# Chapter 9
# Free Generalized van der Pol Oscillators: Overview of the Properties of Oscillatory Responses

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**Abstract** This work is concerned with generalized van der Pol oscillators, the damping-like force of which depends nonlinearly on the displacement and velocity with the powers that can be any positive real numbers, while the restoring force is either linear or purely nonlinear. The cases of small and large values of the damping parameter are considered. In the former case, an overview of contributions related to the amplitude and frequency of free limit cycle oscillations of different forms of generalized van der Pol oscillators are given and then the most general case examined. In the latter case, the jumps, outer curves and period of relaxation oscillations are found.

## 9.1 Introduction

The standard (classical) van der Pol oscillator

$$\ddot{x} + x = \varepsilon\left(1 - x^2\right)\dot{x}, \tag{9.1}$$

represents one of archetypical oscillators. It is named after Balthasar van der Pol (1889–1959), a Dutch physicist, whose achievements and life have attracted the attention of many researchers both from the viewpoint of his scientific contributions and biography [1–4].

Balthasar van der Pol entered the University of Utrecht, where he graduated *cum laude* in Physics. He then studied under John Ambrose Fleming, who was an inventor of a diode, and John Joseph Thomson, who discovered the electron. He was a friend and colleague with Edward Appleton, who was the Nobel Prize laureate for his discovery of a certain layer of the ionosphere. Balthasar van der Pol was assistant to Hendrik Antoon Lorentz, who shared the 1902 Nobel Prize in Physics in recognition

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of the research on the influence of magnetism upon radiation phenomena. Balthasar van der Pol worked for Philips Company and also had an academic career at the Technical University, Delft. He held a temporary professorship at the University of California, Berkeley and the Victor Emanuel Professorship at Cornell in Ithaca, New York.

Balthasar van der Pol pioneered the fields of radio and telecommunications [1]. However, his scientific work did not cover only radio and electrical engineering, but also pure and applied mathematics, which included number theory, special functions, operational calculus and nonlinear differential equations.

He was a theoretician and an experimentalist. While conducting experiments with oscillations in a vacuum tube triode circuit, he concluded that all initial conditions converged to the same periodic orbit of finite amplitude. He proposed a nonlinear differential equation (9.1) as a nondimensional mathematical model for the behaviour observed experimentally [5]. The nonlinear 'damping-like' force that appears on the right-hand side of Eq. (9.1) dissipates energy for large displacements as the expression in the parentheses is negative; it feeds energy for small displacements since this expression is then positive. This behaviour gives rise to self-sustaining/self-exciting oscillations. For small values of the 'damping coefficient' $\varepsilon$ ($0 < \varepsilon \ll 1$), this behaviour is characterized by the appearance of a stable limit cycle with the steady-state amplitude $\left|a_{LC,s}\right| = 2$ (note that the index 's' stans for the 'standard van der Pol oscillator' and this abbreviation will be used through the whole manuscript) and the angular frequency approximately equal to unity (Fig. 9.1).

While investigating the case $\varepsilon \gg 1$, van der Pol discovered the importance of what has become known as relaxation oscillations [6]—the motion consisting of very slow asymptotic behaviour along outer curves followed by a sudden discontinuous jump. The jump-down points $x_{\mathrm{jd},s}$ are located at $x_{\mathrm{jd},s} = 1$, from which the amplitude jumps to $x_{\mathrm{d},s} = -2$. Then the motion proceeds along the outer curve and undergoes a jump-up from $x_{\mathrm{ju},s} = -1$ to $x_{\mathrm{u},s} = 2$ (Fig. 9.2). Ginoux pointed out in [7] that around the same time when van der Pol published the paper [6] in English, he also published

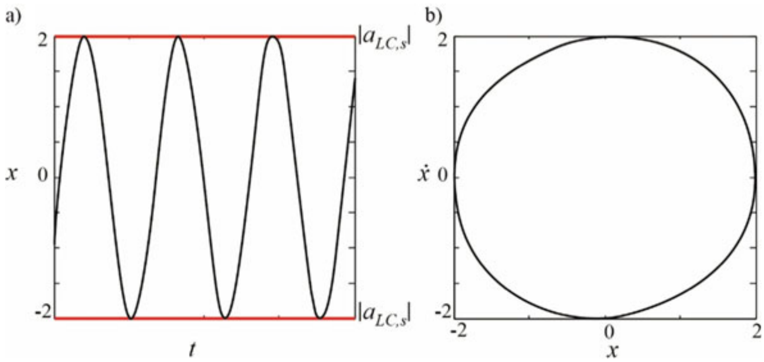


**Fig. 9.1** Characteristic behaviour of the van der Pol oscillator (1) for $0 < \varepsilon \ll 1$: **a** oscillations; **b** phase trajectory

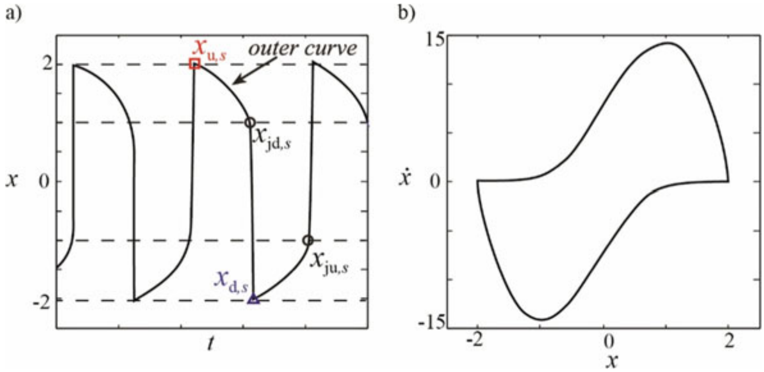


**Fig. 9.2** Characteristic behaviour of the van der Pol oscillator (1) for $\varepsilon \gg 1$: **a** oscillations; **b** phase trajectory

three more contributions in Dutch and German. They were all introducing relaxation oscillations, while 'their conclusions differ in the choice of the devices exemplifying the phenomenon of relaxation oscillations'. A few years later, van der Pol and van der Mark modelled the electric activity of the heart by using relaxation oscillations [8].

In this work, the generalized van der Pol oscillator governed by the following nondimensional equation of motion is considered:

$$\ddot{x} + \mathrm{sgn}(x)|x|^{\alpha} = \varepsilon f(x, \dot{x}),$$
$$f(x, \dot{x}) = \left(1 - |x|^{\beta}\right)|\dot{x}|^{\gamma}\mathrm{sgn}(\dot{x}), \quad (9.2\text{a,b})$$

where $\alpha > 0$, $\beta > 0$, $\gamma \geq 0$ and $\varepsilon > 0$. Here, the nonlinearity appears in both terms of the 'damping-like' force given by Eq. (9.2b) as well as in the restoring force, which is given by the second term on the left side of Eq. (9.2a); the sign and absolute value functions are used in Eqs. (9.2a,b) to assure that these forces have the properties of odd and even functions as in the standard van der Pol oscillator modelled by Eq. (9.1). The aim is to show how the properties of oscillatory responses of this generalized van der Pol oscillator differ with respect to the one described above for the standard van der Pol oscillator. This work also contains a literature survey on previous achievements related to these characteristics of different generalized van der Pol type oscillators.

## 9.2 Small Values of the Damping Coefficient: Limit Cycle

Minorsky [9] examined the generalized van der Pol Eq. (9.2a,b) for $\alpha = \gamma = 1$ and $\beta = 2n$, where $n$ is a positive integer ($n \geq 1$). He used the stroboscopic method, obtained the steady-state amplitude, concluding that this amplitude is smaller than $|a_{LC,s}| = 2$. In addition, he indicated that for $0 < n < 1$, this amplitude is higher than $\left|a_{LC,s}\right| = 2$. Moremedi et al. [10] used a perturbation scheme to conclude that for $\alpha = \gamma = 1$ and $\beta = 2n$, where $n$ is a positive integer, one has $|a_{LC}| \to 1$, when $n \to \infty$. They also concluded that the effect of increasing $n$ is to increase the period of the limit cycle oscillations. Obi [11] analysed the model (9.2a,b) with $\gamma = 1$, but only for the case when the powers $\alpha = 2\text{n} + 1$ and $\beta = 2\text{n} + 2$ ($n \geq 1$) are an odd and even number, respectively. He gave an approximate value of the amplitude of the limit cycle as $|a_{LC}| = (3n + 4)^{\frac{1}{2n+2}}$. By applying the harmonic balance method and the averaging method, Mickens and Oyedeji [12] found that the oscillatory response of a cubic van der Pol oscillator with $\alpha = 3$, $\beta = 2$ and $\gamma = 1$ is characterised by $|a_{LC}| = 2$ and the angular frequency $\omega_{LC} = \sqrt{3}$. This work was the motivation for the subsequent papers [13, 14], where the same oscillator was considered, and its limit cycle described analytically in terms of Jacobi elliptic functions. The approach of elliptic averaging gave $|a_{LC}| = 1.9098$ and the period $T_{LC} = 3.8833$, with an error of less than 1‰ with respect to the corresponding numerical result. The elliptic balancing used in [14] produced the result that depends on $\varepsilon$, which gave more accurate approximate solutions. By developing an elliptic perturbation method, improved accuracy is achieved in [15] even for higher values of $\varepsilon$. Mickens [16] adjusted the averaging method to derive the expression for the limit cycle of the generalized van der Pol oscillator with $\alpha = 1$, $\beta = 2$ and $\gamma = 1/3$, showing that its steady-state amplitude is lower than that of the standard van der Pol oscillator: $|a_{LC}| = 1.82574$, while the frequency stays the same at the level of approximation used. By applying an iterative technique, the same author [17] showed that when $\alpha = 1/3$ and $\beta = 2$, the amplitude of the limit cycle stays the same, but the frequency decreases for about 15% with respect to the standard van der Pol oscillator. Oyedeji [18] considered the quadratic van der Pol oscillator ($\alpha = \beta = 2$ and $\gamma = 1$) and used the first order harmonic balance method to calculate the limit cycle amplitude $a_{LC,s} = 2$ and the frequency $\omega_{LC} = \sqrt{16/(3\pi)} \approx 1.30294$. Waluya and van Horssen constructed asymptotic results on long time-scales $t$ for the periods of the generalized van der Pol Eqs. (9.2a,b) with $\beta = 2$, $\gamma = 1$ and $\alpha = (2m + 1)/(2n + 1)$, $m, n \in \mathrm{N}$ [19]. First, they showed how approximations of first integrals can be obtained and, then, how the existence, stability, and the period of time-periodic solutions can be determined from them. In [20], a more general class of oscillators with $\alpha$ being any positive real number is dealt with. Approximate expressions for the period is obtained for $\alpha \gg 1$, $\alpha \ll 1$ and $\alpha \to 1$. Kovacic [21] applied the averaging method for purely nonlinear systems to determine the amplitude of the limit cycle for $\alpha > 0$, $\beta > 0$, $\gamma = 1$ [21], while Kovacic and Mickens [22] generalized this case to $\gamma \geq 0$. Some of their results are summarised below. They also showed how to calculate the time needed to reach the limit cycle.

### *9.2.1 General Case*

In order to determine the properties of the response of the oscillators governed by Eqs. (9.2a,b) in a general case $\alpha > 0$, $\beta > 0$, $\gamma \geq 0$ and $0 < \varepsilon \ll 1$, the averaging method for purely nonlinear systems is used [21]. When $\varepsilon = 0$, Eq. (9.2a) corresponds to conservative purely nonlinear oscillators. Their energy integral can be used to derive the exact value of their frequency:

$$\omega(a) = c\sqrt{|a|^{\alpha-1}}, \quad c = \sqrt{\frac{\pi(\alpha+1)}{2}}\frac{\Gamma\left(\frac{\alpha+3}{2(\alpha+1)}\right)}{\Gamma\left(\frac{1}{\alpha+1}\right)}, \tag{9.3a,b}$$

where $\Gamma$ is the Euler gamma function and $a$ is the amplitude of motion.

The first approximation to motion of the perturbed systems (9.2a,b) can then be assumed as

$$x = a\cos\psi, \quad \dot{x} = -a\omega\sin\psi, \tag{9.4a,b}$$

where

$$\psi = \int_0^t \omega(a)dt + \theta(t), \tag{9.5}$$

while the frequency $\omega$ depends on the amplitude $a$ and the power $\alpha$ was defined by Eqs. (9.3a,b)

Differentiating Eq. (9.4a) with respect to time, one follows

$$\dot{x} = \dot{a}\cos\psi - a\omega\sin\psi - a\dot{\theta}\sin\psi, \tag{9.6}$$

which, owing to Eq. (9.4b), imposes the following constraint:

$$\dot{a}\cos\psi - a\dot{\theta}\sin\psi = 0. \tag{9.7}$$

Substituting the second time derivative of Eqs. (9.4b) together with Eq. (9.4a) into Eq. (9.2a,b), one can derive:

$$-\dot{a}\omega\sin\psi - a\frac{d\omega}{da}\dot{a}\sin\psi - a\omega\dot{\theta}\cos\psi - a\omega^2\cos\psi + \operatorname{sgn}(a\cos\psi)|a\cos\psi|^{\alpha} = -\varepsilon\left(1 - |a\cos\psi|^{\beta}\right)|-a\omega\sin\psi|^{\gamma}\operatorname{sgn}(-a\omega\sin\psi). \tag{9.8}$$

It should be noted that the last term on the left-hand side of Eq. (9.8) can be approximated by the first term from the corresponding Fourier series expansion [23]

$$\mathrm{sgn}(a\cos\psi)|a\cos\psi|^{\alpha} \approx |a|^{\alpha} b_{1_\alpha}\cos\psi, \quad b_{1_\alpha} = \frac{2}{\sqrt{\pi}}\frac{\Gamma\left(1+\frac{\alpha}{2}\right)}{\Gamma\left(\frac{3+\alpha}{2}\right)}. \tag{9.9a,b}$$

Now, this term can be cancelled by the term in front of it

$$-a\omega^2\cos\psi + |a|^{\alpha} b_{1_\alpha}\cos\psi = 0, \tag{9.10}$$

assuming that $\omega^2 \approx b_{1_\alpha}|a|^{\alpha-1} \approx c|a|^{\alpha-1}$, as given by Eqs. (9.3a,b)

Next, based on Eqs. (9.3a,b), the second term on the left-hand side of Eq. (9.8) can be expressed as:

$$\frac{d\omega}{da} = \frac{\alpha-1}{2a}\omega. \tag{9.11}$$

Substituting Eqs. (9.10) and (9.11) into Eqs. (9.8) and combining it with Eq. (9.7), one can derive

$$\dot{a}\left(1+\frac{\alpha-1}{2}\sin^2\psi\right) = \varepsilon\left(1-|a\cos\psi|^{\beta}\right)|-a\omega\sin\psi|^{\gamma}\,\mathrm{sgn}(-a\omega\sin\psi)\sin\psi, \tag{9.12}$$

$$a\dot{\theta} + \dot{a}\frac{\alpha-1}{2}\sin\psi\cos\psi = \varepsilon\left(1-|a\cos\psi|^{\beta}\right)|-a\omega\sin\psi|^{\gamma}\,\mathrm{sgn}(-a\omega\sin\psi)\cos\psi. \tag{9.13}$$

Averaging Eqs. (9.12) and (9.13), the following first-order differential equations for the amplitude $a$ and the phase shift $\theta$ are obtained:

$$\dot{a} = -\frac{2\varepsilon}{\pi c(\alpha+3)|a|^{\frac{\alpha-1}{2}}}\int_0^{2\pi}\left(1|a\cos\psi|^{\beta}\right)|-a\omega\sin\psi|^{\gamma}\,\mathrm{sgn}(-a\omega\sin\psi)\sin\psi\, d\psi, \tag{9.14}$$

$$a\dot{\theta} = -\frac{\varepsilon}{2\pi c|a|^{\frac{\alpha-1}{2}}}\int_0^{2\pi}\left(1-|a\cos\psi|^{\beta}\right)|-a\omega\sin\psi|^{\gamma}\,\mathrm{sgn}(-a\omega\sin\psi)\cos\psi\, d\psi. \tag{9.15}$$

Solving the integrals on the right-hand sides, yields:

$$\dot{a} = \frac{4\varepsilon c^{\gamma-1}}{\pi(\alpha+3)}\frac{|a|^{\gamma\frac{\alpha+1}{2}-\frac{\alpha-1}{2}}\Gamma\left(1+\frac{\gamma}{2}\right)\left[\sqrt{\pi}\Gamma\left(\frac{3+\beta+\gamma}{2}\right)-|a|^{\beta}\Gamma\left(\frac{1+\beta}{2}\right)\Gamma\left(\frac{3+\gamma}{2}\right)\right]}{\Gamma\left(\frac{3+\gamma}{2}\right)\Gamma\left(\frac{3+\beta+\gamma}{2}\right)}, \tag{9.16}$$

$$\dot{\theta} = 0. \tag{9.17}$$

Based on Eq. (9.17) one concludes that in all generalized van der Pol type oscillators modelled by Eqs. (9.2a,b), the phase shift is constant to terms of order $\varepsilon$. The amplitude of the limit cycle $a_{LC}$ corresponds to $\dot{a} = 0$ and is calculated to be

$$|a_{LC}| = \left[ \frac{\sqrt{\pi}\,\Gamma\left(\frac{3+\beta+\gamma}{2}\right)}{\Gamma\left(\frac{1+\beta}{2}\right)\Gamma\left(\frac{3+\gamma}{2}\right)} \right]^{1/\beta}, \tag{9.18}$$

while Eqs. (9.3a) define the corresponding frequency

$$\omega_{LC} = c\sqrt{|a_{LC}|^{\alpha-1}}. \tag{9.19}$$

The expression (9.18) is also obtained in [21] for the generalized van der Pol with $\gamma = 1$ and indicates that the first approximation for the amplitude of the limit cycle depends on the parameters appearing in the model of the 'damping-like' force.

Equation (9.18) is used to plot how the amplitude of the limit cycle changes with the parameter $\beta$ for two different values of the power $\gamma$ (Fig. 9.3a, b).

In addition, numerically obtained amplitudes of the limit cycle are also presented in this figure for three different values of the parameter $\alpha$ corresponding to the linear $\alpha = 1$, under-linear $\alpha = 2/3$ and over-linear restoring forces $\alpha = 2$. It is seen that the analytical and numerical result agree reasonably well for the whole range of the powers considered.

As the power of the restoring force $\alpha$ increases, the amplitude of the limit cycle decreases. As the power of the geometric term in the 'damping' force $\beta$ increases, the amplitude of the limit cycle decreases as well.

### 9.2.2 Special Case: $\gamma = 1$

If the velocity term in the 'damping' force is linear as in the standard van der Pol oscillator, the amplitude of the limit cycle is [21]:

$$|a_{LC}| = \left[ \frac{\sqrt{\pi}\,\Gamma\left(\frac{4+\beta}{2}\right)}{\Gamma\left(\frac{1+\beta}{2}\right)} \right]^{1/\beta}. \tag{9.20}$$

The way how this amplitude changes with the parameter $\beta$ is plotted in Fig. 9.4. Numerically obtained amplitudes of the limit cycle calculated for different values of the parameter $\alpha$ are also shown.

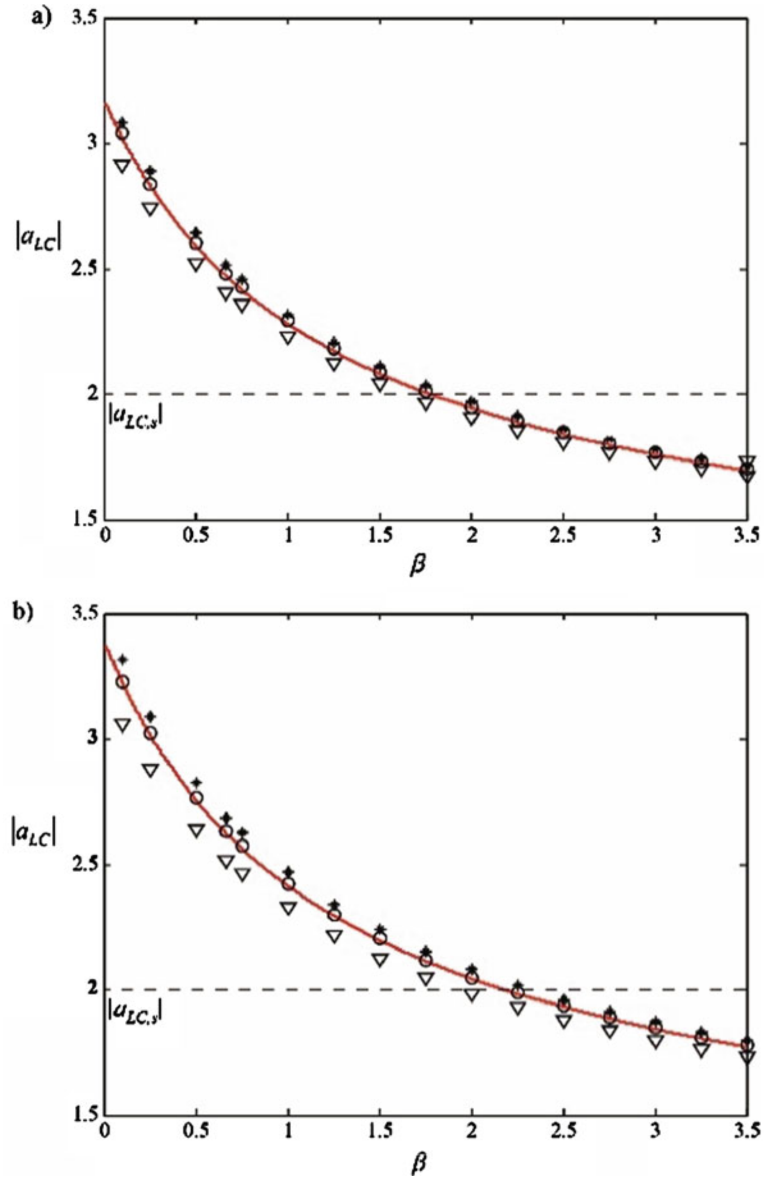


**Fig. 9.3** Amplitude of the limit cycle obtained analytically Eq. (9.18) (solid line) and numerically for $\varepsilon = 0.1$, $\alpha = 2/3$ (stars), $\alpha = 1$ (circles) and $\alpha = 2$ (triangles): **a** $\gamma = 0.8$; **b** $\gamma = 1.2$

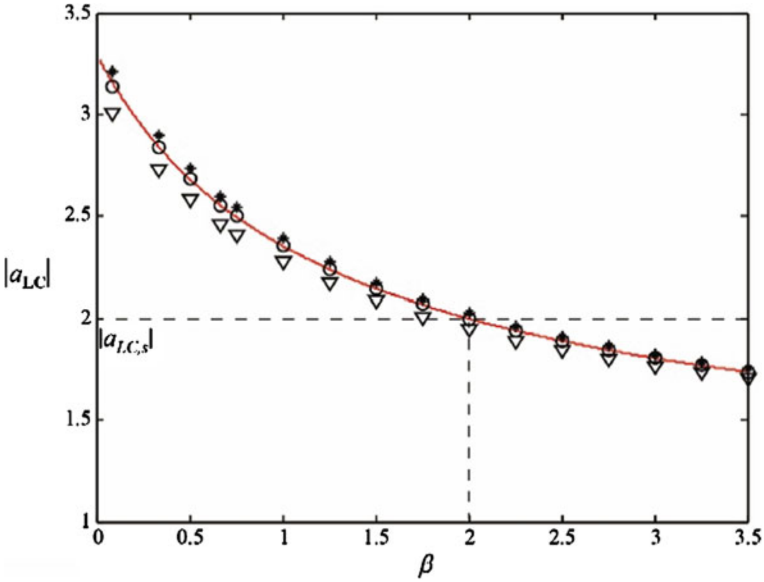


**Fig. 9.4** Amplitude of the limit cycle for $\gamma = 1$, $\varepsilon = 0.1$ obtained analytically Eq. (9.20) (solid line) and numerically: $\alpha = 2/3$ (stars), $\alpha = 1$ (circles) and $\alpha = 2$ (triangles)

Equation (9.20) implies that when $\beta \to 0$, $|a_{LC}| \to 2\sqrt{e}$ as well as when $\beta \to \infty$, $|a_{LC}| \to 1$, which agrees with the results found in [10]. When $\beta = 2$, as is in the standard van der Pol oscillator, the amplitude is obtained $|a_{LC}| = 2$, which is also seen in Fig. 9.1.

Using Eqs. (9.19) and (9.20), the frequency of the limit cycle oscillations is plotted in Fig. 9.5. For the under-linear case ($\alpha < 1$), this frequency is lower than $\omega_{LC,\mathrm{s}}$ and increases with $\beta$; for the over-linear case ($\alpha > 2$), this frequency is higher than $\omega_{LC,\mathrm{s}}$ and decreases with $\beta$.

### 9.2.3 Special Case: $\gamma = 0$

When the parameter $\gamma$ is equal to zero, Eq. (9.18) yields the following amplitude for the limit cycle

$$|a_{LC}| = (1 + \beta)^{1/\beta}. \tag{9.21}$$

Thus, when $\beta \to 0$, one has $|a_{LC}| \to e$. For the case when $\beta = 2$, one can calculate $|a_{LC}| = \sqrt{3}$.

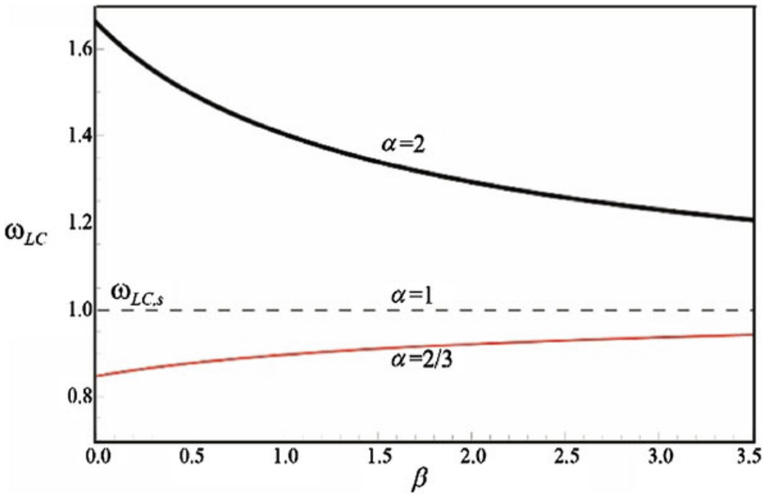


**Fig. 9.5** Frequency of limit cycle oscillations for $\gamma = 1$ and different values of $\alpha$, Eqs. (9.19), (9.20), (9.3a,b)

## 9.3 Large Values of the Damping Coefficient: Relaxation Oscillations

As illustrated in Fig. 9.2, the oscillatory response of the standard van der Pol oscillator (1) corresponding to $\varepsilon \gg 1$ has the form of relaxation oscillations. Besides the characteristic coordinates labelled in Fig. 9.2, their period has also attracted the interest of researchers. Based on geometrical considerations, van der Pol wrote in [6] that 'the period $T$, instead of being $2\pi$ (as was the case when $\varepsilon \ll 1$) increases with increase of $\varepsilon$, and when $\varepsilon \gg 1$ becomes equal to approximately $\varepsilon$ itself'. Later on, he improved this conclusion to $T = 1.61\varepsilon$ [8]. This expression was found not to be accurate for larger values of $\varepsilon$, such as, for example $\varepsilon = 10$, as the experiments had showed $T \approx 20$. Haag [24, 25] and Dorodnitsyn [26] provided more accurate approximations, giving the expressions whose first term coincided with $T_s = 1.61\varepsilon$, while additional ones were either power or log-forms of $\varepsilon$. Stoker's approximation for the period [27] was even more accurate, with the error of 0.8% for $\varepsilon = 5$ and 0.1% for $\varepsilon = 10$. The interested reader is referred to [7] for a rich historical review of the discovery and investigations of relaxation oscillations.

The aim of this section is to analyse relaxation oscillations of the generalized van der Pol oscillator (9.2a,b) and to determine the analytical expressions for the coordinates of jump points, outer curves, and the period. To that end, a perturbation approach with slow and fist time scales will be used. First, time is scaled by setting $t_1 = \varepsilon^{\vartheta} t$, where $\vartheta$ is to be determined. Equations (9.2a,b) turns into

$$\varepsilon^{2\vartheta} \frac{d^2x}{dt_1^2} + \mathrm{sgn}(x)|x|^{\alpha} - \varepsilon^{1+\gamma\vartheta}\left(1 - |x|^{\beta}\right)\left|\frac{dx}{dt_1}\right|^{\gamma} \mathrm{sgn}\left(\frac{dx}{dt_1}\right) = 0. \qquad (9.22)$$

Selecting $\vartheta = -1/\gamma$, the third and the second term are of the same order, while the first term becomes of the order $(1/\varepsilon)^{2/\gamma}$. This term can be neglected as being small for $\gamma$ being around unity. Thus, for the remaining procedure, it will be selected that $\gamma = 1$. As a result, one has $\vartheta = -1$ now. The slow time scale is defined as $t_1 = t/\varepsilon$, and Eq. (9.22) turns into

$$\frac{1}{\varepsilon^2}\frac{d^2x}{dt_1^2} + \operatorname{sgn}(x)|x|^\alpha - \left(1 - |x|^\beta\right)\frac{dx}{dt_1} = 0. \tag{9.23}$$

The first term can be neglected as being small, which yields

$$\frac{dx}{dt_1} = \frac{\operatorname{sgn}(x)|x|^\alpha}{1 - |x|^\beta}. \tag{9.24}$$

Jumps occur when $dx/dt_1$ is infinite, i.e. when the nominator in Eq. (9.24) is zero, giving the jump-down $x_{\mathrm{jd}}$ and jump-up $x_{\mathrm{ju}}$ values of the coordinates: $x_{\mathrm{jd}} = 1$ and $x_{\mathrm{ju}} = -1$. These values are the same as the one in the standard van der Pol oscillator (see Fig. 9.2) and are obtained as independent of the values of the powers $\alpha$ and $\beta$.

In order to define the characteristic amplitudes $x_{\mathrm{d}}$ and $x_{\mathrm{u}}$ labelled in Fig. 9.2 for a generalized van der Pol oscillator, the fast time scale is introduced as $t_2 = \varepsilon t$, resulting in

$$\frac{d^2x}{dt_2^2} + \frac{1}{\varepsilon^2}\operatorname{sgn}(x)|x|^\alpha - \left(1 - |x|^\beta\right)\frac{dx}{dt_2} = 0. \tag{9.25}$$

By neglecting the term of $O\left(1/\varepsilon^2\right)$, the following first integral can be derived

$$\frac{dx}{dt_2} - x + \frac{x|x|^\beta}{\beta + 1} = C, \tag{9.26}$$

where $C$ is a constant. Its value can be calculated by considering jumps for which $dx/dt_2 = 0$. When $x_{\mathrm{jd}} = 1$, one can calculate $C = -\beta/(\beta + 1)$, giving the following implicit equation for the amplitude $x_{\mathrm{d}}$:

$$x_{\mathrm{d}} - \frac{x_{\mathrm{d}}|x_{\mathrm{d}}|^\beta}{\beta + 1} - \frac{\beta}{\beta + 1} = 0. \tag{9.27}$$

In a similar way, the value of $C$ corresponding to $x_{\mathrm{ju}} = -1$ can be obtained, resulting in the following implicit equation for $x_{\mathrm{u}}$:

$$x_{\mathrm{u}} - \frac{x_{\mathrm{u}}|x_{\mathrm{u}}|^\beta}{\beta + 1} + \frac{\beta}{\beta + 1} = 0. \tag{9.28}$$

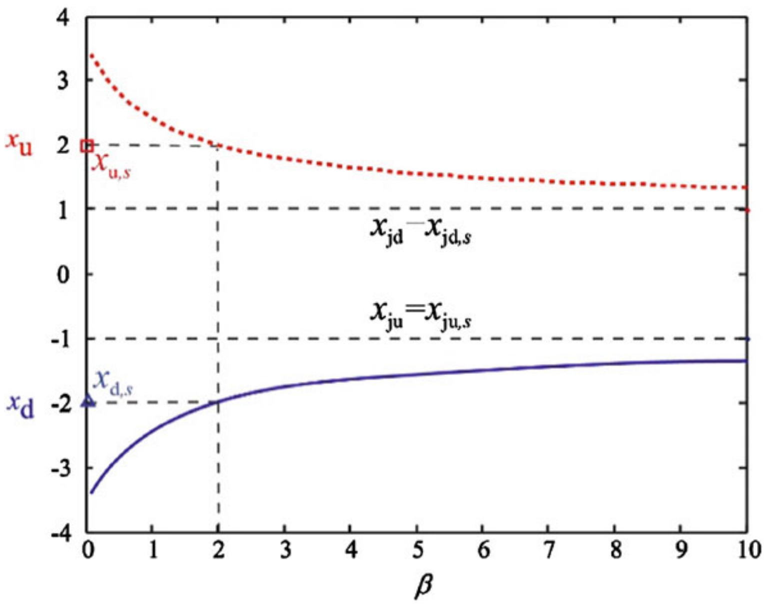


**Fig. 9.6** Change of the coordinates $x_d$ and $x_u$ with the power $\beta$, Eqs. (9.27) and (9.28)

Equations (9.27) and (9.28) imply that $x_d$ and $x_u$ depend on the damping power $\beta$, but do not depend on the power of the restoring force $\alpha$.

The solutions of Eqs. (9.27) and (9.28) are plotted in Fig. 9.6 as a function of the power $\beta$. As $\beta$ increases infinitely, the values of $x_d$ and $x_u$ approach 1 and $-1$, respectively, i.e. $x_{jd}$ and $x_{ju}$. This means that outer curves will be flatter as $\beta$ increases.

Integration of Eq. (9.26) can give analytical expressions for the outer curves. Thus, for $\alpha = 1$, one can derive [21]:

$$\ln|x| - \frac{|x|^{\beta}}{\beta} = t_1 + D, \tag{9.29}$$

where $D$ is a constant.

When $\alpha > 1$ and $\beta = \alpha - 1$, one can obtain

$$\frac{|x|^{1-\alpha}}{1-\alpha} - \ln|x| = t_1 + D. \tag{9.30}$$

If $\alpha \neq 1 \text{and} \beta \neq \alpha - 1$, the integration gives

$$\frac{|x|^{1-\alpha}}{1-\alpha} - \frac{|x|^{\beta-\alpha+1}}{\beta-\alpha+1} = t_1 + D. \tag{9.31}$$

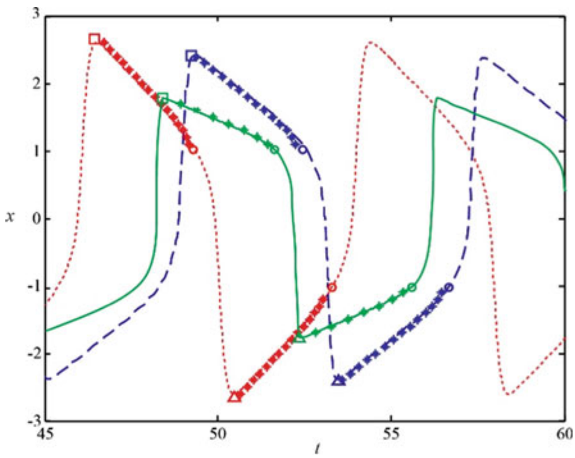


**Fig. 9.7** Relaxation oscillations of the generalized van der Pol oscillators modelled by Eqs. (9.2a,b) for $\gamma = 1$, $\varepsilon = 10$: numerical solution for $\alpha = 5/3$, $\beta = 2/3$ (red dotted line); numerical solution for $\alpha = 2$, $\beta = 1$ (blue dashed line), numerical solution for $\alpha = 5/2$, $\beta = 3/2$ (green solid line); outer curves defined by Eq. (9.30) are depicted by stars; jump-up points $x_{\rm ju}$ and jump-down points $x_{\rm jd}$ by circles; points $x_{\rm d}$ by triangles and points $x_{\rm u}$ by squares

To validate the analytical results obtained, their comparison with the numerical results from direct integration of the equation of motion is carried out for different values of the powers $\alpha$ and $\beta$. Figure 9.7 shows this comparison for the outer curves, and the characteristic coordinates: $x_{\rm u}$, $x_{\rm d}$, $x_{\rm jd}$ and $x_{\rm ju}$.

For the sake of easier visual comparison, the legend used for the characteristics coordinates corresponds to the one used in Fig. 9.2 for the standard van der Pol oscillator. It is seen that the analytical results obtained are in good agreement with the numerical results. These figures illustrate the effects of the powers $\alpha$ and $\beta$ influence on the relaxation oscillations, including their amplitude, i.e. the coordinates $x_{\rm d}$ and $x_{\rm u}$. They also give insight into the time spent moving along the outer curves from $x_{\rm u}$ to $x_{\rm jd}$, which corresponds to the first half of the period. It is seen that this time is affected by the powers $\alpha$ and $\beta$. In order to obtain the analytical expression for the half of the period, one can utilize the previously derived analytical expression for the outer curves and the coordinate $x_{\rm u}$, Eq. (9.27). Thus, the half-period on the slow time scale $t_1$ corresponding to $\alpha = 1$ is

$$\frac{T_1}{2} = \left[\ln|x| - \frac{|x|^{\beta}}{\beta}\right]_{x_{\rm u}}^{x_{\rm jd}=1} = -\frac{1}{\beta} - \ln x_{\rm u} + \frac{x_{\rm u}^{\beta}}{\beta}, \tag{9.32}$$

where the absolute value of $x_{\rm u}$ has been omitted as $0 < x_{\rm u} < 1$. On the original time scale $t$, the period of relaxation oscillations $T$ is

$$T = 2\left[-\frac{1}{\beta} - \ln x_{\rm u} + \frac{x_{\rm u}^{\beta}}{\beta}\right]\varepsilon. \tag{9.33}$$

When $\alpha > 1$, $\beta = \alpha - 1$, this period is

$$T = 2\left[\frac{1}{1-\alpha} + \ln x_{\mathrm{u}} - \frac{x_{\mathrm{u}}^{1-\alpha}}{1-\alpha}\right]\varepsilon. \tag{9.34}$$

If $\alpha \neq 1$, $\beta \neq \alpha - 1$, this period is defined by

$$T = 2\left[\frac{\beta}{(1-\alpha)(\beta-\alpha+1)} - \frac{x_{\mathrm{u}}^{1-\alpha}}{1-\alpha} + \frac{x_{\mathrm{u}}^{\beta-\alpha+1}}{\beta-\alpha+1}\right]\varepsilon. \tag{9.35}$$

Figure 9.8 shows how the ratio of the period of relaxation oscillations and the 'damping' coefficient $\varepsilon$ changes with the power $\beta$. The cases defined by Eqs. (9.33) and (9.35) are plotted in Fig. 9.8a and c, respectively. They illustrate that the ratio $T/\varepsilon$ increases as $\beta$ increases. Figure 9.8b is plotted based on Eq. (9.34) and shows different trends of $T/\varepsilon$ with respect to the value $\beta^* \approx 1.84$, which corresponds to $T^* \approx 0.637\varepsilon$. For $\beta < \beta^*$, the ratio considered increases with the increase of $\beta$, and then decreases.

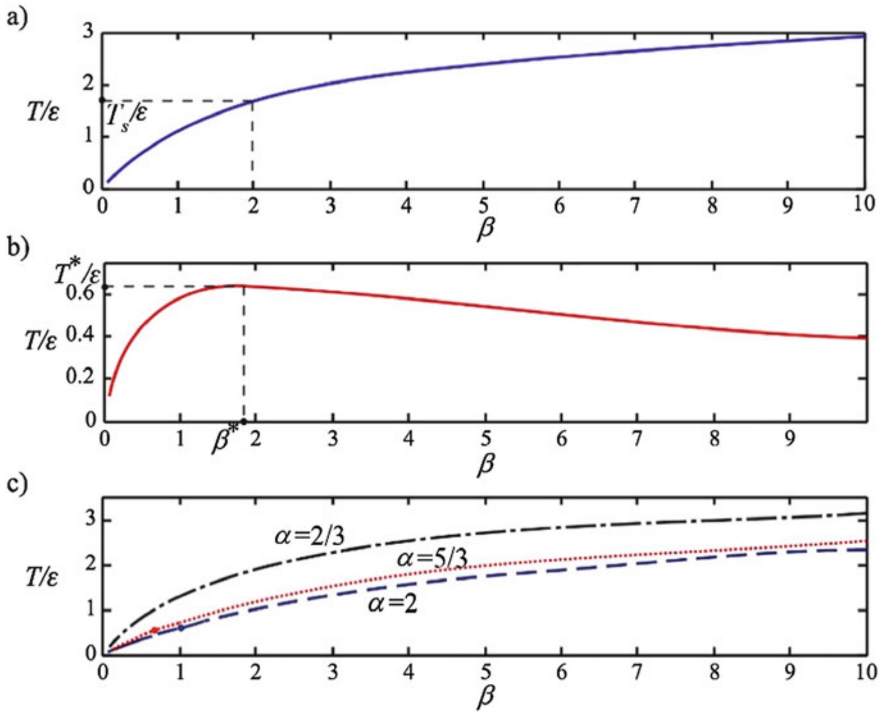


**Fig. 9.8** Ratio of the period of relaxation oscillations $T$ and the 'damping' coefficient $\varepsilon$ versus the damping power $\beta$: **a** Eq. (9.33); **b** Eq. (9.34); **c** Eq. (9.35), $\alpha = 2/3$ (black dashed-dotted line), $\alpha = 5/3$ (red dotted line), $\alpha = 2$ (blue dashed line), discontinuity of the curves due to the condition $\beta \neq \alpha - 1$ is depicted by circles

## 9.4 Conclusions

This work has first given a tribute to Balthasar van der Pol and his contribution related to the standard equation named after him. Two main cases and their properties have been pointed out: (i) the case of small values of the ‘damping’ parameter with the amplitude and frequency of free limit cycle oscillations, (ii) the case of large values of the ‘damping’ parameter and the resulting relaxation oscillations. Then, generalized van der Pol oscillators, have been investigated in both cases. Their restoring force and the ‘damping-like’ force are of power-form. The results have been compared with those for the standard van der Pol oscillator. In the former case, the method of averaging has been used. The expressions for the amplitude and frequency of the limit cycle have been derived, and also simplified for certain special cases related to certain system parameters. In the latter case, the expressions for jumps and outer curves have been obtained by using a perturbation technique for distinguishable combinations of the system parameters. The resulting period of relaxation oscillations has been obtained for these three combinations and the differences between them have been pointed out.

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