{C}, 0)$ relatively to a Whitney stratification $S = \{S_\alpha\}_{\alpha \in A}$ of a reduced equidimensional complex analytic space X. The theory of "polar varieties" was developed more generally by Bernard Teissier and Lê Dũng Tráng in the 1970s in various papers, see for instance [32, 37, 39, 74, 75].

Let $f\colon (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ be a holomorphic function with isolated singularity. For each $a \in \mathbb{C}^{n+1}$ we consider the germ of map $\Phi_a := (f, \ell_a)\colon (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}^2, \underline{0})$, where $\ell_a(z) = \sum_{i=0}^{n} z_i a_i$. We denote by $(\Gamma_a, \underline{0})$ the critical locus of Φ_a, that is, the germ at $\underline{0}$ of the closure of the set of points in $\mathbb{C}^{n+1}$ where Φ_a is not submersive, with the analytic structure given by the 2×2-minors of its Jacobian matrix.

Theorem 6.7.1 ([21]) *There exists a non-empty Zariski open subset $W \subset \mathbb{C}^{n+1}$ such that for any $a \in W$, the map Φ_a satisfies the following conditions:*

1. *Γ_a is a reduced curve,*
2. *The restriction $\Phi_a : (\Gamma_a, \underline{0}) \to (\mathbb{C}^2, 0)$ is one-to-one.*
3. *There exists a neighbourhood U of $\underline{0}$ in $\mathbb{C}^{n+1}$ and a representative of Φ_a defined on U such that for all $p \in U \cap \Gamma_a - \{\underline{0}\}$, Φ_a has an ordinary quadratic singularity at p. That is, the germ of Φ_a at p is equivalent by coordinate changes in the source and target to the Whitney fold singularity $(\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}^2, \underline{0})$ given by $z \mapsto (z_1^2 + \cdots + z_n^2, z_0)$.*

Proof Let us prove the first point (1). Consider the map:

$$F : (\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}, \underline{0}) \to \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$$

given by $F(z, a) = (\nabla f(z), a)$ and for each $a \in \mathbb{C}^{n+1}$, $F_a(z) = F(z, a)$. The differential of F at (z, a) is

$$dF_{(z,a)} = \begin{pmatrix} Hf(z) & 0 \\ 0 & I_{n+1} \end{pmatrix},$$

where $Hf(z)$ is the Hessian matrix of f at z.

A critical point z of Φ_a outside $\underline{0}$ is a point where $\nabla f(z)$ and a are collinear, *i.e.* $\nabla f(z) \wedge a = 0$. It follows that $\Gamma_a = F_a^{-1}(\Sigma)$, where Σ is the closure of the subset of $\mathbb{C}^{*n+1} \times \mathbb{C}^{*n+1}$ of pairs (b, a) such that $b \wedge a = 0$. As before $\mathbb{C}^* := \mathbb{C} - \{0\}$.

It is known that Σ is reduced of codimension n, determinantal (so, Cohen-Macaulay [26]) and its singular subset $\mathrm{Sing}(\Sigma)$ is $\{(\underline{0}, \underline{0})\}$.

At a pair $(b, a) \in \Sigma$ with $b \neq 0$, the algebraic set Σ is parametrized by the map $K : (\mathbb{C}^{n+1} - \{0\}) \times \mathbb{C} \to \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ given by $K(b, t) = (b, tb)$, which has differential:

$$dK_{(b,t)} = \begin{pmatrix} I_{n+1} & 0 \\ tI_{n+1} & b \end{pmatrix}.$$

Thus, the tangent space to $\Sigma^* = \Sigma - \{0\}$ at (b, tb) is $T_{(b,tb)}\Sigma^* = \mathrm{Im}\, dK_{(b,t)}$. Moreover, if $b = \nabla f(z) \neq 0$ and $a = tb$ then:

$$\mathrm{Im}\, dF_{(z,a)} + T_{(b,tb)}\Sigma^* = \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}.$$

Let U be a neighbourhood of $\underline{0}$ in $\mathbb{C}^{n+1}$ such that $\nabla f(z) \neq 0$, for all $z \in U^* = U - \{0\}$. We deduce that F is transverse to Σ^* in $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ at any point of $U^* \times \mathbb{C}^{n+1}$. By the Basic Transversality Lemma [16, Lemma 4.3 p. 53] there exists a non-empty Zariski open subset $W_1 \subset \mathbb{C}^{n+1}$ such that for any $a \in W_1$, F_a is transverse to Σ^* on U^*. In particular, $\Gamma_a \cap U^*$ is smooth of dimension 1. By adding the origin, the germ of Γ_a in $(\mathbb{C}^{n+1}, \underline{0})$ has also dimension 1 and is reduced.

(2) For the second point we proceed in a similar way as above. We consider:

$$G : (\mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}, \underline{0}) \to \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \times \mathbb{C}^4$$

given by:

$$G(z, z', a) = (\nabla f(z), \nabla f(z'), a, f(z), f(z'), \ell_a(z), \ell_a(z')),$$

and the map $G_a(z, z') = G(z, z', a)$. The differential of G at (z, z', a) is

$$dG_{(z,z',a)} = \begin{pmatrix} Hf(z) & 0 & 0 \\ 0 & Hf(z') & 0 \\ 0 & 0 & I_{n+1} \\ \nabla f(z) & 0 & 0 \\ 0 & \nabla f(z') & 0 \\ a & 0 & z \\ 0 & a & z' \end{pmatrix}.$$

We consider the set $G_a^{-1}(\Sigma')$ where now Σ' is the closure in $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \times \mathbb{C}^4$ of the subset of $(\mathbb{C}^{n+1})^* \times (\mathbb{C}^{n+1})^* \times \mathbb{C}^{n+1} \times (\mathbb{C}^4)^*$ of points

(b, c, a, s) such that

$$b \wedge a = 0, \quad c \wedge a = 0, \quad s_1 = s_2, \quad s_3 = s_4.$$

Again Σ' is Cohen-Macaulay of codimension $2(n+1)$ and $\mathrm{Sing}(\Sigma')$ is the subset of points where $a = b = c = 0$. At a point where $a, b \neq 0$, Σ' is parametrized by the map $L : \mathbb{C}^{n+1} \times \mathbb{C}^4 \to \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \times \mathbb{C}^4$ given by

$$L(b, t, u, s_1, s_3) = (b, tb, ub, s_1, s_1, s_3, s_3),$$

whose differential is

$$dL_{(b,t,u,s_1,s_3)} = \begin{pmatrix} I_{n+1} & 0\,0\,0\,0 \\ tI_{n+1} & b\,0\,0\,0 \\ uI_{n+1} & 0\,b\,0\,0 \\ 0 & 0\,0\,1\,0 \\ 0 & 0\,0\,1\,0 \\ 0 & 0\,0\,0\,1 \\ 0 & 0\,0\,0\,1 \end{pmatrix}.$$

We observe that if $G(z, z', a) = L(b, t, u, s_1, s_3)$, with $b \neq 0$ and $z \neq z'$, then:

$$\mathrm{Im}\, dG_{(z,z',a)} + \mathrm{Im}\, dL_{(b,t,u,s_1,s_3)} = \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \times \mathbb{C}^4.$$

This can be checked for each factor separately. It is obvious for the first three factors and for the last one, we can extract a non-zero 4 × 4-minor as follows:

$$\begin{vmatrix} b_i & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ tb_i & z_j & 0 & 1 \\ 0 & z'_j & 0 & 1 \end{vmatrix} = b_i(z'_j - z_j) \neq 0,$$

if $b_i \neq 0$ and $z_j \neq z'_j$. We deduce that G is transverse to Σ' on $(U^*)^{(2)} \times (\mathbb{C}^{n+1})^*$, where $U^* := U - \{0\}$, $(\mathbb{C}^{n+1})^* := \mathbb{C}^{n+1} - \{0\}$, $(U^*)^{(2)}$ is the set of pairs $(z, z') \in U^* \times U^*$ such that $z \neq z'$.

Again by the Basic Transversality Lemma [16, Lemma 4.3 p. 53], there exists a non-empty Zariski open subset $W_2 \subset \mathbb{C}^{n+1}$ such that for any $a \in W_2$, G_a is transverse to Σ' on $(U^*)^{(2)}$. The set $G_a^{-1}(\Sigma')$ has codimension $2(n+1)$ in $(U^*)^{(2)}$ and hence, it has dimension 0. After shrinking the neighbourhood U if necessary, the restriction $\Phi_a : \Gamma_a \cap U \to \mathbb{C}^2$ is one-to-one.

(3) Take $a \in W := W_1 \cap W_2$ and $p \in \Gamma_a \cap U^*$. By items (1) and (2), Γ_a is smooth at $p \neq \underline{0}$ and the restriction $\Phi_a : (\Gamma_a, p) \to \mathbb{C}^2$ is an embedding. By taking translations, we can assume that $p = \underline{0}$ and $\Phi_a(p) = (0, 0)$.
Moreover, we also choose the coordinates in $\mathbb{C}^{n+1}$ in an open neighborhood of $\underline{0}$ in such a way that $\Phi_a(z) = (f(z), z_0)$ and Γ_a is the axis z_0. By definition, Γ_a is defined by the vanishing of $\partial f/\partial z_1, \ldots, \partial f/\partial z_n$ and hence,

$$\frac{\partial f}{\partial z_i} = \sum_{j=1}^{n} \alpha_{ij} z_j,$$

for some functions $\alpha_{ij} \in \mathcal{O}_{n+1,\underline{0}}$ such that $\det(\alpha_{ij}) \neq 0$, with $1 \leq i, j \leq n$. In particular, we must have $\det\left(\partial f^2/\partial z_i \partial z_j\right) \neq 0$, with $1 \leq i, j \leq n$ at any point $(z_0, 0, \ldots, 0)$. By the Splitting Lemma [15, (4.3) p. 125] after a coordinate change in $(\mathbb{C}^{n+1}, \underline{0})$ we may assume that $f(z) = z_1^2 + \cdots + z_n^2 + g(z_0)$, for some function $g : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$. Finally, to arrive to the Whitney fold singularity, just take the coordinate change in $(\mathbb{C}^2, \underline{0})$ given by $(s, t) \mapsto (s - g(t), t)$. □

Remark 6.7.2 If $a \in W$, the restriction of Φ_a to $(\Gamma_a, \underline{0})$ is finite, because in a neighbourhood $\mathcal{U}$ of $\underline{0}$ in $\mathbb{C}^{n+1}$ we have:

$$\Phi_a^{-1}(0, 0) \cap \Gamma_a \cap \mathcal{U} = \{\underline{0}\}.$$

The geometric version of Weierstrass preparation theorem given in [27] implies the local finiteness of Φ_a at $\underline{0}$. Therefore there exist neighborhoods U of $\underline{0}$ in $\mathbb{C}^{n+1}$ and V of $(0, 0)$ in $\mathbb{C}^2$, such that Φ_a induce a finite morphism of $U \cap \Gamma_a$ into V whose image is a curve Δ

In fact, there is a more general theorem, than Theorem 6.7.1. We do not assume any longer that the singularity of f at $\underline{0}$ is isolated:

Theorem 6.7.3 *Let $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ be a non-constant holomorphic function. There is an open dense set of linear forms Ω, such that, for any $\ell \in \Omega$, in an open neighborhood U_ℓ of $\underline{0}$, the critical locus Γ^* of the restriction of (f, ℓ) to $U_\ell - f^{-1}(0)$ has dimension 1 or is empty for $\ell \in \Omega$. If Γ^* has dimension 1, the closure Γ_ℓ of Γ^* in U_ℓ is a reduced curve and the restriction of Φ_ℓ to $\Gamma_\ell \cap U_\ell$ is injective. Furthermore, for any point $z \in \Gamma_\ell - \{\underline{0}\}$ the germ of Φ_ℓ is equivalent up to translation to the germ $(\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ given by $(z_0, z_1, \ldots, z_n) \mapsto (z_1^2 + \ldots + z_n^2, z_0)$.*

One can prove that, for $\ell \in \Omega$, the restriction of Φ_ℓ to $\Gamma_\ell \cap U_\ell$ is finite. So, when the germ $(\Gamma_\ell, \underline{0})$ is not empty one can define its image $(\Delta_\ell, (0, 0))$ by Φ_ℓ.

We are not giving a proof of this Theorem 6.7.3 (see [48, paragraph 3]). Just notice that Γ_ℓ can be empty, for instance when $V = f^{-1}(0)$ is a product of an isolated singularity by a line $\mathbb{C}$.

Definition 6.7.4 A linear form $\ell_a \in \Omega$ as in Theorem 6.7.3 is called *generic* for f. The curve Γ_ℓ is called the *polar curve* of f with respect to the direction defined by ℓ. The image $(\Delta_\ell, (0, 0))$ of $(\Gamma_\ell, \underline{0})$ by Φ_ℓ is also called the *polar discriminant* or *Cerf diagram* of f with respect to ℓ.

In general we have:

Proposition 6.7.5 *Let ℓ be a generic linear form for f. If f has a critical point at $\underline{0}$, then, if Δ_ℓ is not empty, the tangent cone of Δ_ℓ at the origin is $\{0\} \times \mathbb{C}$.*

Proof We may assume that $\ell = z_0$, then the polar curve with respect to the direction z_0 can be seen in the following way. Consider the algebraic set defined by:

$$\partial f/\partial z_1 = \ldots = \partial f/\partial z_n = 0.$$

It obviously contains the singular locus of $V = f^{-1}(0)$ and when z_0 is generic it also contains Γ_{z_0}, if it is not empty. We assume Γ_{z_0} is not empty. Let $p : \overset{\circ}{\mathbb{D}}_\delta \to \Gamma_{z_0}$ be a parametrization of Γ_{z_0}. We have to compare $f(p(t))$ and $z_0(p(t))$ when t tends to 0:

$$\frac{f(p(t))}{z_0(p(t))}$$

By de l'Hôpital rule to obtain the limit, it is enough to compare the derivatives in t of the numerator and of the denominator:

$$\frac{df(p(t))}{dt} = (\partial f/\partial z_0)(p(t))\frac{dp_0(t)}{dt} + \ldots + (\partial f/\partial z_n)(p(t))\frac{dp_n(t)}{dt}$$

where $p_0(t), p_1(t), \ldots, p_n(t)$ are the components of $p(t)$. The quotient:

$$\frac{df(p(t))/dt}{dz_0(p(t))/dt} = (\partial f/\partial z_0)(p(t))$$

because the polar curve lies in the algebraic set with the equation above. If f is singular at $\underline{0}$, the partial derivative $(\partial f/\partial z_0)(p(t))$ tends to 0 when t tends to 0. $\square$

In the situation that we consider in the first paragraph, we have a complex analytic space with an isolated singularity at the point $\underline{0} \in X$ and a germ of holomorphic function $f : (X, \underline{0}) \to (\mathbb{C}, 0)$. We may assume that X has locally at $\underline{0}$ an embedding into $\mathbb{C}^N$, *i.e.*, we have an open neighborhood U of $\underline{0}$ in X such that the representative of the germ $(X, \underline{0})$ is embedded in the open subset $\mathcal{U}$ of $\mathbb{C}^N$ and U is closed in $\mathcal{U}$. We may assume that U has only one isolated singular point $\underline{0}$ and f has a representative defined on U.

In this situation we can still define the polar curve of f with respect to a linear form. Let:

$$\lambda : \mathbb{C}^N \to \mathbb{C}$$

be a linear form. Denote ℓ the restriction of λ to U. One has the analytic morphism:

$$\Phi_\ell : U \to \mathbb{C}^2$$

defined by $\Phi_\ell(z) = (f(z), \ell(z))$ for every $z \in U$. We have the following theorem:

Theorem 6.7.6 *With the notations above, there is an open dense subset Ω of the space of linear forms on $\mathbb{C}^N$, such that there is an open neighborhood U_λ of $\underline{0}$ in X which is contained in U and the restriction ℓ of λ to U_λ defines a restriction $(\Phi_\ell)_{|U_\lambda}$ whose critical locus outside $V = f^{-1}(0) \cap U_\lambda$ is either always empty for all $\lambda \in \Omega$ or non-singular of dimension* 1. *Moreover:*

1. *The closure Γ_λ in U_λ of this critical locus, when it is not empty, is a reduced curve called the polar curve of f with respect to λ.*
2. *If Γ_λ is not empty, the restriction of $(\Phi_\ell)_{|U_\lambda}$ to Γ_λ is finite and injective. The image Δ_λ of Γ_λ by Φ_ℓ is a curve called the polar discriminant with respect to λ or also called the* Cerf diagram *of f with respect to ℓ.*
3. *At each point $z \in \Gamma_\lambda - \{\underline{0}\}$ the fiber of Φ_ℓ has an ordinary quadratic singularity.*

Notice that in this case we do not have a result like Proposition 6.7.5. For instance, when f is given by a generic linear form, the behavior of f and a generic section will be comparable so the tangent cone of Δ_λ would be a line $u = v$ if u and v are the coordinates of $(\mathbb{C}^2, (0, 0))$.

We do not give here a proof of Theorem 6.7.6. The reader can refer to [48, Paragraph 3] for a proof of this theorem.

From now on, in the rest of this section, we assume that X is $\mathbb{C}^{n+1}$ and consider a holomorphic map-germ f with a critical point at $\underline{0} \in \mathbb{C}$.

Given f and a generic linear form ℓ as above in Theorem 6.7.3, we can define the polar curve $\Gamma_{f,\ell}$ as the union of those components in the critical set of (f, ℓ) which are not in Σf, the critical points of f. In other words, assume we have coordinates $(z_0, \cdots, z_n)$ so that the linear function $\ell = z_0$ is "sufficiently general". Then the critical locus of $\Phi_\ell = (f, \ell)$ is $V(\partial f/\partial z_1, \cdots, \partial f/\partial z_n)$, the set of points where $\partial f/\partial z_i = 0$ for all $i = 1, \cdots, n$. In the critical points of Φ_ℓ there is the set of singular points of $V(f)$ and the set of non-critical points z of f where the hyperplane $\ell^{-1}(\ell(z))$ is tangent to the hypersurface $f^{-1}(f(z))$. As we said in Theorem 6.7.3 the contact between $\ell^{-1}(\ell(z))$ and $f^{-1}(f(z))$ is ordinary quadratic, since we have assume $\ell \in \Omega$.

There is a notion a bit less general than our notion of genericity which is called *prepolar* and was introduced by D. Massey (see [59] p. 26).

Definition 6.7.7 A hyperplane $H \subset \mathbb{C}^{n+1}$ through the origin is called a *prepolar slice* for a function $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ at $\underline{0}$ if it intersects transversally all the strata (of a good stratification of f) except perhaps the stratum $\{\underline{0}\}$ itself.

We are going to show how polar curves are used to understand the geometry of hypersurface singularities.

Let us recall S. Smale's classical process of "attaching handles". To attach a p-handle to an m-manifold M we assume one has a smooth embedding ι of $S^{p-1} \times D^{m-p}$ into the boundary ∂M. Set $H^p = D^p \times D^{m-p}$ and define a manifold $M \cup_f H^P$ by taking the disjoint union of M and H^p and identify $S^{p-1} \times D^{m-p}$ with its image by ι. We think of $M \cup_f H^P$ as being obtained from M by attaching a p-handle; the integer p is the index of the handle.

One uses the process of attaching handles in connection with Morse theory. By Morse theory if one has a real smooth function φ which is proper and Morse (see [64]) on a manifold M and if $\varphi - t_0$ is a critical level with one Morse point of index p, when $\varepsilon > 0$ is small enough $\{\varphi \leq t_0 + \varepsilon\}$ is obtained from $\{\varphi \leq t_0 - \varepsilon\}$ by attaching p-handles.

We may now state Lê's attaching theorem:

Theorem 6.7.8 *Let $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ be a germ of holomorphic function, with a possibly non-isolated critical point at $\underline{0}$, consider its local Milnor fiber F_t at $\underline{0}$. If H is a prepolar slice for f at $\underline{0}$, then F_t is obtained from the Milnor fiber of the slice $F_t \cap H$ by attaching a certain number of n-handles. The number of such handles is the intersection number $\left(\Gamma_{f,\ell} \cdot V(f)\right)$, where ℓ is the linear form which defines H. If f has an isolated critical point, this number equals the sum of Milnor numbers $\mu(f) + \mu(f, \ell)$.*

Proof We only give a hint of the proof of this Theorem which is rather technical.

□

Let us consider the Milnor fibration associated to this germ of function f, *i.e.* let $1 \gg \varepsilon \gg \delta > 0$, the Milnor function f defines the locally trivial smooth fibration (see Theorem 6.4.1):

$$\varphi_\varepsilon : \mathbb{B}_\varepsilon \cap f^{-1}(\mathring{\mathbb{D}}_\delta^*) \to \mathring{\mathbb{D}}_\delta^*$$

where $\mathbb{B}_\varepsilon$ is the closed ball of $\mathbb{C}^{n_1}$ centered at $\underline{0}$ with radius ε and $\mathring{\mathbb{D}}_\delta^*$ is the punctured open disc of $\mathbb{C}$ centered at 0 with radius δ.

The hypothesis that the hyperplane H is a prepolar slice for f at the point $\underline{0}$ means that the linear form ℓ which defines H is prepolar for f at $\underline{0}$. This implies that the critical space of $(\ell, f) : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}^2, 0)$ outside $f^{-1}(0)$ is a curve $\Gamma_{(f,\ell)}$. In the case H is a general hyperplane, this curve is reduced or empty. In this case for any $z \in \Gamma_{(f,\ell)} - \{\underline{0}\}$ near to $\underline{0}$ the hyperplane $\ell^{-1}(\ell(z))$ has an ordinary tangency with the hypersurface $f^{-1}(f(z))$.

If we only assume that H is prepolar for f at $\underline{0}$, the critical points of the restriction of ℓ to $f^{-1}(f(z))$ are isolated critical points, but these critical points are in general not ordinary quadratic.

Another property which is corollary of the fact that H is a prepolar slice for f at $\underline{0}$ is that, for $1 \gg \varepsilon > 0$, the fibers of (f, ℓ) over a sufficiently small neighborhood U of $(0, 0)$ outside the line $\{0\} \times \mathbb{C}$ are transverse to the sphere $\mathbb{S}_\varepsilon$.

Now we can apply Morse theory to the manifold with boundary $F_t \cap \mathbb{B}_\varepsilon \cap \ell^{-1}(\mathring{\mathbb{D}}_\delta^*)$. We use the function $|\ell|$. Our starting situation is the space $|\ell| \leq u$ such that ℓ induces a trivial fibration above the disk $\{t\} \times \mathbb{D}_u$. This is possible because H is prepolar for f at $\underline{0}$, because by definition H intersects the strata of a good stratification, *i.e.* a stratification which satisfies A_f-condition (see Definition 6.4.4), of $V = f^{-1}(f(0))$ transversally in $\mathbb{C}^{n+1}$ in a neighborhood U of $\underline{0}$ in $\mathbb{C}^{n+1}$, which implies that $H \cap F_t$ is a manifold in a neighborhood of $\underline{0}$ in $\mathbb{C}^{n+1}$ as we show in the following lemma:

Lemma 6.7.9 *Let H be a prepolar hyperplane for f at $\underline{0}$, defined by the linear form ℓ. Let $\Phi_{f,\ell}$ be the map:*

$$\Phi_{f,\ell} : U \to U_0 \subset \mathbb{C}^2$$

defined on a sufficiently small neighborhood U of $\underline{0}$ in $\mathbb{C}^{n+1}$ by $\Phi_{f,\ell}(z) = (f(z), \ell(z))$. The components of the critical locus of $\Phi_{f,\ell}$ which are not contained in $V = f^{-1}(0) \cap U$ are either empty or a curve $\Gamma_{f,\ell}$, which might not be reduced. When $\Gamma_{f,\ell}$ is not empty, the restriction of $\Phi_{f,\ell}$ to $\Gamma_{f,\ell}$ is finite and its image by $\Phi_{f,\ell}$ is a curve $\Delta_{f,\ell}$ in U_0. With $1 \gg \varepsilon \gg \delta > 0$ the map $\Phi_{f,\ell}$ induces a locally trivial fibration

$$(\Phi_{f,\ell})_| : (\Phi_{f,\ell}^{-1}(\mathbb{B}^2_\delta - (\Delta_{f,\ell} \cup \{0\} \times \mathbb{C})) \cap \mathbb{B}_\varepsilon \to \mathbb{B}^2_\delta - (\Delta_{f,\ell} \cup \{0\} \times \mathbb{C}).$$

Furthermore, the fibers of $\Phi_{f,\ell}$ over the points of $\mathbb{B}^2_\delta - \{0\} \times \mathbb{C}$ are transverse to the sphere $\mathbb{S}_\varepsilon$, boundary of the ball $\mathbb{B}_\varepsilon$.

Proof The hyperplane is prepolar for f at $\underline{0}$ with respect to a good stratification $\mathcal{S}$.

Let us refine the good stratification $\mathcal{S}$ of $U \cap f^{-1}(0)$ into a regular stratification (see Theorem (19.2) p. 540 of [82]) adapted to $H \cap f^{-1}(0) \cap U$. In Lemma 6.4.3 we saw that for $\varepsilon_1 > 0$ small enough the spheres $\mathbb{S}_\varepsilon$, for $\varepsilon_1 > \varepsilon > 0$, are transverse to the strata of $f^{-1}(0) \cap U$ whose closures contain $\underline{0}$, and the strata of $H \cap f^{-1}(0) \cap U$ whose closures also contain $\underline{0}$.

To obtain this former assertion, one applies a Lemma similar to Lemma 6.2.6 but in the case of real analytic functions.

Exercise 6.7.10 Prove the Lemma 6.2.6 for real analytic functions.

Now, to prove the last assertion of the Lemma we are going to use an argument similar to one used in the proof of Theorem 6.4.1.

Assume that the last assertion of the Lemma 6.7.9 is not true. For each $r_n > 0$ of a sequence (r_n) tending to 0 in $\mathbb{R}_+$ we would find $z_n \in \mathbb{S}_\varepsilon \cap \Phi_{f,\ell}^{-1}(\mathring{\mathbb{B}}^2_{r_n})$ for which the Lemma is false. Then we would have a sequence of points z_n of $\mathbb{S}_\varepsilon$ which tends to $z \in V \cap H \cap \mathbb{S}_\varepsilon$ and either $f^{-1}(f(z_n))$ and $\ell^{-1}(\ell(z_n))$ are tangent to each other at z_n and $\Phi_{f,\ell}^{-1}(f(z_n), \ell(z_n))$ is singular at z_n or $f^{-1}(f(z_n)) \cap \ell^{-1}((\ell(z_n))$ is not transverse to $\mathbb{S}_\varepsilon$ at z_n.

However for n big enough $f^{-1}(f(z_n))$ and $\ell^{-1}(\ell(z_n))$ are not tangent at z_n: because of the compactness of the space of linear subspaces of given dimension in $\mathbb{C}^{n+1}$, we can always choose a subsequence so that the tangent space

$T_{z_n}(f^{-1}(f(z_n))$ has a limit $\mathcal{T}$ and the limit of $\ell^{-1}(\ell(z_n))$ is H. Let S be the stratum of the good stratification which contains z. By hypothesis on H, H is transverse to S at z in $\mathbb{C}^{n+1}$ and the hypothesis on good stratification is that $T_z(S) \subset \mathcal{T}$. Hence, at the limit the sequences of tangent spaces are transverse, so they must be transverse when n is big enough. Therefore we assume that $f^{-1}(f(z_n))$ and $\ell^{-1}(\ell(z_n))$ are not tangent at z_n.

As above we may assume the sequence of tangent spaces $(T_{z_n}(f^{-1}(f(z_n)))$ has a limit $\mathcal{T}$. We saw that the hypothesis implies that the limit $\mathcal{T}$ contains $T_z(S)$ where S is the strata of the good stratification $\mathcal{S}$ which contains z. Furthermore, we also may assume that the sequence $(T_{z_n}(f^{-1}(f(z_n)) \cap \ell^{-1}(\ell(z_n)))$ has a limit, which is $\mathcal{T} \cap H$, because of the transversality at the limit. If S' is the stratum of the regular stratification of $V \cap H \cap U$ which is a refinement of the good stratification and which contains z, we have $T_z(S') \subset T_z(S)$.

Notice that if $V \cap H \cap \mathbb{S}_\varepsilon$ is not empty, *i.e.* $n \geq 2$, $\dim S' \geq 1$, and then we have:

$$T_z(S') \subset T_z(S) \subset \mathcal{T}$$

and $T_z(S') \subset H$ because the stratum S' is contained in H. Since $\mathbb{S}_\varepsilon$ is transverse to S' in $\mathbb{C}^{n+1}$, we have that the limit $\mathcal{T} \cap H$ is transverse to $T_z(\mathbb{S}_\varepsilon)$.

This conclusion contradicts that, for any $n \in \mathbb{N}$, the tangent of space $f^{-1}(f(z_n)) \cap \ell^{-1}(\ell(z_n))$ at z_n is contained in $T_{z_n}(\mathbb{S}_\varepsilon)$.

It implies the existence of $\delta > 0$ such that, for any $(t, u) \in \mathring{\mathbb{B}}^2_\delta - \{0\} \times \mathbb{C}$, the space $\{f = t\} \cap \{\ell = u\}$ intersects the sphere $\mathbb{S}_\varepsilon$ transversally in $\mathbb{C}^{n+1}$.

In the case $n = 1$, $V \cap H \cap \mathbb{B}_\varepsilon = \{\underline{0}\}$. The last assertion deals with an empty set. This proves the last assertion of Lemma 6.7.9.

Our reasoning above shows that the intersection of $\mathbb{S}_\varepsilon$ and the set of singular points $\Gamma_{f,\ell} - V$ of the fibers $\Phi^{-1}_{f,\ell}(t, u)$, for any $(t, u) \in \mathring{\mathbb{B}}^2_\delta - \{0\} \times \mathbb{C}$ is empty. It implies that either $\Gamma_{f,\ell} \cap \Phi_{f,\ell}(\mathring{\mathbb{D}}_\delta)$ is empty or it is a curve and the restriction of $\Phi_{f,\ell}$ to this curve is finite using again the geometric version of Weierstrass preparation Theorem in [27]. Therefore the image $\Delta_{f.\ell}$ of $\Gamma_{f,\ell}$ by $\Phi_{f,\ell}$ is a curve.

The fibers of $\Phi_{f,\ell}$ over $\mathring{\mathbb{B}}^2_\delta - (\Delta_{f,\ell} \cup \{0\} \times \mathbb{C})$ are not singular and the last assertion of our Lemma 6.7.9 shows that they are transverse to $\mathbb{S}_e$. The Ehresmann Lemma (see Lemma 6.2.10) shows that $\Phi_{f,\ell}$ induces a locally trivial fibration:

$$(\Phi_{f,\ell})_| : (\Phi^{-1}_{f,\ell}(\mathbb{B}^2_\delta - (\Delta_{f,\ell} \cup \{0\} \times \mathbb{C})) \cap \mathbb{B}_\varepsilon \to \mathbb{B}^2_\delta - (\Delta_{f,\ell} \cup \{0\} \times \mathbb{C}).$$

This ends the proof of Lemma 6.7.9. □

Let us resume the proof of Theorem 6.7.8.

Consider $t \neq 0$ small enough. The image of the Milnor fiber of $F_t = \{f = t\} \cap \mathbb{B}_\varepsilon$ under ℓ lies in the line $\{t\} \times \mathbb{C}$ in $\mathbb{C}^2$. This line intersects $\Delta_{(f,\ell)}$ at the points $y_1, \ldots, y_k$ which tend to $(0, 0)$ when t tends to 0. Let R be the radius of a disc $\{t\} \times \mathbb{D}$ contained in the line $\{t\} \times \mathbb{C}$ and such that the only points of $\Delta_{(f,\ell)}$ that it contains are $\{y_1, \ldots, y_k\}$.

The center of $\{t\} \times \mathbb{D}$ is (t, α). We assume that (t, α) is not one of the points $y_1 \ldots, y_k$. In case it is, we can always slightly change the center of $\{t\} \times \mathbb{D}$.

We are going to apply Morse theory on the Milnor fiber $F_t \cap \mathbb{B}_\varepsilon$ with the smooth function $|\ell - \alpha|$ in the range $\{r \leq |\ell - \alpha| \leq R\}$, where r is chosen such there are not any of the points $y_1, \ldots, y_k$ in the closed disc centered at (t, α) with radius r and the closed disc centered at (t, α) with radius R contains all the points $y_1, \ldots, y_k$ and is contained in the interior $\{t\} \times \overset{\circ}{\mathbb{D}}$.

The ending situation $F_t \cap \mathbb{B}_\varepsilon \cap \{|\ell - \alpha| \leq R\}$ is a manifold with corners (see for instance [61]) which is diffeomorphic (with corners) to the Milnor fiber $F_t \cap \mathbb{B}_\varepsilon$.

However we cannot in general apply Morse theory in this situation. In the case ℓ is sufficiently general, the singularities of ℓ restricted to the Milnor fiber $F_t \cap \mathbb{B}_\varepsilon$ are ordinary quadratic as we saw in Theorem 6.7.3 in which case the function $|\ell - \alpha|$ is a Morse function when restricted to F_t in the domain where $\{r \leq |\ell - \alpha| \leq R\}$.

In general one has to modify the function $|\ell|$ near the critical points of the restriction of ℓ to the space $F_t \cap \{r \leq |\ell| \leq R\}$. Let $z_1, \ldots, z_m$ be these critical points. Consider small spheres B_i and B_i' of $\mathbb{C}^{n+!}$ centered at the points z_i such that $B_i' \subset \overset{\circ}{B}_i$, the spheres B_i are contained in $\overset{\circ}{\mathbb{B}}_\varepsilon$ and are mutually disjoint. We also require that the union of spheres $\cup_1^r B_i$ lies inside the interior of the subspace $\{r \leq |\ell - \alpha| \leq R\}$ of $\mathbb{C}^{n+1}$.

There is a smooth function $\phi : \mathbb{C}^{n+1} \to [0, 1]$ such that ϕ is 0 on $\mathbb{C}^{n+1} - \cup_1^r B_i$ and has value 1 in $\cup_1^r \overset{\circ}{B}_i'$. It is known that for a general linear form λ sufficiently small the restriction of $\ell + \tau\lambda$ to $F_t \cap \mathbb{B}_\varepsilon$ has only ordinary singularities. We choose this linear form λ such that this property is true for all $\tau \neq 0$ in a disc of radius at least, say 2.

Furthermore, we choose the linear form λ small enough such that the functions

$$\psi_\tau = (1 - \phi)\ell + \phi(\ell + \tau\lambda)$$

restricted to $F_t \cap \mathbb{B}_\varepsilon$ have no critical point in $\cup_i^r (B_i - \overset{\circ}{B}_i')$ for $|\tau| < 2|$. This is possible because $\cup_i^r (B_i - \overset{\circ}{B}_i')$ is compact. The smooth functions ψ_τ coincide with ℓ in $(\mathbb{C}^{n+1} - \cup_1^m B_i) \cap F_t \cap \mathbb{B}_\varepsilon$ and with $\ell + \tau\lambda$ in the space $\cup_1^m \overset{\circ}{B}_i' \cap F_t \cap \mathbb{B}_\varepsilon$.

Consider now the smooth unfolding:

$$(\psi_\tau, \tau) : (F_t \cap \{r \leq |\ell - \alpha| \leq R\} \cap \mathbb{B}_\varepsilon) \times \overset{\circ}{D}_2 \to (\{t\} \times \mathbb{D}_R - \{t\} \times \overset{\circ}{\mathbb{D}}_r) \times D_2.$$

Notice that the center of the open disc $\overset{\circ}{\mathbb{D}}_2$ is the origin of $\mathbb{C}$ and its radius is 2.

Since ψ_τ coincide with ℓ in $F_t \cap \mathbb{B}_\varepsilon - (\cup_1^m B_i)$, the fibers of the unfolding intersect the boundary $F_t \cap \mathbb{S}_\varepsilon \cap \{r \leq |\ell - \alpha| \leq R\}$ of $F_t \cap \mathbb{B}_\varepsilon \cap \{r \leq |\ell - \alpha| \leq R\}$ transversally.

Using Ehresmann Lemma, one can prove that the general fiber of this unfolding (ψ_τ, τ) is diffeomorphic to $F_t \cap H \cap \mathbb{B}_\varepsilon$.

We apply Morse theory to the manifold with boundary to the manifold

$$F_t \cap \mathbb{B}_\varepsilon \cap \{r \leq |\ell - \alpha| \leq R\}$$

with the function $|\psi_1 - \alpha|$ restricted to $F_t \cap \mathbb{B}_\varepsilon \cap \{r \leq |\ell - \alpha| \leq R\}$. Since we have required that $\cup_1^m B_i$ lies in the interior of $\{r \leq |\ell - \alpha| \leq R\}$, we have:

$$F_t \cap \mathbb{B}_\varepsilon \cap \{|\psi_1 - \alpha| \leq r\} = F_t \cap \mathbb{B}_\varepsilon \cap \{|\ell - \alpha| \leq r\}$$

and:

$$F_t \cap \mathbb{B}_\varepsilon \cap \{|\psi_1 - \alpha| \leq R\} = F_t \cap \mathbb{B}_\varepsilon \cap \{|\ell - \alpha| \leq R\}$$

Since ℓ restricted to the $F_t \cap \mathbb{B}_\varepsilon$ is a trivial fibration over the disk $\mathbb{D}_r$, we have:

$$F_t \cap \mathbb{B}_\varepsilon \cap \{|\psi_1 - \alpha| \leq r\} = F_t \cap \mathbb{B}_\varepsilon \cap \{|\ell - \alpha| \leq r\} = (F_t \cap H \cap \mathbb{B}_\varepsilon) \times \mathbb{D}_r.$$

We consider the restriction of ψ_1 to $F_t \cap \mathbb{B}_\varepsilon \cap \{r \leq |\ell - \alpha| \leq R\}$. All the critical points of this restriction are ordinary quadratic and there are μ_i ordinary quadratic points for each critical point z_i of the restriction of ℓ, where μ_i is the Milnor number at z_i of the restriction of ℓ to F_t. Now applying Morse theory to the restriction of the function $|\psi_1 - \alpha|$ to $F_t \cap \mathbb{B}_\varepsilon \cap \{r \leq |\ell - \alpha| \leq R\}$, each critical point of ψ_1 contributes to a Morse point of index $\dim F_t = n$.

Computing the total number of critical points of the restriction of ψ_1 to

$$F_t \cap \mathbb{B}_\varepsilon \cap \{r \leq |\ell - \alpha| \leq R\},$$

we obtain that the total number is precisely the intersection number: $\left(\Gamma_{f,\ell} \cdot V(f)\right)$.

We have proved that $F_t \cap \mathbb{B}_\varepsilon \cap \{|\ell - \alpha| \leq R$ is obtained from $(F_t \cap H \cap \mathbb{B}_\varepsilon) \times D_u$ by attaching $\left(\Gamma_{f,\ell} \cdot V(f)\right)$ n-handles.

It remains to prove that the manifold with corners $F_t \cap \mathbb{B}_\varepsilon \cap \{\ell \leq R\}$ is diffeomorphic to $F_t \cap \mathbb{B}_\varepsilon$. We leave this last assertion as an exercise to the reader.

6.8 The Carousel

In this section we focus on holomorphic functions $(\mathbb{C}^2, \underline{0}) \to (\mathbb{C}, 0)$ and denote by (X, Y) the variables in $\mathbb{C}^2$.

6.8.1 Carousel of One Branch

Suppose that the complex analytic function $f \in \mathbb{C}\{X, Y\}$ is an irreducible element of the ring $\mathbb{C}\{X, Y\}$. In this case recall that the germ of a complex plane curve $(C, 0)$ at the point 0 defined by $f = 0$ is called a complex branch.

Suppose that $f(0, Y) \neq 0$. Since we assume that $(0, 0)$ is a singular point, there is a convergent series (Puiseux series in X of the curve $\{f = 0\}$ at the point $(0, 0)$):

$$\varphi(X^{1/n}) \in \mathbb{C}\{X^{1/n}\}$$

with $n > 1$, such that $f(X, \varphi(X^{1/n})) \equiv 0$ and it is known that:

$$\prod_{\zeta, \zeta^n = 1} (Y - \varphi(\zeta X^{1/n})) = \alpha f(X, Y),$$

where α is a function which does not vanish at $(0, 0)$, *i.e.* $\alpha(0, 0) \neq 0$.

We have the Puiseux exponents of f with respect to the variables X and Y, as described in the following:

$$\varphi(X^{1/n}) = P_0(X) + a_1 X^{m_1/n_1} + P_1(X^{1/n_1}) + \ldots + a_h X^{m_h/n_1 \ldots n_h} + P_h(X^{1/n_1 \ldots n_h})$$

where:

1. m_1, n_1 are relatively prime, …, m_k, n_k are relatively prime, and we have $n_1 \ldots n_h = n$;
2. $P_0 \ldots P_{h-1}$ are polynomials and P_h a series;
3. when $k < h$ and $P_k \neq 0$ the degree of P_k divided by $n_1 \ldots n_k$ is less than $m_{k+1}/n_1 \ldots n_{k+1}$ which is less than the valuation, *i.e.* the smallest degree, of P_{k+1} divided by $n_1 \ldots n_{k+1}$, for $0 \leq k \leq h - 1$, when $P_{k+1} \neq 0$;
4. The quotient $\frac{m_1}{n_1}, \ldots, \frac{m_h}{n_1 \ldots n_h}$ are called the Puiseux exponents of the Puiseux series $\varphi(X^{1/n})$;
5. The last Puiseux exponent $m_h/n_1 \ldots n_h$ is less than the valuation of P_h divided by $n_1 \ldots n_h$, when $P_h \neq 0$.

Puiseux expansions give another structure near the singular point that we shall call the carousel associated to the singularity.

Consider the line $X = t$ where $t \neq 0$ and t is small enough.

The distribution of points in the intersection $\{X = t\} \cap \{f(X, Y) = 0\}$ is the following:

1. Let v_0 be the degree of the term of lowest degree of $\varphi(X^{1/n})$.
2. Consider $\alpha_0 X^{v_0}$ the term of $\varphi(X^{1/n})$ of degree v_0.
3. Let r_0 be the radius $5|\alpha_0||t|^{v_0}$ of the closed disk D_0 in the line $X = t$ centered at the point $(t, 0)$.
4.
   - If v_0 is an integer, P_0 is $\neq 0$ and the equation $Y = P_0$ gives a non-singular curve and D_0 contains only one point of the curve $Y = P_0(X)$;
   - otherwise v_0 is the first Puiseux exponent m_1/n_1 of the series $\varphi(X^{1/n})$ with respect to the variables X and Y. In this case $\alpha_0 = a_1$ and D_0 contains n_1 points of the curve with Puiseux expansion $Y = a_1 X^{m_1/n_1}$.
5. Let us call $\mathbb{D}_0$ the disk which is, in the first case, centered at the unique point of the curve $Y = P_0(x)$ contained in D_0 with radius $2|\alpha_0||t|^{v_0}$, and in the second

case the disk $\mathbb{D}_0$ is centered at $(t, 0)$ with radius $2|a_1||t|^{m_1/n_1}$. Notice that in both cases $(D_0 - \mathring{\mathbb{D}}_0) \cap \{f(X, Y) = 0\}$ is empty and $D_0 - \mathring{\mathbb{D}}_0$ is diffeomorphic to an annulus.

6. Assume the number of Puiseux pairs $h \geq 1$. In the second case, in the disk $\mathbb{D}_0$ we consider n_1 disks D_1^j, $j = 1, \dots, n_1$, centered regularly on a circle centered at the center of $\mathbb{D}_0$ with radius $|a_1||t|^{m_1/n_1}$; each disk D_1^j has:
   - radius $r_1 = 5|\alpha_1||t|^{v_1}$, where v_1 is the lowest degree of P_1 and $\alpha_1 t^{v_1}$ is the term of degree v_1, when $P_1 \neq 0$;
   - radius $r_1 = 5|a_2||t^{v_1}$, when there are $h \geq 2$ Puiseux exponents, and $P_1 = 0$, so $v_1 = m_2/n_1 n_2$ and $\alpha_1 = a_2$;
   - any small radius $r_1 > 0$ if $P_2 = 0$ and there is only 1 Puiseux exponent. We assume that r_1 is chosen so that the disks D_1^j do not overlap.

7. Then, we define the disks $\mathbb{D}_1^j$, $1 \leq j \leq n_1$:
   - as the disks of radius $r_1/2$ with the same center as D_1^j for $1 \leq j \leq n_1$, when $P_1 = 0$ and there is only 1 Puiseux exponent;
   - as the disks centered at the points of $C_1 \cap \{X = t\}$, where C_1 is the curve having Puiseux expansion $\varphi_1 = P_0(X) + a_1 X^{m_1/n_1} + P_1(X^{1/n_1})$, with radius $2|\alpha_1||t|^{v_1}$ if $P_1 \neq 0$;
   - as the disks centered at the points of $C_1 \cap \{X = t\}$, where C_1 is the curve given by the Puiseux expansion $\varphi_1 = P_0(X) + a_1 X^{m_1/n_1}$, with radius $2|\alpha_1||t|^{m_2/n_1 n_2}$, if $P_1 = 0$ and $h \geq 2$.

8. Let us suppose that $h > k \geq 1$ and we have defined the disks $D_k^{i_1,\dots,i_k}$, with $1 \leq i_k \leq n_k$, centered at the intersections of the disk $\mathbb{D}_{k-1}^{i_1,\dots,i_{k-1}}$ with the curve having Puiseux expansion:

$$Y = P_0(X) + a_1 X^{m_1/n_1} + \dots + P_{k-1}(X^{m_{k-1}/n_1 \dots n_{k-1}}) + a_k X^{m_k/n_1 \dots n_k} =: \varphi_k$$

   and with radii equal to $5|a_k||t|^{m_k/n_1 \dots n_k}$.

9. Now let $\mathbb{D}_k^{i_1,\dots,i_k}$ be the circle inside $D_k^{i_1,\dots,i_k}$ with radius $2|a_k||t|^{m_k/n_1 \dots n_k}$ and having as center the only point of the intersection of disk $D_k^{i_1,\dots,i_k}$ with the curve with Puiseux expansion:

$$Y = \varphi_k + P_k(X^{m_k/n_1 \dots n_k}) \qquad (\mathrm{C})_k$$

   if $P_k \neq 0$. If $P_k = 0$, $\mathbb{D}_k^{i_1,\dots,i_k}$ has the same center as the disk $D_k^{i_1,\dots,i_k}$ with half the radius.

10. Since $h > k$, the curve C_{k+1} has $k + 1$ Puiseux pairs. Inside the disk $\mathbb{D}_k^{i_1,\dots,i_k}$ we have n_{k+1} disks $D_k^{i_1,\dots,i_{k+1}}$ contained in $\mathbb{D}_k^{i_1,\dots,i_k}$ and centered regularly on a

circle having the same center as the disk $\mathbb{D}_k^{i_1,\ldots,i_k}$ at the points of intersection of the disk $\mathbb{D}_k^{i_1,\ldots,i_k}$ with the following curve with $k+1$ Puiseux pairs:

$$Y = P_0(X) + a_1 X^{m_1/n_1} + \ldots + P_k(X^{m_k/n_1\ldots n_k}) + a_{k+1} X^{m_{k+1}/n_1\ldots n_{k+1}}.$$

The radii are $5|\alpha_{k+1}|t|_{k+1}^{v}$ where $\alpha_{k+1} X^{v_{k+1}}$ is the term of:

$$P_k(X^{m_k/n_1\ldots n_k}) + a_{k+1} X^{m_{k+1}/n_1\ldots n_{k+1}}.$$

of degree equal to the valuation of this sum.

11. We do the preceding construction until $k = h-1$. Then we have n_k disks $D_h^{i_1,\ldots,i_h}$ inside $\mathbb{D}_k^{i_1,\ldots,i_{h-1}}$ of radius $5|\alpha_h||t|^{v_h}$ where $\alpha_h X^{v_h}$ is the term of lowest degree of the Puiseux expansion:

$$P_{h-1}(X^{1/n_1\ldots n+h-1}) + a_h X^{m_h/n_1\ldots n_h}.$$

Then the original curve $f(X,Y) = 0$ intersects each of disks $D_h^{i_1,\ldots,i_h}$ in one point which will be the center of the disk $\mathbb{D}_h^{i_1,\ldots,i_h}$ such that $D_h^{i_1,\ldots,i_h} - \mathring{\mathbb{D}}_h^{i_1,\ldots,i_h}$ is diffeomorphic to an annulus.

The preceding construction gives a special configuration of the points of

$$\{f(X,Y) = 0\} \cap \{X = t\}$$

which are contained in $\cup_{k,1\leq i_1\leq n_1,\ldots,1\leq i_h\leq n_h} \mathbb{D}_h^{i_1,\ldots,i_h}$.

Definition 6.8.1 *The carousel* of the branch $\{f(X,Y) = 0\}$ relatively to the coordinate function X at the point 0 has the isotopy class of a geometric monodromy of the locally trivial smooth fibration of $\{X = |t|\} \times D_0$ over the circle $X = |t|$ relatively to the intersection of the curve $\{f(X,Y) = 0\}$ with $\{X = |t|\} \times D_0$.

In the case of a branch $f(X,Y) = 0$ we can describe a geometric monodromy Φ as follows:

1. Inside $\mathbb{D}_0 - \cup_{1\leq i_1\leq n_1}(D_1^{i_1})$, it is a rotation of angle $2\pi m_1/n_1$.
2. Then, if $h \geq 2$ the monodromy sends $\mathbb{D}_1^{i_1} - \cup_{1\leq i_2\leq n_2}(D_2^{i_1.i_2})$ to $\Phi(\mathbb{D}_1^{i_1} - \cup_{1\leq i_2\leq n_2}(D_2^{i_1.i_2}))$ and is given by a rotation of angle $2\pi m_2/n_2$.
3. Finally the monodromy sends $\mathbb{D}_{h-1}^{i_1,\ldots,i_{h-1}} - \cup_{1\leq i_h\leq n_h}(D_h^{i_1,\ldots,i_h})$ into $\Phi(\mathbb{D}_{h-1}^{i_1,\ldots,i_{h-1}} - \cup_{1\leq i_h\leq n_h}(D_h^{i_1,\ldots,i_h})$ and is given by a rotation $2\pi m_h/n_h$.
4. In the zones $D_k^{i_1,\ldots,i_k} - \mathring{\mathbb{D}}_k^{i_1,\ldots,i_k}$ which are annulus-like, one constructs Φ to fit with the preceding construction.

We may call the map of D_0 into itself obtained in this way an iterated rotation.

Then, the Carousel of the branch $\{f(X,Y) = 0\}$ relatively to the coordinate function X at the point 0 is given by the isotopy class of an iterated rotation.

6.8.2 Carousel of Curves with Several Branches

In the case of several branches the construction is a bit more complicated and involves different levels of monodromy "speed".

Let us consider $(C_1, 0), \dots, (C_r, 0)$ several complex analytic branches whose reduced equations are respectively $f_1(X, Y) = 0, \dots, f_r(X, Y) = 0$.

We suppose that $f_1(0, 0) = \dots = f_r(0, 0) = 0$ and $f_1(0, Y), \dots, f_r(0, Y)$ are all complex analytic series $\neq 0$. We can apply simultaneously Puiseux theorem to all the functions $f_1, \dots, f_r$ with respect to the variables X and Y:

$$\varphi_1(X^{1/n(1)}) = P_{0,1}(X) + a_{1,1}X^{m_{1,1}/n_{1,1}} + P_{1,1}(X^{1/n_{1,1}}) + \dots + a_{h(1)}X^{m_{h(1)}/n_{1,1}\dots n_{h(1)}}$$

$$+ P_{h(1)}(X^{1/n_{1,1}\dots n_{h(1)}}) \qquad (P_1)$$

$$\dots$$

$$\varphi_r(X^{1/n(r)}) = P_{0,r}(X) + a_{1,r}X^{m_{1,r}/n_{1,r}} + P_{1,r}(X^{1/n_{1,r}}) + \dots + a_{h(r)}X^{m_{h(r)}/n_{1,r}\dots n_{h(r)}}$$

$$+ P_{h(r)}(X^{1/n_{1,r}\dots n_{h(r)}}) \qquad (P_r)$$

We order the Puiseux series by the degree of their term of lowest degree v_i, $i = 1, \dots, i = r$. We assume that the ordering is:

$$v_1 \leq v_2 \leq \dots \leq v_r.$$

Then, among the series having the same degree v for the term of lowest degree, consider the term of lowest degree aX^v. Then we order the series so that the term of lowest degree is bigger or smaller in absolute value.

The Puiseux series $P_1, \dots, P_r$ are ordered in such a way that if v_i is the degree of their term of lowest degree and $a_i X_i^v$ their term of smallest degree:

$$|a_1||X|^{v_1} = \dots = |a_{j_1}||X|^{v_1} \geq |a_{j_1+1}||X|^{v_1} = \dots$$

$$\geq \dots = |a_{j_k}||X|^{v_1} > |a_{j_k+1}||X|^{v_2} = \dots$$

with $v_1 > v_2 > \dots$. Then, proceeding by induction we have the following description of the repartition of points of the intersection $\{X = t\} \cap \{f_1(X, Y) = 0\} \cap \dots \cap \{f_r(X, Y) = 0\}$ when t is sufficiently small.

1. Consider an annulus A_1 which contains all the points of the intersection $\{X = t\}$ and the curves $C_1, \dots, C_{i_1}$ for which the terms of lowest degree of Puiseux expansions have degree v_1.

By having t small enough all the points of the intersection of the annulus and the curves $C_1, \dots, C_{i_1}$ are in a neighbourhood contained in A_1 of the points $(t, a_1 t^{v_1}), \dots, (t, a_{i_k} t^{v_1})$.

2. We define in this way a sequence of annuli $A_1, \dots, A_s$ bounded outside by circles of radii R_i and inside by circles of radii r_i and:

$$R_1 > r_1 > \dots > R_s > r_s.$$

3. Now, the points of $\{X = t\} \cap C_j$ are distributed differently in the corresponding A_i according to the fact the valuation v_j of the Puiseux expansion of C_j is an integer n_j or a rational number r_j
In the case v_j is an integer, $(t, a_j t^v)$ is a point in the line $\{X = t\}$.
In the case v_j is a rational number $r_j = p_j/q_j$ we have q_j points in the corresponding A_i regularly distributed on the circle of radius $|a_j||t|^{v_j}$ inside A_i.

Then:

Definition 6.8.2 *The carousel* of the complex analytic plane curve $C_1 \cup \dots \cup C_r$ with respect to the coordinates X and Y at 0 has the isotopy class of a geometric monodromy of the local trivial smooth fibration of $\{X = |t|\} \times D_0$ over the circle $X = |t|$ relatively to intersection of the curve $C_1 \cup C_r$ with $\{X = |t|\} \times D_0$ when t is $\neq 0$ and small enough.

6.8.3 The General Concept of Carousel

In this section we shall give a general definition of carousel.

Let D be a disc and a smaller disc D_0 with the same centre 0 and disjoint annuli $A_1, \dots, A_k$ concentric with D and D_0 and contained in the interior of D, such that the outer radii $r_0, r_1, \dots, r_k$ of $D_0, A_1, \dots, A_k$ and the inner radii of r'_i of A_i, $1 \leq i \leq k$, satisfy:

$$r_0 < r'_1 < r_1 < \dots < r'_i < r_i < \dots < r'_k < r_k.$$

Inside D_0 and the annuli A_i, $1 \leq i \leq k$, we consider ℓ_i points, $0 \leq i \leq k$ on a circle $\mathbb{S}_i$ centred at 0 inside the open disc D_0 or the open annulus A_i. We suppose that these ℓ_i points are regularly distributed on these circles $\mathbb{S}_i$. We assume that these points are centres of discs $D_{i,j}$ inside D_0 and A_i, $1 \leq i \leq k$. For each i, $0 \leq i \leq k$ these discs have the same radius and do not overlap.

We call *elementary configuration* this configuration of a punctured disc and punctured annuli concentric with a disc containing them.

An *elementary carousel* is a map Ψ from $D - \cup_{0 \leq i \leq k, 1 \leq j \leq \ell_i} D_{i,j}$ into itself such that the restrictions of Ψ to $D_0 - \cup_{j=1}^{\ell_0} D_{0,j}$ and $A_i - \cup_{j=1}^{\ell_i} D_{i,j}$, for $1 \leq i \leq k$, are rotations by an angle $2\pi(k_i/\ell_i)$ where the integers k_i and ℓ_i are relatively prime, for $0 \leq i \leq k$, and the restriction to $D - (D_0 \cup_{i=1}^{k} A_i)$ is any map so that Ψ is smooth.

By abuse of language we shall also call elementary carousel a map defined in an obvious way from a punctured disc $D - \cup_{0\leq i\leq k, 1\leq j\leq \ell_i} D_{i,j}$ into another punctured disc $D' - \cup_{0\leq i\leq k, 1\leq j\leq \ell_i} D'_{i,j}$ with an isometric elementary configuration.

Examples The identity $D \to D$ is an elementary carousel. In this case $k = 0$ and the discs $D_{i,j}$ are empty. This elementary carousel is called the *trivial carousel.*

The rotation $D \to D$ of angle $2\pi k_0/\ell_0$ is also an elementary carousel with $k = 0$ and the discs $D_{i,j}$ empty.

Definition 6.8.3 One defines *a carousel* by induction. A 1-carousel is an elementary carousel where the discs $D_{i,j}$ are empty.

A 2-carousel Ψ_2 is given by an elementary carousel Ψ_1, discs $D'_{i,1}$ contained in $D_{i,1}$ and an elementary carousel defined on $D'_{i,1}$. The discs $D'_{i,1}$ generate a family of discs $(D'_{i,j})$ of the same size: the disc $D'_{i,2}$ is contained in the image $D_{i,2}$ of $D_{i,1}$ by the rotation of angle $2\pi k_i/\ell_i$ and so on, D'_{i,ℓ_i} is in D_{1,ℓ_i}. The 2-carousel is given by Ψ_1 on $D - \cup_{0\leq i\leq k, 1\leq j\leq \ell_i} D_{i,j}$, on $D'_{i,1}$ it is given by an elementary carousel $D'_{i,1} \to D'_{i,2}$ which is the same for all the discs $D'_{i,j} \to D'_{i,j+1}$, for $1 \leq j \leq \ell_i - 1$, and $D'_{i,\ell_i} \to D'_{i,1}$. On $D_{i,j} - D'_{i,j}$ the 2-carousel is just an extension of the map already defined such that the extension is smooth.

Suppose that we have defined a k-carousel Ψ_k. Then, Ψ_k is defined on:

$$D - \cup_{i_1,\dots,i_k,j} D_{i_1,\dots,i_k,j}$$

and each disc $D_{i_1,\dots,i_k,j}$ is contained in an annulus $A_{i_1,\dots,i_k}$ of some elementary carousel, with the convention that $A_{i_1,\dots,i_{k-1},0}$ is a disc of that elementary carousel. In each disc $D_{i_1,\dots,i_k,1}$ there is a disc $D'_{i_1,\dots,i_k,1}$. In each of the images $D_{i_1,\dots,i_k,j}$ of $D_{i_1,\dots,i_k,1}$ by the iterations of the rotation $2\pi k_{i_1,\dots,i_k}/\ell_{i_1,\dots,i_k}$ of the annulus $A_{i_1,\dots,i_k}$, we have a disc $D'_{i_1,\dots,i_k,j}$ isometric to $D'_{i_1,\dots,i_k,1}$. Then, all the maps of $D'_{i_1,\dots,i_k,j} \to D'_{i_1,\dots,i_k,j+1}$, for $1 \leq j \leq \ell_{i_1,\dots,i_k} - 1$, and $D'_{i_1,\dots,i_k,\ell_{i_1,\dots,i_k}} \to D'_{i_1,\dots,i_k,1}$ are the same elementary carousel.

The process stops when the last elementary carousel added is the trivial one. Then, we have a carousel.

Examples

1) Consider the elementary carousel given by a disc D containing a concentric disc D_0 and the identity on $D - D_0$. Inside D_0 consider a disc D' and on D' the elementary carousel given by the rotation $2\pi k/\ell$, where k and ℓ are relatively prime, which permutes the discs $D_1, \dots, D_\ell$.
   This defines on $D - \cup_{1\leq i\leq \ell} D_i$ a 2-carousel. Now, inside D_1 consider a disc D'_1 and in each of the images D_2, …, D_ℓ of D_1 by iterating the rotation $2\pi k/\ell$, we suppose that we have discs $D'_2, , \dots, D'_\ell$ isometric to D'_1. Then, we consider the trivial carousel on the D'_i into D'_{i+1}, for $i \leq i \leq \ell - 1$, and the trivial

carousel from D'_ℓ into D'_1. In this way we have define a 3-carousel. Since the last elementary carousel added is the trivial one, we have defined a carousel.
This carousel is the carousel of a curve of one Puiseux pair (k, ℓ).

2) Let us do the same construction as in the first example before, however, instead of considering the trivial carousel in D'_i, we consider the elementary carousel defined by the rotation of $2\pi k'/\ell'$ where k' and ℓ' are relatively prime:

$$D'_i - \cup_{1 \leq j \leq \ell'} D_{ij} \to D'_{i+1} - \cup_{1 \leq j \leq \ell'} D_{i+1,j},$$

for $1 \leq i \leq \ell - 1$. Then, in each D_{ij} we consider discs D'_{ij} isometric to the other $D'_{l.k}$'s, for $1 \leq l \leq \ell$ and $1 \leq k \leq \ell'$, and the trivial carousel on the D'_{ij}'s, for $1 \leq i \leq \ell$ and $1 \leq j \leq \ell'$.
This carousel is the carousel of a curve with two Puiseux pairs.

6.9 Some Theorems Where Carousels Are Used

6.9.1 The Geometric Monodromy of a Hypersurface

Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a non-constant germ of a reduced analytic function. In Proposition 6.2.16 we have shown that, for $1 \gg \varepsilon \gg \delta > 0$, we have a fibration $\varphi : \mathbb{B}_\varepsilon \cap f^{-1}(\partial \mathbb{D}_\delta) \to \partial \mathbb{D}_\delta$. Since $\partial \mathbb{D}_\delta$ is a circle S^1, this fibration is given by a diffeomorphism $h : F := f^{-1}(\delta) \to F$ that we called a geometric monodromy. As we have mentioned, this diffeomorphism is not unique, but its isotopy class is unique. However, any diffeomorphism in the isotopy class of a geometry monodromy of φ allows to reconstruct φ.

In [38] the following theorem is proved using the carousel of the polar discriminant:

Theorem 6.9.1 *Let $f : \mathcal{U} \to \mathbb{C}$ be a non-constant reduced analytic function defined in a neighborhood $\mathcal{U}$ of $\underline{0}$ in $\mathbb{C}^{n+1}$. Suppose that f has a critical point at $\underline{0}$ (not necessarily isolated). There is a geometric monodromy of the Milnor fibration of f at $\underline{0}$ which does not have fixed point.*

As a corollary we have a theorem of N. A'Campo from [1]:

Corollary 6.9.2 *Let $f : \mathcal{U} \to \mathbb{C}$ be a non-constant reduced analytic function defined in a neighborhood $\mathcal{U}$ of $\underline{0}$ in $\mathbb{C}^{n+1}$ with a critical point at $\underline{0}$. Then the Lefschetz number of the monodromy of f at $\underline{0}$ is zero.*

Proof of the Corollary One knows that the Lefschetz number of a diffeomorphism h of X into X is $\sum_k (-1)^k Trace(h_k)$ where $Trace(h_k)$ is the trace of the linear endomorphism induced by h on the k-th homology of X. Then there is a known Theorem due to Lefschetz that if the diffeomorphism h has no fixed point, then its Lefschetz number is 0 (see *e.g.* [23] p. 179). □

Proof of the Theorem The proof of the theorem uses in an essential way the polar curve and the polar discriminant introduced above (see Theorem 6.7.3). For the sake of understanding we repeat here what is needed for our proof. □

Consider the germ of holomorphic map $(f, \ell) : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}^2, 0)$ where ℓ is a general linear form. Then, the critical space of the germ of (f, ℓ) is the germ union of the critical locus of f and maybe a curve germ $(\Gamma, \underline{0})$ which is called the polar curve when it is not empty.

In the case Γ is not empty the restriction of the germ (f, ℓ) to Γ is finite at $\underline{0}$, so by a Theorem of Remmert (see for instance, [53, S5, Chapter V]), the image is a germ $(\Delta, 0)$ of curve in $(\mathbb{C}^2, 0)$. The main observation is that, when $df(\underline{0}) = 0$, the curve Δ is tangent to the line $\mathbb{C} \times \{0\}$ (see Proposition 6.7.5).

Now we proceed to proving Theorem 6.9.1. We make an induction on n. If $n = 0$, the function f is locally at $\underline{0}$ isomorphic to z^k, then f is singular at the point $\underline{0}$ if and only if $k \geq 2$. The Theorem is obvious in this case.

Now let $n \geq 1$. We may make the induction hypothesis that, for any $q \leq n - 1$, there is a geometric monodromy of $g : (\mathbb{C}^{q+1}, \underline{0}) \to (\mathbb{C}, 0)$ at $\underline{0}$ having no fix point if $\underline{0}$ is a critical point of g. Let us consider a general linear form ℓ of $\mathbb{C}^{n+1}$ and the corresponding germ:

$$(f, \ell) : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}^2, 0).$$

Let us assume that the corresponding polar curve Γ is not empty. According to what we prove above the polar discriminant Δ is tangent at 0 to the line $\mathbb{C} \times \{0\}$.

Since the components of Δ are tangent to $\mathbb{C} \times \{0\}$, by denoting (t, u) the coordinates of $\mathbb{C}^2$ the rotations inside the annuli of the carousel of Δ with respect to the coordinates (t, u) (see Definition 6.8.2) have angles $< 2\pi$ and between the annuli the mapping is going increasingly from an angle to another one. Therefore, the only possible fixed point of a geometric monodromy of $f^{-1}(t) \cap \mathbb{B}_\varepsilon$ would lie in $f^{-1}(t) \cap \{\ell = 0\}$ (see [40]). By induction in the generic hyperplane section there is a geometric monodromy which has no fixed point.

When the polar curve Γ is empty a geometric monodromy of the fibration φ is a product of a geometric monodromy of the restriction of φ to a general hyperplane section by the identity of a disc centered at 0. Then, by induction we know that there is a geometric monodromy of the restriction of φ to a general hyperplane section which has no fixed point if $\underline{0}$ is a critical point.

6.9.2 The Monodromy Theorem

There is an important theorem concerning the monodromy of the Milnor fibration of a complex analytic function near a critical point.

Theorem 6.9.3 *Let $f : (\mathbb{C}^n, \underline{0}) \to (\mathbb{C}.0)$ be a germ of complex analytic function at a point $\underline{0}$. Then the eigenvalues of the monodromy of the local Milnor fibration of f at $\underline{0}$ are roots of unity.*

In fact this theorem is true for any germ of function $f : (X, \underline{0}) \to (\mathbb{C}, 0)$ and the proof uses the notion of carousels (see [44]).

6.10 Diffeomorphism Type of the Milnor Fiber

Milnor noticed in [65, Theorem 6.6] that in the isolated singularity case, the fact that the fiber F_f has the homotopy type of a bouquet of n-spheres, together with Smale's h-cobordism theorem, actually imply that in high dimensions F_f is diffeomorphic to a $2n$-ball with μ n-handles attached, where μ is the Milnor number of f which is equal to the intersection number:

$$\mu = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+1,\underline{0}}}{\mathrm{Jac\,f}} ,$$

where $\mathcal{O}_{n+1,\underline{0}}$ is the local ring of germs of holomorphic functions on $\mathbb{C}^{n+1}$ at $\underline{0}$; Jac f is the Jacobian ideal, generated by the derivatives $\big(\partial f/\partial z_0, \dots, \partial f/\partial z_n\big)$.

This claim also holds easily for $n = 1$. Milnor conjectured that the same statement held for $n = 2$ [65, last phrase of p.58]. This was proved by Lê and Perron in [50] by a different method that works in all dimensions, in the vein of Sect. 6.7. That is the first statement in the theorem below, and it is now an immediate consequence of the above Theorem 6.7.8:

Theorem 6.10.1 *Let $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ be a holomorphic map-germ and let $F_f = f^{-1}(0) \cap \mathbb{B}_\varepsilon$ be its Milnor fiber. If f has an isolated critical point, then:*

- *F_f has the homotopy type of a bouquet $\bigvee \mathbb{S}^n$ of spheres of middle dimension, and it is actually diffeomorphic to a closed ball $\mathbb{B}^{2n}$ to which we attach μ handles of index n.*
- *There is in F_f a polyhedron P of middle dimension which is a deformation retract of F_f and, setting $V := (f^{-1}(0) \cap \mathbb{B}_\varepsilon)$, there is a continuous (collapsing) map $F_f \to V$ that carries P into $\underline{0}$ and is a homeomorphism in the complement of P. Hence the collapsing map exhibits the Milnor fiber as being a topological resolution of the singularity of V.*

Recall that Theorem 6.3.1 says that for the Pham-Brieskorn polynomials, the Pham join is a deformation retract of the Milnor fiber. The second statement in the Theorem above extends that construction to all isolated hypersurface germs. Here, by a polyhedron P we mean a compact topological space that is triangulable.

Proof The second statement of Theorem of 6.10.1 is given in [43] with a sketch of its proof. A complete proof is given in [48]. It is worth saying that the same construction of a vanishing polyhedron works for certain families of complex line singularities (see [63]) and also for certain families of real analytic singularities (see [8]).

The proof of the first statement is very technical. So we shall only give here a hint of the proof which make use of the polar curve and polar discriminant.

Since $(\mathbb{C}^{n+1}, 0)$ is non-singular, Theorem 6.7.1 shows that the polar curve Γ_ℓ of f with respect to the linear form ℓ, when ℓ is generic, is reduced and Γ_ℓ coincides with set of the points of U) where (f, ℓ) is not submersive and where U is a small neighborhood of $\underline{0}$ in $\mathbb{C}^{n+1}$. Moreover, if the neighborhood U_0 of $(0, 0)$ in $\mathbb{C}^2$ is small enough, (f, ℓ) induces a locally trivial fibration:

$$(f, \ell)^{-1}(U_0 - \Delta_\ell) \cap U \to U_0 - \Delta_\ell$$

where $(\Delta_\ell, 0)$ is the image of $(\Gamma_\ell, 0)$ by (f, ℓ).

From Theorem 6.7.8 we know that, if $\varepsilon > 0$ is small enough, with t, such that $\varepsilon \gg |t| > 0$, the Milnor fiber $\{f = t\} \cap \mathbb{B}_\varepsilon$ is obtained from a tubular neighborhood of $\{f = t\} \cap \{\ell = 0\}$ by attaching $\mu(f) + \mu(f, \ell)$ n-handles.

We are going to prove here that in this case of a hypersurface with isolated singularity at $\underline{0}$, the Milnor fiber is obtained from a ball $\mathbb{B}^n$ by attaching $\mu(f)$ n-handles.

The idea of the proof is quite simple. Let us choose U_0 to be the open ball $\mathring{\mathbb{B}}_\delta^2$ with $\delta > 0$ small enough. Let us choose $u_0 \neq 0$ small enough such that the number of points $\{\ell = u_0\} \cap \Gamma_\ell \cap (f, \ell)^{-1}(\mathring{\mathbb{B}}_\delta^2)$ equals the multiplicity $m_{(0,0)}(\Delta_\ell)$ of Δ_ℓ at $(0, 0)$, since (f, ℓ) induces an injective map of $\Gamma_\ell \cap (f, \ell)^{-1}(\mathring{\mathbb{B}}_\delta^2) \cap \mathbb{B}_\varepsilon$ into $\mathring{\mathbb{B}}_\delta^2$ and according to Proposition 6.7.5 the line $\{u = u_0\}$, where u is the second coordinate of $\mathbb{C}^2$ is transverse to Δ_ℓ, furthermore all the points of $\{\ell = u_0\} \cap \Gamma_\ell$ lie in $(f, \ell)^{-1}(\mathring{\mathbb{B}}_\delta^2)$.

Now, the Milnor fiber $\{\ell = u_0\} \cap \mathbb{B}_\varepsilon$ of ℓ at $\underline{0}$ is diffeomorphic to an n-ball $\mathbb{B}^n$. Consider in $\mathring{\mathbb{B}}_\delta^2$ the intersection of the line $\{u = u_0\}$. Choose a disc $\mathbb{D} \times \{u_0\}$ in this line which contains all the intersection points $\{u = u_0\} \cap \Delta_\ell \cap \mathring{\mathbb{B}}_0^2$. It is possible if $u_0 \neq 0$ is small enough and the disc $\mathbb{D} \times \{u_0\}$ lies in $\mathring{\mathbb{B}}_\delta^2$ (Fig. 6.1).

Then, one notices that $(f, \ell)^{-1}(\mathbb{D} \times \{u_0\}) \cap \mathbb{B}_\varepsilon$ is diffeomorphic to a Milnor fiber of ℓ at $\underline{0}$, *i.e.*, to a ball $\mathbb{B}^n$.

We apply the swing to the disc $\mathbb{D} \times \{u_0\}$ (see Sect. 4 of [47]) into the line $\{\tau = t\}$ where t is small enough. It gives a dislike domain $\{t\} \times D'$ which intersects $\Delta_\ell \cap \mathring{\mathbb{B}}_\delta^2$ in $m_{(0,0)}(\Delta_\ell)$ points.To simplify we shall assume that the point (t, u_0) belongs to $\{t\} \times \mathring{D}'$

Then choose a disc $\{t\} \times \mathbb{D}_0$ centered at $(t, 0)$ and contained in $\{\tau = t\} \cap \mathring{\mathbb{B}}_\delta^2$ such that the cardinality of $(\{t\} \times \mathbb{D}_0) \cap \Delta_\ell$ is equal to the intersection number $(\{\tau = 0\}.\Delta_\ell)_{(0,0)}$.

We proceed like in the proof of Theorem 6.7.8. First we must find a Morse function between $(f, \ell)^{-1}(\{t\} \times D') \cap \mathbb{B}_\varepsilon$ and $(f.\ell)^{-1}(\{t\} \times \mathbb{D}_0) \cap \mathbb{B}_\varepsilon$.

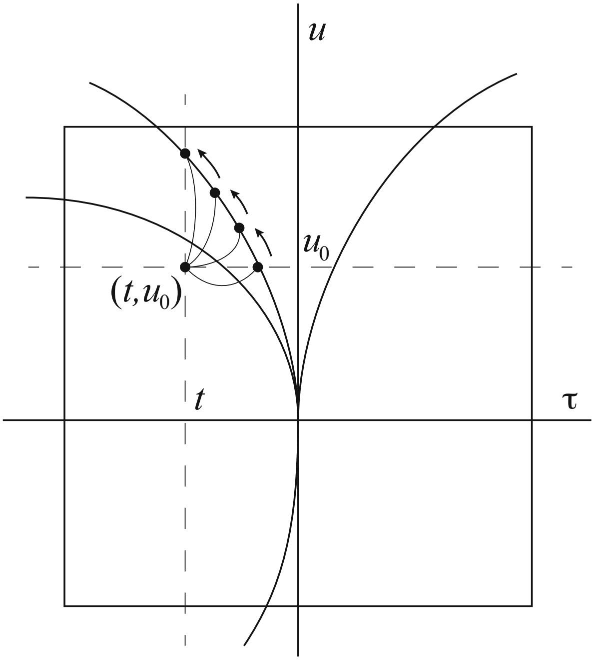


Fig. 6.1 The arrows show the swinging of a point on $u = u_0$ into a point on $\tau = t$. This is realized by a vector field whose lifting gives the swinging of the attached cells or the attached handles

We notice that $\mathbb{D}_0 - \overset{\circ}{D}'$ is diffeomorphic to $\mathbb{D}_b - \overset{\circ}{\mathbb{D}}_a$ where $\mathbb{D}_a$ and $\mathbb{D}_b$ are discs centered at 0 with radii a and b, $b > a > 0$:

$$\varphi : \mathbb{D}_0 - \overset{\circ}{D}' \overset{\simeq}{\to} \mathbb{D}_b - \overset{\circ}{\mathbb{D}}_a$$

On the annulus $\mathbb{D}_b - \overset{\circ}{\mathbb{D}}_a$ we have a natural function which is the distance ρ to the origin. The composition $\rho \circ \varphi$ defines a smooth function on $\mathbb{D}_0 - \overset{\circ}{D}'$.

The composition $\rho \circ \varphi \circ (f, \ell)$ induces on $(f, \ell)^{-1}(\{t\} \times \mathbb{D}_0) \cap \mathbb{B}_\varepsilon$ a Morse function θ:

Lemma 6.10.2

1. *$\theta^{-1}(a) = (f, \ell)^{-1}(\{t\} \times D') \cap \mathbb{B}_\varepsilon$ and $\theta^{-1}(b) = (f, \ell)^{-1}(\{t\} \times \mathbb{D}_0) \cap \mathbb{B}_\varepsilon$;*
2. *the critical values $\leq b$ of θ are the images of the critical points of (f, ℓ) in $(f, \ell)^{-1}(\{t\} \times \mathbb{D}_0)$ by $\rho \circ \varphi \circ (f, \ell)$;*
3. *Since the critical points of $(f.\ell)$ in $\mathbb{B}_\varepsilon - f^{-1}(0)$ are ordinary quadratic, all the critical points of θ are Morse points of indices n;* □

Exercise 6.10.3 Prove Lemma 6.10.2. □

From Lemma 6.10.2, by using Morse theory we obtain that $\{\theta \leq b\} = (f, \ell)^{-1}(\{t\} \times \mathbb{D}_0) \cap \mathbb{B}_\varepsilon$ is obtained from $\{\theta \leq a\} = (f, \ell)^{-1}(\{t\} \times D') \cap \mathbb{B}_\varepsilon$ by attaching n-handles.

The number of handles is the number of critical points of θ in $\{\theta \leq b\} - \{\theta \leq a\}$. This number is the difference between the intersection number $(V(f).\Gamma_\ell)_{(0,0)}$ and the number of critical points of the restriction of (f, ℓ) to $\{f = t\} \cap (f, \ell)^{-1}(\{t\} \times D') \cap \mathbb{B}_\varepsilon$. By the swing this last number is the multiplicity $m_{(0,0)}(\Delta_\ell)$. An algebraic computation gives:

$$(V(f).\Gamma_\ell)_{(0,0)} = \mu(f) + \mu(f, \ell)$$

(see Theorem 6.7.8) and $m_{(0,0)}(\Delta_\ell) = \mu(f, \ell)$. This gives a hint of the proof of Theorem 6.10.1. □

Theorem 6.7.8 also leads naturally to the definition of the Lê numbers, introduced by Massey in [56, 59].

We know from [65, Theorem 5.1] that the Milnor fiber F_f of an arbitrary holomorphic map-germ $(\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ has the homotopy type of a finite CW-complex of middle-dimension n. This also follows from [2] since F_f is a Stein manifold and, perhaps moving the origin $\underline{0}$ slightly if necessary, the square of the function distance to $\underline{0}$ is a strictly plurisubharmonic Morse function on F_f, so one has severe restrictions on the possible Morse indices (see Theorem 6.4.7 above).

If we further assume that f has an isolated critical point at $\underline{0}$, then we already know, from Theorem 6.10.1, that the Milnor fiber F_f is diffeomorphic to a $2n$-ball with μ n-handles attached where $\mu = \mu(f)$ is the Milnor number of f (cf. Sect. 6.10). As we said before, this was proved by Milnor for $n \neq 2$ and by Lê-Perron in general using the technique discussed previously, in Sect. 6.7, *i.e.*, considering an auxiliary linear function $\ell : \mathbb{C}^{n+1} \to \mathbb{C}$, which is "sufficiently general" with respect to f. The two maps together determine a map-germ

$$\varphi = (f, \ell) : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}^2, 0)\,,$$

and the Milnor fiber of f corresponds to the inverse image of an appropriate line in $\mathbb{C}^2$. This allows us to reconstruct F_t by looking at the slices determined by the level hyperplanes of ℓ as described in Theorem 6.7.8.

This brings us to the theory of "polar varieties" developed by Bernard Teissier and Lê Dũng Tráng in the 1970s and briefly discussed in Sect. 6.7.

Recall one has the first the relative polar curve of f with respect to a linear form, $\Gamma^1_{f,\ell}$ (see for instance [32, 37, 39, 74–76]). Given f and ℓ as above, as a set the curve $\Gamma^1_{f,\ell}$ is the union of those components in the critical set of (f, ℓ) which are not in Σf, the critical points of f. In other words, assume we have coordinates $(z_0, \cdots, z_n)$ so that the linear function is $\ell = z_0$ is "sufficiently general". Then the critical locus of (f, ℓ) is $V(\partial f/\partial z_1, \cdots, \partial f/\partial z_n)$, the set of points where $\partial f/\partial z_i = 0$ for all $i = 1, \cdots, n$. Now write the cycle represented by $V(\partial f/\partial z_1, \cdots, \partial f/\partial z_n)$ as a formal sum over the irreducible components:

$$\Big[V(\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n})\Big] = \sum n_i [V_i]$$

Then $\Gamma^1_{f,\ell}$, as a cycle, is defined by:

$$\Gamma^1_{f,\ell} = \sum_{V_i \not\subseteq \Sigma f} n_i [V_i] \, .$$

More generally we may consider a linear functional $\mathbb{C}^{n+1} \to \mathbb{C}^r$. This gives rise to a polar variety relative to f, determined by the points of non-transversality of the fibers of ℓ and f. Massey showed that this gives rise to certain local analytic cycles, that he called the Lê cycles, that depend on the choice of the linear functional ℓ, but they are all equivalent when the form is "general enough". These cycles encode deep topological properties of the Milnor fibration. Let us be more precise.

Let U be an open subset of $\mathbb{C}^{n+1}$ containing the origin, $f : (U, 0) \to (\mathbb{C}, 0)$ the germ of a complex analytic function, $z = (z_0, \cdots, z_n)$ a choice of linear coordinates in $\mathbb{C}^{n+1}$ and $\Sigma(f) = V\left(\frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n}\right)$ the critical set of f. To define the Lê cycles we first need to define the relative polar cycles, which are associated to the relative polar varieties: For each k with $0 < k < n$, the polar variety $\Gamma^k_{f,z}$ is the analytic space $V\left(\frac{\partial f}{\partial z_k}, \dots, \frac{\partial f}{\partial z_n}\right) / \Sigma(f)$, where X/Y means the analytic closure of $X - Y$ in the sense of [13, p. 41]). Hence the analytic structure of $\Gamma^k_{f,z}$ does not depend on the structure of $\Sigma(f)$ as a scheme, but only as an analytic set. At the level of ideals, $\Gamma^k_{f,z}$ consists of those components of $V\left(\frac{\partial f}{\partial z_k}, \dots, \frac{\partial f}{\partial z_n}\right)$ which are not contained in the set $\Sigma(f)$. Massey denotes by $[\Gamma^k_{f,z}]$ the cycle associated with the space $\Gamma^k_{f,z}$ (see [59, p. 9]).

Then, for each $0 < k < n$, Massey defines the k-th Lê cycle $\Lambda^k_{f,z}$ of f with respect to the coordinate system z as the cycle:

$$\Lambda^k_{f,z} := \left[\, \Gamma^{k+1}_{f,z} \cap V\left(\frac{\partial f}{\partial z_k}\right) \right] - [\, \Gamma^k_{f,z} \,].$$

If a point $p = (p_0, \cdots, p_n) \in U$ is an isolated point of the intersection of $\Lambda^k_{f,z}$ with the cycle of $V(z_0 - p_0, \cdots, z_{k-1} - p_{k-1})$, then the Lê number $\lambda^k_{f,z}(p)$ is the intersection number at p:

$$\lambda^k_{f,z}(p) := (\Lambda^k_{f,z} \cdot V(z_0 - p_0, \dots, z_{k-1} - p_{k-1})\,)_p \, .$$

It is proved in [58, Theorem 7.5] (see also [59, Theorem 10.18]) that for a generic choice of coordinates, all the Lê numbers of f at p are defined and they are independent of the choice of coordinates. Hence these are called the (*generic*) *Lê numbers* of f at p.

We consider now a holomorphic map-germ $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$. We denote the generic Lê cycles by $\Lambda^k_{f,\ell}(\underline{0})$.

If the singularity is isolated, then there is only one generic Lê number and it coincides with the Milnor number. Massey's theorem (see [59, Theorem 3.3] or [60, Theorem 3.1]) tells us how to build up the Milnor fiber by successively attaching handles of various dimensions. This is the described in the theorem below.

Theorem 6.10.4 *Let* $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ *be a holomorphic map-germ and let* F_f *be its Milnor fiber.*

- *If the complex dimension* s *of its critical set is* $s \leq n-2$*, then* F_f *is obtained up to diffeomorphism, from a* $2n$*-ball by successively attaching* $\lambda^{n-k}_{f,\ell}(\underline{0})$ k*-handles, where* $n-s \leq k \leq n$ *and* $\lambda^{n-k}_{f,\ell}(\underline{0})$ *is the* $(n-k)^{th}$ *Lê number.*
- *If the complex dimension of its critical set is* $s = n-1$*, then* F_f *is obtained up to diffeomorphism, from a real* $2n$*-manifold with the homotopy type of a bouquet of* $\lambda^{n-1}_{f,\ell}(\underline{0})$ *circles, by successively attaching* $\lambda^{n-k}_{f,\ell}(\underline{0})$ k*-handles, where* $2 \leq k \leq n$*.*

The proof of this remarkable theorem is based on Massey's papers [56, 57].

6.11 A General Fibration Theorem

We now let X be an arbitrary complex analytic subset of an open neighbourhood U of the origin $\underline{0}$ in $\mathbb{C}^N$. Let $f : (X, \underline{0}) \to (\mathbb{C}, 0)$ be holomorphic and set $V = f^{-1}(0)$.

We have seen in Proposition 6.2.9 and Theorem 6.4.1 that under the hypothesis $X - f^{-1}(0)$ is non-singular we can define a local smooth fibration associated to f.

In this section we do not make any assumption on X nor on the holomorphic function germ $f : (X, \underline{0}) \to (\mathbb{C}, 0)$. A general fibration theorem will be given.

Let us recall first well-known material about stratified analytic spaces (see *e.g* [83] sections 18 and 19).

Let X be a subset of a smooth manifold M. A *stratification* of X of is a *locally finite* partition $\{S_\alpha\}$ of X into smooth, connected submanifolds of M (called *strata*) which satisfy that if S_α and S_β are strata with $S_\alpha \cap \bar{S}_\beta \neq \emptyset$, then $S_\alpha \subset \bar{S}_\beta$. When M is a real analytic manifold, the stratification $\{S_\alpha\}$ is *real analytic* if all the strata are real analytic submanifolds whose closure is semianalytic. Analogously, when M is a complex manifold, the stratification $\{S_\alpha\}$ is *complex analytic* if all the strata are complex submanifolds whose closure is complex analytic. Along this chapter, we will assume that all the stratifications are analytic (either real or complex, depending on the context), unless otherwise specified.

Now consider a triple (y, S_α, S_β), where S_α and S_β are strata of X with $y \in S_\alpha \subset \bar{S}_\beta$. We say that the triple (y, S_α, S_β) is *Whitney regular* if it satisfies the *Whitney (b) condition*: every limit of secants is contained in the limit of tangent spaces. More precisely, we may assume that the germ $(\bar{S}_\beta, y)$ is embedded in $(\mathbb{C}^N, y)$, then for every sequence $\{x_n\} \subset S_\beta$ converging in $\mathbb{C}^N$ to $y \in S_\alpha$ such that the sequence of tangent spaces $T_{x_n}S_\beta$ converges to a subspace $T \subset \mathbb{C}^N$, and every sequence $\{y_n\} \subset S_\alpha$ converging to $y \in S_\alpha$ such that the sequence of lines (secants) $l_{x_i y_i}$ from x_i to y_i converges to a line l, one has:

$$l \subset T.$$

The stratification $\{S_\alpha\}$ of X is *Whitney regular* (also called a *Whitney stratification*) if every triple (y, S_α, S_β) as above, is Whitney regular. The existence of Whitney stratifications for every analytic space X was proved by Whitney in [83, Theorem 19.2] for complex varieties, and by Hironaka [25] for real and complex analytic spaces. In fact, every stratification can be refined to become Whitney regular.

In Sect. 6.2 we proved the existence of Milnor balls when X has an isolated singularity. This actually holds in full generality (see [6]):

Theorem 6.11.1 *Let X be a real analytic space in some $\mathbb{R}^N$, $\underline{0}$ a point in X and $\mathcal{S} := \{S_\alpha\}$ a Whitney stratification of X. Assume for simplicity that $\{\underline{0}\}$ is a point-stratum. Then there exists $\varepsilon > 0$ sufficiently small, such that:*

1. *For each ε' such that $\varepsilon \geq \varepsilon' > 0$ one has that the sphere $\mathbb{S}_{\varepsilon'}$ in $\mathbb{R}^N$ centered at $\underline{0}$ with radius ε' intersects transversally every stratum S_α that has $\underline{0}$ in its closure; and*
2. *For each ε' as above, there are homeomorphisms of pairs $(\mathbb{S}_\varepsilon, X \cap \mathbb{S}_\varepsilon) \cong (\mathbb{S}_{\varepsilon'}, X \cap \mathbb{S}_{\varepsilon'})$ and $(\mathbb{B}_\varepsilon, X \cap \mathbb{S}_\varepsilon) \cong \operatorname{Cone}(\mathbb{S}_\varepsilon, X \cap \mathbb{S}_\varepsilon)$, where $\mathbb{B}_\varepsilon$ is the ball bounded by $\mathbb{S}_\varepsilon$.*

Hence one has that the topology of the pair $(\mathbb{S}_\varepsilon, X \cap \mathbb{S}_\varepsilon)$ does not depend on the choice of ε provided this is small enough.

Definition 6.11.2 Every sphere $\mathbb{S}_\varepsilon$ (or ball $\mathbb{B}_\varepsilon$) as above is called a *Milnor sphere (or ball)* for X at $\underline{0}$, and the intersection $L_{X,\underline{0}} := X \cap \mathbb{S}_\varepsilon$ is the link of X at the given point $\underline{0}$.

When it is clear what the point $\underline{0}$ is, for simplicity we denote the link just by L_X.

The proof of Theorem 6.11.1 mimics what we did in the isolated complex singularity case, now using Whitney regularity. However notice that we obtain a continuous vector field which is integrable.

In fact Whitney's condition (b) implies that if (z_n) is a sequence of points in a Whitney stratumS_α of X which tends to $\underline{0}$, and we consider a sequence $(T_{z_n}(X))$ of tangent spaces of X at z_n, which has a limit T and the sequence of lines $l_{\underline{0},z_n}$ has a limit ℓ, then we have:

$$\ell \subset T.$$

This allows us construct on S_α a vector field v_α with no singularities, and such that:

i) At each point $z \in S_\alpha$ it is tangent to S_α, transverse to the sphere centered at $\underline{0}$ and passing through z, pointing outwards; and
ii) it extends to a continuous vector field in the ambient space, with a unique singular point at $\underline{0}$.

We now observe that, as shown by Mather in [62], the Whitney condition (b) implies Whitney's condition (a): given a sequence (z_n) of points in a Whitney stratum S_α of X which tends to a given point x_1 in some stratum S_β, and such that the sequence $(T_{z_n}(X))$ of tangent spaces of X at z_n has a limit T, then T contains the tangent space of S_β at x_1.

This implies that the vector field v_α that we constructed on S_α, extends to a continuous vector field in a neighbourhood of S_α in $\mathbb{R}^N$ in such a way that it is transversal to every sufficiently small sphere centered at $\underline{0}$ and it is a stratified vector field, *i.e.*, at each point it is tangent to the corresponding stratum.

We may now use a partition of unity to glue these vector fields on the various strata that contain x_o in their closure, to obtain a continuous vector field ξ on a neighborhood of $\underline{0}$ in $\mathbb{R}^N$, such that it is stratified, with an isolated zero at $\underline{0}$ and it is radial, *i.e.*, it is transversal to every sufficiently small sphere around x_o, pointing outwards.

Doing this with more care (see for instance [80]) we can furthermore assume that the vector field ξ actually is integrable. Then the integral lines of ξ yield the homeomorphisms stated in Theorem 6.11.1.

In the case the singularities of f are not isolated, there is a fibration theorem similar to Proposition 6.2.9.

From now on we assume again that X is complex analytic space with a singularity at, say, $\underline{0}$ and $f : (X, \underline{0}) \to (\mathbb{C}, 0)$ is a holomorphic function with a possibly non-isolated singularity at $\underline{0}$.

We have the following general fibration theorem from [40]:

Theorem 6.11.3 *Let X be a complex analytic space with a possibly non-isolated singularity at $\underline{0}$ and $f : (X, \underline{0}) \to (\mathbb{C}, 0)$ a non-constant holomorphic function. Then there is $\varepsilon > 0$ small enough and $\delta > 0$ such that $\varepsilon \gg \delta > 0$, so that:*

$$X \cap \mathbb{B}_\varepsilon \cap f^{-1}(\mathbb{D}^*_\delta) \to \mathbb{D}^*_\delta$$

is a locally trivial topological fibration.

The proof of this theorem is essentially like the proof of Theorem 6.4.1 but we now use Hironaka's Theorem 6.4.5 combined with the first Thom-Mather Isotopy Lemma, as we explain below.

In fact, on the one hand, given X and f as above, we know from the theorem above that there is $\varepsilon > 0$ small enough and $\delta > 0$ small enough with respect to $\varepsilon_0 > 0$, so that $\mathbb{B}_{\varepsilon_0}$ is a Milnor ball for both X and $V := f^{-1}(0)$. Then, let (T_β) be a Whitney regular stratification of $X \cap \mathring{B}_{\varepsilon_0}$. By Hironaka's theorem 6.4.5 one can refine this stratification into a stratification of the mapping induced by f from $X \cap \mathring{B}_{\varepsilon_0} \cap f^{-1}(\mathring{\mathbb{D}}_{\delta_0})$ into $\mathring{\mathbb{D}}_{\delta_0}$, where the strata of $\mathring{\mathbb{D}}_{\delta_0}$ are $\mathring{\mathbb{D}}_{\delta_0} - \{0\}$ and $\{0\}$ such that the stratification (S_α) of $X \cap \mathring{B}_{\varepsilon_0} \cap f^{-1}(\mathring{\mathbb{D}}_{\delta_0})$ is Whitney regular and satisfies A_f condition (see Definition 6.4.4). Considering the Milnor ball $\mathbb{S}_\varepsilon$, $\varepsilon_0 > \varepsilon > 0$, transverse to all the strata S_α which contain $\underline{0}$ and its closure, the restriction of every fiber $f^{-1}(t)$ to each of these strata S_α, with $\delta \geq |t| > 0$ intersects transversally the boundary sphere $\mathbb{S}_\varepsilon = \partial \mathbb{B}_\varepsilon$.

We leave as an exercise a generalization of Lemma 6.2.6 to the case of a restriction of f to an analytic stratum S_α.

The first and second Thom-Mather isotopy lemmas were conjectured by Thom and proved by Mather. These are stated in [62, Section 11], where the first of these was proved. This can be stated as follows:

Theorem 6.11.4 (First Thom-Mather Isotopy Lemma) *Let A be a locally closed subset in a smooth manifold M which is equipped with a Whitney stratification $\mathcal{S}$ for which A is a union of strata. Let B be a manifold and $f : M \to B$ a differentiable function such that for each stratum S_α in A satisfies that the restriction $f|_{S_\alpha}$ is a submersion and its restriction to $\overline{S_\alpha} \cap A$ is proper. Then:*

$$f|_A : A \longrightarrow B\,,$$

is a topologically locally trivial fibration in the stratified sense. That is, for each $b \in B$, there exist a neighbourhood U of b in B and a homeomorphism $((f^{-1}(b) \cap A) \times U) \cong (f^{-1}(U) \cap A)$ that carries strata into strata.

Theorem 6.11.4 can be regarded as an extension of Ehresmann's fibration lemma to the case of singular varieties. Using Theorem 6.11.4 similarly to the way we used Ehresmann Lemma in Sect. 6.4, we arrive at Theorem 6.11.3.

Remark 6.11.5 Just as in the isolated singularity case described in Sect. 6.4, one has also two types of fibrations in the general setting envisaged here: one as in Theorem 6.11.3 and another on the link of L_X (minus the link of f) with projection map $f/|f|$. That these two fibrations are equivalent is outlined in [40] and a complete proof is given in [9] (see also [71]). In fact one has from [9, Theorem 2] and its proof, that given X and f as in Theorem 6.11.3, and a Milnor ball $\mathbb{B}_\varepsilon$, there is a locally trivial fibration $\Phi : \mathbb{B}_\varepsilon - f^{-1}(0) \to \mathbb{S}^1$ which has the two previous fibrations as sub-fibrations, and one can deform the fibers of one of these into the fibers of the other fibration along the fibers of Φ.

It is of course interesting to look at the topology of the Milnor fiber F_f of a holomorphic map germ f as in Theorem 6.11.3. The first step in that direction was Theorem 6.6.1 by H. Hamm in [20] (see also the paper of Lê in [42]), proving that if $(X, \underline{0})$ is an complex analytic isolated complete intersection singularity germ and f has an isolated singularity at $\underline{0}$, then F_f has the homotopy type of a bouquet of spheres of middle dimension. Hamm's theorem was improved by Lê in [45] relaxing the conditions on X: its germ at $\underline{0}$ needs to be a local complete intersection but it may have a possibly non-isolated singularity at $\underline{0}$. If f has an isolated singularity at $\underline{0}$ with respect to some Whitney stratification, then F_f has the homotopy type of a bouquet of spheres of middle dimension. The proof uses the techniques introduced in Sect. 6.7.

The next step is also due to Lê in [46], allowing the ambient space X to be more general, not necessarily a complete intersection. He proved that if (X, x) is an equidimensional analytic germ and its *rectified homotopical depth* $rhd(X, \underline{0})$ (see [22, Definition 1.1 and Theorem 1.4]) equals its dimension and if f has isolated singularity in the stratified sense, then the fiber F_f has the homotopy type of a bouquet of spheres of middle dimension.

Later, D. Siersma in [72] proved that if the germ $(X, \underline{0})$ is an isolated singularity (no condition on the $rhd(X, \underline{0})$), but the singularity is isolated), its dimension is $n \neq 3$ and f has an isolated singularity at $\underline{0}$, then F_f has the homotopy type of the wedge of the complex link F_l with a bouquet of spheres of middle dimension:

$$F_f \overset{ht}{\simeq} F_l \vee \bigvee \mathbb{S}^{n-1} ,$$

where F_l is the intersection of X with a general hyperplane section in an ambient $\mathbb{C}^N$ passing near $\underline{0}$.

In the same paper [72], Siersma conjectured a general bouquet theorem, which was proved by M. Tibăr in [78]:

Theorem 6.11.6 *Let $f : (X, \underline{0}) \to (\mathbb{C}, 0)$ be a holomorphic function on a reduced complex analytic germ $(X, \underline{0})$ with dim$(X, \underline{0}) = n$, $n > 2$. Let $\mathcal{S} = \{S_\alpha\}_{\alpha \in A}$ be a Whitney stratification of (X, x) for which f has an isolated singularity at $\underline{0}$. Then the Milnor fiber F_f has the homotopy type of a bouquet of repeated suspensions of complex links of strata.*

We refer to Tibar's paper for the precise statement of this theorem and its proof, which uses the carousel method explained previously.

6.12 Two Open Problems

6.12.1 Lê's Conjecture

In the Problems Section of [81], it appears a long standing open problem, which, according to the first named author, was originally stated by M. Oka as follows:

Conjecture 6.12.1 Let $(X, \underline{0})$ be a surface in $(\mathbb{C}^3, \underline{0})$. If the link of $(X, \underline{0})$ is homeomorphic to a sphere, then $(X, \underline{0})$ is the total space of a Whitney equisingular family of irreducible plane curves.

This question was reformulated by the first named author in terms of the injectivity of a holomorphic map germ from $(\mathbb{C}^2, \underline{0})$ to $(\mathbb{C}^3, \underline{0})$. The following is known as Lê's Conjecture:

Conjecture 6.12.2 If $f : (\mathbb{C}^2, \underline{0}) \to (\mathbb{C}^3, \underline{0})$ is holomorphic and injective, then f has rank ≥ 1 at $\underline{0}$.

Though Lê's conjecture has been intensively studied for the last 30 years, at the moment there are no general proof nor counterexample. An attempt to prove the conjecture in a more general version for maps $(\mathbb{C}^n, \underline{0}) \to (\mathbb{C}^{n+1}, \underline{0})$ but with some genericity restrictions was made by Nemethi in [67]. But Keilen and Mond found a gap in the proof which cannot be repaired (see [29]). A proof of Lê's Conjecture for a particular case of surfaces in $(\mathbb{C}^3, \underline{0})$ is due to Luengo and Pichon [55] and also Bobadilla has given a reformulation of the conjecture in terms of families of plane curves [12].

We prove here that Conjectures 6.12.1 and 6.12.2 are in fact equivalent, which is not obvious at all. Assume first that Conjecture 6.12.1 is true and let $f: (\mathbb{C}^2, \underline{0}) \to (\mathbb{C}^3, \underline{0})$ be holomorphic and injective. We first observe that f is a finite map and hence, its image $(X, \underline{0})$ is an irreducible surface in $(\mathbb{C}^3, \underline{0})$ by the finite mapping theorem. Moreover, the real analytic function $\rho = \| f \|^2$ satisfies that $\rho^{-1}(0) = \{\underline{0}\}$, so $\rho^{-1}(\epsilon) = f^{-1}(\mathbb{S}_\epsilon)$ is homeomorphic to $\mathbb{S}^3$, for $\epsilon > 0$ small enough. Since f is injective, $f(f^{-1}(\mathbb{S}_\epsilon)) = X \cap \mathbb{S}_\epsilon$ is also homeomorphic to $\mathbb{S}^3$. By Conjecture 6.12.1, $(X, \underline{0})$ is the total space of a Whitney equisingular family of irreducible plane curves. That is, we can choose coordinates in $(\mathbb{C}^3, \underline{0})$ such that for each $t \in \mathbb{C}$, $Y_t = (\mathbb{C}^2 \times \{t\}) \cap X$ is a reduced curve and $\mu(Y_t, \underline{0})$ is independent of t (when considered as a plane curve in $\mathbb{C}^2 \times \{t\}$).

This also implies that $(X, 0)$ is a δ-constant family of plane curves. By a theorem of Lê, Lejeune and Teisser [52], the family $(X, \underline{0})$ admits a normalization in family, that is, there exists a normalization map $n: (\mathbb{C}^2, \underline{0}) \to (X, \underline{0})$ such that such that for each $t \in \mathbb{C}$, $n_t(s) := n(s, t)$ is the normalization map of Y_t. In particular, n is written as $n(s, t) = (n_1(s, t), n_2(s, t), t)$ and it has rank ≥ 1 at the origin. On the other hand, the restriction $f: (\mathbb{C}^2, \underline{0}) \to (X, \underline{0})$ is also a normalization map. By the uniqueness of the normalization, there exists a biholomorphism $\varphi: (\mathbb{C}^2, \underline{0}) \to (\mathbb{C}^2, \underline{0})$ such that $n \circ \varphi = f$. By the chain rule, f also has rank ≥ 1 at the origin.

Conversely, assume now that Conjecture 6.12.2 is true and let $(X, \underline{0})$ be a surface in $(\mathbb{C}^3, \underline{0})$ whose link is homeomorphic to $\mathbb{S}^3$. Since the link of $(X, \underline{0})$ is connected, $(X, \underline{0})$ is irreducible and it has a normalization map $n: (\overline{X}, \underline{0}) \to (X, \underline{0})$. By definition, the normalization map is generically injective, but we will prove that in this case, it is in fact injective. If not, we would find analytic arcs $x(t)$ and $y(t)$ in $\overline{X}$ with $x(0) = y(0) = \underline{0}$ and $z(t)$ in X with $z(0) = \underline{0}$ such that $n(x(t)) = n(y(t)) = z(t)$ for all t and $x(t) \neq y(t)$ for all $t \neq 0$. On one hand, the hypothesis on $(X, 0)$ implies that X is a topological manifold in a neighbourhood of $z(t)$ and thus X is irreducible at $z(t)$. On the other hand, the map-germs $n: (\overline{X}, x(t)) \to (X, z(t))$ and $n: (\overline{X}, y(t)) \to (X, z(t))$ are both finite, so they must be both surjective. But this would imply that n is not generically injective, giving a contradiction. It follows that $n: (\overline{X}, \underline{0}) \to (X, \underline{0})$ is injective and hence, a homeomorphism.

The real analytic function $r = \|n\|^2: (\overline{X}, 0) \to (\mathbb{R}, 0)$ defines $\underline{0}$ in $\overline{X}$ in the sense of [54, (2.3)] and hence $r^{-1}(\epsilon) = n^{-1}(\mathbb{S}_\epsilon \cap X)$ is diffeomorphic, for a small enough $\epsilon > 0$, to the link of $\overline{X}$ at $\underline{0}$ with respect to the Euclidean distance, by [54, (2.5)]. In particular, the link of $(\overline{X}, \underline{0})$ is also homeomorphic to $\mathbb{S}^3$. By Mumford's Theorem [66], $(\overline{X}, \underline{0})$ is smooth and there exists a biholomorphism $\varphi: (\overline{X}, \underline{0}) \to (\mathbb{C}^2, \underline{0})$. Let $f: (\mathbb{C}^2, 0) \to (\mathbb{C}^3, \underline{0})$ be the map $f = i \circ n \circ \varphi^{-1}$, where i is the inclusion map of $(X, \underline{0})$ into $(\mathbb{C}^3, \underline{0})$. This map is injective, so f has rank ≥ 1 at the origin, by Conjecture 6.12.2. We choose coordinates in $(\mathbb{C}^2, \underline{0})$ and $(\mathbb{C}^3, \underline{0})$ such that f is written in the form $f(s,t) = (f_1(s,t), f_2(s,t), t)$. If f has rank 2, then $(X, 0)$ is smooth. So we can assume that f has rank 1 and that $f_1, f_2 \in \mathfrak{m}_2^2$.

The restriction $f\colon (\mathbb{C}^2, \underline{0}) \to (X, \underline{0})$ is the normalization map and for each $t \in \mathbb{C}$, $f_t(s) := f(s,t)$ is also the normalization of the reduced plane curve $Y_t = (\mathbb{C}^2 \times \{t\}) \cap X$. That is, we have a normalization in family and hence, the family is δ-constant, by the theorem of Lê, Lejeune and Teisser [52]. Therefore, $\delta(Y_t)$ is independent of t, where

$$\delta(Y_t) := \sum_{x \in S(Y_t)} \delta(Y_t, x),$$

and $S(Y_t)$ is the singular set of Y_t. Moreover, all the curves Y_t are irreducible at any point. By Milnor formula,

$$\mu(Y_t) := \sum_{x \in S(Y_t)} \mu(Y_t, x) = \sum_{x \in S(Y_t)} 2\delta(Y_t, x) = 2\delta(Y_t).$$

Hence, $\mu(Y_t)$ is also constant and we denote this number by μ_0.

Let $H = 0$ be the reduced equation of $(X, 0)$ and let $h_t(z_1, z_2) := H(z_1, z_2, t)$, so that $h_t = 0$ is the reduced equation of Y_t. Let $S(h_t)$ be the singular set of h_t. For $t = 0$, $\mu(h_0, \underline{0}) = \mu(Y_0, \underline{0}) = \mu(Y_0) = \mu_0$ and for $t \neq 0$,

$$um_{x \in S(h_t)} \mu(h_t, x) = \mu_0 = \mu(Y_t) = \sum_{x \in S(h_t) \cap Y_t} \mu(h_t, x).$$

Thus, for all $x \in S(h_t)$, $h_t(x) = 0$. By a result of Gabrielov [14] and Lê [34], this implies that h_t has only one singularity, that is, $S(h_t) = S(Y_t) = \{\underline{0}\}$, with $\mu(Y_t, \underline{0}) = \mu_0$. Therefore, the family is Whitney equisingular.

6.12.2 Lê-Ramanujam

In the 1960s Zariski introduced the concept of *equisingularity*. This refers to a relation of equivalence which formalizes the intuitive idea of singularities of "the same type" in some sense. Lê-Ramanujam's theorem in [51] fits within that general framework. This states that for families of hypersurfaces of dimension $\neq 2$ and with isolated critical point, the invariance of the Milnor number implies (and therefore is equivalent to) the invariance of the topological type. To be precise:

Theorem 6.12.3 (Lê-Ramanujam) *Consider an analytic family $\{f_t\} : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ of germs of holomorphic functions with t in the unit disc* d *in $\mathbb{C}$, each having an isolated critical point at $\underline{0}$. If $n \neq 2$ and the $\{f_t\}$ have constant Milnor number at $\underline{0}$, then, for any t, there is a Milnor ball $\mathbb{B}_{\varepsilon_t}$ for $f_t = 0$ at $\underline{0}$, such that all the pairs $(\mathbb{B}_{\varepsilon_t}, f_t^{-1}(0) \cap \mathbb{B}_{\varepsilon_t})$ are homeomorphic.*

The proof of this theorem shows that under the above conditions, the links L_t of the f_t are all h-cobordant. Then we know from Smale's theorem in [73] that for $n > 2$ the links actually are diffeomorphic, and this leads to the stated theorem. The case $n = 1$ has to be considered separately and was proved in [31].

The Lê-Ramanujam Theorem 6.12.3 is known to be true also for $n = 2$ in some particular cases:

1. When the map-germ f_0 is weighted homogeneous. In this case, a theorem of Varchenko [79] says that the family $\{f_t\}$ must be a deformation of non-negative weight in the sense of Damon [10]: this means that all the monomials in f_t must have weighted degree greater than or equal to that of f_0. Then we can apply a general theorem due to Damon [10] to conclude that the family is topologically trivial.
2. When the family is linear in the parameter t, that is, when $f_t = f_0 + th$, for some fixed function h. This was proved by Parusinski in [68].

It is still an open problem whether the Lê-Ramanujam Theorem holds for $n = 2$. It also remains an open problem to know whether there exists a uniform radius for the Milnor balls $\mathbb{B}_{\varepsilon_t}$ in Theorem 6.12.3 for $n \neq 1$. For $n = 1$, Zariski proved [84, Theorem 8.1] that an analytic family of plane curves is equisingular if and only if it is equimultiple along the singular locus which is non-singular and we have the Whitney condition. On the other hand, in this dimension μ constant implies constant multiplicity and equisingularity (by [31]), and the Whitney condition (by [84, Theorem 8.1]). Hence for $n = 1$ using the Whitney condition, one does have Milnor balls of uniform radii for families with constant Milnor number.

Acknowledgments The second named author acknowledges support from MICINN Grant PGC2018–094889–B–I00 and GVA Grant AICO/2019/024. The third named author acknowledges support from CONACYT (Mexico) project 282937, and UNAM-PAPIIT project IN 110517.

References

1. N. A'Campo, *Le nombre de Lefschetz d'une monodromie,* Indag. Math. **35** (1973), 113–118. 325 and 371
2. A. Andreotti, Th. Frankel, *The Lefschetz theorem on hyperplane sections,* Ann. Math. (2) 69, 713–717 (1959). 324, 340, 345, and 376
3. K. Brauner, *Zur Geometrie der Funktionen zweier komplexen Veränderlicen III und IV,* Abh. Math. Sem. Hamburg **6** (1928), 8–54. 328
4. J. Briançon, P. Maisonobe, M. Merle, Localisation de systèmes différentiels, stratications de Whitney et condition de Thom, Invent. Math. **117**, 531 -550 (1994). 344
5. E. Brieskorn, *Examples of singular normal complex spaces which are topological manifolds,* Proc. Nat. Acad. Sci. U.S.A. **55** (1966) 1395–1397. 328
6. D. Burghelea, A. Verona, *Local homological properties of analytic sets,* Manuscripta Math. **7** (1972), 55–66. 343 and 379
7. T. Bröcker, K. Jänich, Introduction to Differential Topology, Cambridge University Press (1982). 329

8. J. L. Cisneros, A. Menegon, *Lê's vanishing polyhedron for a family of mixed functions*, Bull. London Math. Soc. https://doi.org/10.1112/blms.12299. 374
9. J. L. Cisneros, J. Seade, J. Snoussi, *Refinements of Milnor's fibration theorem*, Adv. Math. 222(3) (2009), 937–970. 347 and 381
10. J. Damon, *Finite determinacy and topological triviality. I.* Invent. Math. 62 (1980/81), no. 2, 299–324. 385
11. A. H. Durfee, *Neighborhoods of algebraic sets,* Trans. Amer. Math. Soc. **276** (1983), 517–530. 322
12. J. Fernández de Bobadilla, *A reformulation of Le's conjecture,* Indag. Math. (N.S.) **17** (2006), no. 3, 345–352. 383
13. G. Fischer, Complex analytic geometry. Lecture Notes in Mathematics, Vol. 538. Springer-Verlag, Berlin-New York, 1976. 377
14. A. M. Gabrielov, *Bifurcations, Dynkin diagrams, and modality of isolated singularities,* Funct. Anal. Appl. **8** (1974), 94–98. 384
15. C. G. Gibson, Singular points of smooth mappings. Research Notes in Mathematics, 25. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1979. 357
16. M. Golubitsky, V. Guillemin, Stable mappings and their singularities, Graduate Texts in Mathematics, Vol. 14. Springer-Verlag, New York-Heidelberg, 1973. 355 and 356
17. M. Goresky, R. MacPherson, Stratified Morse theory, Springer-Verlag, Berlin, 1988. 323
18. G.-M. Greuel, *Der Gauss-Manin Zusammenhang isolierter Singularitäten von vollständigen Durchschnitten,* Dissertation. Göttingen, 1973. Math. Ann. 214 (1975) 235–266. 350 and 351
19. R. C. Gunning, H. Rossi, Analytic function of several complex variables, Prentice-Hall Inc., Englewood Cliffs, N.J. 1965. 324
20. H. Hamm. *Lokale topologische Eigenschaften komplexer Räume, Math. Ann.*, **191** (1971), 235–252. 348, 350, 352, and 381
21. H. Hamm, D. T. Lê, *Un théorème de Zariski du type de Lefschetz,* Ann. Sci. Ec. Norm. Sup., **6** (series 4), 1973, 317–366. 354
22. H. Hamm, D.T. Lê, *Rectified Homotopical Depth and Grothendieck conjectures,* in the Grothendieck Festschrift Vol. II, Birkhaüser (1991). 381
23. A. Hatcher, Algebraic topology, Cambridge University Press, Cambridge, (2002). 371
24. H. Hironaka, *Stratification and flatness,* In "Real and complex singularities" (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), p. 199–265. Sijthoff and Noordhoff, 1977. 343
25. H. Hironaka, *Subanalytic sets,* In *Number theory, algebraic geometry and commutative algebra, in honor of Yasuo Akizuki*, pages 453–493. Kinokuniya, Tokyo, 1973. 379
26. M. Hochster, John A. Eagon, *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci.* Amer. J. Math. **93** (1971), 1020–1058. 352, 353, and 355
27. C. Houzel, *Géométrie analytique locale, I*, in Séminaire H. Cartan 1960–1961, Familles d'espaces complexes, Exposé 18, Secrétariat Mathématique, IHP, 11 rue Pierre et Marie Curie, Paris (1962). 352, 357, and 362
28. T. de Jong, G. Pfister, Local analytic geometry. Basic theory and applications, Advanced Lectures in Mathematics. Friedr. Vieweg & Sohn, Braunschweig, 2000. 349
29. T. Keilen, D. Mond, *Injective Analytic Maps - A Counterexample to the Proof.* arXiv:math/0409426 383
30. M. A. Kervaire, J. W. Milnor, *Groups of homotopy spheres: I*, Annals of Mathematics. Princeton University Press. **77** (1963), 504–537. 339
31. D. T. Lê, *Sur un critère d'équisingularité*, C. R. Acad. Sci. Paris Sér. A-B 272 (1971), A138 - A140. 385
32. D. T. Lê, *Calcul du nombre de cycles évanouissants d'une hypersurface complexe,* Ann. Inst. Fourier **23** (1973), 261–270. 324, 350, 354, and 376
33. D. T. Lê, *Sur les noeuds algébriques,* Compositio Math. **25** (1972), 281–321. 328 and 350
34. D. T. Lê, *Une application d'un théorème d'A'Campo à l'équisingularité,* Nederl. Akad. Wetensch. Proc. Ser. A 76=Indag. Math. **35** (1973), 403–409. 384

35. D. T. Lê. *Calculation of Milnor number of isolated singularity of complete intersection,* Funct. Anal. Appl. **8** (1974), 127–131. 351
36. D. T. Lê, *Vanishing cycles on analytic sets*. Proc. Conf. on Algebraic Analysis, RIMS, Kyoto, 1975. 324 and 354
37. D. T. Lê, *Topological use of polar curves,* A. M. S. Proc. Symp. Pure Maths. **29** (1975), 507–512. 324, 354, and 376
38. D. T. Lê, *La monodromie n'a pas de points fixes,* J. Fac. Sci. Univ. Tokyo Sect. IA Math. **22** (1975), no. 3, 409–427. 325 and 371
39. D. T. Lê, *Vanishing cycles on complex analytic sets,* In "Various problems in algebraic analysis". Proc. Sympos., Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1975, Sûrikaisekikenkyûsho Kókyûroku **266**, 299–318, 1976. 354 and 376
40. D. T. Lê, *Some remarks on relative monodromy,* In P. Holm, editor, in "Real and complex singularities" (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), pages 397–403. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977. 372, 380, and 381
41. D. T. Lê, *The geometry of the monodromy theorem,* C. P. Ramanujam a tribute, pp. 157–173, Tata Inst. Fund. Res. Studies in Math. **8**, Springer, Berlin-New York, 1978. 325
42. D. T. Lê, *Introduction à la théorie des singularités* t. 1 et t. 2, Travaux en cours **36–37**, D. T. Lê (ed.), Hermann Ed., Paris, 1988. 381
43. D. T. Lê, *Polyèdres évanescents et effondrements*. In "A fête of topology", Pap. Dedic. to Itiro Tamura. Academic Press, p. 293–329 (1988). Academic Press, 1988. 373
44. D. T. Lê, *Le théorème de la monodromie singulier,* C. R. Acad. Sci. Paris Sér. A-B **288** (1979), no. 21, A985 - A988. 373
45. D. T. Lê, *Le concept de singularité isolée de function analytique*, Adv. Studies Pure Math. **8** (1986), 213–227. 381
46. D. T. Lê, *Complex analytic functions with isolated singularities*, J. Algebraic Geom. **1** (1992), 83–100. 381
47. D. T. Lê, D. Massey. *Hypersurface singularities and Milnor equisingularity*. Pure Appl. Math. Q. **2**, No. 3, 893–914 (2006). 374
48. D. T. Lê, A. Menegón. *Vanishing polyhedron and collapsing map*. Math. Z. **286** (2017), 1003–1040. 357, 359, and 373
49. D. T. Lê, F. Michel, C. Weber, *Courbes polaires et topologie des courbes planes,* Ann. Sci. Ecole Norm. Sup. (4) **24** (1991), 141–169. 328
50. D. T. Lê, B. Perron. *Sur la fibre de Milnor d'une singularité isolée en dimension complexe trois,* C. R. Acad. Sci. Paris Sér. A-B **289** (1979), no. 2, A115 - A118. 325 and 373
51. D. T. Lê, C. P. Ramanujam. *The invariance of Milnor's number implies the invariance of the topological type,* Am. J. Maths. **98** (1976), 67–78. 384
52. M. Lejeune, D. T. Lê , B. Teissier, *Sur un critère d'équisingularité.* C. R. Acad. Sci. Paris Sér. A-B **271** (1970), A1065–A1067. 383 and 384
53. S. Łojasiewicz, Introduction to complex analytic geometry. Birkhäuser Verlag, Basel, 1991. 372
54. E. J. Looijenga, Isolated singular points on complete intersections, London Mathematical Society Lecture Note Series **77**, Cambridge University Press, Cambridge, 1984. 335, 338, 350, 351, 352, 353, and 383
55. I. Luengo, A. Pichon, *Lê's conjecture for cyclic covers,* Singularités Franco-Japonaises, 163–190, Sémin. Congr. **10**, Soc. Math. France, Paris, 2005. 383
56. D. B. Massey, *The Lê varieties. I.* Invent. Math. **99** (1990), 357–376. 325, 376, and 378
57. D. B. Massey, *The Lê varieties II.* Invent. Math. **104** (1991), 113–148. 325 and 378
58. D. B. Massey, *Numerical invariants of perverse sheaves*, Duke Math. Journal **73**, n. 2, (1994), 307–370. 377
59. D. B. Massey, *Lê Cycles and Hypersurface Singularities*. Lecture Notes in Mathematics **1615**, Springer-Verlag (1995). 325, 359, 376, and 377
60. D. B. Massey, *An introduction to the Lê cycles of a hypersurface singularity*, Lê Dũng Tráng and *al.* (ed.) In "Singularity theory". Proceedings of the symposium, Trieste, Italy, 1991. Singapore: World Scientific. 468–486 (1995). 377

61. J. N. Mather, *Stability of C^∞ mappings. II. Infinitesimal stability implies stability.* Ann. of Math. (2) **89** (1969), 254–291. 363
62. J. Mather, *Notes on Topological stability*, Harvard notes, 1970, published in Bull. A. M. S. **49** (2012), 475–506. 379 and 381
63. A. Menegon, *Lê polyhedron for line singularities*, International Journal of Mathematics Vol. **25**, No. 13 (2014), 374
64. J. W. Milnor, Morse theory, Based on lecture notes by M. Spivak and R. Wells, Annals of Mathematics Studies, **51** Princeton University Press, Princeton, N.J. (1963) vi+153 pp. 360
65. J. W. Milnor, Singular points of complex hypersurfaces, Annals of Mathematics Studies **61**, Princeton University Press, Princeton, N.J., 1968. 322, 326, 328, 333, 335, 339, 340, 345, 346, 347, 348, 373, and 376
66. D. Mumford, *The topology of normal singularities of an algebraic surface and a criterion for simplicity,* Inst. Hautes Etudes Sci. Publ. Mat. **9** (1961), 5–22. 383
67. A. Némethi, *Injective analytic maps,* Duke Math. J. **69** (1993), no. 2, 335–347. 382
68. A. Parusiński, *Topological triviality of μ-constant deformations of type $f(x) + tg(x)$*. Bull. London Math. Soc. 31 (1999), no. 6, 686–692. 385
69. F. Pham, *Formules de Picard-Lefschetz généralisées et ramification des intégrales*, Bull. Soc. Math. France **93** (1965), 333–367. 340 and 341
70. A. Ranicki, High-dimensional knot theory, Springer Verlag Monographs in Mathematics, 1998. With an appendix by E. Winkelnkemper. 348
71. J. Seade, *On Milnor's Fibration theorem and its offspring after 50 years*. Bull. Am. Math. Soc. **56** (2019), 281–348. 323, 347, and 381
72. D. Siersma, *A bouquet theorem for the Milnor fiber,* J. Algebr. Geom. **4**, No.1, 51-66 (1995). 382
73. S. Smale, *On the structure of manifolds*, Am. J. Math. **84** (1962), p. 387–399. 385
74. B. Teissier, *Cycles évanescents, sections planes et conditions de Whitney.* Singularités à Cargése. Astérisque, Nos. 7 et 8, Soc. Math. France, Paris, p. 285–362, 1973. 324, 350, 351, 354, and 376
75. B. Teissier, *Variétés polaires I. Invariants polaires des singularités d'hypersurfaces,* Invent. Math. **40**, 267–292 (1977). 324 and 354
76. B. Teissier, *Variétés polaires. II. Multiplicités polaires, sections planes, et conditions de Whitney*, In *Algebraic geometry (La Rábida, 1981)*, volume **961** of Lecture Notes in Math., pages 314–491. Springer, Berlin, 1982. 376
77. R. Thom, *Ensembles et morphismes stratifiés*, Bull. A. M. S. **75** (1969), 240–284. 324, 343, and 354
78. M. Tibăr, *Bouquet decomposition of the Milnor fiber*, Topology **35**, No.1, 227–241 (1996). 382
79. A. N. Varchenko, *A lower bound for the codimension of the $\mu = const$ stratum in terms of the mixed Hodge structure.* Vestnik Moskov. Univ. Ser. I Mat. Mekh. 1982, no. 6, 28–31, 120. 385
80. J.-L. Verdier, *Stratifications de Whitney et théorème de Bertini-Sard*, Invent. Math. **36** (1976), 295–312. 380
81. C. Weber (Ed.), *Noeuds, tresses et singularités*, Proceedings of the Seminar Held in Plans-sur-Bex, March 27-April 2, 1982, Monographies de L'Enseignement Mathematic, vol. **31**, 1983. 325 and 382
82. H. Whitney, *Local properties of analytic varieties*, Symposium in honor of M. Morse, Princeton Univ. Press, edited by S. Cairns, 1965. 361
83. H. Whitney, *Tangents to an Analytic Variety*, Annals of Math. **81**, No. 3, (1965). 326, 343, 378, and 379
84. O. Zariski. *Studies in equisingularity II.* Am. J. Maths. 87, 972–1006 (1965). 385

Chapter 7
Deformation and Smoothing of Singularities

Gert-Martin Greuel

Contents

Abstract We give a survey on some aspects of deformations of isolated singularities. In addition to the presentation of the general theory, we report on the question of the smoothability of a singularity and on relations between different invariants, such as the Milnor number, the Tjurina number, and the dimension of a smoothing component.

7.1 Introduction

This is a survey on some aspects of deformations of isolated singularities. We present first the most important general constructions and results of deformation theory. Then we report on the question of smoothability of a singularity and on

---

G.-M. Greuel (✉)
Department of Mathematics, University of Kaiserslautern, Kaiserslautern, Germany
e-mail: greuel@mathematik.uni-kl.de

J. L. Cisneros Molina et al. (eds.), *Handbook of Geometry and Topology of Singularities I*, https://doi.org/10.1007/978-3-030-53061-7_7

relations between different invariants, such as e.g. the Milnor number, the Tjurina number, and the dimension of a smoothing component.

In the first chapter we give an overview on the theory of deformations of complex space germs. Although we use the language of functors for precise statements, we provide also explicit descriptions in terms of the defining equations. We give almost no proofs but for every statement we give precise references to sources that contain more details, including proofs, for further reading. Since we confine ourselves to the deformation theory of isolated singularities we avoid the almost unmanageable field of far-reaching generalizations. Thus we give a compact presentation that nevertheless contains all essential fundamental results.

The second chapter is devoted to the study of the nearby fiber, also called Milnor fibre, of a small deformation and provides a short overview of the historically most relevant results on rigidity and smoothability. Moreover, we discuss known results and conjectures about the relationship between the dimension of a smoothing component and the topology of the Milnor fibre over this component. An important question is which invariants of a smoothing are independent of this and depend only on the singularity. The main results concern complete intersections, as well as curve and surface singularities, which we treat separately in different subsections. In addition to known results, we also discuss open problems and conjectures. As a rule, we give no proofs, but we sketch them in cases where the method is particularly interesting. For all results we give precise references.

7.2 Deformation Theory

7.2.1 Deformations of Complex Germs

We give an overview of the deformation theory of isolated singularities of complex space germs. The concepts and theorems for this case may serve as a prototype for deformations of other objects, such as deformations of mappings or, more general, of deformations of diagrams. For the theory of complex spaces and their morphisms, also called holomorphic or analytic maps, we refer to [27] and to [35]. Good references for deformations of algebraic schemes are the books [44] and [85].

A *pointed complex space* is a pair (X, x) consisting of a complex space X and a point $x \in X$. A morphism $f : (X, x) \to (Y, y)$ of pointed complex spaces is a morphism $f : X \to Y$ of complex spaces such that $f(x) = y$. The structure sheaf of X is denoted by $\mathcal{O}_X$, the (analytic) local ring by $\mathcal{O}_{X,x}$ with maximal ideal $\mathfrak{m}$, and the induced map of local rings by $f^\sharp : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. Two morphisms of pointed complex spaces, f resp. g from (X, x) to (Y, y) defined in some open neighbourhood U resp. V of x, are called *equivalent* if they coincide on a neighbourhood $W \subset U \cap V$ of x. A *complex (space) germ* is a pointed complex spaces with morphisms being equivalence classes of morphisms of pointed complex spaces. In particular, a complex germ (X, x) is identified with an arbitrary small open neighbourhood of x. A *singularity* is nothing but a complex space germ.

If $U \subset X$ is an open subset of the complex space X, then $\Gamma(U, \mathcal{O}_X)$ denotes the $\mathbb{C}$-algebra of holomorphic functions on X. If $I = \langle f_1, \dots, f_k \rangle$ is an ideal in $\Gamma(U, \mathcal{O}_X)$, generated by $f_1, \dots, f_k$, we denote by $V(I) = V(f_1, \dots, f_k)$ the (closed) complex subspace of U, being as topological space $\{x \in U | f_1(x) = \ldots = f_k = 0\}$ with structure sheaf $\mathcal{O}_U / I\mathcal{O}_U$.

Definition 7.2.1 Let (X, x) and (S, s) be complex space germs. A *deformation of* (X, x) *over* (S, s) consists of a flat morphism $\phi \colon (\mathscr{X}, x) \to (S, s)$ of complex germs together with an isomorphism $(X, x) \xrightarrow{\cong} (\mathscr{X}_s, x)$. $(\mathscr{X}, x)$ is called the *total space*, (S, s) the *base space*, and $(\mathscr{X}_s, x) := (\phi^{-1}(s), x)$ or (X, x) the *special fibre* of the deformation.

We can write a deformation as a Cartesian diagram

$$\begin{tikzcd}
(X, x) \arrow[r, hook, "i"] \arrow[d] \arrow[dr, phantom, "\square"] & (\mathscr{X}, x) \arrow[d, "\phi \text{ flat}"] \\
\{\text{pt}\} \arrow[r, hook] & (S, s) \, ,
\end{tikzcd}$$

where i is a closed embedding mapping (X, x) isomorphically onto $(\mathscr{X}_s, x)$ and $\{\text{pt}\}$ the reduced point considered as a complex space germ with local ring $\mathbb{C}$. We denote a deformation by

$$(i, \phi) \colon (X, x) \overset{i}{\hookrightarrow} (\mathscr{X}, x) \overset{\phi}{\to} (S, s) \, ,$$

or simply by $\phi : (\mathscr{X}, x) \to (S, s)$ in order to shorten notation. Note that the closed embedding i is part of the data and identifies $(\mathscr{X}_s, x)$ and (X, x). Thus, if $(\mathscr{X}', x) \to (S, s)$ is another deformation of (X, x), we get a *unique* isomorphism of germs $(\mathscr{X}_s, x) \cong (X, x) \cong (\mathscr{X}'_s, x)$.

The essential point here is that ϕ is *flat at* x, that is, $\mathcal{O}_{\mathscr{X},x}$ is a flat $\mathcal{O}_{S,s}$-module via the induced morphism $\phi^\sharp_x : \mathcal{O}_{S,s} \to \mathcal{O}_{\mathscr{X},x}$. A well known theorem of Frisch (cf. [21]) says that for a morphism $\phi : \mathscr{X} \to S$ of complex spaces the set of points in $\mathscr{X}$ where ϕ is flat is analytically open. Hence, a sufficiently small representative $\phi : \mathscr{X} \to S$ of the germ ϕ is everywhere flat and, since flatness implies $\dim(\mathscr{X}_s, x) = \dim(\mathscr{X}, x) - \dim(S, s)$, we have $\dim(\mathscr{X}_t, y) = \dim(\mathscr{X}_s, x)$ for all $t \in S$ and all $y \in \mathscr{X}_t$ if $\mathscr{X}$ and S are pure dimensional. Another important theorem is due to Douady [13], saying that every flat morphism $\phi : \mathscr{X} \to S$ of complex spaces is open, that is, it maps open sets in $\mathscr{X}$ to open sets in S. An important example of flat morphisms are projections: If X, T are complex spaces then the projection $X \times T \to T$ is flat (c.f. [35, Corollary I.1.88]).

A typical example of a non-flat morphism is the projection $(\mathbb{C}^2, \mathbf{0}) \supset V(xy) \to (\mathbb{C}, 0)$, $(x, y) \mapsto x$, since the fibre-dimension jumps (the dimension of the special fibre is 1 and of the other fibres is 0).

The following theoretically and computationally useful criterion for flatness is due to Grothendieck (for a proof see e.g. [35, Proposition I.1.91]).

Proposition 7.2.2 (Flatness by Relations) *Let* $I = \langle f_1, \dots, f_k \rangle \subset \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$ *be an ideal,* (S, s) *a complex space germ and* $\widetilde{I} = \langle F_1, \dots, F_k \rangle \subset \mathcal{O}_{\mathbb{C}^n \times S,(\mathbf{0},s)}$ *a lifting of* I*, i.e.,* F_i *is a preimage of* f_i *under the surjection*

$$\mathcal{O}_{\mathbb{C}^n \times S,(\mathbf{0},s)} \twoheadrightarrow \mathcal{O}_{\mathbb{C}^n \times S,(\mathbf{0},s)} \otimes_{\mathcal{O}_{S,s}} \mathbb{C} = \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}\,.$$

Then the following are equivalent:

(a) $\mathcal{O}_{\mathbb{C}^n \times S,(\mathbf{0},s)}/\widetilde{I}$ *is* $\mathcal{O}_{S,s}$*-flat;*

(b) any relation $(r_1, \dots, r_k)$ *among* $f_1, \dots, f_k$ *lifts to a relation* $(R_1, \dots, R_k)$ *among* $F_1, \dots, F_k$*. That is, for each* $(r_1, \dots, r_k)$ *satisfying*

$$\sum_{i=1}^{k} r_i f_i = 0\,, \quad r_i \in \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}\,,$$

there exists $(R_1, \dots, R_k)$ *such that*

$$\sum_{i=1}^{k} R_i F_i = 0\,, \text{ with } R_i \in \mathcal{O}_{\mathbb{C}^n \times S,(\mathbf{0},s)}$$

and the image of R_i *in* $\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$ *is* r_i*;*

(c) any free resolution of $\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}/I$

$$\dots \to \mathcal{O}^{p_2}_{\mathbb{C}^n,\mathbf{0}} \to \mathcal{O}^{p_1}_{\mathbb{C}^n,\mathbf{0}} \to \mathcal{O}_{\mathbb{C}^n,\mathbf{0}} \to \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}/I \to 0$$

lifts to a free resolution of $\mathcal{O}_{\mathbb{C}^n \times S,(\mathbf{0},s)}/\widetilde{I}$*,*

$$\dots \to \mathcal{O}^{p_2}_{\mathbb{C}^n \times S,(\mathbf{0},s)} \to \mathcal{O}^{p_1}_{\mathbb{C}^n \times S,(\mathbf{0},s)} \to \mathcal{O}_{\mathbb{C}^n \times S,(\mathbf{0},s)} \to \mathcal{O}_{\mathbb{C}^n \times S,(\mathbf{0},s)}/\widetilde{I} \to 0\,.$$

That is, the latter sequence tensored with $\otimes_{\mathcal{O}_{S,s}} \mathbb{C}$ *yields the first sequence.*

Remark 7.2.3 Let us recall some geometric consequences of flatness, for an algebraic proof see e.g. [35, Theorem B.8.11 and B.8.13] and [65, Theorem 15.1].

1. $\phi = (\phi_1, \dots, \phi_k) : (\mathscr{X}, x) \to (\mathbb{C}^k, \mathbf{0})$ is flat iff $\phi_1, \dots, \phi_k$ is a regular sequence.
2. If $(\mathscr{X}, x)$ is Cohen-Macaulay, then $\phi_1, \dots, \phi_k \in \mathfrak{m} \subset \mathcal{O}_{\mathscr{X},x}$ is a regular sequence iff $\dim \mathcal{O}_{\mathscr{X},x}/\langle \phi_1, \dots, \phi_k \rangle = \dim(\mathscr{X}, x) - k$.
3. In particular, $\phi : (\mathbb{C}^m, \mathbf{0}) \to (\mathbb{C}^k, \mathbf{0})$ is flat iff $\dim\big(\phi^{-1}(\mathbf{0}), \mathbf{0}\big) = m - k$.

If (3) holds, then $(X, \mathbf{0}) := (\phi^{-1}(\mathbf{0}), \mathbf{0})$ is called a *complete intersection* and $(i, \phi) : (X, \mathbf{0}) \subset (\mathbb{C}^m, \mathbf{0}) \to (\mathbb{C}^k, \mathbf{0})$ is a deformation of $(X, \mathbf{0})$ over $(\mathbb{C}^k, \mathbf{0})$. If $k = 1$ then $(X, \mathbf{0})$ is called a *hypersurface singularity*.

Note that smooth germs, *hypersurface* and *complete intersection singularities*,[1] reduced curve singularities, and normal surface singularities are Cohen-Macaulay.

Definition 7.2.4 Given two deformations $(i, \phi)\colon (X, x) \hookrightarrow (\mathscr{X}, x) \to (S, s)$ and $(i', \phi')\colon (X, x) \hookrightarrow (\mathscr{X}', x') \to (S', s')$, of (X, x) over (S, s) and (S', s'), respectively. A *morphism of deformations* from (i, ϕ) to (i', ϕ') is a morphism of the diagram after Definition 7.2.1 being the identity on $(X, x) \to \{\mathrm{pt}\}$. Hence, it consists of two morphisms (ψ, φ) such that the following diagram commutes

$$\begin{tikzcd}
 & (X,x) \arrow[dl, hook', "i'"'] \arrow[dr, hook, "i"] & \\
(\mathscr{X}',x') \arrow[rr, "\psi"] \arrow[d, "\phi'"'] & & (\mathscr{X},x) \arrow[d, "\phi"] \\
(S',s') \arrow[rr, "\varphi"] & & (S,s)\,.
\end{tikzcd}$$

Two deformations over the *same* base space (S, s) are *isomorphic* if there exists a morphism (ψ, φ) with ψ an isomorphism and φ the identity map.

It is easy to see that deformations of (X, x) form a category. Usually one considers the (non-full) subcategory of deformations of (X, x) over a fixed base space (S, s) and morphisms (ψ, φ) with $\varphi = \mathrm{id}_{(S,s)}$. The following lemma implies that this category is a *groupoid*, i.e., all morphisms are automatically isomorphims (see e.g. [35, Lemma I.1.86] for a proof).

Lemma 7.2.5 *Let*

$$\begin{tikzcd}
(X,x) \arrow[rr, "f"] \arrow[dr, "\phi"'] & & (Y,y) \arrow[dl, "\psi"] \\
 & (S,s) &
\end{tikzcd}$$

be a commutative diagram of complex germs with ϕ *flat. Then* f *is an isomorphism iff* f *induces an isomorphism of the special fibres,* $f\colon (\phi^{-1}(s), x) \xrightarrow{\cong} (\psi^{-1}(s), y)$ *.*

We introduce now the concept of induced deformations, which give rise, in a natural way, to morphisms between deformations over different base spaces.

Let $(X, x) \hookrightarrow (\mathscr{X}, x) \xrightarrow{\phi} (S, s)$ be a deformation of the complex space germ (X, x) and $\varphi\colon (T, t) \to (S, s)$ a morphism of germs. Then the fibre product is the

---

[1] $(X, x) \subset (\mathbb{C}^N, x)$ is a complete intersection if the minimal number of generators of its ideal $I(X, x) \subset \mathcal{O}_{\mathbb{C}^N,x}$ is $N - \dim(X, x)$. (X, x) is a hypersurface singularity if $\dim(X, x) = N - 1$.

following commutative diagram of germs

$$\begin{tikzcd}
& (X,x) \arrow[dl, hook', "\varphi^* i"'] \arrow[dr, hook, "i"] & \\
(\mathscr{X},x)\times_{(S,s)}(T,t) \arrow[rr, "\widetilde{\varphi}"] \arrow[d, "\varphi^*\phi"'] & & (\mathscr{X},x) \arrow[d, "\phi"] \\
(T,t) \arrow[rr, "\varphi"] & & (S,s)
\end{tikzcd}$$

where $\varphi^*\phi$, resp. $\widetilde{\varphi}$, are induced by the second, resp. first projection, and $\varphi^* i = \big(\widetilde{\varphi}|_{(\varphi^*\phi)^{-1}(t)}\big)^{-1} \circ i$.

Definition 7.2.6 We denote $(\mathscr{X},x)\times_{(S,s)}(T,t)$ by $\varphi^*(\mathscr{X},x)$ and call

$$\varphi^*(i,\phi) := (\varphi^* i, \varphi^*\phi)\colon (X,x) \overset{\varphi^* i}{\hookrightarrow} \varphi^*(\mathscr{X},x) \xrightarrow{\varphi^*\phi} (T,t)$$

the *deformation induced by* φ *from* (i,ϕ), or just the *induced deformation* or *pull-back*; φ is called the *base change map*.

Since flatness is preserved under base change (c.f. [35, Proposition I.187]), $\varphi^*\phi$ is flat. Hence, $\varphi^*(i,\phi)$ is indeed a deformation of (X,x) over (T,t), and $(\widetilde{\varphi},\varphi)$ is a morphism from (i,ϕ) to $(\varphi^* i, \varphi^*\phi)$, and φ^* is a functor from deformations of (X,x) over (S,s) to deformations of (X,x) over (T,t). A typical example of an induced deformation is the restriction to a subspace in the parameter space (S,s).

We introduce the following notations.

Definition 7.2.7 Let (X,x) be a complex space germ.

(1) $\mathcal{D}ef_{(X,x)}$ denotes the *category of deformations of* (X,x), with morphisms as defined in Definition 7.2.4.

(2) $\mathcal{D}ef_{(X,x)}(S,s)$ denotes the *category of deformations of* (X,x) *over* (S,s), whose morphisms satisfy $\varphi = \mathrm{id}_{(S,s)}$.

(3) $\underline{\mathcal{D}ef}_{(X,x)}(S,s)$ denotes the set of *isomorphism classes of deformations* (i,ϕ) *of* (X,x) *over* (S,s).

For a morphism of complex germs $\varphi\colon (T,t)\to(S,s)$, the pull-back $\varphi^*(i,\phi)$ is a deformation of (X,x) over (T,t), inducing a map $\underline{\mathcal{D}ef}_{(X,x)}(S,s)\to \underline{\mathcal{D}ef}_{(X,x)}(T,t)$. It follows that

$$\underline{\mathcal{D}ef}_{(X,x)}\colon (\textit{complex germs}) \longrightarrow \mathcal{S}ets\,,$$

$(S,s)\mapsto \underline{\mathcal{D}ef}_{(X,x)}(S,s)$, is a functor, the *deformation functor* or the *functor of isomorphism classes of deformations* of (X,x).

7.2.2 *Embedded Deformations and Unfoldings*

This section has the goal, to describe the somewhat abstract definitions of the preceding section in more concrete terms, that is, in terms of defining equations and relations.

Let us first recall the notion of unfoldings of a hypersurface singularities and explain its relation to deformations. Given $f \in \mathbb{C}\{x_1, \dots, x_n\}$, $f(\mathbf{0}) = 0$, an *unfolding* of f is a power series $F \in \mathbb{C}\{x_1, \dots, x_n, t_1, \dots, t_k\}$ with $F(\boldsymbol{x}, \mathbf{0}) = f(\boldsymbol{x})$, that is,

$$F(\boldsymbol{x}, \boldsymbol{t}) = f(\boldsymbol{x}) + \sum_{|\boldsymbol{\nu}| \geq 1} g_{\boldsymbol{\nu}}(\boldsymbol{x}) \boldsymbol{t}^{\boldsymbol{\nu}} .$$

We identify the power series f and F with the holomorphic map germs

$$f \colon (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}, 0) , \quad F \colon (\mathbb{C}^n \times \mathbb{C}^k, \mathbf{0}) \to (\mathbb{C}, 0) .$$

Then F induces a deformation of the hypersurface singularity $(X, \mathbf{0}) = (f^{-1}(0), \mathbf{0})$ over $\mathbb{C}^k$ in the following way

$$\begin{tikzcd}
(X, \mathbf{0}) \arrow[r, hook, "i"] \arrow[d] & (\mathscr{X}, \mathbf{0}) \arrow[d, "\phi = pr_2|_{(\mathscr{X}, \mathbf{0})}"] & := (F^{-1}(0), \mathbf{0}) \subset (\mathbb{C}^n \times \mathbb{C}^k, \mathbf{0}) \\
\{0\} \arrow[r, hook] & (\mathbb{C}^k, \mathbf{0}) &
\end{tikzcd}$$

where i is the inclusion and ϕ the restriction of the second projection. By Remark 7.2.3 (i, ϕ) is a deformation of $(X, \mathbf{0})$. In fact, each deformation of a hypersurface singularity $(X, \mathbf{0})$ over some $(\mathbb{C}^k, \mathbf{0})$ is induced in this way by an unfolding of f (even for complete intersections, see Proposition 7.2.11 below).

More generally, we have the following important result.

Proposition 7.2.8 (Embedding of a Morphism) *Given a Cartesian diagram of complex space germs*

$$\begin{tikzcd}
(X_0, x) \arrow[r, hook] \arrow[d, "f_0"'] \arrow[dr, phantom, "\square"] & (X, x) \arrow[d, "f"] \\
(S_0, s) \arrow[r, hook] & (S, s) ,
\end{tikzcd}$$

where the horizontal maps are closed embeddings. Assume that f_0 factors as

$$(X_0, x) \overset{i_0}{\hookrightarrow} (\mathbb{C}^n, \mathbf{0}) \times (S_0, s) \overset{p_0}{\to} (S_0, s)$$

with i_0 a closed embedding and p_0 the second projection.[2] *Then there exists a Cartesian diagram*

$$\begin{tikzcd}
(X_0, x) \arrow[r, hook] \arrow[d, hook, "i_0"'] \arrow[dd, bend right=60, "f_0"'] \arrow[dr, phantom, "\square"] & (X, x) \arrow[d, hook, "i"] \arrow[dd, bend left=60, "f"] \\
(\mathbb{C}^n, \mathbf{0}) \times (S_0, s) \arrow[r, hook] \arrow[d, "p_0"'] \arrow[dr, phantom, "\square"] & (\mathbb{C}^n, \mathbf{0}) \times (S, s) \arrow[d, "p"] \\
(S_0, s) \arrow[r, hook] & (S, s)
\end{tikzcd} \tag{7.1}$$

with i a closed embedding and p the second projection. That is, the embedding of f_0 over (S_0, s) extends to an embedding of f over (S, s).

Note that we do not require that f_0 or f are flat. The proof is not difficult, see [35, Proposition II.1.5].

Applying Proposition 7.2.8 to a deformation of (X, x) we get

Corollary 7.2.9 (Embedding of a Deformation) *Let $(X, \mathbf{0}) \subset (\mathbb{C}^n, \mathbf{0})$ be a closed subgerm. Then any deformation of $(X, \mathbf{0})$,*

$$(i, \phi) : (X, \mathbf{0}) \hookrightarrow (\mathscr{X}, x) \rightarrow (S, s),$$

can be embedded. That is, there exists a Cartesian diagram

$$\begin{tikzcd}
(X, \mathbf{0}) \arrow[r, hook, "i"] \arrow[d, hook] \arrow[dr, phantom, "\square"] & (\mathscr{X}, x) \arrow[d, hook, "J"] \arrow[dd, bend left=60, "\phi"] \\
(\mathbb{C}^n, \mathbf{0}) \arrow[r, hook, "j"] \arrow[d] \arrow[dr, phantom, "\square"] & (\mathbb{C}^n, \mathbf{0}) \times (S, s) \arrow[d, "p"] \\
\{s\} \arrow[r, hook] & (S, s)
\end{tikzcd}$$

where J is a closed embedding, p is the second projection and j the first inclusion.

In particular, the embedding dimension is semicontinuous under deformations, that is, $\operatorname{edim}\big(\phi^{-1}(\phi(y)), y\big) \leq \operatorname{edim}(X, \mathbf{0})$, *for all y in $\mathscr{X}$ sufficiently close to x.*

Remark 7.2.10 We get the following *explicit description of a deformation:*

---

[2] In this situation, we call f_0 an *embedding over* (S_0, s).

Any deformation $(i, \phi)\colon (X, \mathbf{0}) \hookrightarrow (\mathscr{X}, x) \to (S, s)$ of $(X, \mathbf{0})$ can be assumed to be given as follows: Let $I_{X,\mathbf{0}} = \langle f_1, \dots, f_k\rangle \subset \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$ be the ideal of $(X, \mathbf{0}) \subset (\mathbb{C}^n, \mathbf{0})$. The embedding of the total space of the deformation of $(X, \mathbf{0})$ is given as

$$(\mathscr{X}, x) = V(F_1, \dots, F_k) \overset{\mathrm{J}}{\hookrightarrow} \mathbb{C}^n \times S, (\mathbf{0}, s)),$$

with $\mathcal{O}_{\mathscr{X},x} = \mathcal{O}_{\mathbb{C}^n\times S,(\mathbf{0},s)}/I_{\mathscr{X},x}$, $I_{\mathscr{X},x} = \langle F_1, \dots, F_k\rangle \subset \mathcal{O}_{\mathbb{C}^n\times S,(\mathbf{0},s)}$ and f_i being the image of F_i in $\mathcal{O}_{\mathbb{C}^n\times S,(\mathbf{0},s)}/\mathfrak{m}_{S,s} = \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$. Then $(X, 0) = V(f_1, \dots, f_k) \overset{i'}{\hookrightarrow} (\mathscr{X}, x)$ and setting $\phi' = p \circ J$, p the second projection, we get the deformation (i', ϕ'), which coincides with (i, ϕ) up to isomorphism.

Furthermore, let $(S, s) \subset (\mathbb{C}^r, \mathbf{0})$ and denote the coordinates of $\mathbb{C}^n$ by $\boldsymbol{x} = (x_1, \dots, x_n)$ and those of $\mathbb{C}^r$ by $\boldsymbol{t} = (t_1, \dots, t_r)$. Then $f_i = F_i|_{(\mathbb{C}^n,\mathbf{0})}$ and, hence, F_i is of the form[3]

$$F_i(\boldsymbol{x}, \boldsymbol{t}) = f_i(\boldsymbol{x}) + \sum_{j=1}^{r} t_j g_{ij}(\boldsymbol{x}, \boldsymbol{t}), \quad g_{ij} \in \mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^r,\mathbf{0}},$$

that is, F_i is an unfolding of f_i.

In general, F_i as above with g_{ij} arbitrary do not define a deformation, since the flatness condition is not fulfilled. However, if $(X, \mathbf{0})$ is an $(n - k)$-dimensional complete intersection, flatness is automatic.

Proposition 7.2.11 *Let $(X, \mathbf{0}) \subset (\mathbb{C}^n, \mathbf{0})$ be a complete intersection, and let $f_1, \dots, f_k$ be a minimal set of generators of the ideal of $(X, \mathbf{0})$ in $\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$. Then, for any complex germ (S, s) and any lifting $F_i \in \mathcal{O}_{\mathbb{C}^n\times S,(\mathbf{0},s)}$ of f_i, $i = 1, \dots, k$ (i.e., F_i is of the form as in Remark 7.2.10), the diagram*

$$(X, \mathbf{0}) \hookrightarrow (\mathscr{X}, x) \xrightarrow{p} (S, s)$$

with $(\mathscr{X}, x) = V(F_1, \dots, F_k) \subset (\mathbb{C}^n \times S, (\mathbf{0}, s))$ and p the second projection, is a deformation of $(X, \mathbf{0})$ over (S, s).

Proof Since $f_1, \dots, f_k$ is a regular sequence, any relation among the f_i can be generated by the *trivial relations* (also called the *Koszul relations*)

$$(0, \dots, 0, -f_j, 0, \dots, 0, f_i, 0 \dots, 0)$$

with $-f_j$ at place i and f_i at place j. This can be easily shown by induction on k. Another way to see this is to use the Koszul complex of $\boldsymbol{f} = (f_1, \dots, f_k)$: we have

$$H_1(f, \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}) = \{\text{relations between } f_1, \dots, f_k\}/\{\text{trivial relations}\},$$

---

[3] That a system of generators for $I_{\mathscr{X},x}$ can be written in this form follows from the fact that $\mathfrak{m}_{S,s} I_{\mathscr{X},x} = \mathfrak{m}_{S,s}\mathcal{O}_{\mathbb{C}^n\times S,(\mathbf{0},s)} \cap I_{\mathscr{X},x}$, which is a consequence of flatness.

and $H_1(f, \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}) = 0$ if $f_1, \dots, f_k$ is a regular sequence [35, Theorem B.6.3]. Since the trivial relations can obviously be lifted, the result follows from Proposition 7.2.2.

□

Let us finish this section with a concrete example.

Example 7.2.12 $f_1 = xy$, $f_2 = xz$, $f_3 = yz$. We define two unfoldings of (f_1, f_2, f_3), the first does not induce a deformation of $(X, \mathbf{0})$ while the second does.

(1) Consider the unfolding of (f_1, f_2, f_3) over $(\mathbb{C}, 0)$ given by $F_1 = xy - t$, $F_2 = xz$, $F_3 = yz$. It is not difficult to check that the sequence

$$\begin{tikzcd} 0 & \mathcal{O}_{X,\mathbf{0}} \arrow[l] & \mathcal{O}_{\mathbb{C}^3,\mathbf{0}} \arrow[l] & \mathcal{O}^3_{\mathbb{C}^3,\mathbf{0}} \arrow[l, "{(xy,xz,yz)}"'] & \mathcal{O}^2_{\mathbb{C}^3,\mathbf{0}} \arrow[l, "{\begin{pmatrix} 0 & -z \\ -y & y \\ x & 0 \end{pmatrix}}"'] & 0\,, \arrow[l] \end{tikzcd}$$

is exact and, hence, a free resolution of $\mathcal{O}_{X,\mathbf{0}} = \mathcal{O}_{\mathbb{C}^3,\mathbf{0}}/\langle f_1, f_2, f_3 \rangle$. That is, $(0, -y, x)$ and $(-z, y, 0)$ generate the $\mathcal{O}_{\mathbb{C}^3,\mathbf{0}}$-module of relations between xy, xz, yz.

Similarly, we find that $(0, -y, x)$, $(yz, -y^2, t)$, $(xz, t - xy, 0)$ generate the $\mathcal{O}_{\mathbb{C}^3,\mathbf{0}}$-module of relations of F_1, F_2, F_3. The liftable relations for f_1, f_2, f_3 are obtained from the latter by setting $t = 0$, which shows that the relation $(-z, y, 0)$ cannot be lifted. Hence, $\mathcal{O}_{\mathbb{C}^3\times\mathbb{C},\mathbf{0}}/\langle F_1, F_2, F_3 \rangle$ is not $\mathcal{O}_{\mathbb{C},0}$-flat and, therefore, the above unfolding does not define a deformation of $(X, \mathbf{0})$. We check this in the following SINGULAR session:

```
ring R = 0,(x,y,z,t),ds;
ideal f = xy,xz,yz;
ideal F = xy-t,xz,yz;
module Sf = syz(f);    // the module of relations of f
print(Sf);             // shows the matrix of Sf
//-> 0, -z,
//-> -y,y,
//-> x, 0
syz(Sf);               // is 0 iff the matrix of Sf injective
//-> _[1]=0
module SF = syz(F);
print(SF);
//-> 0, yz, xz,
//-> -y,-y2,t-xy,
//-> x, t,  0
```

To show that the relation $(-z, y, 0)$ in `Sf` cannot be lifted to `SF`, we check that `Sf` is not contained in the module obtained by substituting `t` by zero in `SF` (`reduce(Sf,std(subst(SF,t,0)));` does not produce zero).

(2) However, if we consider the unfolding $F_1 = xy - tx$, $F_2 = xz$, $F_3 = yz$ of (f_1, f_2, f_3), we obtain $(-z, -t, x)$, $(-z, y - t, 0)$ as generators of the relations among F_1, F_2, F_3.

Since $(0, -y, x) = (-z, 0, x) - (-z, y, 0)$, it follows that any relation among f_1, f_2, f_3 can be lifted. Hence, $\mathcal{O}_{\mathbb{C}^3\times\mathbb{C},\mathbf{0}}/\langle F_1, F_2, F_3\rangle$ is $\mathcal{O}_{\mathbb{C},\mathbf{0}}$-flat and the diagram

$$\begin{tikzcd}
(X,\mathbf{0}) \arrow[r, hook] \arrow[d] & V(F_1,F_2,F_3) \subset (\mathbb{C}^3\times\mathbb{C},(\mathbf{0},0)) \arrow[d] \\
\{0\} \arrow[r, hook] & (\mathbb{C},0)
\end{tikzcd} .$$

defines a deformation of $(X, 0)$.

7.2.3 Versal Deformations

A versal deformation of a complex space germ is a deformation which contains basically all information about any possible deformation of this germ. A semiuniversal deformation is a minimal versal deformation. It is one of the fundamental facts of singularity theory that any isolated singularity (X, x) has a semiuniversal deformation.

In a little less informal way we say that a deformation (i, ϕ) of (X, x) over (S, s) is *versal* if any other deformation of (X, x) over some base space (T, t) can be induced from (i, ϕ) by some base change $\varphi\colon (T, t) \to (S, s)$. Moreover, if a deformation of (X, x) over some subgerm $(T', t) \subset (T, t)$ is given and induced by some base change $\varphi'\colon (T', t) \to (S, s)$, then φ can be chosen in such a way that it extends φ'. This fact is important, though technical, as it allows us to construct versal deformations by successively extending over bigger and bigger spaces in a formal manner (see [35], Appendix C for general fundamental facts about formal deformations, in particular, Theorem C.1.6, and the sketch of its proof).

The formal definition of a (semiuni-)versal deformation is as follows.

Definition 7.2.13

(1) A deformation $(X, x) \overset{i}{\hookrightarrow} (\mathscr{X}, x) \overset{\phi}{\to} (S, s)$ of (X, x) is called *complete* if, for any deformation $(j, \psi)\colon (X, x) \hookrightarrow (\mathscr{Y}, y) \to (T, t)$ of (X, x), there exists a morphism $\varphi\colon (T, t) \to (S, s)$ such that (j, ψ) is isomorphic to the induced deformation $(\varphi^* i, \varphi^*\phi)$.

(2) The deformation (i, ϕ) is called *versal* (respectively *formally versal*) if, for a given deformation (j, ψ) as above the following holds: for any closed embedding $k\colon (T', t) \hookrightarrow (T, t)$ of complex germs (respectively of Artinian complex germs[4]) and any morphism $\varphi'\colon (T', t) \to (S, s)$ such that $(\varphi'^{*} i, \varphi'^{*}\phi)$ is isomorphic to $(k^* j, k^*\psi)$ there exists a morphism $\varphi\colon (T, t) \to (S, s)$ satisfying

---

[4] A complex germ consisting of one point with local ring an Artinian local ring. It is also called a *fat point*.

(i) $\varphi \circ k = \varphi'$, and
(ii) $(j, \psi) \cong (\varphi^* i, \varphi^* \phi)$.

That is, there exists a commutative diagram with Cartesian squares

$$\begin{tikzcd}
 & (X,x) \arrow[dl, hook', "k^*j"'] \arrow[d, hook, "j"] \arrow[dr, hook, "i"] & \\
k^*(\mathscr{Y},y) \arrow[r, hook] \arrow[d, "k^*\psi"'] \arrow[dr, phantom, "\square"] & (\mathscr{Y},y) \arrow[r, dashed] \arrow[d, "\psi"'] \arrow[dr, phantom, "\square"] & (\mathscr{X},x) \arrow[d, "\phi"] \\
(T',t) \arrow[r, hook, "k"'] \arrow[rr, bend right, "\varphi'"'] & (T,t) \arrow[r, dashed, "\varphi"'] & (S,s)\,.
\end{tikzcd}$$

(3) A (formally) versal deformation is called *semiuniversal* if, with the notations of (2), the Zariski tangent map $T_{(T,t)} \to T_{(S,s)}$ of φ is uniquely determined by (i, ϕ) and (j, ψ).

A semiuniversal deformation is also called *miniversal* because the Zariski tangent space of its base space has the smallest possible dimension among all versal deformations. Note that we do not consider *universal* deformations (i.e., φ in (3) itself is uniquely determined) as this would be too restrictive.

A versal deformation is complete (take as (T', t) the reduced point $\{s\}$), but the converse is not true in general. In the literature the distinction between complete and versal deformations is not always sharp, some authors call complete deformations (in our sense) versal. However, the full strength of versal (and, hence, semiuniversal) deformations comes from the property requested in (2).

If we know a versal deformation of (X, x), we know, at least in principle, all other deformations (up to the knowledge of the base change map φ). In particular, we know all nearby fibres and, hence, all nearby singularities which can appear for an arbitrary deformation of (X, x).

An arbitrary complex space germ may not have a versal deformation. It is a fundamental theorem of Grauert [26] that for isolated singularities a semiuniversal deformation exists.

Recall that (X, x) has an *isolated singularity*, if there exists a representative X with $X \setminus \{x\}$ nonsingular. A point y of X is called *nonsingular* or *smooth* if X is a complex manifold in a neighbourhood of y (equivalently, the local ring $\mathcal{O}_{X,y}$ is regular), otherwise y is called a *singular point* of X. For a proof of the following theorem see [26].

Theorem 7.2.14 (Grauert) *Any complex space germ* (X, x) *with isolated singularity*[5] *has a semiuniversal deformation*

$$(X, x) \overset{i}{\hookrightarrow} (\mathscr{X}, x) \overset{\phi}{\to} (S, s)\,.$$

In Theorem 7.2.22 we describe the semiuniversal deformation explicitly if (X, x) is an isolated complete intersection. For the procedure to construct a formal semiuniversal deformation in general by induction, see the beginning of Sect. 7.2.5. For an equivariant weighted homogeneous version see Theorem 7.3.5 and its proof.

Even knowing that a semiuniversal deformation of an isolated singularity (X, x) exists, in general we cannot say anything in advance about its structure. For instance, we can say nothing about the dimension of the base space of the semiuniversal deformation, which we shortly call the *semiuniversal base space*. It is unknown (but believed), if any complex space germ can occur as a semiuniversal base of an isolated singularity. Further questions are whether (X, x) is *smoothable*, i.e., if there are nearby fibres that are smooth, or if (X, x) is *rigid*, i.e., if it cannot be deformed at all (cf. Sect. 7.3.1 for details).

The following Lemma is an easy consequence of the inverse function theorem.

Lemma 7.2.15 *If a semiuniversal deformation of a complex space germ* (X, x) *exists, then it is uniquely determined up to (non unique) isomorphism.*

We mention some important properties of versal deformations. They hold in a much more general deformation theoretic context (see Remark [35, C.1.5.1]).

Theorem 7.2.16 (Flenner) *If a versal deformation of* (X, x) *exists then there exists also a semiuniversal deformation, and every deformation of* (X, x) *which is formally versal is also versal.*

For the proof see [19, Satz 5.2]. It is based on the following useful result (c.f. [35, Proposition I.1.14]):

Proposition 7.2.17 *Every versal deformation of* (X, x) *differs from the semiuniversal deformation by a smooth factor.*

More precisely, let $\phi : (\mathscr{X}, x) \to (S, s)$ *be the semiuniversal deformation and* $\psi : (\mathscr{Y}, y) \to (T, t)$ *a versal deformation of* (X, x)*. Then there exists a* $p \geq 0$ *and an isomorphism*

$$\varphi\colon (T, t) \xrightarrow{\cong} (S, s) \times (\mathbb{C}^p, \mathbf{0})$$

such that $\psi \cong (\pi \circ \varphi)^* \phi$ *where* $\pi : (S, s) \times (\mathbb{C}^p, \mathbf{0}) \to (S, s)$ *is the projection on the first factor.*

---

[5] More generally, a semiuniversal deformation exists if $\dim_{\mathbb{C}} T^1_{(X,x)} < \infty$ (see Definition 7.2.26).

Remark 7.2.18

(1) A formula for the extra smooth factor $(\mathbb{C}^p, \mathbf{0})$ in Proposition 7.2.17 is given in Corollary 7.3.42.

(2) The statements of 7.2.14–7.2.17 hold also for *multigerms* $(X, x) = \coprod_{\ell=1}^{r}(X_\ell, x_\ell)$, that is, for the disjoint union of finitely many germs (the existence as in Theorem 7.2.14 is assured if all germs (X_ℓ, x_ℓ) are isolated singularities). A *versal*, resp. *semiuniversal,* deformation of the multigerm (X, x) is a multigerm $(\boldsymbol{i}, \boldsymbol{\phi}) = \coprod_{\ell=1}^{r}(i_\ell, \phi_\ell)$ such that, for each $\ell = 1, \dots, r$, (i_ℓ, ϕ_ℓ) is a versal, resp. semiuniversal, deformation of (X_ℓ, x_ℓ) over (S_ℓ, s_ℓ), and the base space of $(\boldsymbol{i}, \boldsymbol{\phi})$ is the cartesian product of base spaces (S_ℓ, s_ℓ).

A semiuniversal deformation has the minimal dimension among all versal deformations (by Proposition 7.2.17). It has also another minimality property, the "economy of the semiuniversal deformation" due to Teissier [95, Theorem 4.8.4].

Theorem 7.2.19 (Teissier) *Let $\phi : (\mathscr{X}, x) \to (S, s)$ be the semiuniversal deformation of an isolated singularity (X, x). Then, for any $y \neq x$ sufficiently close to x no fibre $(\mathscr{X}_{\phi(y)}, y)$ is isomorphic to (X, x).*

This theorem can easily be deduced from the following general result about the trivial locus of a morphism due to Hauser and Müller [46], with special cases proved before in [33, Lemma 1.4] and [20, Corollary 0.2]. Recall that a morphism $f : (X, x) \to (S, s)$ of complex germs is called *trivial* if $(X, x) \cong (f^{-1}(s), x) \times (S, s)$ over (S, s). f is called *smooth* if it is trivial with $(f^{-1}(s), x)$ smooth. Let Z_{red} denote the reduction of the complex space Z.

Theorem 7.2.20 (Hauser, Müller) *For any morphism $f : (X, x) \to (S, s)$ of complex germs there exist complex germs $(Y, x) \subset (X, x)$ and $(T, s) \subset (S, s)$ with the following property for sufficiently small representatives.*

(1) $Y_{red} = \{y \in X \mid (X, y) \cong (X, x)\}$ and $T_{red} = f(Y)$.

(2) The restriction $f_Y : Y \to T$ is a smooth morphism.

(3) $f^{-1}(s) \cong f_Y^{-1}(s) \times Z$ for some complex space Z.

(4) If $\varphi : S' \to S$ is a morphism (of germs), then $\varphi^*(f) : X \times_S S' \to S'$ is trivial iff φ factors through T.

The universal property (7.2.20) implies that (T, s) is uniquely determined, while (Y, x) is only determined up to isomorphism over (T, s).

Denote by $\operatorname{Sing}(f)$ the *singular locus of* f, i.e., set of points in X where f is not smooth. For the proof of the following "openness of versality" theorem, we refer to [3] in the algebraic category, [73] for isolated singularities and [18, 19] in general.

Theorem 7.2.21 (Artin; Pourcin) *Let $f : X \to S$ be a flat morphism of complex spaces such that* $\operatorname{Sing}(f)$ *is finite over S. Then the set of points $s \in S$ such that f induces a versal deformation of the multigerm* $\big(X, \operatorname{Sing}(f^{-1}(s))\big)$ *is analytically open in S.*

Hence, if $\phi : (\mathscr{X}, x) \to (S, s)$ is a versal deformation of $\big(\phi^{-1}(s), x\big)$ then, for a sufficiently small representative $\phi : \mathscr{X} \to S$, the multigerm $\phi : \coprod_{x' \in \phi^{-1}(t)}(\mathscr{X}, x') \to (S, t),\ t \in S$, is a versal deformation of its fibre, the multigerm $\coprod_{x' \in \phi^{-1}(t)}\big(\phi^{-1}(t), x'\big)$. The nearby fibres have only isolated singularities, since $\operatorname{Sing}(f) \cap f^{-1}(s) = \operatorname{Sing}(f^{-1}(s))$ is a finite set by assumption. Note that an analogous statement does not hold for "semiuniversal" in place of "versal".

Although we cannot say anything specific about the semiuniversal deformation of an arbitrary singularity, the situation is different for special classes of singularities. For example, hypersurface singularities or, more generally, complete intersection singularities are never rigid and we can compute explicitly the semiuniversal deformation as in the following theorem (for a proof see [52, 96] or [35, Theorem I.1.16]).

Theorem 7.2.22 (Tjurina; Kas, Schlessinger) *Let $(X, \mathbf{0}) \subset (\mathbb{C}^n, \mathbf{0})$ be an isolated complete intersection singularity, and let $f := (f_1, \dots, f_k)$ be a minimal set of generators for the ideal of $(X, \mathbf{0})$. Let $g_1, \dots, g_\tau \in \mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}}$, $g_i = (g_i^1, \dots, g_i^k)$, represent a basis (respectively a system of generators) for the finite dimensional $\mathbb{C}$-vector space*[6]

$$T^1_{(X,\mathbf{0})} := \mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}} \big/ \big(Df \cdot \mathcal{O}^n_{\mathbb{C}^n,\mathbf{0}} + \langle f_1, \dots, f_k\rangle \mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}}\big)\,,$$

and set $F = (F_1, \dots, F_k)$,

$$\begin{aligned} F_1(\boldsymbol{x}, \boldsymbol{t}) &= f_1(\boldsymbol{x}) + \sum_{j=1}^{\tau} t_j g_j^1(\boldsymbol{x})\,, \\ \vdots \quad & \qquad \vdots \\ F_k(\boldsymbol{x}, \boldsymbol{t}) &= f_k(\boldsymbol{x}) + \sum_{j=1}^{\tau} t_j g_j^k(\boldsymbol{x})\,, \\ (\mathscr{X}, \mathbf{0}) &:= V(F_1, \dots, F_k) \subset (\mathbb{C}^n \times \mathbb{C}^\tau, \mathbf{0})\,. \end{aligned}$$

Then $(X, \mathbf{0}) \overset{i}{\hookrightarrow} (\mathscr{X}, \mathbf{0}) \xrightarrow{\phi} (\mathbb{C}^\tau, \mathbf{0})$ with i, ϕ being induced by the inclusion $(\mathbb{C}^n, \mathbf{0}) \subset (\mathbb{C}^n \times \mathbb{C}^\tau, \mathbf{0})$, resp. the projection $(\mathbb{C}^n \times \mathbb{C}^\tau, \mathbf{0}) \to (\mathbb{C}^\tau, \mathbf{0})$, is a semiuniversal (respectively versal) deformation of $(X, \mathbf{0})$.

Here, Df denotes the Jacobian matrix of f,

$$(Df) = \Big(\frac{\partial f_i}{\partial x_j}\Big) : \mathcal{O}^n_{\mathbb{C}^n,\mathbf{0}} \longrightarrow \mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}}\,,$$

---

[6] The vector space $T^1_{(X,x)}$ will be defined for arbitrary complex space germs (X, x) in Definition 7.2.26. For a definition of T^1 in a general deformation theoretic context see [35, Appendix C].

that is, $(Df) \cdot \mathcal{O}^n_{\mathbb{C}^n,\mathbf{0}}$ is the submodule of $\mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}}$ spanned by the columns of the Jacobian matrix of f.

Note that $T^1_{(X,\mathbf{0})}$ is an $\mathcal{O}_{X,\mathbf{0}}$-module, called the *Tjurina module* of the complete intersection $(X, \mathbf{0})$. If $(X, \mathbf{0})$ is a hypersurface, then $T^1_{(X,\mathbf{0})}$ is an algebra and called the *Tjurina algebra of* $(X, \mathbf{0})$. The number

$$\tau(X, x) := \dim_{\mathbb{C}} T^1_{(X,\mathbf{0})}$$

is called the *Tjurina number* of (X, x).

Since the hypersurface case is of special importance we state it explicitly.

Corollary 7.2.23 *Let* $(X, \mathbf{0}) \subset (\mathbb{C}^n, \mathbf{0})$ *be an isolated singularity defined by* $f \in \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$ *and* $g_1, \ldots, g_\tau \in \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$ *a* $\mathbb{C}$*-basis of the Tjurina algebra*

$$T^1_{(X,\mathbf{0})} = \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}/\big\langle f, \tfrac{\partial f}{\partial x_1}, \ldots, \tfrac{\partial f}{\partial x_n}\big\rangle .$$

If we set

$$F(\boldsymbol{x}, \boldsymbol{t}) := f(\boldsymbol{x}) + \sum_{j=1}^{\tau} t_j g_j(\boldsymbol{x}) , \quad (\mathscr{X}, \mathbf{0}) := V(F) \subset (\mathbb{C}^n \times \mathbb{C}^\tau, \mathbf{0}) ,$$

then $(X, \mathbf{0}) \hookrightarrow (\mathscr{X}, \mathbf{0}) \xrightarrow{\phi} (\mathbb{C}^\tau, \mathbf{0})$*, with* ϕ *the second projection, is a semiuniversal deformation of* $(X, \mathbf{0})$*.*

Remark 7.2.24 Using the notation of Theorem 7.2.22, we can choose the basis $g_1, \ldots, g_\tau \in \mathcal{O}^k_{\mathbb{C}^n,0}$ of $T^1_{(X,\mathbf{0})}$ such that $g_i = -e_i$, $e_i = (0, \ldots, 1, \ldots, 0)$ the i-th canonical generator of $\mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}}$, for $i = 1, \ldots, k$ (assuming that $f_i \in \mathfrak{m}^2_{\mathbb{C}^n,\mathbf{0}}$). Then

$$F_i = f_i - t_i + \sum_{j=k+1}^{\tau} t_j g^i_j ,$$

and we can eliminate $t_1, \ldots, t_k$ from $F_1 = \ldots = F_k = 0$. Hence, the semiuniversal deformation of $(X, \mathbf{0})$ is given by

$$\psi : (\mathbb{C}^n \times \mathbb{C}^{\tau-k}, \mathbf{0}) \to (\mathbb{C}^k \times \mathbb{C}^{\tau-k}, \mathbf{0}) = (\mathbb{C}^\tau, \mathbf{0})$$

with $\psi(\boldsymbol{x}, t_{k+1}, \ldots, t_\tau) = (G_1(\boldsymbol{x}, \boldsymbol{t}), \ldots, G_k(\boldsymbol{x}, \boldsymbol{t}), t_{k+1}, \ldots, t_\tau)$,

$$G_i(\boldsymbol{x}, \boldsymbol{t}) = f_i(\boldsymbol{x}) + \sum_{j=k+1}^{\tau} t_j g_j(\boldsymbol{x}) ,$$

where $g_j = (g_j^1, \dots, g_j^k)$, $j = k+1, \dots, \tau$, is a basis of the $\mathbb{C}$-vector space

$$\big(\mathfrak{m} \cdot \mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}}\big) \big/ \big((Df) \cdot \mathcal{O}^n_{\mathbb{C}^n,\mathbf{0}} + \langle f_1, \dots, f_k \rangle \mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}}\big)\,,$$

assuming $f_1, \dots, f_k \in \mathfrak{m}^2_{\mathbb{C}^n,\mathbf{0}}$.

In particular, if $f \in \mathfrak{m}^2_{\mathbb{C}^n,\mathbf{0}}$ and if $1, h_1, \dots, h_{\tau-1}$ is a basis of the Tjurina algebra T_f, then (setting $\boldsymbol{t} := (t_1, \dots t_{\tau-1})$

$$F\colon (\mathbb{C}^n \times \mathbb{C}^{\tau-1}, \mathbf{0}) \longrightarrow (\mathbb{C}^\tau, \mathbf{0})\,,\ (\boldsymbol{x}, \boldsymbol{t}) \mapsto \Big(f(\boldsymbol{x}) + \sum_{i=1}^{\tau-1} t_i h_i,\ \boldsymbol{t}\Big)$$

is a semiuniversal deformation of the hypersurface singularity $(f^{-1}(0), \mathbf{0})$.

Example 7.2.25 (1) Let $(X, \mathbf{0}) \subset (\mathbb{C}^3, \mathbf{0})$ be the isolated complete intersection curve singularity defined by $f_1(\boldsymbol{x}) = x_1^2 + x_2^3$ and $f_2(\boldsymbol{x}) = x_3^2 + x_2^3$. Then the Tjurina module is $T^1_{(X,\mathbf{0})} = \mathbb{C}\{\boldsymbol{x}\}^2/M$, where $M \subset \mathbb{C}\{\boldsymbol{x}\}^2$ is generated by $\binom{x_1}{0}, \binom{x_2^2}{x_2^2}, \binom{0}{x_3}, \binom{f_1}{0}, \binom{0}{f_1}, \binom{f_2}{0}, \binom{0}{f_2}$. We have $\tau = 9$ and a $\mathbb{C}$-basis for $T^1_{(X,\mathbf{0})}$ is given by $\binom{1}{0}, \binom{0}{1}, \binom{x_2}{0}, \binom{x_3}{0}, \binom{x_2x_3}{0}, \binom{x_2^2}{0}, \binom{0}{x_1}, \binom{0}{x_2}, \binom{0}{x_1x_2}$. By Remark 7.2.24, it follows that a semiuniversal deformation of $(X, \mathbf{0})$ is given by $\psi\colon (\mathbb{C}^{10}, \mathbf{0}) \to (\mathbb{C}^9, \mathbf{0})$,

$$(\boldsymbol{x}, \boldsymbol{t}) \longmapsto \big(f_1(\boldsymbol{x}) + t_1x_2 + t_2x_3 + t_3x_2x_3 + t_4x_2^2,\ f_2(\boldsymbol{x}) + t_5x_1 + t_6x_2 + t_7x_1x_2, \boldsymbol{t}\big)\,.$$

This can easily verified by a computation in SINGULAR.

7.2.4 Infinitesimal Deformations

In this section we develop the infinitesimal deformation theory for *arbitrary singularities*. In particular, we introduce in this generality the vector spaces $T^1_{(X,x)}$ of *first order deformations* that is, the linearization of the semiuniversal deformation of (X, x) and show how it can be computed. In the next section we describe the obstructions for lifting an infinitesimal deformation of a given order to higher order. This and the next section can be considered as a concrete special case of the general theory as described e.g. in [35, Appendix C].

Infinitesimal deformations of first order are deformations over the complex space T_ε, a "point with one tangent direction".

Definition 7.2.26

(1) The complex space germ $T_\varepsilon = (\{pt\}, \mathbb{C}[\varepsilon])$ consists of one point with local ring $\mathbb{C}[\varepsilon] = \mathbb{C}[t]/\langle t^2 \rangle$.

(2) For any complex space germ (X, x) we set

$$T^1_{(X,x)} := \underline{\mathcal{D}ef}_{(X,x)}(T_\varepsilon)\,,$$

the set of isomorphism classes of deformations of (X, x) over T_ε. Objects of $\mathcal{D}ef_{(X,x)}(T_\varepsilon)$ are called *infinitesimal* or *first order deformations* of (X, x).

(3) We shall see in Proposition 7.2.33 that $T^1_{(X,x)}$ carries the structure of a complex vector space. We call $T^1_{(X,x)}$ the *Tjurina module*, and

$$\tau(X, x) := \dim_{\mathbb{C}} T^1_{(X,x)}$$

the *Tjurina number* of (X, x).

Any singularity (X, x) with $\tau(X, x) < \infty$ has a semiuniversal deformation (see [26, 93]); it is not difficult to see that isolated singularities have finite Tjurina number.

The following lemma shows that $T^1_{(X,x)}$ can be identified with the Zariski tangent space to the semiuniversal base of (X, x) (if it exists).

Lemma 7.2.27 *Let (X, x) be a complex space germ and $\phi : (\mathcal{X}, x) \to (S, s)$ a deformation of (X, x). Then there exists a linear map*[7]

$$T_{S,s} \longrightarrow T^1_{(X,x)}\,,$$

called the Kodaira-Spencer map, *which is surjective if ϕ is versal and bijective if ϕ is semiuniversal.*

Moreover, if (X, x) admits a semiuniversal deformation with smooth base space, then ϕ is semiuniversal iff (S, s) is smooth and the Kodaira-Spencer map is an isomorphism.

Proof For any complex space germ (S, s) we have $T_{S,s} = \mathrm{Mor}\big(T_\varepsilon, (S, s)\big)$. Define a map

$$\begin{aligned} \alpha\colon\ \mathrm{Mor}\big(T_\varepsilon, (S, s)\big) &\longrightarrow T^1_{(X,x)}\,, \\ \varphi &\mapsto \big[\varphi^*\phi\big]\,. \end{aligned}$$

Let us see that α is surjective if ϕ is versal: given a class $[\psi] \in T^1_{(X,x)}$ represented by $\psi : (\mathcal{Y}, x) \to T_\varepsilon$, the versality of ϕ implies the existence of a map $\varphi\colon T_\varepsilon \to (S, s)$ such that $\varphi^*\phi \cong \psi$. Hence, $[\psi] = \alpha(\varphi)$, and α is surjective.

If ϕ is semiuniversal, the tangent map $T\varphi$ of $\varphi\colon T_\varepsilon \to (S, s)$ is uniquely determined by ψ. Since φ is uniquely determined by its algebra map $\varphi^\sharp\colon \mathcal{O}_{S,s} \to \mathcal{O}_{T_\varepsilon} = \mathbb{C}[t]/\langle t^2\rangle$ and, since $\varphi^\sharp$ is local, we obtain $\varphi^\sharp(\mathfrak{m}^2_{S,s}) = 0$. That is, φ is uniquely determined by $\underline{\varphi}^\sharp\colon \mathfrak{m}_{S,s}/\mathfrak{m}^2_{S,s} \longrightarrow \langle t\rangle/\langle t^2\rangle$ and hence by the dual map $\big(\underline{\varphi}^\sharp\big)^* = T\varphi$. Thus, α is bijective. We leave the linearity of α as an exercise.

If (T, t) is the smooth base space of a semiuniversal deformation of (X, x) then there is a morphism $\varphi : (S, s) \to (T, t)$ inducing the map $\alpha : T_{S,s} \to T_{T,t} \cong T^1_{(X,x)}$

---

[7] $T_{S,s}$ denotes the Zariski tangent space to S at s.

constructed above. Since (S, s) is smooth, φ is an isomorphism iff α is (by the inverse function theorem). □

Remark 7.2.28 Lemma 7.2.27 shows that $\dim_{\mathbb{C}} T^1_{(X,x)} < \infty$ if (X, x) admits a semiuniversal deformation. Together with Theorem 7.2.14 this shows that $\dim_{\mathbb{C}} T^1_{(X,x)} < \infty$ is necessary and sufficient for the existence of a semiuniversal deformation of (X, x). If (X, x) has an isolated singularity, then $\dim_{\mathbb{C}} T^1_{(X,x)} < \infty$ by Corollary 7.2.34 but the converse does not hold (see Example 7.3.4, below).

We want to describe now $T^1_{(X,x)}$ in terms of the defining ideal of (X, x) if (X, x) is embedded in some $(\mathbb{C}^n, \mathbf{0})$, without knowing a semiuniversal deformation of (X, x). To do this, we need embedded deformations, that is, deformations of the inclusion map $(X, x) \hookrightarrow (\mathbb{C}^n, \mathbf{0})$.

Slightly more general, we define deformations of a morphism, not necessarily an embedding.

Definition 7.2.29 Let $f: (X, x) \to (S, s)$ be a morphism of complex germs.

(1) A *deformation of* f, or a *deformation of* $(X, x) \to (S, s)$, over a germ (T, t) is a Cartesian diagram

$$\begin{tikzcd}
(X,x) \arrow[r, hook, "i"] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & (\mathscr{X},x) \arrow[d, "F"] \arrow[dd, bend left=60, "\phi"] \\
(S,s) \arrow[r, hook, "j"] \arrow[d] \arrow[dr, phantom, "\square"] & (\mathscr{S},s) \arrow[d, "p"] \\
\{pt\} \arrow[r, hook] & (T,t)
\end{tikzcd}$$

such that i and j are closed embeddings, and p and ϕ are flat, hence deformations of (X, x) and of (S, s) over (T, t) (but F is not supposed to be flat). We denote such a deformation by (i, j, F, p) or just by (F, p).

A *morphism* between two deformations (i, j, F, p) and (i', j', F', p') of f is a morphism of diagrams, denoted by $(\psi_1, \psi_2, \varphi)$:

$$\begin{tikzcd}
& (X,x) \arrow[dl, hook', "i"'] \arrow[dr, hook, "i'"] \arrow[dd] & \\
(\mathscr{X},x) \arrow[rr, "\psi_1"] \arrow[dd, "F"'] & & (\mathscr{X}',x') \arrow[dd, "F'"] \\
& (S,s) \arrow[dl, hook', "j"'] \arrow[dr, hook, "j'"] \arrow[dd] & \\
(\mathscr{S},s) \arrow[rr, "\psi_2"] \arrow[dd, "p"'] & & (\mathscr{S}',s') \arrow[dd, "p'"] \\
& \{pt\} \arrow[dl, hook'] \arrow[dr, hook] & \\
(T,t) \arrow[rr, "\varphi"] & & (T',t')
\end{tikzcd}$$

If ψ_1, ψ_2, φ are isomorphisms, then $(\psi_1, \psi_2, \varphi)$ is an isomorphism of deformations of f.

We denote by $\mathcal{D}ef_f = \mathcal{D}ef_{(X,x)\to(S,s)}$ the category of deformations of f, by $\mathcal{D}ef_f(T,t) = \mathcal{D}ef_{(X,x)\to(S,s)}(T,t)$ the (non-full) subcategory of deformations of f over (T,t) with morphisms as in the diagram above and φ the identity on (T,t). Furthermore, we write

$$\underline{\mathcal{D}ef}_f(T,t) = \underline{\mathcal{D}ef}_{(X,x)\to(S,s)}(T,t)$$

for the set of isomorphism classes of such deformations.

(2) A deformation (i, j, F, p) of $(X,x) \to (S,s)$ inducing the trivial deformation of (S,s) is called a *deformation of* $(X,x)/(S,s)$ *over* (T,t) and denoted by (i, F) or just by F. A morphism is a morphism as in (1) of the form $(\psi, \mathrm{id}_{S,s} \times \varphi, \varphi)$; it is denoted by (ψ, φ).

$\mathcal{D}ef_{(X,x)/(S,s)}$ denotes the category of deformations of $(X,x)/(S,s)$, $\mathcal{D}ef_{(X,x)/(S,s)}(T,t)$ the subcategory of deformations of $(X,x)/(S,s)$ over (T,t) with morphisms being the identity on (T,t).

$\underline{\mathcal{D}ef}_{(X,x)/(S,s)}(T,t)$ denotes the set of isomorphism classes of such deformations.

The difference between (1) and (2) is that in (1) we deform (X,x), (S,s) and f, while in (2) we only deform (X,x) and f but not (S,s). Note that $\mathcal{D}ef_{(X,x)/\mathrm{pt}} = \mathcal{D}ef_{(X,x)}$.

The following easy lemma shows that embedded deformations (as in Corollary 7.2.9) are a special case of Definition 7.2.29 (2).

Lemma 7.2.30 *Let* $f\colon (X,x) \to (S,s)$ *be a closed embedding of complex space germs and let* $(\mathscr{X},x) \xrightarrow{F} (\mathscr{S},s) \xrightarrow{p} (T,t)$ *be a deformation of* f. *Then* $F\colon (\mathscr{X},x) \to (\mathscr{S},s)$ *is a closed embedding, too.*

Definition 7.2.31

(1) Let $(X,x) \hookrightarrow (S,s)$ be a closed embedding. The objects of $\mathcal{D}ef_{(X,x)/(S,s)}$ are called *embedded deformations* of (X,x) (in (S,s)).

(2) For an arbitrary morphism $f\colon (X,x) \to (S,s)$ we define

$$T^1_{(X,x)\to(S,s)} := \underline{\mathcal{D}ef}_{(X,x)\to(S,s)}(T_\varepsilon)\,,$$

respectively

$$T^1_{(X,x)/(S,s)} := \underline{\mathcal{D}ef}_{(X,x)/(S,s)}(T_\varepsilon)\,,$$

and call its elements the isomorphism classes of *(first order) infinitesimal deformations of* $(X,x) \to (S,s)$, respectively of $(X,x)/(S,s)$.

The vector space structure on $T^1_{(X,\mathbf{0})/(\mathbb{C}^n,\mathbf{0})}$ and $T^1_{(X,\mathbf{0})}$ is given by the isomorphisms in Proposition 7.2.33 below. We are going to describe $T^1_{(X,\mathbf{0})/(\mathbb{C}^n,\mathbf{0})}$ and $T^1_{(X,0)}$ in terms of the equations defining $(X, 0) \subset (\mathbb{C}^n, \mathbf{0})$. First, we need some preparations.

Definition 7.2.32 Let S be a smooth n-dimensional complex manifold and $X \subset S$ a complex subspace given by the coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_S$.

(1) The sheaf $(\mathcal{I}/\mathcal{I}^2)\big|_X$ is called the *conormal sheaf* and its dual

$$\mathcal{N}_{X/S} := \mathcal{H}om_{\mathcal{O}_X}\big((\mathcal{I}/\mathcal{I}^2)\big|_X, \mathcal{O}_X\big)$$

is called the *normal sheaf* of the embedding $X \subset S$.

(2) Let $\Omega^1_X = \big(\Omega^1_S/(\mathcal{I}\cdot\Omega^1_S + d\mathcal{I}\cdot\mathcal{O}_S)\big)\big|_X$ be the sheaf of holomorphic 1-forms on X. The dual sheaf

$$\Theta_X := \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$$

is called the sheaf of *holomorphic vector fields* on X.

Recall that, for each coherent $\mathcal{O}_X$-sheaf $\mathcal{M}$, there is a canonical isomorphism of $\mathcal{O}_X$-modules

$$\mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{M}) \xrightarrow{\cong} \mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X, \mathcal{M}), \quad \varphi \longmapsto \varphi \circ d,$$

where $d\colon \mathcal{O}_X \to \Omega^1_X$ is the exterior derivation and where $\mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X, \mathcal{M})$ is the *sheaf of $\mathbb{C}$-derivations of $\mathcal{O}_X$ with values in $\mathcal{M}$*. In particular, we have

$$\Theta_X \cong \mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X).$$

Moreover, since S is smooth the sheaf Ω^1_S is locally free with $\Omega^1_{S,s} = \bigoplus_{i=1}^n \mathcal{O}_{S,s}dx_i$ (where $x_1, \ldots, x_n$ are local coordinates of S with center s). As a consequence we have that Θ_S is locally free of rank n and

$$\Theta_{S,s} = \bigoplus_{i=1}^{n} \mathcal{O}_{S,s} \cdot \frac{\partial}{\partial x_i}$$

where $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ is the dual basis of $dx_1, \ldots, dx_n$.

Let $f \in \mathcal{O}_S$ then, in local coordinates, we have $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$. In particular, we can define an $\mathcal{O}_S$-linear map $\alpha\colon \mathcal{I} \to \Omega^1_S,\ f \mapsto df$. Due to the Leibniz rule, α induces a map $\alpha : \mathcal{I}/\mathcal{I}^2 \to \Omega^1_S \otimes_{\mathcal{O}_S} \mathcal{O}_X$ yielding the following exact *Zariski-Jacobi* sequence

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{\alpha} \Omega^1_S \otimes_{\mathcal{O}_S} \mathcal{O}_X \longrightarrow \Omega^1_X \longrightarrow 0,$$

$\alpha([f]) = df$. By $\mathcal{O}_X$-dualizing, we obtain the exact sequence

$$0 \longrightarrow \Theta_X \longrightarrow \Theta_S \otimes_{\mathcal{O}_S} \mathcal{O}_X \xrightarrow{\beta} \mathcal{N}_{X/S} = \mathscr{H}om_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X)\,,$$

where β is the dual of α. In local coordinates, we have for each $x \in X$

$$\Theta_{S,s} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{X,x} = \bigoplus_{i=1}^{n} \mathcal{O}_{X,x} \cdot \tfrac{\partial}{\partial x_i}$$

and the image $\beta\big(\frac{\partial}{\partial x_i}\big) \in \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{I}_x/\mathcal{I}_x^2, \mathcal{O}_{X,x}) = \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{I}_x, \mathcal{O}_{X,x})$ is the map

$$\beta\big(\tfrac{\partial}{\partial x_i}\big) : \mathcal{I}_x/\mathcal{I}_x^2 \to \mathcal{O}_{X,x}, \quad [h] \mapsto \big[\tfrac{\partial h}{\partial x_i}\big].$$

with $[h]$ the residue class of h in $\mathcal{I}_x/\mathcal{I}_x^2$.

Using these notations we can describe the vector space structure of $T^1_{(X,\mathbf{0})/(\mathbb{C}^n,\mathbf{0})}$ and of $T^1_{(X,\mathbf{0})}$:

Proposition 7.2.33 *Let* $(X, \mathbf{0}) \subset (\mathbb{C}^n, \mathbf{0})$ *be a complex space germ and let* $\mathcal{O}_{X,\mathbf{0}} = \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}/I$. *Then*

1. $T^1_{(X,\mathbf{0})/(\mathbb{C}^n,\mathbf{0})} \cong \mathcal{N}_{X/\mathbb{C}^n,\mathbf{0}} \cong \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}}(I, \mathcal{O}_{X,\mathbf{0}})$.
2. $T^1_{(X,\mathbf{0})} \cong \mathrm{coker}(\beta)$, *that is, we have an exact sequence*

$$0 \longrightarrow \Theta_{X,\mathbf{0}} \longrightarrow \Theta_{\mathbb{C}^n,\mathbf{0}} \otimes_{\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}} \mathcal{O}_{X,\mathbf{0}} \xrightarrow{\beta} \mathcal{N}_{X/\mathbb{C}^n,\mathbf{0}} \longrightarrow T^1_{(X,\mathbf{0})} \longrightarrow 0\,,$$

where $\beta\big(\frac{\partial}{\partial x_i}\big) \in \mathrm{Hom}(I, \mathcal{O}_{X,\mathbf{0}})$ *sends* $h \in I$ *to the class of* $\frac{\partial h}{\partial x_i}$ *in* $\mathcal{O}_{X,\mathbf{0}}$.

3. *If* $(X, \mathbf{0})$ *is reduced then* $T^1_{(X,\mathbf{0})} \cong \mathrm{Ext}^1_{\mathcal{O}_{X,x}}(\Omega^1_{X,x}, \mathcal{O}_{X,x})$.[8]

For the proof of (1) and (2) we refer to [35, Proposition I.1.25] or [93]. To see (3) note that $\mathcal{I}/\mathcal{I}^2$ is free on the regular locus of X and hence $ker(\alpha)$ is concentrated on the singular locus of X and hence torsion since X is reduced. It follows that the dual of $\mathcal{I}/\mathcal{I}^2$ coincides with the dual of $(\mathcal{I}/\mathcal{I}^2)/ker(\alpha)$, which implies the claim.

Corollary 7.2.34 $\dim_{\mathbb{C}} T^1_{(X,x)} < \infty$ *if* (X, x) *is an isolated singularity.*

Remark 7.2.35 The proof of Proposition 7.2.33 shows the following:

---

[8] This formula is due to Tjurina [96], who introduced $\mathrm{Ext}^1_{\mathcal{O}_{X,x}}(\Omega^1_{X,x}, \mathcal{O}_{X,x})$ as vector space of infinitesimal deformations in her proof of the existence of a semiuniversal deformation of a normal isolated singularity with $\mathrm{Ext}^2_{\mathcal{O}_{X,x}}(\Omega^1_{X,x}, \mathcal{O}_{X,x}) = 0$. The notation T^i, in particular T^1 resp. T^2 as spaces of infinitesimal deformations resp. obstructions, was used in the paper by Lichtenbaum and Schlessinger on the cotangent complex [64]. The name "Tjurina number" for $\dim_{\mathbb{C}} T^1_{(X,x)}$ was introduced in [29].

(1) If $\mathcal{O}_{X,\mathbf{0}} = \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}/I,\ I = \langle f_1, \ldots, f_k \rangle$, then any embedded deformation of $(X, \mathbf{0}) \subset (\mathbb{C}^n, \mathbf{0})$ over T_ε is given by $f_i + \varepsilon g_i,\ i = 1, \ldots, k,\ g_i \in \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$, hence determined by $(g_1, \ldots, g_k)$. We define a map

$$\gamma \colon T^1_{(X,\mathbf{0})/(\mathbb{C}^n,\mathbf{0})} \longrightarrow \mathcal{N}_{X/\mathbb{C}^n,\mathbf{0}} \cong \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}}(I, \mathcal{O}_{X,\mathbf{0}})\,,$$
$$(g_1, \ldots, g_k) \longmapsto \Bigl(\varphi \colon \sum_{i=1}^{k} a_i f_i \mapsto \sum_{i=1}^{k} \bigl[a_i g_i\bigr]\Bigr)\,,$$

which is well-defined, since any relation $\sum_{i=1}^k r_i f_i = 0$ lifts to a relation $\sum_{i=1}^k (r_i + \varepsilon s_i)(f_i + \varepsilon g_i) = 0$ (by flatness, cf. Proposition 7.2.2) and hence $\sum_i r_i g_i \in I$.

(2) Let $F = (F_1, \ldots, F_k)$ be an embedded deformation of $(X, \mathbf{0})$ over T_ε given by $F_i = f_i + \varepsilon g_i,\ i = 1, \ldots, k$, as in (1) such that $\sum_i r_i g_i \in I$ for each relation $(r_1, \ldots, r_k)$ among $f_1, \ldots, f_k$. Then F and $F' = (F'_1, \ldots, F'_k),\ F'_i = f_i + \varepsilon g'_i$, define isomorphic embedded deformations over T_ε iff $g_i - g'_i \in I$. The vector space structure on the space of embedded deformations is given by

$$F + F' = \bigl(f_1 + \varepsilon(g_1 + g'_1), \ldots, f_k + \varepsilon(g_k + g'_k)\bigr)\,,$$
$$\lambda F = (f_1 + \varepsilon\lambda g_1, \ldots, f_k + \varepsilon\lambda g_k)\,, \quad \lambda \in \mathbb{C}\,.$$

(3) The embedded deformation defined by F as above is trivial as abstract deformation iff there is a vector field $\partial = \sum_{j=1}^n \delta_j \frac{\partial}{\partial x_j} \in \Theta_{\mathbb{C}^n,\mathbf{0}}$ such that

$$g_i = \partial(f_i) \mod I\,, \quad i = 1, \ldots, k\,.$$

In particular, if $I = \langle f \rangle$ defines a hypersurface singularity, then $f + \varepsilon g$ is trivial as abstract deformation iff $g \in \langle f, \frac{\partial f}{\partial x_j} \mid j = 1, \ldots, n \rangle$.

(4) By (2) and (3) the map γ of (1) is an isomorphism. Using γ^{-1}, the morphism β from Proposition 7.2.33(2) maps $\frac{\partial}{\partial x_j}$ to $(\frac{\partial f_1}{\partial x_j}, \ldots, \frac{\partial f_k}{\partial x_j})$ since $\beta(\frac{\partial}{\partial x_j})(\sum_{i=1}^k a_i f_i) = \sum_{i=1}^k \bigl[a_i \frac{\partial f_i}{\partial x_j}\bigr]$.

Proposition 7.2.33 provides an algorithm for computing $T^1_{(X,\mathbf{0})}$. This algorithm is implemented in the SINGULAR library `sing.lib`. The SINGULAR procedure `T_1` computes all relevant information about first order deformations. For details we refer to [35, Section I.1.4].

Infinitesimal deformations are the first step in formal deformation theory as developed by Schlessinger in a very general context (see [35, Appendix C] for a short overview). Schlessinger introduced what is nowadays called the *Schlessinger conditions* (H$_0$) – (H$_4$) in [81]. One can verify that $\underline{\mathcal{D}ef}_{(X,x)}$ satisfies conditions (H$_0$) – (H$_3$) and, therefore, has a formal versal deformation. Moreover, for every deformation functor satisfying the Schlessinger conditions, the corresponding infinitesimal deformations carry a natural vector space structure. For $T^1_{(X,x)}$ this structure coincides with the one defined above. Schlessinger's theory was generalized to groupoids by Rim [77], who studied infinitesimal deformation theory in the presence of automorphisms. A survey of deformations of complex spaces is given in [69], some aspects of deformations of singularities are covered by [93].

7.2.5 Obstructions

We have seen in Remark 7.2.28 that $\dim_{\mathbb{C}} T^1_{(X,x)} < \infty$ is a necessary and sufficient condition for (X,x) to admit a semiuniversal deformation. However, the existence says nothing about the semiuniversal base space. Some information is contained in the vector space $T^2_{(X,x)}$, which we describe below. This vector space contains the obstructions to extend a given deformation of (X,x) over a fat point to a bigger one. The construction of a semiuniversal deformation for a complex germ (X,x) with $\dim_{\mathbb{C}} T^1_{(X,x)} < \infty$ can be carried out as follows (for a $\mathbb{C}^*$-equivariant version see the proof of Theorem 7.3.5):

- We start with *first order deformations* and try to *lift these to second order* deformations. In other words, we are looking for possible liftings of a deformation (i,ϕ), $[(i,\phi)] \in \underline{\mathcal{D}ef}_{(X,x)}(T_\varepsilon) = T^1_{(X,x)}$, to a deformation over the fat point $(T',\mathbf{0})$ containing T_ε, for example to the fat point with local ring $\mathbb{C}[\eta]/\langle\eta^3\rangle$. Or, if we assume the deformations to be embedded (Corollary 7.2.9), this means that we are looking for a lifting of the first order deformation $f_i + \varepsilon g_i$, $\varepsilon^2 = 0$, to a second order deformation $f_i + \eta g_i + \eta^2 g_i'$, $\eta^3 = 0$, $i = 1,\dots,k$.
- This is exactly what we did when we constructed the semiuniversal deformation of a complete intersection singularity. By induction we showed the existence of a lifting to arbitrarily high order. In general, however, this is not always possible, there are *obstructions* against lifting. Indeed, there is an $\mathcal{O}_{X,x}$-module $T^2_{(X,x)}$ and, for each small extension of T_ε, an *obstruction map*

$$\mathrm{ob} : T^1_{(X,x)} \longrightarrow T^2_{(X,x)}$$

  such that the vanishing of $\mathrm{ob}\big([(i,\phi)]\big)$ is equivalent to the existence of a lifting of (i,ϕ) to the small extension, e.g. to second order as above.
- Assuming that the obstruction is zero, we choose a lifting to second order (which is, in general, not unique) and try to lift this to *third order*, that is, to a deformation over the fat point with local ring $\mathbb{C}[t]/\langle t^4\rangle$. Again, there is an obstruction map, and the lifting is possible iff it maps the deformation class to zero.
- Continuing in this manner, in each step, the preimage of 0 under the obstruction map defines homogeneous relations in terms of the elements $t_1,\dots,t_\tau$ of a basis of $(T^1_{(X,x)})^*$, of a given order, which in the limit yield formal power series in $\mathbb{C}[[\boldsymbol{t}]] = \mathbb{C}[[t_1,\dots,t_\tau]]$. If J denotes the ideal in $\mathbb{C}[[\boldsymbol{t}]]$ defined by these power series, the quotient $\mathbb{C}[[\boldsymbol{t}]]/J$ is the local ring of the base space of the (formal)

versal deformation. Then $T^1_{(X,x)} = (\langle \boldsymbol{t}\rangle/\langle \boldsymbol{t}\rangle^2)^*$ is the Zariski tangent space to this base space.

This method works for very general deformation functors having an obstruction theory. A collection of methods and results from general obstruction theory can be found in [35, Appendix C.2].

We give now a concrete description of the module $T^2_{(X,x)}$, containing the obstructions to lift a deformation from a fat point $(T, \mathbf{0})$ to an infinitesimally bigger one $(T', \mathbf{0})$.

Let $\mathcal{O}_{X,x} = \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}/I$, with $I = \langle f_1, \ldots, f_k\rangle$. Consider a presentation of I,

$$0 \longleftarrow I \xleftarrow{\alpha} \mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}} \xleftarrow{\beta} \mathcal{O}^\ell_{\mathbb{C}^n,\mathbf{0}}, \qquad \alpha(e_i) = f_i\,.$$

$\mathrm{Ker}(\alpha) = \mathrm{Im}(\beta)$ is the module of relations for $f_1, \ldots, f_k$, which contains the $\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$-module of *Koszul relations*

$$\mathrm{Kos} := \langle f_i e_j - f_j e_i \mid 1 \le i < j \le k\rangle\,,$$

$e_1, \ldots, e_k$ denoting the standard unit vectors in $\mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}}$. We set $\mathrm{Rel} := \mathrm{Ker}(\alpha)$ and note that Rel/Kos is an $\mathcal{O}_{X,x}$-module: let $\sum_i r_i e_i \in \mathrm{Rel}$, then

$$f_j \cdot \sum_{i=1}^k r_i e_i = f_j \cdot \sum_{i=1}^k r_i e_i - \sum_{i=1}^k r_i f_i e_j = \sum_{i=1}^k r_i \cdot (f_j e_i - f_i e_j) \in \mathrm{Kos}\,.$$

Since $\mathrm{Kos} \subset I\mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}}$, the inclusion $\mathrm{Rel} \subset \mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}}$ induces an $\mathcal{O}_{X,x}$-linear map

$$\mathrm{Rel}/\mathrm{Kos} \longrightarrow \mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}}/I\mathcal{O}^k_{\mathbb{C}^n,\mathbf{0}} = \mathcal{O}^k_{X,x}\,.$$

Definition 7.2.36 We define $T^2_{(X,x)}$ to be the cokernel of Φ, the $\mathcal{O}_{X,x}$-dual of the latter map, that is, we have a defining exact sequence for $T^2_{(X,x)}$:

$$\mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{O}^k_{X,x}, \mathcal{O}_{X,x}) \xrightarrow{\Phi} \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathrm{Rel}/\mathrm{Kos}, \mathcal{O}_{X,x}) \to T^2_{(X,x)} \to 0\,.$$

The following proposition is proved in [35, Proposition II.1.29] and [93, Chapter 3].

Proposition 7.2.37 *Let (X, x) be a complex space germ.*

(1) *Let $j : (T, \mathbf{0}) \hookrightarrow (T', \mathbf{0})$ be an inclusion of fat points, and let J be the kernel of the corresponding map of local rings $\mathcal{O}_{T',\mathbf{0}} \twoheadrightarrow \mathcal{O}_{T,\mathbf{0}}$. Then there is a map, called the* obstruction map,

$$\mathrm{ob} : \underline{\mathcal{D}ef}_{(X,x)}(T, \mathbf{0}) \longrightarrow T^2_{(X,x)} \otimes_{\mathbb{C}} J\,,$$

satisfying: a deformation $(i, \phi) : (X, x) \hookrightarrow (\mathscr{X}, x) \to (T, \mathbf{0})$ *admits a lifting* $(i', \phi') : (X, x) \hookrightarrow (\mathscr{X}', x) \to (T', \mathbf{0})$ *(i.e.,* $j^*(i', \phi') = (i, \phi)$*) iff* $\mathrm{ob}\big([(i, \phi)]\big) = 0$.

(2) If $T^1_{(X,x)}$ *is a finite dimensional* $\mathbb{C}$*-vector space and if* $T^2_{(X,x)} = 0$*, then the semiuniversal deformation of* (X, x) *exists and has a smooth base space (of dimension* $\dim_{\mathbb{C}} T^1_{(X,x)}$*).*

Note that the obstruction map ob is a map between sets (without further structure) as $\underline{\mathcal{D}ef}_{(X,x)}(T, \mathbf{0})$ is just a set.

Definition 7.2.38 We call (X, x) *unobstructed* if it has a semiuniversal deformation with smooth base space.

Hence an isolated singularity is unobstructed if $T^2_{(X,x)} = 0$, but (X, x) may be unobstructed even if $T^2_{(X,x)} \neq 0$.

If (X, x) is a hypersurface or, more generally, a complete intersection, then the Koszul relations are the only existing relations. Hence, in this case $\mathrm{Rel} = \mathrm{Kos}$ and $T^2_{(X,x)} = 0$. In particular, isolated complete intersection singularities are unobstructed.

Statement (2) of Proposition 7.2.37 can be generalized by applying Laudal's theorem [53, Theorem 4.2], which relates the base of a formal semiuniversal deformation of (X, x) with the fibre of a formal power series map:

Theorem 7.2.39 (Laudal) *Let* (X, x) *be a complex space germ such that* $T^1_{(X,x)}$, $T^2_{(X,x)}$ *are finite dimensional complex vector spaces. Then there exists a formal power series map*

$$\Psi : T^1_{(X,x)} \longrightarrow T^2_{(X,x)}$$

such that the fibre $\Psi^{-1}(0)$ *is the base of a formal semiuniversal deformation of* (X, x).

Corollary 7.2.40 *Let* (X, x) *be a complex space germ such that* $T^1_{(X,x)}$ *and* $T^2_{(X,x)}$ *are finite dimensional complex vector spaces, and let* (S, s) *be the base space of the semiuniversal deformation. Then*

$$\dim_{\mathbb{C}} T^1_{(X,x)} \geq \dim(S, s) \geq \dim_{\mathbb{C}} T^1_{(X,x)} - \dim_{\mathbb{C}} T^2_{(X,x)}\,,$$

and $\dim(S, s) = \dim_{\mathbb{C}} T^1_{(X,x)}$ *iff* (S, s) *is smooth.*

This corollary holds in a general deformation theoretic context (see [35, Proposition C.2.3]).

The $\mathcal{O}_{X,x}$-module $T^2_{(X,x)}$ contains the obstructions against smoothness of the base space of the semiuniversal deformation (if it exists), but it may be strictly bigger. That is, in Corollary 7.2.40, the dimension of (S, s) may be strictly larger than the difference $\dim_{\mathbb{C}} T^1_{(X,x)} - \dim_{\mathbb{C}} T^2_{(X,x)}$, as in the following example.

Example 7.2.41 Let us compute the full semiuniversal deformation of the cone $(X, \mathbf{0}) \subset (\mathbb{C}^5, \mathbf{0})$ over the rational normal curve of degree 4, using SINGULAR. We get $\dim_{\mathbb{C}} T^1_{(X,\mathbf{0})} = 4$ and $\dim_{\mathbb{C}} T^2_{(X,\mathbf{0})} = 3$ and that the semiuniversal base space has dimension 3. The total space of the semiuniversal deformation has 4 additional variables A, B, C, D (in the ring `Px`), the unfolding of the 6 defining equations of $(X, \mathbf{0})$ is given by the ideal `Fs` and the base space, which is given by the ideal `Js` in $\mathbb{C}\{A, B, C, D\}$, is the union of the 3-plane $\{D = 0\}$ and the line $\{B = C = D - A = 0\}$ in $(\mathbb{C}^4, \mathbf{0})$:[9]

```
LIB "deform.lib";
ring R = 0,(x,y,z,u,v),ds;
matrix M[2][4] = x,y,z,u,y,z,u,v;
ideal I = minor(M,2);   // rational normal curve in P^4
vdim(T_1(I));
//-> 4
vdim(T_2(I));
//-> 3

list L = versal(I);     // compute semiuniversal deformation
//-> // ready: T_1 and T_2
//-> // start computation in degree 2.
//-> .... (further output skipped) .....

def Px=L[1];
show(Px);
//-> // ring: (0),(A,B,C,D,x,y,z,u,v),(ds(4),ds(5),C);
//-> // minpoly = 0
//-> // objects belonging to this ring:
//-> // Rs                   [0]  matrix 6 x 8
//-> // Fs                   [0]  matrix 1 x 6
//-> // Js                   [0]  matrix 1 x 3

setring Px;
Fs;                     // equations of total space
//-> Fs[1,1]=-u2+zv+Bu+Dv
//-> Fs[1,2]=-zu+yv-Au+Du
//-> Fs[1,3]=-yu+xv+Cu+Dz
//-> Fs[1,4]=z2-yu+Az+By
//-> Fs[1,5]=yz-xu+Bx-Cz
//-> Fs[1,6]=-y2+xz+Ax+Cy
Js;                     // equations of base space
//-> Js[1,1]=BD
//-> Js[1,2]=-AD+D2
//-> Js[1,3]=-CD
```

---

[9] This example is due to Pinkham [70]. It played an important role at the beginning of deformation theory as the first example of a semiuniversal base space with several components. At that time there had been practically no examples of singularities with obstructed deformations.

Hence, the semiuniversal deformation of $(X, \mathbf{0})$ is given by $(\mathscr{X}, \mathbf{0}) \to (S, \mathbf{0})$, induced by the projection onto the first factor of $(\mathbb{C}^4, \mathbf{0}) \times (\mathbb{C}^5, \mathbf{0})$,

$$(\mathbb{C}^4, \mathbf{0}) \times (\mathbb{C}^5, \mathbf{0}) \supset V(\texttt{Fs}) = (\mathscr{X}, \mathbf{0}) \to (S, \mathbf{0}) = V(\texttt{Js}) \subset (\mathbb{C}^4, \mathbf{0}) .$$

Note that the procedure `versal` proceeds by lifting infinitesimal deformations to higher and higher order (as described in the proof of Proposition 7.2.37). In general, this process may be infinite (but `versal` stops at a predefined order). However, in many examples, it is finite (as in the example above).

We can further analyse the base space of the semiuniversal deformation by decomposing it into its irreducible components.

```
ring P = 0,(A,B,C,D),dp;
ideal Js = imap(Px,Js);
minAssGTZ(Js);
//-> [1]:
//->    _[1]=D
//-> [2]:
//->    _[1]=C
//->    _[2]=B
//->    _[3]=A-D
```

The output shows that the base space is reduced (the primary and prime components coincide) and that it has two components: a hyperplane and a transversal line.

Further developments: Abstract deformation theory, basically governed by the Schlessinger's conditions, has been further developed towards "Derived Deformation Theory" (cf.[63]) following the general trend in algebra and algebraic geometry to make everything "derived". While derived algebraic geometry has already become part of the mainstream, this does not yet apply to "Derived Singularity Theory".

7.3 Smoothing of Singularities

7.3.1 Rigidity and Smoothability

We give a brief review of some well-known results on the question of the smoothability and rigidity of singularities. Recall that a complex space germ (X, x) with isolated singularity is not obstructed iff the semiuniversal deformation base space is smooth.

Definition 7.3.1

(1) A singularity (X, x) is called *rigid* if any deformation of (X, x) over some base space (S, s) is *trivial*, that is, isomorphic to the *product deformation*

$$(X, x) \overset{i}{\hookrightarrow} (X, x) \times (S, s) \overset{p}{\to} (S, s)$$

with i the canonical inclusion and p the second projection.

(2) (X, x) is called *smoothable* if there exists a 1-parametric deformation $\phi : (\mathscr{X}, x) \to (\mathbb{C}, 0)$ of (X, x) such that for $t \in \mathbb{C} \setminus \{0\}$ sufficiently close to 0 the fibre $\mathscr{X}_t = \phi^{-1}(t)$ is smooth.

Rigid singularities are unobstructed. Smooth germs are rigid (by the implicit function theorem) and smoothable. For non-smooth singularities the notions are opposite to each other. If (X, x) has an isolated singularity then (X, x) is rigid iff the semiuniversal base is a reduced point while (X, x) is smoothable iff the semiuniversal base has a positive dimensional irreducible component over which the generic fibre is smooth. Such a component is called a *smoothing component.*[10]

Proposition 7.3.2 *A complex space germ is rigid iff* $T^1_{(X,x)} = 0$.

Proof (X, x) is rigid iff the semiuniversal deformation exists and consists of a single, reduced point. By Lemma 7.2.27, together with the existence of a semiuniversal deformation for germs with $\dim_{\mathbb{C}} T^1_{(X,x)} < \infty$, this is equivalent to $T^1_{(X,x)} = 0$. ☐

The existence of rigid singularities in small dimension is still an open problem. Assuming that the singularities are not smooth one may conjecture:

Conjecture 7.3.3 There exist no rigid fat points, no rigid reduced curve singularities and no rigid normal surface singularities.

Example 7.3.4

(1) The simplest known example of an equidimensional (non-smooth) rigid singularity $(X, \mathbf{0})$ is the union of two planes in $(\mathbb{C}^4, \mathbf{0})$, meeting in one point and defined by $\langle x, y\rangle \cap \langle z, w\rangle$ (given by the ideal `I` in the ring `R` below).

(2) The product $(X, \mathbf{0}) \times (\mathbb{C}, 0) \subset (\mathbb{C}^5, \mathbf{0})$ (given by the ideal `I` in the ring `R1 below`) has a non-isolated singularity but is also rigid (hence, has a semiuniversal deformation). We prove these statements using SINGULAR [9]:

```
LIB "deform.lib";
ring R = 0,(x,y,u,v),ds;
ideal I = intersect(ideal(x,y),ideal(u,v));
vdim(T_1(I));          // result is 0 iff V(I) is rigid
//-> 0
ring R1 = 0,(x,y,u,v,w),ds;
ideal I = imap(R,I);
dim_slocus(I);         // dimension of singular locus of V(I)
//-> 1
vdim(T_1(I));
//-> 0
```

(3) An even simpler (but not equidimensional) rigid singularity is the union of the plane $\{x = 0\}$ and the line $\{y = z = 0\}$ in $(\mathbb{C}^3, \mathbf{0})$. This can be checked either by using SINGULAR as above, or, without computer, by showing that the map β in Proposition 7.2.33 is surjective.

---

[10] The term "smoothing component" was coined by Wahl in [99].

Let us first recall some known results on rigidity. In 1964 Grauert and Kerner [25] generalized Thom's example (see below) and showed that the Segre cone over $\mathbb{P}^r \times \mathbb{P}^1$ in $\mathbb{P}^{2r+1}(r \geq 2)$ is rigid and gave thus the first example of a (non-smooth) rigid singularity. Further examples of rigid singularities are due to Schlessinger (isolated quotient singularities of dimension ≥ 3 [82], and to Rim (e.g. the one-point union of two copies of $(\mathbb{C}^n, 0)$ in $(\mathbb{C}^{2n}, 0)$ for $n \geq 2$, [76]). Examples of singularities that are not deformable into rigid singularities (so-called "generic singularities") are due to Schlessinger [83] (dim $\geq$ 3) and Mumford [67] (dim $\geq$ 2) (c.f. also [70]). By Herzog [45] one-dimensional, almost-complete intersections are not rigid. It is also known that monomial (i.e., irreducible, quasihomogenous) curves are not rigid [78, 5.12], [7, 3.1.2]. To date, no examples of rigid curve singularities are known; it is conjectured that they do not exist. A detailed discussion of rigid and smoothable singularities together with references up to 1973 can be found in Hartshorne [43], where also topological conditions for smoothability are derived.

The question whether a singularity (X, x) is smoothable is among others interesting because the smooth nearby fibre is an important topological object associated to the singularity that has been (and still is) a continuous subject of research (see Sect. 7.3.3). Classically, it was even suspected that all singularities are smoothable. In 1909 Severi postulated that each algebraic variety with arbitrary singularities should be the limit of a family of nonsingular algebraic manifolds ([86, p.45] and in [87, p.355] for curves). In fact he conjectured that an irreducible curve can be smoothed in a family of curves with constant degree and arithmetic genus, i.e., in a flat family. It was a guiding principle of Severi in [87] to obtain statements about singular curves from their smoothing.

Of course, hypersurfaces (e.g. plane curves) are smoothable but Severi's general postulate turned out to be wrong. The first example of a non-smoothable singularity, the cone apex of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^1$ in $\mathbb{P}^5$, was found by R. Thom in 1957. Thom gave topological reasons for non-smoothability; his argument was reproduced and worked out in 1974 in [43] (for a strengthening see Theorem 7.3.16). The first rigorous and pure algebraic proof was published anonymously in [105] in 1957 (according to Thom, the author is A. Weil). The author shows that the projective closure of Thom's example in $\mathbb{P}^6$ can not be smoothed in $\mathbb{P}^6$ (which is however weaker than abstract non-smoothability, cf. [71, 2.12]). Rees and Thomas [74, 75] developed Thom's idea further and found refined cobordism invariants of the neighborhood boundary of an isolated singularity (X, x) as a necessary condition for smoothability. They gave also further examples of non-smoothable singularities. Other conditions have been found by Sommese [89].

For a long time the conjecture of Severi that every reduced projective curve is smoothable had not been doubted (cf. [76] and [10, Conjecture 2.30]), until Mumford's, and later Pinkham's examples appeared in 1973. Mumford showed in [68] by an indirect argument that non-smoothable irreducible curve singularities exist. Reducible examples, related to r straight lines in $\mathbb{C}^n$ through 0 in general position, were first found by Pinkham [70]. We consider these examples and generalizations in Sect. 7.3.6.

Further results on smoothability:

1. Complete intersections are smoothable (by Sard's theorem) and non-obstructed [96].
2. A *determinantal singularity* (X, x) is given by the $t \times t$ minors of an $r \times s$ matrix with entries holomorphic functions in an open subset $U \subset \mathbb{C}^N$, such that (X, x) has codimension $(r - t + 1)(s - t + 1)$ in $\mathbb{C}^N$. If (X, x) is an isolated determinantal singularity and $2 \leq t \leq r \leq s$, then (X, x) is smoothable if $\dim(X, x) < s + r - 2t + 3$ [100, 6.2].
3. In particular, if (X, x) is Cohen-Macaulay of codimension[11] 2 (and hence determinantal) then (X, x) is smoothable provided $\dim(X, x) \leq 3$ (Schaps [80]). Note that the semiuniversal base is smooth for Cohen-Macaulay singularities in codimension 2 without any restriction on the dimension (Schlessinger, thesis, and [80]).
4. Moreover, an isolated *Pfaffian singularity* (X, x), defined by the $2m \times 2m$ Pfaffians of a skew-symmetric $(2n + 1) \times (2n + 1)$ matrix of holomorphic functions is smoothable if $\dim(X, x) < 4(n - m) + 7$ [100, 6.3].
5. An irreducible isolated Gorenstein singularity of $\operatorname{codim}(X, x) \leq 3$ (which is Pfaffian) has a smooth semiuniversal base and is moreover smoothable if $\dim(X, x) \leq 6$ (Waldi [104]).

Let us look at dimensions ≤ 2:

1. The first proof for the existence of nonsmoothable normal surface singularities was given by Mumford and later by Pinkham, see Sect. 7.3.2, where the case of surface singularities is treated in more detail.
2. For curves the following is known. In 1975 Mumford showed that non-smoothable curve singularities exist, using similar ideas as Iarrobino for his proof of the existence of non-smoothable fat points.
3. Since reduced curve singularities are Cohen-Macaulay, we get that reduced curves in $(\mathbb{C}^n, 0)$, $n \leq 3$, and, moreover, reduced, irreducible Gorenstein curves in $(\mathbb{C}^n, 0)$, $n \leq 4$, are smoothable and not obstructed. In [78] it is shown that negatively graded monomial curves are smoothable. For reduced quasihomogeneous curves this is not true by Pinkham's examples. Explicit examples of non-smoothable monomial curves were first found by Buchweitz in [7].

   It is interesting to note that the curves of Mumford and Pinkham are not smoothable, since the dimension of the base space of the semiuniversal deformation is "too large". There is no curve singularity known whose semiuniversal base has a smaller dimension than it would have by Deligne's formula (see Sect. 7.3.6) if it were smoothable.
4. Fat points in $\mathbb{C}^2$ are smoothable (c.f. [4]).

---

[11] The codimension of (X, x) is $\operatorname{codim}(X, x) = \operatorname{edim}(X, x) - \dim(X, x)$ with $\operatorname{edim}(X, x) = \dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2$ the embedding dimension.

5. Iarrobino [48] was the first to show (by a dimension count) the existence of non-smoothable points in $\mathbb{C}^n, n \geq 3$. In fact, it seems that up to today no one has found an *explicit example* of a non-smoothable fat point in $\mathbb{C}^3$. For an overview on the Hilbert scheme of points, i.e. the deformation theory of a collection of fat points in some projective space (until 1987), see [49].
6. Since then quite some work concerning smoothability of Artin algebras was done, in particular more examples and methods that show non-smoothability have been found, see e.g. Shafarevich [88], Erman [16], Huibregtse [47], and Jelisiejew [50]. All known examples up to 2018 are summarized in the recent paper [50, Remark 6.10].

7.3.2 Smoothing of Surface Singularities

Smoothability has been an important part of the deformation theory of normal surface singularities. For any smoothing one has a smooth Milnor fibre, a key topological object that has been intensively studied. The first to study systematically the topology of smoothings of normal surface singularities apart from complete intersections was Jonathan Wahl, who discovered in [100] in particular the topological difference of the Milnor fibres of two different types of smoothings.

Before, Mumford [67] and Pinkham [70, 71] had shown the existence of non-smoothable singularities by studying deformations of weighted homogeneous normal surface singularities. This approach was continued by Wahl in [100] and [101].

An arbitrary singularity (X, x) is *weighted homogeneous* if $\mathcal{O}_{X,x}$ is a graded algebra $\mathcal{O}_{X,x} = \mathbb{C}\{x_1, \ldots, x_n\}/I$, where the x_i have positive weights, $wt(x_i) = a_i > 0$, and I is a graded ideal, which is generated by (weighted) homogeneous polynomials f_i. Then (X, x) admits a good $\mathbb{C}^*$-action $\lambda \cdot (x_1, \ldots, x_n) = (\lambda^{a_1}x_1, \ldots, \lambda^{a_n}x_n)$. We call (X, x) *quasihomogeneous* if it is analytically isomorphic to a weighted homogeneous singularity.

The following result, a complement to Grauert's Theorem 7.2.14, was proved by Pinkham in [70, 71].

Theorem 7.3.5 (Pinkham)

1. *A weighted homogeneous isolated singularity (X, x) admits a semiuniversal deformation $\phi : (\mathscr{X}, x) \to (S, s)$ such that the $\mathbb{C}^*$-action extends to $(\mathscr{X}, x)$ and (S, s) with ϕ equivariant.*
2. *Any equivariant deformation $(\mathscr{Y}, y) \to (T, t)$ of (X, x) can be induced from $(\mathscr{X}, x) \to (S, s)$ via an equivariant base change morphism $\varphi : (T, t) \to (S, s)$.*
3. *For any equivariant deformation $(\mathscr{Y}, y) \to (T, t)$ choose homogeneous generators t_j of the maximal ideal of $\mathcal{O}_{T,t}$ and set $(T^-, t) = V\{t_j \mid wt(t_j) < 0\}$ (resp. $(T^0, t) = V\{t_j \mid wt(t_j) \neq 0\}$, resp. $(T^+, t) = V\{t_j \mid wt(t_j) > 0\}$).*

   Then the equivariant morphism $\varphi : (T, t) \to (S, s)$ of (2) restricts to $\varphi^- : (T^-, t) \to (S^-, s)$ (resp. $\varphi^0 : (T^0, t) \to (S^0, s)$, resp. $\varphi^+ : (T^+, t) \to (S^+, s)$).

Proof We sketch only the proof of (1) following [70, 71], who proves the statement in the setting of formal deformation theory. The analytic version for complex germs follows from an appropriate modification of [22, Proposition 1] taking care of the $\mathbb{C}^*$-action.

Choose homogeneous generators $f_1, \dots, f_k$ of I of (weighted) degree d_i. We use the exact sequence

$$\Theta_{\mathbb{C}^n,\mathbf{0}} \otimes_{\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}} \mathcal{O}_{X,\mathbf{0}} \xrightarrow{\beta} \operatorname{Hom}(I/I^2, \mathcal{O}_{X,\mathbf{0}}) = T^1_{X,\mathbf{0}/\mathbb{C}^n,\mathbf{0}} \longrightarrow T^1_{X,\mathbf{0}} \longrightarrow 0\,,$$

from Proposition 7.2.33, where the first module is graded by setting $wt(\frac{\partial}{\partial x_j}) = -a_i$. By Remark 7.2.35 every element in $T^1_{X,\mathbf{0}/\mathbb{C}^n,\mathbf{0}}$ is given by $f_i + \varepsilon g_i$, $i = 1, \dots, k$, i.e., given by a tupel $G = (g_1, \dots, g_k)$ with $g_i \in \mathcal{O}_{\mathbb{C}^n,0}$. We define G to be homogeneous of degree ν if g_i is homogeneous of degree $\nu + d_i$, thus imposing a grading on $T^1_{X,\mathbf{0}/\mathbb{C}^n,\mathbf{0}}$. It follows that β is homogeneous since $\beta(\frac{\partial}{\partial x_j}) = (\frac{\partial f_1}{\partial x_j}, \dots, \frac{\partial f_k}{\partial x_j})$. Therefore $\operatorname{coker}(\beta) = T^1_{X,\mathbf{0}}$ is graded and $T^1_{X,\mathbf{0}}$ decomposes into graded pieces

$$T^1_{X,\mathbf{0}} = \sum_{\nu \in \mathbb{Z}} T^1_{X,\mathbf{0}}(\nu).$$

We choose homogeneous elements $G_j = (g_1^j, \dots, g_k^j) \in T^1_{X,\mathbf{0}/\mathbb{C}^n,\mathbf{0}}$ with $\deg(G_j) = \nu_j$, mapping to a homogeneous basis of $T^1_{X,\mathbf{0}}$, $j = 1, \dots, \tau$. Choose new variables $t = (t_i, \dots, t_\tau)$ and set

$$(f_1', \dots, f_k') = (f_1, \dots, f_k) + \sum_{j=1,\dots,\tau} t_j (g_1^j, \dots, g_k^j) \pmod{\mathfrak{m}^2},$$

$\mathfrak{m}$ the maximal ideal of $\mathbb{C}\{t_1, \dots, t_\tau\}$. Then $(f_1', \dots, f_k')$ defines a first order deformation of $(X, \mathbf{0})$ with total space $(\mathscr{X}', \mathbf{0}) \subset (\mathbb{C}^n \times \mathbb{C}^\tau, \mathbf{0})$ defined by $\langle f_1', \dots, f_k' \rangle \subset \mathbb{C}\{x, t\}$ over $(S', 0) = V(\mathfrak{m}^2) \subset (\mathbb{C}^\tau, 0)$ as base space. Giving t_j the weight $-\nu_j$ then f_j' is homogeneous of degree d_j and $(\mathscr{X}', \mathbf{0}) \to (S', 0)$ is an equivariant deformation of first order.

Now we continue as in Sect. 7.2.5 and lift the first order deformation to second order but in an equivariant way. Continuing by induction, we get finally an equivariant semiuniversal deformation of $(X, \mathbf{0})$.

If the G_j are a system of generators of $T^1_{X,\mathbf{0}}$, we get an equivariant versal deformation. The equivariant base change property is proved in a similar way by induction. □

Remark 7.3.6

(1) It follows from the proof, that the base space of the semiuniversal deformation is given by a subgerm $(S, 0) \subset (\mathbb{C}^\tau, 0)$, $\tau = \dim_{\mathbb{C}} T^1_{X,\mathbf{0}}$, defined by some

homogeneous ideal in $\mathbb{C}\{t_1,\dots,t_\tau\}$, $wt(t_j) \in \mathbb{Z}$ (note that the signs of the weights of the variables t_j are opposite to signs of the weights of the tangent vectors). The total space of the semiuniversal deformation is then a subgerm $(\mathscr{X},\mathbf{0}) \subset (\mathbb{C}^n,\mathbf{0}) \times (S,0)$ and $\phi : (\mathscr{X},\mathbf{0}) \to (S,0)$ is the projection. $(\mathscr{X},\mathbf{0})$ is defined by a homogeneous ideal $J \subset \mathcal{O}_{(\mathbb{C}^n,\mathbf{0})\times(S,0)}$ generated by homogeneous power series

$$F_j(x,t) = f_j(x) + g_j(x,t) \in \mathbb{C}\{x_1,\dots,x_n,t_1,\dots,t_\tau\},\ g_j(x,0) = 0.$$

(2) The restriction $\phi^- : (\mathscr{X}^-,\mathbf{0}) \to (S^-,0)$ of ϕ is defined by $f_j(x) + g_j(x,t)$ with $\deg(g_j) \geq \deg(f_j)$ and any deformation which is induced from a map to $(S^-,0)$ is called a *deformation of non-positive weight*; if it is induced form a map to $(S^-,0) \cap (S^0,0)$ (i.e. $\deg g_j > \deg(fj)$) we call it a *deformation of negative weight*. Similarly we consider the restriction $\phi^+ : (\mathscr{X}^+,\mathbf{0}) \to (S^+,0)$ and speak of *deformations of non-negative* resp. of *positive weight*.

(3) It is easy to see that (X,x) cannot have smoothings of non-positive weight (consider the Jacobian of $(F_1,\dots,F_k$ and use that the total space of a 1-parametric smoothing has an isolated singularity). (X,x) may have smoothings of positive weight, but these are rare as we shall see.

Let us now recall the main results about smoothability for a normal surface singularity (X,x). Besides complete intersections the following is known:

1. *Rational singularities* and especially *quotient singularities* are always smoothable over the *Artin component*, i.e. the component of the semiuniversal base corresponding to deformations of (X,x), induced by blow down, from deformations of the resolution of (X,x) (cf. Artin [3]; see also [71, Proposition 6.10]).
2. A *normal surface singularity in* $(\mathbb{C}^4,0)$ is smoothable with a smooth semiuniversal base space since it is Cohen-Macaulay in codimension 2.
3. Let (X,x) be a *simple elliptic singularity*, i.e. the exceptional divisor of the minimal resolution consists of one elliptic curve with selfintersection number $-d$. Note that d is the multiplicity m of (X,x), except for $d=1$ where $m=2$. Then (X,x) is smoothable if and only if $m \leq 9$ (Pinkham [70]).
4. Let (X,x) be a *cusp singularity* where the exceptional curve of the minimal resolution consists of a cycle of r rational curves meeting transversally. Let m denote again the multiplicity. Then (X,x) is smoothable if $m^2 - m < r$ and is not smoothable if $m > r+9$ (Wahl [100, 5.6], [99, 5.12]).
5. Looijenga proved in [58] that whenever a cusp singularity is smoothable, the minimal resolution of the dual cusp is an anticanonical divisor of some smooth rational surface. He conjectured the converse. The conjecture was proved by Gross, Hacking, and Keel [38] using methods from mirror symmetry. For an alternative proof see [15].
6. If (X,x) is a *Dolgachev (or triangular) singularity*, then (X,x) is not smoothable if the multiplicity is ≥ 14 ([101]; see also [59]).

7. Note that the last three classes are *minimally elliptic singularities* in the sense of Laufer (i.e. Gorenstein and $h^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 1$ for any resolution $\tilde{X}$ of X). Karras proved in [51] that each minimally elliptic singularity (X, x) can be deformed into a simple elliptic singularity with the same multiplicity m. Hence a minimally elliptic singularity can be smoothed if $m \leq 9$.

Important obstructions against smoothability of an isolated singularity come from globalizing the smoothing. A smoothing of the globalized singularity (a projective variety) provides a smooth projective variety in some projective space with properties (coming from the singularity) that cannot exist. The following theorem uses this method and is due to Pinkham [70, Theorem 7.5].

Theorem 7.3.7 (Pinkham) *Let $C \subset \mathbb{P}^n$ be a smooth projectively normal curve of genus $g \geq 1$ and degree $d \geq 10$ if $g = 1$ or $d \geq 4g + 5$ if $g \geq 2$. Let $X \subset \mathbb{C}^{n+1}$ denote the affine cone over C. Then the singularity $(X, 0)$ is not smoothable.*

The proof makes use of the following theorem of [70, Theorem 4.2] that proves globalization for cones.

Theorem 7.3.8 (Pinkham) *Let $Y \subset \mathbb{P}^n$ be a nonsingular, projectively normal subvariety of dimension ≥ 1. Let X be the affine cone over Y in $\mathbb{C}^{n+1}$ and let $\overline{X}$ be its projective closure in $\mathbb{P}^{n+1}$. Assume that the homogeneous singularity $(X, 0)$ has negative grading (i.e., $T^1_{X,0}(\nu) = 0$ for all $\nu > 0$). Then any deformation of $(X, 0)$ lifts to an embedded deformation of $\overline{X} \subset \mathbb{P}^{n+1}$. More precisely, the morphism of deformation functors $\underline{\mathcal{D}ef}_{\overline{X}/\mathbb{P}^{n+1}} \to \underline{\mathcal{D}ef}_{(X,0)}$ is smooth.*

Remark 7.3.9 Pinkham proves the theorem only for infinitesimal deformations, i.e., for deformations over fat points. Let us see how this implies the theorem for deformations over arbitrary complex space germs: $\overline{X} \subset \mathbb{P}^{n+1}$ and $(X, 0)$ have both a convergent semiuniversal deformation. Pinkham's result implies that the induced morphism of the completion of the local rings of their base spaces is smooth, i.e. flat with smooth fibre. This implies that the morphism of their analytic local rings is smooth since completion is faithfully flat.

For a detailed study of deformations of cones over curves see also [93, Ch. 15].

The following theorem is due to Pinkham [71, 6.14] and Wahl [101, 3.9].

Theorem 7.3.10 (Pinkham; Wahl) *Let (X, x) be a normal Gorenstein surface singularity with weighted dual graph of the minimal resolution being star-shaped with n arms and a central rational curve of self-intersection $-(n - 2)$, where the end-vertex of the i-th arm corresponds to a smooth rational curve of self-intersection $-b_i$ ($n \geq 3$, $b_i \geq 2$). If (X, x) is smoothable, then*

$$\sum_{1 \leq i \leq n} (b_i - 1) \leq 19.$$

Such a singularity is weighted homogeneous and Pinkham proved the theorem for deformations of negative weight, where the negativity assumption is used to globalize the smoothing. Then he used hyperplane sections to find obstructions for smoothings.

It would follow right away from [100] that none of these singularities (with opposite inequality) could be smoothed for any deformation if his conjecture on globalization (Theorem 7.3.11) had already been proved. So Wahl used in [101, 3.8] an ad hoc argument for globalization, that under the assumptions of the theorem any deformation of (X, x), in particular any smoothing, can be globalized in the following sense:

Since (X, x) is weighted homogeneous with isolated singularity, it has an affine representative $X \subset \mathbb{C}^n$ with x as its only singularity. Let $\overline{X}$ be the projective closure in the corresponding weighted projective space. Then any deformation of the projective variety $\overline{X}$ induces a deformation of X and Wahl shows that the induced functor of deformation classes $\underline{\mathcal{D}ef}_{\overline{X}} \to \underline{\mathcal{D}ef}_X$ is smooth.

Later Looijenga proved [60, Appendix] that any smoothing of an arbitrary isolated singularity (X, x) can be globalized:

Theorem 7.3.11 (Looijenga) *Let* $f : (\mathscr{X}, x) \to (\mathbb{C}, 0)$ *be a smoothing over* $(\mathbb{C}, 0)$ *of an isolated singularity. Then there is a flat projective morphism* $F : \mathscr{Y} \to \mathbb{C}$*, a point* $y \in Y = F^{-1}(0)$ *and an isomorphism* $h : (\mathscr{X}, x) \to (\mathscr{Y}, y)$ *such that* $F \circ h = f$ *and* F *is smooth along* $Y \setminus \{y\}$*.*

Wahl's paper [100] contains several conjectures which have all been proved shortly after. The above globalization property implies that Wahl's Theorem 3.13 holds for any smoothing of a normal surface singularity. The same is true for his Corollary 4.6 due to the results of Looijenga and the author in [34], while Theorem 4.10 is valid for any smoothing of a Gorenstein surface singularity. The other conjectures made in [100] follow from the results of Steenbrink in [91] and Steenbrink and the author in [37]. See Sects. 7.3.3 and 7.3.5 for a treatment of these conjectures.

Since the 1990s many further examples of smoothable and non-smoothable singularities were found (a search for "smoothable" in zbMATH (Zentralblatt) or Mathematical Reviews lists about 300 articles), often in the global setting for projective varieties and as a result of research on other questions. Moreover, the smoothability assumption is often used in proofs. For a treatment of (formal) smoothing of singularities in the deformation theoretic setting of schemes see [44, Section 29].

7.3.3 Topology of the Milnor Fibre

The main object of research for smoothable surface singularities (X, x) is the topology of the Milnor fiber. For the classical theory of the Milnor fibration and related topics we refer to the textbooks by Milnor [66] (hypersurfaces), Looijenga [61] (complete intersections), Seade [84] (real singularities and index theorems), and Ebeling's article [14] (Milnor lattice, distinguished bases and monodromy). For a computational approach to topological invariants of hypersurfaces we refer to [11].

In general one calls the generic fibre of any 1-parametric deformation the *Milnor fibre of the deformation*. To speak about the topology we need to choose special neighbourhoods.

Let $(X, x) \subset (\mathbb{C}^N, x)$ be an arbitrary singularity. We consider a morphism $\phi : (\mathscr{X}, x) \to (S, s)$ with $(S, s) \subset (\mathbb{C}^k, s)$ and $(X, x) = (\phi^{-1}(s), x)$. We may assume that ϕ is embedded, i.e., $(\mathscr{X}, x)$ is a closed subgerm of $(\mathbb{C}^N, x) \times (S, s)$ and ϕ the projection to the second factor. Let $U \subset \mathbb{C}^N \times \mathbb{C}^k$ be an open neighbouhood of (x, s), $\widetilde{\mathscr{X}} \subset U$ a closed representative of $(\mathscr{X}, x)$ and $\phi : \widetilde{\mathscr{X}} \to S$ a representative of the germ ϕ. We choose now a special representative:

Definition 7.3.12 Let B_ε be an open ball of radius ε around x in $\mathbb{C}^N$ and $\overline{B}_\varepsilon$ the closed ball. Let S_δ be the intersection of S with an open ball D_δ of radius δ around s in $\mathbb{C}^k$ with $\overline{B}_\varepsilon \times D_\delta \subset U$ and $0 < \delta \ll \varepsilon$ sufficiently small. Then

$$\phi : \mathscr{X} := \widetilde{\mathscr{X}} \cap B_\varepsilon \times S_\delta \to S_\delta =: S$$

is called a *good representative* of ϕ. The fibres $\mathscr{X}_t = \phi^{-1}(t),\ t \in S$, are contained in a fixed small ball $B = B_\varepsilon$, also called a *Milnor ball*. Moreover,

$$\partial \mathscr{X}_t := \widetilde{\mathscr{X}_t} \cap \partial \overline{B},\ t \in S,$$

is the boundary of the closed fibre $\overline{\mathscr{X}}_t = \widetilde{\mathscr{X}_t} \cap \overline{B}$ and $\partial X = \partial \mathscr{X}_0$ is called the *neighbourhood boundary* of (X, x).

If $(S, s) = (\mathbb{C}, 0)$ we always write $\phi : \mathscr{X} \to D$ for the good representative and call

$$F := \mathscr{X}_t,\ t \in D \setminus \{0\},$$

the *Milnor fibre* of the 1-parametric deformation ϕ and $X = \mathscr{X}_0$ the *special fibre*.

$\mathscr{X}$ and all fibres $\mathscr{X}_t$ are Stein complex spaces while $\partial \mathscr{X}_t$ is a compact real algebraic subvariety of the sphere $\partial \overline{B}_\varepsilon$. The Milnor fibre depends of course in general on ϕ but for a given 1-parametric deformation its topological type is independent of $t \neq 0$ (cf. Theorem 7.3.15 below).

The following lemma is certainly well known to specialists, but because of missing an explicit reference, I like to sketch a proof (thanks to H. Hamm).

Lemma 7.3.13 *If (X, x) is an isolated singularity then there are only finitely many topologically different Milnor fibres for all deformations $\phi : \mathscr{X} \to D$ of (X, x) .*

Proof Let $\phi : \mathscr{X} \to S$ be a good representative of the semiuniversal deformation of (X, x). Since (X, x) has an isolated singularity, the *singular* or *critical locus* of ϕ,

$$C(\phi) := \{y \in \mathscr{X} \,|\, \mathscr{X}_{\phi(y)} \text{ is singular at } y\}$$

is finite over S and the *discriminant* of ϕ,

$$\Delta(\phi) := \phi(C(\phi))$$

is an analytic subset of S. Consider now a *proper* representative

$$\overline{\phi} : \overline{\mathscr{X}} := \widetilde{\mathscr{X}} \cap \overline{B}_\varepsilon \times S \to S.$$

The fibres $\overline{\mathscr{X}}_t$ of $\overline{\phi}$ meet $\partial \overline{B}_\varepsilon$ transversally such that all boundaries $\partial \mathscr{X}_t$ are differentiable manifolds.

Choose a Whitney stratification of S such that Δ is a union of strata. The restriction $\phi : C \to \Delta$ has a Whitney stratification that refines this stratification of Δ (see [24, I.1.7 Theorem, p. 43]). If one adds to the stratification of C the strata $\overline{\phi}^{-1}(T) \setminus C$, T a stratum of S, one gets a stratification of $\mathscr{X}$. With this stratification and that of S one obtains a Whitney stratification of the proper map $\overline{\phi}$ with finitely many strata. For every stratum T of S the restriction $\overline{\phi}^{-1}(T) \to T$ is a proper stratified submersion. According to Thom's isotopy theorem [24, I.1.5 Theorem, p. 41] the latter defines a topological fiber bundle and the topological type of $\overline{\phi}^{-1}(t)$, $t \in T$, is therefore independent of t. □

We are mainly interested in isolated singularities but let us first recall the following general result from [31].

Theorem 7.3.14 (Bobadilla, Greuel, Hamm) *Let $\phi : (\mathscr{X}, x) \to (\mathbb{C}, 0)$ be a morphism of complex germs and $\phi : \mathscr{X} \to D$ a good representative with special fibre X and Milnor fibre F.*

1. *If $(\mathscr{X}, x)$ is irreducible and X generically reduced then F is irreducible.*
2. *Let $(\mathscr{X}, x)$ be reducible with irreducible components $(\mathscr{X}_i, x)$, $i = 1, \ldots, r$, and assume that the intersection graph $G(\phi)$ is connected. Then F is connected.*
3. *In particular, if (X, x) is reduced then F is connected.*

Here $G(\phi)$ is the graph with vertices $i = 1, \ldots r$, and we join $i \neq j$ by an edge iff there exist points $y \in X \cap \mathscr{X}_i \cap \mathscr{X}_j$ arbitrary close to x ($y = x$ being allowed) such that (X, y) is reduced.

In (1) we need in fact only that at least one irreducible component of X is generically reduced. We do not assume that ϕ is flat, but this is practically irrelevant. Since flatness means that no irreducible component of $(\mathscr{X}, x)$ is mapped to 0, the irreducible components which are mapped to 0 do not contribute to the Milnor fibre and F is the same as the restriction of ϕ to the other components, which is flat.

The proof of Theorem 7.3.14 is somewhat involved and uses the monodromy and the following general fibration theorem of Lê Dũng Tráng (cf. [56], and [55] for a detailed account).

Theorem 7.3.15 (Lê) *Let $\phi : \mathscr{X} \to D$ be a good representative of $\phi : (\mathscr{X}, x) \to (\mathbb{C}, 0)$. Then*

$$\phi : \mathscr{X} \setminus \mathscr{X}_0 \to D \setminus \{0\}$$

is a topological fibre bundle with fibre F.

This theorem has been known before in many special cases, all generalizing Milnor's famous fibration theorem [66] for smoothings of an isolated hypersurface singularity.

If (X, x) has an isolated singularity, then ∂X is a real manifold diffeomorphic to $\partial \mathscr{X}_t$ for all $t \in S$ (by the Ehresmann fibration theorem, see e.g. [84]). Hence ∂F is independent of the deformation ϕ. If moreover ϕ is a smoothing then F is a Stein manifold, and ∂X can be filled by a complex Stein manifold. This imposes the following topological condition on the smoothability of (X, x) (cf. [37, 2.2 Corollary]), which is a strengthening of [43], who proved an analogous result for cohomology instead of homotopy.

Theorem 7.3.16 (Greuel, Steenbrink) *Let (X, x) be an isolated singularity of pure dimension n. If (X, x) is smoothable, then*

$$\pi_i(X \setminus \{x\}) = 0$$

for $0 \leq i \leq min\{n - 2, n - codim(X, x)\}$.

This result and a local Lefschetz-Barth theorem of Hamm [40] is used in the proof of the following result about the homotopy groups of the Milnor fibre (cf. [37, Theorem 1]). Since F is Stein, $\pi_i(F) = 0$ for $i > \dim(X, x)$, and for the other homotopy groups we have:

Theorem 7.3.17 (Greuel, Steenbrink) *Let F be the Milnor fiber of a smoothing of a pure n-dimensional isolated singularity (X, x). Then*

$$\pi_i(F) = 0 \, for \, 0 \leq i \leq n - codim(X, x).$$

The following theorem [37, Theorem 2] is the main result in [37]. It was conjectured by J. Wahl, who proved it when (X, x) is weighted homogeneous and the smoothing has negative weight (cf. [101]).

Theorem 7.3.18 (Greuel, Steenbrink) *Let F be the Milnor fiber of a smoothing of a normal isolated singularity. Then the first Betti number $b_1(F) := \dim_{\mathbb{C}} H^1(F, \mathbb{C}) = 0$.*

The proof considers a good representative $\phi : \mathscr{X} \to D$ of the smoothing of (X, x) and uses a resolution $\pi : \widetilde{\mathscr{X}} \to \mathscr{X}$ of singularities of $\mathscr{X}$, such that $E = \tilde{\phi}^{-1}(x)$, $\tilde{\phi} = \phi \circ \pi$, is a divisor with normal crossings. Using the normality of X, it is proved that $H^1(E, \mathbb{Z}) = H^1(\widetilde{\mathscr{X}}, \mathbb{Z}) = H^1(\widetilde{\mathscr{X}}, \mathcal{O}_{\widetilde{\mathscr{X}}}) = 0$. The hypercohomology sheaves $\mathbf{R}^p\tilde{\phi}_* K^\bullet$, $K^\bullet = \Omega^\bullet_{\widetilde{\mathscr{X}}/D}(\log E)$, of relative logarithmic differential forms are coherent (by [8]) and locally free (by [90]) and satisfy $b_1(F) = b_1(\widetilde{\mathscr{X}_t}) = \dim_{\mathbb{C}} \mathbf{H}^1(E, K^\bullet \otimes \mathcal{O}_E)$. A careful study of the 2nd spectral sequence of hypercohomology, and using $H^1(E, \mathbb{C}) = 0$, leads to the required result.

The following corollary is immediate:

Corollary 7.3.19 *Let X be any compact complex space with at most isolated normal singularities and let $\phi : (\mathscr{X}, X) \to (\mathbb{C}, 0)$, be any smoothing of X. Then $b_1(\mathscr{X}_t)$ is constant for any sufficiently small $t \in \mathbb{C} \setminus \{0\}$.*

The example of Pinkham (Example 7.2.41) shows that $\pi_1(F)$ need not be zero in Theorem 7.3.18. It is shown in [37] that the assumption "normal" is necessary with the following example. Take a smooth n-dimensional projective variety $E \subset \mathbb{P}^{N-1}$ and let $X \subset \mathbb{C}^N$ be the affine cone over E. Let $F_0 \subset \mathbb{P}^{N-1}$ be any smooth connected hypersurface of degree d, which intersects E in a smooth variety $E_0 = E \cap F_0$ and let G_0 be the affine cone over F_0. Consider in $\mathbb{C}^N$ the smoothing of the hypersurface section $X_0 = X \cap G_0$ through the origin by "sweeping out" the hypersurface section away from the origin with X_t the nearby (Milnor) fibre. It is shown that $b_1(X_t) \geq b_1(E)$ and hence X_0 cannot be normal if $b_1(E) \neq 0$ (plenty of such E exist).

In general the following holds by [37, 4.2 Proposition].

Proposition 7.3.20 *A normal connected projective variety E with* $\dim(E) \geq 2$ *admits a projective embedding with projectively normal hypersurface section iff* $b_1(E) = 0$.

Theorem 7.3.18 was generalized by van Straten in [94] using the same method.

Theorem 7.3.21 (van Straten) *Let $\phi : \mathscr{X} \to D$ be a good representative of a smoothing of a reduced equidimensional singularity (X, x) and F its Milnor fibre. Let $X^{[0]}$ denote the disjoint union of the irreducible components of X and $\gamma : H^0(X^{[0]}) \to Cl(\mathscr{X}, x)$ be the map that associates to a divisor supported on X its class in the local class group. Then $b_1(F) \geq rank Ker(\gamma) - 1$, with equality if X is weakly normal.*

For further results on the Milnor fibre of a normal surface singularity, in particular about its diffeomorphism type (up to 2016), see also the survey by Popescu-Pampu [72, Section 6.2].

7.3.4 Milnor Number Versus Tjurina Number

We will now review some results about the Milnor number $\mu(X, x)$, an important topological invariant of the singularity, and in particular its (in some sense mysterious) relation to the Tjurina number $\tau(X, x)$ (Definition 7.2.26), which is an analytic invariant. The Milnor number is defined as follows.

Definition 7.3.22 Let (X, x) be an n-dimensional isolated singularity and $\phi : \mathscr{X} \to D$ a good representative of a 1-parametric deformation of (X, x). The middle Bettti number of the Milnor fibre F of ϕ,

$$\mu_\phi := b_n(F) = \dim_{\mathbb{C}} H^n(F, \mathbb{C}),$$

is called the *Milnor number* of ϕ. If μ_ϕ is independent of the deformation and depends only on (X, x), we denote it by $\mu(X, x)$.

By Lemma 7.3.13 there are only finitely many Milnor numbers of (X, x). The fact that different smoothings of a normal surface singularity can lead to Milnor fibres with different Milnor numbers was first detected by Wahl in [100], where he gave also examples with non-vanishing $H^2(F)$ and $H^3(F)$.

In this section we consider only singularities (e.g. complete intersections) with a unique Milnor number. If (X, x) is a complete intersection or a normal isolated singularity then there are only two non-vanishing Betti numbers ($b_0(F) = 1$ and $b_n(F)$) (see Theorem 7.3.18 for the normal surface case). In general there are more non-vanishing Betti numbers.

Consider first a *hypersurface singularity* $(X, x) = (V(f), x)$ with isolated singularity, $f : (\mathbb{C}^{n+1}, x) \to (\mathbb{C}, 0)$ a holomorphic map germ. Then the Milnor fibre $F = f^{-1}(t)$ is an n-dimensional complex manifold, which is homotopy equivalent to a bouquet of n-dimensional real spheres (Milnor [66]). Therefore the homology groups $H^i(F, \mathbb{Z})$ do all vanish except for $i = 1, n$. The middle Betti number (F has real dimension $2n$)

$$\mu(X, x) = b_n(F) = \dim_{\mathbb{C}} H^n(F, \mathbb{C})$$

is the number of these spheres, is the *Milnor number* of (X, x) or of f. Milnor proved the algebraic formula

$$\mu(X, x) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^{n+1},x} / \left\langle \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle.$$

If (X, x) is an n-dimensional *isolated complete intersection singularity* (ICIS), then the homotopy type of the Milnor fibre is also a bouquet of n-spheres (Hamm [39, Satz 1.7]) and the number of these spheres is again the *Milnor number* of (X, x) and denoted by $\mu(X, x)$.

Since the base space of the semiuniversal deformation $\phi : (\mathscr{X}, x) \to (S, s)$ of an ICIS is smooth (cf. Theorem 7.2.22) there is only one Milnor fibre (up to diffeomorphism). In fact, the semiuniversal deformation is given by a flat morphism $\phi : (\mathbb{C}^{n+k}, x) \to (\mathbb{C}^k, 0)$ such that for a good representative $\phi : \mathscr{X} \to S$ the restriction $\phi : \mathscr{X} \setminus \phi^{-1}(\Delta(\phi)) \to S \setminus \Delta(\phi)$ is a $C^\infty-$ fibre bundle (by [66] for a hypersurface, and [39, Satz 1.6] for an ICIS). Here $C(\phi)$ denotes the critical locus and $\Delta(\phi) = \phi(C(\phi))$ the discriminant of ϕ.

Milnor's algebraic formula for $\mu(X, x)$ has been generalized to complete intersections independently by the author [28] (announced 1973 in [5]) and Lê Dũng Tráng [54]. The following result (cf. [28, Lemma 5.3]) is an important step in the proof and of independent interest in itself.

Proposition 7.3.23 *Let* (X, x) *be an* n*-dimensional ICIS,* $n \geq 0$*, and* $\phi : (\mathscr{X}, x) \to (\mathbb{C}, 0)$ *a deformation of* (X, x) *with* $(\mathscr{X}, x)$ *an ICIS. Then*

$$\mu(\mathscr{X}, x) + \mu(X, x) = \dim_{\mathbb{C}} O_{C(\phi),x},$$

with $O_{C(\phi),x}$ *the local ring of the singular locus of* ϕ *at* x*.*

If $(X, x) \subset (\mathbb{C}^{n+k}, x)$ is defined by $f_1, \ldots, f_k$ and $(\mathscr{X}, x)$ by $f_1, \ldots, f_{k-1}$ (i.e. $\phi = f_k|(\mathscr{X}, x)$), then

$$O_{C(\phi),x} := O_{\mathbb{C}^{n+k},x}/\langle f_1, \ldots, f_{k-1}, k\text{-minors of } Jac(f_1, \ldots, f_{k-1}, f_k)\rangle,$$

where Jac denotes the Jacobian matrix. We can choose the f_i such that $(X_i, x) = V(f_1, \ldots, f_{i-1})$, $i = 1, \ldots, k$, is an ICIS. Applying Proposition 7.3.23 to $f_i : (X_i, x) \to (\mathbb{C}, 0)$ we get

Theorem 7.3.24 (Greuel, Lê)

$$\mu(X, x) = \sum_{i=1}^{k} (-1)^{k-i} \dim_C O_{C(f_i),x}.$$

The proofs in [54] and [28] are very different. While the first is topological the second is algebraic and uses the Poincaré complex $\Omega^\bullet_{X,x}$ of holomorphic differential forms and an index theorem of Malgrange. An important result in [28, Proposition 5.1], from which Theorem 7.3.24 is deduced and which has been extended to Gorenstein curves, is the following.

Theorem 7.3.25 (Greuel) *Let* (X, x) *be an* n*-dimensional ICIS. Then*

$$\mu(X, x) = \begin{cases} \dim_{\mathbb{C}} \Omega^n_{X,x}/d\Omega^{n-1}_{X,x} & if\ n > 0 \\ \dim_{\mathbb{C}} O_{X,x} - 1 & if\ n = 0. \end{cases}$$

If (X, x) is quasihomogeneous, then $\mu(X, x)$ can be expressed purely in terms of the weights and degrees of the defining weighted homogeneous polynomials (c.f. [32]).

For a hypersurface defined by $f \in \mathcal{O}_{\mathbb{C}^{n+1},0}$ we have obviously the inequality $\mu(f) \leq \tau(f)$, which follows from the formulas for μ and τ. By a theorem of Saito [79] we have $\mu(f) = \tau(f)$ iff f is analytically equivalent to a weighted homogeneous polynomial. The same result was conjectured in [29] for complete intersections although the relationship is not at all obvious (τ is the dimension of a vector space while μ is an alternating sum). After proofs in special cases in [29, 30, 36, 97, 102] the final proof follows from results by the author [28, Korollar 5.8], [29, 3.1 Satz], by Looijenga-Steenbrink [62] and by Vosegaard [98].

Theorem 7.3.26 *Let $(X, x) \subset (\mathbb{C}^m, x)$ be an ICIS of positive dimension, defined by $f_1, \dots, f_k$.*

1. (Looijenga-Steenbrink) $\mu(X, x) \geq \tau(X, x)$.
2. (Greuel) *If (X, x) is quasihomogeneous, then* $\mu(X, x) = \tau(X, x) = \tau'(X, x) := \dim_{\mathbb{C}} \mathcal{O}_{C(X),x}$, *with* $\mathcal{O}_{C(X),x} = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^m,x}/\langle f_1, \dots, f_k, k\text{-}minors\ of\ Jac(f_1, \dots, f_k)\rangle$.
3. (Vosegaard) *If $\mu(X, x) = \tau(X, x)$ then (X, x) is quasihomogenous.*

Each item is hard to prove. (1) was proved for $\dim(X, x) = 1$ or if ∂X is a rational homology sphere in [29]. (3) was also proved before in special cases: for $\dim(X, x) = 1$ by Greuel-Martin-Pfister in [36], see also Corollary 7.3.55 (for Gorenstein curves, in the irreducible case already in [30]), for $\dim(X, x) = 2$ by Wahl in [102] and for a purely elliptic ICIS of dimension ≥ 2 by Vosegaard in [97].

Proposition 7.3.27 (Greuel) *Let (X, x) be an n-dimensional ICIS, $n \geq 1$.*

1. *Let $\Omega^\bullet_{X,x}$ be the Poincaré complex and $H^0_{\{x\}}$ local cohomology (in this case $H^0_{\{x\}}(\Omega^\bullet_{X,x})$ is the torsion submodule $T\Omega^\bullet_{X,x}$), then*

$$\tau'(X, x) = \tau''(X, x) := \dim_{\mathbb{C}} H^0_{\{x\}}(\Omega^n_{X,x}),$$

   by [28, Proposition 1.11].
2. *By [28, Proposition 5.7] we have*

$$\mu(X, x) = \tau''(X, x) + \dim_{\mathbb{C}} H^n(\Omega^\bullet_{X,x}/T\Omega^\bullet_{X,x}).$$

   In particular $\tau'(X, x) \leq \mu(X, x)$ with equality if (X, x) is quasihomogeneous (there are however non-quasihomogeneous examples with $\mu = \tau'$ for $n \geq 2$).
3. *Moreover,*

$$\tau(X, x) = \dim_{\mathbb{C}} \mathrm{Ext}^1_{\mathcal{O}_{X,x}}(\Omega^1_{X,x}, \mathcal{O}_{X,x}) = \dim_{\mathbb{C}} H^{n-1}_{\{x\}}(\Omega^1_{X,x}),$$

   where the first equality is due to Tjurina [96] and the second follows from local duality, see [29, 1.2 Satz]. In particular $\tau(X, x) = \tau'(X, x)$ if $n = 1$.

In general no relation between $\tau(X,x)$ and $\tau'(X,x)$ is known. Based on computations with SINGULAR we conjecture:

Conjecture 7.3.28 $\tau(X,x) \le \tau'(X,x)$.

If (X,x) is not an ICIS, the base space of the semiuniversal deformation may have several irreducible components and the topology of a nearby generic fibre depends in general on the component over which the fibre lives. This situation is studied in detail in the next section.

However, there are classes of singularities other than ICIS which have a smooth semiuniversal base space, like Cohen-Macaulay singularities in codim 2 or Gorenstein in codim 3. For these there is a unique (up to homeomorphism) Milnor fiber (the generic fibre over the semiuniversal base) and a unique Milnor number, defined as the middle Betti number of the Milnor fibre (c.f. Definition 7.3.30). A special case are normal surface singularities in $(\mathbb{C}^4, 0)$. They are smoothable, with a smooth semiuniversal base space and, if they are Gorenstein then they are already a complete intersection. For these Wahl offered in [103] the following conjecture:

Conjecture 7.3.29 (Wahl) Let (X,x) be a normal surface singularity in $(\mathbb{C}^4, 0)$, not a complete intersection. Then

$$\mu(X,x) \ge \tau(X,x) - 1,$$

with equality if and only if (X,x) is quasihomogeneous.

7.3.5 Smoothing Components

For an isolated singularity (X,x), which is not a complete intersection, the semiuniversal base space may have several irreducible components (see Pinkham's Example 7.2.41) and the Milnor fibre depends in general on the smoothing. It is interesting to know, which properties are independent of the smoothing and depend only on (X,x). Let

$$\Psi : (\mathscr{Y}, y) \to (S, s)$$

be the semiuniversal deformation of (X,x). Recall that an irreducible component (S', s) of (S,s) is called a *smoothing component*, if the generic fibre F over S' is smooth. The diffeomorphism type of F depends only on (S', s) and F is the Milnor fibre of this component.

Definition 7.3.30 Let (S', s) be a smoothing component of the isolated singularity (X,x) and $\phi : (\mathscr{X}, x) \to (\mathbb{C}, 0)$ a smoothing induced by a morphism $j : (\mathbb{C}, 0) \to (S', s)$. We denote the *dimension of the smoothing component* by

$$e_\phi := \dim(S', s).$$

If (S, s) is smooth (e.g. for (X, x) a complete intersection), then e_ϕ is independent of ϕ and equal to $\tau(X, x)$.

The first to study systematically different smoothings and the corresponding Milnor fibres was Wahl in [100]. It was already mentioned after Theorem 7.3.11 that his conjectures there have all been proved and that several of his statements are now valid in greater generality. Wahl considered in [100] normal surface singularities that are not complete intersections and compares the Milnor number of a smoothing with the dimension of the smoothing component over which the smoothing occurs. In [100, Conjecture 4.2] Wahl made the following interesting conjecture about e_ϕ, which he proved in special cases and which was fully proved by the author and Looijenga in [34].

Theorem 7.3.31 (Greuel, Looijenga) *With the assumptions of Definition 7.3.30 we have*

$$\dim(S', s) = \dim_{\mathbb{C}} \operatorname{coker}(\Theta_{\mathscr{X}/\mathbb{C},x} \to \Theta_{X,x}),$$

with $\Theta_{\mathscr{X}/\mathbb{C}}$ the sheaf of relative derivations.

We will comment on the proof at the end of this section.

The more recent paper [103] is partly an updated survey on old results, but it contains also new results and conjectures on normal surface singularities, which we like to recall. Let $\phi : (\mathscr{X}, x) \to (\mathbb{C}, 0)$ be a smoothing of a normal surface singularity (X, x). Then Wahl introduced another invariant,

$$\alpha_\phi := \dim_{\mathbb{C}} \operatorname{coker}(\omega^*_{\mathscr{X}/\mathbb{C},x} \otimes \mathcal{O}_{X,x} \to \omega^*_{X,x}),$$

with $\omega^*_{\mathscr{X}/\mathbb{C}}$ the $\mathcal{O}_{\mathscr{X}}$-dual of the relative dualizing sheaf.

Using Theorems 7.3.18 and 7.3.31 Wahl relates μ_ϕ, e_ϕ and α_ϕ with resolution invariants of (X, x) and proves ([100, Theorem 3.13] and [103, Theorem 1.1] where Wahl denotes our e_ϕ by τ_ϕ[12]):

Theorem 7.3.32 (Wahl) *Let $\phi : (\mathscr{X}, x) \to (\mathbb{C}, 0)$ be a smoothing of a normal surface singularity (X, x) and let $(Y, E) \to (X, x)$ be a good resolution. Then (with χ_T the topological Euler characteristic)*

1. $1 + \mu_\phi = \alpha_\phi + 13h^1(\mathcal{O}_Y) + \chi_T(E) - h^1(-K_Y)$.
2. $e_\phi = 2\alpha_\phi + 12h^1(\mathcal{O}_Y) + h^1(\Theta_Y) - 2h^1(-K_Y)$.

If (X, x) is Gorenstein, then $\alpha_\phi = 0$, so μ and e are independent of the smoothing.

---

[12] Wahl calls the dimension of a smoothing component Tjurina number of the smoothing. We stay however with the widely accepted terminology, and denote the dimension of T^1 as the Tjurina number for an arbitrary singularity (Definition 7.2.26, see also the footnote to Proposition 7.2.33).

For (X, x) Gorenstein (but not necessary smoothable) denote by $\tilde{\mu}(X, x)$ resp. $\tilde{e}(X, x)$ the expressions for μ_ϕ resp. e_ϕ given by the above theorem (these are independent of ϕ since $\alpha = 0$ and hence invariants of (X, x) that may be negative if (X, x) is not smoothable). Then Wahl proves in [100, Theorem 3.13] and [103, Theorem 1.2]:

Theorem 7.3.33 (Wahl) *If* (X, x) *is a Gorenstein surface singularity, then* $\tilde{\mu}(X, x) - \tilde{e}(X, x) \geq 0$, *with equality if and only if* (X, x) *is quasihomogeneous.*

Wahl's new main conjecture in [103] uses the sheaf of logarithmic derivations $S_Y := (\Omega^1_Y(log(E))^*$ on the resolution.

Conjecture 7.3.34 (Wahl) For $(Y, E) \to (X, x)$ the minimal good resolution of a non-Gorenstein normal surface singularity (X, x)one has

$$h^1(\mathcal{O}_Y) - h^1(S_Y) + h^1(\Lambda^2 S_Y) \geq 0,$$

with equality if and only if (X, x) is quasi-homogeneous.

The quasihomogeneous case is settled by Wahl himself [103, Theorem 3.3].

Theorem 7.3.35 (Wahl) *If* (X, x) *is quasihomogeneous and not Gorenstein, then* $h^1(\mathcal{O}_Y) - h^1(S_Y) + h^1(\Lambda^2 S_Y) = 0$.

For further conjectures and results concerning smoothings of (special classes of) normal surface singularities, in particular for different formulas for μ_ϕ and e_ϕ, we refer to [103].

At the end of this section, let us sketch the main steps in the proof of Theorem 7.3.31 because the method is valid in a very general deformation theoretic setting [34, Section 3] and some aspects in our special situation are interesting in itself. For details of what follows we refer to [34, Section 1 and 2].

Let $\phi : (\mathscr{X}, x) \to (\mathbb{C}, 0)$ be a deformation of an isolated singularity (X, x). Deformations of a morphism were considered in Definition 7.2.29 and we consider now deformations of ϕ. In particular we have (Definition 7.2.31)

$$T^1_{(\mathscr{X},x)/(\mathbb{C},0)} = \underline{\mathcal{D}ef}_{(\mathscr{X},x)/(\mathbb{C},0)}(T_\varepsilon).$$

Let us give an explicit description of $T^1_{(\mathscr{X},x)/(\mathbb{C},0)}$. We may assume that ϕ is embedded (Corollary 7.2.9), i.e. $(\mathscr{X}, x) \subset (\mathbb{C}^N \times \mathbb{C}, 0)$ is an embedding such that its composite with the projection on $(\mathbb{C}, 0)$ yields ϕ. Choose a good representative $\phi : \mathscr{X} \subset B \times D \to D$ and let $\mathcal{I} \subset \mathcal{O}_{B\times D}$ be the ideal sheaf defining $\mathscr{X}$. Then we have an exact sequence of $\mathcal{O}_{\mathscr{X}}$-modules

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{\alpha} \Omega^1_{B\times D/D} \otimes_{\mathcal{O}_{B\times D}} \mathcal{O}_{\mathscr{X}} \longrightarrow \Omega^1_{\mathscr{X}/D} \longrightarrow 0\,,$$

and dualizing with $\mathscr{H}om_{\mathcal{O}_{\mathscr{X}}}(-, \mathcal{O}_{\mathscr{X}})$ we get the exact sequence

$$0 \longrightarrow \Theta_{\mathscr{X}/D} \longrightarrow \Theta_{B\times D/D} \otimes_{\mathcal{O}_{B\times D}} \mathcal{O}_{\mathscr{X}} \xrightarrow{\beta} \mathscr{H}om_{\mathcal{O}_{\mathscr{X}}}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_{\mathscr{X}}),$$

with $\Theta_{\mathscr{X}/D} = \mathscr{H}om_{\mathcal{O}_{\mathscr{X}}}(\Omega^1_{\mathscr{X}/D}, \mathcal{O}_{\mathscr{X}})$ the sheaf of relative holomorphic vector fields of $\mathscr{X}/D$.

Lemma 7.3.36 *Setting* $T^1_{\mathscr{X}/D} := \operatorname{coker}(\beta)$, *we have an exact sequence of sheaves*

$$0 \to \Theta_{\mathscr{X}/D} \to \Theta_{B\times D/D} \otimes \mathcal{O}_{\mathscr{X}} \xrightarrow{\beta} \mathscr{H}om_{\mathcal{O}_{\mathscr{X}}}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_{\mathscr{X}}) \to T^1_{\mathscr{X}/D} \to 0.$$

Moreover, $T^1_{\mathscr{X}/D,x} = T^1_{(\mathscr{X},x)/(\mathbb{C},0)}$, *i.e. an element of* $T^1_{\mathscr{X}/D,x}$ *may be regarded as a deformation* $\Phi : (\mathscr{Y}, x) \to (D, 0) \times T_\varepsilon \to T_\varepsilon$ *of* ϕ, *which induces* $\phi : (\mathscr{X}, x) \to (D, 0) \to \{0\}$ *by restricting to* $\{0\} \subset T_\varepsilon$ *(up to isomorphism).*

By Theorem 7.3.31 we have to consider the cokernel of the map $\Theta_{\mathscr{X}/D,x} \to \Theta_{X,x}$, which appears in the following exact sequence.

Lemma 7.3.37 *For any deformation of an isolated singularity as above there is an exact sequence of* $\mathcal{O}_{\mathscr{X}}$*-modules,*

$$0 \to \Theta_{\mathscr{X}/D} \xrightarrow{\phi} \Theta_{\mathscr{X}/D} \to \Theta_X \to T^1_{\mathscr{X}/D} \xrightarrow{\phi} T^1_{\mathscr{X}/D} \to T^1_X,$$

with ϕ *the multiplication by* $\phi \in \mathcal{O}_{\mathscr{X},x}$.

Now let (S, s) be a complex germ and $j : (\mathbb{C}, 0) \to (S, s)$ a morphism. We set

$$\Theta(j) := Der_{\mathbb{C}}(\mathcal{O}_{S,s}, \mathcal{O}) \text{ with } \mathcal{O} := \mathcal{O}_{\mathbb{C},0} = \mathbb{C}\{t\}.$$

For $\zeta \in \Theta(j)$ define $j^*_\zeta : \mathcal{O}_{S,s} \to \mathcal{O}_{\mathbb{C},0}[\varepsilon]/\varepsilon^2$ by $j^*_\zeta = j^* + \varepsilon\zeta$. This ring map defines a morphism of complex germs $j_\zeta : (\mathbb{C}, 0) \times T_\varepsilon \to (S, s)$, which extends j. Hence j_ζ is a deformation of j. Applying the left-exact functor $Der_{\mathbb{C}}(\mathcal{O}_{S,s}, -)$ to $0 \to \mathcal{O} \xrightarrow{t} \mathcal{O} \to \mathbb{C} \to 0$, we have an exact sequence

$$0 \to \Theta(j) \xrightarrow{t} \Theta(j) \to Der_{\mathbb{C}}(\mathcal{O}_{S,s}, \mathbb{C}) \cong T_{S,s} \to 0,$$

where $T_{S,s}$ is the Zariski tangent space of (S, s). Hence $\Theta(j)$ is a free $\mathcal{O}_{\mathbb{C},0}$ module of rank $\dim_{\mathbb{C}}(\Theta(j)/t\Theta(j))$ and $\Theta(j)/t\Theta(j) = \Theta(j) \otimes \mathbb{C}$ maps injectively onto a subspace V of $T_{S,s}$. Since $\Theta(j)$ is free, it follows

Lemma 7.3.38

$$\begin{aligned}\dim_{\mathbb{C}} V &= \dim_{\mathbb{C}} \Theta(j) \otimes \mathbb{C} = rk_{\mathcal{O}}\Theta(j)\\ &= \textit{dim of the Zariski tangent space of } S\\ &\quad\ \textit{at the generic point of the image of } j.\end{aligned}$$

Remark 7.3.39 The following geometric interpretation of V may be helpful. Embed (S, s) in some $(\mathbb{C}^k, 0)$. The Zariski tangent spaces $T_{S,j(t)} \subset \mathbb{C}^k$ fit together to form an analytic vector bundle over the punctured disk $D_\delta \setminus \{0\} \subset \mathbb{C}$ for sufficiently small δ. Then V is limit of the Zariski tangent spaces $T_{S,j(t)}$ for $t \to 0$, taken in the Grassmannian of subspaces in $\mathbb{C}^k$.

Now let (S, s) be the base space of the semiuniversal deformation of (X, x). $\zeta \in \Theta(j)$ determines a morphism $j_\zeta : (\mathbb{C}, 0) \times T_\varepsilon \to (S, s)$ extending j as above and hence by pullback a deformation of (X, x) over $(\mathbb{C}, 0) \times T_\varepsilon$ extending ϕ. Thus we get an element of $T^1_{\mathscr{X}/\mathbb{C},x}$ and the corresponding map $\Theta(j) \to T^1_{\mathscr{X}/\mathbb{C},x}$ is surjective by versality (Definition 7.2.13). We get

Lemma 7.3.40 *Let (S, s) be the base space of the semiuniversal deformation of (X, x).*

1. *The natural $\mathcal{O}$-homomorphism $\Theta(j) \to T^1_{\mathscr{X}/\mathbb{C},x}$ is onto.*
2. *The image of $T^1_{\mathscr{X}/\mathbb{C},x} \to T^1_{X,x}$ coincides with the image of $\Theta(j) \otimes \mathbb{C} \to T_{S,s}$ under the identification $T^1_{X,x} \cong T_{S,s}$.*

Now we can derive easily the main result from [34], which implies Wahl's conjecture (Theorem 7.3.31).

Theorem 7.3.41 (Greuel, Looijenga) *Let $\Psi : (\mathscr{Y}, y) \to (S, s)$ be the semiuniversal deformation of (X, x) and $\phi : (\mathscr{X}, x) \to (\mathbb{C}, 0)$ induced by $j : (\mathbb{C}, 0) \to (S, s)$. Then the dimension of the Zariski tangent space of (S, s) at the generic point of the image of j equals*

$$rk_{\mathcal{O}} T^1_{\mathscr{X}/\mathbb{C},x} + \dim_{\mathbb{C}} \operatorname{coker}(\Theta_{\mathscr{X}/\mathbb{C},x} \to \Theta_{X,x}).$$

In particular, if the generic point of the image of j is nonsingular (e.g. if the fibre over the generic point is smooth or rigid), then this is the dimension of the irreducible component of (S, s) to which j maps.

By openness of versality (Theorem 7.2.21) $\Psi : \mathscr{Y} \to S$ is a joint versal deformation of $(\mathscr{X}_t, z)$ for any point $t \in S$ close to s and any $z \in Sing(\mathscr{X}_t)$. Therefore the germ (S, t) is isomorphic to the cartesian product of the germs of the semiuniversal base spaces of $(\mathscr{X}_t, z)$, which we denote by $(S_{\mathscr{X}_t}, t)$, and an extra smooth factor (Proposition 7.2.17) which we denote by (T, t). Since $\phi_* T^1_{\mathscr{X}/\mathbb{C}}$ is free at a generic point t in the image of j, we see that

$$rk_{\mathcal{O}} T^1_{\mathscr{X}/\mathbb{C},x} = \sum_{z \in Sing(\mathscr{X}_t)} \dim_{\mathbb{C}} T^1_{\mathscr{X}_t,z}.$$

Which is equal to the embedding dimension of $(S_{\mathscr{X}_t}, t)$ and differs from the embedding dimension of (S, s) by $\dim(T, t)$. Theorem 7.3.41 implies therefore

Corollary 7.3.42 *If t is a generic point of the image of j then*

$$\dim(S, t) = \dim(S_{\mathscr{X}_t}, t) + \dim_{\mathbb{C}} \operatorname{coker}(\Theta_{\mathscr{X}/\mathbb{C},x} \to \Theta_{X,x}).$$

The first general formula for the dimension of a smoothing component was obtained by Deligne [10] in the case of reduced curve singularities. Although his formula is local, Deligne's proof uses global methods. As an application of Corollary 7.3.42 a purely local proof of Deligne's formula was given in [34].

Let (X,x) be a reduced curve singularity and $\overline{(X,x)}$ its normalization. Since any derivation on (X,x) lifts uniquely (in characteristic 0) to $\overline{(X,x)}$ (cf. [10]), we get natural inclusions $\Theta_{X,x} \subset \Theta_{\overline{X,x}}$.

Theorem 7.3.43 (Deligne) *Any smoothing component of a reduced curve singularity (X,x) has dimension*

$$3\delta(X,x) - \dim_{\mathbb{C}} \Theta_{\overline{X,x}}/\Theta_{X,x},$$

with $\delta(X,x) = \dim_{\mathbb{C}} \mathcal{O}_{\overline{X,x}}/\mathcal{O}_{X,x}$.

Note that Deligne's formula depends only on (X,x), and hence all smoothing components of a reduced curve singularity have the same dimension. First examples of reduced curve singularities with more than one smoothing component were found by Stevens (c.f. [93, Ch. 13]).

A normal surface singularity (X,x) may have smoothing components of different dimensions. If (X,x) is rational, the dimension of the Artin component is maximal among all components by [100, Corollay 3.18]. For more details on smoothings of surface singularities and many examples see [93, Ch. 14 and Ch. 15].

Smoothing questions for curve singularities will be treated in the next section.

7.3.6 Curve Singularities

This section is about smoothing components of reduced curve singularities, their smoothability, and the topology of the Milnor fibre.

At first glance the situation is different from that for singularities of bigger dimension. There are no topological obstructions to smoothability: a small perturbation of the parameterization of the curve singularity (X,x) by linear terms parametrizes a smooth curve. However, the particular fiber of a family defined by "deformation of the parametrization" has in general embedded components and only the reduction of the special fiber agrees with (X,x) (cf. [43, 1.2]).

A point of interest is Deligne's formula (Theorem 7.3.43) for the dimension of the smoothing component of a reduced curve singularity. In the following we reformulate it by using more common invariants of (X,x), and relate it to the Milnor number. Through our reformulation this formula is effectively computable and provides a useful criterion for non-smoothability of a curve singularity. This is shown by applying it to examples of Pinkham.

Let us first fix the notations for the most common invariants of a reduced curve singularity $(X, x) \subset (\mathbb{C}^n, 0)$. Let $\mathcal{O} = \mathcal{O}_{X,x}$ be the local ring, $\mathfrak{m}$ the maximal ideal of $\mathcal{O}$ and

$$n : (\overline{X, x}) \to (X, x)$$

the normalization with semilocal ring $\overline{\mathcal{O}} = n_* \mathcal{O}_{\overline{X,x}}$ and its Jacobson radical $\overline{\mathfrak{m}}$. Moreover, we set

$$\begin{aligned}
\mathscr{C} &= \mathrm{Ann}_{\overline{\mathcal{O}}}(\overline{\mathcal{O}}/\mathcal{O}) &&= \text{conductor ideal,}\\
\Omega &= \Omega^1_{X,x} &&= \text{holomorphic (Kähler) 1-forms on } (X, x),\\
T\Omega &= H^0_{\{x\}}(\Omega) &&= \text{torsion submodule of } \Omega,\\
\overline{\Omega} &= n_* \Omega^1_{\overline{X,x}} &&= \text{holomorphic 1-forms on } (\overline{X, x}),\\
\omega &= \omega_{X,x} &&= \text{dualizing module of } (X, x),\\
\Theta &= \mathrm{Hom}_{\mathcal{O}}(\Omega, \mathcal{O}) &&= \text{module of derivations on } (X, x),\\
\overline{\Theta} &= \mathrm{Hom}_{\overline{\mathcal{O}}}(\overline{\Omega}, \overline{\mathcal{O}}) &&= \text{module of derivations on } (\overline{X, x}).
\end{aligned}$$

Recall that $\omega = \mathrm{Ext}^{n-1}_{\mathcal{O}_{\mathbb{C}^n,0}}(\mathcal{O}, \Omega^n_{\mathbb{C}^n,0})$ can be identified with $\omega = \{\gamma \in \overline{\Omega} \otimes K \mid \mathrm{res}(\gamma) = 0\}$, with K the total ring of fractions of $\mathcal{O}$. Let $d : \mathcal{O} \to \omega$ be defined as the composition

$$\mathcal{O} \xrightarrow{d} \Omega \xrightarrow{j} \overline{\Omega} \hookrightarrow \omega,$$

with $d : \mathcal{O} \to \Omega$ the exterior derivation. We use the following classical numerical invariants (and μ from [8]) of (X, x):

$$\begin{aligned}
\delta &= \delta(X, x) &&= \dim_{\mathbb{C}} \overline{\mathcal{O}}/\mathcal{O} &&= \text{delta-invariant,}\\
\mu &= \mu(X, x) &&= \dim_{\mathbb{C}} \omega/d\mathcal{O} &&= \text{Milnor number,}\\
\lambda &= \lambda(X, x) &&= \dim_{\mathbb{C}} \omega/j\Omega &&= \text{lambda-invariant,}\\
\tau' &= \tau'(X, x) &&= \dim_{\mathbb{C}} T\Omega &&= \text{length of the torsion,}\\
\tau &= \tau(X, x) &&= \dim_{\mathbb{C}} T^1_{X,x} &&= \text{Tjurina number,}\\
m &= m(X, x) &&= \dim_{\mathbb{C}} \overline{\mathcal{O}}/\mathfrak{m}\overline{\mathcal{O}} &&= \text{multiplicity of } \mathcal{O},\\
r &= r(X, x) &&= \dim_{\mathbb{C}} \overline{\mathcal{O}}/\overline{\mathfrak{m}} &&= \text{number of branches,}\\
t &= t(X, x) &&= \dim_{\mathbb{C}} \omega/\mathfrak{m}\omega &&= \text{Cohen-Macaulay type,}\\
c &= c(X, x) &&= \dim_{\mathbb{C}} \overline{\mathcal{O}}/\mathscr{C} &&= \text{multiplicity of conductor.}
\end{aligned}$$

Note that $j\Omega \cong \Omega/T\Omega$ and $\tau' = \tau$ by Proposition 7.2.33 (3) and duality (see [29, 1.2 Satz]). Moreover, (X, x) is Gorenstein iff $t = 1$.

We introduce

$$m_1 = m_1(X, x) := \dim_{\mathbb{C}} \overline{\Theta}/\Theta,$$

$$e = e(X, x) := 3\delta - m_1,$$

and call e the *Deligne number* of (X, x). Recall two important relations among these invariants from [8] and [10].

Theorem 7.3.44 *Let (X, x) be a reduced curve singularity.*

1. (Buchweitz, Greuel) $\mu = 2\delta - r + 1$.
2. (Deligne) $\dim E = e$ *for every smoothing component* E *of* (X, x).

Before we consider the smoothing problem for curve singularities, let us recall the main properties of μ from [8, Theorem 4.2.2, 4.2.4].

Theorem 7.3.45 (Buchweitz, Greuel) *Let $\phi : \mathscr{X} \to D$ be a good representative of a deformation of the reduced curve singularity (X, x) with Milnor fibre $F = \mathscr{X}_t, t \neq 0$.*

1. $\mu(X, x) - \mu(F) = b_1(F)$,
2. $\mu(X, x) - \mu(F) \geq \delta(X, x) - \delta(F) \geq 0$.

Here

$$\mu(F) := \sum_{y \in Sing(F)} \mu(F, y)$$

and similar for δ. Note that $\mu(X, x) = 0$ iff (X, x) is smooth.

This and the following corollary show in particular the topological meaning of μ.

Corollary 7.3.46

1. *If F is smooth, then $\mu(X, x) = b_1(F)$.*
2. *The following are equivalent:*
   - (i) $\mu(\mathscr{X}_t)$ *is constant for all* $t \in D$,
   - (ii) $\delta(\mathscr{X}_t)$ *and* $\sum_{y \in Sing(\mathscr{X}_t)}(r(\mathscr{X}_t, y) - 1)$ *are constant for* $t \in D$,
   - (iii) $b_1(\mathscr{X}_t) = 0$ *for all* $t \in D$,
   - (iv) $\mathscr{X}_t$ *is contractible for all* $t \in D$.

Note that the constancy of $\mu(\mathscr{X}_t)$ in a flat family as above does not imply that $\mathscr{X}_t$ has only one singularity if the embedding dimension of (X, x) is ≥ 3 (in contrast to the case of plane curves). However, if we assume that $\mathscr{X}_t$ has only one singularity then $\mu(\mathscr{X}_t) =$ constant implies *topological equisingularity* [8, Theorem 5.2.2]:

Theorem 7.3.47 (Buchweitz, Greuel) *In addition to the assumptions of Theorem 7.3.45 let $\sigma : D \to \mathscr{X}$ be a section of ϕ such that $\mathscr{X}_t \setminus \sigma(t)$ is smooth for each $t \in D$. The following conditions are equivalent.*

1. $\mu(\mathscr{X}_t, \sigma(t))$ *is constant for* $t \in D$,
2. $\delta(\mathscr{X}_t, \sigma(t))$ *and* $r(\mathscr{X}_t, \sigma(t))$ *are constant for* $t \in D$,
3. $\phi : \mathscr{X} \to D$ *is topologically trivial, i.e., there is a homeomorphism* $h : \mathscr{X} \xrightarrow{\sim} X \times D$ *such that* $\phi = \pi \circ h$ *where* $\pi : X \times D \to D$ *is the projection.*

The Milnor number and the delta-invariant have been generalized to non-reduced curve singularities (X, x) with an embedded component at x in [6] with similar topological properties (however, the Milnor fibre need not be connected in this case). See also a generalization to singularities of arbitrary dimension with $X \setminus \{x\}$ normal in the survey article on equisingularity and equinormalizability [31].

Equisingularity has been and still is an important research topic in singularity theory, see Chaps. 5 and 6 of this volume. Let us just mention a famous result by Lê and Ramanujam [57].

Theorem 7.3.48 (Lê, Ramanujam) *In a μ-constant family of hypersurface singularities $f_t : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$, $t \in \mathbb{C}$, of dimension n the homotopy type of the Milnor fibration of f_t is constant (for sufficiently small t). If further $n \neq 2$, these fibrations are diffeomorphic and the topological types of the singularities are the same.*

Different aspects of equisingularity are treated in [17] (with emphasis on Zariski's multiplicity conjecture and non-isolated singularities) and in [41, 42].

In order to get criteria for smoothability of reduced curve singularities we need further relations among the above invariants. It is convenient to work with fractional ideals, i.e. $\mathcal{O}$-ideals in K. $\overline{\mathcal{O}}$ is the integral closure of $\mathcal{O}$ in K and choosing an $\overline{\mathcal{O}}$-generator α of $\overline{\Omega}$ we get an isomorphism

$$\phi_\alpha : \overline{\Omega} \otimes K \overset{\cong}{\to} K,$$

with $\phi_\alpha(\overline{\Omega}) = \overline{\mathcal{O}} \subset K$. We denote the image of ω in K under ϕ_α again by ω. It is shown in [30, 2.3 Lemma] that there exists an $\overline{\mathcal{O}}$-generator f of $\mathscr{C}$ such that $\tilde{\omega} := f\omega$ satisfies

$$\mathcal{O} \subset \tilde{\omega} \subset \overline{\mathcal{O}}, \quad \mathscr{C} = \tilde{\omega} : \overline{\mathcal{O}} = \{h \in K \,|\, h\overline{\mathcal{O}} \subset \tilde{\omega}\},$$

with $\mathcal{O} = \tilde{\omega}$ iff (X, x) is Gorenstein. Moreover, if $\bar{t}$ is an $\overline{\mathcal{O}}$-generator of $\overline{\mathfrak{m}}$ we set $\tilde{\Omega} := \bar{t} \cdot \phi_\alpha(j\Omega)$. Obviously $\tilde{\Omega} \cong \Omega/T\Omega$. For the proof of the following lemma see [30, Section 2.4, 2.5].

Lemma 7.3.49

1. $\dim_{\mathbb{C}} \omega/\overline{\Omega} = \delta$,
2. $\dim_{\mathbb{C}} \overline{\mathcal{O}}/\tilde{\omega} = c - \delta$,
3. $\dim_{\mathbb{C}} \tilde{\omega}/\mathcal{O} = 2\delta - c$,
4. $\dim_{\mathbb{C}} \tilde{\omega}\mathfrak{m}/\mathfrak{m} = 2\delta - c - t + 1$,

These formulas are used to prove the following relation between the Deligne number e and the other invariants [30, 2.5 Theorem].

Theorem 7.3.50 (Greuel) *Let (X, x) be a reduced curve singularity.*

1. $e = \mu + t - 1 + \dim_{\mathbb{C}} \overline{\mathcal{O}}/\tilde{\omega} \cdot \tilde{\Omega} - \dim_{\mathbb{C}} \overline{\mathcal{O}}/\tilde{\omega} \cdot \mathfrak{m}$.

2. *$\delta \leq \delta + t - 1 + m - r \leq 3\delta - c + m - r \leq\ e\ \leq \mu + 2\delta - c \leq 3\delta - r + 1$, $\mu + 2\delta - c \leq 3\delta - r < 3\delta$ if (X, x) is singular.*
3. *If (X, x) is quasihomogeneous then $e = \mu + t - 1$.*
4. *Let (X, x) be smoothable. If (X, x) is quasihomogeneous then $\tau \geq \mu + t - 1$ and equality holds iff (X, x) is unobstructed.*

Note that (4) gives a useful criterion to decide whether a smoothable curve is obstructed. The inequalities in (2) imply the following estimates for e.

Corollary 7.3.51 *Any smoothing component E of a non-smooth reduced curve singularity (X, x) satisfies*

$$\mu + 2\delta - c \geq \dim E = e > \frac{1}{2}\,\mu.$$

In particular, $\mu \geq e > \frac{1}{2}\,\mu$ for (X, x) Gorenstein.

Proof Since $2\delta \geq c$ we get

$$e \geq 3\delta - c + m - r \geq \delta + m - r = \frac{\mu + r - 1}{2} + m - r = \frac{\mu}{2} + \frac{m - r}{2} + \frac{m - 1}{2} > \frac{\mu}{2}$$

if (X, x) is singular. □

This implies that the generic fibre over an irreducible component of the semiuniversal base space of (X, x) is not smoothable, if the dimension of the component is not in the range of Corollary 7.3.51.

By the above formulas we get an easy proof of the following theorem by Dimca and the author in [12].

Theorem 7.3.52 (Dimca, Greuel) *Let (X, x) be a reduced complete intersection curve singularity. Then the following hold.*

1. *$\tau = \tau' = \lambda \geq \delta + m - r$,*
2. *$\tau - \delta = \dim_{\mathbb{C}}(\bar{\Omega}/\Omega)$. In particular, one has the equality*

$$\dim_{\mathbb{C}}(\bar{\Omega}/\Omega) = \delta - r + 1$$

if and only if the singularity (X, x) is weighted homogeneous.

3. *$\tau > \frac{1}{2}\,\mu$ if (X, x) is not smooth.*

Moreover in [12] the authors pose the following question.

Question 7.3.53 Is it true that

$$\tau(X, x) > \frac{3}{4}\,\mu(X, x)$$

for any isolated singular plane curve singularity (X, x)?

The answer to this question is positive for semi-quasihomogeneous singularities (X, x) (see [2]) and for curves with one branch (see [1] and [23]). Examples in [12] show that the inequality is sharp.

For Gorenstein curves Theorem 7.3.50 was complemented in [36], giving a numerical characterization of quasi homogeneity for Gorenstein curves.

Theorem 7.3.54 (Greuel, Martin, Pfister) *If (X, x) is Gorenstein, then $e \leq \mu$ with equality if and only if (X, x) is quasihomogeneous.*

The following corollary generalizes Theorem 7.3.26 in dimension 1.

Corollary 7.3.55 *If (X, x) is Gorenstein and unobstructed, then $\tau \leq \mu$ with equality if and only if (X, x) is quasihomogeneous.*

In [36] there is an example of a complete intersection with several branches, satisfying (a): $\tau = e < \mu$, (b): all branches are non-Gorenstein and non-quasihomogeneous and (c): $\tau = e = \mu$ for each branch. This shows that the assumption "Gorenstein" in Theorem 7.3.54 is necessary and that we cannot conclude from the branches to their union.

Some problems remain however open.

Problem 7.3.56 Is the converse of Theorem 7.3.50(3) also true, i.e. does $e = \mu + t - 1$ imply that (X, x) is quasihomogeneous?

Problem 7.3.57 Does the inequality $e \leq \mu + t - 1$ always hold? We conjecture this at least for (X, x) smoothable.

The answer to both problems is "yes" in the following cases: (a): (X, x) is Gorenstein and (b): (X, x) is irreducible and the monomial curve of (X, x) has Cohen-Macaulay type ≤ 2.

Note the obvious similarity between the Problems 7.3.56 and 7.3.57, and Conjecture 7.3.34 in the surface case.

We turn now to non-smoothable curves. We want to apply the following criterion [30, 3.1 Proposition] for non smoothability.

Proposition 7.3.58 *Let $\phi : \mathscr{X} \to T$ a sufficiently small representative of a deformation with section $\sigma : T \to \mathscr{X}$ of a reduced curve singularity (X, x). Assume:*

(i) $(\mathscr{X}_t, \sigma(t))$ is singular and not isomorphic to (X, x) for $t \neq 0$,
(ii) T is irreducible and $\dim T \geq e(X, x)$.

Then there is an analytic open dense subset $T_0 \subset T$, such that $(\mathscr{X}_t, \sigma(t))$ is not smoothable if $t \in T_0$.

The proof is easy: ϕ can be induced from the semiuniversal deformation with base $(S, 0)$ by a map $\varphi : (T, 0) \to (S, 0)$. By (i) φ is finite and $\varphi(T)$ has dimension $\geq e = \dim E$ for every smoothing component E of S. φ cannot map T to any E since there are no smooth fibres over T. (X, x) may be smoothable but the generic point of the image of ϕ cannot be smoothable by openness of versality.

Let us finish with proving non-smoothability for the curve singularity L_r^n consisting of r lines through the origin in $\mathbb{C}^n$ in generic position. The following was proved in [30, 3.4 Theorem], generalizing Pinkham [70, Theorem 11.10], who proved it for the range $n < r < 2n$ by global methods.

Theorem 7.3.59 (Pinkham; Greuel) *The curve singularity L_r^n is not smoothable in the following ranges:*

1. $n < r \le \binom{n+1}{2}$ and $(r-n-2)(n-5) \ge 7$,
2. $\binom{n+d-1}{d} < r \le \binom{n+d}{d+1}$, $d \ge 2$ and $r(n-3-3d) + 3\binom{n+d}{d} \ge n^2 - 1$.

E.g., L_r^n is not smoothable if r is in the following intervals:

n	6	7	8	9	10
r	[15, 21]	[13, 30]	[13, 72]	[13, 193]	[14, 419]

The intervals for the case $n < r \le \binom{n+1}{2}$ have been slightly enlarged by Stevens in [92]. For $n = 2$ and 3 arbitrary many lines are smoothable. It is also known that $n, n+1$ and $n+2$ lines in $(\mathbb{C}^n, 0)$ are always smoothable, but L_{n+3}^n not if $n \ge 12$ by the theorem. Note that for fixed n, the theorem shows the existence of non-smoothable curves of lines only for r within some finite interval (which is growing with n). Also, we obtain nothing for $n = 4, 5$.

Problem 7.3.60 Do there exist for fixed $n \ge 4$ non-smoothable curves L_r^n if r goes to infinity? It seems unlikely that this is not the case.

The formula of Deligne cannot only be used to show that certain curve singularities are not smoothable but also that the semiuniversal base for a smoothable curve is not smooth, namely if $e < \tau$. Examples are n (resp. $n+1$) general lines in $(\mathbb{C}^n, 0)$, which are obstructed if $n \ge 4$ (resp. $n \ge 5$), cf. [30].

Note Added in Proof

Question 7.3.53 has meanwhile been answered positively by Patricio Almiron: On the Quotient of Milnor and Tjurina Numbers for Two-Dimensional Isolated Hypersurface Singularities, arXiv:1910.12843.

Acknowledgments I would like to thank Helmut Hamm, Toni Iarrobino, Joachim Jelisiejew, Patrick Popescu-Pampu and in particular Jonathan Wahl for useful comments.

References

1. Alberich-Carramiñana, M.; Almirón, P.; Blanco, G.; Melle-Hernández, A.: *The Minimal Tjurina Number of Irreducible Germs of Plane Curve Singularities.* Preprint, arXiv:1904.02652 (2019) 442
2. Almirón, P.; Blanco, G.: A note on a question of Dimca and Greuel. C. R. Acad. Sci. Paris, Ser. I 357 (2019) 205–208, DOI: 10.1016/j.crma.2019.01.002. 442
3. Artin, M.: *Algebraic construction of Brieskorn's resolutions.* J. Algebra **29**, 330–348 (1974). 402 and 422
4. Briançon, J.: *Description de* $Hilb^n\mathbb{C}\{x, y\}$. Invent. Math. **41**, 45–89 (1977). 419
5. Brieskorn, E.; Greuel, G.-M.: *Singularities of complete intersections.* Manifolds, Proc. int. Conf. Manifolds relat. Top. Topol., Tokyo 1973, 123–129 (1975). 430
6. Brücker, C.; Greuel, G.-M.: *Deformationen isolierter Kurvensingularitäten mit eingebetteten Komponenten.* Manuscr. math. **70**, 93–114 (1990). 440
7. Buchweitz, R.-O.: *On deformations of monomial curves.* Sémin. sur les singularités des surfaces, Cent. Math. Ec. Polytech., Palaiseau 1976–77, Lect. Notes Math. **777**, 205–220 (1980). 418 and 419
8. Buchweitz, R.-O.; Greuel, G.-M.: *The Milnor number and deformations of complex curve singularities.* Invent. Math. **58**, 241–281 (1980). 428, 438, and 439
9. Decker, W; Greuel, G.-M.; Pfister, G.; Schönemann, H.: SINGULAR *4.1.1. A Computer Algebra System for Polynomial Computations*. University of Kaiserslautern (2018). http://www.singular.uni-kl.de. 417
10. Deligne, P.: *Intersections sur les surfaces régulières.* Sém. Géom. Algébrique Bois-Marie 1967–1969, SGA 7 II, Lect. Notes Math. **340**, Exposé X, 1–38 (1973). 418, 437, and 439
11. Dimca, A.: *Singularities and topology of hypersurfaces.* Universitext. New York etc.: Springer-Verlag. xvi, 263 p. (1992). 425
12. Dimca, A.; Greuel, G.-M.: *On 1-forms on isolated complete intersection curve singularities.* J. Singul. **18**, 114–118 (2018). 441 and 442
13. Douady, A.: *Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné.* Ann. Inst. Fourier **16**, 1–98 (1966). 391
14. Ebeling, W.: *Distinguished Bases and Monodromy of Complex Hypersurface Singularities.* This handbook. 425
15. Engel, P.: *Looijenga's conjecture via integral-affine geometry.* J. Differ. Geom. **109**, No. 3, 467–495 (2018). 422
16. Erman, D.: *Murphy's law for Hilbert function strata in the Hilbert scheme of points.* Math. Res. Lett. **19**, 1277–1281 (2012). 420
17. Eyral, C.: *Topics in equisingularity theory.* IMPAN Lecture Notes 3. Warsaw: Polish Academy of Science, Institute of Mathematics. 100 p. (2016). 440
18. Flenner, H.: *Über Deformationen holomorpher Abbildungen.* Habilitationsschrift, Universität Osnabrück (1978). 402
19. Flenner, H.: *Ein Kriterium für die Offenheit der Versalität.* Math. Z. **178**, 449–473 (1981). 401 and 402
20. Flenner, H. ; Kosarev S.: *On locally trivial deformations.* Publ. Res. Inst. Math. Sci., **23**, 627–665 (1987). 402
21. Frisch, J.: *Points de platitude d'un morphisme d'espaces analytiques complexes.* Invent. Math. **4**, 118–138 (1967). 391
22. Galligo, A.; Houzel, C.: *Module des singularités isolées d'après Verdier et Grauert.* Astérisque **7–8** (1973), 139–163 (1974). 421
23. Genzmer, Y.; Hernandes, M. E.: *On the Saito Basis and The Tjurina Number for Plane Branches.* Preprint arXiv:1904.03645v3 (2019) 442
24. Goresky, M.; MacPherson, R.: *Stratified Morse theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Bd. 14. Berlin etc.: Springer-Verlag. 272 p. (1988). 426

25. Grauert, H; Kerner, H.: *Deformationen von Singularitäten komplexer Räume*, Math. Ann. 153, 236–260 (1964). 418
26. Grauert, H.: *Über die Deformation isolierter Singularitäten analytischer Mengen.* Invent. Math. **15**, 171–198 (1972). 400 and 406
27. Grauert, H.; Remmert, R: *Coherent Analytic Sheaves*. Grundlehren der mathematischen Wissenschaften **265**, Springer-Verlag (1984). 390
28. Greuel, G.-M.: *Der Gauß-Manin-Zusammenhang isolierter Singularitäten von vollständigen Durchschnitten.* Math. Ann. **214**, 235–266 (1975). 430 and 431
29. Greuel, G.-M.: *Dualität in der lokalen Kohomologie isolierter Singularitäten.* Math. Ann. **250**, 157–173 (1980). 410, 431, and 438
30. Greuel, G.-M.: *On deformation of curves and a formula of Deligne.* Algebraic geometry, Proc. int. Conf., La Rabida/Spain 1981, Lect. Notes Math. **961**, 141–168 (1982). 431, 440, 442, and 443
31. Greuel, G.-M.: *Equisingular and equinormalizable deformations of isolated non-normal singularities.* Methods Appl. Anal. **24**, No. 2, 215–276 (2017). 426 and 440
32. Greuel, G.-M.; Hamm, H.: *Invarianten quasihomogener vollständiger Durchschnitte.* Invent. Math. **49**, 67–86 (1978). 430
33. Greuel, G.-M.; Karras, U.: *Families of varieties with prescribed singularities.* Compos. Math., **69**, 83–110 (1989). 402
34. Greuel, G.-M.; Looijenga, E.: *The dimension of smoothing components.* Duke Math. J. **52**, 263–272 (1985). 424, 433, 434, 436, and 437
35. Greuel, G.-M.; Lossen, C.; Shustin, E.: *Introduction to Singularities and Deformations*. Springer (2007). 390, 391, 392, 393, 394, 396, 398, 399, 401, 403, 405, 410, 411, 413, and 414
36. Greuel, G.-M.; Martin, B.; Pfister, G.: *Numerische Charakterisierung quasihomogener Gorenstein-Kurvensingularitäten.* Math. Nachr. **124**, 123–131 (1985). 431 and 442
37. Greuel, G.-M.; Steenbrink, J.: *On the topology of smoothable singularities.* Singularities, Summer Inst., Arcata/Calif. 1981, Proc. Symp. Pure Math. **40**, Part 1, 535–545 (1983). 424, 427, and 428
38. Gross, M.; Hacking, P.; Keel, S.: *Mirror symmetry for log Calabi-Yau surfaces. I.* Publ. Math., Inst. Hautes Étud. Sci. **122**, 65–168 (2015). 422
39. Hamm, H.: *Lokale lopologische Eigenschaften komplexer Räume.* Math. Ann. **191**, 235–252 (1971). 429 and 430
40. Hamm, H.: *On the vanishing of local homotopy groups for isolated singularities of complex spaces.* J. Reine Angew. Math. **323**, 172–176 (1981). 427
41. Hamm, H.: *Complements of hypersurfaces and equisingularity.* Chéniot, Denis (ed.) et al., Singularity theory. Proceedings of the 2005 Marseille singularity school and conference, CIRM, Marseille, France, January 24-February 25, 2005. Dedicated to Jean-Paul Brasselet on his 60th birthday. Singapore: World Scientific (ISBN 978-981-270-410-8/hbk). 625–649 (2007). 440
42. Hamm, H.: *On local equisingularity.* Blanlœil, Vincent (ed.) et al., Singularities in geometry and topology. Proceedings of the 5th Franco-Japanese symposium on singularities, Strasbourg 2009. European Mathematical Society, IRMA Lectures in Mathematics and Theoretical Physics **20**, 19–37 (2012). 440
43. Hartshorne, R.: *Topological conditions for smoothing algebraic singularities.* Topology 13, 241–253 (1974). 418, 427, and 437
44. Hartshorne, R.: *Deformation Theory.* Graduate Text in Mathematics, **257**, Springer (2010). 390 and 424
45. Herzog, J.: *Eindimensionale fast-vollständige Durchschnitte sind nicht starr.* Manuscr. Math. **30**, 1–19 (1979). 418
46. Hauser, H.; Müller, G.: *The trivial locus of an analytic map germ.* Ann. Inst. Fourier **39**, 831–844 (1989). 402
47. Huibregtse, M. E.: *Some elementary components of the Hilbert scheme of points.* Rocky Mountain J. Math. **47**, 1169–1225 (2017) . 420

48. Iarrobino, A.: *Reducibility of the families of 0-dimensional schemes on a variety.* Invent. Math. **15**, 72–77 (1972). 420
49. Iarrobino, A.: *Hilbert scheme of points: Overview of last ten years.* Algebraic geometry, Proc. Summer Res. Inst., Brunswick/Maine 1985, part 2, Proc. Symp. Pure Math. **46**, 297–320 (1987). 420
50. Jelisiejew, J.: *Elementary components of Hilbert schemes of points.* J. London Math. Soc. (2) 00, 1–24 (2019). 420
51. Karras, U.: *Normally flat deformations of rational and minimally elliptic singularities.* Singularities, Summer Inst., Arcata/Calif. 1981, Proc. Symp. Pure Math. **40**, Part 1, 619–639 (1983). 423
52. Kas, A.; Schlessinger, M.: *On the versal deformation of a complex space with an isolated singularity.* Math. Ann. **196**, 23–29 (1972). 403
53. Laudal, A.: *Formal moduli of algebraic structures.* SLN **754**, Springer Verlag (1979). 414
54. Lê, D.-T.: *Calculation of Milnor number of isolated singularity of complete intersection.* Funct. Anal. Appl. **8**, 127–131 (1974). 430
55. Lê, D.-T.: *Vanishing Cycles on Complex Analytic Sets*, Lecture delivered at RIMS in 1975, published in 1976. Full text available at http://repository.kulib.kyoto-u.ac.jp/dspace/handle/2433/105857. 427
56. Lê, D.-T.: *Some remarks on relative monodromy,* Real and compl. Singul., Proc. Nordic Summer Sch., Symp. Math., Oslo 1976, pp. 397–403 (1977). 427
57. Lê, D.-T.; ; Ramanujam, C. P.: *The invariance of Milnor's number implies the invariance of the topological type.* Am. J. Math. **98**, 67–78 (1976). 440
58. Looijenga, E.: *Rational surfaces with an anticanonical cycle*. Ann. Math. (2), **114**, 267–322, (1981). 422
59. Looijenga, E.: *The smoothing components of a triangle singularity. I.* Singularities, Summer Inst., Arcata/Calif. 1981, Proc. Symp. Pure Math. **40**, Part 2, 173–183 (1983). 422
60. Looijenga, E.: *Riemann-Roch and smoothings of singularities.* Topology **25**, 293–302 (1986). 424
61. Looijenga, E.: *Isolated singular points on complete intersections.* 2nd revised ed. Surveys of Modern Mathematics 5. Somerville, MA: International Press; Beijing: Higher Education Press. 136 p. (2013). 424
62. Looijenga, E.; Steenbrink, J. *Milnor number and Tjurina number of a complete intersection.* Math. Ann. **271**, 121–124 (1985). 431
63. Lurie, J.: *Derived Algebraic Geometry IV: Deformation Theory,* May 4, 2009, arXiv:0709.3091 416
64. Lichtenbaum, S.; Schlessinger, M.: *The cotangent complex of a morphism.* Trans. AMS **128**, 41–70 (1967). 410
65. Matsumura, H.: *Commutative ring theory.* Cambridge Univ. Press (1986). 392
66. Milnor, J.: *Singular points of complex hypersurfaces.* Princeton Univ. Press (1968). 424, 427, 429, and 430
67. Mumford, D.: *A remark on the paper of M. Schlessinger*, Rice University Studies **59**, 113–117 (1973). 418 and 420
68. Mumford, D.: *Pathologies IV*, Am. J. of Math. **97**, 847–849, (1975). 418
69. Palamodov, V.P.: *Deformations of complex spaces.* In: Several complex variables IV, Springer, Encyclopedia of math. sciences, **v.10**, (1990). 412
70. Pinkham, H.: *Deformation of Algebraic Varieties with G_m-Action*, Astérisque **20**, (1974). 415, 418, 420, 421, 422, 423, and 443
71. Pinkham, H.: *Deformations of normal surface singularities with $\mathbb{C}^*$-action.* Math. Ann. **232**, 65–84 (1978). 418, 420, 421, 422, and 423
72. Popescu-Pampu, P.: *Complex singularities and contact topology.* Winter Braids Lecture Notes Vol.3, 1–74 Course n^o III (2016). 428
73. Pourcin, G.: *Déformations de singularités isolées.* Astérisque **16**, 161–173 (1974). 402
74. Rees, E; Thomas, E.: *Smoothings of isolated singularities.* Algebr. geom. Topol., Stanford/Calif. 1976, Proc. Symp. Pure Math., Vol. **32**, Part 2, 111–117 (1978). 418

75. Rees, E; Thomas, E.: *Cobordism Obstructions to Deforming Isolated Sinpularities*, Math. Ann. **232**, 31–54 (1978). 418
76. Rim, D. S.: *Torsion Differentials and Deformation*, Trans. AMS 169, 257–278 (1972). 418
77. Rim, D. S.: *Formal deformation theory.* Groupes de monodromie en géométrie algébrique (SGA 7I), exp. 6. Springer SLN **288**, 32–132 (1972). 412
78. Rim, D. S.; Vitulli, M. A.: *Weierstrass points and monomial curves.* J. Algebra **48**, 454–476 (1977). 418 and 419
79. Saito, K.: *Quasihomogene isolierte Singularitäten von Hyperflächen.* Invent. Math. **14**, 123–142 (1971). 431
80. Schaps, M.: *Deformations of Cohen-Macaulay Schemes of Codimension 2 and Non-Singular Deformations of Space Curves*, Am. J. of Math. **99**, 669–685 (1977). 419
81. Schlessinger, M.: *Functors of Artin rings.* Trans. Amer. Math. Soc. **130**, 208–222 (1968). 411
82. Schlessinger, M.: *Rigidity of Quotient Singularities*, Inv. Math. **14**, 17–26 (1971). 418
83. Schlessinger, M.: *On rigid singularities.* Rice Univ. Studies 59, No. 1, (Proc. Conf. Complex Analysis 1972, I), 147–162 (1973). 418
84. Seade, J.:*On the topology of isolated singularities in analytic spaces.* Progress in Mathematics 241. Basel: Birkhäuser (ISBN 3-7643-7322-9/hbk; 3-7643-7395-4/ebook). xiv, 238 p. (2006). 424 and 427
85. Sernesi, E.: *Deformation of Algebraic Schemes.* Springer (2006). 390
86. Severi, F.: *Fondamenti per la geometria sulle varietà algebriche.* Palermo Rend. 28, 33–87 (1909). 418
87. Severi, F.: *Vorlesungen liber Algebraische Geometrie.* Teubner (1921). 418
88. Shafarevich, I. R.: *Deformations of commutative algebras of class 2.* Leningr. Math. J. **2**, No. 6, 1335–1351 (1991); translation from Algebra Anal. 2, No. 6, 178–194 (1990). 420
89. Sommese, A. J.: *Non-smoothable varieties.* Comment. Math. Helv. **54**, 140–146 (1979). 418
90. Steenbrink, J. H. M.: *Limits of Hodge structures.* Invent. Math. **31**, 229–257 (1976). 428
91. Steenbrink, J. H. M.: *Mixed Hodge structures associated with isolated singularities.* Singularities, Summer Inst., Arcata/Calif. 1981, Proc. Symp. Pure Math. **40**, Part 2, 513–536 (1983). 424
92. Stevens, J.: *On the number of points determining a canonical curve.* Indag. Math. **51**, No. 4, 485–494 (1989). 443
93. Stevens, J.: *Deformations of singularities.* LNM 1811, Springer (2003). 406, 410, 412, 413, 423, and 437
94. van Straten, D.: *On a theorem of Greuel and Steenbrink.* In: Decker W., Pfister G., Schulze M. (eds) Singularities and Computer Algebra. Based on the conference, Lambrecht (Pfalz), June 2015. Cham: Springer (ISBN 978-3-319-28828-4/hbk; 978-3-319-28829-1/ebook). 353–364 (2017). 428
95. Teissier, B.: *The hunting of invariants in the geometry of discriminants.* In: P. Holm (ed.): Real and Complex Singularities, Oslo 1976, Northholland (1978). 402
96. Tjurina, G. N.: *Locally Flat Deformations of Isolated Singularities of Complex Spaces*. Math. of the USSR- Izvestia 3, 967–999 (1969). 403, 410, 419, and 431
97. Vosegaard, H.: *A characterization of quasi-homogeneous purely elliptic complete intersection singularities.* Compos. Math. **124**, No. 1, 111–121 (2000). 431
98. Vosegaard, H.: *A characterization of quasi-homogeneous complete intersection singularities.* J. Algebr. Geom. **11**, No. 3, 581–597 (2002). 431
99. Wahl, J.: *Elliptic deformations of minimally elliptic singularities.* Math. Ann. **253**, 241–262 (1980). 417 and 422
100. Wahl, J.: *Smoothings of normal surface singularities.* Topology **20**, 219–246 (1981). 419, 420, 422, 424, 429, 433, 434, and 437
101. Wahl, J.: *Derivations of negative weight and non-smoothability of certain singularities.* Math. Ann. **258**, 383–398 (1982). 420, 422, 423, 424, and 427
102. Wahl, J.: *A characterization of quasi-homogeneous Gorenstein surface singularities.* Compos. Math. **55**, 269–288 (1985). 431

103. Wahl, J.: *Milnor and Tjurina numbers for smoothings of surface singularities.* Algebr. Geom. **2**, No. 3, 315–331 (2015). 432, 433, and 434
104. Waldi, R.: *Deformation von Gorenstein-Singularitäten der Kodimension 3.* Math. Ann. **242**, 201–208 (1979). 419
105. Anonymous (A. Weil): *Correspondence,* Am. J. Math. 79, 951–952 (1957). 418

Chapter 8
Distinguished Bases and Monodromy of Complex Hypersurface Singularities

Wolfgang Ebeling

Contents

Abstract We give a survey on some aspects of the topological investigation of isolated singularities of complex hypersurfaces by means of Picard-Lefschetz theory. We focus on the concept of distinguished bases of vanishing cycles and the concept of monodromy.

8.1 Introduction

The pioneering fibration theorem of J. Milnor [118] opened the way to study the topology of isolated complex hypersurface singularities. To study the topology of real smooth manifolds one can use Morse theory. The idea of Morse theory is that the topological type of the level set of a real function changes when passing through a critical value. In order to study the topology of the singularity defined by a complex analytic function one can investigate the level sets of this function. The complex analogue of Morse theory is Picard-Lefschetz theory. It is older than Morse theory and goes back to E. Picard and S. Simart [122] and to S. Lefschetz [104].

W. Ebeling (✉)
Institut für Algebraische Geometrie, Leibniz Universität Hannover, Hannover, Germany
e-mail: ebeling@math.uni-hannover.de

J. L. Cisneros Molina et al. (eds.), *Handbook of Geometry and Topology of Singularities I*, https://doi.org/10.1007/978-3-030-53061-7_8

Around 1967–1969, the Picard-Lefschetz theory experienced a revival when it was brought into an algebraic form by A. Grothendieck, P. Deligne, and N. Katz in [45]. On a more modest scale, the theory was applied in the late 1960s and early 1970s to the analysis of isolated singularities of complex hypersurfaces. The first fundamental contributions were made by F. Pham [121], Lê Dũng Tráng [101, 102], E. Brieskorn [28, Appendix], K. Lamotke [97], and A. M. Gabrielov [65–68]. Gabrielov coined the notion of "distinguished bases". Instead of passing through a critical value, the fundamental principle of Picard-Lefschetz theory is going around a critical value in the complex plane. Roughly speaking, to the critical values there corresponds a distinguished basis of vanishing cycles and the change of the topology of the level set is given by the "monodromy". This article is a survey of these fundamental concepts and the further developments.

Nowadays, there are good references for this subject. There is a survey article by S. M. Gusein-Zade [73] and a later one by Brieskorn [34]. A very good reference is the second volume of the book of V. I. Arnold, Gusein-Zade, and A. Varchenko [13]. The book of E. Looijenga [113] is devoted to isolated complete intersection singularities, but it also contains relevant information about hypersurface singularities which are a special case. Moreover, there are also textbooks by D. Bättig and H. Knörrer [16] (in German) and by the author [58]. The author has already written a survey on the classical monodromy [57]. We keep the intersection with this survey to a minimum. We give almost no proofs, but provide precise references to these books as well as to the original articles for details, including proofs.

Let me outline the contents of this article. In the first section, we introduce the notion of a distinguished basis of vanishing cycles. More precisely, we define distinguished and weakly distinguished bases. In the second section, we consider the intersection form, the classical monodromy, and the Seifert form and we show how matrices of these invariants with respect to distinguished bases are related to one another. Moreover, we define the concept of Coxeter-Dynkin diagram. In Sect. 8.4, we consider the change of basis and introduce the action of the braid group on the set of distinguished bases. In Sect. 8.5, we collect together results about the computation of intersection matrices and Seifert matrices with respect to distinguished bases. In Sect. 8.6, we discuss the implication of the irreducibility of the discriminant to properties of the invariants and we introduce the Lyashko-Looijenga map. In Sect. 8.7, we review Arnold's classification of singularities and compile explicit results for the simple, unimodal, and bimodal singularities. Sect. 8.8 is devoted to an algebraic description of the monodromy group. Finally, in Sect. 8.9, we consider the question to which extent the invariants determine the topological type of the singularities. We conclude with some open problems.

The notion of distinguished bases can also be generalized to isolated complete intersection singularities, see [55]. We shall not discuss this case in this survey, we restrict ourselves to isolated complex hypersurface singularities.

There are many further generalizations and applications of the theory, even outside of singularity theory. We mention some of the results, but mainly indicate references to the corresponding articles. We do not claim to be complete.

8.2 Distinguished Bases of Vanishing Cycles

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function with an isolated singularity at the origin. This means that

$$\operatorname{grad} f(a) = \left(\frac{\partial f}{\partial z_0}(a), \ldots, \frac{\partial f}{\partial z_n}(a) \right) \neq 0$$

for all points $a \neq 0$ in a small neighborhood of the origin, $(z_0, \ldots, z_n)$ denote the coordinates of $\mathbb{C}^{n+1}$. For short, we call f a singularity.

One has the famous result of Milnor [118]: Let $\varepsilon > 0$ be small enough such that the closed ball $B_\varepsilon \subset \mathbb{C}^{n+1}$ of radius ε around the origin in $\mathbb{C}^{n+1}$ intersects the fiber $f^{-1}(0)$ transversely. Let $0 < \eta \ll \varepsilon$ be such that for t in the closed disc $\Delta \subset \mathbb{C}$ of radius η around the origin, the fiber $f^{-1}(t)$ intersects the ball B_ε transversely. Let

$$\begin{aligned} X_t &:= f^{-1}(t) \cap B_\varepsilon \text{ for } t \in \Delta, \\ X &:= f^{-1}(\Delta) \cap B_\varepsilon, \\ X^* &:= X \setminus X_0, \\ \Delta^* &:= \Delta \setminus \{0\}. \end{aligned}$$

By a result of J. Milnor [118], the mapping $f|_{X^*} : X^* \to \Delta^*$ is the projection of a (locally trivial) (C^∞)-differentiable fiber bundle. The fiber X_η over the point $\eta \in \Delta^*$ is a $2n$-dimensional differentiable manifold with boundary which has the homotopy type of a bouquet of μ n-spheres where μ is the Milnor number of the singularity. This differentiable fiber bundle $(X^*, f|_{X^*}, \Delta^*, X_\eta)$ is called the *Milnor fibration* and the typical fiber X_η is called the *Milnor fiber*. The only non-trivial reduced homology group is the group $\widetilde{H}_n(X_\eta; \mathbb{Z})$. It is equipped with the intersection form $\langle \, , \, \rangle$. This bilinear form is symmetric if n is even and skew-symmetric if n is odd. We shall only consider homology with integral coefficients and we shall write $\widetilde{H}_n(X_\eta)$ for $\widetilde{H}_n(X_\eta; \mathbb{Z})$ in the sequel.

Definition 8.2.1 The group $\widetilde{H}_n(X_\eta)$ together with the intersection form $\langle \, , \, \rangle$ is called the *Milnor lattice* of f and denoted by M.

The Milnor lattice M is a lattice, i.e., a free $\mathbb{Z}$-module of finite rank equipped with a symmetric or skew-symmetric bilinear form $\langle \, , \, \rangle$. The rank of the Milnor lattice is the Milnor number μ.

Let ω be the loop

$$\begin{aligned} \omega : [0, 1] &\to \mathbb{C} \\ t &\mapsto \eta e^{2\pi\sqrt{-1}t}. \end{aligned}$$

Then parallel translation along this path induces a diffeomorphism $h = h_\omega : X_\eta \to X_\eta$ which is called the *geometric monodromy* of the singularity f.

Definition 8.2.2 The induced homomorphism $h_* : M \to M$ on the Milnor lattice M is called the *(classical) monodromy* (or the *(classical) monodromy operator*) of the singularity f.

Our aim is to study the Milnor fibration, the Milnor lattice M, and the monodromy.

For this purpose, we shall consider a morsification of the function f. This is defined as follows. An *unfolding* of f is a holomorphic function germ $F : (\mathbb{C}^{n+1} \times \mathbb{C}^k, 0) \to (\mathbb{C}, 0)$ with $F(z, 0) = f(z)$ (see [70, 1.2]). A *morsification* is a representative $F : V \times U \to \mathbb{C}$ of an unfolding

$$\begin{aligned} F : (\mathbb{C}^{n+1} \times \mathbb{C}, 0) &\to (\mathbb{C}, 0) \\ (z, \lambda) \quad &\mapsto f_\lambda(z) \end{aligned}$$

of f such that for almost all $\lambda \in U \setminus \{0\}$ (everywhere except from a Lebesgue null set) the function $f_\lambda : V \to \mathbb{C}$ is a Morse function, i.e., has only non-degenerate critical points with distinct critical values. The Morse function f_λ is itself often called a morsification of f. One can show that f has a morsification (see, e.g., [58, Proposition 3.18]).

Let λ be chosen so that f_λ is a Morse function. Let $Y := f_\lambda^{-1}(\Delta) \cap B_\varepsilon$ and $Y_t := f_\lambda^{-1}(t) \cap B_\varepsilon$ for $t \in \Delta$. Assume that $\lambda \neq 0$ is chosen so small that all the critical points are contained in the interior of Y and the fiber $f_\lambda^{-1}(t)$ for $t \in \Delta$ intersects the ball B_ε transversely. Denote the critical points by $p_1, \ldots, p_\mu$ and the critical values by $s_1, \ldots, s_\mu$. Assume that $\eta \in \partial\Delta$ is a non-critical value of f_λ. Let $\Delta' := \Delta \setminus \{s_1, \ldots, s_\mu\}$ and $Y' := Y \cap f_\lambda^{-1}(\Delta')$. Then the mapping $f_\lambda|_{Y'} : Y' \to \Delta'$ is the projection of a differentiable fiber bundle. The fiber Y_t for $t \in \Delta' \cap \Delta^*$ is diffeomorphic to X_t. In particular, Y_η is diffeomorphic to X_η. We therefore identify these fibers.

For a fixed s_i let $\gamma : I = [0, 1] \to \Delta$ be a piecewise differentiable path which connects the critical value s_i with η and does not pass through any other critical value, i.e. $\gamma(0) = s_i,\ \gamma(1) = \eta$ and $\gamma((0, 1]) \subset \Delta'$. By the complex Morse lemma there exists a neighborhood B_i of the non-degenerate critical point p_i over s_i and local coordinates $(z_0, \ldots, z_n)$ centered at the point p_i such that f_λ can be written in B_i in the form

$$f_\lambda(z_0, \ldots, z_n) = s_i + z_0^2 + \ldots + z_n^2$$

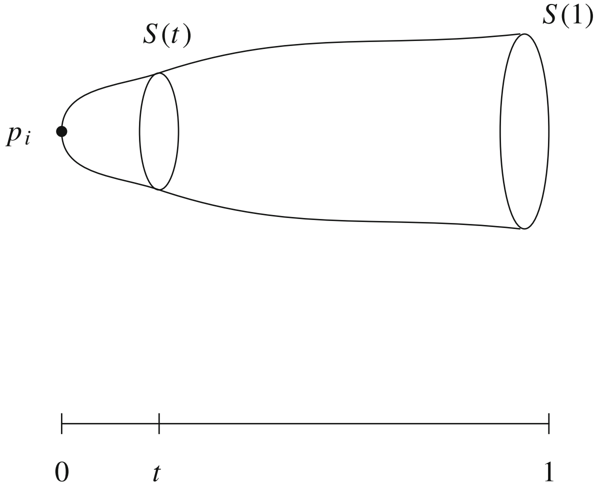
and B_i is a ball of radius ε centered at 0 in these coordinates. For sufficiently small $t > 0$ the fiber $X_{\gamma(t)}$ contains an n-sphere

$$S(t) := \sqrt{\gamma(t) - s_i}\, S^n$$

where S^n is the n-dimensional unit sphere

$$S^n = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \mid \operatorname{Im} z_i = 0,\ \sum z_i^2 = 1\}.$$

Fig. 8.1 Vanishing cycle



By parallel translation along γ one obtains an n-sphere $S(t) \subset X_{\gamma(t)}$ for each $t \in (0, 1]$. For $t = 0$ the sphere $S(t)$ shrinks to the critical point p_i (cf. Fig. 8.1). We now choose an orientation of $S(1)$. Then $S(1)$ is an n-cycle and represents a homology class δ in the Milnor lattice $M = \widetilde{H}_n(X_\eta)$.

Definition 8.2.3 The homology class $\delta \in M$ is called a *vanishing cycle* of f_λ (along γ). Denote by $\Lambda^* \subset M$ the set of vanishing cycles of f (for all possible choices of a morsification, a critical point, a path γ, and an orientation).

A vanishing cycle is well defined up to orientation.

For the self-intersection number of the vanishing cycle δ in the Milnor fiber X_η one has the following result (see also [13, Lemma 1.4], [58, Proposition 5.3]).

Proposition 8.2.4 *The vanishing cycle $\delta \in M$ has the self-intersection number*

$$\langle \delta, \delta \rangle = (-1)^{n(n-1)/2}(1 + (-1)^n) = \begin{cases} 0 & \textit{for } n \textit{ odd,} \\ 2 & \textit{for } n \equiv 0 \ (\textit{mod}\ 4), \\ -2 & \textit{for } n \equiv 2 \ (\textit{mod}\ 4). \end{cases}$$

Proof In order to compute the self-intersection number $\langle \delta, \delta \rangle$ of the vanishing cycle δ, it suffices to compute the self-intersection number of the sphere S^n in the complex manifold

$$Z = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \mid z_0^2 + \cdots + z_n^2 = 1\}.$$

It is easy to see that the manifold Z is diffeomorphic to the total space TS^n of the tangent bundle of the sphere S^n which can be described as follows:

$$TS^n = \left\{ u + \sqrt{-1}v \in \mathbb{C}^{n+1} \;\middle|\; \sum u_i^2 = 1, \sum u_i v_i = 0 \right\}.$$

A diffeomorphism from the manifold Z to TS^n can be defined by

$$z_i = x_i + \sqrt{-1}y_i \mapsto u_i + \sqrt{-1}v_i = \frac{x_i}{|x|} + \sqrt{-1}y_i$$

where $|x| = \sqrt{\sum x_i^2}$. This diffeomorphism sends the unit sphere $S^n \subset Z$ to the zero section of the tangent bundle TS^n. The self-intersection number of the zero section S^n in the total space of the tangent bundle TS^n is equal to the Euler characteristic $\chi(S^n) = 1 + (-1)^n$. However, the natural orientations of the manifolds Z (as a complex analytic manifold) and TS^n (as the total space of a tangent bundle) differ by the sign $(-1)^{n(n-1)/2}$. □

The path $\gamma : I \to \Delta$ from s_i to η defines a closed path around the critical value s_i in the following way: Let Δ_i be a disc of sufficiently small radius η_i around s_i such that $\gamma(I)$ intersects the boundary $\partial\Delta_i$ of Δ_i exactly once, namely at time $t = \theta$ at the point $s_i + u_i$. Let $\tau : I \to \Delta_i$, $t \mapsto s_i + u_i e^{2\pi\sqrt{-1}t}$, be the path starting at $s_i + u_i$ which goes once around s_i on the boundary of Δ_i in counterclockwise direction. Moreover, set $\widetilde{\gamma} := \gamma|_{[\theta,1]}$. A suitable small deformation ω (avoiding self-intersections) of the closed path $\widetilde{\gamma}^{-1}\tau\widetilde{\gamma}$ with starting and end point η is called the *simple loop* associated to γ (cf. Fig. 8.2). The monodromy

$$h_\delta := h_{\omega *} : M \longrightarrow M$$

corresponding to the simple loop ω associated to γ is called the *Picard-Lefschetz transformation* corresponding to the vanishing cycle δ.

The following theorem is the basic result of the Picard-Lefschetz theory. It goes back to Picard and Simart [122, p. 95ff.] and Lefschetz [104, Théorème fondamental, p. 23 & p. 92]. For a proof see [97, §5], [113, Chapter 3], and [13, 1.3]. A proof following the proof in Looijenga's book [113, Chapter 3] is also given in [58, §5.3]. For a modern account of Picard-Lefschetz theory see also the article of Lamotke [98].

Theorem 8.2.5 (Picard-Lefschetz Formula) *For $\alpha \in M$ we have*

$$h_\delta(\alpha) = \alpha - (-1)^{\frac{n(n-1)}{2}}\langle\alpha, \delta\rangle\delta.$$

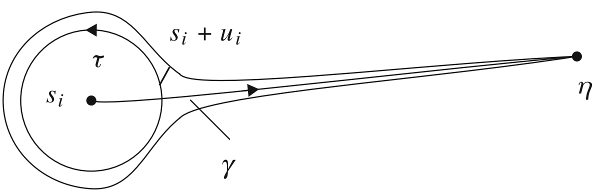


Fig. 8.2 Simple loop ω associated to γ

When n is even, the intersection form $\langle\ ,\ \rangle$ is a symmetric bilinear form and we can combine the formulas from Proposition 8.2.4 and Theorem 8.2.5 together as

$$h_\delta(\alpha) = \alpha - \frac{2\langle\alpha, \delta\rangle}{\langle\delta, \delta\rangle}\delta.$$

This means that the operator $h_\delta : M \to M$ is a *reflection* in the hyperplane of M orthogonal to δ. Such a reflection is also denoted by s_δ, so in this case $h_\delta = s_\delta$. When n is odd, the intersection form $\langle\ ,\ \rangle$ is skew symmetric and Theorem 8.2.5 means that h_δ is a *symplectic transvection*.

We now assume that ε and η are chosen so small that all the balls B_i and all the discs Δ_i are disjoint. We consider an ordered system $(\gamma_1, \ldots, \gamma_\mu)$ of paths $\gamma_i : I \to \Delta$ with $\gamma_i(0) = s_i,\ \gamma_i(1) = \eta$ and $\gamma_i((0, 1]) \subset \Delta'$.

Definition 8.2.6 The system $(\gamma_1, \ldots, \gamma_\mu)$ of paths is called *distinguished* if the following conditions are satisfied:

(i) The paths γ_i are non-selfintersecting.
(ii) The only common point of γ_i and γ_j for $i \neq j$ is η.
(iii) The paths are numbered in the order in which they arrive at η where one has to count clockwise from the boundary of the disc (cf. Fig. 8.3).

A system $(\delta_1, \ldots, \delta_\mu)$ of vanishing cycles $\delta_i \in \Lambda^*$ is called *distinguished*, if there exists a distinguished system $(\gamma_1, \ldots, \gamma_\mu)$ of paths such that δ_i is a cycle vanishing along γ_i.

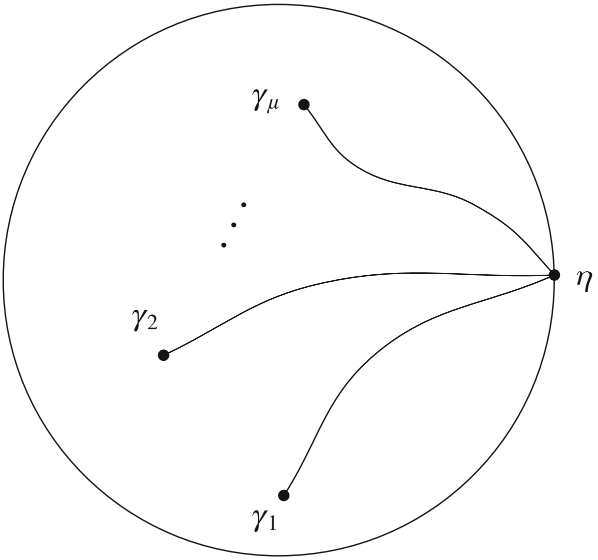


Fig. 8.3 Distinguished system of paths

Since Δ' is a disc from which μ points have been deleted, its fundamental group $\pi_1(\Delta', \eta)$ is the free group on μ generators. If $(\gamma_1, \dots, \gamma_\mu)$ is a distinguished system of paths, then $\pi_1(\Delta', \eta)$ is the free group on the generators $\omega_1, \dots, \omega_\mu$, where ω_i is the simple loop associated to γ_i.

Definition 8.2.7 The system $(\gamma_1, \dots, \gamma_\mu)$ of paths is called *weakly distinguished* if $\pi_1(\Delta', \eta)$ is the free group on the generators $(\omega_1, \dots, \omega_\mu)$, where ω_i is the simple loop belonging to γ_i.

A system $(\delta_1, \dots, \delta_\mu)$ of vanishing cycles $\delta_i \in \Lambda^*$ is called *weakly distinguished* if δ_i is a vanishing cycle along a path γ_i of a weakly distinguished system $(\gamma_1, \dots, \gamma_\mu)$ of paths.

Note that the numbering is important for a distinguished system of paths, but of no significance for a weakly distinguished system of paths. A distinguished system of paths is of course also weakly distinguished.

Brieskorn proved the following theorem [28, Appendix] (see also [13, Theorem 2.1], [58, Proposition 5.5]).

Theorem 8.2.8 (Brieskorn) *A distinguished system* $(\delta_1, \dots, \delta_\mu)$ *of vanishing cycles is a basis of the lattice* M*, i.e.,* $\langle \delta_1, \dots, \delta_\mu \rangle_{\mathbb{Z}} = M$*, where* $\langle \delta_1, \dots, \delta_\mu \rangle_{\mathbb{Z}}$ *denotes the* $\mathbb{Z}$*-span of* $(\delta_1, \dots, \delta_\mu)$.

From this theorem, one can derive the following corollary (see [13, Theorem 2.8], [58, Proposition 5.6]).

Corollary 8.2.9 *A weakly distinguished system* $(\delta_1, \dots, \delta_\mu)$ *of vanishing cycles also forms a basis of* M.

Definition 8.2.10 A basis $(\delta_1, \dots, \delta_\mu)$ of M is called *distinguished* (resp. *weakly distinguished*) if $(\delta_1, \dots, \delta_\mu)$ is a distinguished (resp. weakly distinguished) system of vanishing cycles.

By Theorem 8.2.8 and Corollary 8.2.9 every distinguished or weakly distinguished system of vanishing cycles forms a basis.

The concepts "distinguished" and "weakly distinguished" are due to Gabrielov. In order to distinguish both concepts better, one sometimes says, following a suggestion of Brieskorn, "strongly distinguished" instead of "distinguished". The term "geometric basis" is also used for a distinguished basis.

The group of all automorphisms of a lattice M, i.e., isomorphisms $M \to M$ which respect the bilinear form, will be denoted by $\mathrm{Aut}(M)$.

Definition 8.2.11 The image Γ of the homomorphism

$$\begin{aligned} \rho : \pi_1(\Delta', \eta) &\longrightarrow \mathrm{Aut}(M) \\ [\gamma] \quad &\longmapsto h_{\gamma *} \end{aligned}$$

is called the *monodromy group* of the singularity f.

If $(\delta_1, \dots, \delta_\mu)$ is a weakly distinguished basis, then the monodromy group of f is generated by the Picard-Lefschetz transformations h_{δ_i} corresponding to the vanishing cycles δ_i. Therefore the monodromy group of f is a group with μ generators. Indeed, the monodromy group is independent of the morsification of f, see Theorem 8.6.2 below.

Example 8.2.12 (For this example see also [13, 2.9] and [58, Example 5.4].) We consider the function $f : \mathbb{C} \to \mathbb{C}$ with $f(z) = z^{k+1}$. (This is the singularity A_k, see Sect. 8.7.) The Milnor fiber X_η consists of $k + 1$ points, namely the $(k + 1)$-th roots of η. As a morsification of f we consider the function $f_\lambda(z) = z^{k+1} - \lambda z$ for $\lambda \neq 0$. Fix $\lambda \in \mathbb{R},\ \lambda > 0$. The critical points of the function f_λ are given by the equation

$$f'_\lambda(z) = (k + 1)z^k - \lambda = 0.$$

Therefore they are the points

$$p_i = \sqrt[k]{\frac{\lambda}{k+1}}\xi_i, \quad \xi_i = e^{-\frac{2\pi i\sqrt{-1}}{k}},$$

with the critical values

$$s_i = -\frac{\lambda k}{k+1}\sqrt[k]{\frac{\lambda}{k+1}}\xi_i, \quad i = 1, \dots, k.$$

As a noncritical value we choose $-\eta$, where $\eta \in \mathbb{R},\ \eta > 0$ and

$$\eta \gg \frac{\lambda k}{k+1}\sqrt[k]{\frac{\lambda}{k+1}}.$$

Let $\gamma_i : [0, 1] \to \bar{\Delta},\ t \mapsto (1 - t)s_i$, and let τ be a path from 0 to $-\eta$ which runs along the real axis and goes once around the critical value

$$s_k = -\frac{\lambda k}{k+1}\sqrt[k]{\frac{\lambda}{k+1}}\xi_k \in \mathbb{R}$$

in the positive direction.

We consider the path system $(\gamma_1\tau, \dots, \gamma_k\tau)$. This system is homotopic to a distinguished path system. (For the notion of homotopy of path systems see Sect. 8.4 below.) Let $(\delta_1, \dots, \delta_k)$ be a corresponding distinguished system of vanishing cycles in $\widetilde{H}_0(X_{-\eta})$.

In order to compute the intersection numbers $\langle \delta_i, \delta_j \rangle$ of the vanishing cycles in $\widetilde{H}_0(X_{-\eta})$ we transport the system $(\delta_1, \dots, \delta_k)$ by parallel transport along the path τ^{-1} to $\widetilde{H}_0(X_0)$. We thus consider a system of vanishing cycles in $\widetilde{H}_0(X_0)$, which we again denote by $(\delta_1, \dots, \delta_k)$, and which is defined by the path system $(\gamma_1, \dots, \gamma_k)$.

The fiber X_0 consists of the $k+1$ points

$$x_0 = 0, x_1 = \sqrt[k]{\lambda}\xi_1, \dots, x_k = \sqrt[k]{\lambda}\xi_k.$$

Then up to orientation δ_i is represented by the cycle $x_i - x_0$. It is easy to calculate that $x_i - x_0$ vanishes along γ_i, i.e., that the points x_i and x_0 fall together along γ_i. Let

$$\delta_i = [x_i - x_0],\ i = 1, \dots, k.$$

Then

$$\langle \delta_i, \delta_j \rangle = \begin{cases} 2 & \text{for } i = j, \\ 1 & \text{for } i \neq j. \end{cases}$$

In this case, the Milnor lattice M, the set of vanishing cycles Λ^*, and the monodromy group Γ can be described as follows. Let $e_1, \dots, e_{k+1}$ be the standard basis of $\mathbb{R}^{k+1}$ and $\langle\ ,\ \rangle$ the Euclidean scalar product on $\mathbb{R}^{k+1}$. Denote by $\mathcal{S}_{k+1}$ the symmetric group in $k+1$ symbols. Then

$$\begin{aligned} M &= \{(v_1, \dots, v_{k+1}) \in \mathbb{Z}^{k+1} \mid v_1 + \cdots + v_{k+1} = 0\}, \\ \Lambda^* &= \{e_i - e_j \mid 1 \leq i, j \leq k+1, i \neq j\} = \{v \in M \mid \langle v, v \rangle = 2\}, \\ \Gamma &= \mathcal{S}_{k+1}. \end{aligned}$$

8.3 Coxeter-Dynkin Diagram and Seifert Form

Definition 8.3.1 Let $(\delta_1, \dots, \delta_\mu)$ be a weakly distinguished basis of M. The matrix

$$S := (\langle \delta_i, \delta_j \rangle)_{j=1,\dots,\mu}^{i=1,\dots,\mu}$$

is called the *intersection matrix* of f with respect to $(\delta_1, \dots, \delta_\mu)$.

By Proposition 8.2.4, the diagonal entries of the intersection matrix satisfy

$$\langle \delta_i, \delta_i \rangle = (-1)^{\frac{n(n-1)}{2}}(1 + (-1)^n) \text{ for all } i.$$

It is usual to represent the intersection matrix by a graph called the Coxeter-Dynkin diagram.

Definition 8.3.2 Let $(\delta_1, \dots, \delta_\mu)$ be a weakly distinguished basis of M. The *Coxeter-Dynkin diagram* of the singularity f with respect to $(\delta_1, \dots, \delta_\mu)$ is the graph D defined as follows:

(i) The vertices of D are in one-to-one correspondence with the elements $\delta_1, \dots, \delta_\mu$.
(ii) For $i < j$ with $\langle \delta_i, \delta_j \rangle \neq 0$ the i-th and the j-th vertex are connected by $|\langle \delta_i, \delta_j \rangle|$ edges, weighted with the sign $+1$ or -1 of $\langle \delta_i, \delta_j \rangle \in \mathbb{Z}$. We indicate the weight

$$w = \begin{cases} (-1)^{\frac{n}{2}} & \text{for } n \text{ even,} \\ (-1)^{\frac{n+1}{2}} & \text{for } n \text{ odd} \end{cases}$$

by a dashed line, the weight $-w$ by a solid line.

These diagrams are usually called Dynkin diagrams. However, according to A. J. Coleman [40, p. 450], they first appeared in mimeographed notes written by H. S. M. Coxeter (around 1935). Therefore we call them Coxeter-Dynkin diagrams.

Example 8.3.3 We continue Example 8.2.12. The Coxeter-Dynkin diagram with respect to $(\delta_1, \dots, \delta_k)$ is a complete graph with only dashed edges (i.e., each two vertices are joined by a dashed edge).

If $(\delta_1, \dots, \delta_\mu)$ is a distinguished basis then the classical monodromy operator of f can be expressed as follows:

$$h_* = h_{\delta_1} \cdots h_{\delta_\mu}.$$

We call this product the *Coxeter element* corresponding to the distinguished basis. This follows from the fact that the loop ω corresponding to h_* is homotopic to the combination $\omega_\mu \omega_{\mu-1} \cdots \omega_1$ of the simple loops associated to $h_{\delta_\mu}, h_{\delta_{\mu-1}}, \dots, h_{\delta_1}$.

We have the following algebraic proposition (cf. [26, Ch. V, §6, Exercice 3]).

Proposition 8.3.4 *Let M be a free $\mathbb{Z}$-module of rank ℓ with a basis $(e_1, \dots, e_\ell)$ and $A = (a_{ij})$ an $\ell \times \ell$-matrix with integral coefficients. Consider the operator $s_i : M \to M$ defined by*

$$s_i(e_j) = e_j - a_{ij} e_i$$

and let $c = s_1 \cdots s_\ell$. Let C be the matrix of c with respect to the basis $(e_1, \dots, e_\ell)$, I the $\ell \times \ell$ unit matrix, and let $U = (u_{ij})$ and $V = (v_{ij})$ be the matrices defined by

$$u_{ij} = \begin{cases} a_{ij} & \text{if } i < j, \\ 0 & \text{otherwise,} \end{cases} \qquad v_{ij} = \begin{cases} 0 & \text{if } i < j, \\ a_{ij} & \text{otherwise.} \end{cases}$$

Then

$$C = (I + U)^{-1}(I - V).$$

Let S_ε^{2n+1} be the boundary of the ball B_ε. The set $K = f^{-1}(0) \cap S_\varepsilon^{2n+1}$ is called the *link* of the singularity f. Let T be an (open) tubular neighborhood of K in S_ε^{2n+1}. Milnor [118] has shown that the map

$$\begin{array}{rccc} \Phi : & S_\varepsilon^{2n+1} \setminus T & \longrightarrow & S^1 \subset \mathbb{C} \\ & z & \longmapsto & \frac{f(z)}{|f(z)|} \end{array}$$

is the projection of a differentiable fiber bundle. Moreover, this fibration is equivalent to the restriction of the fibration $f|_{X^*} : X^* \to \Delta^*$ to the boundary S^1_η of $\overline{\Delta}$ [118, §5]. In particular, the fiber $Z_{w/|w|} := \Phi^{-1}(w/|w|)$ is diffeomorphic to X_w for $w \in S^1_\eta$. Let $g_t : Z_1 \to Z_{e^{2\pi it}}$ be the parallel transport along $\omega(t) = e^{2\pi it}$. For the definition of the linking number see [13, 2.3], [58, 4.7].

Definition 8.3.5 The *Seifert form* of f is the bilinear form L on $\widetilde{H}_n(Z_1) \cong \widetilde{H}_n(X_\eta)$ defined by $L(a, b) = l(a, g_{1/2*}(b))$ where $l(\, , \,)$ is the linking number.

Let $(\delta_1, \dots, \delta_\mu)$ be a distinguished basis of f and let

- $S := (\langle \delta_i, \delta_j \rangle)^{i=1,\dots,\mu}_{j=1,\dots,\mu}$ be the intersection matrix,
- $L := (L(\delta_i, \delta_j))^{i=1,\dots,\mu}_{j=1,\dots,\mu}$ be the matrix of the Seifert form, and
- H be the matrix of the monodromy h_* with respect to the basis $(\delta_1, \dots, \delta_\mu)$.

Then one has the following theorem.

Theorem 8.3.6 *The following holds:*

(i) *The matrix L is a lower triangular matrix with* $-(-1)^{n(n-1)/2}$ *on the diagonal.*
(ii) $S = -L - (-1)^n L^t$.
(iii) $H = (-1)^{n+1}(L^t)^{-1}L$.

Proof (i) This is [13, Lemma 2.5]. (Note that, according to [13, Remark in 2.5], the matrix of the bilinear form in [13] is written down as the matrix of the corresponding operator and hence corresponds to the transpose matrix in our convention. See also [58, Corollary 5.3 (i)], where, unfortunately, there is a misprint: “upper” should be “lower”.)

For the proof of (ii) see [13, Theorem 2.4] (see also [58, Corollary 5.3 (ii)]).

(iii) follows from (i) and (ii) by applying Proposition 8.3.4. (Note that the formula of [58, Proposition 5.9] has to be modified correspondingly.) □

It follows from Theorem 8.3.6 that each of these matrices determines the other two. It is clear that S and L determine the matrix H. That the matrix H of the classical monodromy operator with respect to a distinguished basis determines the intersection matrix S was first proved by F. Lazzeri [100] and follows from

Theorem 8.3.6 and a simple fact in linear algebra [13, Lemma 2.6] (see also [58, Lemma 5.5]).

A. B. Givental [69] introduced q-analogues of the invariants and formulas above. He considered a Picard-Lefschetz theory with "twisted" coefficients. As an upshot, he obtained a bilinear form on the free module $M_q := M \otimes \mathbb{Z}[q, q^{-1}]$ defined by the matrix $S_q := -L - qL^t$ with respect to a distinguished basis of f, where q is a variable, which can take non-zero complex values, and $\mathbb{Z}[q, q^{-1}]$ denotes the ring of Laurent polynomials in q. Proposition 8.3.4 (with the ring $\mathbb{Z}$ replaced by $\mathbb{Z}[q, q^{-1}]$) yields a matrix $H_q = -q(L^t)^{-1}L = qH$. This interpolates between the symmetric ($q = 1$) and the skew symmetric ($q = -1$) versions of these invariants. The q-analogue of the monodromy group was studied by G. G. Il'yuta [86].

8.4 Change of Basis

A distinguished or weakly distinguished system $(\gamma_1, \ldots, \gamma_\mu)$ of paths can be chosen in many various ways. Next we consider elementary operations on path systems which preserve the property of being distinguished or weakly distinguished.

Let $(\gamma_1, \ldots, \gamma_\mu)$ be a distinguished system of paths from the critical values $s_1, \ldots, s_\mu$ to the non-critical value η and let $(\delta_1, \ldots, \delta_\mu)$ be a corresponding distinguished system of vanishing cycles. Furthermore, let $(\omega_1, \ldots, \omega_\mu)$ be a corresponding system of simple loops.

Definition 8.4.1 The operation α_j for $1 \leq j < \mu$ is defined as

$$\alpha_j : (\gamma_1, \ldots, \gamma_\mu) \mapsto (\gamma_1, \ldots, \gamma_{j-1}, \widetilde{\gamma}_j, \widetilde{\gamma}_{j+1}, \gamma_{j+2}, \ldots, \gamma_\mu),$$

where $\widetilde{\gamma}_{j+1} = \gamma_j$ and $\widetilde{\gamma}_j$ is a small homotopic deformation of $\gamma_{j+1}\omega_j$ such that $\widetilde{\gamma}_j$ has no self-intersection points and intersects the other paths only at η, for $t = 1$ (see Fig. 8.4).

Then $(\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_\mu)$ is again a distinguished system of paths.

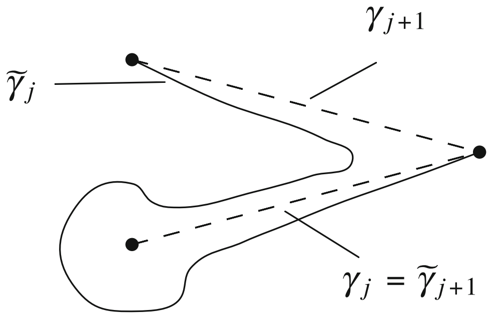


Fig. 8.4 The operation α_j

This induces the following operation on the corresponding system $(\delta_1, \ldots, \delta_\mu)$ of vanishing cycles which will be denoted by the same symbol:

$$\alpha_j : (\delta_1, \ldots, \delta_\mu) \mapsto (\delta_1, \ldots, \delta_{j-1}, h_{\delta_j}(\delta_{j+1}), \delta_j, \delta_{j+2}, \ldots, \delta_\mu)$$

where

$$h_{\delta_j}(\delta_{j+1}) = \delta_{j+1} - (-1)^{n(n-1)/2}\langle \delta_{j+1}, \delta_j \rangle \delta_j.$$

Definition 8.4.2 The operation β_{j+1} for $1 \leq j < \mu$ is defined as

$$\beta_{j+1} : (\gamma_1, \ldots, \gamma_\mu) \mapsto (\gamma_1, \ldots, \gamma_{j-1}, \gamma'_j, \gamma'_{j+1}, \gamma_{j+2}, \ldots, \gamma_\mu)$$

where $\gamma'_j = \gamma_{j+1}$ and γ'_{j+1} is a small homotopic deformation of $\gamma_j \omega_{j+1}^{-1}$ with the properties above (see Fig. 8.5). Then $(\gamma'_1, \ldots, \gamma'_\mu)$ is again a distinguished system of paths.

This induces the following operation on the corresponding system $(\delta_1, \ldots, \delta_\mu)$ of vanishing cycles which will also be denoted by the same symbol:

$$\beta_{j+1} : (\delta_1, \ldots, \delta_\mu) \mapsto (\delta_1, \ldots, \delta_{j-1}, \delta_{j+1}, h_{\delta_{j+1}}^{-1}(\delta_j), \delta_{j+2}, \ldots, \delta_\mu)$$

where

$$h_{\delta_{j+1}}^{-1}(\delta_j) = \delta_j - (-1)^{n(n-1)/2}\langle \delta_{j+1}, \delta_j \rangle \delta_{j+1}$$

is the inverse Picard-Lefschetz transformation.

Two distinguished systems $(\gamma_1, \ldots, \gamma_\mu)$ and $(\tau_1, \ldots, \tau_\mu)$ of paths are called *homotopic* if there are homotopies $\phi_i : I \times I \to \bar{\Delta}$ between γ_i and τ_i, $i = 1, \ldots, \mu$, such that for all $u \in I$ and paths $\phi_i^u : I \to \bar{\Delta}$, $t \mapsto \phi_i(u, t)$, $i = 1, \ldots, \mu$, the following properties are satisfied:

(i) $\phi_i^u(0) = s_i$, $\phi_i^u(1) = \eta$.
(ii) The paths ϕ_i^u are double point free.
(iii) Each two paths ϕ_i^u and ϕ_j^u have, for $i \neq j$, only the end point η in common.

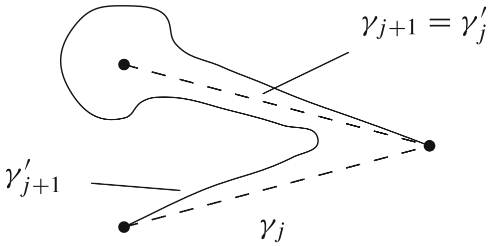


Fig. 8.5 The operation β_{j+1}

One can easily show (see [58, Lemma 6]):

Lemma 8.4.3 *The operations α_j and β_{j+1} are mutually inverse, i.e., the application of $\alpha_j \beta_{j+1}$ and $\beta_{j+1}\alpha_j$ to a distinguished path system $(\gamma_1, \dots, \gamma_\mu)$ yields a homotopic distinguished path system.*

Up to homotopy of distinguished path systems we have

(i) $\alpha_i \alpha_j = \alpha_j \alpha_i$ for i, j with $|i - j| \geq 2$,
(ii) $\alpha_j \alpha_{j+1} \alpha_j = \alpha_{j+1} \alpha_j \alpha_{j+1}$ for $1 \leq j < \mu - 1$.

These are the relations of Artin's braid group [14, 15] (see also [25]). Therefore we have an action of the braid group Br_μ on μ strings on the set of the homotopy classes of distinguished path systems and so also on the set of all distinguished systems of vanishing cycles. One can show the following result ([73], see also [58, Proposition 5.15]).

Proposition 8.4.4 *The braid group* Br_μ *acts transitively on the set of all homotopy classes of distinguished path systems, i.e., any two distinguished path systems can be transformed one to the other by iteration of the operations α_j and β_{j+1} and a succeeding homotopy.*

Definition 8.4.5 Let

- $\mathcal{B}$ be the set of all distinguished bases of vanishing cycles of f,
- $\mathcal{D}$ be the set of Coxeter-Dynkin diagrams of distinguished bases of f.

One also has a braid group action on the sets $\mathcal{B}$ and $\mathcal{D}$. Moreover, one can change the orientation of a cycle. Let H_μ be the direct product of μ cyclic groups of order two with generators $\kappa_1, \dots, \kappa_\mu$, where κ_i acts on $\mathcal{B}$ by

$$\kappa_i : (\delta_1, \dots, \delta_i, \dots, \delta_\mu) \mapsto (\delta_1, \dots, -\delta_i, \dots, \delta_\mu).$$

The braid group Br_μ acts on H_μ by permutation of the generators $\kappa_1, \dots, \kappa_\mu$: α_j corresponds to the transposition of κ_j and κ_{j+1}. Let $\mathrm{Br}_\mu^{\rtimes} = H_\mu \rtimes \mathrm{Br}_\mu$ be the semidirect product. It follows from Proposition 8.4.4 that the action of the group $\mathrm{Br}_\mu^{\rtimes}$ on $\mathcal{B}$ is transitive.

The set $\mathcal{B}$ depends on the chosen morsification. In order to get an invariant of the singularity, Brieskorn [33] proposed a more general notion of distinguished bases. Namely, he considered the natural action of the monodromy group Γ on the set $\mathcal{B}$: An element $h \in \Gamma$ acts as follows:

$$h : (\delta_1, \dots, \delta_\mu) \mapsto (h(\delta_1), \dots, h(\delta_\mu)).$$

Brieskorn called a basis B of M *geometric* if it is obtained by any choice of a distinguished path system, of orientations, and of $h \in \Gamma$. He introduced the notions

- $\mathcal{B}^*$ for the set of all geometric bases of f,
- $\mathcal{D}^*$ for the set of Coxeter-Dynkin diagrams of geometric bases of f.

The sets $\mathcal{B}^*$ and $\mathcal{D}^*$ are invariants of the singularity. In fact, the set $\mathcal{D}^*$ coincides with $\mathcal{D}$. The action of Γ commutes with the action of the group $\mathrm{Br}_\mu^\rtimes$. It follows from Proposition 8.4.4 that the action of the group $\Gamma \times \mathrm{Br}_\mu^\rtimes$ on $\mathcal{B}^*$ is transitive. One can derive from this that the invariants $\mathcal{B}^*$ and $\mathcal{D}^*$ determine each other, see [33].

Note that, unfortunately, in [60] the set $\mathcal{B}$ was considered but denoted by $\mathcal{B}^*$.

The braid group action above first appeared in a paper of A. Hurwitz [80] from 1891 where he describes a braid group action on certain sets of Riemann surfaces (cf. [94]). It was also studied by Brieskorn and his students, see [34]. In [34], Brieskorn introduced a simple unifying concept, the notion of an automorphic set.

Definition 8.4.6 An *automorphic set* is a set Λ with a product $* : \Lambda \times \Lambda \to \Lambda$ such that all left translations are automorphisms, i.e., one has the following properties:

(i) For all $a, c \in \Lambda$ there is a unique $b \in \Lambda$ such that $a * b = c$.
(ii) For all $a, b, c \in \Lambda$ one has $(a * b) * (a * c) = a * (b * c)$.

The set Λ^* of vanishing cycles of f is an automorphic set with the product $a * b := h_a(b)$ for $a, b \in \Lambda^*$.

If Λ is an automorphic set, then one has a canonical braid group action on the n-fold cartesian product Λ^n of Λ:

$$\alpha_i : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{i-1}, x_i * x_{i+1}, x_i, x_{i+2}, \dots, x_n).$$

The concept of an automorphic set is a basic concept which is also studied under the following names: left self-distributive system, self-distributive groupoid, quandle, wrack, and rack, see, e.g., the book of P. Dehornoy [42].

This braid group action is also considered in the representation theory of algebras, see, e.g., [41, 92, 124]. It has also been applied in mathematical physics, see, e.g., [37, 63, 64].

Example 8.4.7 We continue Example 8.3.3. By the transformations

$$\alpha_{k-1}, \alpha_{k-2}, \dots, \alpha_1; \alpha_{k-1}, \alpha_{k-2}, \dots, \alpha_2; \dots; \alpha_{k-1}, \alpha_{k-2}; \alpha_{k-1},$$

the distinguished basis $(\delta_1, \dots, \delta_k)$ is transformed to a distinguished basis with the Coxeter-Dynkin diagram depicted in Fig. 8.6. This is the classical Coxeter-Dynkin diagram of type A_k.

Finally, we consider operations that transform weakly distinguished path systems again into weakly distinguished path systems.

Fig. 8.6 Standard Coxeter-Dynkin diagram A_k

Let $(\gamma_1, \ldots, \gamma_\mu)$ now be a weakly distinguished path system from the points $s_1, \ldots, s_\mu$ to η, let $(\omega_1, \ldots, \omega_\mu)$ be a corresponding system of simple loops and let $(\delta_1, \ldots, \delta_\mu)$ be a corresponding weakly distinguished system of vanishing cycles.

Definition 8.4.8 We define operations $\alpha_i(j)$ and $\beta_i(j)$ for $i, j \in \{1, \ldots, \mu\}$, $i \neq j$, as follows:

$$\alpha_i(j) : (\gamma_1, \ldots, \gamma_\mu) \mapsto (\gamma_1, \ldots, \gamma_{j-1}, \gamma_j\omega_i, \gamma_{j+1}, \ldots, \gamma_\mu)\,,$$

$$\beta_i(j) : (\gamma_1, \ldots, \gamma_\mu) \mapsto (\gamma_1, \ldots, \gamma_{j-1}, \gamma_j\omega_i^{-1}, \gamma_{j+1}, \ldots, \gamma_\mu).$$

These operations induce the following operations on the corresponding systems of simple loops, and we denote them by the same symbols:

$$\alpha_i(j) : (\omega_1, \ldots, \omega_\mu) \mapsto (\omega_1, \ldots, \omega_{j-1}, \omega_i^{-1}\omega_j\omega_i, \omega_{j+1}, \ldots, \omega_\mu)\,,$$

$$\beta_i(j) : (\omega_1, \ldots, \omega_\mu) \mapsto (\omega_1, \ldots, \omega_{j-1}, \omega_i\omega_j\omega_i^{-1}, \omega_{j+1}, \ldots, \omega_\mu).$$

If $\omega_1, \ldots, \omega_\mu$ forms a generating system for $\pi_1(\Delta', \eta)$, then $\pi_1(\Delta', \eta)$ is also generated by the new simple loops that arise from application of the operations $\alpha_i(j)$ and $\beta_i(j)$. Hence $\alpha_i(j)$ and $\beta_i(j)$ transfer weakly distinguished path systems again to weakly distinguished path systems.

These operations thus induce operations on the corresponding weakly distinguished systems of vanishing cycles too, which we denote by the same symbols, and they appear as follows:

$$\alpha_i(j) : (\delta_1, \ldots, \delta_\mu) \mapsto (\delta_1, \ldots, \delta_{j-1}, h_{\delta_i}(\delta_j), \delta_{j+1}, \ldots, \delta_\mu),$$

$$\beta_i(j) : (\delta_1, \ldots, \delta_\mu) \mapsto (\delta_1, \ldots, \delta_{j-1}, h_{\delta_i}^{-1}(\delta_j), \delta_{j+1}, \ldots, \delta_\mu).$$

The operations $\alpha_i(j)$ and $\beta_i(j)$ are again mutually inverse in the sense above. For even n they even agree.

If $(\gamma_1, \ldots, \gamma_\mu)$ is a distinguished path system and if $\tau_{j,j+1} \in \mathcal{S}_\mu$ denotes the transposition of j and $j+1$, then, up to homotopy,

$$\alpha_j = \tau_{j,j+1} \circ \alpha_j(j+1),$$

$$\beta_{j+1} = \tau_{j,j+1} \circ \beta_{j+1}(j).$$

We now also have the following proposition:

Proposition 8.4.9 *Let $(\omega_1, \ldots, \omega_\mu)$ and $(\omega'_1, \ldots, \omega'_\mu)$ be two free generating systems of the free group $\pi_1(\Delta', \eta)$ such that ω_i and ω'_i are conjugate to one another for $i = 1, \ldots, \mu$. Then one can obtain $(\omega'_1, \ldots, \omega'_\mu)$ from $(\omega_1, \ldots, \omega_\mu)$ by the application of a sequence of operations of type $\alpha_i(j)$ or $\beta_i(j)$.*

This proposition was conjectured by Gusein-Zade [73] and proved by S. P. Humphries [79] in 1985. It also follows, as remarked by R. Pellikaan, from an old result of J. H. C. Whitehead from the year 1936 (cf. [116, Proposition 4.20]). We refer to [79].

It follows from Proposition 8.4.9 that any two weakly distinguished systems of vanishing cycles can be transformed one to the other by iteration of the operations $\alpha_i(j)$ and $\beta_i(j)$ and a succeeding change of orientation of some of the cycles.

J. McCool [117] found a presentation of the subgroup of the automorphism group of a free group generated by the operations $\alpha_i(j)$ and $\beta_i(j)$.

8.5 Computation of Intersection Matrices

The *Sebastiani-Thom sum* of the singularities $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ and $g : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ is the singularity of the function germ $f \oplus g : (\mathbb{C}^{n+m+1}, 0) \to (\mathbb{C}, 0)$ defined by the formula

$$(f \oplus g)(x, y) = f(x) + g(y)$$

($x \in \mathbb{C}^{n+1}$, $y \in \mathbb{C}^m$, $(x, y) \in \mathbb{C}^{n+m+1} \cong \mathbb{C}^{n+1} \oplus \mathbb{C}^m$).

M. Sebastiani and R. Thom [130] proved that the monodromy operator of the singularity $f \oplus g$ is equal to the tensor product of the monodromy operators of the singularities f and g. If L_f, L_g, and $L_{f\oplus g}$ denote the Seifert form of f, g, and $f \oplus g$ respectively, then by a result of Deligne (see [46], see also [13, Theorem 2.10])

$$L_{f\oplus g} = (-1)^{(n+1)m} L_f \otimes L_g.$$

Gabrielov [65] showed how to calculate an intersection matrix of $f \oplus g$ from the intersection matrices of f and g with respect to distinguished bases (see also [13, Theorem 2.11]). As a corollary, he obtained certain intersection matrices for singularities of the form

$$f(x) = z_0^{a_0} + \cdots + z_n^{a_n}, \quad \text{for } a_i \in \mathbb{Z},\ a_i \geq 2,\ i = 0, \ldots, n.$$

These singularities are called *Brieskorn-Pham singularities*. They were considered by Brieskorn [27] and Pham [121] (see also [77]). For such a singularity, already Pham [121] had found a basis and calculated the intersection matrix with respect to this basis. Gabrielov showed that Pham's basis can be deformed to a distinguished basis and the intersection matrix is given by the same formulas which Gabrielov obtained. Independently, these intersection matrices with respect to distinguished bases were also calculated by A. Hefez and Lazzeri [75].

A special case of the Sebastiani-Thom sum of f and g is the case when $g(y) = y_1^2 + \cdots + y_m^2$. This is called a *stabilization* of f. The following theorem is a special case of Gabrielov's result (see also [13, Theorem 2.14]).

Theorem 8.5.1 *Let f_λ be a morsification of the singularity f, let $(\gamma_1, \ldots, \gamma_\mu)$ be a distinguished path system for f_λ, and let $(\delta_1, \ldots, \delta_\mu)$ be a corresponding distinguished basis.*

Then $f_\lambda(x) + y_1^2 + \ldots + y_m^2$ is a morsification of the singularity $f(x) + y_1^2 + \ldots + y_m^2$, with the same critical values, $(\gamma_1, \ldots, \gamma_\mu)$ is also a distinguished path system for this singularity, and for a corresponding distinguished basis $(\widetilde{\delta}_1, \ldots, \widetilde{\delta}_\mu)$ we have

$$\langle \widetilde{\delta}_i, \widetilde{\delta}_j \rangle = [\mathrm{sign}(j-i)]^m (-1)^{(n+1)m+\frac{m(m-1)}{2}} \langle \delta_i, \delta_j \rangle \textit{ for } i \neq j.$$

It follows from Theorem 8.5.1 that, by taking a suitable stabilization, one can assume that $n \equiv 2 \bmod 4$. In this case, the intersection form is symmetric and the vanishing cycles have self intersection number -2. The Picard-Lefschetz transformation h_{δ_i} acts on M by the formula

$$h_{\delta_i}(\alpha) = s_{\delta_i}(\alpha) = \alpha + \langle \alpha, \delta_i \rangle \delta_i .$$

This is a reflection in the hyperplane orthogonal to the vanishing cycle δ_i. In accordance with the definition in Sect. 8.3, in the Coxeter-Dynkin diagram, edges of weight $+1$ are depicted by solid lines and edges of weight -1 are depicted by dashed lines. Note that the definition of a Coxeter-Dynkin diagram in [13, 2.8] is slightly different: It encodes the intersection matrix in the case $n \equiv 2 \bmod 4$, and the i-th and j-th vertices are joined by an edge of multiplicity $\langle \delta_i, \delta_j \rangle$.

Example 8.5.2 Consider the germ of the function $f : \mathbb{C}^2 \to \mathbb{C}$ defined by $f(x, y) = x^5 + y^3$. (This is the singularity E_8, see Sect. 8.7.) By Example 8.4.7 and the result of Gabrielov [65] there is a distinguished basis of f with a Coxeter-Dynkin diagram of the shape of Fig. 8.7. By the transformations

$$\alpha_7, \alpha_6, \alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1; \beta_5, \beta_4; \beta_7, \beta_6, \beta_5; \beta_7, \beta_6, \beta_5; \beta_8, \beta_7, \beta_6; \kappa_2, \kappa_7, \kappa_8,$$

the Coxeter-Dynkin diagram is transformed to the classical Coxeter-Dynkin diagram of type E_8, see Fig. 8.8. It follows from Theorem 8.7.1 below that the numbering can be changed by braid group transformations to an arbitrary numbering.

Fig. 8.7 Gabrielov diagram of E_8

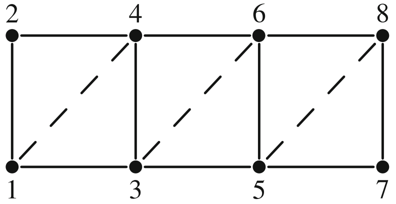
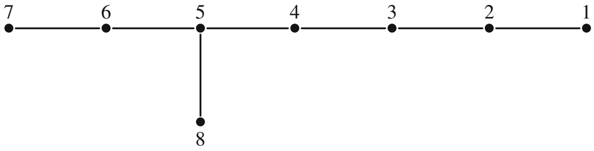


Fig. 8.8 Standard Coxeter-Dynkin diagram E_8



Another method to compute an intersection matrix with respect to a distinguished basis of f is the polar curve method of Gabrielov [68].

If $n = 1$, so $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ defines a curve singularity, there is an especially nice method to compute an intersection matrix with respect to a distinguished basis using a real morsification of the singularity. This method is independently due to N. A'Campo [2] and Gusein-Zade [71, 72].

P. Orlik and R. Randell [120] computed the classical monodromy operator for weighted homogeneous polynomials of the form

$$f(z_0, \ldots, z_n) = z_0^{a_0} + z_0 z_1^{a_1} + \ldots + z_{n-1} z_n^{a_n}, \quad n \geq 1.$$

Moreover, they formulated the following conjecture. Let $r_k = a_0 a_1 \cdots a_k$ for $k = 0, 1, \ldots, n,\ r_{-1} = 1$, and define integers $c_0, c_1, \ldots, c_\mu$ by

$$\prod_{i=-1}^{n} (t^{r_i} - 1)^{(-1)^{n-i}} = c_\mu t^\mu + \cdots + c_1 t + c_0.$$

Conjecture 8.5.3 (Orlik-Randell) There exists a distinguished basis of f such that the Seifert matrix L of f is given by

$$L = -(-1)^{n(n+1)/2} \begin{pmatrix} c_0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ c_1 & c_0 & 0 & \cdots & \cdots & 0 & 0 \\ c_2 & c_1 & c_0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ c_{\mu-3} & c_{\mu-4} & c_{\mu-5} & \cdots & c_0 & 0 & 0 \\ c_{\mu-2} & c_{\mu-3} & c_{\mu-4} & \cdots & c_1 & c_0 & 0 \\ c_{\mu-1} & c_{\mu-2} & c_{\mu-3} & \cdots & c_2 & c_1 & c_0 \end{pmatrix}.$$

This conjecture is still open. However, recently D. Aramaki and A. Takahashi [7] proved an algebraic analogue of this conjecture. Namely, they considered the so called Berglund-Hübsch transpose ([20], see also [59])

$$\widetilde{f}(z_0, \ldots, z_n) = z_0^{a_0} z_1 + z_1^{a_1} z_2 + \ldots + z_{n-1}^{a_{n-1}} z_n + z_n^{a_n}$$

of the polynomial f. They showed that the triangulated category of maximally-graded matrix factorizations for $\widetilde{f}$ admits a full exceptional collection with this matrix.

8.6 The Discriminant and the Lyashko-Looijenga Map

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a holomorphic function germ with an isolated singularity at 0, $\operatorname{grad} f(0) = 0$. Then one obtains a universal unfolding F of f as follows (see [70, 1.3] or [58, Proposition 3.17]): Let $g_0 = -1, g_1, \ldots, g_{\mu-1}$ be representatives of a basis of the $\mathbb{C}$-vector space

$$\mathcal{O}_{n+1} \Big/ \left(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n} \right) \mathcal{O}_{n+1},$$

which has dimension μ. Then put

$$\begin{aligned} F : (\mathbb{C}^{n+1} \times \mathbb{C}^{\mu}, 0) &\longrightarrow (\mathbb{C}, 0) \\ (z, u) \qquad &\longmapsto f(z) + \sum_{j=0}^{\mu-1} g_j(z) u_j . \end{aligned}$$

Let

$$F : V \times U \to \mathbb{C}$$

be a representative of the unfolding F, where V is an open neighborhood of 0 in $\mathbb{C}^{n+1}$ and U is an open neighborhood of 0 in $\mathbb{C}^{\mu}$. We put

$$\begin{aligned} \mathcal{Y} &:= \{(z, u) \in V \times U \mid F(z, u) = 0\}, \\ \mathcal{Y}_u &:= \{z \in V \mid F(z, u) = 0\}. \end{aligned}$$

Since $F(z, 0) = f(z)$, there is an $\varepsilon > 0$ such that every sphere $S_\rho \subset V$ around 0 of radius $\rho \leq \varepsilon$ intersects the set $\mathcal{Y}_0$ transversally. Let $\varepsilon > 0$ be so chosen. Then there is also an $\theta > 0$ such that for $|u| \leq \theta$ the set $\{u \in \mathbb{C}^{\mu} \mid |u| \leq \theta\}$ lies entirely in U and $\mathcal{Y}_u$ intersects the sphere S_ε transversally. Let θ be so chosen. We put

$$\begin{aligned} \mathcal{X}^{\circ} &:= \{(z, u) \in \mathcal{Y} \bigm| |z| < \varepsilon, |u| < \theta\}, \\ \mathcal{X} &:= \{(z, u) \in \mathcal{Y} \bigm| |z| \leq \varepsilon, |u| < \theta\}, \\ \partial \mathcal{X} &:= \{(z, u) \in \mathcal{Y} \bigm| |z| = \varepsilon, |u| < \theta\}, \\ S &:= \{u \in U \bigm| |u| < \theta\}, \end{aligned}$$

$$\begin{aligned} p: \quad \mathcal{X} &\longrightarrow S \\ (z,u) &\longmapsto u. \end{aligned}$$

Let C be the set of critical points and $D = p(C) \subset S$ the *discriminant* of p. We have the following result (see, e.g., [58, Proposition 3.21]).

Theorem 8.6.1 *For a suitable* $\theta > 0$ *we have:*

(i) *The map* $p : \mathcal{X} \to S$ *is proper.*
(ii) C *is a nonsingular analytic subset of* $\mathcal{X}^\circ$ *and is closed in* $\mathcal{X}$.
(iii) *The restriction* $p|_C : C \to S$ *is finite (i.e., proper with finite fibers).*
(iv) *The discriminant* D *is an irreducible hypersurface in* S.

By the Ehresmann fibration theorem the map

$$p' := p|_{\mathcal{X} \setminus p^{-1}(D)} : \mathcal{X} \setminus p^{-1}(D) \longrightarrow S \setminus D$$

is then the projection of a differentiable fiber bundle, and the fibers of p' are diffeomorphic to a Milnor fiber X_η of f.

Let $s \in S \setminus D$ and $X_s := p^{-1}(s) = (p')^{-1}(s)$. Then p' defines a representation

$$\rho : \pi_1(S \setminus D, s) \longrightarrow \mathrm{Aut}(\widetilde{H}_n(X_s)).$$

One has the following theorem (see [13, Theorem 3.1] or [58, Proposition 5.17]).

Theorem 8.6.2 *The image* Γ *of the homomorphism*

$$\rho : \pi_1(S \setminus D, s) \longrightarrow \mathrm{Aut}(\widetilde{H}_n(X_s))$$

coincides with the monodromy group of the singularity.

In particular, Theorem 8.6.2 implies that the monodromy group is independent of the chosen morsification.

As a corollary of the irreducibility of the discriminant (Theorem 8.6.1(iv)) and Theorem 8.6.2 we obtain the following result which was first proved by Gabrielov [66] and independently by Lazzeri [99, 100].

Corollary 8.6.3 (Gabrielov, Lazzeri) *The Coxeter-Dynkin diagram with respect to a weakly distinguished system* $(\delta_1, \ldots, \delta_\mu)$ *of vanishing cycles is a connected graph.*

One can show that if 0 is neither a regular nor a non-degenerate critical point of f, then there are two vanishing cycles δ, δ' of f with $\langle \delta, \delta' \rangle = 1$. This follows from the following result due to G. N. Tyurina [136, Theorem 1] and D. Siersma [133, Proposition (8.9)] (see also [13, Theorem 3.23], [58, 5.9]) and the fact that, if 0 is neither a regular nor a non-degenerate critical point of f, then f deforms to the singularity g with $g(z) = z_0^3 + z_1^2 + \cdots + z_n^2$.

Theorem 8.6.4 (Tyurina, Siersma) *Let* $f_t : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$, $t \in [0, 1]$, *be a continuous deformation of the singularity* $f_0 = f$ *with* $\mu(f_0) = \mu$, $\mu(f_t) = \mu'$ *for* $0 < t \leq 1$. *Then one has* $\mu \geq \mu'$, *one has a natural inclusion of the Milnor lattice* M_{f_t} *of* f_t *in the Milnor lattice* M_{f_0} *of* f_0, *and a distinguished basis of* f_t *can be extended to a distinguished basis of* f_0.

From these results one obtains another corollary of the irreducibility of the discriminant (cf. [13, Theorem 3.4], [58, Proposition 5.20]):

Corollary 8.6.5 *If not both* (i) n *is odd and* (ii) 0 *is a non-degenerate critical point of* f, *then the set of vanishing cycles* Λ^* *is the only* Γ*-orbit, i.e., the monodromy group* Γ *acts transitively on* Λ^*.

Using these results, K. Saito [128] showed that the monodromy group Γ determines the Milnor lattice M.

We can also deduce from these results that the classical monodromy operator acts irreducibly (cf. [13, Theorem 3.5]). (An earlier result for curves was obtained by C. H. Bey [23, 24].)

Corollary 8.6.6 *Let* $(\delta_1, \ldots, \delta_\mu)$ *be a distinguished basis of* f *and let* I *be a subset of the set of indices* $I_0 = \{1, \ldots, \mu\}$ *such that the linear span of the basis elements* δ_i *with* $i \in I$ *is invariant under the classical monodromy operator* h_*. *Then either* $I = \emptyset$ *or* $I = I_0$.

Corollary 8.6.7 *If the classical monodromy operator of a singularity is the multiplication by* ± 1, *then the singularity is non-degenerate.*

This was first proved by A'Campo [1, Théorème 2] as an answer to a question of Sebastiani. It was deduced from the following result.

Theorem 8.6.8 (A'Campo) *The trace of the classical monodromy operator of* f *is*

$$\operatorname{tr} h_* = (-1)^n.$$

The *corank* of a singularity f is the corank of the Hesse matrix of f. Using a result of Deligne (see [4]), the author proved the following result [56, Proposition 5].

Proposition 8.6.9 *Let* $n \equiv 2 \bmod 4$ *and let* $c(f)$ *denote the corank of* f. *Then*

$$\operatorname{tr} h_*^2 = (-1)^{c(f)}.$$

A very important result on the classical monodromy is the following theorem.

Theorem 8.6.10 (Monodromy Theorem) *The classical monodromy of* f *is quasi-unipotent, i.e., its eigenvalues are roots of unity.*

For the history of this theorem and further properties of the classical monodromy see the survey article [57]. The usual proofs of Theorem 8.6.10 use a resolution of the singularity, see e.g. [61] for an instructive one. For a proof which does not use a resolution see [102].

The *bifurcation variety* Bif is the set of all $\lambda \in S$ such that f_λ does not have μ distinct critical values. Looijenga [109] in 1974 and independently Lyashko (in the same year, but his work was only published later in [114, 115]) introduced the following mapping: The *Lyashko-Looijenga mapping* LL sends a point $\lambda \in S$ to the unordered collection of critical values of the function f_λ or, what amounts to the same thing but is sometimes more convenient, to the polynomial which has these critical values as roots. If $\mathcal{P}^\mu$ denotes the set of monic polynomials of degree μ, then this is the mapping

$$\begin{aligned} \mathrm{LL} : S &\longrightarrow \mathcal{P}^\mu \\ \lambda &\longmapsto \textstyle\prod_{i=1}^{\mu}(t - s_i) \end{aligned}$$

where $s_1, \ldots, s_\mu$ are the critical values of the function f_λ. Let $\Sigma \subset \mathcal{P}^\mu$ denote the discriminant variety in $\mathcal{P}^\mu$. Then there exists a neighborhood $U \subset S$ of $0 \in S$ such that $\mathrm{LL}|_{U \setminus \mathrm{Bif}} : U \setminus \mathrm{Bif} \to \mathcal{P}^\mu \setminus \Sigma$ is locally biholomorphic [109, Theorem (1.4)].

8.7 Special Singularities

We shall now consider what is known about these invariants for special classes of singularities.

Let $f, g : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be holomorphic function germs with an isolated singularity at 0. The germs f and g are called *right equivalent* if f is taken to g under (the germ of) a biholomorphic mapping of the domain space which leaves the origin invariant. The *modality* (or *module number*) of f is the smallest number m for which there exists a representative $p : \mathcal{X} \to S$ of the universal unfolding $F : (\mathbb{C}^{n+1} \times \mathbb{C}^\mu, 0) \to (\mathbb{C}, 0)$ of f such that for all $(z, u) \in \mathcal{X}$ the function germs $F_u : (\mathbb{C}^{n+1}, z) \to (\mathbb{C}, F(z, u))$ given by $F_u(z') = F(z', u)$ fall into finitely many families of right equivalence classes depending on at most m (complex) parameters. Singularities of modality 0,1 and 2 are called *simple*, *unimodal* (or *unimodular*), and *bimodal* (or *bimodular*), respectively.

V. I. Arnold classified the singularities up to modality 2 [10]. He listed certain normal forms. A normal form determines a class of singularities. This class corresponds to a *μ=const stratum*: Any two singularities of a μ=const stratum are μ-equivalent, see Sect. 8.9 below. By Proposition 8.9.7 below, the class $\mathcal{D}$ is the same for all singularities of a μ=const stratum. Gabrielov [66] proved that the dimension of the μ=const stratum is equal to the modality of the singularity. Arnold found that in the lists of classes, all the classes are split into series which are now called the Arnold series. However, as Arnold writes in [10], "although the series undoubtedly exist, it is not at all clear what a series is". Let us look at Arnold's classification.

Table 8.1 Simple singularities

Notation	Function	Notation	Function
A_k	$x^{k+1},\ k \geq 1$	D_k	$x^2y + y^{k-1},\ k \geq 4$
E_6	$x^3 + y^4$	E_7	$x^3 + xy^3$
E_8	$x^3 + y^5$		

Let us first assume that $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ defines a simple singularity. Up to stabilization, the simple singularities are given by the germs of the functions of Table 8.1. There are many characterizations of simple singularities, see [49]. They are the only singularities where, for $n \equiv 2 \bmod 4$, the intersection form is negative definite [49, Characterization B5]. They are also the only singularities where the set $\mathcal{D}$ contains a tree [49, Characterization B7], [3, 5]. Moreover, this is also the only case where the monodromy group Γ is finite [49, Characterization B8]. The author [60] has recently shown that they are the only singularities where the set $\mathcal{B}$ is finite.

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ define a simple singularity and $n \equiv 2 \bmod 4$. Then Λ^* is a root system of type A_k, D_k, E_k. (Note that the usual bilinear form of [26] has to be multiplied by -1.) The Milnor lattice M is the corresponding root lattice, the group Γ is the corresponding Weyl group, and the classical monodromy operator h_* is a Coxeter element of the corresponding root system. Let $c \in \Gamma$ be a Coxeter element. Define

$$\Xi_c := \{(s_1, \ldots, s_k) \mid s_i \in \Gamma \text{ reflection},\ s_1 \cdots s_k = c\}.$$

Deligne [44] in a letter to Looijenga (with the help of J. Tits and D. Zagier) showed the following theorem.

Theorem 8.7.1 (Deligne) *The braid group* Br_k *acts transitively on* Ξ_c.

From this we obtain the following result.

Corollary 8.7.2 *One has*

$$\mathcal{B} = \{(\delta_1, \ldots, \delta_k) \in (\Lambda^*)^k \mid \langle \delta_1, \ldots, \delta_k \rangle_{\mathbb{Z}} = M, s_{\delta_1} \cdots s_{\delta_k} = h_*\}.$$

The sets Ξ_c, $\mathcal{B}$, and $\mathcal{D}$ are finite sets. The cardinality of these sets was calculated in the letter of Deligne (see also [95, 139]).

Example 8.7.3 For E_8 one has $|\mathcal{D}_{E_8}| = 2^8 3^4 5^6 = 324\,000\,000$.

The first published proof of Theorem 8.7.1 is due to D. Bessis and can be found in [21]. This theorem has been generalized and it has also applications outside of singularity theory, see [17]. K. Igusa and R. Schiffler generalized this result to arbitrary Coxeter groups of finite rank [82, Theorem 1.4] (see also [17, 18]). Recently, B. Baumeister, P. Wegener, and S. Yahiatene [19] generalized it to certain extended Weyl groups (see below). The theorem has applications in the theory of Artin groups, see [21, 47], and in the representation theory of algebras, see [78, 81, 82].

Let $\mathbb{R}^k$ be a vector space on which the Weyl group $W = \Gamma$ acts in a canonical way and let $\mathbb{C}^k = \mathbb{R}^k \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification. The action of W on $\mathbb{R}^k$ extends in a natural way to an action of W on $\mathbb{C}^k$. Let H be the union of the complexifications of the reflection hyperplanes of W. Let S be the base space and D the discriminant of the universal unfolding of the simple singularity f. Then the pair (S, D) is analytically isomorphic (in a neighborhood of the origin) to the pair $(\mathbb{C}^k/W, H/W)$ [8]. Brieskorn [29] proved that the fundamental group of the space $\mathbb{C}^k/W \setminus H/W$ is the generalized Brieskorn braid group π [30, 35] of the Weyl group W. He conjectured [30] and Deligne [43] proved that this space is in fact a $K(\pi, 1)$-space. (A $K(\pi, 1)$-space is a topological space with fundamental group π and trivial higher homotopy groups.) From this it follows that the complement $S \setminus D$ is a $K(\pi, 1)$-space as well (see also [13, Theorem 3.9]). Brieskorn asked [31, Problème 15] whether this is true in general.

Now we consider the Lyashko-Looijenga map in the case of the simple singularities. Looijenga [109] and Lyashko [114, 115] showed that the mapping $\mathrm{LL}|_{U\setminus \mathrm{Bif}} : U \setminus \mathrm{Bif} \to \mathcal{P}^k \setminus \Sigma$ is a covering of degree

$$d = \frac{k!N^k}{|\Gamma|}$$

where N is the Coxeter number of the corresponding root system. I. S. Livshits [106] determined the Galois group of this covering (see also [142]). Let $p \in \mathcal{P}^k \setminus \Sigma$. It is well known that

$$\pi_1(\mathcal{P}^k \setminus \Sigma, p) \cong \mathrm{Br}_k.$$

Therefore the complement of the bifurcation variety of a simple singularity is a $K(\pi, 1)$, where π is a subgroup of index d in the braid group Br_k.

Similar questions were also answered for complex reflection groups, see [22, 125, 126].

Looijenga already proved Theorem 8.7.1 in the case A_k [109, Corollary (3.8)]. Moreover, in this case, he established a correspondence between generic polynomial coverings of the complex sphere and trees with totally ordered edges. By considering a generalized version of the Lyashko-Looijenga mapping, more general combinatorial results were obtained by Arnold [11], D. Zvonkine and S. K. Lando [143], and B. S. Bychkov [36].

By studying the Lyashko-Looijenga mapping, Jianming Yu [141] determined the number of Seifert matrices with respect to distinguished bases of a simple singularity.

Gusein-Zade [74] gave a characterization of distinguished bases for simple singularities. Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ define a simple singularity of Milnor number μ. He showed that, if $(\delta_1, \dots, \delta_\mu)$ is an integral basis of the homology group M in which the matrix of the Seifert form is lower triangular, then $(\delta_1, \dots, \delta_\mu)$ is a distinguished basis of vanishing cycles. The proof is based on the following result:

Table 8.2 Simple elliptic singularities

Notation	Function	Restrictions
$\widetilde{E}_6$	$x^3 + y^3 + z^3 + axyz$	$a^3 + 27 \neq 0$
$\widetilde{E}_7$	$x^2 + y^4 + z^4 + ay^2z^2$	$a^2 \neq 4$
$\widetilde{E}_8$	$x^2 + y^3 + z^6 + ay^2z^2$	$4a^3 + 27 \neq 0$

Let n be even. For any vanishing cycle δ and any distinguished basis $(\delta_1, \ldots, \delta_\mu)$ for f, there exists a distinguished basis $(\delta'_1, \ldots, \delta'_\mu)$ with the first element $\delta'_1 = \pm\delta$. H. Serizawa [131] showed that the latter result is false for a non-simple singularity.

The next case are the *simple elliptic singularities* (see also [127]). These are the singularities $\widetilde{E}_6$, $\widetilde{E}_7$, and $\widetilde{E}_8$. Up to stabilization, they are given by the one-parameter families of Table 8.2. These singularities can be characterized as follows: For $n \equiv 2 \bmod 4$, the intersection form is not negative definite but negative semi-definite [49, Characterization C5]. Therefore, the simple elliptic singularities are also called the *parabolic singularities*. The monodromy group is not finite but has polynomial growth [49, Characterization C6]. (For the notions of polynomial and exponential growth, see e.g. [119]. See also Remark 8.8.7 below.) The set $\mathcal{B}$ is infinite but the set $\mathcal{D}$ is finite [60].

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ define a simple elliptic singularity of type $\widetilde{E}_k$, $k = 6, 7, 8$, and let $n \equiv 2 \bmod 4$. If M is a lattice, denote by $M^\# = \mathrm{Hom}(M, \mathbb{Z})$ the dual module and by

$$\begin{aligned} j : M &\longrightarrow M^\# \\ v &\longmapsto l_v \text{ with } l_v(x) = \langle v, x \rangle, x \in M, \end{aligned}$$

the canonical homomorphism. The Milnor lattice M is the orthogonal direct sum of the root lattice E_k and the radical $\ker j$ which is two-dimensional. The set Λ^* of vanishing cycles is an extended affine root system of type $E_k^{(1,1)}$ in the sense of Saito [129]. The monodromy group Γ is the semi-direct product of the group $\ker j \otimes j(M)$ and the Weyl group $W(E_k)$ of E_k, where the group $\ker j \otimes j(M)$ acts on M by $(v \otimes w^\#)(x) = x + w^\#(x)v$ and the action of $W(E_k)$ on $\ker j \otimes j(M)$ is trivial on the first factor and canonical on the second one, see [110, Proposition (6.7)]. It follows from this description that the monodromy group has polynomial growth.

P. Kluitmann extended Corollary 8.7.2 to the simple elliptic singularities [93]. He also calculated the cardinality of $\mathcal{D}$ for $\widetilde{E}_6$ and $\widetilde{E}_7$. In [90, 91], P. Jaworski considered the Lyashko-Looijenga map for the simple elliptic singularities and showed that the complement of the bifurcation variety of a simple elliptic singularity is again a $K(\pi, 1)$ for a certain subgroup of the braid group Br_μ [90, Corollary 2]. Recently, C. Hertling and C. Roucairol [76] used a different approach to study the Lyashko-Looijenga map for the simple and simple elliptic singularities and refined and extended the results of Kluitmann and Jaworski.

For the remaining singularities, the sets $\mathcal{B}$ and $\mathcal{D}$ are infinite [60]. Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be such a singularity. We assume $n \equiv 2 \bmod 4$. The only singularities with a hyperbolic intersection form, i.e., an indefinite form with only one positive eigenvalue, are the singularities of the series $T_{p,q,r}$ with $2 \leq p \leq q \leq r$ and $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} < 1$ [9]. Up to stable equivalence, they are given by the one parameter families

$$T_{p,q,r} : x^p + y^q + z^r + axyz, \ a \neq 0.$$

They are also called the *hyperbolic singularities*. The simple elliptic and hyperbolic singularities are unimodal singularities. Gabrielov [67] calculated Coxeter-Dynkin diagrams with respect to distinguished bases for the unimodal singularities. According to [67], the simple elliptic and hyperbolic singularities have Coxeter-Dynkin diagrams with respect to distinguished bases in the form of Fig. 8.9. Here $(p, q, r) = (3, 3, 3), (2, 4, 4), (2, 3, 6)$ for $\widetilde{E}_6$, $\widetilde{E}_7$, and $\widetilde{E}_8$ respectively. The Milnor lattice M of a hyperbolic singularity has a one-dimensional radical $\ker j$ generated by the vector $\delta_2 - \delta_1$. By [67], the monodromy group Γ is the semi-direct product of the group $\ker j \otimes j(M)$ and the Coxeter group corresponding to the graph of Fig. 8.9 with the vertex δ_2 removed. It can also be described as the extended Weyl group of a generalized root system as defined by Looijenga [111].

As an application of [19], one obtains an extension of Corollary 8.7.2 to the hyperbolic singularities.

Looijenga ([111], [112, Chapter III.3]) gave a description of the complement of the discriminant of a simple elliptic or hyperbolic singularity as an orbit space Y/Γ. Using this, H. van der Lek [105] gave a presentation of the fundamental group of

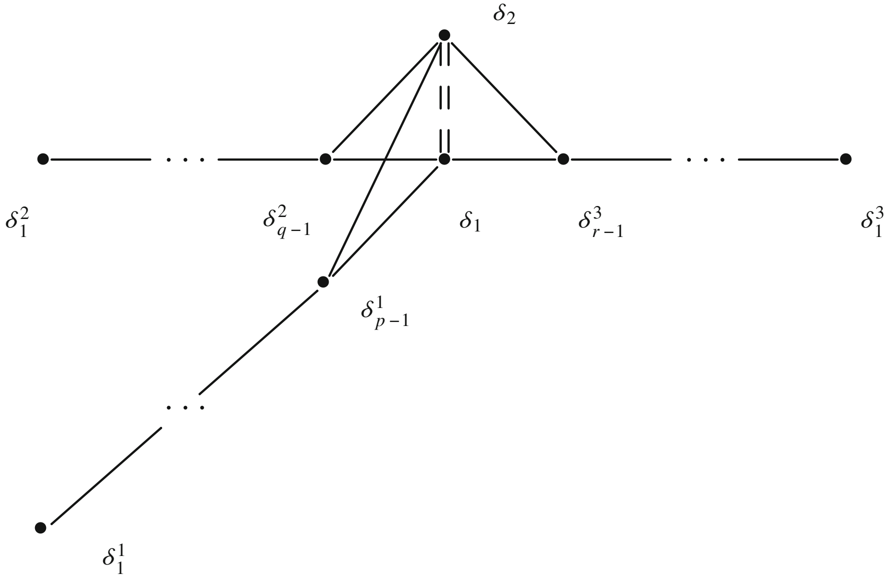


Fig. 8.9 The graph $T_{p,q,r}$

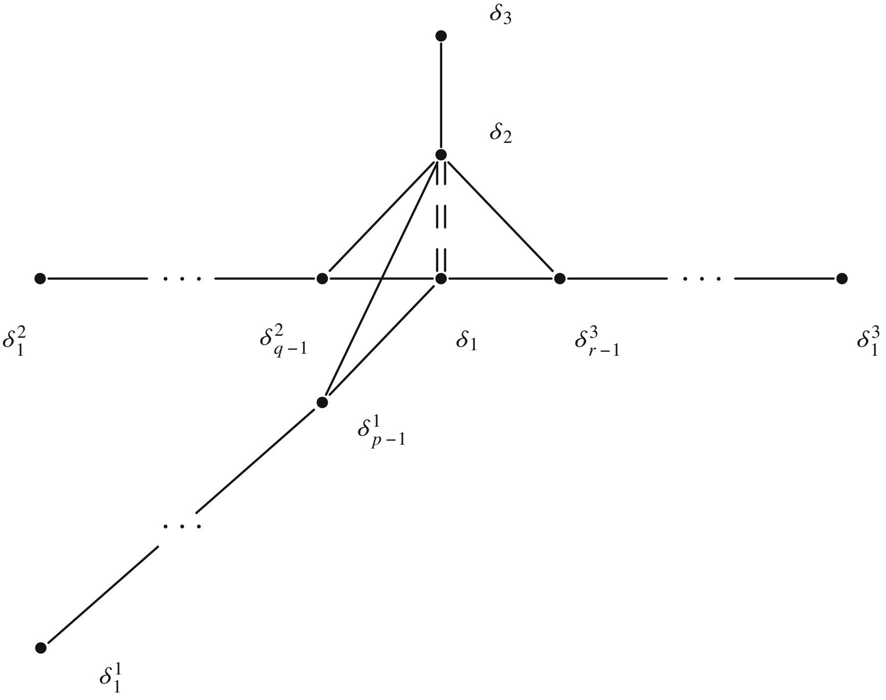


Fig. 8.10 The graph $S_{p,q,r}$

the discriminant complement for such singularities generalizing the results for the simple singularities.

Besides these singularities, there are 14 exceptional unimodal singularities. Equations of these singularities can be found in [10, 12]. Coxeter-Dynkin diagrams for these singularities were also calculated by Gabrielov [67]. He claimed that by change-of-basis transformations $\alpha_i(j)$ and $\beta_i(j)$, the Coxeter-Dynkin diagrams can be reduced to the "normal form" depicted in Fig. 8.10 for certain triples (p, q, r). The author showed in his PhD thesis [50] that these diagrams are in fact Coxeter-Dynkin diagrams with respect to distinguished bases. The necessary braid group transformations are indicated in the appendix of the thesis which is not published in [51]. However, they can be found in the paper [56]. Arnold observed a strange duality between the 14 exceptional unimodal singularities [10]. The relation to homological mirror symmetry is explained in the survey article [59]. For a description of the monodromy groups see Sect. 8.8.

V. I. Arnold also classified the bimodal singularities [10, 12]. Gabrielov computed Coxeter-Dynkin diagrams with respect to distinguished bases for the singularities of all the series of Arnold, including the bimodal singularities [68]. The author suggested a "normal form" for the Coxeter-Dynkin diagrams with respect to distinguished bases for the bimodal singularities [51, 53]. Jointly with

D. Ploog [62], he gave a geometric construction of these diagrams. Moreover, he suggested a "normal form" for the Coxeter-Dynkin diagrams with respect to weakly distinguished bases of all the singularities of Arnold's series and calculated their intersection forms [51].

Il'yuta [83] formulated two conjectures relating the shape of Coxeter-Dynkin diagrams to the modality of the singularity. He used the definition of the Coxeter-Dynkin diagram of [13, 2.8]: It is a graph with simple edges where the edge between δ_i and δ_j has the weight $\langle \delta_i, \delta_j \rangle$. (We assume $n \equiv 2 \bmod 4$.) A *monotone cycle* in a Coxeter-Dynkin diagram is a sequence of vertices $(\delta_{i_1}, \ldots, \delta_{i_k})$ where $i_1 < i_2 < \ldots < i_k$ and δ_{i_j} is connected to $\delta_{i_{j+1}}$ for $j = 1, \ldots, k$ and $j+1$ taken modulo k. The *weight* of a monotone cycle is the product $\prod \langle \delta_{i_j}, \delta_{i_{j+1}} \rangle$ where the product is over $j = 1, \ldots, k$ and $j+1$ taken modulo k. Now Il'yuta conjectured:

Conjecture 8.7.4 (Il'yuta) The minimum over all $D \in \mathcal{D}$ of the smallest number of edges that have to be deleted in order that D does not contain monotone cycles is equal to the modality of the singularity.

Conjecture 8.7.5 (Il'yuta) The minimum over all $D \in \mathcal{D}$ of the number of edges of D of negative weight is equal to the modality of the singularity.

Il'yuta showed that these conjectures hold for the unimodal singularities. The author [56] showed that both conjectures are even true for the unimodal singularities if one counts an edge of weight $\langle \delta_i, \delta_j \rangle$ as $|\langle \delta_i, \delta_j \rangle|$ edges as in the definition of the Coxeter-Dynkin diagram in Sect. 8.3. However, he gave counterexamples to both conjectures for the bimodal singularities. Il'yuta also found other characterizations of Coxeter-Dynkin diagrams of the simple singularities [84, 85, 87].

Using the Lyashko-Looijenga mapping, M. Lönne [107, 108] gave a presentation of the fundamental group of the discriminant complement for Brieskorn-Pham singularities which is related to the intersection matrix with respect to a distinguished basis considered in Sect. 8.5.

V. A. Vassiliev listed some problems about the Lyashko-Looijenga mapping for non-simple singularities in [137].

8.8 The Monodromy Group

In this section we give a description of the monodromy group in the general case.

Let M be a lattice which is either symmetric and even or skew symmetric. Let $\varepsilon \in \{+1, -1\}$ and let Λ be a subset of M. If M is symmetric we demand that $\langle \delta, \delta \rangle = 2\varepsilon$ for all $\delta \in \Lambda$. We define an automorphism $s_\delta \in \mathrm{Aut}(M)$ by

$$s_\delta(v) := v - \varepsilon \langle v, \delta \rangle \delta$$

for all $v \in M$. Then s_δ is a reflection in the symmetric case and a symplectic transvection in the skew symmetric case. Let $\Gamma_\Lambda \subset \mathrm{Aut}(M)$ be the subgroup of $\mathrm{Aut}(M)$ generated by the transformations $s_\delta,\ \delta \in \Lambda$.

Definition 8.8.1 The pair (M, Λ) is called a *vanishing lattice*, if it satisfies the following conditions:

(i) Λ generates M.
(ii) Λ is an orbit of Γ_Λ in M.
(iii) If $\operatorname{rank} M > 1$, then there exist $\delta, \delta' \in \Lambda$ such that $\langle \delta, \delta' \rangle = 1$.

It follows from Corollary 8.6.5 that, if it is not true that both n is odd and 0 is a non-degenerate critical point of f, then the pair $(\widetilde{H}_n(X_s), \Lambda^*)$ is a vanishing lattice with $\varepsilon = (-1)^{n(n-1)/2}$ and Γ is the corresponding monodromy group.

We introduce some more algebraic notions. Let $M^\# = \mathrm{Hom}(M, \mathbb{Z})$ be the dual module and $j : M \to M^\#$ be the canonical homomorphism. A homomorphism $h : M \to M$ induces a homomorphism $h^t : M^\# \to M^\#$ of the dual modules. If h leaves the bilinear form $\langle\ ,\ \rangle$ invariant, then $h^t(j(M)) \subset j(M)$. An automorphism $h \in \mathrm{Aut}(M)$ thus induces a homomorphism $h^t : M^\#/j(M) \to M^\#/j(M)$. Let $\mathrm{Aut}^\#(M) \subset \mathrm{Aut}(M)$ be the subgroup of those automorphisms $h \in \mathrm{Aut}(M)$ with $h^t = \mathrm{id}_{M^\#/j(M)}$.

Now let M be a skew symmetric lattice. It has a basis

$$(e_1, f_1, \ldots, e_m, f_m, g_1, \ldots, g_k)$$

such that

$$\langle e_i, f_i \rangle = -\langle f_i, e_i \rangle = d_i \text{ for } d_i \in \mathbb{Z},\ d_i \geq 1,\ i = 1, \ldots m,$$

all other inner products are equal to zero, and d_{i+1} is divisible by d_i for $i = 1, \ldots, m-1$. Such a basis is called a *symplectic* basis.

Let $(e_1, f_1, \ldots, e_m, f_m, g_1, \ldots, g_k)$ be a symplectic basis of M. Let η_2 be the exponent of 2 in the prime factor decomposition of d_m. Let $\mu = 2m + k$. We identify M with $\mathbb{Z}^\mu$ through the symplectic basis $(e_1, f_1, \ldots, e_m, f_m, g_1, \ldots, g_k)$. A subgroup $G \subset \mathrm{Aut}^\#(M)$ then corresponds to a subgroup $\rho(G) \subset \mathrm{Sp}^\#(\mu, \mathbb{Z})$, where $\mathrm{Sp}^\#(\mu, \mathbb{Z})$ is the corresponding subgroup of the symplectic group

$$\mathrm{Sp}(\mu, \mathbb{Z}) = \{A \in \mathrm{GL}(\mu, \mathbb{Z}) \mid A^t J A = J\}.$$

Let $r \in \mathbb{N} \setminus \{0\}$. A subgroup $G \subset \mathrm{Aut}^\#(M)$ is called a *congruence subgroup modulo* r if

$$\rho(G) = \{A \in \mathrm{Sp}^\#(\mu, \mathbb{Z}) \mid A \equiv E \bmod r\}.$$

Here E is the unit matrix and $A \equiv E \bmod r$ means that $a_{ij} \equiv \delta_{ij} \bmod r$ for all $1 \leq i, j \leq \mu$, where $A = (a_{ij})$.

A congruence subgroup is obviously of finite index in the group $\mathrm{Aut}^{\#}(M) = \mathrm{Sp}^{\#}(M)$.

The following theorem was proved by W. A. M. Janssen [88] based on previous work of A'Campo [6], B. Wajnryb [140], and S. V. Chmutov [38, 39]. The notation $\langle v, M\rangle = \mathbb{Z}$ means that there is a $y \in M$ with $\langle v, y\rangle = 1$. We write $a \in \Lambda \bmod 2$ if there is an element $b \in M$ with $a - 2b \in \Lambda$.

Theorem 8.8.2 (Janssen) *Let* (M, Λ) *be a skew symmetric vanishing lattice. Then*

(i) Γ_Λ *contains the congruence subgroup modulo* 2^{η_2+1} *of* $\mathrm{Sp}^{\#}(M)$,
(ii) $\Lambda = \{v \in M \mid \langle v, M\rangle = \mathbb{Z}$ *and* $v \in \Lambda \bmod 2\}$.

As a corollary, we get the following result:

Corollary 8.8.3 *Let* $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ *be a holomorphic function germ with an isolated singularity at* 0 *and let n be odd. Then*

(i) Γ *contains the congruence subgroup modulo* 2^{η_2+1} *of* $\mathrm{Sp}^{\#}(M)$,
(ii) $\Lambda^* = \{v \in M \mid \langle v, M\rangle = \mathbb{Z}$ *and* $v \in \Lambda^* \bmod 2\}$.

In (ii) *it is assumed that* 0 *is not a non-degenerate critical point of* f.

Janssen also proved a version of Theorem 8.8.2 for skew symmetric vanishing lattices over the field $\mathbb{F}_2$ and classified skew symmetric vanishing lattices over $\mathbb{F}_2$ [88] and over $\mathbb{Z}$ [89]. B. Shapiro, M. Shapiro, and A. Vainshtein [132] applied these results to certain enumeration problems.

Now let M be symmetric and let $\varepsilon \in \{-1, +1\}$. We put

$$\overline{M} := M/\ker j, \quad \overline{M}_{\mathbb{R}} := \overline{M} \otimes \mathbb{R}.$$

Then $\overline{M}_{\mathbb{R}}$ is a finite-dimensional real vector space with a nondegenerate symmetric bilinear form. Let $h \in \mathrm{Aut}(M)$ and $\bar{h}$ the induced element in $O(\overline{M}_{\mathbb{R}})$. The transformation $\bar{h}$ can be written as a product of reflections

$$\bar{h} = s_{v_1} \circ \ldots \circ s_{v_r}$$

with $v_i \in \overline{M}_{\mathbb{R}},\ \langle v_i, v_i\rangle \neq 0,\ i = 1, \ldots, r$. We define

$$\nu_\varepsilon(h) := \begin{cases} +1 & \text{if } \varepsilon\langle v_i, v_i\rangle < 0 \text{ for an even number of indices,} \\ -1 & \text{otherwise.} \end{cases}$$

The homomorphism $\nu_\varepsilon : \mathrm{Aut}(M) \to \{-1, +1\}$ is called the *real ε-spinor norm.*

Definition 8.8.4 We define a subgroup $O^*_\varepsilon(M) \subset O(M)$ as follows:

$$O^*_\varepsilon(M) := \mathrm{Aut}^{\#}(M) \cap \ker \nu_\varepsilon.$$

If M is non-degenerate, then $M^{\#}/j(M)$ is a finite group and hence $O^{\#}(M) = \mathrm{Aut}^{\#}(M)$ is a subgroup of finite index in $O(M) = \mathrm{Aut}(M)$. The subgroup $\ker \nu_\varepsilon \subset O(M)$ is of index ≤ 2 in $O(M)$. Thus if M is non-degenerate, then $O^*_\varepsilon(M)$ is a subgroup of finite index in $O(M)$.

Using a result of M. Kneser [96], the author proved the following theorem [54]. A unimodular hyperbolic plane is a two-dimensional lattice with the bilinear form given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Theorem 8.8.5 (Ebeling) *Let* (M, Λ) *be an even symmetric vanishing lattice. Assume that* M *contains a six-dimensional sublattice* $K \subset M$ *which is the orthogonal direct sum of two unimodular hyperbolic planes and a lattice of type* εA_2*. Assume moreover* $\{v \in K \mid \langle v, v\rangle = 2\varepsilon\} \subset \Lambda$*. Then*

(i) $\Gamma_\Lambda = O^*_\varepsilon(M)$,
(ii) $\Lambda = \{v \in M \mid \langle v, v\rangle = 2\varepsilon$ *and* $\langle v, M\rangle = \mathbb{Z}\}$.

From Theorem 8.8.5 one can derive the following theorem [54]. The statement for the exceptional unimodal singularities was already proven by H. Pinkham [123] (see also [52] for the history of the problem and previous results). It was noticed by Looijenga that (ii) is a consequence of (i).

Theorem 8.8.6 (Ebeling) *Let* $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ *be a holomorphic function germ with an isolated singularity at* 0 *and let* n *be even,* $\varepsilon = (-1)^{n(n-1)/2}$*. Suppose that* f *is not of type* $T_{p,q,r}$ *with* $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ *and* $(p, q, r) \neq (2, 3, 7), (2, 4, 5), (3, 3, 4)$*. Then*

(i) $\Gamma = O^*_\varepsilon(M)$,
(ii) $\Lambda^* = \{v \in M \mid \langle v, v\rangle = 2\varepsilon$ *and* $\langle v, M\rangle = \mathbb{Z}\}$.

In (ii) *it is assumed that* 0 *is not a non-degenerate critical point of* f.

Remark 8.8.7 Theorem 8.8.6 follows for the simple and simple elliptic singularities by the results stated in Sect. 8.7. It is false for the singularities of type $T_{p,q,r}$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ and $(p, q, r) \neq (2, 3, 7), (2, 4, 5), (3, 3, 4)$, see [51, §3]. This follows from the fact that the graph of Fig. 8.9 with the vertex δ_2 removed and with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ defines a Coxeter system of hyperbolic type if and only if $(p, q, r) = (2, 3, 7), (2, 4, 5), (3, 3, 4)$ [26, Ch. V, § 4, Exercice 12]. The three singularities $T_{p,q,r}$ with these values of (p, q, r) are the *minimal hyperbolic singularities* . Theorem 8.8.6 was proved for these singularities by Brieskorn ([32, Theorem 2], but no proof is given). A proof following Brieskorn's proof can be found in [55, 5.5]. A'Campo (unpublished) and Looijenga showed that the monodromy groups of these singularities have exponential growth. Looijenga's proof is published in [49, Appendix II].

8.9 Topological Equivalence

We shall now consider the question to which extent the invariants determine the topological type of the singularity.

The topological type of a singularity $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is described by the (local) embedding of the variety $f^{-1}(0)$ in a neighborhood of the singular point $0 \in \mathbb{C}^{n+1}$.

Definition 8.9.1 Two singularities $f, g : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ are *topologically equivalent* if there is a homeomorphism of neighborhoods U and V of the origin which maps $f^{-1}(0) \cap U$ to $g^{-1}(0) \cap V$.

By [118, Theorem 2.10], the variety $f^{-1}(0)$ is locally the cone over its link K. By [118], the link is a fibered knot. A. Durfee [48] proved the following theorem.

Theorem 8.9.2 (Durfee) *Let $n \geq 3$. There is a one-to-one correspondence of isotopy classes of fibered knots in S^{2n+1} and equivalence classes of integral unimodular bilinear forms given by associating to each fibered knot its Seifert form.*

In view of the preceding remarks and Theorem 8.3.6 we obtain the following corollary.

Corollary 8.9.3 *For $n \neq 2$ the set $\mathcal{D}$ of $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ determines f up to topological equivalence.*

This corollary was also proved by S. Szczepanski [134]. Moreover, she showed in [135] the following theorem.

Theorem 8.9.4 (Szczepanski) *Two singularities $f, g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ are topologically equivalent if*

(i) *the singularities have a common Coxeter-Dynkin diagram with respect to distinguished bases, and*
(ii) *the Milnor fibers have homeomorphic boundaries and the algebraic isomorphism of the Milnor lattices induced by the common Coxeter-Dynkin diagram is realized geometrically by either an inclusion of one Milnor fiber into the other or a homotopy equivalence of the Milnor fibers which induces a homeomorphism of the boundaries.*

There is also the notion of μ-homotopy or μ-equivalence (see [34]).

Definition 8.9.5 Two singularities $f_0, f_1 : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ are *μ-equivalent* if there is a family $f_t : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ of analytic function germs with isolated singularities at the origin continuously depending on the parameter $t \in [0, 1]$ with constant Milnor number $\mu(f_t)$.

Lê Dũng Tráng and C. P. Ramanujam [103] proved the following theorem.

Theorem 8.9.6 (Lê-Ramanujam) *If $n \neq 2$, then μ-equivalent singularities are topologically equivalent.*

The following proposition was proved by Gabrielov [67, Proposition 1].

Proposition 8.9.7 (Gabrielov) *For two μ-equivalent singularities there exist distinguished bases whose Coxeter-Dynkin diagrams coincide.*

Using this proposition, one obtains the Lê-Ramanujam theorem as a consequence of Corollary 8.9.3. Moreover, one can derive from Theorem 8.9.4 a Lê-Ramanujam theorem for $n = 2$, see [135].

Now let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be an isolated singularity satisfying the conditions of Theorem 8.8.6. It follows from Theorem 8.8.6 that the invariants Γ and Λ^* are completely determined by M. The author [51] has found examples of pairs of singularities (e.g. the bimodal singularities Z_{17} and Q_{17} in Arnold's notation) which have the same Coxeter-Dynkin diagrams with respect to weakly distinguished bases and the invariants M, Γ, and Λ^* are the same, but the invariants $\mathcal{B}^*$ and $\mathcal{D}^*$ are different, the classical monodromy operators are not conjugate to each other, and the singularities are not topologically equivalent.

We conclude the article with some open questions which were posed to the author by late Brieskorn (around 1982?). We keep the condition that $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is a singularity satisfying the conditions of Theorem 8.8.6. Let $n \equiv 2 \bmod 4$ and let μ be the Milnor number of f.

Question 1 (Brieskorn) Let M be the Milnor lattice (of rank μ) and Γ be the monodromy group of f. Let

$$\Lambda := \{v \in M \mid \langle v, v\rangle = -2\}.$$

Then Γ acts on Λ. Are there only finitely many orbits?

Question 2 (Brieskorn) Let

$$\mathcal{B}_0 := \{(e_1, \dots, e_\mu) \in \Lambda^\mu \mid \langle e_1, \dots, e_\mu\rangle_{\mathbb{Z}} = M\}.$$

Then the group $\mathrm{Br}_\mu^{\rtimes}$ acts on $\mathcal{B}_0$. Are there only finitely many orbits? Alternatively, one can consider the set

$$\widetilde{\mathcal{B}}_0 := \{(e_1, \dots, e_\mu) \in (\Lambda^*)^\mu \mid \langle e_1, \dots, e_\mu\rangle_{\mathbb{Z}} = M\}.$$

Question 3 (Brieskorn) The group Γ acts on $\mathcal{B}_0$ (or $\widetilde{\mathcal{B}}_0$) by

$$\gamma(e_1, \ldots, e_\mu) = (\gamma e_1, \ldots, \gamma e_\mu).$$

This action commutes with the action of $\mathrm{Br}_\mu^{\rtimes}$. Are there only finitely many Γ-equivalence classes of $\mathrm{Br}_\mu^{\rtimes}$-orbits?

Very little is known about these questions. The answers to these questions are trivially yes for the simple singularities, since the sets Λ, $\mathcal{B}_0$, and $\widetilde{\mathcal{B}}_0$ are finite in this case. We have $\Lambda = \Lambda^*$ (and hence there is only one Γ-orbit) for the simple, simple elliptic, and minimal hyperbolic singularities (for the latter ones see [55, Proposition 5.5.1]).

An element $c \in \Gamma$ for which there exists a basis $(e_1, \ldots, e_\mu) \in \mathcal{B}_0$ such that $c = s_{e_1} \cdots s_{e_\mu}$ is called a *quasi Coxeter element*.

Question 4 (Brieskorn) Let $c \in \Gamma$ be a quasi Coxeter element and let

$$\mathcal{B}_{0,c} := \{(e_1, \ldots, e_\mu) \in \mathcal{B}_0 \,|\, s_{e_1} \cdots s_{e_\mu} = c\}.$$

The set $\mathcal{B}_{0,c}$ is invariant under the action of the group $\mathrm{Br}_\mu^{\rtimes}$. What is the relation between the orbits of $\mathrm{Br}_\mu^{\rtimes}$ on $\mathcal{B}_0$ and the sets $\mathcal{B}_{0,c}$?

For the simple singularities, the quasi Coxeter elements were determined up to conjugacy by E. Voigt [138, 139] and he showed that the group $\mathrm{Br}_\mu^{\rtimes}$ acts transitively on $\mathcal{B}_{0,c}$ for each quasi Coxeter element c. For c being the classical monodromy operator, it is known for the simple (Corollary 8.7.2), the simple elliptic [93], and the hyperbolic singularities [19] that the group $\mathrm{Br}_\mu^{\rtimes}$ acts transitively on $\mathcal{B}_{0,c}$ (see Sect. 8.7 above). To the author's knowledge, this is all what is known about Question 4.

Acknowledgments This work is partially supported by DFG. The author is grateful to C. Hertling for drawing his attention to an error in his book [58] (see the proof of Theorem 8.3.6) and for useful discussions. I would like to thank the referee for useful comments.

References

1. A'Campo, N.: Le nombre de Lefschetz d'une monodromie. Indag. Math. **35**, 113–118 (1973) 471
2. A'Campo, N.: Le groupe de monodromie du déploiement des singularités isolées de courbes planes. I. Math. Ann. **213**, 1–32 (1975) 468

3. A'Campo, N.: Le groupe de monodromie du déploiement des singularités isolées de courbes planes. II. In: Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1, pp. 395–404. Canad. Math. Congress, Montreal, Que. (1975) 473
4. A'Campo, N.: La fonction zêta d'une monodromie. Comment. Math. Helv. **50**, 233–248 (1975) 471
5. A'Campo, N.: Sur les valeurs propres de la transformation de Coxeter. Invent. Math. **33**, no. 1, 61–67 (1976) 473
6. A'Campo, N.: Tresses, monodromie et le groupe symplectique. Comment. Math. Helv. **54**, no. 2, 318–327 (1979) 480
7. Aramaki, D., Takahashi, A.: Maximally-graded matrix factorizations for an invertible polynomial of chain type. Preprint arXiv:1903.02732 (2019) 468
8. Arnold, V. I.: Normal forms of functions near degenerate critical points, the Weyl groups A_k, D_k, E_k and Lagrangian singularities. Funktsional. Anal. i Prilozen. **6**, no. 4, 3–25 (1972) (Engl. translation in Funct. Anal. Appl. **6**, no. 4, 254–272 (1972)) 474
9. Arnold, V. I.: Remarks on the method of stationary phase and on the Coxeter numbers. Usp. Mat. Nauk **28**:5, 17–44 (1973) (Engl. translation in Russ. Math. Surv. **28**:5, 19–48 (1973)) 476
10. Arnold, V. I.: Critical points of smooth functions and their normal forms. Usp. Mat. Nauk. **30**:5, 3–65 (1975) (Engl. translation in Russ. Math. Surv. **30**:5, 1–75 (1975)) 472 and 477
11. Arnold, V. I.: Topological classification of complex trigonometric polynomials and the combinatorics of graphs with an identical number of vertices and edges. Funktsional. Anal. i Prilozhen. **30**, no. 1, 1–17 (1996) (Engl. translation in Funct. Anal. Appl. **30**, no. 1, 1–14 (1996)) 474
12. Arnold, V. I., Gusein-Zade, S. M., Varchenko, A. N.: Singularities of Differentiable Maps, Volume I. Birkhäuser, Boston–Basel–Berlin (1985) 477
13. Arnold, V. I., Gusein-Zade, S. M., Varchenko, A. N.: Singularities of Differentiable Maps, Volume II. Birkhäuser, Boston–Basel–Berlin (1988) 450, 453, 454, 456, 457, 460, 461, 466, 467, 470, 471, 474, and 478
14. Artin, E.: Theorie der Zöpfe. Abh. Math. Sem. Univ. Hamburg **4**, 101–126 (1925) 463
15. Artin, E.: Theory of braids. Ann. of Math. (2) **48**, 101–126 (1947) 463
16. Bättig, D., Knörrer, H.: Singularitäten. Birkhäuser-Verlag, Basel (1991) 450
17. Baumeister, B., Dyer, M., Stump, Ch., Wegener, P.: A note on the transitive Hurwitz action on decompositions of parabolic Coxeter elements. Proc. Amer. Math. Soc. Ser. B **1**, 149–154 (2014) 473
18. Baumeister, B., Gobet, Th., Roberts, K., Wegener, P.: On the Hurwitz action in finite Coxeter groups. J. Group Theory **20**, no. 1, 103–131 (2017) 473
19. Baumeister, B., Wegener, P., Yahiatene, S.: Extended Weyl groups and Hurwitz transitivity. Preprint (2019) 473, 476, and 484
20. Berglund, P., Hübsch, T.: A generalized construction of mirror manifolds. Nuclear Phys. B **393**, 377–391 (1993) 468
21. Bessis, D.: The dual braid monoid. Ann. Sci. École Norm. Sup. (4) **36**, no. 5, 647–683 (2003) 473
22. Bessis, D.: Finite complex reflection arrangements are $K(\pi, 1)$. Ann. of Math. (2) **181**, no. 3, 809–904 (2015) 474
23. Bey, C. H.: Sur l'irréductibilité de la monodromie locale. C. R. Acad. Sci. Paris Sér. A–B **275**, A21–A24 (1972) 471
24. Bey, C. H.: Sur l'irréductibilité de la monodromie locale; application à l'équisingularité. C. R. Acad. Sci. Paris Sér. A–B **275**, A105–A107 (1972) 471
25. Birman, J. S.: Braids, links, and mapping class groups. Annals of Mathematics Studies, No. 82. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo (1974) 463
26. Bourbaki, N: Lie groups and Lie algebras. Chapters 4–6. Translated from the 1968 French original by Andrew Pressley. Elements of Mathematics (Berlin). Springer-Verlag, Berlin (2002) 459, 473, and 481

27. Brieskorn, E.: Beispiele zur Differentialtopologie von Singularitäten. Invent. Math. **2**, 1–14 (1966) 466
28. Brieskorn, E.: Die Monodromie der isolierten Singularitäten von Hyperflächen. Manuscripta math. **2**, 103–160 (1970) 450 and 456
29. Brieskorn, E.: Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe. Invent. Math. **12**, 57–61 (1971) 474
30. Brieskorn, E.: Sur les groupes de tresses [d'après V. I. Arnol'd]. In: Séminaire Bourbaki, 24ème année (1971/1972), Exp. No. 401, pp. 21–44. Lecture Notes in Math., Vol. 317, Springer, Berlin (1973) 474
31. Brieskorn, E.: Vue d'ensemble sur les problèmes de monodromie. In: Singularités à Cargèse (Rencontre sur les Singularités en Géométrie Analytique, Inst. Études Sci. de Cargèse, 1972), pp. 393–413. Astérisque Nos. 7 et 8, Soc. Math. France, Paris (1973) 474
32. Brieskorn, E.: The unfolding of exceptional singularities. In: Leopoldina Symposium: Singularities (Thüringen, 1978). Nova Acta Leopoldina (N.F.) **52**, no. 240, 65–93 (1981) 481
33. Brieskorn, E.: Milnor lattices and Dynkin diagrams. In: Singularities, Part 1 (Arcata, Calif., 1981), pp. 153–165, Proc. Sympos. Pure Math., 40, Amer. Math. Soc., Providence, RI (1983) 463 and 464
34. Brieskorn, E.: Automorphic sets and braids and singularities. In: Braids (Santa Cruz, CA, 1986), pp. 45–115, Contemp. Math., 78, Amer. Math. Soc., Providence, RI (1988) 450, 464, and 482
35. Brieskorn, E., Saito, K.: Artin-Gruppen und Coxeter-Gruppen. Invent. Math. **17**, 245–271 (1972) 474
36. Bychkov, B. S.: On decompositions of a cyclic permutation into a product of a given number of permutations. Funktsional. Anal. i Prilozhen. **49**, no. 2, 1–6 (2015) (Engl. translation in Funct. Anal. Appl. **49**, no. 2, 81–85 (2015)) 474
37. Cecotti, S., Vafa, C.: On classification of $N = 2$ supersymmetric theories. Comm. Math. Phys. **158**, no. 3, 569–644 (1993) 464
38. Chmutov, S. V.: Monodromy groups of critical point of functions. Invent. Math. **67**, no. 1, 123–131 (1982) 480
39. Chmutov, S. V.: The monodromy groups of critical points of functions. II. Invent. Math. **73**, no. 3, 491–510 (1983) 480
40. Coleman, A. J.: Killing and the Coxeter transformation of Kac-Moody algebras. Invent. Math. **95**, no. 3, 447–477 (1989) 459
41. Crawley-Boevey, W.: Exceptional sequences of representations of quivers. In: Representations of algebras (Ottawa, ON, 1992), pp. 117–124, CMS Conf. Proc., 14, Amer. Math. Soc., Providence, RI (1993) 464
42. Dehornoy, P.: Braids and self-distributivity. Progress in Mathematics, 192. Birkhäuser Verlag, Basel (2000) 464
43. Deligne, P.: Les immeubles des groupes de tresses généralisés. Invent. Math. **17**, 273–302 (1972) 474
44. Deligne, P.: Letter to Looijenga on March 9, 1974 (French). Available at https://homepage.rub.de/christian.stump/Deligne_Looijenga_Letter_09-03-1974.pdf. Cited 5 Sep 2019 473
45. Deligne, P., Katz, N.: Groupes de monodromie en géométrie algébrique. II. In: Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II). Lecture Notes in Mathematics, Vol. 340. Springer-Verlag, Berlin–New York (1973) 450
46. Demazure, M.: Classification des germes à point critique isolé et à nombres de modules 0 ou 1 (d'après V. I. Arnold). In: Séminaire Bourbaki, Vol. 1973/1974, 26ème année, Exp. No. 443, pp. 124–142. Lecture Notes in Math., Vol. 431, Springer, Berlin (1975) 466
47. Digne, F.: Présentations duales des groupes de tresses de type affine $\widetilde{A}$. Comment. Math. Helv. **81**, no. 1, 23–47 (2006) 473
48. Durfee, A. H.: Fibered knots and algebraic singularities. Topology **13**, 47–59 (1974) 482
49. Durfee, A. H.: Fifteen characterizations of rational double points and simple critical points. Enseign. Math. (2) **25**, no. 1–2, 131–163 (1979) 473, 475, and 481

50. Ebeling, W.: Quadratische Formen und Monodromiegruppen von Singularitäten. Dissertation, Rheinische Friedrich-Wilhelms-Universität, Bonn (1980) 477
51. Ebeling, W.: Quadratische Formen und Monodromiegruppen von Singularitäten. Math. Ann. **255**, 463–498 (1981) 477, 478, 481, and 483
52. Ebeling, W.: On the monodromy groups of singularities. In: Singularities, Part 1 (Arcata, Calif., 1981), pp. 327–336, Proc. Sympos. Pure Math., 40, Amer. Math. Soc., Providence, RI (1983) 481
53. Ebeling, W.: Milnor lattices and geometric bases of some special singularities. In: Nœuds, tresses et singularités (Ed. C.Weber), Monographie Enseign. Math. 31, pp. 129–146, Genève (1983) and Enseign. Math. (2) **29**, 263–280 (1983) 477
54. Ebeling, W.: An arithmetic characterisation of the symmetric monodromy groups of singularities. Invent. Math. **77**, no. 1, 85–99 (1984) 481
55. Ebeling, W.: The Monodromy Groups of Isolated Singularities of Complete Intersections. Lect. Notes in Math., Vol. 1293, Springer-Verlag, Berlin etc. (1987) 450, 481, and 484
56. Ebeling, W.: On Coxeter-Dynkin diagrams of hypersurface singularities. J. Math. Sciences **82**, 3657–3664 (1996) 471, 477, and 478
57. Ebeling, W.: Monodromy. In: Singularities and computer algebra, pp. 129–155, London Math. Soc. Lecture Note Ser., 324, Cambridge Univ. Press, Cambridge (2006) 450 and 471
58. Ebeling, W.: Functions of several complex variables and their singularities. Graduate Studies in Math. Vol. 83, American Mathematical Society, Providence RI (2007) 450, 452, 453, 454, 456, 457, 460, 461, 463, 469, 470, 471, and 484
59. Ebeling, W.: Homological mirror symmetry for singularities. In: Representation theory – current trends and perspectives, pp. 75–107, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich (2017) 468 and 477
60. Ebeling, W.: A note on distinguished bases of singularities. Topology Appl. **234**, 259–268 (2018) 464, 473, and 475
61. Ebeling, W., Gusein-Zade, S. M.: Lectures on monodromy. In: Singularities in geometry and topology, pp. 234–252, World Sci. Publ., Hackensack, NJ (2007) 472
62. Ebeling, W., Ploog, D.: A geometric construction of Coxeter-Dynkin diagrams of bimodal singularities. Manuscripta Math. **140**, no. 1–2, 195–212 (2013) 478
63. Feng, B., Hanany, A., He, Y., Iqbal, A.: Quiver theories, soliton spectra and Picard-Lefschetz transformations. J. High Energy Phys. 2003, no. 2, 056, 33 pp (2003) 464
64. Franco, S., Hanany, A.: Toric duality, Seiberg duality and Picard-Lefschetz transformations. In: Proceedings of the 35th International Symposium Ahrenshoop on the Theory of Elementary Particles (Berlin-Schmöckwitz, 2002). Fortschr. Phys. **51**, no. 7–8, 738–744 (2003) 464
65. Gabrielov, A. M.: Intersection matrices for certain singularities. Funktsional. Anal. i Prilozen. **7**, no. 3, 18–32 (1973) (Engl. translation in Funct. Anal. Appl. **7**, no. 3, 182–193 (1974)) 450, 466, and 467
66. Gabrielov, A. M.: Bifurcations, Dynkin diagrams and the modality of isolated singularities. Funktsional. Anal. i Prilozen. **8**, no. 2, 7–12 (1974) (Engl. translation in Funct. Anal. Appl. **8**, no. 2, 94–98 (1974)) 470 and 472
67. Gabrielov, A. M.: Dynkin diagrams of unimodal singularities. Funktsional. Anal. i Prilozen. **8**, no. 3, 1–6 (1974) (Engl. translation in Funct. Anal. Appl. **8**, no. 3, 192–196 (1974)) 476, 477, and 483
68. Gabrielov, A. M.: Polar curves and intersection matrices of singularities. Invent. math. **54**, 15–22 (1979) 450, 468, and 477
69. Givental, A. B.: Twisted Picard-Lefschetz formulas. Funktsional. Anal. i Prilozhen. **22**, no. 1, 12–22 (1988) (Engl. translation in Funct. Anal. Appl. **22**, no. 1, 10–18 (1988)) 461
70. Greuel, G.-M.: Deformation and smoothing of singularities. This handbook. 452 and 469
71. Gusein-Zade, S. M.: Intersection matrices for certain singularities of functions of two variables. Funktsional. Anal. i Prilozen. **8**, no. 1, 1–15 (1974) (Engl. translation in Funct. Anal. Appl. **8**, no. 1, 10–13 (1974)) 468
72. Gusein-Zade, S. M.: Dynkin diagrams of the singularities of functions of two variables. Funktsional. Anal. i Prilozen. **8**, no. 4, 23–30 (1974) (Engl. translation in Funct. Anal. Appl. **8**, no. 4, 295–300 (1975)) 468

73. Gusein-Zade, S. M.: Monodromy groups of isolated singularities of hypersurfaces. Uspehi Mat. Nauk **32**, no. 2, 23–65 (1977) (Engl. translation in Russian Math. Surveys **32**, no. 2, 23–69 (1977)) 450, 463, and 465
74. Gusein-Zade, S. M.: Distinguished bases of simple singularities. Funktsional. Anal. i Prilozhen. **14**, no. 4, 73–74 (1980) (Engl. translation in Funct. Anal. Appl. **14**, no. 4, 307–308 (1980)) 474
75. Hefez, A., Lazzeri, F.: The intersection matrix of Brieskorn singularities. Invent. Math. **25**, 143–157 (1974) 466
76. Hertling, C., Roucairol, C.: Distinguished bases and Stokes regions for the simple and the simple elliptic singularities. Preprint arXiv: 1806.00996 (2018) 475
77. Hirzebruch, F., Mayer, K. H.: O(n)-Mannigfaltigkeiten, exotische Sphären und Singularitäten. Lecture Notes in Mathematics, No. 57, Springer-Verlag, Berlin-New York (1968) 466
78. Hubery, A., Krause, H.: A categorification of non-crossing partitions. J. Eur. Math. Soc. (JEMS) **18**, no. 10, 2273–2313 (2016) 473
79. Humphries, St.: On weakly distinguished bases and free generating sets of free groups. Quart. J. Math. Oxford Ser. (2) **36**, no. 142, 215–219 (1985) 465 and 466
80. Hurwitz, A.: Ueber Riemann'sche Flächen mit gegebenen Verzweigungspunkten. Math. Ann. **39**, no. 1, 1–60 (1891) 464
81. Igusa, K.: Exceptional sequences, braid groups and clusters. In: Groups, algebras and applications, pp. 227–240, Contemp. Math., **537**, Amer. Math. Soc., Providence, RI (2011) 473
82. Igusa, K., Schiffler, R.: Exceptional sequences and clusters. J. Algebra **323**, no. 8, 2183–2202 (2010) 473
83. Il'yuta, G. G.: On the Coxeter transformation of an isolated singularity. Usp. Mat. Nauk **42** (1987), no. 2, 227–228 (Engl. translation in Russ. Math. Surveys **42**, no. 2, 279–280 (1987)) 478
84. Il'yuta, G. G.: A'Campo's theorem on the discriminant. Funktsional. Anal. i Prilozen. **28**, no. 2, 12–20 (1994) (Engl. translation in Funct. Anal. Appl. **28**, no. 2, 85–91 (1994)) 478
85. Il'yuta, G. G.: Characterization of simple Coxeter-Dynkin diagrams. Funktsional. Anal. i Prilozen. **29**, no. 3, 72–75 (1995) (Engl. translation in Funct. Anal. Appl. **29**, no. 3, 205–207 (1995)) 478
86. Il'yuta, G. G.: On q-monodromy groups of singularities. Izv. Ross. Akad. Nauk Ser. Mat. **60**, no. 1, 115–132 (1996) (Engl. translation in Izv. Math. **60**, no. 1, 119–136 (1996)) 461
87. Il'yuta, G. G.: Coxeter-Dynkin diagrams of partially ordered sets and A'Campo-Gusein-Zade diagrams of simple singularities. Funktsional. Anal. i Prilozen. **31**, no. 1, 12–19 (1997) (Engl. translation in Funct. Anal. Appl. **31**, no. 1, 10–15 (1997)) 478
88. Janssen, W. A. M.: Skew-symmetric vanishing lattices and their monodromy groups. Math. Ann. **266**, no. 1, 115–133 (1983) 480
89. Janssen, W. A. M.: Skew-symmetric vanishing lattices and their monodromy groups. II. Math. Ann. **272**, no. 1, 17–22 (1985) 480
90. Jaworski, P.: Distribution of critical values of miniversal deformations of parabolic singularities. Invent. Math. **86**, no. 1, 19–33 (1986) 475
91. Jaworski, P.: Decompositions of parabolic singularities. Bull. Sci. Math. (2) **112**, no. 2, 143–176 (1988) 475
92. Keller, B., Yang, D.: Derived equivalences from mutations of quivers with potential. Adv. Math **226**, 2118–2168 (2011) 464
93. Kluitmann, P.: Ausgezeichnete Basen erweiterter affiner Wurzelgitter. Dissertation, Rheinische Friedrich-Wilhelms-Universität, Bonn, 1986. Bonner Mathematische Schriften, 185. Universität Bonn, Mathematisches Institut, Bonn (1987) 475 and 484
94. Kluitmann, P.: Hurwitz action and finite quotients of braid groups. In: Braids (Santa Cruz, CA, 1986), pp. 299–325, Contemp. Math., 78, Amer. Math. Soc., Providence, RI (1988) 464
95. Kluitmann, P.: Addendum zu der Arbeit: "Ausgezeichnete Basen von Milnorgittern einfacher Singularitäten" von E. Voigt. Abh. Math. Sem. Univ. Hamburg **59**, 123–124 (1989) 473

96. Kneser, M.: Erzeugung ganzzahliger orthogonaler Gruppen durch Spiegelungen. Math. Ann. **255**, no. 4, 453–462 (1981) 481
97. Lamotke, K.: Die Homologie isolierter Singularitäten. Math. Z. **143**, 27–44 (1975) 450 and 454
98. Lamotke, K.: The topology of complex projective varieties after S. Lefschetz. Topology **20**, no. 1, 15–51 (1981) 454
99. Lazzeri, F.: A theorem on the monodromy of isolated singularities. In: Singularités à Cargèse (Rencontre Singularités Géom. Anal., Inst. Études Sci., Cargèse, 1972), pp. 269–275. Astérisque, Nos. 7 et 8, Soc. Math. France, Paris (1973) 470
100. Lazzeri, F.: Some remarks on the Picard-Lefschetz monodromy. In: Quelques journées singulières, i+9 pp. Centre Math. École Polytechnique, Paris (1974) 460 and 470
101. Lê Dũng Tráng: Topologie des singularités des hypersurfaces complexes. In: Singularités à Cargèse (Rencontre Singularités Géom. Anal., Inst. Études Sci., Cargèse, 1972), pp. 171–182. Astérisque, Nos. 7 et 8, Soc. Math. France, Paris (1973) 450
102. Lê Dũng Tráng: The geometry of the monodromy theorem. In: C. P. Ramanujam – a tribute, pp. 157–173, Tata Inst. Fund. Res. Studies in Math., 8, Springer, Berlin-New York (1978) 450 and 472
103. Lê Dũng Tráng, Ramanujam, C. P.: The invariance of Milnor's number implies the invariance of the topological type. Amer. J. Math. **98**, no. 1, 67–78 (1976) 482
104. Lefschetz, S.: L'analysis situs et la géométrie algébrique. Gauthier-Villars, Paris (1924) 449 and 454
105. van der Lek, H.: Extended Artin groups. In: Singularities, Part 2 (Arcata, Calif., 1981), pp. 117–121, Proc. Sympos. Pure Math., 40, Amer. Math. Soc., Providence, RI (1983) 476
106. Livshits, I. S.: Automorphisms of the complement to the bifurcation set of functions for simple singularities. Funktsional. Anal. i Prilozhen. **15**, no. 1, 38–42 (1981) (Engl. translation in Funct. Anal. Appl. **15**, no. 1, 29–32 (1981)) 474
107. Lönne, M.: Fundamental group of discriminant complements of Brieskorn-Pham polynomials. C. R. Math. Acad. Sci. Paris **345**, no. 2, 93–96 (2007) 478
108. Lönne, M.: Braid monodromy of some Brieskorn-Pham singularities. Internat. J. Math. **21**, no. 8, 1047–1070 (2010) 478
109. Looijenga, E.: The complement of the bifurcation variety of a simple singularity. Invent. Math. **23**, 105–116 (1974) 472 and 474
110. Looijenga, E.: On the semi-universal deformation of a simple-elliptic hypersurface singularity. II. The discriminant. Topology **17**, no. 1, 23–40 (1978) 475
111. Looijenga, E.: Invariant theory for generalized root systems. Invent. Math. **61**, no. 1, 1–32 (1980) 476
112. Looijenga, E.: Rational surfaces with an anticanonical cycle. Ann. of Math. (2) **114**, no. 2, 267–322 (1981) 476
113. Looijenga, E.: Isolated singular points on complete intersections. London Mathematical Society Lecture Note Series, 77. Cambridge University Press, Cambridge (1984) 450 and 454
114. Lyashko, O. V.: The geometry of bifurcation diagrams. Uspekhi Mat. Nauk **34**, no. 3, 205–206 (1979) (Engl. translation in Russian Math. Surveys **34**, no. 3, 209–210 (1979)) 472 and 474
115. Lyashko, O. V.: The geometry of bifurcation diagrams. J. Soviet Math. **27**, 2736–2759 (1984) 472 and 474
116. Lyndon, R. C., Schupp, P. E.: Combinatorial group theory. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89. Springer-Verlag, Berlin-New York (1977) 466
117. McCool, J.: On basis-conjugating automorphisms of free groups. Canad. J. Math. **38**, no. 6, 1525–1529 (1986) 466
118. Milnor, J.: Singular Points of Complex Hypersurfaces. Ann. of Math. Studies Vol. 61, Princeton University Press, Princeton (1968) 449, 451, 460, and 482
119. Milnor, J.: A note on curvature and fundamental group. J. Differential Geometry **2**, 1–7 (1968) 475
120. Orlik, P., Randell, R.: The monodromy of weighted homogeneous singularities. Invent. Math. **39**, 199–211 (1977) 468

121. Pham, F.: Formules de Picard-Lefschetz généralisées et ramification des intégrales. Bull. Soc. Math. France **93**, 333–367 (1965) 450 and 466
122. Picard, E., Simart, S.: Traité des fonctions algébriques de deux variables. Vol. I. Gauthier-Villars, Paris (1897) 449 and 454
123. Pinkham, H.: Groupe de monodromie des singularités unimodulaires exceptionnelles. C. R. Acad. Sci. Paris Sér. A–B **284**, no. 23, A1515–A1518 (1977) 481
124. Ringel, C. M.: The braid group action on the set of exceptional sequences of a hereditary Artin algebra. In: Abelian group theory and related topics (Oberwolfach, 1993), pp. 339–352, Contemp. Math., 171, Amer. Math. Soc., Providence, RI (1994) 464
125. Ripoll, V.: Orbites d'Hurwitz des factorisations primitives d'un élément de Coxeter. J. Algebra **323**, no. 5, 1432–1453 (2010) 474
126. Ripoll, V.: Lyashko-Looijenga morphisms and submaximal factorizations of a Coxeter element. J. Algebraic Combin. **36**, no. 4, 649–673 (2012) 474
127. Saito, K.: Einfach-elliptische Singularitäten. Invent. Math. **23**, 289–325 (1974) 475
128. Saito, K.: A characterization of the intersection form of a Milnor's fiber for a function with an isolated critical point. Proc. Japan Acad. Ser. A Math. Sci. **58**, no. 2, 79–81 (1982) 471
129. Saito, K.: Extended affine root systems. I. Coxeter transformations. Publ. Res. Inst. Math. Sci. **21**, no. 1, 75–179 (1985) 475
130. Sebastiani, M., Thom, R.: Un résultat sur la monodromie. Invent. Math. **13**, 90–96 (1971) 466
131. Serizawa, H.: Distinguished bases of non-simple singularities. Tokyo J. Math. **24**, no. 1, 19–38 (2001) 475
132. Shapiro, B., Shapiro, M., Vainshtein, A.: Skew-symmetric vanishing lattices and intersections of Schubert cells. Internat. Math. Res. Notices **1998**, no. 11, 563–588 (1998) 480
133. Siersma, D.: Classification and deformation of singularities. Doctoral dissertation, University of Amsterdam. University of Amsterdam, Amsterdam, ii+115 pp. (1 foldout) (1974) 470
134. Szczepanski, S.: Geometric bases and topological equivalence. Comm. Pure Appl. Math. **40**, no. 3, 389–399 (1987) 482
135. Szczepanski, S.: Criteria for topological equivalence and a Lê-Ramanujam theorem for three complex variables. Duke Math. J. **58**, no. 2, 513–530 (1989) 482 and 483
136. Tyurina, G. N.: The topological properties of isolated singularities of complex spaces of codimension one. Izv. Akad. Nauk SSSR Ser. Mat. **32**, no. 3, 605–620 (1968) (Engl. translation in Math. USSR-Izv. **2**:3, 557–571 (1968)) 470
137. Vassiliev, V. A.: A few problems on monodromy and discriminants. Arnold Math. J. **1**, no. 2, 201–209 (2015) 478
138. Voigt, E.: Ausgezeichnete Basen von Milnorgittern einfacher Singularitäten. Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 1984. Bonner Mathematische Schriften, 160. Universität Bonn, Mathematisches Institut, Bonn (1985) 484
139. Voigt, E.: Ausgezeichnete Basen von Milnorgittern einfacher Singularitäten. Abh. Math. Sem. Univ. Hamburg **55**, 183–190 (1985) 473 and 484
140. Wajnryb, B.: On the monodromy group of plane curve singularities. Math. Ann. **246**, no. 2, 141–154 (1979/80) 480
141. Yu, Jianming: Combinatorial structure of Stokes regions of a simple singularity. Math. Ann. **305**, no. 2, 355–368 (1996) 474
142. Yu, Jianming: Galois group of Looijenga-Lyashko mapping. Math. Z. **232**, no. 2, 321–330 (1999) 474
143. Zvonkine, D., Lando, S. K.: On multiplicities of the Lyashko-Looijenga mapping on strata of the discriminant. Funktsional. Anal. i Prilozhen. **33**, no. 3, 21–34 (1999) (Engl. translation in Funct. Anal. Appl. **33**, no. 3, 178–188 (1999)) 474

Chapter 9
The Lefschetz Theorem for Hyperplane Sections

Helmut A. Hamm and Dũng Tráng Lê

Contents

H. A. Hamm (✉)
Mathematisches Institut, Universität Münster, Münster, Germany
e-mail: hamm@uni-muenster.de

D. T. Lê
Centre de Mathématiques et Informatique, Université d’Aix-Marseille, Marseille, France
e-mail: ledt@ictp.it

J. L. Cisneros Molina et al. (eds.), *Handbook of Geometry and Topology of Singularities I*, https://doi.org/10.1007/978-3-030-53061-7_9

Abstract In these notes we consider different theorems of the Lefschetz type. We start with the classical Lefschetz Theorem for hyperplane sections on a non-singular projective variety. We show that this extends to the cases of a non-singular quasi-projective variety and to singular varieties. We also consider local forms of theorems of the Lefschetz type.

9.1 Introduction

One of the basic problems of algebraic geometry is to extract topological information from the equations which define an algebraic variety. The theorem of Lefschetz for hyperplane sections shows that to some extent one can compare the topology of a given projective variety and the one of a hyperplane section of this variety, when the base field is the field of complex numbers and the projective variety is non-singular.

The theorem of Lefschetz for hyperplane sections appears in the memoir of Lefschetz [33] which received the Bordin Prize of the French Academy of Sciences in 1919 and the Bôcher memorial prize by the American Mathematical Society in 1924. This memoir was written in French and translated into English with some minor modifications and published in the *Trans. Amer. Math. Soc.* **22** (1921). It appears also in his famous book, “L’Analysis Situs et la Géométrie Algébrique”, published in the collection of E. Borel, “Monographies sur la Théorie des Fonctions”, in 1924 (facsimile in 1950, *cf.* [34]). In his memoir, Lefschetz mentioned that his theorem was known for cycles of dimension one by Castelnuovo, Enriques [5] and E. Picard. The case of dimension 2, which Lefschetz actually generalizes to higher dimension, was already treated by H. Poincaré in a paper published in *Journal Math. Pures et Appliquées* **6** (1906).

The proofs by S. Lefschetz of his theorem on hyperplane sections in [33] and [34] are very sketchy and not easy to understand. A first complete proof was proposed by A. Wallace in his book [50]. Then, following an idea of R. Thom to use the theory of Morse, A. Andreotti and T. Frankel gave a very elegant proof in [1] and R. Bott had a homotopy version in [3]. A very clean account of these proofs using Morse theory was given by J. Milnor in [37].

Several generalizations of the Lefschetz theorem were given. Algebraic generalizations were mainly obtained by A. Grothendieck in [13] and Michèle Raynaud (see [13] Chapitre XIV). Their interest lies in the use of geometric methods over a field of positive characteristic.

Here in these notes we are mainly interested in the topology of complex varieties and the generalizations obtained by the authors, and also by M. Goresky and R. MacPherson on the topology of complex algebraic varieties.

In particular one finds out that the classical theorem of Bertini and a theorem stated by O. Zariski on the fundamental group of the complement of a projective hypersurface are in fact particular cases of topological generalizations of the Lefschetz theorem.

The aim of these notes is to give a quick survey of the known topological generalizations of the Lefschetz theorem and to extend the original theorem of Lefschetz to the case of singular quasi-projective varieties. We shall use transcendental methods and, though some results might be true on a field of any characteristic (or at least on an algebraically closed field of characteristic zero), we shall assume in all these notes that the base field is the field $\mathbb{C}$ of complex numbers.

Beware that we will not speak about the so-called Hard Lefschetz theorem.

Often, we shall not give any proof but refer to the original paper where the reader can have a proof of the given theorems.

In the first chapter we shall comment on the original theorem of Lefschetz and indicate a topological proof. Then we prove the theorem of O. Zariski mentioned above and we show how it can be understood as a generalization of the Lefschetz theorem. It leads us naturally to several generalizations of the Lefschetz theorem which are related to the local study of singularities.

In the third chapter we define the notion of homotopical depth introduced by Grothendieck. We show that indications and conjectures of A. Grothendieck in [13], Exposé XIII, are mostly true and lead to the generalization of the theorem of Lefschetz for hyperplane sections that we have in mind.

We shall see that different local topological data about the singularities can be used to calculate the homotopical depth.These data are related to a Morse type theory for stratified spaces due to M. Goresky and R. MacPherson [12]. We end these notes with a characterization of spaces with maximal homotopical depth and we show that on these spaces a natural generalization of the Theorem of Milnor on the Milnor fibre of an isolated critical point of a complex analytic function is true. These lectures present and follow the works of the authors [19, 20, 22, 31].

We assume that the reader is accustomed to the language of singularity theory. We can refer to the article contained in this book by Lê D.T., J.J. Nuño Ballesteros and J. Seade, *The topology of Milnor fibration*.

9.2 The Classical Theorem of Lefschetz for Hyperplane Sections

9.2.1 Original Statement

Let us state the theorem of Lefschetz for hyperplane sections. First let us introduce the notations.

Let $\mathbb{P}^N$ be the complex projective space of dimension N. Let V be a non-singular complex projective subvariety of $\mathbb{P}^N$. Recall that a family (L_t) of complex projective hyperplanes is a pencil if it is the (1-dimensional) family of all hyperplanes containing a given projective subspace A of codimension 2, called the axis of the pencil. Thus the space $\mathcal{P}$ of pencils of hyperplanes of $\mathbb{P}^N$ can be identified with the Grassmann space of projective spaces of codimension 2 in $\mathbb{P}^N$.

Since we can index the hyperplanes of the pencil (L_t) by their intersections with a projective line $\mathbb{P}^1$ of $\mathbb{P}^N$ which does not meet the axis A, a pencil of hyperplanes also defines a projective line in the space $\check{\mathbb{P}}^N$ of projective hyperplanes of $\mathbb{P}^N$.

Definition 9.2.1 A pencil of hyperplanes is said to be a Lefschetz pencil for V if:

1. the axis of the pencil intersects V transversally;
2. all the hyperplanes of the pencil, but a finite number, intersect the non-singular variety V transversally;
3. the hyperplanes which do not intersect V transversally are tangent to V only at one unique point and the tangency at this point is simple (or equivalently ordinary quadratic).

One can prove that general pencils are Lefschetz's pencils of V (see [25, 2.3 of p. 216 and 2.5.2 of Theorem 2.5 p. 217]):

Lemma 9.2.2 *There is a non-empty Zariski open subset of the space $\mathcal{P}$ of pencils of hyperplanes of $\mathbb{P}^N$ which are Lefschetz pencils for the variety V.*

The idea of the proof is quite simple. In the space $\check{\mathbb{P}}^N$ of projective hyperplanes of $\mathbb{P}^N$, let us consider the subspace of projective hyperplanes tangent to the variety V. One can show that this subspace is a projective subvariety $\check{V}$ of $\check{\mathbb{P}}^N$, which is called the dual variety of V.

In fact in $\mathbb{P}^N \times \check{\mathbb{P}}^N$, we consider the set $\mathcal{V}$ of pairs (x, L) of a non-singular point x of V and a projective hyperplane L tangent to V at x, *i.e.* which contains the projective tangent space $T_x(V)$. Of course, when V is a hypersurface of $\mathbb{P}^N$ there is only one tangent hyperplane at x, but the situation can be different when V is not a hypersurface. Remember that we only consider a non-singular V.

This set $\mathcal{V}$ is an algebraic subset $\mathcal{V}$ of $\mathbb{P}^N \times \check{\mathbb{P}}^N$. Actually the space $\mathcal{V}$ is the critical subspace of the map:

$$p\colon I(V) \to \check{\mathbb{P}}^N$$

defined on the incidence variety $I(V)$ of V, *i.e.* the subvariety of $\mathbb{P}^N \times \check{\mathbb{P}}^N$ of the pairs (x, L) of points x of V and hyperplanes L of $\check{\mathbb{P}}^N$ such that $x \in L$. The projection onto $\mathbb{P}^N$ maps $\mathcal{V}$ onto V and the projection onto $\check{\mathbb{P}}^N$ maps $\mathcal{V}$ onto $\check{V}$ which is algebraic, because the projection is proper. It is clear that, since V is a non-singular variety, the incidence variety $I(V)$ is non-singular, so that $\dim \mathcal{V} = N - 1$ or the space $\mathcal{V}$ is empty. Therefore $\dim \check{V} \leq N - 1$. It is known (*cf.* [25, Corollaire 3.2.4 p. 219]) that the set of hyperplanes in $\check{V}$ which do not intersect V with one and only one ordinary quadratic singularity has codimension at least 2 in $\check{\mathbb{P}}^N$.

In fact, this is obvious if $\dim \check{V} \leq N - 2$ and, if $\dim \check{V} = N - 1$, it means that the dual variety $\check{V}$ is reduced in the following sense that a non-singular point of this variety represents a projective hyperplane of $\mathbb{P}^N$ which is tangent at only one point of V and at this point the tangency is ordinary. Thus a general projective line in $\check{\mathbb{P}}^N$, *i.e.* a pencil of hyperplanes in $\mathbb{P}^N$, is intersecting this dual variety $\check{V}$ at $\check{m}$ non-singular points, where $\check{m}$ is the degree of the dual variety $\check{V}$ of V (which is 0 if $\dim \check{V} \leq N - 2$). Therefore such a pencil is a Lefschetz pencil for V if its axis intersects V transversally.

This shows that the space of Lefschetz's pencils for the variety V, *i.e.* the intersection of the space of projective lines in $\check{\mathbb{P}}^N$ which define pencils whose axis intersects V transversally and of the space of projective lines in $\check{\mathbb{P}}^N$ which intersect the dual variety $\check{V}$ transversally at $\check{m}$ non-singular points, is a non-empty Zariski open subset of the space $\mathcal{P}$ of pencils of hyperplanes of $\mathbb{P}^N$.

Let us suppose that (L_t) is a Lefschetz pencil for V and that the dimension of the variety V is d. We give the Lefschetz Theorem for hyperplane sections not quite in the same form as it is found in [34] (Chapitre IV, p. 89). The original terminology is somehow heavy. We propose the following statement which closely follows the original version:

Theorem 9.2.3 *Suppose that V is a non-singular complex projective variety. Let (L_t) be a Lefschetz pencil of hyperplanes of V. Then we have, whenever L_t intersects V transversally:*

1. *Any k-dimensional cycle Γ_k of $V \cap L_t$ $(k < d - 1)$ is invariant;*
2. *Any k-dimensional cycle Γ_k of V $(k < d)$ is homologous to a cycle in $V \cap L_t$;*
3. *For $k \leq d - 2$, a k-dimensional cycle Γ_k of V and a cycle of $V \cap L_t$ to which it is homologous are simultaneously boundaries in their respective varieties;*
4. *For $k \leq d - 2$ the k-th Betti number of V and $V \cap L_t$ are equal.*

Moreover in [34] (Chapter 1, §1, 2 (b)) S. Lefschetz quotes another assertion about the d-dimensional cycles. Namely:

Proposition 9.2.4 *Any d-dimensional cycle of V is the sum of two others of which one is wholly within an L_t and the other is composed of a d-dimensional manifold contained in L_0 plus the loci of certain $(d - 1)$ dimensional cycles of $L_t \cap V$ when t describes the lines from* 0 *to the critical values t_i.*

9.2.2 Recent Statements

One can notice that Lefschetz' formulation of his theorem speaks about cycles. Nowadays we formulate his theorem in terms of singular homology with integer coefficients (see [50]). If we do so, the assertion (2) of the Lefschetz theorem says that the inclusion of $V \cap L_t$ into V induces a surjection of the k-th homology groups, for $k \leq d-1$, and assertion (3) says that the same inclusion induces an injection of the k-th homology groups, for $k \leq d-2$. This automatically gives the assertion (4). By using the exact sequence of the homology of the pair $(V, V \cap L_t)$, we can replace assertions (2), (3), (4) by the vanishing of the k-th relative homology of $(V, V \cap L_t)$ for $k < d$:

$$(*) \quad \text{if } 0 \leq k < d : \qquad H_k(V, V \cap L_t) = 0$$

It remains to understand the assertion (1) in the Lefschetz theorem. We shall see that assertion (1) means that, for any $k \leq d-2$, the inclusion of $V \cap A$ in V, where A is the axis of the pencil, induces an isomorphism of the k-th homology. If so, assertion (1) is a consequence of the vanishing of $H_k(V, V \cap L_t)$ and $H_k(V \cap L_t, V \cap A)$ for any $k \leq d-2$, which is consequence of the assertion $(*)$ itself and the assertion $(*)$ applied to $V \cap L_t$, when $V \cap L_t$ is non-singular.

As in the Lefschetz theorem we suppose that (L_t) is a Lefschetz pencil. In particular this implies that the axis A of the pencil intersects the non-singular variety V transversally. We shall show that there is a fibration on the complement of a finite number of points in the projective line associated to this situation.

Namely, let z be a point of the ambient projective space $\mathbb{P}^N$ outside the axis A. Evidently there is a unique hyperplane L_t of the pencil containing both the point z and the axis A. Therefore this defines a mapping θ of $\mathbb{P}^N - A$ to $\mathbb{P}^1$ which associates the index t of L_t to the point z. The map θ is algebraic. The graph of θ is a subset of $\mathbb{P}^N \times \mathbb{P}^1$ which is not closed. The closure Z of this graph can be shown to be an algebraic subvariety of $\mathbb{P}^N \times \mathbb{P}^1$ (use the Remmert-Stein Theorem [44] and the Theorem of Chow [41, Corollary (4.6)], or the fact that the graph is not only analytic but even algebraic). The projection onto $\mathbb{P}^N$ induces an algebraic map e_A from Z onto $\mathbb{P}^N$. This map e_A is the blowing-up of the subspace A in $\mathbb{P}^N$:

$$e_A \colon Z \to \mathbb{P}^N$$

The definition of the map e_A shows that it induces an isomorphism of $Z - e_A^{-1}(A)$, which is the graph of θ, onto $\mathbb{P}^N - A$.

The inverse image $e_A^{-1}(A)$ is called the exceptional divisor of the blowing-up e_A. On the other hand one can check that the restriction to Z of the projection onto $\mathbb{P}^1$ defines a map $\tilde{\theta} \colon Z \to \mathbb{P}^1$ which is a locally trivial fibration of Z on $\mathbb{P}^1$ with a fibre isomorphic to $\mathbb{P}^{N-1}$. The projection of $e_A^{-1}(A)$ onto $\mathbb{P}^1$ is a trivial fibration.

Now one can consider the inverse image $e_A^{-1}(V - A)$. As we notice above, e_A induces an algebraic isomorphism of $e_A^{-1}(V - A)$ onto $V - A$. Using *e.g.* the theorem of Remmert-Stein, one can prove that the topological closure $\tilde{V}$ of $e_A^{-1}(V - A)$ in Z is an algebraic subvariety of Z.

Because the axis A intersects the variety V transversally (and V is non-singular), the variety $\tilde{V}$ is non-singular. The mapping e_A induces a mapping $\pi : \tilde{V} \to V$ which is called the blowing-up of V along $V \cap A$ (or the blowing-up of $V \cap A$ in V).

On the other hand the map $\tilde{\theta} : Z \to \mathbb{P}^1$ induces a map

$$\tilde{\theta}_V : \tilde{V} \to \mathbb{P}^1$$

this map is a locally trivial fibration on neighbourhoods of points of $\mathbb{P}^1$ which correspond to hyperplanes of the considered Lefschetz pencil and which intersect the variety V transversally , *i.e.* $\tilde{\theta}_V$ is a locally trivial fibration over $\mathbb{P}^1 - \{t_1, \dots, t_k\}$. Since we have assumed that the pencil with axis A is a Lefschetz pencil, fibres over the points t_i are singular at exactly one point. This fibration induced by $\tilde{\theta}_V$ over $\mathbb{P}^1 - \{t_1, \dots, t_k\}$ is smooth and locally trivial, but it is not a trivial fibration. Considering a loop around one of the points t_i, $1 \leq i \leq k$, and a point t on this loop, one can define a diffeomorphism of $V \cap L_t$ onto itself, which is not the identity, and we shall call it a geometric monodromy of the local trivial fibration induced by $\tilde{\theta}_V$ around the point t_i.

The restriction of the map $\tilde{\theta}_V$ to $e_A^{-1}(A) \cap \tilde{V}$ is a trivial fibration, so that the monodromy around any t_i, $1 \leq i \leq k$, of the locally trivial fibration of $\tilde{\theta}_V$ onto $\mathbb{P}^1 - \{t_1, \dots, t_k\}$ is the identity when restricted to $e_A^{-1}(A) \cap \tilde{V}$. Therefore cycles of V in $V \cap A$ or homologous to cycles in $V \cap A$ are called invariant, because lifted into $\tilde{V}$ they are invariant by these monodromies. The assertion (1) of Lefschetz theorem is actually a consequence of the fact that cycles of V of dimension $k \leq d - 2$ are homologous to cycles contained in $V \cap A$.

The theorem of Lefschetz can now be summarized by:

Theorem 9.2.5 *Let V be a complex non-singular projective variety of dimension d in $\mathbb{P}^N$ and let (L_t) be a Lefschetz pencil of hyperplanes of V. Then we have a vanishing of relative homologies:*

$$H_k(V, V \cap L_t) = 0 \qquad \textit{for any } k \leq d - 1$$

where L_t is a general hyperplane of the Lefschetz pencil (L_t) which intersects V transversally.

Actually A. Andreotti and T. Frankel [1] proved a stronger theorem (see [37] Theorem 7.1):

Theorem 9.2.6 *Let V be a projective non-singular variety in $\mathbb{P}^N$ and let L be any hyperplane of $\mathbb{P}^N$. Then:*

$$H_k(V, V \cap L) = 0 \qquad \textit{for any } k \leq d - 1.$$

Note that the hyperplane L in this theorem might not belong to a Lefschetz pencil.

Following an idea of R. Thom the proof of this theorem uses Morse theory.

By the Lefschetz duality one sees that the integral homology $H_k(V, V \cap L)$ is isomorphic to the integral cohomology $H^{2d-k}(V - L)$. Now $V - L$ is an affine variety and we have:

Lemma 9.2.7 *A complex non-singular affine subvariety of complex dimension d of $\mathbb{C}^N$ has the homology type of a CW-complex of (real) dimension d.*

Then, by the universal coefficient theorem for cohomology (see [23, Theorem 3.2 p. 195]), we have:

$$H^{2d-k}(V - L) = 0 \text{ for } k \leq d - 1,$$

which implies Theorem 9.2.5.

In fact the preceding lemma is consequence of a more general statement about Stein manifolds. Any complex affine non-singular variety is a Stein manifold because any closed complex analytic submanifold of some $\mathbb{C}^N$ is a Stein manifold. We have:

Lemma 9.2.8 *A Stein submanifold V of complex dimension d of $\mathbb{C}^N$ has the homotopy type of a CW-complex of (real) dimension d.*

Proof We can assume that V is closed in $\mathbb{C}^N$. The proof of this lemma uses the following observation on the distance function:

Assume that the distance function δ_x in $\mathbb{C}^N$ to a point x restricted to V has only non-degenerate critical points. Then, the indices of these critical points are $\leq d$.

In [37, Chapter 1] J. Milnor gives a way to compute the indices of these critical points.

Consider a smooth manifold M embedded in $\mathbb{R}^n$. Let a point $x \in \mathbb{R}^n$. The function δ_x is the restriction to M of the distance function to x. There is a relation between the critical points of δ_x and the geometry of M embedded in $\mathbb{R}^n$.

Let us define N as the subspace of $M \times \mathbb{R}^n$ which is the set of pairs (y, v), where y is a point of M and v is a vector of $\mathbb{R}^n$ orthogonal to M at y. Notice that $\dim N = n$. We have a map $E : N \to \mathbb{R}^n$ defined by $E(y, v) = y + v$.

Definition 9.2.9 A point c of $\mathbb{R}^n$ is called a focal point of (M, y) with multiplicity ν, if it is a critical value of E and if $c = y + v$ and the nullity of the Jacobian of E at (y, v) is ν. A point c is called a focal point of M, if it is the focal point of (M, y) for some $y \in M$.

Since by the Baire Theorem (see *e.g.* [38]) critical values form a set of measure 0 of the target space, we have:

Proposition 9.2.10 *For all points x of $\mathbb{R}^n$ but a subset of measure* 0, *x is not a focal point of M.*

Then, we have (see [37, Lemma 6.5 Chapter 1]):

Lemma 9.2.11 *A point y of M is a degenerate critical point of δ_x if and only if x is a focal point of (M, y).*

This shows that for almost all $x \in \mathbb{R}^n$ the function $\delta_x : M \to \mathbb{R}$ is a Morse function.

There is a result which allows to compute the index of the non-degenerate points of δ_x (see [37, Lemma 6.9 of Chapter 1]):

Lemma 9.2.12 *The index of δ_x at a non-degenerate point $y \in M$ is the sum of the multiplicities of the focal points on the segment from x to y.*

It remains to show that the indices of δ_x when δ_x is a non-degenerate function are $\leq d = \dim_{\mathbb{C}} V$ when V is a complex submanifold of $\mathbb{C}^N$. Before considering the complex situation, we give another characterization of focal points of (M, y) given by Differential Geometry.

Let k be the dimension of M. Consider a unit vector v orthogonal to M at y. Let $u_1, \dots, u_k$ be local coordinates of M at the point y. The embedding of M into $\mathbb{R}^n$ is given locally at y by the vector function:

$$x(u_1, \dots, u_k) = (x_1(u_1, \dots, u_k), \dots, x_n(u_1, \dots, u_k))$$

with smooth entries. The vector $\partial^2 x/\partial u_i \partial u_j$ is the sum of a tangent vector to M at y and a normal vector $n_{i,j}$ to M at y.

The $k \times k$ matrix of scalar products $(v.n_{i,j})$ is symmetric, since we have assumed that M is a smooth manifold. This matrix is called the second fundamental form of M at y in the direction of v.

One defines the principal curvatures $K_1, \dots, K_k$ of M at y in the direction v as the eigenvalues of this matrix (see [37, §6 of Chapter 1]). Then, we have (see [37, Lemma 6.3 of Chapter 1]):

Proposition 9.2.13 *The focal points of (M, y) on the line $\ell = \{y + \lambda v,\ \lambda \in \mathbb{R}\}$ are the points $y + K_i^{-1} v$ where $K_i \neq 0$. So on the line ℓ there are at most k focal points each counted with its multiplicity.*

In the case of a complex submanifold V of dimension d analytically embedded in $\mathbb{C}^N$ we have the remarkable result (see [37, Assertion 7.4 of Chapter 1]):

Lemma 9.2.14 *The focal points of (V, y) on any normal line to V at y are distributed symmetrically about y and symmetric focal points have the same multiplicity ν.*

On the whole line from x to y there are at most $2d$ focal points counted with their multiplicity. Therefore, because of the symmetry, on the segment form x to y there are at most d focal points counted with their multiplicity, so Lemma 9.2.12 shows that the index of δ_x at $y \in V$ is at most d.

Now we can apply Morse Theory (see [40] or [37]) using the Morse function δ_x, since $\{\delta_x \leq \alpha\}$ is compact for any $\alpha \in \mathbb{R}$. The indices of δ_x at every non-degenerate point are $\leq d$, which shows that V has the homotopy type of a CW-complex of dimension $\leq d$ and proves Lemma 9.2.8.

In [3], R. Bott shows that Morse theory provides a homotopy statement. Namely if $x \in V \cap H_t$, we have:

$$\pi_k(V, V \cap H_t, x) = 0 \qquad \text{for any } k \leq d-1.$$

Using Morse theory as given in [37], we already obtain this result.

We will see in Chap. 3 that the Lefschetz theorem cannot be, in a straightforward way, transferred to the case of a singular space. The situation is simpler for the homotopy type of Stein manifolds. In fact, H.A.Hamm has shown that Lemma 9.2.8 goes over to the singular case, see [15, Satz 1], the proof being corrected in [17]:

Theorem 9.2.15 *Let X be a Stein space of dimension n. Then X has the homotopy type of a CW complex of dimension $\leq n$.*

As for the Lefschetz theorem, the problem is that the Lefschetz duality does not extend to the singular case.

9.3 Generalizations of the Lefschetz Theorem

As above $\mathbb{P}^N$ is the complex projective space of dimension N. We want to generalize the Lefschetz theorem to the cases

1. the variety V is quasi-projective;
2. the variety V is singular.

First, the classical theorem of Bertini can be understood as a consequence of a Lefschetz type theorem for quasi-projective non-singular varieties, when one considers the 0-th homology groups. Then, a theorem of Zariski [52] can be viewed as a consequence of a similar Lefschetz type theorem, by comparing the fundamental groups.

9.3.1 Bertini's Theorem

We can formulate Bertini's theorem in the following way:

Theorem 9.3.1 *Let V be a complex projective variety in $\mathbb{P}^N$ of dimension d. If $d \geq 2$, there is a non-empty open Zariski subset Ω of the space of projective hyperplanes of $\mathbb{P}^N$ such that, for any $L \in \Omega$, the intersection $V \cap L$ is irreducible.*

This theorem is a consequence of the following ones:

1. From [41, Corollary (4.16) p. 68] one can obtain the following result:

   Theorem 9.3.2 *A complex projective algebraic set E is irreducible if and only if the subset E^0 of non-singular points of E is connected.*

2. the following generalization of the Lefschetz theorem, consequence of Theorem 1.1.1 and 1.1.3 of [20]:

   Theorem 9.3.3 *Let V be a complex quasi-projective variety in $\mathbb{P}^N$. There is a non-empty open Zariski subset Ω_1 of the space of projective hyperplanes of $\mathbb{P}^N$ such that, for any $L \in \Omega_1$,*

   $$H_k(V, V \cap L) = 0 \qquad \textit{for any } k < d - 1$$

3. and finally the following lemma which can be proved as an easy consequence of stratification theory:

   Lemma 9.3.4 *Let E be a complex projective algebraic subset of $\mathbb{P}^N$. There is a non-empty open Zariski subset Ω_2 of the space of projective hyperplanes of $\mathbb{P}^N$ such that, for any $L \in \Omega_2$, the subset of non-singular points of $E \cap L$ is the intersection $E^0 \cap L$ of the subset E^0 of non-singular points of E and the hyperplane L.*

Obviously the Zariski open set Ω mentioned in Bertini's theorem is $\Omega_1 \cap \Omega_2$.

Proof of (3) Since E is an algebraic subset of $\mathbb{P}^N$, there is a finite algebraic stratification which satisfies Whitney regular condition and in which the open subset E^0 of non-singular points of E is a stratum.

Now, the Zariski open set of hyperplanes of the space of hyperplanes of $\mathbb{P}^N$ that we consider is the Zariski open set of hyperplanes L which are transverse to the strata of the Whitney stratification of E.

In particular $E^0 \cap L$ is non-singular.

Since $E^0 = E - \Sigma$, where Σ is the subset of singular points of E, E^0 is a quasi-projective subspace of $\mathbb{P}^N$. So we can apply Theorems 1.1.1 or 1.1.3 of [20].

If V is irreducible, the subset of non-singular points of V^0 is connected and with $L \in \Omega$, $L \cap V^0$ is connected because of Theorem 9.3.3. The space $L \cap V^0$ being the subset of non-singular points of $L \cap V$, $L \cap V$ is irreducible because of Theorem 9.3.2.

9.3.2 The Zariski-Lefschetz Theorem

In [52] O. Zariski wanted to calculate the fundamental group of the complement of a projective complex hypersurface. He stated the following theorem:

Theorem 9.3.1 *Let* X *be a complex hypersurface in* $\mathbb{P}^N$. *If* $N \geq 3$, *there is a non-empty open Zariski subset* Ω *of the space of projective hyperplanes of* $\mathbb{P}^N$ *such that, for any* $L \in \Omega$, *the fundamental group of the complement of* $X \cap L$ *in* L *equals the fundamental group of the complement of* X *in* $\mathbb{P}^N$.

By induction the theorem of Zariski reduces the calculation of the fundamental group of the complement of a hypersurface in $\mathbb{P}^N$ to the calculation of the complement of a plane curve in a general plane section.

Unfortunately the original proof of O. Zariski relies on an isotopy theorem which could not be completely proved in those days. This isotopy theorem and therefore Zariski's theorem were proved by A. N. Varchenko in [49]. D. Cheniot in [6] gave another proof using an isotopy theorem consequence of the first isotopy theorem of Thom and Mather. In [19] the authors showed that in fact Zariski's theorem can be understood as a generalization of the Lefschetz theorem. Namely they proved:

Theorem 9.3.2 *Let* X *be a complex hypersurface in* $\mathbb{P}^N$. *Denote* $V := \mathbb{P}^N - X$. *If* $N \geq 3$, *there is a non-empty open Zariski subset* Ω *of the space of projective hyperplanes of* $\mathbb{P}^N$ *such that, for any* $L \in \Omega$ *and* $x \in V \cap L$, *the inclusion* $V \cap L \subset V$ *induces mappings:*

$$\pi_k(V \cap L, x) \to \pi_k(V, x) \qquad \textit{which are isomorphisms, for any } k < N - 1$$

and a surjective map for $k = N - 1$.

In the preceding theorem it is often more convenient to say that the relative homotopy groups

$$\pi_k(V, V \cap L, x)$$

vanish for $k < N$ or to say that the pair of spaces $(V, V \cap L)$ is $(N-1)$-connected. Note that this is by abuse of language: $\pi_k(V, V \cap L, x)$ is a group for $k \geq 2$, the vanishing of $\pi_1(V, V \cap L, x)$ means that this set consists of one element, and $\pi_0(V, V \cap L, x)$ is not defined, but $\pi_0(V, V \cap L, x) = 0$ means that $\pi_0(V \cap L, x) \to \pi_0(V, x)$ is surjective.

Notice that Theorem 9.3.2 is a corollary of a theorem similar to Theorem 9.3.3, but involving homotopy groups, instead of homology groups. This theorem is also proved in [19].

We now give another proof of Theorem 9.3.2 which uses a local version of the Lefschetz theorem. Consider the canonical map:

$$\lambda \colon \mathbb{C}^{N+1} - \{0\} \to \mathbb{P}^N$$

If L is a projective hyperplane in $\mathbb{P}^N$, we denote the corresponding hyperplane in $\mathbb{C}^{N+1}$ by $\tilde{L}$. Let $f = 0$ be an equation of X. Then f is a homogeneous polynomial. Let t be a non-zero complex number. The affine hypersurface $F_t := \{f = t\}$ is non-singular and contained in $\mathbb{C}^{N+1} - \{0\}$. The map λ restricted to F_t induces a cyclic covering of $V := \mathbb{P}^N - X$ of degree m which is the degree of the polynomial f. Therefore we have:

Lemma 9.3.3 *We have:*

$$\pi_k(V, V \cap L, x) = 0 \qquad \textit{for any } k < N$$

if and only if, for $y \in F_t \cap \lambda^{-1}(x)$*, we have:*

$$\pi_k(F_t, F_t \cap \tilde{L}, y) = 0 \qquad \textit{for any } k < N$$

To prove this lemma we compare the homotopy exact sequences of the pairs:

$$(F_t, F_t \cap \tilde{L}) \text{ and } (V, V \cap L).$$

As λ induces a cyclic covering of F_t and $F_t \cap \tilde{L}$ onto V and $V \cap L$ respectively, we have the isomorphisms of the homotopy groups:

$$\pi_k(F_t \cap \tilde{L}, y) \overset{\simeq}{\to} \pi_k(V \cap L, y)$$

and

$$\pi_k(F_t, y) \overset{\simeq}{\to} \pi_k(V, y)$$

for $k \neq 1$ and for $k = 1$ we have the exact sequences of groups:

$$0 \to \pi_1(F_t \cap \tilde{L}, y) \to \pi_1(V \cap L, y) \to \mathbb{Z}/m\mathbb{Z} \to 0$$

$$0 \to \pi_1(F_t, y) \to \pi_1(V, y) \to \mathbb{Z}/m\mathbb{Z} \to 0$$

Now a diagram chasing and the Five Lemma give the answer.

Theorem 9.3.2 is proved if one can prove that

$$\pi_k(F_t, F_t \cap \tilde{L}, y) = 0 \qquad \text{for any } k < N$$

This result can be obtained directly, by considering on F_t the function distance to the hyperplane $\tilde{L}$. Then one shows that, if L is general enough, this function is a locally trivial fibration on $F_t - \tilde{L}$ near infinity and its critical points on $F_t - \tilde{L}$ are Morse critical points, *i.e.* non-degenerate, of index N because F_t is a complex manifold of dimension N and the restriction to F_t of the linear form which defines $\tilde{L}$ has only ordinary quadratic singularities. Another way to obtain this result is through a reduction to a local theorem and again the use of Morse theory as we do in the following results:

Lemma 9.3.4 *For any hyperplane* $\tilde{L}$ *through the origin* 0 *and any* $\varepsilon > 0$*, there is* $\tau > 0$*, such that, for any* t*,* $\tau \geq |t| > 0$*, the pair* $(F_t, F_t \cap \tilde{L})$ *is diffeomorphic to* $(F_t \cap \overset{\circ}{B}_\varepsilon, F_t \cap \tilde{L} \cap \overset{\circ}{B}_\varepsilon)$*, where* $\overset{\circ}{B}_\varepsilon$ *is the open ball centered at* 0 *with radius* ε*.*

This lemma is a consequence of the homogeneity of f.
Now Theorem 9.3.2 is a consequence of the following local result proved by one of the authors in [26]:

Theorem 9.3.5 *Let* $f : U \to \mathbb{C}$ *be a complex analytic function defined on an open neighbourhood of* 0 *in* $\mathbb{C}^{N+1}$ *such that* $f(0) = 0$*. Let* F_t *be the hypersurface* $f = t$*. There is a non-empty open Zariski subset* Ω_0 *of the space of hyperplanes of* $\mathbb{C}^{N+1}$ *through the origin* 0*, such that, for any hyperplane* $\tilde{L} \in \Omega_0$*, there is* ε_0*, such that for any* ε*,* $\varepsilon_0 \geq \varepsilon > 0$*, there is* $\tau > 0$*, such that, for any* t*,* $\tau \geq |t| > 0$*, the pair of spaces*

$$(F_t \cap \overset{\circ}{B}_\varepsilon, F_t \cap \tilde{L} \cap \overset{\circ}{B}_\varepsilon)$$

is $(N-1)$*-connected.*

This theorem can be understood as a Lefschetz theorem on an open analytic set.
This proves Theorem 9.3.2 and consequently Theorem 9.3.1.

9.3.3 Quasi-projective Theorem of Lefschetz Type

In the preceding section we saw that the Theorem of Zariski can be understood as a theorem of Lefschetz type.

More generally let V be a subvariety of the projective space $\mathbb{P}^N$. Let W be a Zariski closed subspace of V. Then $V - W$ is a quasi-projective variety.

We have the following theorem (see [20, Theorem 1.1.3]):

Theorem 9.3.1 *We assume that* $V - W$ *is a non-singular quasi-projective variety of dimension* d*. There is a non-empty open Zariski subset* Ω *of the space of hyperplanes of* $\mathbb{P}^N$*, such that, for any* $L \in \Omega$*, the pair* $(V - W, (V - W) \cap L)$ *is* $(d-1)$*-connected.*

If one wants to avoid to consider generic sections by hyperplane, one can consider good neighbourhoods $\mathcal{V}(L)$ of L with respect to W (see Definition 9.4.5). This idea was introduced by P. Deligne in [7]. Then. we have:

Theorem 9.3.2 *We assume that* $V - W$ *is a non-singular quasi-projective variety of dimension* d*. Let* L *be a hyperplane of* $\mathbb{P}^N$*. Then for any good neighbourhood* $\mathcal{V}(L)$ *of* L *in* $\mathbb{P}^N$ *with respect to* W*, we have that the pair of spaces* $(V - W, (V - W) \cap \mathcal{V}(L))$ *is* $(d-1)$*-connected.*

Remark 9.3.3

1. In the case W is empty, *i.e.* in the case one considers a non-singular projective variety, one does not need to choose a generic hyperplane (compare with Theorem 9.2.6).
2. In the case W is not empty, the statement of the Lefschetz Theorem is nearly the same as the classical one, except that one intersects the quasi-projective variety with a generic hyperplane or with the good neighbourhood of a hyperplane with respect to W.

There are local versions of Theorems 9.3.1 and 9.3.2:

Theorem 9.3.4 *Let X be an equidimensional locally closed complex analytic subspace of $\mathbb{C}^N$ of dimension d. Assume that x is a point of X and Y is a closed complex analytic subspace of X which contains x and that $X-Y$ is non-singular. Let $\mathbb{B}_\epsilon$ be a ball centered at x with radius ϵ. There is a non-empty open Zariski subset Ω of affine hyperplanes containing x such that, for ϵ small enough and $L \in \Omega$, the pair $(\mathbb{B}_\epsilon \cap (X-Y), \mathbb{B}_\epsilon \cap (X-Y) \cap L)$ is $(d-2)$-connected.*

See a more general statement as Theorem 1.6 in [22].

Theorem 9.3.5 *Let X be an equidimensional locally closed complex analytic subspace of $\mathbb{C}^N$ of dimension d. Assume that x is a point of X and Y is a closed complex analytic subspace of X which contains x and that $X-Y$ is non-singular. Let $\mathbb{B}_\epsilon$ be a ball centered at x with radius ϵ. For ϵ small enough and any sufficiently small good neighbourhood $\mathcal{V}(L)$ of a hyperplane L which contains x, the pair $(\mathbb{B}_\epsilon \cap (X-Y), \mathbb{B}_\epsilon \cap (X-Y) \cap \mathcal{V}(L))$ is $(d-2)$-connected..*

See Theorem I.1.1 of [21] or Lemma 1.7 of [22] for a more general result.

9.3.4 Local Lefschetz Theorems

In fact we have several local theorems that we can call local Lefschetz theorems. First notice that locally the situation is conelike as the following lemma teaches us:

Lemma 9.3.1 *Let the notations be the ones of Theorem 9.3.5. There is ε_0, such that for any ε, $\varepsilon_0 \geq \varepsilon > 0$, the pair of spaces $(B_\varepsilon - \{0\}, B_\varepsilon - F_0)$ is homeomorphic to the pair $(S_\varepsilon \times (0,1], (S_\varepsilon - F_0) \times (0,1])$ and furthermore the homeomorphism class of this pair does not depend on ε, $\varepsilon_0 \geq \varepsilon > 0$.*

This lemma is a consequence of results proved in [4]. In the case 0 is an isolated singularity, this lemma has been proved by Milnor in [39].

When $\varepsilon > 0$ is small enough, in [39, Theorem 5.11 p. 53] J. Milnor shows that the space $S_\varepsilon - F_0$ fibres onto the circle $\mathbb{S}^1$ by a map induced by $f/|f|$ and the general fibre is diffeomorphic to the space $F_t \cap \overset{\circ}{B}_\varepsilon$, when t is small enough and $\neq 0$. Namely:

Theorem 9.3.2 *Let* $f: U \to \mathbb{C}$ *be a complex analytic function defined on an open neighbourhood of* 0 *in* $\mathbb{C}^{N+1}$ *such that* $f(0) = 0$*. Let* F_t *be the hypersurface* $f = t$*. There is* ε_0*, such that for any* ε, $\varepsilon_0 \geq \varepsilon > 0$*, the map* $f/|f|$ *induces a locally trivial smooth fibration* φ_ε *of* $S_\varepsilon - F_0$ *onto the circle* $\mathbb{S}^1$ *and there is* $\tau > 0$*, such that, for any* t, $\tau \geq |t| > 0$*, the fibre of the fibration* φ_ε *is diffeomorphic to* $F_t \cap \overset{\circ}{B}_\varepsilon$*.*

Using the exact homotopy sequence of the fibration φ_ϵ, we obtain the following local Lefschetz type theorem in [19, Part (a) of Théorème (0.2.1)]:

Theorem 9.3.3 *Let* $f: U \to \mathbb{C}$ *be a complex analytic function defined on an open neighbourhood of* 0 *in* $\mathbb{C}^{N+1}$ *such that* $f(0) = 0$*. Let* F_t *be the hypersurface* $f = t$*. There is a non-empty open Zariski subset* Ω_1 *of the space of hyperplanes of* $\mathbb{C}^{N+1}$ *through the origin* 0*, such that, for any hyperplane* $\tilde{L} \in \Omega_1$*, there is* ε_0*, such that for any* ε, $\varepsilon_0 \geq \varepsilon > 0$*, the pair of spaces*

$$(B_\epsilon - F_0, (B_\varepsilon - F_0) \cap \tilde{L})$$

is $(N - 1)$*-connected.*

Just notice that we may have expected that the pair of spaces in Theorem 9.3.3 is N-connected, as we are embedded in an $(N + 1)$-dimensional space. There is a result in this direction, but instead of considering a general hyperplane through the origin 0, we have to consider a general hyperplane near the origin but not passing through it. However this type of result also proved in [19] is more difficult to obtain. To distinguish these two situations we call Theorem 9.3.3 a weak local Lefschetz type theorem and the other one a strong local Lefschetz type theorem .

Warning This notion has nothing to do with the one of "hard" Lefschetz theorem which is not at all subject of the present notes!

The strong version of Theorem 9.3.3 (see (b) of Theorem (0.2.1) of [19]) is:

Theorem 9.3.4 *If* $f: U \to \mathbb{C}$ *is a complex analytic function defined on an open neighbourhood of* 0 *in* $\mathbb{C}^{N+1}$ *such that* $f(0) = 0$*. Let* F_t *be the hypersurface* $f = t$*. There is a non-empty open Zariski subset* Ω_1 *of the space of hyperplanes of* $\mathbb{C}^{N+1}$ *through the origin* 0*, such that, for any hyperplane* $\tilde{L} \in \Omega_1$ *defined by the equation* $\ell = 0$*, there is* ϵ_0*, such that for any* ϵ, $\epsilon_0 \geq \epsilon > 0$*, there is* υ*, such that, for any* u, $\upsilon \geq |u| > 0$*, the pair of spaces*

$$(B_\epsilon - F_0, (B_\epsilon - F_0) \cap \tilde{L}_u)$$

where $\tilde{L}_u$ *is the hyperplane* $\ell = u$*, is* N*-connected.*

As for Theorem 9.3.3 indicating that Theorem 9.3.4 is true, we have the following theorem:

Theorem 9.3.5 *Let* X *be an equidimensional reduced complex analytic space. Assume that* x *is a point of* X *and* Y *is a closed complex analytic subspace of* X *which contains* x*. Let* $\mathbb{B}_\epsilon$ *be a ball centered at* x *with radius* ϵ*. There is a non-empty* *open Zariski subset* Ω *of affine hyperplanes containing* x *such that, for* ϵ *small enough and* $L \in \Omega$ *defined by* $\ell = 0$*, the pair* $(\mathbb{B}_\epsilon \cap (X - Y), \mathbb{B}_\epsilon \cap (X - Y) \cap L_u)$ *is* $(d - 1)$*-connected, where* $\{\ell = u\} = L_u$ *with* $u \neq 0$ *small enough.*

This was proved by [21, Main Theorem II.1.4] or by [22, Theorem 2.12] in a more general setting.

9.3.5 Other Local Lefschetz Theorems

There are other local generalizations of Lefschetz theorem. In [39], J. Milnor gives a connectivity theorem for the local link $S_\epsilon \cap F_0$ of the hypersurface F_0 at the point 0:

Theorem 9.3.1 *Let* $f : U \to \mathbb{C}$ *be a complex analytic function defined on an open neighbourhood of* 0 *in* $\mathbb{C}^{N+1}$ *such that* $f(0) = 0$*. Let* F_t *be the hypersurface* $f = t$*. There is* ϵ_0*, such that for any* ϵ*,* $\epsilon_0 \geq \epsilon > 0$*, the local link* $K_\epsilon := S_\epsilon \cap F_0$ *of* F_0 *at* 0 *is* $(N - 2)$*-connected.*

This theorem can be understood as a local version of Lefschetz theorem. In [14] and in [27], the authors proved:

Theorem 9.3.2 *Let* $f : U \to \mathbb{C}$ *be a complex analytic function defined on an open neighbourhood of* 0 *in a complex analytic subspace* X *of* $\mathbb{C}^{N+1}$ *and such that* $f(0) = 0$*. Let* Y *be* $\{f = 0\} \cap X$*. Assume that* $X - Y$ *is non-singular. There is* ϵ_0*, such that for any* ϵ*,* $\epsilon_0 \geq \epsilon > 0$*, the pair of spaces*

$$(S_\epsilon \cap X, S_\epsilon \cap Y)$$

is $(\dim X - 2)$*-connected.*

Considering the cone on a projective variety, the local link of the cone at its vertex is a locally trivial smooth fibration over the projective variety with fibre diffeomorphic to $\mathbb{S}^1$. Another diagram chasing shows that Theorem 9.3.2 implies Theorem 9.2.5 and its homotopy version. Notice that in Theorem 9.3.2 the important fact is that $X - Y$ is non-singular. As J. Milnor already noticed in [37] Corollary 7.5 and Theorem 7.6, in the original version of Lefschetz' theorem, it is enough to assume that all possible singularities lie inside the hyperplane section.

9.4 Homotopical Depth

When a variety contains singularities which are not contained in the hyperplane section, a defect in the Lefschetz theorems comes from the nature of the singularities.

In [13] A. Grothendieck, in analogy with the notion of depth in Commutative algebra, introduced the notion of homotopical depth.Using this notion of depth, we shall give below theorems of Lefschetz type.

9.4.1 Good Neighbourhoods

In what follows it is very convenient to introduce the concept of good neighbourhoods in the sense of D. Prill [43, Section B].
Let X be a topological space and let Y be a topological subspace. Consider a point $x \in X$.

Definition 9.4.1 A neighbourhood U of x in X is called a good neighbourhood of x with respect to Y, if there is a family $(U_\alpha)_\alpha \in A$ of neighbourhoods of x in X such that:

1. The family $(U_\alpha)_{\alpha \in A}$ is a neighbourhood basis of the point x in X;
2. Each set $U_\alpha - Y$ $(\alpha \in A)$ is a deformation retract of $U - Y$.

The interest of good neighbourhoods lies in the following:

Lemma 9.4.2 *Let U and V be good neighbourhoods of x in X with respect to Y, then $U - Y$ and $V - Y$ have the same homotopy type.*

In the case of simplicial complexes, if Y is a subcomplex, the star of a point is a natural good neighbourhood of x in X with respect to Y. In the case of real analytic spaces (or more generally subanalytic sets) the following result gives us natural good neighbourhoods:

Proposition 9.4.3 *Let X be a subanalytic set and x be a point of X. Consider a subanalytic subset Y of X. Suppose that X is embedded in $\mathbb{R}^N$. There is $\epsilon_0 > 0$, such that, for any ϵ, $\epsilon_0 \geq \epsilon > 0$, the intersection $X \cap B_\epsilon(x)$ of X with the open ball $B_\epsilon(x)$ of $\mathbb{R}^N$ centered at x with radius $\epsilon > 0$, is a good neighbourhood of x in X with respect to Y.*

Look at the paper of [4] for a proof.

Remark 9.4.4 A subanalytic set can always be locally embedded in an affine space $\mathbb{R}^N$. Therefore the preceding proposition provides us with many good neighbourhoods which are easy to find. When we consider the local homotopy type of $X - Y$ at a point x of X, we mean the homotopy type of $U - Y$, where U is a good neighbourhood of x in X with respect to Y. Therefore we can in particular consider the homotopy type of $X \cap B_\epsilon(x) - Y$, where $B_\epsilon(x)$ is a sufficiently small open ball of an affine space $\mathbb{R}^N$ in which X is locally embedded in a neighbourhood of x.

Similarly if Z is a topological subspace of X, we can define:

Definition 9.4.5 A neighbourhood U of Z in X is called a good neighbourhood of Z with respect to Y, if there are subsets $(U_\alpha)_{\alpha \in A}$ which satisfy the following conditions:

1. The family $(U_\alpha)_{\alpha \in A}$ is a neighbourhood basis of the Z in X;
2. Each set $U_\alpha - Y$ $(\alpha \in A)$ is a deformation retract of $U - Y$.

In this case a similar result as Lemma 9.4.2 holds.

One has the existence of good neighbourhoods of a subanalytic subset Z of a subanalytic set X with respect to a subanalytic set $Y \subset X$ by using the triangulability of subanalytic sets [24].

9.4.2 *Rectified Homotopical Depth*

Let X be a complex analytic space. Let Y be a non-empty closed analytic subset of X. First we give the following definition of A. Grothendieck (*cf.* [13], see also [22])

Definition 9.4.1 The space X has homotopical depth $hd_Y(X) \geq n$ along Y if, for any $y \in Y$, there is a fundamental system (U_α) of neighbourhoods of y in X such that the pairs $(U_\alpha, U_\alpha - Y)$ are $(n-1)$-connected. The integer $hd_Y(X)$ is the maximum of the set of integers as in the definition above.

As it is defined in [Sw], a pair of topological spaces (B, A), such that $A \subset B$, is 0-connected if the path connected components of B meet all the path components of A. For $k \geq 1$, the pair (A, B) is k-connected if it is 0-connected and $\pi_i(B, A, a)$ vanishes for $1 \leq i \leq k$ and any point $a \in A$. In particular if $B \neq \emptyset$ and $A = \emptyset$, the pair $(B, \emptyset)$ is not 0-connected, but it is, if $B = \emptyset$.

Remark 9.4.2 The definition given by Grothendieck in Exp. XIII, p. 26 p. 197 of [13] is more general. For instance, it applies to the case when the topological space X is locally triangulable along a subspace Y, *i.e.* for any point y of Y there is a fundamental system of neighbourhoods U_α of y in X such that the pair $(U_\alpha, U_\alpha \cap Y)$ is triangulable.

Of course, a complex analytic space is triangulable along any closed analytic subspace, because of the result of S. Łojasiewicz [35] about the triangulability of analytic spaces.

Note that, if X is locally triangulable along a subspace Y, for any point y of Y, there is a good neighbourhood V of y in X with respect to Y in the sense of Prill [43]. In practice we shall often use the following lemma:

Lemma 9.4.3 *Let V be a good neighbourhood of y in X with respect to Y. There is a fundamental system of neighbourhoods U_α of y in X such that the pairs $(U_\alpha, U_\alpha - Y)$ are $(n-1)$-connected, if and only if the pair $(V, V - Y)$ is $(n-1)$-connected.*

It is convenient to introduce the local definition of the homotopical depth:

Definition 9.4.4 Let X be a complex analytic space and x be a point of X. Let Y be a complex analytic subspace of X. Assume that x is in Y. We say that the space X (or the germ (X, x)) has homotopical depth $hd_Y(X, x) \geq n$ along Y at the point x if there is an open neighbourhood U of x in X, such that the homotopical depth $hd_{Y \cap U}(X \cap U)$ of $X \cap U$ along $Y \cap U$ is $\geq n$.

We have the following lemma which will be useful:

Lemma 9.4.5 *Let $\mathcal{S} = (X_i)_{i \in I}$ be an analytic Whitney stratification of the complex analytic space X, then the function $hd_{X_i}(X, x)$ is constant along a stratum X_i of $\mathcal{S}$.*

Remember that a Whitney stratification (or regular stratification) of an analytic space is analytic if the strata and their closures are complex analytic subspaces (see [51, p. 536 and §19])

The proof of this Lemma is left to the reader. It is consequence of the topological triviality along the strata of the Whitney stratification [36, Consequence of Proposition 11.1].

Along a Whitney stratum it is often more convenient to consider the normal slice of the stratum (*cf.* [12]) to calculate the homotopical depth. Namely let X_i be a stratum of the Whitney stratification $\mathcal{S}$ of X. Consider $y \in X_i$. Embed X in $\mathbb{C}^M$ locally at y. Consider an affine space N of codimension $\dim X_i$ of $\mathbb{C}^M$ transverse to X_i at y. Let $\mathring{B}_\epsilon(y)$ be the open ball of $\mathbb{C}^M$. We call the intersection $X \cap \mathring{B}_\epsilon(y) \cap N$ a normal slice of the stratum X_i in X at y.

We have the following general result which is also consequence of the topological triviality along the strata of the Whitney stratification:

Lemma 9.4.6 *Let $\mathcal{N}$ be a normal slice of the stratum X_i in X at y and V be a good neighbourhood of y in X with respect to X_i. Then the pairs $(V, V - X_i)$ and $(\mathcal{N}, \mathcal{N} - \{y\})$ have the same homotopy type.*

Therefore it is often convenient to use normal slices to calculate the homotopical depth along a Whitney stratum.

In general if the dimension of the subspace Y of X is big, the connectivity of the pairs $(U_\alpha, U_\alpha - Y)$ is low. In connection with this fact A. Grothendieck introduced the notion of rectified homotopical depth:

Definition 9.4.7 We say that the rectified homotopical depth $rhd(X, x)$ of X at the point x is $\geq n$, if, for any locally closed analytic subspace Y of X with $x \in Y$, there is an open neighbourhood U of x in X, such that the homotopical depth of $X \cap U$ along $Y \cap U$ is $\geq n - \dim Y$.

Of course, the integer $rhd(X, x)$ is the maximum of the set of integers n as in the definition above.

Remark 9.4.8 We always have $rhd(X, x) \leq dim(X, x)$. In fact if $Y = X$ the pair $(U, U - X)$ is not 0-connected as it has been noticed above, because $U - X = \emptyset$.

Definition 9.4.9 The rectified homotopical depth of X is:

$$rhd(X) := \inf_{x \in X} rhd(X, x).$$

9.4.3 Comments

Why can one hope that the hypothesis that $rhd \geq n$ is a good substitute for the condition that dim $\geq n$ when extending the Lefschetz theorems to the singular case?

First let us suppose that (X, x) is a complete intersection of dimension n in $(\mathbb{C}^N, x)$.

Let L be an affine subspace of $\mathbb{C}^N$ of codimension i such that dim $X \cap L = n - i$. Then, at least when L is chosen in an adequate Zariski open subset of affine subspaces of codimension i through x, the pair $(B_\epsilon(x) \cap X - \{x\}, B_\epsilon(x) \cap X \cap L - \{x\})$ is $(n - i - 1)$-connected (see [29]), without smoothness assumption, because $(B_\epsilon(x) - \{x\}, B_\epsilon(x) \cap X \cap L - \{x\})$ in $(n - i - 1)$-connected and $(B_\epsilon(x) - \{x\}, B_\epsilon(x) \cap X - \{x\})$ is $(n - 1)$-connected by Satz 1.2 of [14].

Since the result concerns homotopy groups we want to replace the hypothesis that X is a complete intersection by a hypothesis about homotopy groups. Fix a Whitney stratification $(X_i)_{i \in I}$ of X. Let $\mathcal{N}$ be a normal slice at y to X_i in X. Then $\mathcal{N}$ is a complete intersection of dimension $n -$dim X_i, so $(\mathcal{N}, \mathcal{N} - \{y\})$ is $(n -$dim $X_i - 1)$-connected, hence $(V, V - X_i)$, too, where V is a good neighbourhood of y in X with respect to X_i. As we will see in the following theorem, this implies that $rhd(X) = n$.

It was shown by Hamm in [16] that the notion of rectified homotopical depth enables Lefschetz theorems. This paper is very concise. Here we will follow our detailed and comprehensive paper [22] which deals with Grothendieck's conjectures.

9.4.4 The Main Result

In the following theorem we assume that all the Whitney stratifications are analytic. In [22] Theorem 1.4 we give the following theorem which allows us to compute the rectified homotopical depth:

Theorem 9.4.1 *Let X be a reduced complex analytic space and x be a point of X. Let $\mathcal{S} = (X_i)_{i \in I}$ be a Whitney stratification of X. The following conditions are equivalent:*

1. $rhd(X, x) \geq n$;
2. *for any locally closed irreducible complex analytic subspace Y of dimension i in X, there is an open neighbourhood U of the point x in X and an open dense*

analytic subset Y_0 *of* Y *such that, for any point* y *in* $Y_0 \cap U$, *there is a fundamental system of neighbourhoods* U_α *of* y *in* U *such that the pair* $(U_\alpha, U_\alpha - Y)$ *is* $(n - 1 - i)$*-connected;*

3. *for any* $i \in I$, *such that the point* x *belongs to the closure of the stratum* X_i, *the homotopical depth* $hd_{X_i}(X)$ *is* $\geq n - \dim X_i$.

This theorem shows that one can compute the rectified homotopical depth with a given analytic Whitney stratification of X.

The main difficulty in the proof lies in the equivalence of the conditions (1) and (3).

Proof The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious. The inverse implications will be shown in the following two subsections.

9.4.5 Proof of (3) ⇒ (2)

It is sufficient to show that, if (3) holds for the given Whitney stratification, it holds for any other as well. For then we may pass to a stratification such that if $y \in Y \setminus Y_0$, Y coincides with some X_i near y. Take Y_0 to be the union of all maximal strata of Y, maximal with respect to the relation: $X_i \leq X_j$ if $X_i \subset \overline{X_j}$. The rest is obvious.

So we have to show:

Lemma 9.4.1 *Let* $\mathcal{S} = (X_i)_{i \in I}$ *be an analytic Whitney stratification of* X *such that, for any stratum* X_i, *the homotopical depth* $hd_{X_i}(X)$ *is* $\geq n - \dim X_i$. *Then, for any other Whitney stratification* $\mathcal{T} = (Y_j)_{j \in J}$ *of* X, *the homotopical depth* $hd_{Y_j}(X)$ *is* $\geq n - \dim Y_j$, *for any stratum* Y_j.

First notice that the assertion of the lemma is true for any analytic Whitney stratification $\mathcal{S}'$ of X finer than $\mathcal{S}$.

Let X'_k be a stratum of $\mathcal{S}'$. Let y be a point of X'_k. This point is contained in a unique stratum X_i of $\mathcal{S}$. As $\mathcal{S}'$ is finer than $\mathcal{S}$, we have that $X'_k \subset X_i$. Therefore there are good neighbourhoods V of y in X homeomorphic to $\mathcal{N} \times U_1 \times U_2$ where $\mathcal{N}$ is the normal slice of X_i in X (see Lemma 9.4.6), U_1 is a good neighbourhood of y in X'_k and U_2 is the normal slice of X'_k in X_i. In this homeomorphism the image of $V \cap X'_k$ is $\{y\} \times U_1 \times \{y\}$ and the image of a normal slice of X'_k in X is $\mathcal{N} \times \{y\} \times U_2$. We have to estimate the connectivity of the pair $(\mathcal{N} \times U_1 \times U_2, \mathcal{N} \times U_1 \times U_2 - \{y\} \times U_1 \times \{y\})$. It is the same as the one of $(\mathcal{N} \times U_2, \mathcal{N} \times U_2 - \{(y, y)\})$.

We know that U_2 is homeomorphic to a ball of real dimension $2 \dim X_i - 2 \dim X'_k$ and that the pair $(\mathcal{N}, \mathcal{N} - \{y\})$ is $(n - \dim X_i - 1)$-connected by Lemma 9.4.6 and the hypothesis. We shall conclude by using the following lemma (Lemma 1.8 of [22]):

Lemma 9.4.2 *Let* (E, E') *be a pair of locally trivial topological fibrations over a connected CW-complex* B, *let* $\pi : E \to B$ *be the corresponding projection. Let*

(N, N') be a pair of fibres of these fibrations. Consider a subcomplex B' of B. Assume that the pair (B, B') is $(r-1)$-connected, $r \geq 1$, the space N is contractible and $H_i(N, N'; \mathbb{Z}) = 0$, for $i \leq m-r-1$, then the pair $(E, E' \cup \pi^{-1}(B'))$ is $(m-1)$-connected.

We apply Lemma 9.4.2 to the proof of Lemma 9.4.1 with $E := \mathcal{N} \times U_2,\ E' := (\mathcal{N} - \{y\}) \times U_2,\ B := U_2$ and $B' := U_2 - \{y\}$. Therefore π is the projection of the product $\mathcal{N} \times U_2$ onto $U_2,\ N = \mathcal{N},\ N' = \mathcal{N} - \{y\},\ r = 2 \dim X_i - 2 \dim X'_k,\ m = n + (2 \dim X_i - 2 \dim X'_k)$ and Lemma 9.4.2 implies that the pair:

$$(\mathcal{N} \times U_2, \mathcal{N} \times U_2 - \{(y, y)\}$$

is $(n - \dim X_i + (2 \dim X_i - 2 \dim X'_k) - 1)$-connected, because:

$$E' \cup \pi^{-1}(B') = ((\mathcal{N} - \{y\}) \times U_2) \cup (\mathcal{N} \times (U_2 - \{y\})) = \mathcal{N} \times U_2 - \{(y, y)\}.$$

As $n - \dim X_i + (2 \dim X_i - 2 \dim X'_k) - 1 \geq n - \dim X'_k - 1$, we obtain:

$$hd_{X'_k}(X) \geq n - \dim X'_k.$$

To end the proof of Lemma 9.4.1 it remains to prove that for any Whitney stratification $\mathcal{T} = (Y_j)_{j \in J}$ of X, the homotopical depth $hd_{Y_j}(X)$ is $\geq n - \dim Y_j$. We can always find a Whitney stratification $\mathcal{S}'$ which is finer than both $\mathcal{S}$ and $\mathcal{T}$. For any stratum Y_j of $\mathcal{T}$ there is a stratum X'_k which is dense in Y_j. Therefore $\dim Y_j = \dim X'_k$ and, for any point y of X'_k, there is a good neighbourhood V of y in X with respect to X'_k such that $(V, V - X'_k)$ is $(n - \dim X'_k - 1)$-connected. But if V is small enough, we have $V - X'_k = V - Y_j$. Lemmas 9.4.3 and 9.4.5 imply that $hd_{Y_j}(X) \geq n - \dim Y_j$.

We still have to prove Lemma 9.4.2: We may assume that B is obtained from B' by adding cells of dimension $\geq r$, because of 6.13 in [48]. Therefore it is enough to consider the case $(B, B') = (D^j, S^{j-1})$ with $j \geq r$; then E and E' are trivial fibrations,*i.e.* $E = D^j \times N$ and $E' = D^j \times N'$. As E is contractible, we have to show that $E' \cup \pi^{-1}(B')$ is $(m-2)$-connected. Using the Mayer-Vietoris sequence, Künneth formula and Hurewicz isomorphism, it is enough to show that $E' \cup \pi^{-1}(B')$ is simply connected if $m \geq 3$. To prove this fact we use the homotopy excision theorem Blakers and Massey (see 6.21 of [48]), namely:

Theorem 9.4.3 *Let U and V be subspaces of W and u be a point of $U \cap V$, such that $W = U \cup V$, the pair $(U, U \cap V)$ is a k-connected relative CW-complex, $k \geq 1$, and $(V, U \cap V)$ is a l-connected relative CW-complex. Then the inclusion of $(U, U \cap V)$ into (W, V) induces isomorphisms of $\pi_i(U, U \cap V, u)$ and $\pi_i(W, V, u)$ for $1 \leq i < k + l$ and an epimorphism for $i = k + l$.*

In our case we consider $W = E' \cup \pi^{-1}(B'),\ V = E'$ and $U = \pi^{-1}(B')$. So $U \cap V = S^{j-1} \times N'$. We have isomorphisms of $\pi_i(\pi^{-1}(B'), S^{j-1} \times N', u)$ and $\pi_i(E' \cup \pi^{-1}(B'), E', u)$ for any point u in $S^{j-1} \times N'$ and $1 \leq i < m - (r - 1) +$

$(j-1)$, and an epimorphism for $i = m-(r-1)+(j-1)$. As $j \geq r$, we have isomorphisms for $i \leq 2$, if $m \geq 3$. We consider now the following commutative diagram where vertical arrows are induced by inclusions and horizontal lines are the homotopy exact sequences of the corresponding pairs (choosing always the same base point u for the homotopy groups):

$$\begin{tikzcd}
\pi_{i+1}(N_b, N'_b) \arrow[r] \arrow[d] & \pi_i(N'_b) \arrow[r] \arrow[d] & \pi_i(N_b) \arrow[r] \arrow[d] & \pi_i(N_b, N'_b) \arrow[d] \\
\pi_{i+1}(E' \cup \pi^{-1}(B'), E') \arrow[r] & \pi_i(E') \arrow[r] & \pi_i(E' \cup \pi^{-1}(B')) \arrow[r] & \pi_i(E' \cup \pi^{-1}(B'), E')
\end{tikzcd}$$

where $N_b = \{b\} \times N,\ N'_b = \{b\} \times N',\ b = \pi(u)$. The homotopy excision theorem tells that the vertical arrows on the right and on the left are isomorphisms for $i = 0, 1$, because $\pi_j(N_b, N'_b, u) = \pi_j(B' \times N, B' \times N', u)$, for any integer $j \geq 0$, and any point $u \in B' \times N'$. As $E' = D^j \times N'$ and $j \geq 1$, we have that $\pi_i(N'_b, u) = \pi_i(E', u)$, for any integer $i \geq 0$.

Therefore the comparison between the two lines of the diagram, when $i = 1$, gives that $E' \cup \pi^{-1}(B')$ is simply connected, if $m \geq 3$. We can compare this argument with the one that H. Hamm already used in [16] (p. 552). This ends the proof of Lemma 9.4.2.

9.4.6 Proof of (2) ⇒ (1)

Note that this implication is trivial if the subspace Y to be considered is of dimension 0, we reduce to this case by intersecting with an affine subspace L of codimension $\dim_x Y$. This means that we use a local theorem of Lefschetz type which is interesting in itself.

Let Y be an closed analytic subspace of X. We can assume that X is embedded in $\mathbb{C}^N$. For ϵ small enough, we saw that $B_\epsilon(x) \cap X$ is a good neighbourhood of x in X with respect to Y (Proposition 9.4.3).

We have to prove that, assuming (2) or (3), for $\epsilon > 0$ small enough, the pair:

$$(B_\epsilon(x) \cap X, B_\epsilon(x) \cap X - Y)$$

is $(n - \dim_x Y - 1)$-connected.

We first show that this fact is consequence of a local theorem of Lefschetz type. Precisely, let L be a general affine subspace of $\mathbb{C}^N$ of codimension $\dim_x Y$ passing through x. In particular we have $L \cap Y \cap B_\epsilon = \{x\}$.

Then, we notice that the pair:

$$(B_\epsilon(x) \cap X, B_\epsilon(x) \cap X - Y)$$

is $(n - \dim_x Y - 1)$-connected, if the pairs:

$$(B_\epsilon(x) \cap X - Y, (B_\epsilon(x) \cap X - Y) \cap L)$$

and:

$$(B_\epsilon(x) \cap X - \{x\}, (B_\epsilon(x) \cap X - \{x\}) \cap L)$$

are $(n - \dim_x Y - 1)$-connected.

In fact the space $B_\epsilon(x) \cap X$ is contractible, because of the local conic structure theorem ([4], or see Lemma 9.3.1). So we only have to prove that:

$$B_\epsilon(x) \cap X - Y$$

is $(n - \dim_x Y - 2)$-connected. This is a consequence of the $(n - \dim_x Y - 1)$-connectivity of the pair

$$(B_\epsilon(x) \cap X - Y, (B_\epsilon(x) \cap X - Y) \cap L)$$

and the $(n - \dim_x Y - 2)$-connectivity of the space $(B_\epsilon(x) \cap X - Y) \cap L$. This last assertion comes from the equality:

$$(B_\epsilon(x) \cap X - Y) \cap L = (B_\epsilon(x) \cap X - \{x\}) \cap L,$$

the $(n - \dim_x Y - 1)$-connectivity of the pair of spaces:

$$(B_\epsilon(x) \cap X - \{x\}, (B_\epsilon(x) \cap X - \{x\}) \cap L)$$

and the $(n - 2)$-connectivity of the space $B_\epsilon(x) \cap X - \{x\}$ which is a consequence of the assumption $rhd(X, x) \geq n$, as we shall see below.

Therefore to end the proof of Theorem 9.4.1 it remains to prove that for a general affine subspace L of $\mathbb{C}^N$ of codimension $\dim_x Y$ passing through x the pair of spaces:

$$(B_\epsilon(x) \cap X - Y, (B_\epsilon(x) \cap X - Y) \cap L)$$

and:

$$(B_\epsilon(x) \cap X - \{x\}, (B_\epsilon(x) \cap X - \{x\}) \cap L)$$

are $(n - \dim_x Y - 1)$-connected and that the space $B_\epsilon(x) \cap X - \{x\}$ is $(n - 2)$-connected because $rhd(X, x) \geq n$.

The first assertions are consequences of a local theorem of Lefschetz type. Namely (see [22, Theorem 1.6]):

Theorem 9.4.1 *Let X be a reduced complex analytic space embedded in $\mathbb{C}^N$. Consider a closed complex analytic subspace Z of X. Fix an analytic Whitney stratification $(X_i)_{i\in I}$ of X such that Z is a union of strata. Suppose that for all i such that $X_i \subset X - Z$, $hd_{X_i}X \geq n - \dim X_i$. Then for any point x in Z, for any general affine space L through x of codimension i, and for any $\epsilon > 0$ small enough, the pair:*

$$(B_\epsilon(x) \cap X - Z, B_\epsilon(x) \cap X \cap L - Z)$$

is $(n-1-i)$-connected.

We shall prove this theorem below. First, we show the second assertion above, *i.e.* that $rhd(X, x) \geq n$ implies that the space $B_\epsilon(x) \cap X - \{x\}$ is $(n-2)$-connected. We observe that we can always refine the Whitney stratification $\mathcal{S}$ of X in such a way that $\{x\}$ is a stratum.
By Lemma 9.4.1, the homotopical depth $hd_{\{x\}}(X)$ is $\geq n$ and therefore by definition the space $B_\epsilon(x) \cap X - \{x\}$ is $(n-2)$-connected, because $B_\epsilon(x)$ is a good neighbourhood of the stratum $\{x\}$ in X. This ends the proof of (2) $\Rightarrow$ (1) and therefore of Theorem 9.4.1.
It remains to prove Theorem 9.4.1.

9.4.7 Proof of Theorem 9.4.1

Instead of proving Theorem 9.4.1 directly, we prove a more general statement which is a generalization of a local Lefschetz theorem.

The local conic structure theorem ([B-V]; see Lemma 9.3.1) implies that, for any $\epsilon > 0$ small enough, the spaces $B_\epsilon(x) \cap X - Z$ and $B_\epsilon(x) \cap L \cap X - Z$ are homeomorphic to the products of the spaces $S_\epsilon(x) \cap X - Z$ and $S_\epsilon(x) \cap L \cap X - Z$ by the interval $(0, 1]$. Therefore Theorem 9.4.1 is proved if we show that the pair:

$$(S_\epsilon(x) \cap X - Z, S_\epsilon(x) \cap L \cap X - Z)$$

is $(n-1-i)$-connected. Instead of considering L, following an original idea of P. Deligne (*cf.* [7]), we shall consider good neighbourhoods of L defined in the following way. Let $g_1 = \ldots = g_i = 0$ be i affine equations which define the affine subspace L in $\mathbb{C}^N$. Define:

$$\psi(x) = \sum |g_j(x)|^2$$

and:

$$V_\alpha(L) := \{x \in \mathbb{C}^N \quad |\psi(x)| \leq \alpha\}$$

when $\alpha > 0$. Then we shall actually prove:

Theorem 9.4.1 *Let X be a reduced complex analytic space embedded in $\mathbb{C}^N$ and x be a point of X. Consider a closed complex analytic subspace Z of X containing x. Assume that $rhd(X - Z) \geq n$. For any affine subspace L of $\mathbb{C}^N$ containing x and of codimension $i \geq \dim_x Z$, there is $\epsilon_0 > 0$ such that for any ϵ, $\epsilon_0 > \epsilon > 0$, there is α_ϵ such that for any α, $\alpha_\epsilon > \alpha > 0$, the space $S_\epsilon(x) \cap X - Z$ has the homotopy type of a space obtained from $S_\epsilon(x) \cap X \cap V_\alpha(L)$ by adding cells of dimension $\geq n - i$.*

Now, we show how Theorem 9.4.1 implies Theorem 9.4.1.
First, notice that if α is small enough, the space $V_\alpha(L) \cap S_\epsilon \cap X$ is a good neighbourhood of $S_\epsilon \cap X \cap L$ in $S_\epsilon \cap X$ with respect to $S_\epsilon \cap Z$.
Theorem 9.4.1 is clearly a consequence of Theorem 9.4.1 and the following lemma:

Lemma 9.4.2 *Let X be a reduced complex analytic space embedded in $\mathbb{C}^N$ and x be a point of X. Consider a closed complex analytic subspace Z of X containing x. There is a Zariski open dense subset Ω of the space of affine subspaces L of codimension ℓ in $\mathbb{C}^N$ containing x, such that, for any affine subspace L in Ω, there is $\epsilon_0 > 0$ such that for any ϵ, $\epsilon_0 > \epsilon > 0$, there is α_ϵ such that for any α, $\alpha_\epsilon > \alpha > 0$, the space $S_\epsilon(x) \cap (X - Z) \cap V_\alpha(L)$ retracts onto $S_\epsilon(x) \cap (X - Z) \cap L$.*

Note that in this lemma there is no condition on the codimension ℓ of L.
In what follows we are giving a proof of Theorem 9.4.1 and Lemma 9.4.2.

9.4.8 Proof of Theorem 9.4.1

This part is technically much involved, so the reader may skip the proof at first reading.

We assume that the case when $X - Z$ is non-singular is proved (*cf* [20] Theorem I.1.1.) However here we give a local version of the same theorem. Namely:

Theorem 9.4.1 *Let X be a reduced complex analytic space embedded in $\mathbb{C}^N$ and x be a point of X. Consider a closed complex analytic subspace Z of X containing x. Assume that $X - Z$ is non-singular and purely n-dimensional. For any affine subspace L of $\mathbb{C}^N$ of codimension i containing x, there is $\epsilon_0 > 0$ such that for any ϵ, $\epsilon_0 > \epsilon > 0$, there is α_ϵ such that for any α, $\alpha_\epsilon > \alpha > 0$, the space $S_\epsilon(x) \cap X - Z$ has the homotopy type of a space obtained from $S_\epsilon(x) \cap (X - Z) \cap V_\alpha(L)$ by adding cells of dimension $\geq n - i$.*

We shall not prove here this theorem (See [21, Theorem I.1.1] for a proof).

We are proceeding by decreasing induction on the dimension of the strata.

Now, as above, we stratify the space X with an analytic Whitney stratification $\mathcal{S}$ adapted to Z. We call Z_k the union of Z and the strata of X of dimension $\leq k$.

We shall prove that the space

$$M := S_\epsilon(x) \cap [(X - Z_k) \cup ((X - Z) \cap V_\alpha(L))]$$

has the homotopy type of a space obtained from

$$N := S_\epsilon(x) \cap [(X - Z_{k+1}) \cup ((X - Z) \cap V_\alpha(L))]$$

by adding cells of dimension $\geq n - i$.

The case of highest dimension $\dim X$ of the strata is supposed to be solved because of Theorem 9.4.1.

Since the spaces are CW-complexes, Theorem 6.13 of [48] tells us that it will be enough to prove that the pair (M, N) is $(n - i - 1)$-connected. Then Theorem 9.4.1 is obtained by comparing the spaces for the cases $k = 0$ and $k = \dim_x X$.

Now let us fix an integer k, $\dim_x X - 1 \geq k \geq 0$.

Consider equations $f_1 = \ldots = f_r = 0$ of Z_{k+1} and $h_1 = \ldots = h_s = 0$ of Z_k in a neighbourhood U of x in $\mathbf{C}^N$. Writing $\phi(x) = \sum |f_j(x)|^2$ and $\chi(x) = \sum |h_j(x)|^2$, we define

$$T_\beta(Z_{k+1}) := \{x \in X \cap U \mid \phi(x) < \beta\}$$

and

$$T_\gamma(Z_k) := \{x \in X \cap U \mid \chi(x) < \gamma\}.$$

We denote

$$\partial T_\beta(Z_{k+1}) := \{x \in X \cap U \mid \phi(x) = \beta\}$$

and

$$\partial T_\gamma(Z_k) := \{x \in X \cap U \mid \chi(x) = \gamma\}$$

Now fix $\epsilon > 0$ and $\alpha > 0$ small enough, so that Theorem 9.4.1 is true for $Z_{k+1} - Z_k$, *i.e.* the space $S_\epsilon(x) \cap (Z_{k+1} - Z_k)$ has the homotopy type of a space obtained from $S_\epsilon(x) \cap (Z_{k+1} - Z_k) \cap V_\alpha(L)$ by adding cells of dimension $\geq k + 1 - i$.

We have the lemma:

Lemma 9.4.2 *Let ϵ and α fixed as before. There are β_0 and γ_0 such that, for any β, $\beta_0 \geq \beta > 0$ and any γ, $\gamma_0 \geq \gamma > 0$, the space*

$$M_\gamma := S_\epsilon(x) \cap [(X - T_\gamma(Z_k)) \cup \{(X - Z) \cap V_\alpha(L)\}]$$

is a deformation retract of M and the space

$$N_\beta := S_\epsilon(x) \cap [(X - T_\beta(Z_{k+1})) \cup \{(X - Z) \cap V_\alpha(L)\}]$$

is a deformation retract of N*. Furthermore if* $\beta,\ \beta_0 \geq \beta > 0$*, there is* γ_1 *such that, for any* $\gamma,\ \gamma_1 \geq \gamma > 0$ *the space*

$$N_{\gamma,\beta} := S_\epsilon(x) \cap [(X - T_\gamma(Z_k) \cap T_\beta(Z_{k+1})) \cup (X - Z) \cap V_\alpha(L)]$$

is a deformation retract of M*.*

Proof First notice that the analytic Whitney stratification $\mathcal{S}$ of X induces a subanalytic (actually real analytic) Whitney stratification $\mathcal{S}_\epsilon$ of $S_\epsilon(x) \cap X$ which is adapted to $S_\epsilon(x) \cap Z,\ S_\epsilon(x) \cap X \cap \overset{\circ}{V}_\alpha(L),\ S_\epsilon(x) \cap X \cap \partial V_\alpha(L)$, where

$$\partial V_\alpha(L) := V_\alpha(L) - \overset{\circ}{V}_\alpha(L).$$

Because $\epsilon > 0$ has been chosen small enough so that the sphere $S_\epsilon(x)$ intersects the strata of $\mathcal{S}$ transversally, the stratification $\mathcal{T}_\epsilon$ induced by $\mathcal{S}$ on $S_\epsilon(x) \cap X$ is a Whitney stratification and is evidently adapted to $S_\epsilon(x) \cap Z$.

Then, the function ψ which defines the closed neighbourhoods of L (see the beginning of Sect. 9.4.7), is real analytic, therefore the restrictions of this function to the strata of $\mathcal{T}_\epsilon$ have no critical values in intervals $(0, \alpha)$, with $\alpha > 0$ small enough. Therefore, there is α_0 such that, for any $\alpha,\ \alpha_0 \geq \alpha > 0$ the stratification $\mathcal{T}_\epsilon$ can be refined to a Whitney stratification $\mathcal{S}_\epsilon$ adapted to $S_\epsilon(x) \cap X \cap \overset{\circ}{V}_\alpha(L)$ and $S_\epsilon(x) \cap X \cap \partial V_\alpha(L)$.

Remember that for relative CW complexes, the notions: weak deformation retract, deformation retract, and strong deformation retract coincide, see [47, Cor. 1.4.10, Theorem 1.4.11].

Since χ is real analytic, there is γ_0 such that any $\gamma > 0,\ \gamma_1 \geq \gamma > 0$ is not a critical value of the restrictions of χ to the strata of $\mathcal{S}_\epsilon$. Therefore a positive non-zero vector field v of the interval $(0, \gamma]$ can be lifted to a continuous and integrable vector field w of:

$$S_\epsilon(x) \cap (X - Z_k) - \overset{\circ}{V}_\alpha(L)$$

tangent to the strata of $\mathcal{S}_\epsilon$ which gives a retraction onto:

$$S_\epsilon(x) \cap (X - T_\gamma(Z_k)) - \overset{\circ}{V}_\alpha(L)$$

Considering only the restriction of the vector field w on $S_\epsilon(x) \cap (X - Z_k) \cap \partial V_\alpha(L)$, we obtain that:

$$S_\epsilon(x) \cap (X - T_\gamma(Z_k)) \cap \partial V_\alpha(L)$$

is a deformation retract of:

$$S_\epsilon(x) \cap (X - Z_k) \cap \partial V_\alpha(L).$$

Therefore the space:

$$S_\epsilon(x) \cap (X - T_\gamma(Z_k)) - \overset{\circ}{V}_\alpha(L)$$

is a deformation retract of

$$S_\epsilon(x) \cap [\{(X - Z_k) \cap \partial V_\alpha(L)\} \cup (X - T_\gamma(Z_k))] - \overset{\circ}{V}_\alpha(L)$$

as well as of

$$S_\epsilon(x) \cap (X - Z_k) - \overset{\circ}{V}_\alpha(L)$$

see above. So

$$S_\epsilon(x) \cap [\{(X - Z_k) \cap \partial V_\alpha(L)\} \cup (X - T_\gamma(Z_k))] - \overset{\circ}{V}_\alpha(L)$$

is a (strong) deformation retract of

$$S_\epsilon \cap (X - Z_k) - \overset{\circ}{V}_\alpha(L).$$

This implies that M_γ is a deformation retract of M.

Using the real analytic function ϕ, we prove in the same way that there is β_0 such that, for any $\beta,\ \beta_0 \geq \beta > 0$ the space N_β is a deformation retract of N.

Finally, we repeat the proof that M_γ is a deformation retract of M with $T_\gamma(Z_k) \cap T_\beta(Z_{k+1})$ instead of $T_\gamma(Z_k)$ in order to show that $N_{\gamma,\beta}$ is a deformation retract of M. We start with a vector field w on $S_\epsilon \cap (X \setminus Z_k)$ such that near Z_k, on each stratum $S,\ d\chi(w|S) > 0$ and - if $S \subset X \setminus Z_{k+1}$ - $d\psi(w|S) > 0$; this is possible because $d(\chi|S)_x,\ d(\psi|S)_x$ cannot be $\mathbb{R}$-collinear with a negative ratio, by the Curve Selection Lemma; cf. [14, Lemma 2.13].

In fact we have also proved that:

For any $k,\ 0 \leq k \leq dim_x X$, and, for any $\gamma > 0$ small enough, the space

$$S_\epsilon(x) \cap (X - Z_k) - \overset{\circ}{V}_\alpha(L)$$

retracts onto the space

$$S_\epsilon(x) \cap (X - T_\gamma(Z_k)) - \overset{\circ}{V}_\alpha(L).$$

In particular Lemma 9.4.2 tells us that the pair (M, N) is $(n-i-1)$-connected if and only if the pair $(N_{\gamma,\beta}), N_\beta)$ is $(n - i - 1)$-connected. An immediate consequence of Blakers-Massey homotopy excision theorem (see Theorem 9.4.3) gives:

Lemma 9.4.3 *For* $0 < \beta \leq \beta_0$ *and* $0 < \gamma \leq \gamma_1$, *the pair* $(N_{\gamma,\beta}, N_\beta)$ *is* $(n-i-1)$-*connected if the pair of spaces*

$$(N_{\gamma,\beta} - \overset{\circ}{V}_\alpha(L), N_\beta - \overset{\circ}{V}_\alpha(L))$$

is $(n-i-1)$-*connected.*

Proof of Lemma 9.4.3 Using the notation of Theorem 9.4.3, we consider:

$$U := N_{\gamma,\beta} - \overset{\circ}{V}_\alpha(L)$$

$$V := N_\beta$$

$$W = U \cup V = N_{\gamma,\beta}$$

therefore we have:

$$U \cap V = N_\beta - \overset{\circ}{V}_\alpha(L)$$

and we can apply Blakers-Massey homotopy excision theorem.

Now we observe that:

$$N_{\gamma,\beta} - \overset{\circ}{V}_\alpha(L) = (N_\beta - \overset{\circ}{V}_\alpha(L)) \cup E$$

with $E := S_\epsilon(x) \cap X \cap \{\overline{T}_\beta(Z_{k+1}) - (T_\gamma(Z_k) \cup \overset{\circ}{V}_\alpha(L))\}$ where:

$$\overline{T}_\beta(Z_{k+1}) = T_\beta(Z_{k+1}) \cup \partial T_\beta(Z_{k+1})$$

On the other hand, we have:

$$(N_{\gamma,\beta} - \overset{\circ}{V}_\alpha(L)) \cap E$$

$$= [S_\epsilon(x) \cap X \cap \{\partial T_\beta(Z_{k+1}) - (T_\gamma(Z_k) \cup \overset{\circ}{V}_\alpha(L))\}]$$

$$\cup [S_\epsilon(x) \cap X \cap \{\overline{T}_\beta(Z_{k+1}) - T_\gamma(Z_k)\} \cap \partial V_\alpha(L))]$$

Therefore the Blakers-Massey theorem again with:

$$U := E$$

$$V := N_{\gamma,\beta} - \overset{\circ}{V}_\alpha(L)$$

$$W := U \cup V = M_\gamma - \overset{\circ}{V}_\alpha(L)$$

gives that the pair:

$$(N_{\gamma,\beta} - \overset{\circ}{V}_\alpha(L), N_\beta - \overset{\circ}{V}_\alpha(L))$$

is $(n - i - 1)$-connected if the pair

$$(E, (N_\beta - \overset{\circ}{V}_\alpha(L)) \cap E)$$

is $(n - i - 1)$-connected.

To end the proof of Theorem 9.4.1, it remains to prove that:

Lemma 9.4.4 *The pair*

$$(E, (N_\beta - \overset{\circ}{V}_\alpha(L)) \cap E)$$

is $(n - i - 1)$*-connected.*

Proof of Lemma 9.4.4 We first observe that, because the stratification induced on E by the analytic stratification of X is a subanalytic Whitney stratification, the space E fibres topologically over:

$$B := S_\epsilon(x) \cap [Z_{k+1} - (T_\gamma(Z_k) \cup \overset{\circ}{V}_\alpha(L))]$$

with fibres which are normal slices $\mathcal{N}$ of Z_{k+1} in X and, therefore, contractible. See the Appendix of [22]. Let us call $\pi : E \to B$ this locally trivial topological fibration. This fibration induces a subfibration of:

$$E' := S_\epsilon(x) \cap X \cap \{\partial T_\beta(Z_{k+1}) - (T_\gamma(Z_k) \cup \overset{\circ}{V}_\alpha(L))\}$$

over B. The fibres of this subfibration are the spaces $\partial\mathcal{N} := \mathcal{N} \cap \partial T_\beta(Z_{k+1})$.

The hypothesis about the rectified homotopical depth, $rhd(X - Z) \geq n$, implies that the pairs $(\mathcal{N}, \partial\mathcal{N})$, are $(n - (k + 1) - 1)$-connected (see Lemma 9.4.6).

Define:

$$B' := S_\epsilon(x) \cap (Z_{k+1} - T_\gamma(Z_k)) \cap \partial V_\alpha(L)$$

Now the pair:

$$(B, B') = (S_\epsilon(x) \cap [Z_{k+1} - (T_\gamma(Z_k) \cup \overset{\circ}{V}_\alpha(L))], S_\epsilon(x) \cap (Z_{k+1} - T_\gamma(Z_k)) \cap \partial V_\alpha(L))$$

is $((k + 1) - i - 1)$-connected, when $k \geq i$, because it has the same homotopy type as the pair:

$$(S_\epsilon(x) \cap [Z_{k+1} - (Z_k \cup \overset{\circ}{V}_\alpha(L))], S_\epsilon(x) \cap (Z_{k+1} - Z_k) \cap \partial V_\alpha(L))$$

as γ is small enough (compare to Lemma 9.4.2), and this latter pair is $((k+1)-i-1)$-connected, because of the local Lefschetz type Theorem 9.4.1 applied to $Z_{k+1} - Z_k$ which is non-singular of dimension $k+1$. Theorem 9.4.1 gives the result in our case because, if the space $S_\epsilon(x) \cap (Z_{k+1} - Z_k)$ has the homotopy type of a space obtained from $S_\epsilon(x) \cap (Z_{k+1} - Z_k) \cap V_\alpha(L)$ by adding cells of dimension $\geq (k+1) - i$, then $S_\epsilon(x) \cap [Z_{k+1} - (Z_k \cup \overset{\circ}{V}_\alpha(L))]$ has the homotopy type of a space obtained from $S_\epsilon(x) \cap (Z_{k+1} - Z_k) \cap \partial V_\alpha(L)$ by adding cells of dimension $\geq (k+1) - i$.

Now we apply Lemma 9.4.2.

The inverse image of B' by π is precisely:

$$S_\epsilon(x) \cap X \cap \{\overline{T}_\beta(Z_{k+1}) - T_\gamma(Z_k)\} \cap \partial V_\alpha(L))$$

which implies that:

$$(N_{\gamma,\beta} - \overset{\circ}{V}_\alpha(L)) \cap E = E' \cup \pi^{-1}(B').$$

Thus the pair:

$$(E, (N_\beta - \overset{\circ}{V}_\alpha(L)) \cap E)$$

is $(n-i-1)$-connected, when $k \geq i$, because our data satisfy the conditions of Lemma 9.4.2.

If $k < i$, then either, the space $B' = S_\epsilon(x) \cap (Z_{k+1} - T_\gamma(Z_k)) \cap \partial V_\alpha(L)$ is non-empty and the pair (B, B') is 0-connected, because the components of B' evidently meet the components of B, or it is empty and the space:

$$\pi^{-1}(B') = S_\epsilon(x) \cap X \cap \{\overline{T}_\beta(Z_{k+1}) - T_\gamma(Z_k)\} \cap \partial V_\alpha(L))$$

is empty, so that the pair $(E, N_{\gamma,\beta} - \overset{\circ}{V}_\alpha(L)) \cap E) = (E, E')$ is $(n-(k+1)-1)$-connected, because $(\mathcal{N}, \partial\mathcal{N})$ is $(n-(k+1)-1)$-connected and we have $n-(k+1)-1 \geq n-i-1$. This ends the proof of Lemma 9.4.4 and therefore the proof of Theorem 9.4.1.

Now we have to prove Lemma 9.4.2.

9.4.9 Proof of Lemma 9.4.2

Let $\mathcal{S}$ be an analytic Whitney stratification of X adapted to Z. Consider a sufficiently small neighbourhood U of x in $\mathbb{C}^N$ such that only the strata $X_1, \ldots, X_k$ of $\mathcal{S}$ are meeting U and x belongs to the closure of these strata. Now consider the Zariski open dense subset Ω of the space of affine subspaces L of $\mathbb{C}^N$ containing x which consists of affine subspaces L which intersect all the X_i $(1 \leq i \leq k)$

in a neighbourhood $V \subset U$ transversally except possibly at the point x. The stratification $\mathcal{S}$ induces a Whitney stratification $\mathcal{S}_L$ on $V \cap L \cap X - \{x\}$. There is $\epsilon_0 > 0$, such that, for any $\epsilon_0 \geq \epsilon > 0$, the sphere $S_\epsilon(x)$ is contained in the neighbourhood V and intersects the strata of $\mathcal{S}$ and $\mathcal{S}_L$ transversally in $\mathbb{C}^N$. Therefore for such ϵ, the Whitney stratification $\mathcal{S}$ induces a Whitney stratification $\mathcal{S}_\epsilon$ on $S_\epsilon(x) \cap X$.

Now let $g_1 = \ldots = g_\ell = 0$ be ℓ affine equations which define the affine subspace L in $\mathbb{C}^N$. The map $G := (g_1, \ldots, g_\ell)$ restricted to $S_\epsilon(x) \cap X$ defines a map G_0 from $S_\epsilon(x) \cap X$ into $\mathbb{C}^\ell$. The fibre of G over 0 intersects the strata of $\mathcal{S}_\epsilon$ transversally in $\mathbb{C}^N$, because L belongs to the open set Ω and S_ϵ intersects the strata of $\mathcal{S}$ transversally. By continuity there is a neighbourhood $\mathcal{U}$ of 0 in $\mathbb{C}^\ell$, such that all the fibres of G over points of $\mathcal{U}$ intersect the strata of $\mathcal{S}_\epsilon$ transversally in $\mathbb{C}^N$. The first isotopy theorem of Thom-Mather (see *e.g.* [36, Proposition 11.1]) implies that G_0 induces a trivial topological fibration of:

$$S_\epsilon(x) \cap X \cap G^{-1}(D_\beta(0))$$

onto an open ball $D_\beta(0)$ in $\mathbb{C}^\ell$ of radius β centered at 0 and contained in $\mathcal{U}$, because it is proper and its restrictions to the strata of $\mathcal{S}_\epsilon$ have maximal rank. As the stratification is adapted to Z, this fibration induces on:

$$S_\epsilon(x) \cap (X - Z) \cap G^{-1}(D_\beta(0))$$

a trivial subfibration over the same ball. Therefore we have a trivial fibration:

$$S_\epsilon(x) \cap (X - Z) \cap G^{-1}(\overline{D}_\alpha(0)) \to \overline{D}_\alpha(0)$$

induced by G on any closed ball $\overline{D}_\alpha(0)$, such that $0 < \alpha < \beta$. It means that, for such α, the space $S_\epsilon(x) \cap (X - Z) \cap V_\alpha(L)$ is homeomorphic to the product of $S_\epsilon(x) \cap (X - Z) \cap L$ by $\overline{D}_\alpha(0)$. Then choose $\alpha_\epsilon := \beta$.

This proves Lemma 9.4.2, hence the proof of Theorem 9.4.1 is completed.

9.4.10 Application to a Lefschetz Type Theorem

Let V be a complex subvariety of $\mathbb{P}^N$ of dimension d. Let Z be a subvariety of V. We assume that L is a hyperplane in $\mathbb{P}^N$, then:

Proposition 9.4.1 *Assume that the rectified homotopical depth of $V - Z$ is $\geq n$. Let $\mathcal{V}(L)$ be a good neighbourhood of $L \cap V$ in X relatively to Z. Then the pair*

$$(V - Z, (V - Z) \cap \mathcal{V}(L))$$

is $(n-1)$-connected.

This proposition is an immediate consequence of Theorem 3.4.1 of [22] that we are stating here:

Theorem 9.4.2 *Let X be a compact reduced complex analytic space, Y and Z be closed complex analytic subspaces of X such that $X - Y$ is the union of $c + 1$ open Stein subsets. Assume that the rectified homotopical depth $rhd(X - Y \cup Z)$ is $\geq n$. Let $V(Y)$ be a good neighbourhood of Y in X relatively to Z. Then, the pair $(X - Z, V(Y) - Z)$ is $(n - c - 1)$-connected.*

Notice that $V - L$ is a Stein open subset of V.

Let $\mathcal{S} = (S_\alpha)$ be an analytic Whitney stratification of V adapted to Z. In the case that L is transverse to the strata of $\mathcal{S}$, by using Thom-Mather first isotopy theorem (see [36, Proposition 11.1]), one can prove using Lemma 9.4.2 that the subset $(V - Z) \cap \mathcal{V}(L)$ retracts by deformation onto $(V - Z) \cap L$ (compare with Lemma 9.4.2).

One can try to weaken the hypothesis of transversality. For the case of a Lefschetz theorem for π_0, see [18].

Note that a local Lefschetz theorem has been already proved above, namely Theorem 9.4.1! The hypothesis about homotopical depth there just says that $rhd(X - Z) \geq n$.

Remark 9.4.3

1. In [9] K.-H. Fieseler and L. Kaup consider theorems of Lefschetz type for intersection homology.
2. In [8] C. Eyral uses the notion of homotopical depth to solve one of the conjectures of A. Grothendieck in [13, Exposé 13]. It would be interesting to compare it with the rectified homotopical depth defined above.
3. In [45] J. Schürmann considers rectified homological depth instead of rectified homotopical depth.
4. There are many generalizations of the Lefschetz theorem for hyperplane sections. The reader who is interested in knowing them might look to the bibliography of [10]. For example, one can consult the papers of [46] and [42]. W. Barth obtained other generalizations of the Lefschetz Theorem by different techniques from the ones we use here (see [2], for instance).
5. Many results concern questions of connectivity as we did here when we consider the theorem of Bertini (see Sect. 9.3.1).
6. Of course, there are many other mathematicians who have considered generalizations of the Lefschetz Theorem for hyperplane sections. In these notes we have only considered a topological viewpoint.

9.5 Spaces with Maximal Depth

9.5.1 Definition

We saw that the rectified homotopical depth $rhd(X, x)$ of a complex analytic space X at a point x is bounded by the complex dimension $\dim_x X$ (see the Remark 9.4.8). From Theorem 9.4.1 we obtain evidently:

Lemma 9.5.1 *If the complex analytic space X is non-singular at the point x, we have:*

$$rhd(X, x) = \dim_x X$$

Proof It suffices to consider locally the trivial stratification of X with the one stratum X.

In this chapter we are going to characterize complex analytic spaces X for which the rectified homotopical depth is maximal, *i.e.* at each point x of X, we have $rhd(X, x) = \dim_x X$.

Notice that this hypothesis implies that each connected component of the space X is equidimensional since by definition the dimension of X is the biggest dimension of the irreducible components, so $x \mapsto \dim(X, x)$ is upper semicontinuous, whereas $x \mapsto rhd(X, x)$ is lower continuous, so $x \mapsto \dim(X, x)$ is continuous in our case.

The key result which leads us to a nice topological characterization of complex analytic spaces X with maximal rectified homotopical depth is a theorem of [22] (Theorem 4.1.2) which expresses the rectified homotopical depth in terms of connectivity of normal Morse data that M. Goresky and R. MacPherson introduce to give a stratified Morse theory (*cf.* [12]).

9.5.2 Stratified Morse Data

Let X be a complex analytic space. We consider an analytic Whitney stratification $\mathcal{S} = (X_i)_{i\in I}$ of X. In Sect. 9.4.2 (see Lemma 9.4.6) we have associated a normal slice in X to each stratum of such a stratification $\mathcal{S}$.

Following M. Goresky and R. MacPherson [12, §1.5 Chap.1 p.15] we shall define the complex link in X of a stratum of $\mathcal{S}$.

Consider a stratum X_i of $\mathcal{S}$, a point x in X_i and a local embedding of an open neighbourhood of x in X in $\mathbb{C}^N$. Let d_i be the complex dimension of the stratum X_i. We have (compare with (2.3.5) of [32]):

Theorem 9.5.1 *In the space of linear projections of $\mathbb{C}^N$ onto $\mathbb{C}^{d_i+1}$, there is an open dense set Ω, such that, for any $p \in \Omega$, there is $\epsilon_0 > 0$, such that, for any ϵ, $\epsilon_0 > \epsilon > 0$, there is α_ϵ, such that, for any α, $\alpha_0 > \alpha > 0$, the projection p induces a map p_0 of $U := B_\epsilon(x) \cap X \cap p^{-1}(D_\alpha(p(x)))$ into the open ball $V :=$*

$D_\alpha(p(x))$ and p_0 is a locally trivial fibration over $V - p_0(X_i \cap U)$. Furthermore the homotopy type of the general fibre $\mathcal{L}$ of this fibration is an analytic invariant of the germ (X, x).

Then Theorem 9.5.1 leads to the following definition:

Definition 9.5.2 A general fibre $\mathcal{L}$ as defined in Theorem 9.5.1 is called a complex link of the stratum X_i in X.

Notice that, for a general projection p of $\mathbb{C}^N$ onto $\mathbb{C}^{d_i+1}$, the inverse image $p^{-1}(D)$ by p of a general affine line D through $p(x)$ transverse to $p(X_i)$ contained in $\mathbb{C}^{d_i+1}$ is transverse to the stratum X_i. Therefore we can always find normal slices of a stratum X_i in X (cf 3.2.5) which contain a complex link of X_i in X. We define (compare with [12] §2.4 chap.2, Corollary 1 p.166):

Definition 9.5.3 The pair $(\mathcal{N}, \mathcal{L})$ of a normal slice $\mathcal{N}$ of a Whitney stratum X_i in X and of a complex link $\mathcal{L}$ of X_i in X which is contained in $\mathcal{N}$ is called a normal Morse data of the stratum X_i in X.

One can show (see [12] Part II, Chap.2 §2.3; compare to [32] §3) that the homotopy type of a complex link of a Whitney stratum is the same along the stratum and Theorem 9.5.1 tells us that it is an analytic invariant of the germ of X at any point of the stratum.

We can now state a theorem which tells us how to get the rectified homotopical depth in terms of connectivity of normal Morse data of M. Goresky and R. MacPherson.

Theorem 9.5.4 *Let X be a complex analytic space and x be a point in X. Let $\mathcal{S}$ be an analytic Whitney stratification of X. The following conditions are equivalent:*

1. *$rhd(X, x) \geq n$;*
2. *for any stratum S which contains x in its closure, the normal Morse data $(\mathcal{N}, \mathcal{L})$ of S in X is $(n - \dim S - 1)$-connected.*

Proof We first show the implication (2) $\Rightarrow$ (1). It is proved by induction on $\dim_x X$. If $\dim_x X = 0$, there is nothing to be proved. So suppose that $\dim_x X \geq 1$, and that the theorem is true for any germ of a complex analytic space of dimension $< \dim_x X$. Let X_i be the union of strata of X of dimension $\leq i$ which contain the point x in their closures. Consider an open neighbourhood U of x in X which meets only these strata adherent to x. If $\mathcal{N}$ is a normal slice of a stratum S_j of X_i at a point x_i in $U \cap X_i$, we prove that $(\mathcal{N}, \mathcal{N} - \{x_i\})$ is $(n - i - 1)$-connected and, by Lemma 9.4.6, it will imply $rhd(X, x) \geq n$.

First, notice that the normal slice $\mathcal{N}$ has a Whitney stratification induced by $\mathcal{S}$, one can prove that complex links in $\mathcal{N}$ of induced strata $X_j \cap \mathcal{N}$ by $\mathcal{S}$ in $\mathcal{N}$ are complex links of X_j in X.

Therefore, if $i \geq 1$, we have $rhd(\mathcal{N}, x_i) \geq n - i$, by the induction hypothesis applied to the germ $(\mathcal{N}, x_i)$. This implies that, for $i \geq 1$, the pair $(\mathcal{N}, \mathcal{N} - \{x_i\})$ is $(n - i - 1)$-connected and this shows that $rhd(X - X_0) \geq n$.

It remains to consider the slice and the complex link at $x \in X_0$. Embed locally X into $\mathbb{C}^N$, the assertion (2) implies that, for any ϵ small enough, the pair:

$$(B_\epsilon(x) \cap X, \mathcal{L})$$

is $(n-1)$-connected. We have the following strong local Theorem of Lefschetz type, proved by the authors in [22] (Theorem 2.12):

Theorem 9.5.5 (Strong Local Lefschetz Type Theorem) *Let X be a complex analytic space embedded in $\mathbb{C}^N$. Consider a closed complex analytic subspace Z of X and a point z of X and suppose that $rhd(X - Z) \geq n$. Then there is a Zariski open dense subset Ω in the projective space of linear hyperplanes of $\mathbb{C}^N$, such that, for any $H \in \Omega$ and any affine function g such that Ker $dg_z = H$, there is $\epsilon_0 > 0$ such that, for any ϵ, $\epsilon_0 > \epsilon > 0$, there is η_ϵ, such that, for and any t, $0 < |t| < \eta_\epsilon$, the pair*

$$(B_\epsilon(z) \cap (X - Z), B_\epsilon(z) \cap (X - Z) \cap H_t),$$

with $H_t := \{g = t\}$, is $(n-1)$-connected.

As the complex link $\mathcal{L}$ of $\{x\}$ in X is given by a general linear form, we can apply this theorem with $Z := \{x\}$. As $rhd(X - X_0) \geq n$, it gives that the pair $(B_\epsilon(x) \cap X - \{x\}, \mathcal{L})$ is $(n-1)$-connected.

As $B_\epsilon(x) \cap X$ is contractible, together with the hypothesis (2), it implies that the pair $(B_\epsilon(x) \cap X, B_\epsilon(x) \cap X - \{x\})$ is $(n-1)$-connected. Thus we obtained $rhd(X, x) \geq n$.

Now let us prove the implication (1) $\Rightarrow$ (2). Consider S a stratum of $\mathcal{S}$. The rectified homotopy depth of a normal slice $\mathcal{N}$ of S in X at a point s is $\geq n - \dim S$, because $\mathcal{N}$ is obviously defined by $\dim S$ holomorphic equations in a neighbourhood of s in X and we can apply the following Purity Theorem, conjectured by A. Grothendieck and proved in [22] (Theorem 3.2.1):

Theorem 9.5.6 *Let X be a reduced complex analytic space, such that $rhd(X) \geq n$, and let Y be a complex analytic subspace locally defined on X by i complex analytic equations. Then $rhd(Y) \geq n - i$.*

Now it suffices to apply Theorem 9.5.5 to obtain the desired result, because a complex link of $\{s\}$ in $\mathcal{N}$ is a complex link of S in X. This ends the proof of Theorem 9.5.4.

9.5.3 Characterization of Maximal Rectified Depth

Now we can give the main result of this chapter which is a consequence of Theorem 9.5.4:

Theorem 9.5.1 *Let X be a complex analytic space and x be a point in X. Let $\mathcal{S}$ be an analytic Whitney stratification of X. The following conditions are equivalent:*

1. $rhd(X, x) = \dim_x X = n$;
2. *for any stratum S which contains x in its closure, a complex link $\mathcal{L}$ of S in X has the homotopy type of a bouquet of spheres of real dimension $n - \dim S - 1$.*

Proof Theorem 9.5.4 gives the equivalence of $rhd(X, x) = \dim_x X$ and, for each integer k, $0 \leq k \leq \dim_x X$, the $(\dim_x X - k - 2)$-connectivity of the complex links in X of the strata of dimension k in $\mathcal{S}$. These complex links have the homotopy type of closed analytic subspaces of euclidian balls, therefore they are Stein spaces and have the homotopy type of a CW-complex of real dimension $\leq \dim_x X - k - 1$ (see [15]).

Then, Theorem 9.5.1 is a consequence of the following Lemma (compare with [39] Proof of Theorem 6.5):

Lemma 9.5.2 *A $(d - 1)$-connected CW-complex E of dimension d has the homotopy type of a bouquet of spheres of (real) dimension d.*

Proof of Lemma 9.5.2 The integral homology groups $H_d(E, \mathbb{Z})$ must be free abelian, since any torsion elements would contribute to non-zero cohomology classes in dimension $d + 1$ which would contradict the hypothesis that E is a CW-complex of dimension d. If $d \geq 2$, the space E is simply connected, therefore the Hurewicz Theorem shows that the homotopy group $\pi_d(E, e)$ is isomorphic to $H_d(E, \mathbb{Z})$. We can choose finitely many maps

$$(\mathbb{S}^d, e_0) \to (E, e)$$

which represent a basis of the free abelian homotopy group $\pi_d(E, e)$. These maps define a map

$$\mathbb{S}^d \vee \ldots \vee \mathbb{S}^d \to E$$

which induces an isomorphism of integral homology groups. By a theorem of Whitehead it is a homotopy equivalence. This proves Lemma 9.5.2 when $d \geq 2$. The cases $d \leq 1$ are elementary and left to the reader.

In [32, §3] we give the definition of the local vanishing homotopy type of a complex analytic germ (X, x) which is the finite family of homotopy types of vanishing fibres of (X, x). We can translate Theorem 9.5.1 in terms of the homotopy type of these vanishing fibres:

Corollary 9.5.3 *Let X be a complex analytic space and x be a point in X. The following conditions are equivalent:*

1. $rhd(X, x) = \dim_x X$;

2. *there is an open neighbourhood U of x in X, such that, for any $y \in U$, vanishing fibres of (X, y) have the homotopy type of bouquets of spheres of (real) dimension equal to the complex dimension of the corresponding fibre.*

Proof Let us fix a Whitney stratification $\mathcal{S}$ of X. Let $S_1, \dots, S_k$ be the strata of $\mathcal{S}$ which contain the point x in their closure. Consider an open neighbourhood V of x in X, such that a stratum of $\mathcal{S}$ meets V if and only if it contains x in its closure.

First we prove the implication (2) ⇒ (1). According to Theorem 9.5.1, we have to prove that, for any i, $1 \leq i \leq k$, the complex links of S_i in X are bouquets of spheres of dimension $\dim_x X - \dim_x S_i - 1$. Let x_i be a point of the stratum S_i. By definition a vanishing fibre of (X, x_i) of codimension $\dim S_i + 1$ is a complex link of S_i in X. Therefore, by hypothesis, it is a bouquet of spheres of dimension $\dim X - \dim S_i - 1$.

Now let us prove (1) ⇒ (2). So assume (1). According to Theorem 9.5.1, for any i, $1 \leq i \leq k$, the complex links of S_i in X are bouquets of spheres of dimension $\dim_x X - \dim_x S_i - 1$. Let y be a point of V. There is a unique S_j, $1 \leq j \leq k$, such that $y \in S_j$. Vanishing fibres of X at y of codimension $\leq \dim S_j$ are contractible, because they are homeomorphic to the product of a vanishing fibre of X at y of codimension $\dim S_j$ by an euclidean ball of adequate dimension and, when $y \in S_j$, a vanishing fibre of (X, y) of codimension $\dim S_j$ is contractible. By Theorem 9.5.1, a vanishing fibre of X at y of codimension $\dim S_j + 1$, which is a complex link of S_j in X, is a bouquet of spheres of (real) dimension $\dim X - \dim S_j - 1$.

Recall that a vanishing fibre of (X, x) of codimension r $(1 \leq r \leq \dim X)$ is obtained by considering a general linear projection p from $\mathbb{C}^N$ onto $\mathbb{C}^r$ and its restriction to X.

Let $\mathcal{S} = (S_1, \dots, S_k)$ be a Whitney stratification of X. We assume that each stratum S_i of $\mathcal{S}$ contains x in its closure. If ϵ is small enough, the restriction of p to $S_i \cap B_\epsilon(x)$ has a critical space Γ_i which is either empty or reduced of dimension $\inf(r - 1, \dim X_i)$ and the restriction of p to Γ_i is finite. Call Δ_i its image by p. Then there is $\eta > 0$ small enough, such that p induces a topological fibration:

$$\varphi_r \colon X \cap B_\epsilon(x) \cap p^{-1}(D_\eta(p(x)) - \cup_{1 \leq i \leq k} \Delta_i) \to D_\eta(p(x)) - \cup_{1 \leq i \leq k} \Delta_i,$$

where $D_\eta(p(x))$ is the open ball of $\mathbb{C}^r$ centered at the point $p(x)$ and of radius η. The general fibre of φ_r is a vanishing fibre of codimension r.

One can obtain from this description the vanishing fibre of codimension $r - 1$, if $2 \leq r \leq \dim X$. Precisely let

$$q \colon \mathbb{C}^r \to \mathbb{C}^{r-1}$$

a general projection. There is $\theta > 0$ small enough and a hypersurface Θ of the open ball $D'_\theta(q(p(x)))$, such that, for any point u in $D'_\theta(q(p(x))) - \Theta$, the space

$$X \cap B_\epsilon(x) \cap (q \circ p)^{-1}(u)$$

is a vanishing fibre of codimension $r-1$. The following lemma allows us to do an induction:

Lemma 9.5.4 *Suppose that* $rhd(X, x) = \dim_x X$. *Let* r *be an integer,* $2 \leq r \leq \dim_x X$. *The vanishing fibre of codimension* $r-1$ *of* (X, x) *has the homotopy type of a space obtained from the vanishing fibre of codimension* r *by adding cells of dimension* $\dim X - r + 1$.

First let us show how the lemma gives us (2). We operate by descending induction on the integer r. For $r = 1$, either $\{x\}$ is a stratum of dimension 0, and the complex link of $\{x\}$ in X is a bouquet of spheres of dimension $\dim_x X - 1$ by Theorem 9.5.1, or x belongs to a stratum of dimension ≥ 1 and the vanishing fibre of (X, x) of codimension 1 is contractible, according to what we have observed above. In both cases a vanishing fibre of codimension 1 is $(\dim_x X - 2)$-connected.

By descending induction on the integer $r \geq 2$, a vanishing fibre $F^{(r-1)}$ of (X, x) of codimension $r-1$ being $(\dim_x X - r)$-connected, Lemma 9.5.4 says that the pair $(F^{(r-1)}, F^{(r)})$ is $(\dim X - r)$-connected which implies that a vanishing fibre $F^{(r)}$ of (X, x) of codimension r is $(\dim X - r - 1)$-connected. As $F^{(r)}$ is a Stein space, Theorem 9.2.15 tells us that $F^{(r)}$ has the homotopy type of a CW-complex of dimension $\dim X - r$. Lemma 9.5.2 shows that $F^{(r)}$ has the homotopy type of a bouquet of spheres of dimension $\dim X - r$. To end the proof of Corollary 9.5.3, it remains to prove Lemma 9.5.4.

9.5.4 Proof of Lemma 9.5.4

We consider the description of vanishing fibres of codimension r and $r-1$ that we did before the statement of Lemma 9.5.4. The inverse image by q of the point u is an affine line L_u in $\mathbb{C}^r$. We have:

$$X \cap B_\epsilon(x) \cap (q \circ p)^{-1}(u) = X \cap B_\epsilon(x) \cap p^{-1}(L_u)$$

Let L be the affine line $q^{-1}(q(p(x)))$. Because it is a general line of $\mathbb{C}^r$, if the point $p(x)$ belongs to the intersection of L and the space $\cup_{1 \leq i \leq k} \Delta_i$, it is isolated in this intersection. Let us first suppose that the point $p(x)$ belongs to the intersection of L and the space $\cup_{1 \leq i \leq k} \Delta_i$. Then there is an open disk D centered at $p(x)$, contained in the line L, which does not contain other intersection points of L and the space $\cup_{1 \leq i \leq k} \Delta_i$ but $p(x)$. Let us fix a linear section s of q which contains $p(x)$, *i.e.* $s(q(p(x))) = p(x)$. Then, if u is sufficiently near to $q(p(x))$, the translated disk $D + s(u)$ is contained in L_u and contains all the points of the intersection of L_u and $\cup_{1 \leq i \leq k} \Delta_i$ which specialize to $p(x)$. One can prove the following Lemma:

Lemma 9.5.1 *The inclusion*

$$X \cap B_\epsilon(x) \cap p^{-1}(D + s(u)) \subset X \cap B_\epsilon(x) \cap p^{-1}(L_u)$$

is a homotopy equivalence.

The proof of this lemma proceeds as for the proof of (2.3.10) in [32].

Now let us end the proof of Lemma 9.5.4. If u is sufficiently general, the intersection points of the affine line L_u and $\cup_{1\leq i\leq k}\Delta_i$ are non-singular points of the reduced hypersurface components of $\cup_{1\leq i\leq k}\Delta_i$. Therefore above a non-singular point of $L_u \cap \Delta_i$, there is a non-singular point of Γ_i, which is the critical space of the restriction of p to the stratum X_i. This means that the projection p induces a function p_1 from $X \cap B_\epsilon(x) \cap p^{-1}(D + s(u))$ onto $D + s(u)$, whose restrictions to the strata of $X \cap B_\epsilon(x) \cap p^{-1}(D + s(u))$ induced by $\mathcal{S}$ have maximal rank except at the points of Γ_i which project onto the points of $L_u \cap \Delta_i$. By the choice made on ϵ the fibres of p_1 are transverse to the strata induced on $X \cap S_\epsilon(x) \cap p^{-1}(D + s(u))$.

Now let t in $(D + s(u)) - \cup_{1\leq i\leq k}\Delta_i$. The fibre of φ_r above the point t, i.e.

$$X \cap B_\epsilon(x) \cap p^{-1}(t)$$

is a vanishing fibre of codimension r of (X, x). Let t be chosen in such a way that the affine segments E_j which join t and the points of $L_u \cap (\cup_{1\leq i\leq k}\Delta_i)$ have only the point t in common.

Let us call E the union $\cup E_j$ of these segments. The Thom-Mather first isotopy theorem gives that the space $X \cap B_\epsilon(x) \cap p_1^{-1}(D + s(u))$ which has the same homotopy type as a vanishing fibre of codimension $r - 1$ retracts by deformation onto the space $X \cap B_\epsilon(x) \cap p_1^{-1}(E)$. To prove Lemma 9.5.4 it is enough to prove that for each segment E_j in E the pair of spaces

$$(X \cap B_\epsilon(x) \cap p_1^{-1}(E_j), X \cap B_\epsilon(x) \cap p_1^{-1}(t))$$

is $(\dim X - r)$-connected. To prove this fact we use the Stratified Morse Theory of M. Goresky and R. MacPherson and this result is actually an immediate consequence of Theorem 3.2 of Part II, Chapter 3 of [G-M], which states that the local Morse data (J, K) of p_1 at a singular point on the stratum S_i is homeomorphic to the product of $(\mathbb{B}^\lambda, \mathbb{S}^{\lambda-1})$, where $\lambda = \dim S_i - r$, and $(cone(\mathcal{L}), \mathcal{L})$, where $\mathcal{L}$ is the complex link of S_i in X.

Theorem 9.5.1 tells us that if $rhd(X, x) = \dim_x X$, the complex links of a Whitney stratifications have the homotopy type of bouquets of spheres of middle dimension. As $\dim \mathcal{L} = \dim X - \dim S_i - 1$, the pair $(cone(\mathcal{L}), \mathcal{L})$ is $(\dim X - \dim S_i - 1)$-connected. The pair $(\mathbb{B}^\lambda, \mathbb{S}^{\lambda-1})$ is obviously $(\lambda-1)$-connected, therefore Lemma 9.4.2 implies that the local Morse data (J, K) is $(\dim X - r - 1)$-connected. This ends the proof of Lemma 9.5.4 and the proof of Corollary 9.5.3.

9.6 Spaces with Milnor Property

In this chapter we follow [31] and give a generalization of Milnor's result on the fibres of holomorphic functions with isolated singularities. First we define functions with isolated singularities as in [30].

9.6.1 Basic Results

Let X be a complex analytic space and x be a point of X.

Definition 9.6.1 We say that a complex analytic function f defined on X has an isolated singularity at the point $x \in X$ if there is a Whitney stratification $\mathcal{S}$ of X and an open neighbourhood U of x, such that the restriction of f to the intersection of U with the strata of $\mathcal{S}$ has rank 1 at every point of $U - \{x\}$.

Of course this definition generalizes the definition of an isolated critical point for a function defined on a non-singular space. In [39], J. Milnor shows that near an isolated critical point x of a complex analytic function f defined on a non-singular complex analytic space of dimension n, one can define the general fibre of f at the point x, which we call the Milnor fibre of f at x, and this fibre has the homotopy type of a bouquet of spheres of real dimension $n - 1$. The main result of this section is to generalize this result of Milnor to complex analytic functions defined on an arbitrary complex analytic space.

The first problem is to show the existence of a general fibre for a complex analytic function. In fact the hypothesis of isolated singularity is not needed to define this general fibre. In [28] D. T. Lê (see the Chap. 6 "The Topology of the Milnor Fibration" by Lê D. T., J. J. Nuño Ballesteros and J. Seade in this book) proved:

Theorem 9.6.2 *Let X be a complex analytic space embedded in $\mathbb{C}^N$ and x be a point of X. Consider a complex analytic function f defined on X. There is ϵ_0, such that, for any ϵ, $0 < \epsilon < \epsilon_0$, there is η_ϵ, such that for any η, $0 < \eta < \eta_\epsilon$, the function f induces a map:*

$$\psi_{\epsilon,\eta} : X \cap B_\epsilon(x) \cap f^{-1}(D_\eta(f(x))) \to D_\eta(f(x))$$

which is a locally trivial topological fibration above the punctured disc:

$$D^*_\eta(f(x)) := D_\eta(f(x)) - \{f(x)\}.$$

Furthermore in [32, (2.3.3)] we notice that these fibrations are isomorphic for any $(\epsilon, \eta) \in A$, where A is a semi-analytic subset of $\mathbb{R}^2_+$ of non-empty interior which contains $[0, \epsilon_0] \times \{0\}$ in its closure. As we prove in [32, (2.3.2)], the homotopy type of the fibre of the fibration described in the Theorem 9.6.2 is an analytic invariant of the germ of f at x.

Definition 9.6.3 For any $(\epsilon, \eta) \in A$, we call general fibre of f at x the fibre of $\psi_{\epsilon,\eta}$ over a point of the punctured disk $D^*_\eta(f(x))$.

A general fibre of a complex analytic function $f : X \to \mathbb{C}$ is the Milnor fibre of f at x, when X is non-singular at x. By definition we call general fibre of a germ of a complex analytic function at x a general fibre at this point of a representative of this germ.

In general there is no reason to hope that the general fibre of the germ of a complex analytic function has the homotopy type of a bouquet of spheres. But on a space with maximal depth, general fibres of germs of complex analytic functions with isolated singularity have the homotopy type of a bouquet of spheres of middle dimension. Precisely we have:

Theorem 9.6.4 *Let X be a complex analytic space and x be a point of X. Consider a complex analytic function f defined on X and suppose that it has an isolated singularity at x. Then if $rhd(X, x) = \dim_x X$, a general fibre of f at x has the homotopy type of a bouquet of spheres of real dimension* $\dim_x X - 1$.

Proof The proof of this theorem is similar to the proof of the result announced in [29] for local complete intersections. Assume that the function f has an isolated singularity at x relatively to a Whitney stratification $\mathcal{S}$ of X. First, notice, as it has been done in [30], that the function f has an isolated singularity at a point x relatively to the Whitney stratification $\mathcal{S}$ if and only if there are a local embedding of X in $\mathbb{C}^N$, a complex analytic function F defined on an open neighbourhood V of x in $\mathbb{C}^N$ whose restriction to $V \cap X$ equals the restriction of f to $V \cap X$ and such that the image $Im(dF)$ of the differential dF in the cotangent space $\mathbb{T}^*(\mathbb{C}^N)$ of $\mathbb{C}^N$ intersects the union of conormal spaces $\mathbb{T}^*_{X_i}(\mathbb{C}^N)$ of the strata X_i of $\mathcal{S}$ at an isolated point (x, ℓ) above x:

$$(\cup_i \mathbb{T}^*_{X_i}(\mathbb{C}^N)) \cap Im(dF) = \{(x, \ell)\}$$

In [30], we show that, if l is a general linear form of $\mathbb{C}^N$, in a neighbourhood $\mathcal{U}$ of the point (x, ℓ) in the cotangent space $\mathbb{T}^*(\mathbb{C}^N)$, for any $\tau \neq 0$ small enough, the image of the differential $Im(dF + \tau l)$ is transverse to the union of conormal spaces $\cup_i \mathbb{T}^*_{X_i}(\mathbb{C}^N)$.

Precisely, if $\epsilon > 0$ and $\eta > 0$ are chosen as in Corollary 9.5.3 for the fibration φ_r, there is $\boldsymbol{\tau}_0 > 0$, such that, for any $\tau \in \mathbb{C}$, $\boldsymbol{\tau}_0 > |\tau| > 0$, the fibres of the restriction of the function $F + \tau l$ to X over the points of $D_\eta(f(x))$ are transverse to the Whitney strata induced on $X \cap S_\epsilon(x)$ by $\mathcal{S}$ and, using Thom-Mather's first isotopy theorem, it can be shown that, with τ small enough, the spaces $X \cap B_\epsilon(x) \cap F^{-1}(u)$ and $X \cap B_\epsilon(x) \cap (F + \tau l)^{-1}(t)$, with t and u general in the disk $D_\eta(f(x))$, are homeomorphic and the spaces:

$$X \cap B_\epsilon(x) \cap F^{-1}(D_\eta(f(x))$$

and:

$$X \cap B_\epsilon(x) \cap (F + \tau l)^{-1}(D_\eta(f(x))$$

are homeomorphic and in fact contractible. On another hand the isolated singular points of the restriction of the function $F + \tau l$ to X are points of the Whitney strata where the restrictions of $F + \tau l$ to these strata have ordinary quadratic points. This last assertion translates the transversality in $\mathcal{U}$ of $Im(dF + \tau l)$ and the conormal spaces $\cup_i \mathbb{T}^*_{X_i}(\mathbb{C}^N))$. Let

$$\mathcal{U} \cap Im(dF + \tau l) \cap (\cup_i \mathbb{T}^*_{X_i}(\mathbb{C}^N)) = \{(x_1, \ell_1), \ldots, (x_\nu, \ell_\nu)\}.$$

Now the function $F + \tau l$ induces a map:

$$\psi_\tau : X \cap B_\epsilon(x) \cap (F + \tau l)^{-1}(D_\eta(f(x)) \to D_\eta(f(x))$$

This map has isolated singularities at $x_1, \ldots, x_\nu$. As in the proof of Lemma 9.5.4, let t be a general point of $D_\eta(f(x))$, such that the segments E_i which join the point t to the points $y_i := (F + \tau l)(x_i)$ $(1 \leq i \leq \nu)$ have only the point t in common. Call E the union of these segments. The space

$$X \cap B_\epsilon(x) \cap (F + \tau l)^{-1}(D_\eta(f(x)))$$

retracts by deformation onto $X \cap B_\epsilon(x) \cap (F + \tau l)^{-1}(E)$. As in the proof of Lemma 9.5.4, the pair

$$(X \cap B_\epsilon(x) \cap (F + \tau l)^{-1}(E), X \cap B_\epsilon(x) \cap (F + \tau l)^{-1}(t))$$

is $(\dim X - 1)$-connected if the pairs

$$(X \cap B_\epsilon(x) \cap (F + \tau l)^{-1}(E_i), X \cap B_\epsilon(x) \cap (F + \tau l)^{-1}(t))$$

are $(\dim X - 1)$-connected for $1 \leq i \leq \nu$.

This again reduces to the local study near each isolated singular point x_i of the restriction to X of the function $F + \tau l$, the hypothesis about the transversality of the image of the differential $d(F + \tau l)$ with the union of conormal spaces at (x_i, ℓ_i) means that the restriction of the function $F + \tau l$ to the stratum containing x_i has a Morse point at x_i.

We make again use of the Stratified Morse Theory of M. Goresky and R. MacPherson and apply Theorem 3.2 of Part II, Chapter 3 of [12] to prove that the general fibre at x_i of the restriction to X of the function $F + \tau l$ is $(\dim X - 2)$-connected. It is enough to check that at each point x_i the local Morse data is $(\dim X - 1)$-connected.

There is a locally trivial fibration π of a neighbourhood U of x_i in X onto $U \cap S_j$, where S_j is the stratum which contains x_i, with slices of S_j in X as fibres and a subfibration ∂U onto $U \cap S_j$ with complex links of S_j in X as fibres.

The local Morse data at x_i is a pair $(E, E' \cap \pi^{-1}(\mathbb{S}))$, where E is the inverse image by π of a closed ball $\mathbb{B}$ in S_j centered at x_i and of real dimension $\dim S_j$, E' is the intersection of E with ∂U and $\mathbb{S}$ is the boundary of $\mathbb{B}$.

Lemma 9.4.2 shows that, complex links of S_j in X being $(\dim X - \dim S_j - 2)$-connected, and $(\mathbb{B}, \mathbb{S})$ being $(\dim S_j - 1)$-connected, if $\dim S_j \geq 1$, we have that $(E, E' \cap \pi^{-1}(\mathbb{S}))$ is $(\dim X - 1)$-connected. If $\dim S_j = 0$, *i.e.* $S_j = \{x\}$, the pair $(E, E' \cap \pi^{-1}(\mathbb{S})) = (E, E')$ is $(\dim X - 1)$-connected, as the complex link of $\{x\}$ in X is $(\dim X - 2)$-connected. This implies that the pair:

$$(X \cap B_\epsilon(x) \cap (F + \tau l)^{-1}(E_i), X \cap B_\epsilon(x) \cap (F + \tau l)^{-1}(t))$$

is $(\dim X - 1)$-connected. Therefore the pair:

$$(X \cap B_\epsilon(x) \cap (F + \tau l)^{-1}(E), X \cap B_\epsilon(x) \cap (F + \tau l)^{-1}(t))$$

is also $(\dim X - 1)$-connected as well as the pair:

$$(X \cap B_\epsilon(x) \cap (F + \tau l)^{-1}(D_\eta(f(x))), X \cap B_\epsilon(x) \cap (F + \tau l)^{-1}(t))$$

With what we have observed above, this implies that, with $u \in D_\eta(f(x))$ general, the pair:

$$(X \cap B_\epsilon(x) \cap F^{-1}(D_\eta(f(x)), X \cap B_\epsilon(x) \cap F^{-1}(u))$$

is $(\dim X - 1)$-connected. The contractibility of $X \cap B_\epsilon(x) \cap F^{-1}(D_\eta(f(x))$ and the fact that the space $X \cap B_\epsilon(x) \cap F^{-1}(u)$ is a CW-complex of dimension $\leq \dim X - 1$, because of Theorem 9.2.15, imply that $X \cap B_\epsilon(x) \cap F^{-1}(u)$ is $(\dim X - 2)$-connected, so that it has the homotopy type of a bouquet of spheres of dimension $\dim X - 1$ by Lemma 9.5.2. This ends the proof of Theorem 9.6.4.

9.6.2 Main Result

To be complete, it remains to understand on which spaces a theorem similar to the one of Milnor is true. We end this chapter by defining the spaces on which we have a theorem similar to Theorem 9.6.4.

In the proof of Theorem 9.6.4 the main point is to consider the connectivity of the local Morse data $(E, E' \cap \pi^{-1}(\mathbb{S}))$. We obtain:

Theorem 9.6.1 *Let X be a complex analytic space and x be a point of X. There is an open neighbourhood U of x in X such that, for any y in U, a general fibre* *of a complex analytic function f defined in a neighbourhood of y having at y an isolated singularity has the homotopy type of a bouquet of spheres of real dimension* $\dim_x X - 1$ *if and only if the complex link of $\{x\}$ in X has the homotopy type of a bouquet of spheres of middle dimension and the complex links of the strata of dimension ≥ 1 of a Whitney stratification of X which contain x in their closures have the homology type of a bouquet of spheres of middle dimension.*

Proof The condition is sufficient as we apply Lemma 9.4.2 to estimate the connectivity of the local Morse data and, from Lemma 9.4.2, it is only needed that the complex links are homologically like bouquets of spheres.

The condition is necessary, because the local Morse data $(E, E' \cap \pi^{-1}(S))$ coming from strata of dimension ≥ 1 are $(\dim X - 1)$-connected if and only if the complex links of these strata in X have the homology of a bouquet of spheres of middle dimension, because, in the case $\dim X \geq 3$, the first non-zero relative homotopy group is isomorphic to the first non-zero relative homology group which is obtain by tensor product, by the Künneth formula. If $\dim X \leq 2$, the argument is left to the reader.

In the case that we are interested in the rectified homological depth instead of the rectified homotopical depth, we have the following nice formulation:

Corollary 9.6.2 *Let X be a complex analytic space and x be a point of X. There is an open neighbourhood U of x in X such that, for any y in U, a general fibre of a complex analytic function f defined in a neighbourhood of y having at y an isolated singularity has the homology of a bouquet of spheres of real dimension* $\dim_x X - 1$ *if and only if the rectified homological depth X at x equals the complex dimension of X at x.*

Proof This is an immediate consequence of Theorem 9.6.1 and the version of Theorem 9.5.1 for rectified homological depth.

9.6.3 Remarks and Problems

When there is an analytic Whitney stratification of the space X and a stratum of this Whitney stratification which has dimension ≥ 1 which contains x, the complex link of $\{x\}$ in X is contractible. In this case the conditions in Theorem 9.6.1 are only homological. For instance, this is the case, when a space X has maximal rectified homological depth, for the product of X with an open subset of the complex line.

For a hypersurface in $\mathbb{C}^{n+2}$ with one dimensional singular locus which is a family of hypersurfaces of $\mathbb{C}^{n+1}$ with Milnor number constant, but for which the stratification by the non-singular part and the singular locus is not a Whitney stratification, the complex link in the hypersurface of any point of the singular locus is contractible.

Let X be a reduced complex analytic space with maximal rectified homotopical (resp. homological) depth. Suppose that x is singular and is an isolated singular point in X, is it true that the complex link in X of x is never contractible? This is true if X is a local complete intersection or X is a normal surface at x according to a theorem of Gonzalez-Sprinberg [11]. How can one calculate algebraically the number of spheres in the different bouquets which appear when the rectified depth is maximal?

In view of theorems of Lefschetz type, spaces with maximal rectified homotopical depth or maximal rectified homological depth, have the best level of comparison for the homotopy and the homology between the space and its hyperplane sections. It is remarkable that on these spaces a theorem of Milnor type for functions with isolated singularity holds. If one searches theorems of Lefschetz type for the cohomology of a constructible sheaf or complex, the condition of maximal depth in this case means that the constructible complex satisfies a certain cosupport condition. For instance, one has a theorem of Lefschetz type up to the maximal level for perverse sheaves with respect to middle perversity. This generalizes the Lefschetz type Theorem for Intersection Homology of M. Goresky and R. MacPherson in [G-M] (Part II, Chapter 6, §6.11). On the other hand we have seen in [22] (Corollary 1.10) that a complex analytic space has maximal rectified rational homological depth if and only if the constant sheaf $\mathbb{Q}$ is perverse. For instance, on spaces with maximal rectified homotopical or homological depth, the constant sheaf $\mathbb{Q}$ is perverse.

References

1. A. Andreotti - T. Frankel, The Lefschetz theorem on hyperplane sections, Ann. of Math. (2) **69** (1959) 713–717. 492 and 497
2. W. Barth - M. Larsen, On the homotopy groups of complex projective algebraic manifolds, Math. Scand. **30** (1972), 88–94. 525
3. R. Bott, On a theorem of Lefschetz, Michigan Math. J. **6** (1959), 211–216. 492 and 500
4. D. Burghelea - A. Verona, Local homological properties of analytic sets, Manuscripta Math. **7** (1972), 55–66. 505, 508, and 515
5. G. Castelnuovo - F. Enriques, Sur les intégrales simples de première espèce d'une surface ou d'une variété algébrique à plusieurs dimensions, Ann. Sci. École Norm. Sup. (3) **23** (1906), 339–366. 492
6. D. Cheniot, Topologie du complémentaire d'un ensemble algébrique projectif, Enseign. Math. (2) **37** (1991), no. 3–4, 293–402. 502
7. P. Deligne, Le groupe fondamental du complémentaire d'une courbe plane n'ayant que des points doubles ordinaires est abélien (d'après Fulton) in Sém. Bourbaki, exposé 543 Lect. Notes Math. **842**, pp. 1–10, Berlin Heidelberg New York, Springer (1981). 504 and 516
8. C. Eyral, Profondeur homotopique et conjecture de Grothendieck, Ann. Sci. École Norm. Sup. **33** (2000), 823–836. 525
9. K.-H. Fieseler - L. Kaup, Theorems of Lefschetz type in intersection homology. I. The hyperplane section theorem, Rev. Roumaine Math. Pures Appl. **33** (1988), 175–195. 525
10. W. Fulton - R. Lazarsfeld, Connectivity and its applications in algebraic geometry. In: Algebraic geometry (Chicago, Ill., 1980), Lecture Notes in Math. **862**, pp. 26–92, Springer, Berlin (1981). 525
11. G. Gonzalez-Sprinberg, Une formule pour les singularités isolées de surfaces, C. R. Acad. Sci. Paris Sér. A-B 290 (1980), A475 - A478. 538

12. M. Goresky - R. MacPherson, Stratified Morse theory, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], **14**, Springer-Verlag, Berlin, (1988) xiv + 272 pp. 493, 510, 526, 527, and 535
13. A. Grothendieck, Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2). augmenté d'un exposé par Michèle Raynaud, Séminaire de Géométrie Algébrique du Bois-Marie, 1962. Advanced Studies in Pure Mathematics, Vol. 2. North-Holland Publishing Co., Amsterdam; Masson & Cie, Éditeur, Paris, (1968). vii+287 pp. 492, 493, 508, 509, and 525
14. H. A. Hamm, Lokale topologische Eigenschaften komplexer Räume, Math. Ann. **191** (1971) 235–252. 507, 511, and 520
15. H. A. Hamm, Zum Homotopietyp Steinscher Räume, J. Reine Angew. Math. **338** (1983), 121–135. 500 and 529
16. H. A. Hamm, Lefschetz theorems for singular varieties, in Proc. Summer Inst. on Singularities, Arcata 1981, p. 1, pp. 547–557. Proc. Symp. Pure Math. **40** (1983). 511 and 514
17. H. A. Hamm, Zum Homotopietyp q-vollständiger Räume, J. Reine Angew. Math. **364** (1983), 1–9. 500
18. H. A. Hamm, Connectedness of the Milnor fibre and Stein factorization of compactifiable holomorphic functions, Preprint (2019). 525
19. H. A. Hamm - Lê Dũng Tráng, Un théorème de Zariski du type de Lefschetz, Ann. Sci. École Norm. Sup. (4) **6** (1973), 317–355. 493, 502, and 506
20. H. A. Hamm - Lê Dũng Tráng, Lefschetz theorems on quasi-projective varieties, Bull. de la S. M. F. **113** (1985), 123–142. 493, 501, 504, and 517
21. H. A. Hamm - Lê Dũng Tráng, Local generalizations of Lefschetz-Zariski theorems, J. reine angew. Math. **389** (1988), 157–189. 505, 507, and 517
22. H. A. Hamm - Lê Dũng Tráng, Rectified homotopical depth and Grothendieck conjectures, The Grothendieck Festschrift, Vol. II, 311–351, Progr. Math., 87, Birkhäuser Boston, Boston, MA (1991). 493, 505, 507, 509, 511, 512, 515, 522, 525, 526, 528, and 538
23. A. Hatcher, Algebraic topology, Cambridge University Press, Cambridge, (2002). 498
24. H. Hironaka, Subanalytic sets, Number theory, algebraic geometry and commutative algebra, in honor of Yasuo Akizuki, pp. 453–493, Kinokuniya, Tokyo (1973). 509
25. N. Katz, Pinceaux de Lefschetz: Théorème d'existence, in SGA 7 Tome 2, Springer Lecture Notes 390, Springer Verlag. 494 and 495
26. Lê Dũng Tráng, Calcul du nombre de cycles évanouissants d'une hypersurface complexe, Ann. Inst. Fourier **23** (1973), no. 4, 261–270. 504
27. Lê Dũng Tráng, Singularités isolées des intersections complètes, Séminaire Shih Wei Shu 1969–1970 in "Introduction à la théorie des singularités, I", 1–48, Travaux en Cours **36**, Hermann, Paris (1988). 507
28. Lê Dũng Tráng, Vanishing cycles on complex analytic sets, in Various problems in algebraic analysis (Proc. Sympos., Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1975). Sûrikaisekikenkyûsho Kókyûroku No. 266 (1976), 299–318. 533
29. Lê Dũng Tráng, Sur les cycles évanouissants des espaces analytiques, C. R. Acad. Sci. Paris Sér. A-B 288 (1979), no. 4, A283 - A285. 511 and 534
30. Lê Dũng Tráng, Le concept de singularité isolée de fonction analytique, Complex analytic singularities, 215–227, Adv. Stud. Pure Math., 8, North-Holland, Amsterdam, (1987). 533 and 534
31. Lê Dũng Tráng, Depth and perversity, in Algebraic geometry and analytic geometry (Tokyo, 1990), 111–125, ICM-90 Satell. Conf. Proc., Springer, Tokyo (1991). 493 and 533
32. Lê Dũng Tráng - B. Teissier, Cycles évanescents et conditions de Whitney II, Proc. Symp. Pure Math. , Amer. Math. Soc. 40, part 2, 65–103, (1983). 526, 527, 529, 532, and 533
33. S. Lefschetz, Mémoire pour le prix Bordin de l'Académie Française des Sciences (1919). 492
34. S. Lefschetz, L'Analysis Situs et la Géométrie Algébrique, Ed. Gauthier Villars (1924) 154 pp. 492 and 495
35. S. Łojasiewicz, Triangulation of semi-analytic sets, Ann. Scuola Norm. Sup. Pisa (3) **18** (1964), 449–474. 509

36. J. Mather, Notes on topological stability. Bull. Amer. Math. Soc. **49** (2012), 475–506. 510, 524, and 525
37. J. Milnor, Morse theory (Based on lecture notes by M. Spivak and R. Wells), Annals of Mathematics Studies **51**, Princeton University Press, Princeton, N.J. (1963) 153 pp. 492, 497, 498, 499, 500, and 507
38. J. Milnor, Topology from the differentiable viewpoint, Based on notes by David W. Weaver, The University Press of Virginia, Charlottesville, Va. (1965) ix+65 pp. 499
39. J. Milnor, Singular points of complex hypersurfaces, Annals of Mathematics Studies **68**, Princeton University Press, Princeton, N.J. (1968). 505, 507, 529, and 533
40. M. Morse, The calculus of variations in the large, Reprint of the 1932 original, American Mathematical Society Colloquium Publications, **18**, American Mathematical Society, Providence, RI (1996). xii+368 pp. 500
41. D. Mumford, Algebraic geometry. I. Complex projective varieties. Grundlehren der Mathematischen Wissenschaften **221**. Springer-Verlag, Berlin-New York (1976) x+186 pp. 496 and 501
42. C. Okonek, Barth-Lefschetz theorems for singular spaces. J. Reine Angew. Math. **374** (1987), 24–38. 525
43. D. Prill, Local classification of quotients of complex manifolds by discontinuous groups, Duke Math. J. **34** (1967) 375–386. 508 and 509
44. R. Remmert, K. Stein, Über die wesentlichen Singularitäten analytischer Mengen, Mathematische Annalen, **126** (1953) 263–306. 496
45. J. Schürmann, Topology of Singular spaces and Constructible sheaves, Birkhäuser (2003), x+452 pp. 525
46. A. Sommese - A. Van de Ven, Homotopy groups of pullbacks of varieties, Nagoya Math. J. **102** (1986), 79–90. 525
47. E. H. Spanier, Algebraic Topology, McGraw-Hill Inc. (1966), xv+528 pp. 519
48. R. M. Switzer, Algebraic topology: homotopy and homology, Die Grundlehren der mathematischen Wissenschaften, Band 212. Springer-Verlag, New York-Heidelberg, (1975) xii+526 pp. 513 and 518
49. A. Varchenko, The connection between the topological and the algebraic-geometric equisingularity in the sense of Zariski [Russian] Funkcional. Anal. i Priložen. **7** (1973), no. 2, 1–5. English transl.: Functional Anal. Appl. **7** (1973), 87–90. 502
50. A.H. Wallace, Homology theory on algebraic varieties, International Series of Monographs on Pure and Applied Mathematics. Vol. 6 Pergamon Press, New York-London-Paris-Los Angeles (1958) viii+115 pp. 492 and 496
51. H. Whitney, Tangents to an Analytic Variety, Annals of Math. (2) **81** (1965), 496–549. 510
52. O. Zariski, A theorem on the Poincaré group of an algebraic hypersurface. Ann. of Math. (2) **38** (1937), no. 1, 131–141. 500 and 502

Chapter 10
Finite Dimensional Lie Algebras in Singularities

José Luis Cisneros Molina and Meral Tosun

Contents

J. L. Cisneros Molina (✉)
Unidad Cuernavaca, Instituto de Matemáticas, Universidad Nacional Autónoma de México, Cuernavaca, Mexico

Unidad Mixta Internacional 2001 CNRS, Laboratorio Solomon Lefschetz, Cuernavaca, Mexico
e-mail: jlcisneros@im.unam.mx

M. Tosun
Galatasaray University, Çirağan Caddesi, Ortaköy, Istanbul, Turkey
e-mail: mtosun@gsu.edu.tr

J. L. Cisneros Molina et al. (eds.), *Handbook of Geometry and Topology of Singularities I*, https://doi.org/10.1007/978-3-030-53061-7_10

Abstract Complex simple Lie algebras with simply laced root systems are classified by Dynkin diagrams of type A_n, D_n, E_6, E_7, and E_8. Also the dual graphs of the minimal resolution of Kleinian singularities are precisely the same aforementioned Dynkin diagrams. In this work, we recall the basic definitions and some results of the theory of complex Lie algebras and of Kleinian singularities, in order to present a relation between finite dimensional complex simple Lie algebras and the Kleinian singularities, given by a theorem by Brieskorn. We also present the extension of Brieskorn's theorem to the simple elliptic singularity $\tilde{D}_5$.

10.1 Introduction

Let us start by the first phrase in [16] by E. B. Dynkin "Both in algebra and in geometry, the study of [...] semi-simple Lie algebras [...] is important". Indeed, the theory of Lie algebras has contact with other areas of mathematics such as group theory, differential geometry, differential equations and topology, and applications on physics, in particular in quantum mechanics and particle physics. Thus, it is not surprising that they are also related to singularity theory.

In his research on the solutions of algebraic equations of degree five, Klein in his book [35] classified the finite subgroups Γ of $SL(2, \mathbb{C})$, these groups act naturally in $\mathbb{C}^2$ with the origin as its unique fixed point. Klein showed that the quotients $\mathbb{C}^2/\Gamma$ can be seen as complex hypersurfaces in $\mathbb{C}^3$ with an isolated singularity corresponding to the origin. Such hypersurfaces are now called Kleinian singularities. After the work [12–14] by Du Val, Kleinian singularities are characterized as the isolated singularities of surfaces which do not affect the condition of adjunction; then these singularities are also known as Du Val singularities. Du Val described the minimal resolution of Kleinian singularities in the following way: the preimage of the singularity, called the exceptional set of the resolution, is a connected union of projective lines, they meet transversely, not three of them meet at a point, the pairwise intersection of them is a single point or empty and their self-intersection is -2. One encodes this information associating to the exceptional set a graph, called the dual graph of the minimal resolution. It turns out that the dual graphs of the minimal resolution of Kleinian singularities are the Dynkin diagrams of type A_n, D_n, E_6, E_7 and E_8 which classify complex simple Lie algebras with simply laced root systems. This was the first relation found between Kleinian singularities and simple Lie algebras of type ADE. This relation may look as a coincidence and, a natural question is to ask if there is a direct relation between them.

Brieskorn proved in [4, 6] the existence of simultaneous resolutions for Kleinian singularities. After reading Brieskorn's work, Grothendieck conjectured that Kleinian singularities can be obtained from the corresponding complex simple Lie algebra of type A, D or E intersecting its nilpotent variety with a transverse slice to the orbit of a subregular element, or analogously, from the corresponding complex simple Lie group of type A, D or E but using the unipotent variety instead of the nilpotent variety. Brieskorn announced a proof of Grothendieck's conjecture at the International Congress of Mathematicians at Nice in 1970 and sketched it in its Proceedings [7]. Following a geometric idea by Grothendieck, H. Esnault gave a proof in her thesis [17], and later, the details of Brieskorn's proof were given by P. Slodowy in [65]. See [23] for more details on this historical achievement.

The aim of this chapter is to present the relation between the finite dimensional complex simple Lie algebras and Kleinian singularities given by Brieskorn's theorem, and its generalization in [48] to simple elliptic singularities which are non-hypersurface complete intersections. The hypersurface simple elliptic singularities have been studied by many authors [24, 42, 43, 56] but, until now, no result extending Brieskorn's theorem to this type of singularities, using finite dimensional Lie algebras, have been found.

10.2 Lie Algebras

In this section we define complex Lie algebras and we give some examples, we describe its relation with complex Lie groups, we define simple complex Lie algebras and we briefly describe their classification.

A vector space $\mathfrak{g}$ over $\mathbb{C}$ is called a *(complex) Lie algebra* if there is an $\mathbb{C}$-bilinear map $[\ ,\ ]\colon \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$, called the *bracket*, which satisfies the following:

1. **Anti-commutativity:** For all $x \in \mathfrak{g}$ we have $[x, x] = 0$ and,
2. **Jacobi identity:** For all $x, y, z \in \mathfrak{g}$, we have

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

A linear map $\varphi : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ between two Lie algebras that preserves the Lie bracket is called a *Lie algebra homomorphism*. If, in addition, it is bijective, φ is a Lie algebra *isomorphism*. An isomorphism of a Lie algebra with itself is an *automorphism*.

Example 10.2.1 Any complex vector space V becomes a complex Lie algebra by setting $[x, y] = 0$ for every $x, y \in V$. Such a Lie algebra is called *abelian*.

Example 10.2.2 The set $\mathfrak{gl}(n, \mathbb{C})$ of $n \times n$ matrices with coefficients in $\mathbb{C}$ is a complex Lie algebra by defining $[X, Y] = XY - YX$ for all $X, Y \in \mathfrak{g}l(n, \mathbb{C})$; it is of complex dimension n^2.

Example 10.2.3 Let M be a complex manifold and let $p \in M$. Denote by $\mathcal{F}_M(p)$ the $\mathbb{C}$-algebra of *germs of analytic functions at* p. Recall that the *tangent space* to M at p, denoted by T_pM, can be identified with the complex vector space of derivations of $\mathcal{F}_G(g)$ into $\mathbb{C}$, that is, $\mathbb{C}$-linear maps $X\colon \mathcal{F}_G(g) \to \mathbb{C}$ such that for any $f_1, f_2 \in \mathcal{F}_G(g)$

$$X(f_1 f_2) = X(f_1)f_2(g) + f_1(g)X(f_2).$$

A *vector field* X on a complex manifold M is a map which assigns to each point $p \in M$ a tangent vector $X_p \in T_pM$. Let U be an open subset of M and let $f : U \to \mathbb{C}$ be a complex analytic function. For $p \in M$ we define $X(f)(p) := X_p(f)$, then $p \mapsto X(f)(p)$ defines a function $X(f) : U \to \mathbb{C}$. A vector field X is *complex analytic* if, for each complex analytic function f (on some open subset U), the function $X(f)$ is also complex analytic on U. The vector space of all complex analytic vector fields on a complex manifold M forms a complex Lie algebra under the Lie bracket operation on vector fields: Let X and Y be complex analytic vector fields and f a smooth function on M. Define the (complex analytic) vector field $[X, Y]$, called the *Lie bracket* of X and Y by setting

$$[X, Y]_p(f) = X_p(Y(f)) - Y_p(X(f)).$$

Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra. A subset $\mathfrak{h}$ of $\mathfrak{g}$ is a *subalgebra* of $\mathfrak{g}$ if, for all $x, y \in \mathfrak{h}$, we have $[x, y] \in \mathfrak{h}$.

Example 10.2.4 The set of matrices of trace zero $\mathfrak{sl}(n, \mathbb{C})$ is a subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ of dimension $n^2 - 1$. The set of diagonal matrices in $\mathfrak{sl}(n, \mathbb{C})$ is a subalgebra of $\mathfrak{sl}(n, \mathbb{C})$ of dimension $n - 1$.

Example 10.2.5 There are two important subalgebras of a Lie algebra $\mathfrak{g}$ associated to a subset $S \subset \mathfrak{g}$: the normalizer and centralizer of S, analogous to the homonymous notions in group theory. Let S be a subset of a Lie algebra $\mathfrak{g}$. The set

$$N_{\mathfrak{g}}(S) = \{x \in \mathfrak{g} \mid [x, y] \in S \text{ for all } y \in S\} \tag{10.1}$$

is called the *normalizer* of S in $\mathfrak{g}$; the set

$$Z_{\mathfrak{g}}(S) = \{x \in \mathfrak{g} \mid [x, y] = 0 \text{ for all } y \in S\} \tag{10.2}$$

is called the *centralizer* of S in $\mathfrak{g}$. By the Jacobi identity, we have $[y, [x, z]] = -[x, [z, y]] - [z, [y, x]]$ for $x, z \in \mathfrak{g}$; so $N_{\mathfrak{g}}(S)$ and $Z_{\mathfrak{g}}(S)$ are subalgebras of $\mathfrak{g}$.

10.2.1 Lie Algebras and Lie Groups

A Lie algebra is an algebraic structure that can be studied in its own right and that has many applications. But most of its power comes from its relation with Lie groups, which are a very important class of differentiable manifolds which play a major role in modern geometry. For further details on this subsection see for instance [40, 63].

A *complex Lie group* G is a complex manifold with a group structure such that the map $\mu\colon G \times G \to G$ defined by $(g_1, g_2) \mapsto g_1 g_2^{-1}$ is holomorphic.

A map $\phi\colon G \to H$ between two complex Lie groups is a (*complex Lie group*) *homomorphism* if it is both holomorphic and a homomorphism of the abstract groups. Moreover, if ϕ is a biholomorphism, we call it an *isomorphism*.

Example 10.2.6 The set $GL(n, \mathbb{C})$ of all $n \times n$ invertible matrices with coefficients in $\mathbb{C}$ is an open complex submanifold of $\mathbb{C}^{n^2}$ which is a complex Lie group under matrix multiplication. It is called the *complex general linear group*.

A *complex Lie subgroup* H of a complex Lie group G is a subgroup of G which is also a complex submanifold of G, and H is a Lie group with respect to its complex structure.

Let $h \in G$. The *left translation by* h and the *right translation by* h are respectively, the biholomorphisms L_h and R_h of G given by

$$L_h(g) = hg, \quad R_h(g) = gh \quad \text{for all } g \in G.$$

A vector field X on G is called *left invariant* if it satisfies

$$DL_h \circ X = X \circ L_h \quad \text{for all } h \in G.$$

Any left invariant vector field on a complex Lie group is complex analytic [40, Lemma 1.6]. The set of left invariant vector fields on a complex Lie group G is usually denoted by the corresponding lowercase *Fraktur* letter $\mathfrak{g}$. It is a complex vector space and the map

$$\begin{aligned} \alpha\colon \mathfrak{g} &\to T_e G, \\ \alpha(X) &\mapsto X_e \end{aligned} \tag{10.3}$$

is a linear isomorphism of $\mathfrak{g}$ with the tangent space $T_e G$ to G at the identity element e. Hence $\dim_{\mathbb{C}} \mathfrak{g} = \dim_{\mathbb{C}} T_e G = \dim_{\mathbb{C}} G$. The Lie bracket of two invariant vector fields is again an invariant vector field, so $\mathfrak{g}$ is a Lie algebra with the Lie bracket of vector fields.

Definition 10.2.7 We define the *Lie algebra of the complex Lie group* G to be the complex Lie algebra $\mathfrak{g}$ of left invariant vector fields on G, or alternatively, the tangent space $T_e G$ at the identity with Lie algebra structure induced by requiring the vector space isomorphism (10.3) to be a Lie algebra isomorphism.

Example 10.2.8 The following are the examples of complex Lie groups and their corresponding Lie algebras, that we will use in relation with singularities.

1. The Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ of Example 10.2.2 is the Lie algebra of the Lie group $GL(n, \mathbb{C})$ of Example 10.2.6.
2. The *complex special linear group*

$$SL(n, \mathbb{C}) = \{A \in GL(n, \mathbb{C}) \mid \det A = 1\},$$

has as Lie algebra the *matrices of trace* 0 given in Example 10.2.4

$$\mathfrak{sl}(n, \mathbb{C}) = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid \operatorname{tr} X = 0\},$$

3. The *complex special orthogonal group*

$$SO(n, \mathbb{C}) = \{A \in GL(n, \mathbb{C}) \mid A^{-1} = A^t, \det A = 1\},$$

where X^t denotes the transpose of X, has as Lie algebra the *skew-symmetric matrices of trace* 0

$$\mathfrak{so}(n, \mathbb{C}) = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid X + X^t = 0, \operatorname{tr} X = 0\}.$$

4. Let V be a complex vector space of dimension n. Denote by $\operatorname{End}(V)$ the set of all complex linear transformations on V (endomorphisms) and let $\operatorname{Aut}(V) \subset \operatorname{End}(V)$ denote the subset of non-singular complex transformations (automorphisms). $\operatorname{End}(V)$ is a complex vector space of dimension n^2 and becomes a Lie algebra setting $[l_1, l_2] = l_1 \circ l_2 - l_2 \circ l_1$ for every $l_1, l_2 \in \operatorname{End}(V)$. A basis of V gives a diffeomorphism between $\operatorname{End}(V)$ and $\mathfrak{gl}(n, \mathbb{C})$ which maps $\operatorname{Aut}(V)$ onto $GL(n, \mathbb{C})$. Hence $\operatorname{Aut}(V)$ inherits a manifold structure as an open subset of $\operatorname{End}(V)$, and it is a Lie group under the composition, with Lie algebra $\operatorname{End}(V)$.

Note that the complex Lie groups (respectively Lie algebras) in items (2) and (3) of Example 10.2.8 are contained in $GL(n, \mathbb{C})$ (respectively in $\mathfrak{gl}(n, \mathbb{C})$), they are in fact complex Lie subgroups (respectively Lie subalgebras).

Let G be a complex Lie group and let H be a complex Lie subgroup, then the Lie algebra $\mathfrak{h} \cong T_eH$ of H is a subspace of the Lie algebra $\mathfrak{g} \cong T_eG$ of G. It is not difficult to prove that $\mathfrak{h}$ is actually a Lie subalgebra.

The following theorem says that any abstract Lie algebra can be seen concretely as a Lie algebra of matrices (see for instance [30, Chapter VI 2.]):

Theorem 10.2.9 (Ado's Theorem) *Every finite dimensional complex Lie algebra* $\mathfrak{g}$ *is isomorphic to a subalgebra of* $\mathfrak{gl}(n, \mathbb{C})$.

To every complex Lie group we associate a Lie algebra. Ado's theorem can be used to prove the converse (see for instance [63, II.V.8 Theorem 3]):

Theorem 10.2.10 (Lie's Third Theorem) *For any finite dimensional complex Lie algebra $\mathfrak{g}$, there exists a complex Lie group G with Lie algebra $\mathfrak{g}$.*

Remark 10.2.11 Let $\phi \colon G \to H$ be a homomorphism of complex Lie groups. Since ϕ sends the identity element of G to the identity element of H, its differential at the identity $d\phi_e \colon T_eG \cong \mathfrak{g} \to T_eH \cong \mathfrak{h}$ is a linear map between the corresponding complex Lie algebras which turns out to be also a Lie algebra homomorphism [40, Theorem 1.8].

Given a connected complex Lie group G its universal covering space $\tilde{G}$ (its simply connected covering space) can be endowed with a complex structure and a group structure which makes it a complex Lie group, such that the covering map is a homomorphism of complex Lie groups [40, Corollary 1.14], and the Lie algebra homomorphism induced by the covering map is an isomorphism. Hence G and $\tilde{G}$ have isomorphic Lie algebras.

If two simply connected complex Lie groups G and H have isomorphic Lie algebras $\mathfrak{g} \cong \mathfrak{h}$, then they are isomorphic, thus, there is a one-to-one correspondence between isomorphism classes of simply connected complex Lie groups and isomorphism classes of complex Lie algebras [63, II.V.8 Theorem 2]. Hence, one can classify simply connected Lie groups classifying the corresponding Lie algebras.

10.2.2 Simple Lie Algebras and Simple Lie Groups

We are interested in complex simple Lie algebras, a particular type of complex Lie algebras which are some of the building blocks that make up all (finite-dimensional) complex Lie algebras.

Let $\mathfrak{g}$ be a complex Lie algebra. A subspace I of $\mathfrak{g}$ is called an *ideal* of $\mathfrak{g}$ if $x \in \mathfrak{g}$ and $y \in I$ imply that $[x, y] \in I$.

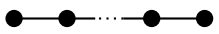
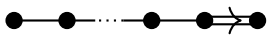
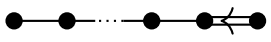
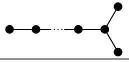
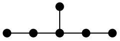
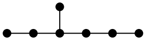
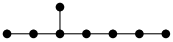
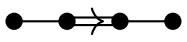
A complex Lie algebra $\mathfrak{g}$ is called *simple* if it is non-abelian and has no non-trivial ideals.

A *complex simple Lie group* is a connected non-abelian complex Lie group G which does not have non-trivial connected normal subgroups.

Proposition 10.2.12 *A connected complex Lie group is simple if and only if its Lie algebra is simple.*

Example 10.2.13 The Lie algebra $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ of Example 10.2.8 (2) is a complex simple Lie algebra [57, §2.14 Theorem A].

Table 10.1 Dynkin diagrams of simple complex Lie algebras

Name	Dynkin diagram	Complex Lie algebra
A_n		$\mathfrak{sl}(n+1,\mathbb{C})$
B_n		$\mathfrak{so}(2n+1,\mathbb{C})$
C_n		$\mathfrak{sp}(2n,\mathbb{C})$
D_n		$\mathfrak{so}(2n,\mathbb{C})$
E_6		
E_7		
E_8		
F_4		
G_2		

Hence, isomorphism classes of simply connected simple complex Lie groups correspond to isomorphism classes of simple complex Lie algebras. These are classified by their fundamental root systems, and the root systems are described by their Dynkin diagrams (see Sect. 10.4.3). The classification of Dynkin diagrams coming from complex simple Lie algebras give four infinite series A_n ($n \geq 1$), B_n ($n \geq 2$), C_n ($n \geq 3$), D_n ($k \geq 4$) which correspond to matrix Lie algebras and five exceptional cases E_6, E_7, E_8, F_4 and G_2, they are showed in Table 10.1. Among these diagrams, we are interested in the ones which have *simply laced root systems* (with roots of equal length), which are A_n, D_n, E_6, E_7, E_8.

10.3 Kleinian Singularities

In this section we present an important class of surface singularities embedded in $\mathbb{C}^3$, depending on the context that they have been studied, they have received several names: Kleinian Singularities, Du Val singularities, rational double point singularities or simple singularities. In [15] one can find fifteen characterizations of this class of singularities. See also [66] for further references on the material in this section.

Let U be a neighbourhood of the origin in $\mathbb{C}^2$ and let G be a properly discontinuous group of holomorphic automorphisms of U fixing 0. H. Cartan proved in [8] that the quotient space U/G is a normal analytic surface with an isolated singularity and the quotient map $U \to U/G$ is analytic. A *quotient surface singularity* is a singularity which is isomorphic to a singularity of a quotient U/G for some U and G as before. By a linearization argument [8, p. 97] (see also [5, Lemma 2.2]) every two-dimensional quotient singularity is isomorphic to $\mathbb{C}^2/G$ for some finite subgroup of $GL(2,\mathbb{C})$. In [53] Prill call a subgroup G of $GL(2,\mathbb{C})$

small if no $g \in G$ has 1 as an eigenvalue of multiplicity precisely $(n-1)$), in other words, no element of G is a pseudo-reflexion (leaves a hyperplane fixed). He proved that every two-dimensional quotient singularity is isomorphic to $\mathbb{C}^2/G$ for a small finite subgroup of $GL(2,\mathbb{C})$. If G and G' are two small finite subgroups of $GL(2,\mathbb{C})$ then $\mathbb{C}^2/G$ and $\mathbb{C}^2/G'$ are analytically isomorphic if and only if G and G' are conjugate. By a theorem of Chevalley [9, Theorem A], if a finite subgroup $G \leq GL(2,\mathbb{C})$ is generated by pseudo-reflexions then $\mathbb{C}^2/G$ is biholomorphic to $\mathbb{C}^2$, so it does not have singularities, this rules out the dihedral groups. The list of finite small subgroups of $GL(2,\mathbb{C})$ is given in [5, Satz 2.9] (see also [51]). If G is a small finite subgroup of $GL(2,\mathbb{C})$, then $G \leq SL(2,\mathbb{C})$ if and only if $\mathbb{C}^2/G$ embeds in $\mathbb{C}^3$ [15, Corollary 5.3]. Moreover, all finite subgroups of $SL(2,\mathbb{C})$ are small.

The finite subgroups of $SL(2,\mathbb{C})$ (which are conjugated to the finite subgroups of $SU(2)$) were classified (up to conjugation) by F. Klein in his book [35] as follows. Let $SO(3)$ be the group of rotations of $\mathbb{R}^3$, there is a surjective homomorphism $\rho\colon SU(2) \to SO(3)$ with kernel of order 2. The finite subgroups of $SO(3)$ are very well known, they are the cyclic groups C_k of order k $(k \geq 2)$, the dihedral groups D_k of order $2k$ $(k \geq 2)$, and the rotation groups of the Platonic solids: the *tetrahedral group* T of order 12, the *octahedral group* O of order 24 and the *icosahedral group* I of order 60. If G' is a finite subgroup of $SO(3)$, then $G = \rho^{-1}(G')$ is a finite subgroup of $SU(2)$, we say that G is the *binary group* of G' since its order is twice the order of G'. Hence, the finite subgroups of $SU(2)$ are the cyclic groups C_k of order k $(k \geq 2)$, the *binary dihedral* groups $\mathcal{D}_k$ of order $4k$ $(k \geq 2)$, the *binary tetrahedral* group $\mathcal{T}$ of order 24, the *binary octahedral* group $\mathcal{O}$ of order 48 and the *binary icosahedral* group $\mathcal{I}$ of order 120.

Let Γ be a finite subgroup of $SL(2,\mathbb{C})$. The group Γ acts naturally on $\mathbb{C}^2$ by matrix multiplication; the action is free on $\mathbb{C}^2 \setminus \{0\}$ and the origin 0 is a fixed point. Klein proved that the quotient $\mathbb{C}^2/\Gamma$ is a complex surface with an isolated singularity corresponding to 0 as follows.

The group Γ acts on the $\mathbb{C}$-algebra $\mathbb{C}[z_1, z_2]$ of polynomials by

$$(Pg)(\mathbf{z}) = P(g \cdot \mathbf{z}), \quad \text{for } P \in \mathbb{C}[z_1, z_2],\ g \in \Gamma \text{ and } \mathbf{z} \in \mathbb{C}^2.$$

The Γ-invariant polynomials $\mathbb{C}[z_1, z_2]^\Gamma$ form a subalgebra, Klein proved that this subalgebra is generated by three invariant polynomials X, Y and Z which satisfy only one non-trivial polynomial relation $f(X,Y,Z) = 0$. In Table 10.2 we show such relation for each of the finite subgroups Γ of $SL(2,\mathbb{C})$.

Table 10.2 Relation $f(X,Y,Z) = 0$ of the generators of $\mathbb{C}[z_1, z_2]^\Gamma$

Group Γ	Relation $f(X,Y,Z)$
C_k	$X^k + YZ$
$\mathcal{D}_k$	$X^{k+1} + XY^2 + Z^2$
$\mathcal{T}$	$X^4 + Y^3 + Z^2$
$\mathcal{O}$	$X^3Y + Y^3 + Z^2$
$\mathcal{I}$	$X^5 + Y^3 + Z^2$

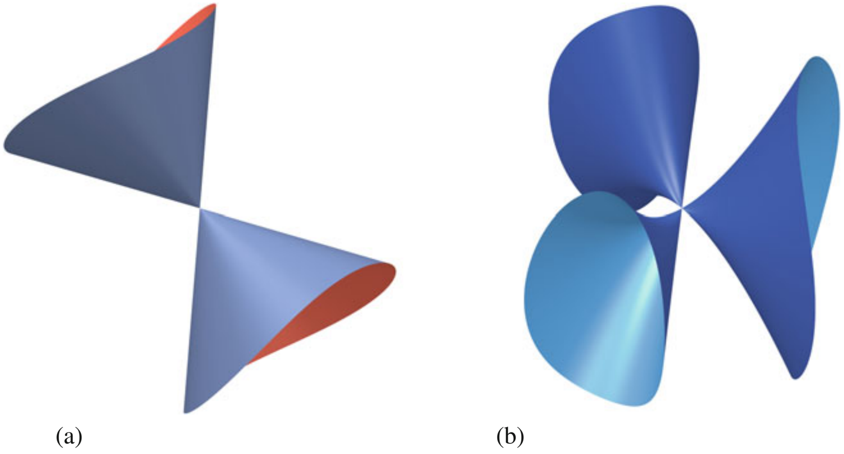


Fig. 10.1 Some Kleinian singularities. (**a**) Singularity $X^2 + XY = 0$. (**b**) Singularity $-X^3 + XY^2 + Z^2 = 0$

Using the invariant polynomials $X,\ Y$ and Z one defines the following map

$$F \colon \mathbb{C}^2 \to \mathbb{C}^3$$
$$F(z_1, z_2) = (X(z_1, z_2), Y(z_1, z_2), Z(z_1, z_2)).$$

It is invariant under the action of Γ on $\mathbb{C}^2$, so it factors over the quotient

$$\begin{tikzcd}
\mathbb{C}^2 \arrow[r, "F"] \arrow[d] & \mathbb{C}^3 \\
\mathbb{C}^2/\Gamma \arrow[ur, "\tilde{F}"'] &
\end{tikzcd}$$

thus $\tilde{F}$ embeds $\mathbb{C}^2/\Gamma$ in $\mathbb{C}^3$ and its image is precisely the zero-set of the polynomial f, since $X,\ Y$ and Z satisfy the relation $f(X, Y, Z) = 0$.

Definition 10.3.1 The *Kleinian singularities* are the surfaces in $\mathbb{C}^3$ given as the zero-locus of the polynomials listed in Table 10.2.

Figure 10.1 shows the real surfaces[1] of some of the polynomials of Table 10.2.

Recall that the link L of an isolated singularity p of an analytic space V is the intersection $L := V \cap \mathbb{S}_\varepsilon$ of V with a sphere $\mathbb{S}_\varepsilon$ centered at p of sufficiently small radius, such that it is transverse to the manifold $V \setminus \{p\}$. Thus, L is a differentiable manifold. Notice that by Klein's result, the link of the Kleinian singularities are

---

[1] Figures created with SURFER.

diffeomorphic to the quotients $\mathbb{S}^3/\Gamma$, where, as before, Γ is a finite subgroup of $SU(2) \cong \mathbb{S}^3$.

In [45] Milnor gives analogous results proving that the links of the singularities of Brieskorn-Pham varieties $V(p,q,r) = \{z_1^p + z_2^q + z_3^r = 0\}$ are diffeomorphic to homogeneous spaces G/Γ, where G is one of the following 3-dimensional Lie groups: $SU(2)$, $\widetilde{SL(2,\mathbb{R})}$, the universal cover of $SL(2,\mathbb{R})$, and N, the Heisenberg group; and Γ is a discrete subgroup. In fact, for the case of $SU(2)$, Milnor's result is a refinement of Klein's theorem.

More generally, let G be one of the groups $SU(2)$, $\widetilde{SL(2,\mathbb{R})}$ or N. Given a quasi-homogeneous Gorenstein surface singularity, its link is diffeomorphic to an homogeneous space of the form G/Γ, where Γ is a discrete subgroup of G. Conversely, given a discrete subgroup Γ of G with compact quotient G/Γ, then this quotient is diffeomorphic to the link of a normal, Gorenstein quasi-homogeneous surface singularity, see [59, Chapter 3] for details and related results. Expressing the links as the quotients G/Γ endow them with a geometric structure in the sense of Thurston [58]. By [49] such links can only have six of Thurston's eight geometries and the deformations of the geometric structures correspond to the deformations of the complex structures on the corresponding singularity.

10.3.1 Resolution of Kleinian Singularities

Kleinian singularities can also be characterized by their minimal resolutions. For a survey on the problem of resolution of singularities see [67]. Let S be a complex surface with exactly one singular point at $p \in S$. A *resolution of* S is a surjective proper morphism $\pi : \tilde{S} \to S$ (in the category of algebraic or analytic varieties) from a smooth variety $\tilde{S}$ to S, such that its restriction $\pi|_{\tilde{S}\setminus\pi^{-1}(p)} : \tilde{S} \setminus \pi^{-1}(p) \to S \setminus \{p\}$ is an isomorphism. The set $E = \pi^{-1}(p)$ is called the *exceptional set*. A resolution $\pi : \tilde{S} \to S$ is called *minimal* if any other resolution $\pi' : \tilde{S}' \to S$ factors through π, that is, there exists a unique map $\rho : \tilde{S}' \to \tilde{S}$ such that $\pi' = \pi \circ \rho$.

A resolution can be seen as a finite sequence of blowing-ups and normalizations. Let $B = \{(x,\ell) \in \mathbb{C}^3 \times \mathbb{CP}^2 \mid x \in \ell\}$, the *blowing-up* of $\mathbb{C}^3$ at the origin $0 \in \mathbb{C}^3$ is the natural projection $\beta : B \to \mathbb{C}^3$. We have that $\beta^{-1}(0)$ is isomorphic to $\mathbb{CP}^1$ and the restriction $\beta| : B \setminus \beta^{-1}(0) \to \mathbb{C}^3 \setminus \{0\}$ is an isomorphism. Let X be a subvariety of $\mathbb{C}^3$ with an isolated singularity at 0 and $\tilde{X}$ the closure of $\beta^{-1}(X)$ in B, then $\tilde{X}$ is the *blowing-up* of X at 0. This construction does not depend on the embedding of X in $\mathbb{C}^3$ and can be applied again to points of $\tilde{X}$. Kleinian singularities are the only surface singularities of multiplicity two that can be resolved by performing a finite number of blowing-ups of points, without normalizations [34, 41, 71]. Figure 10.2 shows the minimal resolutions of the real surfaces given in Fig. 10.1.

Let $\pi : \tilde{S} \to S$ be the minimal resolution of a Kleinian singularity S. The exceptional set $E = \pi^{-1}(0)$ consists of a union of finitely many components E_i

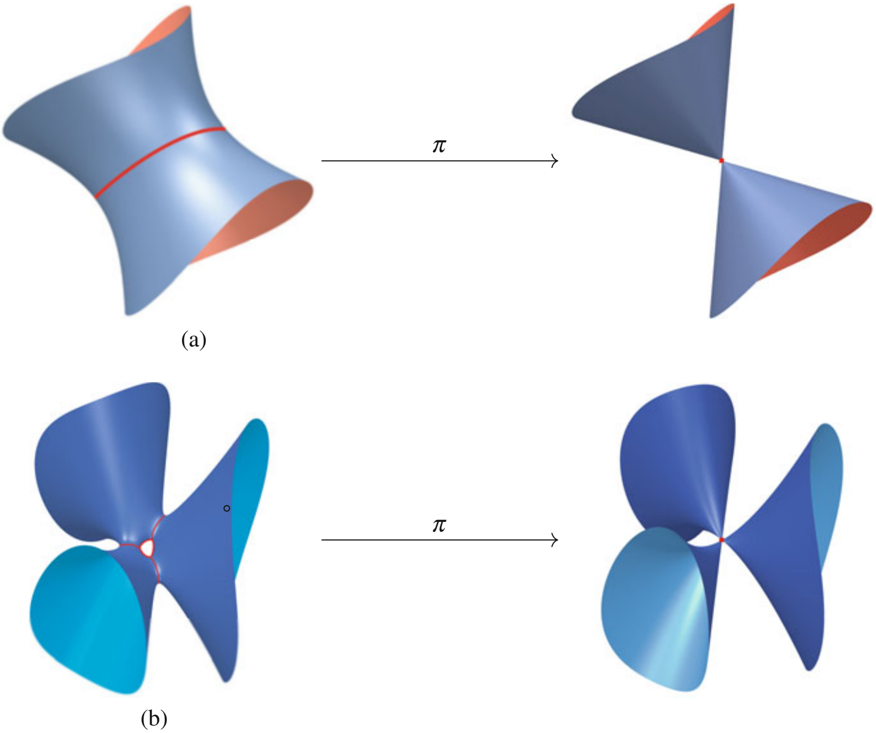


Fig. 10.2 Minimal resolution of some Kleinian singularities, with red exceptional set. (**a**) Minimal resolution of the singularity $X^2+YZ=0$. (**b**) Minimal resolution of the singularity $-X^3+XY^2+Z^2=0$

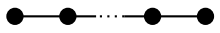
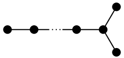
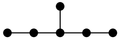
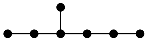
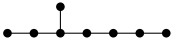
isomorphic to projective lines $E_i \cong \mathbb{CP}^1$ with

$$E=\pi^{-1}(0)=E_1\cup\cdots\cup E_r,$$

where each E_i has self-intersection -2, i.e., the normal bundle of each E_i in $\tilde{S}$ is isomorphic to the cotangent bundle $T^*\mathbb{CP}^1$ of the projective line $\mathbb{CP}^1$.The intersection of E_i and E_j is empty or one point, and if they intersect, they intersect transversely. We can assign to π a graph $\Delta(S)$, called the *dual graph of the resolution*, in the following way: to each E_i we associate a vertex, and two vertices are connected by an edge if the corresponding components intersect. The dual graphs of the minimal resolution of Kleinian singularities are showed in Table 10.3 and we can see that they exactly correspond to the Dynkin diagrams of the simple Lie algebras of type A, D, E given in Table 10.1. This property also characterizes Kleinian singularities and for this reason they are also called *ADE-singularities*.

The real resolutions of Fig. 10.2 illustrate what happens in the complex situation. In Fig. 10.2a the exceptional set is the red circle (isomorphic to $\mathbb{RP}^1$), thus its dual

Table 10.3 Dual graph of the minimal resolution of Kleinian singularities

Group	Singularity	Dual graph	Dynkin diagram
$\mathcal{C}_k$	$X^k + YZ$		A_{k-1}
$\mathcal{D}_k$	$X^{k+1} + XY^2 + Z^2$		D_{k+2}
$\mathcal{T}$	$X^4 + Y^3 + Z^2$		E_6
$\mathcal{O}$	$X^3Y + Y^3 + Z^2$		E_7
$\mathcal{I}$	$X^5 + Y^3 + Z^2$		E_8

graph is of type A_1. In Fig. 10.2b the exceptional set consists of the four red circles, the central one, and the other three intersecting it in different points, its dual graph is of type D_4.

So far, this relation looks as a coincidence, a natural question is to ask if there is a direct relation between the simple Lie algebras of type ADE and Kleinian singularities. Our aim for the following sections is to explain one of this direct relations.

In [44] McKay obtained the Dynkin diagrams of type ADE from the irreducible representations of the corresponding finite subgroups of $SL(2, \mathbb{C})$ given in Table 10.3. This gives a one-to-one correspondence between (non-trivial) irreducible representations of the group and the components of the exceptional set of the minimal resolution of the corresponding Kleinian singularity. This correspondence is now called the McKay correspondence which we briefly explain in Sect. 10.8.

10.3.2 Deformation of Kleinian Singularities

One way to study resolution of singularities is through their deformations. Here we define semi-universal deformations and give the example of a semi-universal deformation of a Kleinian singularity which we shall later construct using Lie algebras. For an introduction to deformation of singularities see [22, 50].

A holomorphic map $\phi\colon \mathcal{X} \to B$ of complex spaces is called *flat* at $x \in \mathcal{X}$ if the structure sheaf $\mathcal{O}_{\mathcal{X},x}$ is flat as an $\mathcal{O}_{B,b}$-module, with $b = \phi(x)$. It is called *flat*, if it is flat at every $x \in \mathcal{X}$. If $(\mathcal{X}, x)$ and (B, b) are singularities, flatness refers to flatness at x of a representing map between representatives of the two germs.

Let (S, x) be the germ of an analytic variety at x. In our case S is a Kleinian singularity in $\mathbb{C}^3$. A *deformation* (ϕ, i) of (S, x) is a flat morphism of germs of analytic spaces $\phi\colon (\mathcal{X}, x) \to (B, b)$ together with an isomorphism $i\colon (S, x) \to (\phi^{-1}(b), x)$. The space (B, b) is called the *base* or the *parameter space* of the deformation (ϕ, i).

An *isomorphism* $\chi\colon (\phi, i) \to (\phi', i')$ of two deformations $\phi\colon (\mathcal{X}, x) \to (B, b)$ and $\phi'\colon (\mathcal{X}', x) \to (B, b)$ over (B, b) is an isomorphism $\chi\colon (\mathcal{X}, x) \to (\mathcal{X}', x)$ such that the following diagram commutes

$$\begin{tikzcd}
 & (S,x) \arrow[dl, "i"'] \arrow[dr, "i'"] & \\
(\mathcal{X},x) \arrow[rr, "\chi"] \arrow[dr, "\phi"'] & & (\mathcal{X}',x) \arrow[dl, "\phi'"] \\
 & (B,b) &
\end{tikzcd}$$

If $\phi\colon (\mathcal{X},x) \to (B,b)$ is a deformation of (S,x) and $\psi\colon (A,a) \to (B,b)$ is a morphism, then the pull-back $\phi_A\colon (A,a) \times_{(B,b)} (\mathcal{X},x) \to (A,a)$ given in the following diagram

$$\begin{tikzcd}
(A,a) \times_{(B,b)} (\mathcal{X},x) \arrow[r] \arrow[d, "\phi_A"'] & (\mathcal{X},x) \arrow[d, "\phi"] \\
(A,a) \arrow[r, "\psi"] & (B,b)
\end{tikzcd}$$

is again flat, and therefore a deformation of (S,x) over (A,a), which it is called the *deformation induced by* ψ *from* ϕ.

Definition 10.3.2 A deformation $\phi\colon (\mathcal{X},x) \to (B,b)$ of (S,x) is called a *semi-universal deformation* if and only if

(i) Any other deformation $\phi'\colon (\mathcal{X}',x) \longrightarrow (A,a)$ of (S,x) is isomorphic to a deformation induced from ϕ by a base change $\psi\colon (A,a) \to (B,b)$.
(ii) The differential $d\psi$ at a is uniquely determined.

It follows that the semi-universal deformations are unique up to isomorphism.

Theorem 10.3.3 ([33]) *A semi-universal deformation of any isolated singularity exists.*

For the case of isolated hypersurface singularities (or more generally isolated complete intersection singularities) by [72] it is possible to compute explicitly a semi-universal deformation as follows (see [22, Corollary 7.2.23 and Remark 7.2.24]).

Let $(S,0) \subset (\mathbb{C}^n,0)$ be an isolated singularity defined by $f \in \mathcal{O}_{\mathbb{C}^n,0}$. The *Tjurina algebra* of $(S,0)$ is given by

$$T^1_{(S,0)} := \mathcal{O}_{\mathbb{C}^n,0} \Big/ \left\langle f, \frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \dots, \frac{\partial f}{\partial z_n} \right\rangle.$$

Proposition 10.3.4 *Let* $(S,0)\subset(\mathbb{C}^n,0)$ *be an isolated singularity defined by* $f\in\mathcal{O}_{\mathbb{C}^n,0}$ *and* $g_1,\dots,g_\tau\in\mathcal{O}_{\mathbb{C}^n,0}$ *a* $\mathbb{C}$*-basis of the Tjurina algebra* $T^1_{(S,0)}$*. If we set*

$$F(\mathbf{x},\mathbf{t}) := f(\mathbf{x})+\sum_{j=1}^{\tau} t_j g_j(\mathbf{x}), \quad \textit{and} \quad (\mathcal{X},0) := V(F)\subset(\mathbb{C}^n\times\mathbb{C}^\tau,0),$$

then $(S,0)\hookrightarrow(\mathcal{X},0)\xrightarrow{\phi}(\mathbb{C}^\tau,0)$*, with* ϕ *the second projection, is a semi-universal deformation of* $(S,0)$*.*

Remark 10.3.5 If $f\in\mathfrak{m}^2_{\mathbb{C}^n,0}$, on the basis $g_1,\dots,g_\tau\in\mathcal{O}_{\mathbb{C}^n,0}$ of the Tjurina algebra $T^1_{(S,0)}$ one can choose $g_1=-1$ and we can eliminate t_1 from $F(\mathbf{x},\mathbf{t})=0$. Then setting $\mathbf{t}=(t_1,\dots,t_{\tau-1})$

$$\begin{aligned}(\mathbb{C}^n\times\mathbb{C}^\tau,0)&\to(\mathbb{C}^\tau,0)\\ (\mathbf{x},\mathbf{t})&\mapsto\Big(f(\mathbf{x})+\sum_{j=1}^{\tau-1}t_j g_j(\mathbf{x}),\mathbf{t}\Big)\end{aligned}$$

is a semi-universal deformation of the hypersurface singularity $\big(f^{-1}(0),0\big)$.

Example 10.3.6 Let $(S,0)$ be the A_2-singularity, that is, the surface defined by $f(x,y,z)=x^3+yz=0$ in $\mathbb{C}^3$ (see Table 10.3) . The *Tjurina algebra*

$$T^1 := \mathbb{C}\{x,y,z\}\Big/\Big\langle f,\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\Big\rangle$$

is generated by $\{-1,x\}$. The map

$$\begin{aligned}\mathbb{C}^3\times\mathbb{C}&\to\mathbb{C}^2\\ (x,y,z,t)&\mapsto(x^3+yz+tx,t).\end{aligned}\tag{10.4}$$

is a semi-universal deformation of $(S,0)$.

In [11] Crawley-Boevey and Holland construct deformations of Kleininan singularities in terms of skew group algebras and deformed preprojective algebras.

10.4 More on Lie Algebras

In this section we give the necessary definitions and results of Lie algebras that we will need later. We only consider complex Lie algebras even when we do not say it explicitly. The reader can see all the details about the results announced below in a basic textbook of Lie algebras, for instance [26, 30, 57].

One way to construct new Lie algebras from given ones is via the *direct sum of Lie algebras*. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie algebras. Their direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is defined as $\mathfrak{g}_1 \oplus \mathfrak{g}_2 = \{(x, y) \mid x \in \mathfrak{g}_1, y \in \mathfrak{g}_2\}$ and the Lie bracket on it is given by $[(x, y), (x', y')] := ([x, x'], [y, y'])$.

Example 10.4.1 It is easy to see that $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{so}(4, \mathbb{C})$.

On the other hand, given a Lie algebra $\mathfrak{g}$ we would like to decompose it in "canonical" pieces which we can classify later. In the following subsection we will see such a decomposition called a *Levi decomposition of* $\mathfrak{g}$.

10.4.1 Levi Decomposition

Let $\mathfrak{g}$ be a Lie algebra. For two subspaces A, B of $\mathfrak{g}$ the symbol $[A, B]$ denotes the linear span of the set of all $[x, y]$ with $x \in A$ and $y \in B$. Occasionally this notation is also used for arbitrary subsets A, B of $\mathfrak{g}$. Given two ideals I and J of $\mathfrak{g}$, then the set $[I, J]$ is again an ideal of $\mathfrak{g}$.

An important example of an ideal of a Lie algebra $\mathfrak{g}$ is its *derived (sub)algebra* $\mathfrak{g}'$ defined by the ideal $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$.

Remark 10.4.2 The quotient $\mathfrak{g}/\mathfrak{g}'$ is abelian, and $\mathfrak{g}'$ is the unique minimal ideal of $\mathfrak{g}$ with abelian quotient.

Let us define two sequences of ideals of a Lie algebra $\mathfrak{g}$ which will allow us to define two special types of Lie algebras.

The *derived series* is the sequence of ideals: $\mathfrak{g},\ \mathfrak{g}',\ \mathfrak{g}'' = [\mathfrak{g}', \mathfrak{g}'], \dots, \mathfrak{g}^{(r)} := [\mathfrak{g}^{(r-1)}, \mathfrak{g}^{(r-1)}], \dots$. We have $\mathfrak{g} \supset \mathfrak{g}' \supset \cdots \supset \mathfrak{g}^{(n)} \supset \dots$. We say that $\mathfrak{g}$ is a *solvable* Lie algebra if there exists a positive integer n such that $\mathfrak{g}^{(n)} = 0$.

Example 10.4.3 The Lie algebra of upper-triangular matrices ($a_{ij} = 0$ for $i > j$) is solvable.

The *lower central series* is the sequence of ideals given by: $\mathfrak{g}^0 := \mathfrak{g},\ \mathfrak{g}^1 := [\mathfrak{g}, \mathfrak{g}^0],\ \mathfrak{g}^2 := [\mathfrak{g}, \mathfrak{g}^1],\ \dots, \mathfrak{g}^n := [\mathfrak{g}, \mathfrak{g}^{(n-1)}], \dots$. We have $\mathfrak{g}^0 \supset \mathfrak{g}^1 \supset \dots \supset \mathfrak{g}^n \supset \dots$. We say that $\mathfrak{g}$ is a *nilpotent* Lie algebra if there exists a positive integer n such that $\mathfrak{g}^n = 0$.

Notice that the derived and lower central series of an ideal of $\mathfrak{g}$ consists of ideals of $\mathfrak{g}$.

Example 10.4.4 Any abelian Lie algebra is nilpotent.

Example 10.4.5 The Lie algebra of *triangular nilpotent matrices*, that is, the triangular matrices with 0's in the diagonal, is nilpotent.

Remark 10.4.6 Nilpotency implies solvability. Using induction it is easy to see that $\mathfrak{g}^{(r)} \subset \mathfrak{g}^r$. Also a subalgebra of a solvable (respectively nilpotent) Lie algebra is itself solvable (respectively nilpotent), and similar for quotients.

A Lie algebra contains a maximal solvable ideal $\mathfrak{r}$, i.e., an ideal which contains all solvable ideals, called the *radical* of $\mathfrak{g}$. A Lie algebra $\mathfrak{g}$ is called *semisimple*, if its radical is 0 and its dimension is positive.

Remark 10.4.7 By Remark 10.4.2 the last term of the derived series is an abelian ideal, thus the radical is zero if and only if there is no non-zero abelian ideals in $\mathfrak{g}$.

Remark 10.4.8 If $\mathfrak{g}$ is a simple Lie algebra, then it is semisimple: if $\dim \mathfrak{g} > 1$, it is not abelian (otherwise it would have non-trivial ideals). If it is not abelian, it is not solvable (the absence of non-trivial ideals would make it abelian); thus the radical is a proper ideal and so equal to 0, making $\mathfrak{g}$ semisimple.

Theorem 10.4.9 *Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{r}$ be its radical. Then we have:*

1. *The quotient $\mathfrak{g}/\mathfrak{r}$ is semisimple.*
2. *There is a subalgebra $\mathfrak{s} \cong \mathfrak{g}/\mathfrak{r}$ of $\mathfrak{g}$ which is a complement for $\mathfrak{r}$.*

Hence, $\mathfrak{g}$ is the *semidirect sum* (see [57, §1.6] for the definition) of a semisimple Lie algebra $\mathfrak{s}$ and a solvable Lie algebra $\mathfrak{r}$. This is called a *Levi decomposition* of $\mathfrak{g}$.

10.4.2 The Cartan-Killing Criteria

There is a useful characterization of solvable and semisimple Lie algebras in terms of a symmetric bilinear form called the Killing form. In order to define it we need first to define the adjoint representation of a Lie algebra $\mathfrak{g}$. This can be defined using the Lie bracket of $\mathfrak{g}$, but it can also be defined via the adjoint representation of a complex Lie group G with Lie algebra $\mathfrak{g}$. Since we are also going to use the latter in the sequel we define both representations now.

Let V be a finite-dimensional complex vector space and let G be a complex Lie group (respectively $\mathfrak{g}$ a complex Lie algebra). A homomorphism of complex Lie groups $G \to \mathrm{Aut}(V)$ (respectively a homomorphism of complex Lie algebras $\mathfrak{g} \to \mathrm{End}(V)$) is called a *representation of the Lie group G* (respectively, a *representation of the Lie algebra* $\mathfrak{g}$).

Let G be a complex Lie group. Its Lie algebra $\mathfrak{g}$ is a complex vector space and there is a special representation $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ of G, called the *adjoint representation* or the *adjoint action of G on* $\mathfrak{g}$, defined as follows. Let $g \in G$, conjugation by g in G defines the inner automorphism

$$\begin{aligned} c_g\colon G &\to G \\ h &\mapsto ghg^{-1}. \end{aligned} \tag{10.5}$$

Since $c_g(e) = e$, its differential gives a linear automorphism of the Lie algebra $\mathfrak{g}$ of G

$$d(c_g)_e \colon T_eG \cong \mathfrak{g} \to T_eG \cong \mathfrak{g}, \qquad d(c_g)_e \in \operatorname{Aut}(\mathfrak{g}),$$

and we get the holomorphic map

$$\begin{aligned} \operatorname{Ad} \colon G &\to \operatorname{Aut}(\mathfrak{g}) \\ g &\mapsto d(c_g)_e. \end{aligned} \tag{10.6}$$

Since $c_{g_1g_2} = c_{g_1} \circ c_{g_2}$ for every $g_1, g_2 \in G$, Ad is a representation of G. Let us denote by ad $= d(\operatorname{Ad})_e$ the differential of the adjoint representation of G at e. By Remark 10.2.11 ad_e is a homomorphism of Lie algebras

$$\operatorname{ad} = d(\operatorname{Ad})_e \colon \mathfrak{g} \cong T_eG \to \operatorname{End}(\mathfrak{g}) \cong T_{\mathrm{Id}} \operatorname{Aut}(\mathfrak{g}), \tag{10.7}$$

thus we get a representation called the *adjoint representation of the Lie algebra* $\mathfrak{g}$. We denote $\operatorname{Ad}(g)$ by Ad_g and $\operatorname{ad}(x)$ by ad_x.

Proposition 10.4.10 *Let G be a Lie group with Lie algebra* $\mathfrak{g}$. *Let* $x, y \in \mathfrak{g}$. *Then*

$$\operatorname{ad}_x y = [x, y]. \tag{10.8}$$

Remark 10.4.11 Notice that given a Lie algebra $\mathfrak{g}$ one can define the adjoint representation $\operatorname{ad} \colon \mathfrak{g} \to \operatorname{End}(\mathfrak{g})$ by (10.8) without referring to the Lie group.

Using the adjoint representation a symmetric bilinear form can be defined.

Definition 10.4.12 The symmetric bilinear form

$$\begin{aligned} \mathcal{K} \colon \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathbb{C} \\ (x, y) &\mapsto \operatorname{tr}(\operatorname{ad}_x \circ \operatorname{ad}_y) \end{aligned}$$

is called the *Killing form* of $\mathfrak{g}$, where tr denotes the trace.

Example 10.4.13 The Killing forms of some matrix Lie algebras are the following:

1. On $\mathfrak{gl}(n, \mathbb{C})$, we have $\mathcal{K}(X, Y) = 2n \operatorname{tr}(XY) - 2 \operatorname{tr}(X) \operatorname{tr}(Y)$.
2. On $\mathfrak{sl}(n, \mathbb{C})$, $\mathcal{K}(X, Y) = 2n \operatorname{tr}(XY)$.
3. On $\mathfrak{so}(n)$, $\mathcal{K}(X, Y) = (n - 2) \operatorname{tr}(XY)$.
4. On $\mathfrak{sp}(2n, \mathbb{C})$, $\mathcal{K}(X, Y) = (2n + 2) \operatorname{tr}(XY)$.

A crucial property of the Killing form is the following identity:

$$\mathcal{K}([x, y], z) = \mathcal{K}(x, [y, z]), \tag{10.9}$$

for every $x, y, z \in \mathfrak{g}$. Using the Killing form, we can check whether $\mathfrak{g}$ is solvable [57, §1.9 Theorem A]:

Theorem 10.4.14 (Cartan's 1st Criterion) *A Lie algebra* $\mathfrak{g}$ *is solvable if and only if its Killing form* $\mathcal{K}$ *vanishes identically on the derived Lie algebra* $\mathfrak{g}'$.

There is also another criterion using the Killing form to determine whether $\mathfrak{g}$ is semisimple [57, §1.10 Theorem A]:

Proposition 10.4.15 (Cartan's 2nd Criterion) *A Lie algebra* $\mathfrak{g}$ *is semisimple if and only if its dimension is positive and its Killing form* $\mathcal{K}$ *is non-degenerate.*

Corollary 10.4.16 *A Lie algebra* $\mathfrak{g}$ *is semisimple if and only if it can be written as*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_1 \oplus \dots \mathfrak{g}_r$$

such that $\mathfrak{g}_i$ *is simple for each* $i = 1, \dots, r$.

Proof Suppose $\mathfrak{g}$ is semisimple. For any ideal $\mathfrak{h}$ of $\mathfrak{g}$ the annihilator

$$\mathfrak{h}^{\perp} = \{x \in \mathfrak{g} \,|\, \mathcal{K}(x, y) = 0, \text{ for all } y \in \mathfrak{h}\}$$

is an ideal by (10.9). By Cartan's first criterion $\mathfrak{h} \cap \mathfrak{h}^{\perp}$ is solvable, hence zero, so $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$. The decomposition follows by induction on the dimension of $\mathfrak{g}$.

Conversely, semisimplicity is preserved by direct sum and by Remark 10.4.8 simple implies semisimple. □

Example 10.4.17 The Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ is semisimple since its Killing form $\mathcal{K}$ is non-degenerate. This also follows from Example 10.2.13 and Remark 10.4.8.

10.4.3 Classification of Complex Semisimple Lie Algebras

In this subsection, $\mathfrak{g}$ denotes a finite dimensional complex semisimple Lie algebra and its goal is to sketch their classification.

A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is *self-normalizing* if it is equal to its own normalizer (10.1), i.e., $\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h})$. A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called a *Cartan subalgebra* if $\mathfrak{h}$ is nilpotent and self-normalizing.

Proposition 10.4.18 ([62, III Theorem 1]) *Every Lie algebra admits a Cartan subalgebra.*

Proposition 10.4.19 ([62, III Theorem 2]) *For any two Cartan subalgebras* $\mathfrak{h}$, $\mathfrak{h}'$ *of* $\mathfrak{g}$, *there exits an automorphism* $\sigma : \mathfrak{g} \longrightarrow \mathfrak{g}$ *such that* $\sigma(\mathfrak{h}) = \mathfrak{h}'$.

Definition 10.4.20 The (common) dimension of Cartan subalgebras is called the *rank* of $\mathfrak{g}$ and it is denoted by $\operatorname{rk}(\mathfrak{g})$.

Example 10.4.21 Let $\mathfrak{g} := \mathfrak{sl}(n+1, \mathbb{C})$. Since we have $[X, Y] := XY - YX$ on $\mathfrak{g}$ a Cartan subalgebra $\mathfrak{h}$ is given by the set of diagonal matrices

$$\mathfrak{h} = \left\{ \begin{pmatrix} h_1 & 0 & \dots \\ 0 & h_2 & \dots \\ & \dots & \\ 0 & \dots & h_{n+1} \end{pmatrix} \in \mathfrak{sl}(n, \mathbb{C}) \middle| \sum_{i=1}^{n+1} h_i = 0 \right\}.$$

We have that $\dim \mathfrak{h} = n = \operatorname{rk}(\mathfrak{g})$.

Cartan subalgebras have the following important properties (see [62, III Theorem 3]):

Theorem 10.4.22 *Let* $\mathfrak{h}$ *be a Cartan subalgebra of a complex semisimple Lie algebra* $\mathfrak{g}$. *Then:*

(a) $\mathfrak{h}$ *is abelian,*
(b) ad_h *is diagonalizable for every* $h \in \mathfrak{h}$,
(c) the centralizer of $\mathfrak{h}$ *is* $\mathfrak{h}$ *itself,*
(d) the restriction of the Killing form of $\mathfrak{g}$ *to* $\mathfrak{h}$ *is non-degenerate.*

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let $\mathfrak{h}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ be its dual vector space. By Theorem 10.4.22-(a)–(b), $\{\mathrm{ad}_h\}_{h \in \mathfrak{h}}$ is a commuting family of diagonalizable endomorphisms of $\mathfrak{g}$, and by a standard result in linear algebra, $\{\mathrm{ad}_h\}_{h \in \mathfrak{h}}$ is *simultaneously diagonalizable*, i.e., $\mathfrak{g}$ is the direct sum of the subspaces

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid \mathrm{ad}_h\, x = [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\},$$

where α ranges over $\mathfrak{h}^*$. Since $\mathbb{C}$ is algebraically closed $\alpha(H) \in \mathbb{C}$ exists. In particular, $\mathfrak{g}_0$ is the set of elements $x \in \mathfrak{g}$ commuting with $\mathfrak{h}$, by Theorem 10.4.22-(c) we have that $\mathfrak{g}_0 = \mathfrak{H}$.

Proposition 10.4.23 *One has the following direct sum decomposition:*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha\,.$$

Definition 10.4.24 Any element $\alpha \in \mathfrak{h}^*$ is called a *root* of $\mathfrak{g}$ (relative to $\mathfrak{h}$) if $\mathfrak{g}_\alpha \neq 0$ and $\alpha \neq 0$. We denote by Φ the set of roots of $\mathfrak{g}$.

Given a non-zero element α in $\mathfrak{h}^*$ a *symmetry with vector* α is an automorphism $\tilde{s}$ of $\mathfrak{h}^*$ satisfying the following conditions:

1. $\tilde{s}(\alpha) = -\alpha$.
2. The set U of elements of $\mathfrak{h}^*$ fixed by $\tilde{s}$ is a hyperplane of $\mathfrak{h}^*$.

Let $(\mathfrak{h}^*)^* \cong \mathfrak{h}$ be the dual space of $\mathfrak{h}^*$, and let α^* be the unique element in $(\mathfrak{h}^*)^*$ which vanishes on U and $\alpha^*(\alpha) = 2$. We have

$$\tilde{s}(x) = \lambda - \alpha^*(\lambda)\alpha, \qquad \text{for all } \lambda \in \mathfrak{h}^*.$$

Identifying $\operatorname{End}(\mathfrak{h}^*)$ and $(\mathfrak{h}^*)^* \otimes \mathfrak{h}^*$ we can write $\tilde{s}$ as $\tilde{s} = 1 - \alpha^* \otimes \alpha$. Conversely, if $\alpha \in \mathfrak{h}^*$ and $\alpha^* \in (\mathfrak{h}^*)^*$ are such that $\alpha^*(\alpha) = 2$, then $\tilde{s} = 1 - \alpha^* \otimes \alpha$ is a symmetry with vector α.

Theorem 10.4.25 ([62, VI Theorem 2]) *The set Φ of roots of $\mathfrak{g}$ relative to the Cartan subalgebra $\mathfrak{h}$ is a (reduced) complex root system in $\mathfrak{h}^*$, that is, it satisfies:*

1. *Φ is finite, spans $\mathfrak{h}^*$ and $0 \notin \Phi$.*
2. *For each $\alpha \in \Phi$ there is a symmetry $\tilde{s}_\alpha = 1 - \alpha^* \otimes \alpha$ with vector α on $\mathfrak{h}^*$ which leaves Φ invariant.*
3. *If $\alpha, \beta \in \Phi$ we have that $\tilde{s}_\alpha(\beta) - \beta$ is an integer multiple of α.*
4. *For any $\alpha \in \Phi$, α and $-\alpha$ are the only roots proportional to α (reduced root system).*

The dimension of $\mathfrak{h}^*$ is called the *rank* of the root system, which in this case is equal to the rank of $\mathfrak{g}$. Proposition 10.4.19 shows that the root system of a complex semisimple Lie algebra is independent (up to isomorphism) of the chosen Cartan algebra.

Example 10.4.26 The root system for $\mathfrak{sl}(n+1, \mathbb{C})$. Let E_{ij} be the matrix with 1 on the i-th row and j-th column and 0 elsewhere. Let $H \in \mathfrak{h}$ as in (10.4.21), then we have $[H, E_{ij}] = HE_{ij} - E_{ij}H = h_i E_{ij} - h_j E_{ij} = (h_i - h_j)E_{ij}$. If $i \neq j$, then $E_{ij} \in \mathfrak{sl}(n+1, \mathbb{C})$, and this shows that E_{ij} is a simultaneous eigenvector for each ad_H with $H \in \mathfrak{h}$ with eigenvalue $h_i - h_j$. Thus, the roots are the elements $\alpha_{ij} \in \mathfrak{h}^*$, with $i \neq j$, defined by $\alpha_{ij}(H) = h_i - h_j$ for all $H \in \mathfrak{h}$.

The importance of root systems is that they classify complex semisimple Lie algebras [62, VI Theorem 8 and 9]:

Theorem 10.4.27 *Let Φ be a reduced root system. Then there exists a semisimple Lie algebra whose root system is isomorphic to Φ. Moreover, two semisimple Lie algebras corresponding to isomorphic root systems are isomorphic.*

Definition 10.4.28 The *Weyl group* of the root system Φ is the subgroup W of $GL(\mathfrak{h}^*)$ generated by the symmetries $\tilde{s}_\alpha$ with $\alpha \in \Phi$.

Remark 10.4.29 By Theorem 10.4.25-2 every element of the Weyl group W fixes Φ, and in fact, it is a normal subgroup of the group $\operatorname{Aut}(\Phi)$ of automorphisms of $\mathfrak{h}^*$ leaving Φ invariant. Since Φ spans $\mathfrak{h}^*$ every element of $\operatorname{Aut}(\Phi)$ is completely determined by the permutation of the elements of Φ determined by it; thus, $\operatorname{Aut}(\Phi)$, and therefore also W, can be identified with subgroups of the group of permutations of Φ, and since Φ is finite, also $\operatorname{Aut}(\Phi)$ and W are finite groups.

Since by Theorem 10.4.22-(d) the Killing form $\mathcal{K}$ on $\mathfrak{h}$ is non-degenerate, we have the usual isomorphism of $\mathfrak{h}$ with its dual, that is, for each $\lambda \in \mathfrak{h}^*$ there exists a unique element h_λ in $\mathfrak{h}$ with $\mathcal{K}(h_\lambda, y) = \lambda(y)$ for every $y \in \mathfrak{h}$. We transfer the Killing form to $\mathfrak{h}^*$ in the usual way by setting $\mathcal{K}(\lambda, \mu) := \mathcal{K}(h_\lambda, h_\mu)$. Let $\mathfrak{h}_0^*$ be the *real* subspace of $\mathfrak{h}^*$ spanned by Φ. The Killing form on $\mathfrak{h}^*$ restricted to $\mathfrak{h}_0^*$ is a (real) positive definite bilinear form [57, §2.6 Proposition A] which gives $\mathfrak{h}_0^*$ the structure of an Euclidean space. We denote this inner product on $\mathfrak{h}_0^*$ by $(\cdot,\cdot) := \mathcal{K}(\cdot,\cdot)|_{\mathfrak{h}_0^*}$.

We have that Φ is a (reduced) *real* root system in $\mathfrak{h}_0^*$, that is, it satisfies Axioms 1–4 in Theorem 10.4.25 but replacing the complex vector space $\mathfrak{h}^*$ by the real vector space $\mathfrak{h}_0^*$ and the (complex) symmetries $\tilde{s}_\alpha$ with vector α by *real* symmetries s_α with vector α of $\mathfrak{h}_0^*$. We have that $\mathfrak{h}^*$ is isomorphic to the complexification $\mathfrak{h}_0^* \otimes \mathbb{C}$ of $\mathfrak{h}_0^*$ and $\tilde{s}_\alpha$ is the linear extension of s_α. In fact, any complex root system can be obtained as the complexification of a real root system [62, V Theorem 5]. Hence, it is equivalent to classify complex or real root systems.

Using the Euclidean structure on $\mathfrak{h}_0^*$ given by the Killing form, we can express the symmetries s_α as

$$s_\alpha(\lambda) = \lambda - 2\frac{(\lambda,\alpha)}{(\alpha,\alpha)}\alpha, \qquad \text{for all } \lambda \in \mathfrak{h}_0^*, \tag{10.10}$$

and see that they are orthogonal transformations, i.e., $(s_\alpha(\lambda), s_\alpha(\mu)) = (\lambda, \mu)$ for every $\lambda, \mu \in \mathfrak{h}_0^*$. Define

$$n(\beta,\alpha) := 2\frac{(\beta,\alpha)}{(\alpha,\alpha)},$$

then Axiom 3 of the real root system in $\mathfrak{h}_0^*$ can be written as $n(\beta,\alpha) \in \mathbb{Z}$.

Denote by $\|\alpha\| := (\alpha,\alpha)^{\frac{1}{2}}$ the length of $\alpha \in \mathfrak{h}_0$. Recall that the cosine of the angle ϕ between two vectors $\alpha, \beta \in \mathfrak{h}_0^*$ is given by $(\alpha,\beta) = \|\alpha\|\|\beta\|\cos\phi$, therefore $n(\beta,\alpha) = 2\frac{\|\beta\|}{\|\alpha\|}\cos\phi$, and from this we get

$$n(\alpha,\beta)n(\beta,\alpha) = 4\cos^2\phi.$$

Remark 10.4.30 Axiom 3 restricts the possible angles occurring between pairs of roots, since $n(\beta,\alpha) \in \mathbb{Z}$ and $0 \le \cos^2\phi \le 1$ we have that $4\cos^2\phi$ can only take the values 0, 1, 2, 3, 4, the last case being when α and β are proportional. Table 10.4 shows the only possibilities when $\alpha \neq \pm\beta$ and $\|\beta\| > \|\alpha\|$:

Example 10.4.31 We continue Example 10.4.26 of the root system of $\mathfrak{sl}(n+1,\mathbb{C})$. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{sl}(n+1,\mathbb{C})$ given in Example 10.4.21. Let $H = \operatorname{diag}(h_1,\dots,h_{n+1})$, $K = \operatorname{diag}(k_1,\dots,k_{n+1}) \in \mathfrak{h}$, by Example 10.4.13–2 the restriction of the Killing form to $\mathfrak{h}$ is given by $\mathcal{K}(H,K) = 2(n+1)\sum_{i=1}^{n+1} h_i k_i$. Thus, consider $\mathbb{C}^{n+1}$ with the symmetric bilinear form given by the $(n+1)\times(n+1)$ identity matrix and identify $\mathfrak{h}$ with the subspace V of $\mathbb{C}^{n+1}$ consisting of vectors whose components sum zero. Using this bilinear form we transfer the roots from $\mathfrak{h}^*$

Table 10.4 Possible angles between pairs of roots

$n(\alpha, \beta)$	$n(\beta, \alpha)$	ϕ	$\|\beta\|^2/\|\alpha\|^2$
0	0	$\pi/2$	
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

to $\mathfrak{h}$ to get

$$\alpha_{ij} = e_i - e_j, \qquad \text{with } i \neq j.$$

It is easy to see that for any i and j with $i \neq j$, the symmetry $\tilde{s}_{\alpha_{ij}}$ given by (10.10) acts on $\mathbb{C}^{n+1}$ by interchanging the ith and jth entries of each vector. It follows that the Weyl group is the symmetric group S_{n+1}. Now, let V_0 be the real subspace of V spanned by the roots α_{ij}. The restriction of the bilinear form to V_0 is the restriction of the standard Euclidean inner product $\langle\cdot, \cdot\rangle$ in $\mathbb{R}^{n+1}$ to V_0. Each root has length $\sqrt{2}$ and $\langle\alpha_{ij}, \alpha_{kl}\rangle$ has value 0, ± 1 or ± 2, depending on whether $\{i, j\}$ and $\{k, l\}$ have zero, one or two elements in common. Thus $n(\alpha_{ij}, \alpha_{kl}) \in \{0, \pm 1, \pm 2\}$. If α and β are roots, with $\alpha \neq \beta$ and $\beta \neq -\alpha$ then the angle between α and β is either $\pi/3$, $\pi/2$ or $2\pi/3$ depending on whether $\langle\alpha, \beta\rangle$ is 1, 0 or -1 (see Table 10.4). When $n = 2$ we have 6 roots α_{ij}, $i \neq j$ with $i, j \in \{1, 2, 3\}$. Taking $\alpha = \alpha_{1,2}$ and $\beta = \alpha_{3,1}$ we obtain the root system A_2 in Fig. 10.3b.

Figure 10.3 shows the reduced root systems of rank 2.

A subset Δ of the root system Φ of $\mathfrak{h}_0^*$ is a *base* if:

(i) Δ is a basis for $\mathfrak{h}_0^*$,
(ii) Each $\beta \in \Phi$ can be written as $\beta = \sum_{\alpha \in B} m_\alpha \alpha$ where the coefficients m_α are integers all with the same sign (all $m_\alpha \geq 0$ or all $m_\alpha \leq 0$).

In the literature, a base is also called a *simple root system* or a *fundamental root system* and its elements are called *simple roots*. A base for Φ always exists [62, V Theorem 1]. In Fig. 10.3 the simple roots are indicated by arrows. An important property of a base is the following [62, V Lemma 3]:

Lemma 10.4.32 *If Δ is a base for the root system Φ, then $(\alpha, \beta) \leq 0$ for all roots $\alpha \neq \beta$ in Δ.*

A base Δ of the root system Φ allows us to assign a matrix to Φ. Choose an ordering $\alpha_1, \ldots, \alpha_l$ of the simple roots in Δ.

Definition 10.4.33 The *Cartan matrix* of Φ (with respect to the base Δ) is the matrix $(n(\alpha_i, \alpha_j))_{\alpha_i, \alpha_j \in B}$. Its entries are called *Cartan integers*.

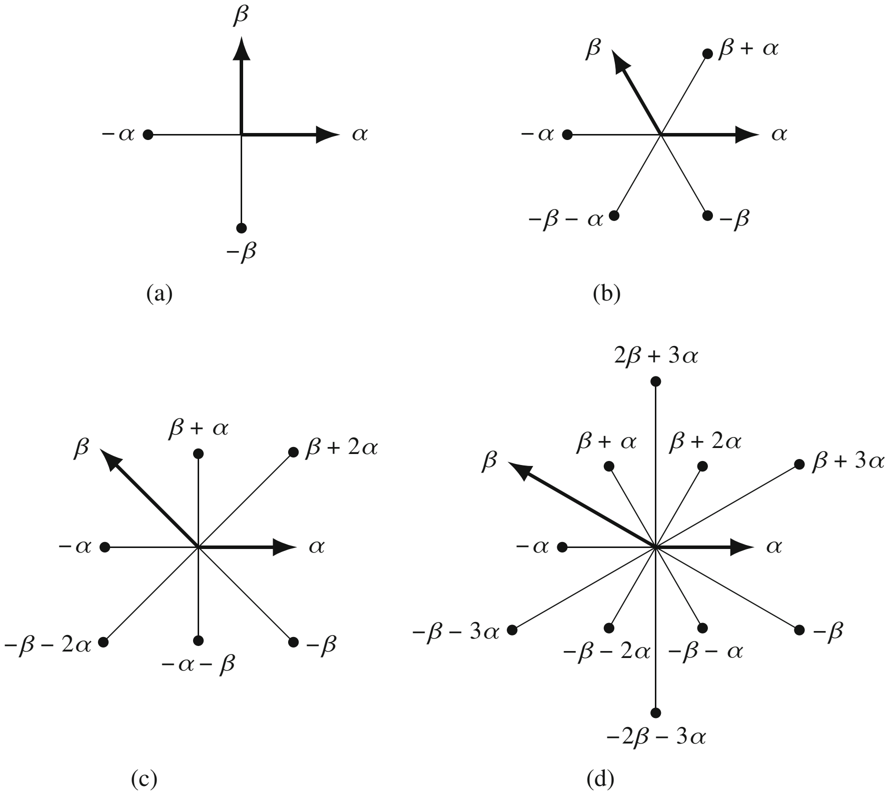


Fig. 10.3 Reduced root systems of rank 2. (**a**) Type $A_1 \times A_1$. (**b**) Type A_2. (**c**) Type B_2. (**d**) Type G_2

By Axiom 3 $n(\alpha_i, \alpha_j)$ is an integer, we have $n(\alpha_i, \alpha_i) = 2$ and if $i \neq j$, by Lemma 10.4.32 $n(\alpha_i, \alpha_j) \leq 0$, so we have $n(\alpha_i, \alpha_j) = 0, -1, -2$ or -3. The Cartan matrix is independent of the choice of base, since the Weyl group acts transitively on the set of all bases of Φ [62, V Theorem 2]. The importance of the Cartan matrix is given by the following proposition [62, V Proposition 8]:

Proposition 10.4.34 *A reduced root system is determined, up to isomorphism, by its Cartan matrix.*

Remark 10.4.35 Let $\tilde{F}$ be the group of permutations of Φ which leave invariant the Cartan matrix, it can be identified with the subgroup of $\mathrm{Aut}(\Phi)$ which leaves the base Δ invariant. In fact, $\mathrm{Aut}(\Phi)$ is the semidirect product of $\tilde{F}$ and the Weyl group W and $\tilde{F} \cong \mathrm{Aut}(\Phi)/W$.

We can also associate a graph to the root system Φ from its Cartan matrix. The *Coxeter graph* of Φ (with respect to the base Δ) is defined as follows: the vertices are the elements of Δ and two distinct vertices α and β are joined by $0, 1, 2$ or 3

Table 10.5 Coxeter graphs of reduced root systems of rank 2

Type	Coxeter graph
$A_1 \times A_1$	
A_2	
B_2	
G_2	

Table 10.6 Dynkin diagrams of root systems of type B_2 and G_2

Type	Dynkin diagram
B_2	
G_2	

vertices as $n(\alpha, \beta)n(\beta, \alpha)$ is equal to $0,\ 1,\ 2$ or 3 (see Remark 10.4.30). Table 10.5 shows the Coxeter graphs of root systems of rank 2.

The information given by the Coxeter graph is not enough to determine the Cartan matrix (and hence the root system), it gives only the angles between the pairs of simple roots (see Table 10.4) without indicating which one is the longer of the two. To remedy this, whenever a double or triple edge occurs in the Coxeter graph of Φ we can add an arrow pointing to the shorter of the two roots. The resulting figure is called the *Dynkin diagram* of Φ. For instance, in Table 10.6 the Dynkin diagrams of the root systems of type B_2 and G_2 are presented.

The extra information allows us to recover the Cartan matrix as follows:

- If $\alpha = \beta$ then $n(\alpha, \beta) = 2$.
- If $\alpha \neq \beta$ and α and β are not joined by any edge, we have $n(\alpha, \beta) = 0$.
- If $\alpha \neq \beta$ and α and β are joined by one edge, then $n(\alpha, \beta) = n(\beta, \alpha) = -1$.
- If $\alpha \neq \beta$ and α and β are joined by i edges, with $i = 2, 3$, then $n(\alpha, \beta) = -1$ and $n(\beta, \alpha) = -i$ if the arrow point to α.

Remark 10.4.36 Specifying a Dynkin diagram is equivalent to specifying a Cartan matrix. They determine the root system up to isomorphism [62, V Proposition 13]. It follows that the group $\tilde{F}$ of automorphisms of the Coxeter matrix (Remark 10.4.35) is isomorphic to the group of automorphisms of the Dynking diagram.

A root system Φ is called *irreducible* if it cannot be partitioned into the union of two proper subsets such that each root in one set is orthogonal to each root on the other. Let Δ be a base of Φ, then Φ is irreducible if and only if Δ cannot be partitioned in two proper subsets Δ_1 and Δ_2 such that each simple root in Δ_1 is orthogonal to each simple root in Δ_2 [26, 10.4]. If a root system is not irreducible, then we have a partition of a basis $\Delta = \Delta_1 \sqcup \Delta_2$ where each root in Δ_1 is orthogonal to each root in Δ_2, thus, their corresponding Cartan integers are zero (see Table 10.4) and its Dynkin diagram is disconnected. In fact we have [62, V Proposition 12]:

Proposition 10.4.37 *A root system Φ is irreducible if and only if its Dynkin diagram is connected and non-empty.*

In general, let $\Phi = \Phi_1 \sqcup \cdots \sqcup \Phi_t$ be the partition of the root system Φ into mutually orthogonal subsets. If V_i is the span of Φ_i we have that $\mathfrak{h}_0^* = V_1 \oplus \cdots \oplus V_t$ and Φ_i is a root system of V_i. We say that Φ is the *sum* of the subsystems Φ_i. Every root system is a sum of irreducible systems [62, V Proposition 11], thus it is enough to classify irreducible root systems.

Theorem 10.4.38 *Each nonempty connected Dynkin diagram is isomorphic to one of the diagrams in Table 10.1.*

One can construct explicitly the root systems corresponding to the Dynkin diagrams of Table 10.1 (see [62, V.16]). A complex Lie algebra is simple if and only if its root system Φ is irreducible, hence the Dynkin diagrams in Table 10.1 correspond to the complex simple Lie algebras. By Corollary 10.4.16, classifying the complex simple Lie algebras we also classify the complex semisimple ones.

Remark 10.4.39 By Remark 10.4.36 the subgroup $\tilde{F}$ of Aut(Φ) can be identified with the group of automorphisms of the Dynkin diagram. From Table 10.1 we have:

- $\tilde{F} = \{1\}$ for types A_1, B_n, C_n, G_2, F_4, E_7 and E_8.
- $\tilde{F}$ is a group of order 2 for A_n ($n \geq 2$), D_n ($n \geq 5$), and E_6.
- $\tilde{F}$ is isomorphic to the symmetric group S_3 for type D_4.

In [52] H. Pinkham defined the root system for a Dynkin diagram in a purely geometric way, and in [73] and [46] the authors generalized that construction for the dual graph of the minimal resolution, of rational singularities of surfaces with reduced fundamental divisors.

10.4.4 Folding of Simply Laced Root Systems

Let Φ be an irreducible reduced root system. Then there are roots of at most two different lengths in Φ corresponding to *short* and *long* roots. All roots of a given length are conjugate under the Weyl group W [26, §10.4 Lemma C]. If all roots have the same length they are taken to be long by definition and the root system is said to be *simply laced* or *homogeneous*. Thus, the irreducible reduced simply laced root systems correspond to the Dynkin diagrams in Table 10.1 with no double or triple edges, that is, of types A_n, D_n, E_6, E_7, E_8. In this subsection we will describe how to obtain the non-simply laced root systems from the simply laced ones, by a procedure known as *folding*. Here we follow [70].

Let $\Delta = \{\alpha_i\}_{i \in I}$ be a base of the irreducible reduced simply laced root system Φ in the real vector space V with inner product $\langle \cdot, \cdot \rangle$. Assume that the roots have been normalize so that $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in \Phi$. Let $\sigma \in \tilde{F}$ be a diagram automorphism, it is equivalent to have a permutation of the index set I such that $\langle \alpha_{\sigma(i)}, \alpha_{\sigma_j} \rangle = \langle \alpha_i, \alpha_j \rangle$ for every $i, j \in I$. Since Δ spans V, by extending linearly the map $\alpha_i \mapsto \alpha_{\sigma(i)}$, we get an isometry of V. Let s_i be the symmetry of V with vector α_i. We have that $\sigma s_i \sigma^{-1} = s_{\sigma(i)}$, thus σ acts as an automorphism $w \mapsto w^\sigma$ of the Weyl group $W = W(\Phi)$.

Now suppose the following condition is satisfied: *simple roots in the same σ-orbit must be orthogonal*, or equivalently, σ-orbits are *edge free sets* in the Dynkin diagram. Let $I^\sigma = \{B_1, \ldots, B_l\}$ denote the set of σ-orbits on I, it is a partition of the set I into disjoint blocks of the form $\{\sigma^k(i) \mid k \in \mathbb{Z}\}$ for various $i \in I$. For each block define $\beta_j = \sum_{i \in B_i} \alpha_i$. Note that each β_j has squared length $2b_j$, with $b_j := |B_j|$. Moreover

$$2\frac{\langle \beta_i, \beta_j \rangle}{\langle \beta_j, \beta_j \rangle} = \frac{N_{ij}}{b_j}, \tag{10.11}$$

where N_{ij} denotes the number of edges in the Dynkin diagram between block B_i and block B_j with $i \neq j$. Since σ acts transitively on each block, if follows that each of the b_j nodes in B_j have the same number of neighbours in B_i. Hence, for $i \neq j$

$$-2\frac{\langle \beta_i, \beta_j \rangle}{\langle \beta_j, \beta_j \rangle} = \# \text{ of nodes in } B_i \text{ adjacent to any fixed member of } B_j.$$

In particular, $\Delta^\sigma := \{\beta_1, \ldots, \beta_l\}$ forms a set of simple roots for some root system Φ^σ on V but probably non-simply laced. The root system Φ^σ is called the *folding* of the simply laced root system Φ. The relation between the root systems Φ and Φ^σ are given by the following theorem:

Theorem 10.4.40 *Let Φ be an irreducible reduced simply laced root system in the real vector space V and let $\sigma \in \tilde{F}$. Then*

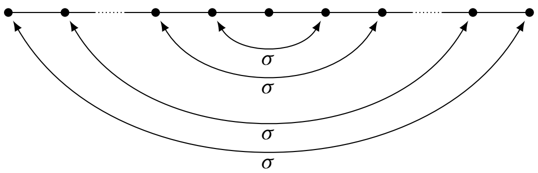
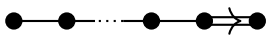
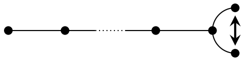
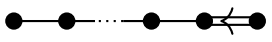
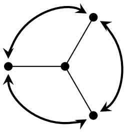
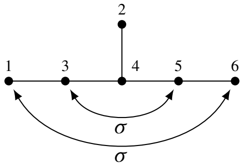
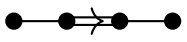
1. *If β is a sum of pairwise orthogonal roots in Φ comprising a single σ-orbit, then β is a root in Φ^σ. Conversely, all roots of Φ^σ have this form.*
2. *For each block B_j, let t_j be the symmetry with vector β_j and β_j is a simple root of Φ^σ. Let $\bar{s}_j := \prod_{i \in B_j} s_i$. Then $\bar{s}_j$ is fixed under conjugation by σ and the map $t_j \mapsto \bar{s}_j$ extends to an isomorphism from the Weyl group $W(\phi^\sigma)$ to W^σ, the subgroup of W fixed by σ.*
3. *Every root system may be realized as the folding Φ^σ of a simply laced root system Φ by some diagram automorphism σ.*

Table 10.7 shows the foldings of the irreducible simply laced root systems.

10.5 From ADE Lie Algebras to Kleinian Singularities

In this section we state Brieskorn's theorem, which relates the complex simple Lie algebras of type ADE with the corresponding Kleinian singularities. We recommend the survey article [39] which complements what we present here.

Table 10.7 Foldings of simply laced root systems

Simply laced root system		Folding
A_{2n-1} 	$\Longrightarrow$	B_n 
D_{n+1} 	$\Longrightarrow$	C_n 
D_4 	$\Longrightarrow$	G_2 
E_6 	$\Longrightarrow$	F_4 

In this section all the Lie algebras are complex Lie algebras and all dimensions are complex dimensions.

10.5.1 Nilpotent Varieties of Lie Algebras

We start defining a special algebraic variety inside a complex simple Lie algebra. Before defining the nilpotent variety of a Lie algebra, we need first to recall the Jordan-Chevalley decomposition (see [26, §4.2]).

Definition 10.5.1 Let V be a finite dimensional complex vector space and let $x \in \mathrm{End}(V)$. We call x *semisimple* if it is diagonalizable. On the other hand, we call x *nilpotent* if $x^k = 0$ for some $k > 0$.

Theorem 10.5.2 (Jordan-Chevalley Decomposition) *Let V a finite dimensional complex vector space and $x \in \mathrm{End}(V)$. There exist unique $x_s, x_n \in \mathrm{End}(V)$ such that $x = x_s + x_n$, with x_s semisimple, x_n nilpotent and x_s and x_n commute.*

The elements $x_s, x_n \in \mathrm{End}(V)$ are called respectively the *semisimple part* and the *nilpotent part* of x.

It is possible to introduce an *abstract Jordan-Chevalley decomposition* on an arbitrary complex semisimple Lie algebra $\mathfrak{g}$ (see [26, §5.4]). Let $x \in \mathfrak{g}$ and consider its image $\mathrm{ad}_x \in \mathrm{End}(\mathfrak{g})$ under the adjoint representation given in (10.7). Take its Jordan-Chevalley decomposition

$$\mathrm{ad}_x = (\mathrm{ad}_x)_s + (\mathrm{ad}_x)_n.$$

If $\mathfrak{g}$ is semisimple, $\mathfrak{g} \to \mathrm{ad}\,\mathfrak{g}$ is an isomorphism of Lie algebras and therefore there exists unique elements $x_s, x_n \in \mathfrak{g}$ such that $\mathrm{ad}_{x_s} = (\mathrm{ad}_x)_s$ and $\mathrm{ad}_{x_n} = (\mathrm{ad}_x)_n$, hence $x = x_s + x_n$ with $[x_s, x_n] = 0$. We call x_s and x_n respectively, the *semisimple* and *nilpotent* parts of x. In case $\mathfrak{g}$ is a linear complex Lie algebra, i.e., a matrix Lie algebra, the usual Jordan-Chevalley and the abstract Jordan-Chevalley decompositions coincide.

Definition 10.5.3 Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{C}$. The subset

$$\mathcal{N}(\mathfrak{g}) := \{x \in \mathfrak{g} \mid \mathrm{ad}_x \text{ is nilpotent.}\}$$

is called the *nilpotent variety* of $\mathfrak{g}$.

For a Lie algebra $\mathfrak{g}$, we say that the nilpotent variety $\mathcal{N}(\mathfrak{g})$ is *trivial* if for some k, $\mathcal{N}(\mathfrak{g}) \cong \mathbb{C}^k$.

Proposition 10.5.4 ([26, §3.2]) *The Lie algebra* $\mathfrak{g}$ *is nilpotent if and only if for every* $x \in \mathfrak{g}$ *the endomorphism* $\mathrm{ad}\,x$ *is nilpotent.*

Corollary 10.5.5 *If* $\mathfrak{g}$ *is a nilpotent Lie algebra, then* $\mathcal{N}(\mathfrak{g})$ *is trivial.*

Proof By Proposition 10.5.4 for any $x \in \mathfrak{g}$, ad_x is nilpotent. Hence $\mathcal{N}(\mathfrak{g}) = \mathfrak{g}$. □

Proposition 10.5.6 ([48]) *If* $\mathfrak{g}$ *is a solvable Lie algebra, then* $\mathcal{N}(\mathfrak{g})$ *is trivial.*

Hence we only consider the non-solvable Lie algebras $\mathfrak{g}$ for finding non-trivial nilpotent varieties. The following characterizations of nilpotent matrices are useful to compute nilpotent varieties.

Proposition 10.5.7 *Let* $X \in \mathfrak{gl}(n, \mathbb{C})$. *The following are equivalent:*

- *(a) X is nilpotent.*
- *(b) The characteristic polynomial of X is* t^n.
- *(c)* $\mathrm{tr}(X^k) = 0$ *for all* $k > 0$.

The next example shows that considering the nilpotent variety of a complex simple Lie algebra is taking us into the right direction for our purpose.

Example 10.5.8 Let $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$, the simple Lie algebra of type A_n (see Table 10.1). By Proposition 10.5.7-(c) we have

$$\mathcal{N}(\mathfrak{g}) = \{X \in \mathfrak{g} \mid tr(X^2) = tr(X^3) = \ldots = tr(X^{n+1}) = 0\}.$$

When $n = 1$, $\mathfrak{sl}(2, \mathbb{C})$ is the simple Lie algebra of type A_1, we have that $\dim \mathfrak{sl}(2, \mathbb{C}) = 3$ and

$$\mathcal{N}(\mathfrak{g}) = \left\{ X = \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \,\middle|\, x^2 + yz = 0 \right\}.$$

Hence the nilpotent variety of $\mathfrak{g}$ is defined by one equation, so it is a surface with singularity of type A_1 (see Table 10.3) as we wanted.

However, for $n > 1$, we have that $\dim_{\mathbb{C}} \mathfrak{sl}(n+1, \mathbb{C}) = (n+1)^2 - 1$ and $\mathcal{N}(\mathfrak{g})$ is given by n equations, so $\dim_{\mathbb{C}} \mathcal{N}(\mathfrak{g}) = n(n+1) > 2$. In order to get a surface, we need to intersect $\mathcal{N}(\mathfrak{g})$ with an appropriate subvariety $\mathcal{S}$ of $\mathfrak{g}$.

10.5.2 The Adjoint Quotient

In this subsection we will see the nilpotent variety as a fibre of a map called the adjoint quotient and we will describe the nilpotent variety as union of orbits of the adjoint action.

Let G be a complex simple Lie group, let $\mathfrak{g}$ be its Lie algebra. Consider the adjoint action $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ of G on $\mathfrak{g}$ defined in (10.6). Given $x \in \mathfrak{g}$ we denote its orbit by $\mathcal{O}_x$. The stabilizer of x is the Lie subgroup of G defined by

$$Z_G(x) = \{g \in G \mid \mathrm{Ad}_g(x) = x\}.$$

Hence we have that the orbit $\mathcal{O}_x$ of x is isomorphic to the homogeneous space $G/Z_G(x)$ and therefore $\dim_{\mathbb{C}} \mathcal{O}_x = \dim_{\mathbb{C}} G - \dim_{\mathbb{C}} Z_G(x)$. The Lie algebra of $Z_G(x)$ is precisely $Z_{\mathfrak{g}}(x)$, the centralizer of x defined in (10.2). Since $\dim_{\mathbb{C}} G = \dim_{\mathbb{C}} \mathfrak{g}$ and $\dim_{\mathbb{C}} Z_G(x) = \dim_{\mathbb{C}} Z_{\mathfrak{g}}(x)$ we have that

$$\dim_{\mathbb{C}} \mathcal{O}_x = \dim_{\mathbb{C}} \mathfrak{g} - \dim_{\mathbb{C}} Z_{\mathfrak{g}}(x). \tag{10.12}$$

The following theorem by B. Kostant gives a more precise description of the dimension of the orbits.

Theorem 10.5.9 ([37]) *For all $x \in \mathfrak{g}$,*

1. *The dimension of the orbit $\mathcal{O}_x$ is even for every $x \in \mathfrak{g}$.*
2. $\dim_{\mathbb{C}} Z_{\mathfrak{g}}(x) - \mathrm{rk}(\mathfrak{g}) \in 2\mathbb{Z} \geq 0$, *where* $\mathrm{rk}(\mathfrak{g})$ *is the rank of* $\mathfrak{g}$.

By (10.12) the dimension of the orbit O_x of an element $x \in \mathfrak{g}$ is maximal if the dimension of its centralizer $Z_{\mathfrak{g}}(x)$ is minimal. By Theorem 10.5.9 the minimal dimension of a centralizer $Z_{\mathfrak{g}}(X)$ is $\mathrm{rk}(\mathfrak{g})$.

Corollary 10.5.10 *The maximal dimension of the orbits of the adjoint action is* $\dim_{\mathbb{C}} \mathfrak{g} - \mathrm{rk}(\mathfrak{g})$.

Definition 10.5.11 An orbit of maximal dimension $\dim_{\mathbb{C}} \mathfrak{g} - \mathrm{rk}(\mathfrak{g})$ is called a *regular orbit* and the elements in a regular orbit are called *regular elements*.

Definition 10.5.12 An element $x \in \mathfrak{g}$ such that $\dim_{\mathbb{C}} Z_{\mathfrak{g}}(x) = \mathrm{rk}(\mathfrak{g}) + 2$ is called a *subregular element* and its orbit is called a *subregular orbit*.

Proposition 10.5.13 ([69, §3.10 Theorem 1]) *There is a unique subregular nilpotent G-orbit in* $\mathfrak{g}$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, let $r = \dim_{\mathbb{C}} \mathfrak{h} = \mathrm{rk}(\mathfrak{g})$, and let W be the Weyl group corresponding to $\mathfrak{h}$. By Definition 10.4.28 the Weyl group acts on $\mathfrak{h}^*$ and this action in turn induces an action on $\mathfrak{h}$. Let $\mathbb{C}[\mathfrak{g}]$ (respectively $\mathbb{C}[\mathfrak{h}]$) be the algebra of polynomial functions on $\mathfrak{g}$ (respectively on $\mathfrak{h}$). Let $\mathbb{C}[\mathfrak{g}]^G$ (respectively $\mathbb{C}[\mathfrak{h}]^W$) denote the subalgebra of G-invariant polynomial functions on $\mathfrak{g}$ (respectively W-invariant polynomial functions on $\mathfrak{h}$). It is well known that a function on $\mathfrak{g}$ is invariant under the action of G on $\mathfrak{g}$ if and only if the function is constant on every orbit.

Proposition 10.5.14 *Let $x \in \mathfrak{g}$ and let $x = x_s + x_n$ be its (abstract) Jordan-Chevalley decomposition. For every $f \in \mathbb{C}[\mathfrak{g}]^G$ we have that $f(x) = f(x_s)$, in other words, an invariant polynomial function on $\mathfrak{g}$ is determined by the semisimple elements of* $\mathfrak{g}$.

Proof The algebra $\mathbb{C}[\mathfrak{g}]^G$ is generated by polynomial functions of the form $x \to \mathrm{tr}(\phi(x)^r)$ where $\phi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is a finitely dimensional representation of $\mathfrak{g}$ [2, Ch. VIII, §8, Theorem 1 (ii)]. If $x = x_s + x_n$ is the (abstract) Jordan-Chevalley decomposition of x, then $\phi(x) = \phi(x_s) + \phi(x_n)$ is the (usual) Jordan-Chevalley decomposition [26, §6.4 Corollary]. Since $\phi(x_s)$ and $\phi(x_n)$ commute, all terms except $\phi(x_s)^k$ in the expansion of $(\phi(x_s) + \phi(x_n))^k$ are nilpotent, hence of trace 0. □

The following theorem compares the previous algebras of invariant polynomials, its proof can be found in [2, Ch. VIII, §8, Theorem 1].

Theorem 10.5.15 (Grothendieck-Chevalley Theorem) *Consider the restriction homomorphism*

$$\iota\colon \mathbb{C}[\mathfrak{g}] \longrightarrow \mathbb{C}[\mathfrak{h}].$$

(i) The restriction $\iota|_{\mathbb{C}[\mathfrak{g}]^G}$ is an isomorphism $\iota|_{\mathbb{C}[\mathfrak{g}]^G} : \mathbb{C}[\mathfrak{g}]^G \longrightarrow \mathbb{C}[\mathfrak{h}]^W$.

(ii) Let r be the rank of $\mathfrak{g}$. There are r algebraically independent homogeneous polynomials $\gamma_1, \ldots, \gamma_r \in \mathbb{C}[\mathfrak{g}]^G$ that generate the algebra $\mathbb{C}[\mathfrak{g}]^G$.

Definition 10.5.16 By Theorem 10.5.15 the inclusion $\mathbb{C}[\mathfrak{h}]^W \cong \mathbb{C}[\mathfrak{g}]^G \hookrightarrow \mathbb{C}[\mathfrak{g}]$ induces the morphism

$$\gamma\colon \mathfrak{g} \longrightarrow \mathfrak{h}/W$$

called the *adjoint quotient map* of $\mathfrak{g}$.

Proposition 10.5.17 ([37]) *The adjoint quotient map $\gamma\colon \mathfrak{g} \longrightarrow \mathfrak{h}/W$ has the following properties:*

1. *The morphism γ is flat; all fibers of γ have dimension $\dim_{\mathbb{C}} \mathfrak{g} - r$.*
2. *All fibres are normal and they are a union of finitely many G-orbits in $\mathfrak{g}$.*
3. *Each fibre contains a unique regular orbit and hence it is dense in the fibre.*
4. *The adjoint quotient map can be realized using the generators of $\mathbb{C}[\mathfrak{g}]^G$ given in Theorem 10.5.15, that is,*

$$\begin{aligned}\gamma\colon \mathfrak{g} &\longrightarrow \mathfrak{h}/W \cong \mathbb{C}^r,\\ x &\longmapsto (\gamma_1(x), \dots, \gamma_r(x)).\end{aligned}$$

5. *An element $x \in \mathfrak{g}$ is a regular element if and only if is a regular point of γ, i.e., the differential of γ at x has maximal rank.*
6. *If $x = x_s + x_n$ is the Jordan-Chevalley decomposition of $x \in \mathfrak{g}$ we have $\gamma(x) = \gamma(x_s)$ for every $x \in \mathfrak{g}$.*

Corollary 10.5.18 *The fibre of γ which contains* 0*, i.e., $\gamma^{-1}(\gamma(0))$ is the nilpotent variety $\mathcal{N}(\mathfrak{g})$.*

Example 10.5.19 Consider Example 10.2.8-2. Let $G = SL(n+1, \mathbb{C})$ and let $\mathfrak{g}$ be its Lie algebra $\mathfrak{sl}(n+1, \mathbb{C})$. By Example 10.4.21 a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is given by the zero-trace diagonal matrices, thus $\operatorname{rk}(\mathfrak{g}) = \dim_{\mathbb{C}} \mathfrak{h} = n$. By Example 10.4.31 the Weyl group is the symmetric group S_{n+1} and it acts on $\mathfrak{h}$ permuting the entries in the diagonal. The adjoint representation is given by matrix conjugation

$$\begin{aligned}G \times \mathfrak{g} &\to \mathfrak{g},\\ (g, x) &\mapsto gxg^{-1}.\end{aligned}$$

Let $X \in \mathfrak{g}$. Consider the characteristic polynomial of X

$$P_X(t) = \det(tI - X) = t^{n+1} + a_n(X)t^n + a_{n-1}(X)t^{n-1} + \cdots + a_1(X)t + a_0(X).$$

The coefficients a_i with $i = 0, \dots, n$ are polynomial functions on $\mathfrak{g}$ and they are clearly invariant under the adjoint action. Since $a_n(X) = \operatorname{tr}(X) = 0$ for every $X \in \mathfrak{g}$, we have $\operatorname{rk}(\mathfrak{g}) = n$ non-zero invariant polynomials a_i, $i = 0, \dots, n-1$ and in fact they are a set of generators of $\mathbb{C}[\mathfrak{sl}(n, \mathbb{C})]^{SL(n,\mathbb{C})}$ as in Theorem 10.5.15. Hence the adjoint quotient is given by

$$\begin{aligned} \gamma \colon \mathfrak{g} &\longrightarrow \mathbb{C}^n, \\ X &\longmapsto (a_0(X), \ldots, a_{n-1}(X)). \end{aligned} \tag{10.13}$$

If $X \in \mathfrak{g}$ is a nilpotent element by Proposition 10.5.7-(b) $a_i(X) = 0$ for $i = 1, \ldots, n$, and in this case the nilpotent variety $\mathcal{N}(\mathfrak{g})$ is the fibre over zero $\gamma^{-1}(0)$. Hence, $a_i(X) = 0$ with $i = 1, \ldots, n$ are the n equations that define $\mathcal{N}(\mathfrak{g})$ in $\mathfrak{g}$ (compare with Example 10.5.8).

For the case $n = 2$, the adjoint quotient for $\mathfrak{sl}(3, \mathbb{C})$ is given by

$$\gamma \colon \mathfrak{g} \longrightarrow \mathbb{C}^2, \tag{10.14}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & -a-e \end{pmatrix} \mapsto (-abd+a^2e-bde+ae^2+ceg-bfg-cdh+afh, -a^2-bd-ae-e^2-cg-fh).$$

10.5.3 Slodowy Slices for Kleinian Singularities

As in the previous subsection, let G be a complex simple Lie group of dimension n, let $\mathfrak{g}$ be its Lie algebra of rank r. By Proposition 10.5.17 the nilpotent variety $\mathcal{N}(\mathfrak{g})$ of $\mathfrak{g}$ is a fibre of the adjoint quotient (say $\mathcal{N}(\mathfrak{g}) = \gamma^{-1}(0)$) which contains a unique regular nilpotent orbit $\mathcal{O}_{\text{reg}}$ and a unique subregular nilpotent orbit $\mathcal{O}_{\text{sreg}}$. This description of $\mathcal{N}(\mathfrak{g})$ will allow us to find a subvariety $\mathcal{S}$ of $\mathfrak{g}$ such that $\mathcal{N}(\mathfrak{g}) \cap \mathcal{S}$ is a surface with an isolated singularity.

Definition 10.5.20 Let $x \in \mathfrak{g}$ and let $\mathcal{O}_x$ be its G-orbit. A *transverse slice* to $\mathcal{O}_x$ at x is a smooth locally closed subvariety $\mathcal{S} \subset \mathfrak{g}$, such that $x \in \mathcal{S}$, $\dim_{\mathbb{C}} \mathcal{S} = \operatorname{codim} \mathcal{O}_x = \dim_{\mathbb{C}} Z_{\mathfrak{g}}(x)$, and the map $G \times \mathcal{S} \to \mathfrak{g}$ defined by $(g, s) \mapsto \mathrm{Ad}_g(s)$ is a smooth submersion.

A transverse slice can be obtained taking an affine subspace in $\mathfrak{g}$ complementary to the affine tangent space of $\mathcal{O}_x$ at x. We are interested in a transverse slice $\mathcal{S}_x$ to the subregular nilpotent orbit $\mathcal{O}_{\text{sreg}}$ at a subregular nilpotent element $x \in \mathcal{O}_{\text{sreg}}$ since we have the following:

- We have $\mathcal{S}_x \cap \mathcal{N}(\mathfrak{g}) = (\mathcal{S}_x \cap \mathcal{O}_{\text{sreg}}) \sqcup (\mathcal{S}_x \cap \mathcal{O}_{\text{reg}})$.
- $\mathcal{S}_x$ intersects $\mathcal{O}_{\text{sreg}}$ only at x.
- The slice $\mathcal{S}_x$ is also transverse to the regular nilpotent orbit $\mathcal{O}_{\text{reg}}$, thus $\mathcal{S}_x \cap \mathcal{O}_{\text{reg}}$ is a complex submanifold of $\mathcal{O}_{\text{reg}}$.
- $\operatorname{codim}_{\mathbb{C}} \mathcal{S}_x = n - r - 2$, $\operatorname{codim}_{\mathbb{C}} \mathcal{O}_{\text{reg}} = r$, so $\operatorname{codim}_{\mathbb{C}}(\mathcal{S}_x \cap \mathcal{O}_{\text{reg}}) = n - 2$, therefore $\dim_{\mathbb{C}} \mathcal{S}_x \cap \mathcal{O}_{\text{reg}} = 2$.

Hence, $\mathcal{S}_x \cap \mathcal{N}(\mathfrak{g})$ is a complex surface with an isolated singularity at x. Another way to see this is to consider the restriction of the adjoint quotient to $\mathcal{S}_x$

$$\tilde{\gamma} = \gamma|_{\mathcal{S}_x} : \mathcal{S}_x \to \mathbb{C}^r.$$

Then, $\tilde{\gamma}^{-1}(0) = \mathcal{S}_x \cap \mathcal{N}(\mathfrak{g})$ is an algebraic surface since $\dim_{\mathbb{C}} \mathcal{S}_x = r + 2$. Also, by Proposition 10.5.17-(5), the points in $\mathcal{S}_x \cap \mathcal{O}_{\text{reg}}$ are regular points of $\tilde{\gamma}$ and x is a critical point of $\tilde{\gamma}$, therefore $\tilde{\gamma}^{-1}(0) = \mathcal{S}_x \cap \mathcal{N}(\mathfrak{g})$ is an algebraic surface with an isolated singularity.

Now we can state the theorem which relates directly the simple Lie algebras with Kleinian singularities, conjectured by A. Grothendieck. As we mentioned in the Introduction, E. Brieskorn announced a proof of Grothendieck's conjecture at the International Congress of Mathematicians at Nice in 1970 and sketched a proof in its Proceedings [7]. A proof of this theorem was given in H. Esnault's thesis [17] following a geometric idea by Grothendieck. Later, the details of Brieskorn's proof were given by P. Slodowy in [65].

Theorem 10.5.21 *Let $\mathfrak{g}$ be a simple Lie algebra of type $\Delta = A_r,\ D_r$ or E_r. Let $x \in \mathfrak{g}$ be a subregular nilpotent element of $\mathfrak{g}$. Let $\mathcal{S}_x \subset \mathfrak{g}$ be a transverse slice at x to the G-orbit of x (the subregular nilpotent orbit). Then*

1. *the intersection $\mathcal{S}_x \cap \mathcal{N}(\mathfrak{g})$ is a surface with a Kleinian singularity of type Δ.*
2. *The restriction $\gamma|_{\mathcal{S}_x} : \mathcal{S}_x \to \mathfrak{h}/W$ is a semi-universal deformation of the singularity $\mathcal{S}_x \cap \mathcal{N}(\mathfrak{g})$.*

There is a canonical way to construct transverse slices by P. Slodowy in [65, §7.4]. To do this we need the following theorem:

Theorem 10.5.22 (Jacobson-Morozov Theorem) *Let $\mathfrak{g}$ be a semisimple Lie algebra. Let $x \in \mathfrak{g}$ be a nonzero nilpotent element in $\mathfrak{g}$. There exist elements $y, h \in \mathfrak{g}$ such that $[h, x] = 2x,\ [h, y] = -2y,\ [x, y] = h$.*

The triple (x, y, h) of such elements in $\mathfrak{g}$ is called an $\mathfrak{sl}_2$-*triple* in $\mathfrak{g}$.

Let $x \in \mathfrak{g}$ be a nonzero nilpotent element in $\mathfrak{g}$. Computing the differential of the orbit map $G \to \mathfrak{g}$ given by $g \mapsto \text{Ad}_g(x)$ shows that the tangent space to the orbit $\mathcal{O}_x$ at x is given by $T_x(\mathcal{O}_x) = x + [\mathfrak{g}, x]$. A transverse slice $\mathcal{S}$ has the form $\mathcal{S} = x + V$, where V is any linear complement to $[\mathfrak{g}, x]$.

Definition 10.5.23 For a fixed $\mathfrak{s}l_2$-triple (x, y, h) in $\mathfrak{g}$, the affine space

$$\mathcal{S}_x = x + Z_{\mathfrak{g}}(y)$$

is called a *Slodowy slice at x*.

Though $\mathfrak{sl}_2$-triples in $\mathfrak{g}$ are not unique, they are all $\mathfrak{s}l_2$-conjugate and hence they give isomorphic Slodowy slices.

Example 10.5.24 Let $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$. The nilpotent variety is given by $\mathcal{N}(\mathfrak{g}) = \gamma^{-1}((0, 0))$, where γ is the adjoint quotient map given in (10.14), thus

$$\mathcal{N}(\mathfrak{g}) = \left\{ X = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & -a-e \end{pmatrix} \;\middle|\; \begin{matrix} abd-a^2e+bde-ae^2-ceg+bfg+cdh-afh=0 \\ a^2+bd+ae+e^2+cg+fh=0 \end{matrix} \right\}$$

and it is defined by two equations.

A regular and a subregular elements are respectively

$$X' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence the regular and subregular orbits are

$$\mathcal{O}_{\text{reg}} = SL(3, \mathbb{C}) \cdot X', \qquad \text{and} \qquad \mathcal{O}_{\text{sreg}} = SL(3, \mathbb{C}) \cdot X.$$

The subregular element X lies in an $\mathfrak{sl}_2$-triple (X, Y, H) with

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The corresponding transverse Slodowy slice $\mathcal{S}_X$ is given by

$$\mathcal{S}_X = X + Z_{\mathfrak{g}}(Y),$$

where $Z_{\mathfrak{g}}(Y)$ is the centralizer of Y in $\mathfrak{g}$ which is given by

$$Z_{\mathfrak{g}}(Y) = \left\{ \begin{pmatrix} x & 0 & 0 \\ w & x & y \\ z & 0 & -2x \end{pmatrix} \right\}$$

Hence

$$\mathcal{S}_X = \left\{ \begin{pmatrix} x & 1 & 0 \\ w & x & y \\ z & 0 & -2x \end{pmatrix} \right\}$$

Notice that $\mathcal{S}_X$ is isomorphic to $\mathbb{C}^4$. The intersection of $\mathcal{N}(\mathfrak{g})$ with the slice $\mathcal{S}_X$ consists of the elements in $\mathcal{S}_X$ which satisfy the equations defining $\mathcal{N}(\mathfrak{g})$, i.e., whose coefficients of its characteristic polynomial vanishes. So they satisfy

$$-2x^3 + 2xw + yz = 0. \tag{10.15}$$

$$3x^2 + w = 0, \tag{10.16}$$

By (10.16) we have that $w = -3x^2$ and substituting in (10.15) we get

$$8x^3 - yz = 0 \tag{10.17}$$

which is the equation defining an A_2-singularity (see Table 10.3).

Now, restricting the adjoint quotient (10.14) to the Slodowy slice we obtain the map

$$\begin{aligned} \gamma \colon \mathcal{S}_X &\cong \mathbb{C}^4 \longrightarrow \mathbb{C}^2, \\ X = \begin{pmatrix} x & 1 & 0 \\ w & x & y \\ z & 0 & -2x \end{pmatrix} &\mapsto (2x^3 - yz - 2wx, -w - 3x^2), \end{aligned} \tag{10.18}$$

setting $t = -w - 3x^2$ we get

$$w = 3x^2 - t,$$

and substituting in (10.18) we get

$$\begin{aligned} \gamma \colon \mathcal{S}_X \cong \mathbb{C}^4 &\longrightarrow \mathbb{C}^2, \\ (x, y, z, t) &\mapsto (8x^3 - yz + 2tx, t), \end{aligned}$$

the semi-universal deformation of the A_2-singularity (10.17) given in (10.4).

10.6 Further Extensions

Slodowy extended Theorem 10.5.21 for complex Lie algebras of type B_n, C_n, F_4 and G_2 which are the ones with non-simply laced root systems (with roots of different lengths). Their corresponding Dynkin diagrams are showed in Table 10.1.

Roughly speaking, these diagrams are quotients of the dual graphs of the minimal resolutions of Kleinian singularities by certain symmetries. Slodowy defines a *simple singularity of type* B_n, C_n, F_4 or G_2 as a couple (S, F) of a simple (Kleinian) singularity S and a group F of automorphisms of S according to the following list, in all cases $F = \Gamma'/\Gamma$ operates naturally on $\mathbb{C}^2/\Gamma$ (see Table 10.8 and compare with Table 10.3).

Table 10.8 Simple singularities of type B_n, C_n, F_4 and G_2

Type (S, F)	Type S	Γ	Γ'	F
B_n	A_{2n-1}	C_{2n}	$\mathcal{D}_n$	$\mathbb{Z}_2$
C_n	D_{n+1}	$\mathcal{D}_{n-1}$	$\mathcal{D}_{2(n-1)}$	$\mathbb{Z}_2$
F_4	E_6	$\mathcal{T}$	$\mathcal{O}$	$\mathbb{Z}_2$
G_2	D_4	$\mathcal{D}_2$	$\mathcal{O}$	S_3

The action of F lifts in a unique way to an action on the minimal resolution $\tilde{S}$ of S. Since F fixes the singular point of S, the exceptional set E of $\tilde{S}$ will be stable under the action of F and behaves as a permutation group of its components. Note that the group F corresponds to the group $\tilde{F}$ given in Remark 10.4.39 for the simply laced root systems. The diagram of (S, F) is obtained as an "F-quotient" of the dual graph of the minimal resolutions of S given in Table 10.3, i.e., by the folding of ADE-Dynkin diagrams given in Table 10.7, see [65, §6.2] for details.

Slodowy also extended these results for arbitrary fields (not necessarily algebraically closed) under some mild conditions on the characteristic of the field (see [64, 65]). In [60, 61] the authors obtain deformations of simple singularities of dimensions different from two, for instance, of the simple curve singularities. In [68], the author introduced a natural analogue of the slice $\mathcal{S}_x$ for regular nilpotent x. Note that, extending these results to other nilpotent orbits may not give an isolated singularity. But, it will be interesting to classify all singularities in this way.

10.7 Simple Elliptic Singularities

Citing Looijenga in [42] "In V. I. Arnold's hierarchy of isolated hypersurface singularities the simple-elliptic singularities come next to the simple singularities and this property actually characterizes them".

Let E be an elliptic curve and let L be a holomorphic line bundle over E with negative Chern class c. Contracting the zero-section to a point one obtains a surface singularity S called a *simple elliptic singularity*. The surface S is a complete intersection if and only if $c \geq -4$. In [56] K. Saito proved that for $c = -3, -2, -1$ the surface S is an *hypersurface simple elliptic singularity* of type $\tilde{E}_{9-c}$ with equation:

$$\tilde{E}_6: \quad x^3 + y^3 + z^3 + \lambda xyz = 0,$$

$$\tilde{E}_7: \quad x^4 + y^4 + z^2 + \lambda xyz = 0,$$

$$\tilde{E}_8: \quad x^6 + y^3 + z^2 + \lambda xyz = 0,$$

where $\lambda \in \mathbb{C}$ is such that the singularity is isolated. The simple elliptic singularity with $c = -4$ is an isolated complete intersection singularity in $\mathbb{C}^4$ of type $\tilde{D}_5$

$$\tilde{D}_5 : \begin{cases} x^2 + y^2 + \lambda z w = 0, \\ xy + z^2 + w^2 = 0, \end{cases} \tag{10.19}$$

where $\lambda \in \mathbb{C}$ with $\lambda \neq 0, 1/4$.

As we mentioned in the Introduction, instead of using a complex simple Lie algebra, one can obtain the Kleinian singularities from the corresponding simple complex Lie group, using the unipotent variety instead of the nilpotent variety. In [24] Helmke and Slodowy obtained the complete intersection simple elliptic singularities from the adjoint quotient of the infinite-dimensional loop group of a simple complex Lie group. In contrast with the finite-dimensional group, some elements of its loop group do not have a Jordan-Chevalley decomposition and the exponential map is not surjective. Therefore it is not possible to work in the Lie algebra as in the finite-dimensional case (see [25]).

In [47, 48] the authors construct the simple elliptic singularity $\tilde{D}_5$ from the 6-dimensional Lie algebra $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ by the same method given in Sect. 10.5 generalizing Slodowy slices. In the rest of this section, we present this construction. Set $\mathfrak{g} := \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$. The nilpotent variety of $\mathfrak{g}$ is given by

$$\mathcal{N}(\mathfrak{g}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \middle| a^2 + bc = 0 \right\} \times \left\{ \begin{pmatrix} d & e \\ f & -d \end{pmatrix} \middle| d^2 + ef = 0 \right\}$$

We can also see the nilpotent variety of $\mathfrak{g}$ as the inverse image of the origin by the adjoint quotient map

$$\gamma \colon \mathfrak{g} \longrightarrow \mathfrak{h}/W \cong \mathbb{C}^2$$
$$\left(\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right), \left(\begin{smallmatrix} d & e \\ f & -d \end{smallmatrix}\right) \right) \mapsto (-a^2 - bc, -d^2 - ef).$$

where a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is

$$\mathfrak{h} := \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} d & 0 \\ 0 & -d \end{pmatrix} \right\}$$

and the Weyl group W is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

We need to choose a special slice as the Slodowy slices are not suitable in this case. The reader can see [47, 48] for more details. Now we construct a generic linear subspace of $\mathfrak{g}$, that we call *generalized Slodowy slice*: Let $x \in \mathfrak{g}$ and let $x = x_s + x_n$ be its Jordan-Chevalley decomposition (Theorem 10.5.2) where x_s is semisimple and x_n is nilpotent such that $[x_s, x_n] = 0$.

Definition 10.7.1 A 2-dimensional subspace $V \subset \mathfrak{g}$ is said to be a *good subspace* if for a basis $x, y \in V$, we have $x = (x_s, x_n)$ and $y = (y_n, y_s)$, where $x_s, y_s \in \mathfrak{sl}(2, \mathbb{C})$ are non-zero semisimple elements and $x_n, y_n \in \mathfrak{sl}(2, \mathbb{C})$ are non-zero nilpotent elements in $\mathfrak{g}$. We denote it by $V_{(x,y)}$.

Definition 10.7.2 A 4-dimensional subspace $\mathcal{S}_{(x,y)} \subset \mathfrak{g}$ is called a *generalized Slodowy slice* if there exists a 2-dimensional good subspace $V_{(x,y)}$ in $\mathfrak{g}$ such that

$$\mathcal{S}_{(x,y)} = V_{(x,y)}^{\perp} = \{z \in \mathfrak{g} \mid \mathcal{K}(z, v) = 0 \text{ for each } v \in V_{(x,y)}\}$$

where $z = (Z_1, Z_2)$, $v = (V_1, V_2) \in \mathfrak{g}$ and $\mathcal{K}(\ ,\ )$ is the Killing form of $\mathfrak{g}$ given by

$$\mathcal{K}(z, v) = \mathcal{K}\big((Z_1, Z_2), (V_1, V_2)\big) = 4(\mathrm{tr}(Z_1 V_1) + \mathrm{tr}(Z_2 V_2)).$$

Hence we have:

Theorem 10.7.3 ([48]) *Let $\mathcal{S}_{(x,y)} \subset \mathfrak{g}$ be a generalized Slodowy slice in $\mathfrak{g}$. Then $(X_{(x,y)}, 0) = (\mathcal{N}(\mathfrak{g}) \cap \mathcal{S}_{(x,y)}, 0)$ is a surface in $\mathbb{C}^4$ with a $\tilde{D}_5$-singularity.*

The normalization of good subspaces gives simple coordinates $(p, q) \in \mathbb{C}^2$ which parameterize them:

Lemma 10.7.4 ([48, Lemma 2.3]) *Let $V \subset \mathfrak{g}$ be a 2-dimensional good subspace. Then there exists $g \in SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ such that $g^{-1} V g$ has a basis $x = (x_s, x_n)$ and $y = (y_n, y_s)$ with the following properties:*

1. $x_s = y_s = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.
2. $x_n = \left(\begin{smallmatrix} p & 1 \\ -p^2 & -p \end{smallmatrix}\right)$ *for some* $p \in \mathbb{C}$ *or* $x_n = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$.
3. $y_n = \left(\begin{smallmatrix} q & 1 \\ -q^2 & -q \end{smallmatrix}\right)$ *for some* $q \in \mathbb{C}$ *or* $y_n = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$.

Hence, if we take a generalized Slodowy slice $\mathcal{S}_{(x,y)}$ with $x = (x_s, x_n(p))$ and $y = (y_n, y_s(q))$ given by Lemma 10.7.4, then $(X_{(x,y)}, 0)$ is obtained as:

$$X_{(x,y)} = \left\{ (a, b, d, e) \in \mathbb{C}^4 \;\middle|\; \begin{array}{l} g_1 := a^2 - 2qab + q^2 b^2 - 2bd = 0 \\ g_2 := -2ae + d^2 - 2pde + p^2 e^2 = 0 \end{array} \right\}. \tag{10.20}$$

More generally, let A and B be 4×4 symmetric matrices and consider the singularity defined by two quadratic equations in $\mathbb{C}^4$, say

$$S_{(A,B)} := \{v = (x, y, z, w) \in \mathbb{C}^4 \mid f(v) = g(v) = 0\}$$

where

$$f(v) = vAv^t, \quad g(x, y, z, w) = vBv^t,$$

and v^t is the transpose vector. We obtain that:

Proposition 10.7.5 ([48, Lemma A.11]) *With the preceding notation, let* $(A, B) \in \mathrm{Sym}_4(\mathbb{C}) \times \mathrm{Sym}_4(\mathbb{C})$ *with* $A \in GL(4, \mathbb{C})$. *Then* $(S_{(A,B)}, 0)$ *has an isolated surface singularity if and only if the characteristic polynomial of* $A^{-1}B$ *has no multiple root.*

Note that:

Remark 10.7.6 ([48]) $(S_{(A,B)}, 0)$ has an isolated surface singularity if and only if it is a type $\tilde{D}_5$-singularity.

10.7.1 Semi-universal Deformation

Now we present the construction of a semi-universal deformation $\phi\colon (\mathcal{X}, 0) \to (B, 0)$ of the $\tilde{D}_5$-singularity $(X_{(x,y)}, 0)$ using the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$. By [72] the base B of a semi-universal deformation is of dimension 7.

Consider $(X_{(x,y)}, 0)$ given by (10.20). By [48], when $\lambda \neq 0, 1/4$ in (10.19), the Tjurina algebra T^1 is generated by the vectors $(1, 0),\ (b, 0),\ (e, 0),\ (0, 1),\ (0, b),\ (0, ae),\ (0, e)$. Hence we have that the base of the deformation is given by $B := \mathbb{C}^2 \times \mathbb{C}^3 \times \mathfrak{h}/W \cong \mathbb{C}^7$. To construct the semi-universal deformation we deform the adjoint quotient and the generalized Slodowy slice as follows. Consider the adjoint quotient map

$$\gamma : \mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}/W \cong \mathbb{C}^2$$
$$(X_1, X_2) \mapsto (\det X_1, \det X_2).$$

Set $x_\infty = y_\infty := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and let $(\alpha, \beta) \in \mathbb{C}^2$, we deform γ by

$$\gamma_{(\alpha,\beta)}\colon \mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}/W \cong \mathbb{C}^2$$
$$X = (X_1, X_2) \mapsto \Big(\det X_1 + \alpha\mathcal{K}\big(z, (0, x_\infty)\big), \det X_2 + \beta\mathcal{K}\big(z, (y_\infty, 0)\big)\Big).$$

Deform the generalized Slodowy slice $\mathcal{S}_{(x,y)} = \{z \in \mathfrak{g} \mid \langle z, x\rangle = \langle z, y\rangle = 0\}$ by

$$\mathcal{S}_{(x,y)}(\gamma, \delta, \varepsilon) := \{z \in \mathfrak{g} \mid \langle z, x\rangle + \gamma\langle z, (x_s, 0)\rangle = 4\delta,\ \langle z, y\rangle = 4\varepsilon\}$$

where $(\gamma, \delta, \varepsilon) \in \mathbb{C}^3$. Take as the total space of the deformation

$$\mathcal{X} := \big\{(Z, \alpha, \beta, \gamma, \delta, \varepsilon, \lambda, \mu) \in \mathfrak{g} \times S \mid \gamma_{(\alpha,\beta)}(Z) = (\lambda, \mu), Z \in \mathcal{S}_{(x,y)}(\gamma, \delta, \varepsilon)\big\}$$

and let $\phi\colon \mathcal{X} \to S$ be the second projection.

Theorem 10.7.7 ([48]) *With previous notation, the projection* $\phi\colon (\mathcal{X},0) \to (B,0)$ *is a semi-universal deformation of* $(X_{(x,y)},0)$ *for* $pq \neq 0, 1/4$.

10.8 McKay Correspondence

Recall from Sect. 10.3.1 that given a finite subgroup Γ of $SL(2,\mathbb{C})$, the dual graph of the minimal resolution of the corresponding Kleinian singularity $\mathbb{C}^2/\Gamma$ is a Dynkin diagram of type ADE. In [44] McKay obtained the same diagrams directly from the irreducible representations of Γ, this gives a one-to-one correspondence between (non-trivial) irreducible representations of the group and the components of the exceptional set of the minimal resolution of the corresponding Kleinian singularity. In this section, we briefly describe this correspondence, known as the McKay correspondence. Even though this correspondence is not related to Lie algebras, it completes the picture of the beautiful relation between simple Lie algebras, finite subgroups of $SL(2,\mathbb{C})$ and Kleinian singularities, via Dynkin diagrams of type ADE. We recommend the articles by Reid [54] and Riemenschneider [55] for a more complete account on different approaches of McKay correspondence.

Let Γ be a finite subgroup of $GL(2,\mathbb{C})$ (see Sect. 10.3). The groups Γ has a finite number of irreducible representations, in fact, its number is equal to the number of conjugacy classes of Γ. Let $\mathrm{Irr}(\Gamma) = \{\rho_0, \rho_1, \ldots, \rho_r\}$ be the set of complex *irreducible representations* of Γ, where ρ_0 denotes the *trivial representation*. Let ρ be the *natural representation* on $\mathbb{C}^2$ given by the inclusion $\Gamma \subset GL(2,\mathbb{C})$. Consider the following tensor products of representations and their corresponding decomposition as direct sum of irreducible representations

$$\rho \otimes \rho_i = \bigoplus_{j=0}^{r} a_{ij}\rho_j, \qquad j = 0, \ldots, r, \tag{10.21}$$

where a_{ij} is the multiplicity of ρ_j in $\rho \otimes \rho_i$. With this information we construct the *McKay quiver of* Γ, denoted by $Q(\Gamma)$, as follows: associate a vertex to each irreducible representation ρ_i, and join the i-th vertex to the j-th vertex by a_{ij} arrows. We take the convention that an undirected edge between to vertices, represents a pair of arrows between those vertices pointing in opposite directions. For every finite subgroup Γ of $SL(2,\mathbb{C})$ one has $a_{ij} = a_{ji} \in \{0,1\}$, so in this case the McKay quiver is just a graph. In [44] McKay made the following remarkable observation computing the McKay quivers case by case:

McKay Correspondence *Let* Γ *be a finite subgroup of* $SL(2,\mathbb{C})$. *If in the McKay quiver* $Q(\Gamma)$ *of* Γ *we remove the vertex corresponding to the trivial representation* ρ_0, *then one obtains the dual graph of the minimal resolution* $\Delta(S)$ *of the Kleinian singularity* $S = \mathbb{C}^2/\Gamma$, *which in turn corresponds to a Dynkin diagram of type* ADE *(see Table 10.3).*

Slodowy [65] generalized the construction of the McKay quiver to obtain the Dynkin diagrams of type B_n, C_n, F_4 and G_2 given in Table 10.1. Let (Γ, Γ') be a pair of finite subgroups of $SL(2, \mathbb{C})$ such that Γ is normal in Γ'. If ρ is a representation of Γ' we denote by $\rho^{\downarrow}$ its restriction to the subgroup Γ. Let $\operatorname{Irr}(\Gamma') = \{\rho_0, \rho_1, \dots, \rho_r\}$ be the set of complex irreducible representations of Γ', where ρ_0 denotes the trivial representation. Let $\rho_i^{\downarrow}$, $i = 0, \dots, r$ be the restricted representations of the subgroup Γ. Let ρ be the natural representation of Γ on $\mathbb{C}^2$ given by the inclusion $\Gamma \subset GL(2, \mathbb{C})$ which can be considered as the restriction of the natural representation of Γ'. Consider the following tensor products of representations and their corresponding decomposition as direct sum of the representations $\rho_i^{\downarrow}$

$$\rho \otimes \rho_i^{\downarrow} = \bigoplus_{j=0}^{r} a_{ij} \rho_j^{\downarrow}, \qquad j = 0, \dots, r, \tag{10.22}$$

and construct the corresponding quiver as before. Considering the pairs (Γ, Γ') of finite subgroups of $SL(2, \mathbb{C})$ given in Table 10.9 one obtains the Dynkin diagrams of type B_n, C_n, F_4 and G_2 (compare with Table 10.8).

Notice that with respect to Table 10.8, in Table 10.9 for G_2, the group $\mathcal{O}$ was replaced by the smaller group $\mathcal{T}$. This simplifies the description and the generalization of Theorem 10.5.21 for the simple Lie algebra G_2 remains valid when reformulated accordingly (see [65, Appendix III]).

A conceptual geometric construction of the McKay correspondence was obtained in the series of articles by Gonzalez-Springberg and Verdier [21], Knörrer [36], Artin and Verdier [1] and Esnault and Knörrer [19]. Let $\pi : \tilde{S} \to S$ be the minimal resolution of a Kleinian singularity $S = \mathbb{C}^2/\Gamma$. To an irreducible representation ρ_i of Γ one associates an indecomposable *reflexive* $\mathcal{O}_S$-module M. The module $\pi^*M/\text{Torsion}$ is proved to be locally free and its first Chern class $c_1(\pi^*M/\text{Torsion})$ is the Poincaré dual of a curvette hitting transversely an unique irreducible component E_i of the exceptional set E of $\tilde{S}$. The component E_i is the image of the representation ρ_i under McKay correspondence. The first Chern class determines the module M along with the representation ρ_i. Conversely, for any irreducible component of the exceptional set there is an irreducible representation and a module realizing it.

Other interesting interpretations of McKay correspondence have been found, for instance, in terms of Hilbert schemes [10, 27, 28] or equivalence of derive categories [32].

Table 10.9 Dynkin diagrams B_n, C_n, F_2 and G_2 from pairs of subgroups $\Gamma \lhd \Gamma'$

Γ	Γ'	Dynkin diagram
$\mathcal{C}_{2n}$	$\mathcal{D}_n$	B_n
$\mathcal{D}_{n-1}$	$\mathcal{D}_{2(n-1)}$	C_n
$\mathcal{T}$	$\mathcal{O}$	F_4
$\mathcal{D}_2$	$\mathcal{T}$	G_2

The McKay correspondence was generalized for arbitrary *rational surface singularities* in the articles by Esnault [18] and Wunram [74]. Esnault notice that for quotient singularities by subgroups of $GL(2, \mathbb{C})$ which are not contained in $SL(2, \mathbb{C})$, the first Chern class $c_1(\pi^*M/\text{Torsion})$ and the rank of the module are not enough to determine the reflexive module M, so in this case the correspondence given by Artin and Verdier is not a bijection. To fix this problem Wunram use reflexive modules which satisfy certain condition which he called *special* reflexive modules. He proved that the construction by Artin and Verdier of the first Chern class $c_1(\pi^*M/\text{Torsion})$ gives a bijection between the set of special indecomposable reflexive $\mathcal{O}_S$-modules and the set of irreducible components of the exceptional set of the minimal resolution of the singularity. Moreover, the first Chern class $c_1(\pi^*M/\text{Torsion})$ determines special reflexive $\mathcal{O}_S$-modules.

Khan in [31] studied the case of normal surface singularities, focusing on the *minimally elliptic singularities* introduced by Laufer in [38], that is, Gorenstein surface singularities with geometric genus $p_g = 1$. Let S be a minimally elliptic singularity, $\pi : \tilde{S} \rightarrow S$ its minimal resolution and Z the fundamental cycle. He gave a correspondence between isomorphism classes of reflexive $\mathcal{O}_S$-modules and some locally free sheaves over $\mathcal{O}_Z$ which satisfy certain conditions. A particular case of minimal elliptic singularities are the simple elliptic singularities described in Sect. 10.7, for this, Kahn gave a complete classification of indecomposable reflexive modules.

Recently, a generalization of McKay correspondence for *Gorenstein surface singularities* was given by Fernández de Bobadilla and Romano-Velázquez in [20]. The authors generalize the notion of special reflexive module defined by Wunram in [74], and, given a Gorenstein surface singularity (X, x), they give a bijection between the set of special indecomposable reflexive $\mathcal{O}_X$-modules up to isomorphism, and the set of irreducible divisors E over x, such that any resolution of X where E appears, the Gorenstein form has neither zeros nor poles along E. This bijection specializes to the classical McKay correspondence in the case when (X, x) is a Kleinian singularity.

There have been also generalizations of McKay correspondence for the higher dimensional case, i.e., for subgroups Γ of $SL(n, \mathbb{C})$, see for instance [3, 29, 54].

Let us finish with the last phrase in [44] by McKay "Would not the Greeks appreciate the result that the simple Lie algebras may be derived from the platonic solids?"

Acknowledgments The first author is Regular Associate of the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy. This work was supported by UNAM-DGAPA-PASPA sabbatical scholarship, CONACYT 253506, TUBITAK 2221-109T667, Galatasaray University BAP 16.504.001.

References

1. Artin, M., Verdier, J.L.: Reflexive modules over rational double points. Math. Ann. **270**(1), 79–82 (1985). http://dx.doi.org/10.1007/BF01455531 582
2. Bourbaki, N.: Lie groups and Lie algebras. Chapters 7–9. Elements of Mathematics (Berlin). Springer-Verlag, Berlin (2005). Translated from the 1975 and 1982 French originals by Andrew Pressley. 571
3. Bridgeland, T., King, A., Reid, M.: The McKay correspondence as an equivalence of derived categories. J. Amer. Math. Soc. **14**(3), 535–554 (2001). https://doi.org/10.1090/S0894-0347-01-00368-X. 583
4. Brieskorn, E.: Über die Auflösung gewisser Singularitäten von holomorphen Abbildungen. Math. Ann. **166**, 76–102 (1966). https://doi.org/10.1007/BF01361440. 542
5. Brieskorn, E.: Rationale Singularitäten komplexer Flächen. Invent. Math. **4**, 336–358 (1967/68). URL https://doi.org/10.1007/BF01425318. 548 and 549
6. Brieskorn, E.: Die Auflösung der rationalen Singularitäten holomorpher Abbildungen. Math. Ann. **178**, 255–270 (1968). https://doi.org/10.1007/BF01352140. 542
7. Brieskorn, E.: Singular elements of semi-simple algebraic groups. In: Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, pp. 279–284 (1971). 543 and 574
8. Cartan, H.: Quotient d'un espace analytique par un groupe d'automorphismes. In: Algebraic geometry and topology, pp. 90–102. Princeton University Press, Princeton, N. J. (1957). A symposium in honor of S. Lefschetz. 548
9. Chevalley, C.: Invariants of finite groups generated by reflections. Amer. J. Math. **77**, 778–782 (1955). https://doi.org/10.2307/2372597. 549
10. Crawley-Boevey, W.: On the exceptional fibres of Kleinian singularities. Amer. J. Math. **122**(5), 1027–1037 (2000). http://muse.jhu.edu/journals/american_journal_of_mathematics/v122/122.5crawley-boevey.pdf. 582
11. Crawley-Boevey, W., Holland, M.P.: Noncommutative deformations of Kleinian singularities. Duke Math. J. **92**(3), 605–635 (1998). https://doi.org/10.1215/S0012-7094-98-09218-3. 555
12. Du Val, P.: On isolated singularities of surfaces which do not affect the conditions of adjunction (Part I). Mathematical Proceedings of the Cambridge Philosophical Society **30**(4), 453–459 (1934). https://doi.org/10.1017/S030500410001269X. 542
13. Du Val, P.: On isolated singularities of surfaces which do not affect the conditions of adjunction (Part II). Mathematical Proceedings of the Cambridge Philosophical Society **30**(4), 460–465 (1934). https://doi.org/10.1017/S0305004100012706.
14. Du Val, P.: On isolated singularities of surfaces which do not affect the conditions of adjunction (Part III). Mathematical Proceedings of the Cambridge Philosophical Society **30**(4), 483–491 (1934). https://doi.org/10.1017/S030500410001272X. 542
15. Durfee, A.H.: Fifteen characterizations of rational double points and simple critical points. Enseign. Math. (2) **25**(1–2), 131–163 (1979). 548 and 549
16. Dynkin, E.B.: Semisimple subalgebras of semisimple lie algebra. Math. Sbornik **30**(72), 349–462 (1952). English Translation in Amer. Math. Soc. Trans. , Series 2, Vol. 6, 1957, 245–378. 542
17. Esnault, H.: Sur l'identification de singularités apparaissant dans des groupes algébriques complexes. In: Seminar on Singularities (Paris, 1976/1977), *Publ. Math. Univ. Paris VII*, vol. 7, pp. 31–59. Univ. Paris VII, Paris (1980). 543 and 574
18. Esnault, H.: Reflexive modules on quotient surface singularities. J. Reine Angew. Math. **362**, 63–71 (1985). http://dx.doi.org/10.1515/crll.1985.362.63. 583
19. Esnault, H., Knörrer, H.: Reflexive modules over rational double points. Math. Ann. **272**(4), 545–548 (1985). http://dx.doi.org/10.1007/BF01455865. 582
20. Fernández de Bobadilla, J., Romano-Velázquez, A.: Reflexive modules on normal gorenstein stein surfaces, their deformations and moduli (2019). Preprint arXiv:1812.06543v2 [math.AG]. 583

21. Gonzalez-Sprinberg, G., Verdier, J.L.: Construction géométrique de la correspondance de McKay. Ann. Sci. École Norm. Sup. (4) **16**(3), 409–449, (1983). 582
22. Greuel, G.M.: Deformation and smoothing of singularities. This volume, 2020. 553 and 554
23. Greuel, G.M., Purket, W.: Life and work of Egbert Brieskorn (1936–2013). Journal of Singularities **18**, 7–34. Special volume in honor of E. Brieskorn. (See also arXiv:1711.09600). 543
24. Helmke, S., Slodowy, P.: Loop groups, elliptic singularities and principal bundles over elliptic curves. In: Geometry and topology of caustics—CAUSTICS '02, *Banach Center Publ.*, vol. 62, pp. 87–99. Polish Acad. Sci. Inst. Math., Warsaw (2004). 543 and 578
25. Helmke, S., Slodowy, P.: Singular elements of affine Kac-Moody groups. In: European Congress of Mathematics, pp. 155–172. Eur. Math. Soc., Zürich (2005). 578
26. Humphreys, J.E.: Introduction to Lie algebras and representation theory, *Graduate Texts in Mathematics*, vol. 9. Springer-Verlag, New York-Berlin (1978). Second printing, revised. 555, 565, 566, 568, 569, and 571
27. Ito, Y., Nakamura, I.: McKay correspondence and Hilbert schemes. Proc. Japan Acad. Ser. A Math. Sci. **72**(7), 135–138 (1996). http://projecteuclid.org/euclid.pja/1195510273. 582
28. Ito, Y., Nakamura, I.: Hilbert schemes and simple singularities. In: New trends in algebraic geometry (Warwick, 1996), *London Math. Soc. Lecture Note Ser.*, vol. 264, pp. 151–233. Cambridge Univ. Press, Cambridge (1999). https://doi.org/10.1017/CBO9780511721540.008. 582
29. Ito, Y., Reid, M.: The McKay correspondence for finite subgroups of $SL(3, \mathbb{C})$. In: Higher-dimensional complex varieties (Trento, 1994), pp. 221–240. de Gruyter, Berlin (1996). 583
30. Jacobson, N.: Lie algebras. Dover Publications, Inc., New York (1979). Republication of the 1962 original. 546 and 555
31. Kahn, C.P.: Reflexive modules on minimally elliptic singularities. Math. Ann. **285**(1), 141–160 (1989). http://dx.doi.org/10.1007/BF01442678. 583
32. Kapranov, M., Vasserot, E.: Kleinian singularities, derived categories and Hall algebras. Math. Ann. **316**(3), 565–576 (2000). https://doi.org/10.1007/s002080050344. 582
33. Kas, A., Schlessinger, M.: On the versal deformation of a complex space with an isolated singularity. Math. Ann. **196**, 23–29 (1972). https://doi.org/10.1007/BF01419428. 554
34. Kirby, D.: The structure of an isolated multiple point of a surface. II, III. Proc. London Math. Soc. (3) **7**, 1–18, 19–28 (1957). https://doi.org/10.1112/plms/s3-7.1.1. 551
35. Klein, F.: Lectures on the icosahedron and the solution of equations of the fifth degree. Dover (1956). English translation from the 1884 original. 542 and 549
36. Knörrer, H.: Group representations and the resolution of rational double points. In: Finite groups—coming of age (Montreal, Que., 1982), *Contemp. Math.*, vol. 45, pp. 175–222. Amer. Math. Soc., Providence, RI (1985). https://doi.org/10.1090/conm/045/822239. 582
37. Kostant, B.: Lie group representations on polynomial rings. Amer. J. Math. **85**, 327–404 (1963). https://doi.org/10.2307/2373130. 570 and 572
38. Laufer, H.B.: On minimally elliptic singularities. Amer. J. Math. **99**(6), 1257–1295 (1977). https://doi.org/10.2307/2374025. 583
39. Lê, D.T., Tosun, M.: Simple singularities and simple Lie algebras. TWMS J. Pure Appl. Math. **2**(1), 97–111 (2011). 567
40. Lee, D.H.: The structure of complex Lie groups, *Chapman & Hall/CRC Research Notes in Mathematics*, vol. 429. Chapman & Hall/CRC, Boca Raton, FL (2002). 544, 545, and 547
41. Lipman, J.: Rational singularities, with applications to algebraic surfaces and unique factorization. Inst. Hautes Études Sci. Publ. Math. (36), 195–279 (1969). http://www.numdam.org/item?id=PMIHES_1969__36__195_0. 551
42. Looijenga, E.: On the semi-universal deformation of a simple-elliptic hypersurface singularity: Part i: Unimodularity. Topology **16**(3), 257–262 (1977). https://doi.org/10.1016/0040-9383(77)90006-4. http://www.sciencedirect.com/science/article/pii/0040938377900064. 543 and 577

43. Looijenga, E.: On the semi-universal deformation of a simple-elliptic hypersurface singularity. II. The discriminant. Topology **17**(1), 23–40 (1978). https://doi.org/10.1016/0040-9383(78)90010-1. 543
44. McKay, J.: Graphs, singularities, and finite groups. In: The Santa Cruz Conference on Finite Groups, *Proc. Sympos. Pure Math.*, vol. 37, pp. 183–186. Amer. Math. Soc., Providence, R.I. (1980). 553, 581, and 583
45. Milnor, J.: On the 3-dimensional Brieskorn manifolds $M(p, q, r)$. In: Knots, groups, and 3-manifolds (Papers dedicated to the memory of R. H. Fox), pp. 175–225. Ann. of Math. Studies, No. 84. Princeton Univ. Press, Princeton, N. J. (1975). 551
46. Nakamoto, K., Sharland, A., Tosun, M.: Triple root systems, rational quivers and examples of linear free divisors. Internat. J. Math. **29**(3), 1850017, 29 (2018). https://doi.org/10.1142/S0129167X18500179. 566
47. Nakamoto, K., Tosun, M.: Some surface singularities obtained via Lie algebras. In: Singularities in Geometry and Topology, pp. 779–786. World Scientific (2007). 578
48. Nakamoto, K., Tosun, M.: A new construction of $\widetilde{D}_5$-singularities and generalization of Slodowy slices. J. Algebra **376**, 139–151 (2013). https://doi.org/10.1016/j.jalgebra.2012.11.016. 543, 569, 578, 579, 580, and 581
49. Neumann, W.D.: Geometry of quasihomogeneous surface singularities. In: Singularities, Part 2 (Arcata, Calif., 1981), *Proc. Sympos. Pure Math.*, vol. 40, pp. 245–258. Amer. Math. Soc., Providence, RI (1983). 551
50. Palamodov, V.P.: Deformations of complex spaces. Russian Math. Surveys **31**(3), 129–197 (1976). 553
51. Pe Pereira, M.: On Nash problem for quotient surface singularities. Ph.D. thesis, Universidad Complutense de Madrid (2011). 549
52. Pinkham, H.: Singularites rationnelles de surfaces. In: M. Demazure, H.C. Pinkham, B. Teissier (eds.) Séminaire sur les Singularités des Surfaces, pp. 147–178. Springer Berlin Heidelberg, Berlin, Heidelberg (1980). 566
53. Prill, D.: Local classification of quotients of complex manifolds by discontinuous groups. Duke Math. J. **34**, 375–386 (1967). http://projecteuclid.org/euclid.dmj/1077377006. 548
54. Reid, M.: La correspondance de McKay. 276, pp. 53–72 (2002). Séminaire Bourbaki, Vol. 1999/2000. 581 and 583
55. Riemenschneider, O.: Special representations and the two-dimensional McKay correspondence. Hokkaido Math. J. **32**(2), 317–333 (2003). 581
56. Saito, K.: Einfach-elliptische Singularitäten. Invent. Math. **23**, 289–325 (1974). https://doi.org/10.1007/BF01389749. 543 and 577
57. Samelson, H.: Notes on Lie algebras, second edn. Universitext. Springer-Verlag, New York (1990). https://doi.org/10.1007/978-1-4613-9014-5. 547, 555, 557, 559, and 562
58. Scott, P.: The geometries of 3-manifolds. Bull. London Math. Soc. **15**(5), 401–487 (1983). 551
59. Seade, J.: On the topology of isolated singularities in analytic spaces, *Progress in Mathematics*, vol. 241. Birkhäuser Verlag, Basel (2006). 551
60. Sekiguchi, J.: The nilpotent subvariety of the vector space associated to a symmetric pair. Publ. Res. Inst. Math. Sci. **20**(1), 155–212 (1984). https://doi.org/10.2977/prims/1195181836. 577
61. Sekiguchi, J., Shimizu, Y.: Simple singularities and infinitesimally symmetric spaces. Proc. Japan Acad. Ser. A Math. Sci. **57**(1), 42–46 (1981). http://projecteuclid.org/euclid.pja/1195516591. 577
62. Serre, J.P.: Complex Semisimple Lie Algebras. Springer (1987). 559, 560, 561, 562, 563, 564, 565, and 566
63. Serre, J.P.: Lie algebras and Lie groups, *Lecture Notes in Mathematics*, vol. 1500. Springer-Verlag, Berlin (2006). 1964 lectures given at Harvard University, Corrected fifth printing of the second (1992) edition. 544 and 547
64. Slodowy, P.: Four lectures on simple groups and singularities, *Communications of the Mathematical Institute, Rijksuniversiteit Utrecht*, vol. 11. Rijksuniversiteit Utrecht, Mathematical Institute, Utrecht (1980). 577

65. Slodowy, P.: Simple singularities and simple algebraic groups, *Lecture Notes in Mathematics*, vol. 815. Springer, Berlin (1980). 543, 574, 577, and 582
66. Slodowy, P.: Platonic solids, Kleinian singularities, and Lie groups. In: Algebraic geometry (Ann Arbor, Mich., 1981), *Lecture Notes in Math.*, vol. 1008, pp. 102–138. Springer, Berlin (1983). https://doi.org/10.1007/BFb0065703. 548
67. Spivakovsky, M.: Resolution of singularities: an introduction. This volume, 2020. 551
68. Steinberg, R.: Regular elements of semisimple algebraic groups. Inst. Hautes Études Sci. Publ. Math. (25), 49–80 (1965). http://www.numdam.org/item?id=PMIHES_1965__25__49_0. 577
69. Steinberg, R.: Conjugacy classes in algebraic groups. Lecture Notes in Mathematics, Vol. 366. Springer-Verlag, Berlin-New York (1974). Notes by Vinay V. Deodhar. 571
70. Stembridge, J.R.: Folding by automorphisms (2008). http://www.math.lsa.umich.edu/~jrs/papers/folding.pdf. Unpublished. 566
71. Tjurina, G.N.: Absolute isolation of rational singularities, and triple rational points. Funkcional. Anal. i Priložen. **2**(4), 70–81 (1968). 551
72. Tjurina, G.N.: Locally semi-universal flat deformations of isolated singularities of complex spaces. Izv. Akad. Nauk SSSR Ser. Mat. **33**, 1026–1058 (1969). 554 and 580
73. Tosun, M.: ADE surface singularities, chambers and toric varieties. In: Singularités Franco-Japonaises, *Sémin. Congr.*, vol. 10, pp. 341–350. Soc. Math. France, Paris (2005). 566
74. Wunram, J.: Reflexive modules on quotient surface singularities. Math. Ann. **279**(4), 583–598 (1988). http://dx.doi.org/10.1007/BF01458530. 583

Correction to: The Combinatorics of Plane Curve Singularities

How Newton Polygons Blossom into Lotuses

Evelia R. García Barroso, Pedro D. González Pérez, and Patrick Popescu-Pampu

Correction to:
Chapter 1 in: J. L. Cisneros Molina et al. (eds.), *Handbook of Geometry and Topology of Singularities I*, https://doi.org/10.1007/978-3-030-53061-7_1

The original version of the book was inadvertently published with errors in the family names of these author names: **Evelia R. García Barroso**, and **Pedro D. González Pérez**. However, the chapter has now been corrected.

Corrections:

In Chapter 1, "The Combinatorics of Plane Curve Singularities", the below author names have been corrected as requested by the authors:

The incorrect author names: Barroso E.R.G., Pérez P.D.G. have now been corrected as García Barroso E.R., González Pérez P.D.

The updated version of this chapter can be found at
https://doi.org/10.1007/978-3-030-53061-7_1

J. L. Cisneros Molina et al. (eds.), *Handbook of Geometry and Topology of Singularities I*, https://doi.org/10.1007/978-3-030-53061-7_11

Index

J. L. Cisneros Molina et al. (eds.), *Handbook of Geometry and Topology of Singularities I*, https://doi.org/10.1007/978-3-030-53061-7