Optimal Investment–Consumption Decisions with Partially Observed Inflation: A Discrete-Time Formulation

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Abstract We consider a discrete-time optimal consumption and investment problem of an investor who is interested in maximizing his utility from consumption and terminal wealth subject to a random inflation in the consumption basket price over time. We consider two cases: (i) when the investor observes the basket price and (ii) when he receives only noisy signals on the basket price. We derive the optimal policies and show that a modified Mutual Fund Theorem consisting of three funds holds in both cases, as it does in the continuous-time setting. The compositions of the funds in the two cases are the same but, in general, the investor's allocations of his wealth into these funds differ.

1 Introduction

We study a discrete-time optimal investment and consumption decision problem of an investor when the consumption basket and real (inflation adjusted) asset prices are partially observed. Traditionally, the investment literature has assumed that the basket price, a measure of inflation, is fully observed. In reality, the basket price is difficult to assess, as it requires collecting the prices of all the consumption goods in the basket and their weights. Moreover, these prices may not be unique as discussed in Borenstein and Rose (1994). In other words, inflation is not fully observed and, as a consequence, the real asset prices are also incompletely observed.

As a benchmark case, we first consider fully observed inflation. In this case, the real asset market is complete, and the optimal policy can be obtained by solving

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the dynamic programming equation for the problem. The real optimal consumption process is the discrete-time equivalent of the optimal policy in the classical case considered in the continuous-time formulations of Karatzas et al. (1986), Merton (1971), and Sethi (1997). However, since the consumption basket price is also stochastic, its presence affects the optimal portfolio selection. Whereas the optimal portfolio in the classical case can be stated in terms of the risk-free fund and growth optimal risky fund, a result known as a Mutual Fund Theorem, and we show that the optimal portfolio with uncertain inflation can be characterized as a combination of three funds: the risk-free fund, the growth optimal fund (of the classical case), and a fund that arises from the correlation between the inflation uncertainty and the market risk. Every investor uses the first two funds, but the composition of the third fund may be different for different investors. However, if two investors have perfectly correlated consumption baskets, then they both will use the same third fund. Furthermore, in general, the amount invested in each of the three funds depends on their respective wealth, consumption basket prices, and utility functions. Henceforth, we will use the terms nominal consumption and consumption interchangeably. When we mean real consumption, it will be specified as such. The same convention will apply to the terms asset prices, wealth, savings, etc.

Following the analysis of the benchmark case, we study the situation when the investor receives noisy signals on inflation. Given the signal observations, the investor obtains the conditional probability distribution of the current basket price and, in turn, the conditional distribution of the current real asset prices. In general, the new risk due to the partial observability of the basket price affects the optimal policy. Interestingly enough, the characterization of the optimal portfolio in the partially observed case is the same as in the fully observed case. Thus, in both cases, the optimal portfolio is a linear combination of the risk-free fund, growth optimal fund, and the fund that arises from the correlation between the inflation uncertainty and the market risk. As before, the composition of the last fund for an investor depends on the nature of his consumption basket, and his allocation in the three funds depends on his wealth, utility function, and consumption basket price filter, which represents his best estimate given the observations.

There have been several studies on consumption measurement. Klenow (2003) discusses how the U.S. government measures consumption growth and how it considers the fact that the consumption basket changes over time. Inflation measurement and the problems with that are considered in Alchian and Klein (1973), Bradley (2001), and Shapiro and Wilcox (1997). Many of the social costs of inflation are caused by its unpredictability. The unpredictability is studied, e.g., by Ungar and Zilberhard (1993). The results of these studies are consistent with the present paper in the sense that our investor, due to noisy signals measurements, does not completely observe the consumption goods prices and, therefore, updates his belief about inflation from different consumption basket price signals.

The connection between inflation and asset prices is studied by Basak and Yan (2010), Campbell and Vuolteenaho (2004), and Cohen et al. (2005). According to them, the stock market suffers from money illusion, i.e., it incorrectly discounts real cash flows with nominal discount rates. Thus, when the inflation is high, the equity premium is also high and vice versa. In this paper, we do not consider money illusion. Optimal portfolio selection under inflation is studied, e.g., Brennan and Xia (2002), Chen and Moore (1985), Manaster (1979), Munk et al. (2004), and Solnik (1978). Brennan and Xia (2002) consider a more complicated inflation process but assume perfectly observed inflation. In our paper, we emphasize the fact that inflation signals are noisy and, therefore, the current consumption basket price is not completely observed. Portfolio selection with learning is also considered in Xia (2001) and Brennan (1998). In these papers, the investor learns about the stock returns, i.e., about the parameters of the price processes. As explained earlier, in the present paper the investor does not observe the consumption basket price directly, but infers it from the observed inflation signal. Thus, without the perfect information, the current real asset prices are also incompletely observed. In this way, our model differs from the above papers and also answers a different economic question: What is the effect of the noisy observations of inflation on the optimal portfolio selection?

Much more related to our paper is that of Bensoussan et al. (2009) that presents the continuous-time counterpart of our model. Also the results presented here are consistent with their continuous-time counterparts. However, the mathematical analysis of the discrete-time formulation, presented here for the first time, is different. More importantly, our discrete-time formulation paves the road for future researchers to perform related empirical studies since the data in practice can only be collected in discrete time.

The rest of the paper is organized as follows. Section 2 formulates the discrete-time model under considerations along with the underlying information sets and stochastic processes. The optimal policy under the fully observed inflation is derived in Sect. 3. Section 4 formulates the model in the partially observed case. Section 5 concludes the paper.

2 Discrete-Time Model

Let us consider a discrete-time model with the length of period h. Then the time period can be represented by

$$t \in \{0, h, 2h, \dots, Nh = T\},$$

where N is the number of periods. We introduce notation:

$$\delta f(t) = f(t+h) - f(t).$$

2.1 Evolution of Prices of Stocks

Let us consider a probability space $(\Omega, \mathcal{A}, P)$. Let $\alpha_i(t)$ and $\sigma_{ij}(t)$ denote deterministic and bounded expected returns and volatility functions of time, respectively. Let $\delta w_j(t) = w_j(t+h) - w_j(t)$ denote independent Gaussian variables with 0 mean and variance h. Let $\mathcal{G}_t$ be the σ-algebra generated by $\delta w(s), s = 0, \ldots, t$, where $w(s)$ is an n-dimensional Gaussian random vector, i.e., $w(s) = (w_1(s), \ldots, w_n(s))^{\mathsf{T}}$. Then the evolution of the stock price of security $i, i = 1, 2, \ldots, n$, can be described by

$$Y_i(t+h) = Y_i(t) \exp\left[\left(\alpha_i(t) - \frac{1}{2}\sigma_{ii}(t)\right) h + \sum_{j=1}^{n} \sigma_{ij}(t) \cdot \delta w_j(t)\right]. \tag{1}$$

Let us introduce the following process

$$\theta(t) = \sigma^{-1}(t)\left(\alpha(t) - r\mathbf{1}\right), \tag{2}$$

known as the market price risk, where $\mathbf{1}$ denotes the unit column vector, and we assume that the market is complete so that the matrix $\sigma(t) = \left(\sigma_{ij}(t)\right)$ is invertible.

The dynamics of the nominal value of the risk-free asset is given by

$$Y_0(t+h) = Y_0(t)e^{rh}. \tag{3}$$

Let us define the process $Q(t)$ by

$$Q(t+h) = Q(t)\exp\left[-\theta(t)\cdot\delta w(t) - \frac{1}{2}h\,|\theta(t)|^2\right]; \quad Q(0) = 1. \tag{4}$$

The processes $Q(t)$ and $Q(t)Y_i(t)e^{-rt}$ are $(P, \mathcal{G}_t)$ martingales.

2.2 Risk-Neutral Probability

Define on $(\Omega, \mathcal{A})$ a probability $\hat{P}$ as follows

$$\left.\frac{d\hat{P}}{dP}\right|_{\mathcal{G}_t} = Q(t).$$

Now let us set

$$\delta\tilde{w}(t) = \delta w(t) + h\theta(t).$$

Then on $(\Omega, \mathcal{A}, \hat{P})$, the $\delta\tilde{w}(t)$ forms a sequence of independent Gaussian random variables with mean 0 and variance h.

We can also write (1) as

$$Y_i\ (t+h) = Y_i(t) \exp\left[\left(r - \frac{1}{2}\sigma_{ii}(t)\right)h + \sum_{j=1}^{n} \sigma_{ij}(t) \cdot \delta\tilde{w}_j(t)\right], \tag{5}$$

and $Y_i(t)e^{-rt}$ is a $\left(\hat{P}, \mathcal{G}_t\right)$ martingale.

2.3 Evolution of Basket Price

We consider another Wiener process $w_I(t)$, which is one-dimensional and correlated with $w(t)$. Then we have

$$E\left[\delta w_i(t)\delta w_I(t)\right] = \rho_i h,$$

for all $i \in \{1, \ldots, n\}$, and redefine $\mathcal{G}_t$ as the σ-algebra generated by $\delta w(s)$ and $\delta w_I(s)$, $s = 0, \ldots, t$. Let $\rho^* = (\rho_1, \rho_2, \ldots, \rho_n)$ with $*$ denoting the transpose operation.

The dynamics of the basket price process $B(t)$ is given by

$$B\ (t+h) = B(t) \exp\left[\left(I - \frac{1}{2}\zeta^2\right)h + \zeta \cdot \delta w_I(t)\right]; \quad B\ (0) = B_0, \tag{6}$$

where $I > 0$ represents the expected periodic inflation and $\zeta > 0$ denotes the inflation volatility. The initial basket price B_0 is known when there is full observation, whereas it can be a random variable independent of $\mathcal{G}_t$ in the case of partial information.

Let us define the log basket price $L(t) = \log B(t)$, so that

$$L\ (t+h) = L(t) + \left(I - \frac{1}{2}\zeta^2\right)h + \zeta \cdot \delta w_I(t);\ L(0) = L_0 = \log B_0. \tag{7}$$

2.4 Self-financing Wealth Process

In the discrete-time setting, the nominal wealth at time t is defined by

$$X(t) = C(t)h + \varpi_0(t)e^{rt} + \varpi(t)Y(t), \tag{8}$$

where $\varpi_0(t)$ and $\varpi(t)$ denote the amount of riskless and risky assets owned by the investor, and $C(t)$ is the consumption process.

The self-financing condition implies

$$X(t+h) = \varpi_0(t)e^{rt}e^{rh} + \varpi(t)Y(t+h)$$

and, therefore, we have

$$\delta\left(X(t)e^{-rt}\right) = \varpi(t)\cdot\delta\left[Y(t)e^{-rt}\right] - C(t)h\cdot e^{-rt}, \tag{9}$$

where, by (5),

$$\delta\left[Y_i(t)e^{-rt}\right] = Y_i(t)e^{-rt}\left[\exp\left(-\frac{1}{2}\sigma_{ii}(t)h + \sum_{j=1}^{n}\sigma_{ij}(t)\cdot\delta\tilde{w}_j(t)\right) - 1\right].$$

Set

$$\delta\mu_i(t) = -\frac{1}{2}\sigma_{ii}(t)h + \sum_{j=1}^{n}\sigma_{ij}(t)\cdot\delta\tilde{w}_j(t);\quad \mu_i(0) = 0. \tag{10}$$

Then

$$\delta\left[Y_i(t)e^{-rt}\right] = Y_i(t)e^{-rt}\left[\exp\left(\delta\mu_i(t)\right) - 1\right]. \tag{11}$$

Let us define $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ with

$$\pi_i(t) = \frac{\varpi_i(t)Y_i(t)}{X(t)}, \tag{12}$$

representing the proportion of the wealth invested in security i. Then the evolution of wealth is given by

$$\delta\left[X(t)e^{-rt}\right] = X(t)e^{-rt}\sum_{i=1}^{n}\pi_i(t)\left[\exp\left(\delta\mu_i(t)\right) - 1\right] - C(t)h\cdot e^{-rt}. \tag{13}$$

3 Fully Observed Inflation Case

We consider a problem starting at t, with $X(t) = x$, $L(t) = L$, and dynamics

$$\delta\left[X(s)e^{-rs}\right] = X(s)e^{-rs}\sum_{i=1}^{n}\pi_i(s)\left[\exp\left(\delta\mu_i(s)\right) - 1\right] - C(s)h\cdot e^{-rs}, \tag{14}$$

$$\delta L(s) = \left(I - \frac{\zeta^2}{2}\right)h + \zeta\delta w_I(s). \tag{15}$$

With $U_1(\cdot)$ and $U_2(\cdot)$ denoting the utility of real consumption and real wealth, respectively, and $U_i'(0) = \infty$, $U_i'(\infty) = 0$, $i = 1, 2$, the performance is given by

$$J(\pi(\cdot), C(\cdot); x, L, t) = E\left[\sum_{s=t}^{T-h} h \cdot U_1\left(C(s)e^{-L(s)}\right) \cdot e^{-\beta(s-t)} + U_2\left(X(t)e^{-L(t)}\right) e^{-\beta(T-t)} | L(t) = L, X(t) = x\right]. \tag{16}$$

Here the wealth process $X(t)$ follows (13) and β as the utility discount rate. We define the *value function* as

$$V(x, L, t) = \sup_{C(\cdot),\pi(\cdot)} J(C(\cdot), \pi(\cdot); x, L, t). \tag{17}$$

3.1 Dynamic Programming

Now we have the following dynamic programming problem:

$$V(x, L, t) = \max_{\pi, C} h \cdot U_1\left(Ce^{-L}\right) + e^{-\beta h} E\left[V\left(X(t+h), L(t+h), t+h\right)\right], \tag{18}$$

where $V(x, L, T) = U_2\left(x \cdot e^{-L}\right)$. From (13) and (7), we have

$$\begin{aligned} X(t+h) &= x \cdot e^{rh}\left[1 + \sum_{i=1}^{n} \pi_i \cdot (\exp(\delta\mu_i(t)) - 1)\right] - C \cdot h \cdot e^{rh} \\ &= x \cdot e^{rh}\left[1 - \sum_{i=1}^{n} \pi_i + \sum \pi_i \exp\left(\alpha_i(t) - r - \frac{1}{2}\sigma_{ii}(t)\right)h \right. \\ &\quad \left. \cdot \exp\left(\sum_{j=1}^{n} \sigma_{ij}\delta w_j(t)\right)\right] - C \cdot h \cdot e^{rh} \end{aligned}$$

and

$$L(t+h) = L + \left(I - \frac{\zeta^2}{2}\right)h + \zeta\delta w_I(t).$$

We define

$$\delta\tilde{w}_I = \frac{\delta w_I - \rho^* \delta w}{\sqrt{1 - |\rho|^2}}.$$

Then $\delta\tilde{w}_I$ and δw are independent and the variance of $\delta\tilde{w}_I$ is h. Hence,

$$L(t+h) = L + \left(I - \frac{\zeta^2}{2}\right)h + \zeta\sqrt{1-|\rho|^2}\delta\tilde{w}_I + \zeta\rho^*\delta w$$

and

$$\begin{aligned}
&E\left[V\left(X(t+h), L(t+h), t+h\right)\right] \\
&= (2n)^{-\frac{n+1}{2}} \int \cdots \int V\left(xe^{rh}\left(1-\sum_{i=1}^{n}\pi_i\right) - C\cdot h\cdot e^{rh}\right. \\
&\quad + xe^{rh}\sum_{i=1}^{n}\pi_i \exp\left(\alpha_i(t) - r - \frac{1}{2}\sigma_{ii}(t)\right)h \exp\left(\sqrt{h}\sum_{j=1}^{n}\sigma_{ij}\xi_j\right), L + \left(I - \frac{\zeta^2}{2}\right)h \\
&\quad \left. + \zeta\sqrt{h}\sum_{j=1}^{n}\rho_j\xi_j + \zeta\sqrt{h}\sqrt{1-|\rho|^2}\psi, t+h\right) \\
&\quad \cdot \exp\left(-\frac{1}{2}\left(\sum_{j=1}^{n}\xi_j^2 + \psi^2\right)\right) d\xi_1\cdots d\xi_n d\psi.
\end{aligned}$$

So, (18) becomes

$$\begin{aligned}
&V(x, L, t) = \max_{\pi, C} hU_1\left(Ce^{-L}\right) + e^{-\beta h}\int \cdots \int \\
&V\left(xe^{rh}\left(1-\sum_{i=1}^{n}\pi_i\right) - C\cdot h\cdot e^{rh}\right. \\
&+ xe^{rh}\sum_{i=1}^{n}\pi_i \cdot \exp\left(\left(\alpha_i(t) - r - \frac{1}{2}\sigma_{ii}(t)\right)h + \sqrt{h}\sum_{j=1}^{n}\sigma_{ij}\xi_j\right), \\
&\left. L + \left(I - \frac{\zeta^2}{2}\right)h + \zeta\sqrt{h}\left(\sum_{j=1}^{n}\rho_j\xi_j + \sqrt{1-|\rho|^2}\psi\right), t+h\right) \\
&\cdot \frac{\exp\left(-\frac{1}{2}\left(\sum_{j=1}^{n}\xi_j^2 + \psi^2\right)\right)}{(2n)^{\frac{n+1}{2}}} d\xi_1\cdots d\xi_n d\psi.
\end{aligned} \tag{19}$$

3.2 Optimal Feedback Policy

Let $(\hat{C}(x, L, t), \hat{\pi}(x, L, t))$ be the optimal feedback consumption and investment policy corresponding to (19). It is convenient to introduce the notation:

$$\begin{aligned}\hat{X}_h\left(x,t;\xi\right) &= xe^{rh}\left(1-\sum_{i=1}^{n}\hat{\pi}_i\right)-\hat{C}\cdot h\cdot e^{rh}\\ &\quad+xe^{rh}\sum_{i=1}^{n}\hat{\pi}_i\exp\left(\left(\alpha_i(t)-r-\frac{1}{2}\sigma_{ii}(t)\right)h+\sqrt{h}\left(\sigma\xi\right)_i\right)\end{aligned} \tag{20}$$

and

$$\hat{L}_h\left(L,\xi,\psi\right)=L+\left(I-\frac{\zeta^2}{2}\right)h+\zeta\sqrt{h}\cdot\left(\rho^*\xi+\sqrt{1-|\rho|^2}\psi\right).$$

The necessary conditions of optimality are

$$\begin{aligned}&hU_1'\left(\hat{C}e^{-L}\right)e^{-L}+e^{-\beta h}\int\cdots\int\frac{\partial V}{\partial x}\left(\hat{X}_h\left(x,t;\xi\right),\hat{L}_h\left(L,\xi,\psi\right),t+h\right)\\ &\quad\cdot\frac{\exp\left(-\frac{1}{2}\left(|\xi|^2+\psi^2\right)\right)}{(2n)^{\frac{n+1}{2}}}d\xi_1\cdots d\xi_n d\psi\left(-he^{rh}\right)=0\end{aligned} \tag{21}$$

and

$$\begin{aligned}&\int\cdots\int\frac{\partial V}{\partial x}\left(\hat{X}_h\left(x,t,\xi\right),\hat{L}_h\left(L,\xi,\psi\right),t+h\right)\Big[-xe^{rh}\\ &+xe^{rh}\exp\left(\left(\alpha_i(t)-r-\frac{1}{2}\sigma_{ii}(t)\right)h+\sqrt{h}\left(\sigma\xi\right)_i\right)\Big]\frac{\exp\left(-\frac{1}{2}\left(|\xi|^2+\psi^2\right)\right)}{(2n)^{\frac{n+1}{2}}}\\ &d\xi_1\cdots d\xi_n d\psi=0,\quad i=1,\ldots,n.\end{aligned} \tag{22}$$

We divide (21) by h and (22) by xe^{rh} to get

$$\begin{aligned}&U_1'\left(\hat{C}e^{-L}\right)e^{-L}=e^{-(\beta-r)h}\int\cdots\int\frac{\partial V}{\partial x}\left(\hat{X}_h\left(x,t,\xi\right),\hat{L}_h\left(L,\xi,\psi\right),t+h\right)\\ &\quad\cdot\frac{\exp\left(-\frac{1}{2}\left(|\xi|^2+\psi^2\right)\right)}{(2n)^{\frac{n+1}{2}}}d\xi_1\cdots d\xi_n d\psi\end{aligned} \tag{23}$$

and

$$\begin{aligned}&\int\cdots\int\frac{\partial V}{\partial x}\left(\hat{X}_h\left(x,t,\xi\right),\hat{L}_h\left(L,\xi,\psi\right),t+h\right)\\ &\left[-1+\exp\left(\left(\alpha_i(t)-r-\frac{1}{2}\sigma_{ii}(t)\right)h+\sqrt{h}\left(\sigma\xi\right)_i\right)\right]\end{aligned}$$

$$\cdot \frac{\exp\left(-\frac{1}{2}\left(|\xi|^2+\psi^2\right)\right)}{(2n)^{\frac{n+1}{2}}} d\xi_1 \cdots d\xi_n d\psi = 0, \quad i = 1, \ldots, n. \tag{24}$$

But, by (19), we also get by differentiating with respect to x:

$$\frac{\partial V}{\partial x}(x, L, t) = e^{-\beta h} \int \cdots \int \frac{\partial V}{\partial x}\left(\hat{X}_h(x, t, \xi), \hat{L}_h(L, \xi, \psi), t+h\right)$$
$$\left[e^{rh}\left(1-\sum_{i=1}^n \pi_i\right) + e^{rh} \sum_{i=1}^n \hat{\pi}_i \exp\left(\left(\alpha_i(t) - r - \frac{1}{2}\sigma_{ii}(t)\right) h + \sqrt{h} \sum_{j=1}^n \sigma_{ij}\xi_j\right)\right]$$
$$\cdot \frac{\exp\left(-\frac{1}{2}\left(|\xi|^2+\psi^2\right)\right)}{(2n)^{\frac{n+1}{2}}} d\xi_1 \cdots d\xi_n d\psi.$$

Multiplying (24) by $\hat{\pi}_i e^{rh}$ and summing up, we get from the above equation

$$\frac{\partial V}{\partial x} = e^{-(\beta-r)h} \int \cdots \int \frac{\partial V}{\partial x}\left(\hat{X}_h(x, t, \xi), \hat{L}_h(L, \xi, \psi), t+h\right) \tag{25}$$
$$\cdot \frac{\exp\left(-\frac{1}{2}\left(|\xi|^2+\psi^2\right)\right)}{(2n)^{\frac{n+1}{2}}} d\xi_1 \cdots d\xi_n d\psi.$$

Now comparing with (23), we have proven

$$U_1'\left(\hat{C}e^{-L}\right)e^{-L} = \frac{\partial V}{\partial x}(x, L, t), \tag{26}$$

which yields the optimal feedback $\hat{C}(x, L, t)$. It corresponds with the solution (3.3) obtained in the continuous-time model of Bensoussan et al. (2009).

To obtain $\hat{\pi}_i$, we must use relation (24), replacing $\hat{X}_h(x, t, \xi)$ and $\hat{L}_h(L, \xi, \psi)$ by formulas in (20). For convenience, we first write the integrand of the system (24), plug (20) into that integrand, and transform it by using the integration by substitution:

$$\frac{\partial V}{\partial x}\left(\hat{X}_h(x, t, \xi), \hat{L}_h(L, \xi, \psi), t+h\right)$$
$$= \frac{\partial V}{\partial x}\left(\left(x - \hat{C}h\right)e^{rh}, \hat{L}_h(L, \xi, \psi), t+h\right)$$
$$+ \int_0^1 \frac{\partial^2 V}{\partial x^2}\left(\left(x-\hat{C}h\right)e^{rh} + \theta x e^{rh} \sum_{k=1}^n \hat{\pi}_k \left(\exp\left(\left(\alpha_k(t) - r - \frac{1}{2}\sigma_{kk}(t)\right)h\right.\right.\right.$$
$$\left.\left.\left. + \sqrt{h}(\sigma\xi)_k\right) - 1\right), \hat{L}_h(L, \xi, \psi), t+h\right)$$
$$x e^{rh} \sum_{k=1}^n \hat{\pi}_k \left(\exp\left(\left(\alpha_k(t) - r - \frac{1}{2}\sigma_{kk}(t)\right)h + \sqrt{h}(\sigma\xi)_k\right) - 1\right) d\theta,$$

and write the system (24) as in (27) which gives $\hat{\pi}_i(x, L, t)$ with small h (see below).

$$0 = \int \cdots \int \frac{\partial V}{\partial x}\left(\left(x - \hat{C}h\right)e^{rh}, \hat{L}_h\left(L, \xi, \psi\right), t + h\right)\Big[\exp\left(\left(\alpha_i(t) - r - \frac{1}{2}\sigma_{ii}(t)\right)h + \sqrt{h}\left(\sigma\xi\right)_i\right) - 1\Big]\frac{\exp\left(-\frac{1}{2}\left(|\xi|^2 + \psi^2\right)\right)}{(2n)^{\frac{n+1}{2}}}d\xi_1 \cdots d\xi_n d\psi$$
$$+ xe^{rh}\int \cdots \int \int_0^1 d\theta \frac{\partial^2 V}{\partial x^2}\left(\left(x - \hat{C}h\right)e^{rh} + \theta x e^{rh}\sum_k \hat{\pi}_k\left(\exp\left(\left(\alpha_k(t) - r - \frac{1}{2}\sigma_{kk}(t)\right)h + \sqrt{h}\left(\sigma\xi\right)_k\right) - 1\right), \hat{L}_h\left(L, \xi, \psi\right), t + h\right)$$
$$\cdot \sum_{k=1}^n \hat{\pi}_k\left(\exp\left(\left(\alpha_k(t) - r - \frac{1}{2}\sigma_{kk}(t)\right)h + \sqrt{h}\left(\sigma\xi\right)_k\right) - 1\right)\left(\exp\left(\left(\alpha_i(t) - r - \frac{1}{2}\sigma_{ii}(t)\right)h + \sqrt{h}\left(\sigma\xi\right)_i\right) - 1\right)\frac{\exp\left(-\frac{1}{2}\left(|\xi|^2 + \psi^2\right)\right)}{(2n)^{\frac{n+1}{2}}}d\xi_1 \cdots d\xi_n d\psi,$$
$$\forall \quad i = 1, \ldots, n. \tag{27}$$

3.3 Approximation for Small h

The system (27) is highly non-linear. We can simplify it for small h to obtain the same formulas as in the continuous time. We make the following three approximations by (20) and exponential function with small h:

$$\left(x - \hat{C}h\right)e^{rh} \sim x,$$
$$\hat{L}_h\left(L, \xi, \psi\right) \sim L + \zeta\sqrt{h}\left(\rho^*\xi + \sqrt{1 - |\rho|^2}\psi\right),$$
$$\exp\left(\left(\alpha_i(t) - r - \frac{1}{2}\sigma_{ii}(t)\right)h + \sqrt{h}\left(\sigma\xi\right)_i\right) - 1 \sim \sqrt{h}\left(\sigma\xi\right)_i + \left(\alpha_i(t) - r - \frac{1}{2}\sigma_{ii}(t) + \frac{1}{2}\left(\sigma\xi\right)_i^2\right)h.$$

Then, by plugging the above approximations into (27), we obtain

$$0 = \int \cdots \int \frac{\partial V}{\partial x}\left(x, L + \zeta\sqrt{h}\left(\rho^*\xi + \sqrt{1 - |\rho|^2}\psi\right), t + h\right)\left(\sqrt{h}\left(\sigma\xi\right)_i + \left(\alpha_i(t) - r - \frac{1}{2}\sigma_{ii}(t) + \frac{1}{2}\left(\sigma\xi\right)_i^2\right)h\right)\frac{\exp\left(-\frac{1}{2}\left(|\xi|^2 + \psi^2\right)\right)}{(2n)^{\frac{n+1}{2}}}d\xi_1 \cdots d\xi_n d\psi$$

$$+x\int_0^1\cdots\int\int_0^1 d\theta\frac{\partial^2 V}{\partial x^2}\left(x+\theta x\sum_k\hat{\pi}_k\left(\sqrt{h}\,(\sigma\xi)_k\right.\right.$$
$$\left.\left.\cdot\left(\alpha_k(t)-r-\frac{1}{2}\sigma_{kk}(t)\right)h\right),\hat{L}_h\left(L,\xi,\psi\right),t+h\right)$$
$$\cdot\sum_{k=1}^n\hat{\pi}_k\left(\left(\sqrt{h}\,(\sigma\xi)_k+\alpha_k(t)-r-\frac{1}{2}\sigma_{kk}(t)\right)h\right)\left(\left(\sqrt{h}\,(\sigma\xi)_i+\alpha_i(t)\right.\right.$$
$$\left.\left.-r-\frac{1}{2}\sigma_{ii}(t)\right)h\right)\frac{\exp\left(-\frac{1}{2}\left(|\xi|^2+\psi^2\right)\right)}{(2n)^{\frac{n+1}{2}}}d\xi_1\cdots d\xi_n d\psi,\ i=1,\ldots,n.$$

Next we apply Taylor's expansion to obtain

$$0\sim h\cdot(\alpha_i(t)-r)\frac{\partial V}{\partial x}(x,L,t)+\frac{\partial^2 V}{\partial x\partial L}(x,L,t)\,\zeta h$$
$$\cdot\int\cdots\int(\sigma\xi)_i\left(\rho^*\xi+\sqrt{1-|\rho|^2}\psi\right)\frac{\exp\left(-\frac{1}{2}\left(|\xi|^2+\psi^2\right)\right)}{(2n)^{\frac{n+1}{2}}}d\xi_1\cdots d\xi_n d\psi$$
$$+hx\frac{\partial^2 V}{\partial x^2}(x,L,t)\int\cdots\int\sum_k\hat{\pi}_k(\sigma\xi)_k(\sigma\xi)_i\frac{\exp\left(-\frac{1}{2}\left(|\xi|^2+\psi^2\right)\right)}{(2n)^{\frac{n+1}{2}}}$$
$$d\xi_1\cdots d\xi_n d\psi.$$

Hence, finally after dividing by h, we have

$$(\alpha_i(t)-r)\frac{\partial V}{\partial x}(x,L,t)+\frac{\partial^2 V}{\partial x\partial L}(x,L,t)\,\zeta\,(\sigma\rho)_i$$
$$+x\frac{\partial^2 V}{\partial x^2}(x,L,t)\left(\sigma\sigma^*\hat{\pi}\right)_i=0. \tag{28}$$

Recalling that

$$\alpha_i(t)-r=(\sigma\theta)_i\,,$$

we deduce

$$\theta\frac{\partial V}{\partial x}(x,L,t)+\zeta\rho\frac{\partial^2 V}{\partial x\partial L}(x,L,t)+x\frac{\partial^2 V}{\partial x^2}(x,L,t)\,\sigma^*\hat{\pi}=0. \tag{29}$$

Thus, we have

$$\hat{\pi}\,(x,L,t)=-\frac{(\sigma^*(t))^{-1}\left[\theta\frac{\partial V}{\partial x}(x,L,t)+\zeta\rho\frac{\partial^2 V}{\partial x\partial L}(x,L,t)\right]}{x\frac{\partial^2 V}{\partial x^2}(x,L,t)}, \tag{30}$$

which can be compared with the formula (3.4) obtained in the continuous-time model of Bensoussan et al. (2009)

We can see that the first term on the right-hand side of (30) represents the risky mutual fund of classical Mutual Fund Theorem that states that the investor can limit his portfolio to investing simply in the risk-free fund and this risky mutual fund. The presence of the second term on the right-hand side of (30) requires the investor to consider a third fund due to the effect of uncertainty in inflation. Note that if the inflation is uncorrelated with all the risky assets, then the second term is zero. The inflation effect depends also on the inflation volatility $\zeta > 0$ and on V_{Lx}, i.e., on the sensitivity of the marginal value V_x with respect to the ln-basket price. If the marginal value of nominal wealth rises (falls) in the basket price, then the higher the correlation, the more (less) funds the investor allocates to the stock market.

Now we state the following extension of the classical Mutual Fund Theorem.

Theorem 1 *With fully observed inflation, the optimal portfolio involves an allocation between the risk-free fund F_1 and two risky funds that consist only of risky assets: $F_2(t) = (\sigma^*(t))^{-1}\,\theta(t)$ and $F_3(t) = (\sigma^*(t))^{-1}\,\rho$, where the vector $F_k(t)$ represents the k^{th} portfolio's weights of the risky assets at time t, $k = 2, 3$. Furthermore, the optimal proportional allocations $\mu_k(t)$ of wealth in the fund $F_k(t)$, $k = 1, 2, 3$, at time t are given by*

$$\mu_2(t) = \frac{-V_x\,(x, L, t)}{x(t)V_{xx}\,(X, L, t)},$$
$$\mu_3(t) = \frac{-\zeta\,V_{Lx}\,(x, L, t)}{x(t)V_{xx}\,(X, L, t)},$$

and

$$\mu_1(t) = 1 - \mu_2(t) - \mu_3(t).$$

According to Theorem 1, the optimal portfolio can consist of investments in three funds, whereas the classical problem (without uncertain inflation) requires only two funds. The first fund is the risk-free asset and the second one is the growth optimum portfolio fund as in the classical problem. The third fund arises from the correlation between the inflation uncertainty and the market risk.

Three-fund theorems are not new. They arise, e.g., in the continuous-time portfolio models of Zhao (2007) and Brennan and Xia (2002). Zhao (2007) considers an optimal asset allocation policy for an investor concerned with the performance of his investment relative to a benchmark. In his case, one of the two risky funds replicates the benchmark portfolio. In the three-fund theorem obtained by Brennan and Xia (2002), one fund replicates real interest rate uncertainty, another one is the classical growth optimal fund, and the last one replicates the fully observed inflation uncertainty. They do not consider partially observed inflation as in the present paper.

Before we take up the case of partially observed inflation in Sect. 4, next, let us illustrate the special case of one period where we can obtain explicitly the optimal consumption and portfolio policies.

3.4 One-Period Problem

Now, let us take $T = h$ and $N = 1$. We call $V(x, L, 0) = V(x, L)$, $\alpha_i(0) = \alpha_i$, and $\sigma_i(0) = \sigma_{ij}$. Also, let $U_2(x, L) = U_2\left(xe^{-L}\right)$. Then, we get from (19)

$$V(x, L) = \max_{\pi, C}\Bigg[hU_1\left(Ce^{-L}\right) + e^{-\beta h}\int\cdots\int U_2\Bigg(xe^{rh}\Bigg(1 - \sum_{i=1}^{n}\pi_i\Bigg) \quad (31)$$
$$-C\cdot h\cdot e^{rh} + xe^{rh}\sum_{i=1}^{n}\pi_i\cdot\exp\Bigg(\left(\alpha_i - r - \frac{1}{2}\sigma_{ii}\right)h + \sqrt{h}\sum_{j=1}^{n}\sigma_{ij}\xi_j\Bigg),$$
$$L + \left(x - \frac{\zeta^2}{2}\right)h + \zeta\sqrt{h}\left(\rho^*\xi + \sqrt{1 - |\rho|^2}\psi\right)\Bigg)\frac{\exp\left(-\frac{1}{2}\left(|\xi|^2 + \psi^2\right)\right)}{(2n)^{\frac{n+1}{2}}}$$
$$d\xi_1\cdots d\xi_n d\psi\Bigg].$$

We still have (26). Now (22) becomes

$$0 = \int\cdots\int\frac{\partial U_2}{\partial x}\left(\left(x - \hat{C}h\right)e^{rh}, \hat{L}_h(x, \xi, \psi)\right)\Big[\exp\Big(\Big(\alpha_i - r \quad (32)$$
$$-\frac{1}{2}\sigma_{ii}\Big)h + \sqrt{h}\,(\sigma\xi)_i\Big) - 1\Big]\frac{\exp\left(-\frac{1}{2}\left(|\xi|^2 + \psi^2\right)\right)}{(2n)^{\frac{n+1}{2}}}d\xi_1\cdots d\xi_n d\psi$$
$$+xe^{rh}\int\cdots\int\int_0^1 d\theta\frac{\partial^2 U_2}{\partial x^2}\Big(\left(x - \hat{C}h\right)e^{rh} + \theta xe^{rh}$$
$$\cdot\sum_k\hat{\pi}_k\left(\exp\left(\left(\alpha_k(t) - r - \frac{1}{2}\sigma_{kk}(t)\right)h + \sqrt{h}\,(\sigma\xi)_k\right) - 1\right),$$
$$\hat{L}_h(x, \xi, \psi)\Big)\sum_k\hat{\pi}_k\left(\exp\left(\left(\alpha_k - r - \frac{1}{2}\sigma_{kk}\right)h + \sqrt{h}\,(\sigma\xi)_k\right) - 1\right)$$
$$\cdot\left(\exp\left(\left(\alpha_i - r - \frac{1}{2}\sigma_{ii}\right)h + \sqrt{h}\,(\sigma\xi)_i\right) - 1\right)\frac{\exp\left(-\frac{1}{2}\left(|\xi|^2 + \psi^2\right)\right)}{(2n)^{\frac{n+1}{2}}}$$
$$d\xi_1\cdots d\xi_n d\psi.$$

We use the small h approximation to solve the system (32). It amounts to using (30) with replacing

$$\frac{\partial V}{\partial x}(x,L,t) \quad \text{by} \quad U_2'\left(xe^{-L}\right)e^{-L},$$
$$\frac{\partial^2 V}{\partial x \partial L}(x,L,t) \quad \text{by} \quad -U_2''\left(xe^{-L}\right)\left(e^{-L}\right)^2 - U_2'\left(xe^{-L}\right)e^{-L},$$

and

$$\frac{\partial^2 V}{\partial x^2}(x,L,t) \quad \text{by} \quad U_2''\left(xe^{-L}\right)\left(e^{-L}\right)^2.$$

Therefore, we get from (30),

$$\hat{\pi}(x,L) = -\frac{(\sigma^*)^{-1}\left[(\theta-\zeta\rho)\,U_2'\left(xe^{-L}\right) - \zeta\rho U_2''\left(xe^{-L}\right)e^{-L}\right]}{xU_2''\left(xe^{-L}\right)e^{-L}}. \tag{33}$$

To obtain $\hat{C}(x,L)$, we use (26), which implies the calculation of $\frac{\partial V}{\partial x}(x,L)$. We use (25) to obtain

$$\frac{\partial V}{\partial x}(x,L) = e^{-(\beta-r)h}\int\cdots\int \frac{\partial U_2}{\partial x}\left(\hat{X}_h(x,\xi),\hat{L}_h(L,\xi,\psi)\right) \cdot \frac{\exp\left(-\frac{1}{2}\left(|\xi|^2+\psi^2\right)\right)}{(2n)^{\frac{n+1}{2}}}d\xi_1\cdots d\xi_n d\psi, \tag{34}$$

where

$$\hat{X}_h(x,\xi) = \left(x-\hat{C}h\right)e^{rh} + xe^{rh}\sum_{i=1}^{n}\hat{\pi}_i\left(\exp\left(\left(\alpha_i - r - \frac{1}{2}\sigma_{ii}\right)h + \sqrt{h}\,(\sigma\xi)_i\right) - 1\right)$$
$$\hat{L}_h(x,\xi,\psi) = L + \zeta\sqrt{h}\left(\rho^*\xi + \sqrt{1-|\rho|^2}\psi\right) + \left(I - \frac{\zeta^2}{2}\right)h.$$

Using the small h approximation, we get

$$\frac{\partial V}{\partial x}(x,L) \sim \frac{\partial U_2}{\partial x}(x,L) = U_2'\left(xe^{-L}\right)e^{-L},$$

and thus we obtain $\hat{C}$ by solving

$$U_1'\left(\hat{C}e^{-L}\right) = U_2'\left(xe^{-L}\right). \tag{35}$$

So, if $U_1 = U_2$, we get $\hat{C}(x,L) = x$. Note that the real consumption on the period is $h\hat{C}(x,L)$, so we can consider it as negligible.

4 Partially Observed Inflation Case

We now consider that $L(t)$ is not observable, but we observe the signal

$$\delta Z(t) = L(t)h + m \cdot \delta w_Z(t); \quad Z\,(0) = 0, \tag{36}$$

where $w_Z(t)$ is independent from $w(t)$ and $w_I(t)$. In this case, we extend (7), by

$$\delta L(t) = \left(I - \frac{\zeta^2}{2}\right) h + \zeta \cdot \delta w_I(t); \quad L(0) = N(L_0, S_0). \tag{37}$$

where $L(0)$ is gaussian with mean L_0 and standard deviation S_0.

Let us define

$$\mathcal{G}^t = \sigma\left(\delta w(s), \delta Z(s), s = 0, \ldots, t-h\right).$$

We look for the Kalman filter

$$\hat{L}(t) = E\left[L(t)|\mathcal{G}^t\right]; \quad \hat{L}\,(0) = L_0.$$

Consider the mean $\bar{L}(t)$ evolving as

$$\delta \bar{L}(t) = \left(I - \frac{\zeta^2}{2}\right) h; \quad \bar{L}\,(0) = L_0.$$

On account of linearity, it is sufficient to consider

$$\hat{L}(t) = \bar{L}(t) + \sum_{s=0}^{t-h} K_1(s) \cdot \delta w(s) + \sum_{s=0}^{t-h} K_2(s)\left(\delta Z(s) - \bar{L}(s)h\right), \tag{38}$$

where $K_1(t)$ and $K_2(t)$ are deterministic functions.

Let $\hat{L}^-(t) = E\left[L(t)|\mathcal{G}^{t-h}\right]$. Then from (37), we get

$$\hat{L}^-\,(t+h) = \hat{L}(t) + \left(I - \frac{\zeta^2}{2}\right) h. \tag{39}$$

Now by (38), we have

$$\begin{aligned}\hat{L}^-(t) &= E\left[\hat{L}(t)|G^{t-h}\right] \\ &= \bar{L}(t) + \sum_{s=0}^{t-2h} K_1(s) \cdot \delta w(s) + \sum_{s=0}^{t-2h} K_2(s)\left(\delta Z(s) - \hat{L}(s)h\right) \\ &\quad + K_2\,(t-h)\left(\hat{L}\,(t-h) - \bar{L}\,(t-h)\right) h.\end{aligned}$$

Hence,

$$\hat{L}^{-}(t+h) = \bar{L}(t+h) + \sum_{s=0}^{t-h} K_1(s) \cdot \delta w(s) + \sum_{s=0}^{t-h} K_2(s) \left(\delta Z(s) - \bar{L}(s)h\right) + K_2(t) \left(\hat{L}(t) - \bar{L}(t)\right) h. \tag{40}$$

However, from (38) we get

$$\hat{L}(t+h) = \bar{L}(t+h) + \sum_{s=0}^{t-h} K_1(s) \cdot \delta w(s) + \sum_{s=0}^{t-h} K_2(s) \left(\delta Z(s) - \bar{L}(s)\right) h + K_1(t) \cdot \delta w(t) + K_2(t) \left(\delta Z(t) - \bar{L}(t)h\right),$$

and from (40)

$$\hat{L}(t+h) = \bar{L}^{-}(t+h) + K_1(t) \cdot \delta w(t) + K_2(t) \left(\delta Z(t) - \hat{L}(t)h\right).$$

Using (39), we deduce

$$\hat{L}(t+h) = \hat{L}(t) + \left(I - \frac{\zeta^2}{2}\right) h + K_1 \cdot \delta w(t) + K_2(t) \left(\delta Z(t) - \hat{L}(t)h\right). \tag{41}$$

Calling $\varepsilon(t) = L(t) - \hat{L}(t)$, we get

$$\varepsilon(t+h) = \varepsilon(t) + \zeta \cdot \delta w_I(t) - K_1(t) \cdot \delta w(t) - K_2(t) \left(\delta Z(t) - \hat{L}(t)h\right),$$

$$\varepsilon(t+h) = \varepsilon(t) + \zeta \cdot \delta w_I(t) - K_1(t) \cdot \delta w(t) - K_2(t) \left(\varepsilon(t)h + m \cdot \delta w_Z(t)\right). \tag{42}$$

Set $S(t) = E[\varepsilon(t)^2]$. Then we get

$$\begin{aligned}
E\left[\varepsilon\left(t+h\right)^2\right] &= S(t)\left(1-hK_2(t)\right)^2+m^2\left(K_2(t)\right)^2 h \\
&\quad +\zeta^2 h+|K_1(t)|^2 h-2\zeta K_1(t)\rho h \\
&= S(t)+\zeta^2\left(1-|\rho|^2\right)h+h\left|K_1(t)-\zeta\rho\right|^2 \\
&\quad +\left(h^2S(t)+m^2h\right)\left(K_2(t)-\frac{S(t)}{hS(t)+m^2}\right)^2-\frac{hS^2(t)}{hS(t)+m^2}.
\end{aligned}$$

In order to minimize the error, the best choices are

$$K_1(t)=\zeta\rho,\quad K_2(t)=\frac{S(t)}{hS(t)+m^2}, \tag{43}$$

where $S(t)$ is the solution of

$$S\left(t+h\right)=S(t)+\zeta^2\left(1-|\rho|^2\right)h-\frac{hS^2(t)}{hS(t)+m^2};\quad S\left(0\right)=S_0. \tag{44}$$

The Kalman filter is given by

$$\begin{aligned}
\hat{L}\left(t+h\right) &= \hat{L}(t)+\left(I-\frac{\zeta^2}{2}\right)h+\zeta\rho\cdot\delta w(t) \\
&\quad +\frac{S(t)}{hS(t)+m^2}\left(\delta Z(t)-\hat{L}(t)h\right);\quad \hat{L}\left(0\right)=L_0.
\end{aligned} \tag{45}$$

It is standard to check that the conditional probability of $L(t)$ given $\mathcal{G}^t$ is gaussian with mean $\hat{L}(t)$ and variance $S(t)$ (deterministic).

4.1 *Objective Function for Partially Observed Case*

Consider again the cost function (16). This time the consumption process $C(t)$ and the portfolio $\pi(t)$ are adapted to $\mathcal{G}^t$. Hence, the wealth process $X(t)$ is observable.

Introducing the function

$$\begin{aligned}
\tilde{U}_1\left(C,\hat{L},s\right) &= \frac{1}{\sqrt{2n}}\int U_1\left(C\exp\left(-\left(\hat{L}+y\sqrt{S(s)}\right)\right)\right)e^{-\frac{1}{2}y^2}dy, \\
\tilde{U}_2\left(x,\hat{L},s\right) &= \frac{1}{\sqrt{2n}}\int U_2\left(x\exp\left(-\left(\hat{L}+y\sqrt{S(s)}\right)\right)\right)e^{-\frac{1}{2}y^2}dy,
\end{aligned}$$

the cost function (16) can be written as

$$\tilde{J}_{x,\hat{L},t}\left(\pi\left(\cdot\right),C\left(\cdot\right)\right)=E\left[\sum_{s=t}^{T-h}h\tilde{U}_1\left(C(s),\hat{L}(s),s\right)e^{-\beta(s-t)}\right. \quad (46)$$
$$\left.+\tilde{U}_2\left(X(t),\hat{L}(t),T\right)e^{-\beta(T-t)}|X(t)=x,\hat{L}(t)=\hat{L}\right],$$

with evolution

$$\delta\hat{L}(s)=\left(I-\frac{\zeta^2}{2}\right)h+\zeta\rho\cdot\delta w(s) \quad (47)$$
$$+\frac{S(s)}{hS(s)+m^2}\left(\delta Z(s)-\hat{L}(s)h\right);\quad \hat{L}(t)=t,$$

$$\delta\left(X(s)e^{-rs}\right)=X(s)e^{-rs}\sum_{i=1}^{n}\pi_i(s)\left(e^{\delta\mu_h(s)}-1\right) \quad (48)$$
$$-C(s)he^{-rh};\qquad X(t)=x.$$

The innovation

$$\delta\tilde{w}_Z(t)=\frac{\delta Z(t)-\hat{L}(t)h}{m} \quad (49)$$

is independent from $\mathcal{G}^t$ and is gaussian with mean 0 and variance

$$E\left[\left(\delta\tilde{w}_Z(t)\right)^2\right]=E\left[\left(\frac{\varepsilon(t)h}{m}+\delta w_Z(t)\right)^2\right]$$
$$=\frac{h^2}{m^2}S(t)+h=h\frac{\left(m^2+hS(t)\right)}{m^2}.$$

Hence,

$$\frac{S(t)}{hS(t)+m^2}\left(\delta Z(t)-\hat{L}(t)h\right)=\frac{S(t)m}{hS(t)+m^2}\delta\tilde{w}_Z(t)$$

is gaussian with mean 0 and variance $\frac{hS^2(t)}{m^2+hS(t)}$.

Since

$$\delta\tilde{w}_Z(t)=\frac{\varepsilon(t)h}{m}+\delta w_Z(t),$$

we see that $\delta w(t)$ and $\delta\tilde{w}_Z(t)$ are independent.

So, we can write

$$\delta\hat{L}(s)=\left(I-\frac{\zeta^2}{2}\right)h+\zeta\rho*\cdot\delta w(s)+\delta\tilde{w}_I, \quad (50)$$

where

$$\delta \tilde{w}_I = \frac{S(t)m}{hS(t)+m^2}\delta \tilde{w}_Z(t)$$

is gaussian independent of $\delta w(t)$ and has a variance $\frac{hS^2(t)}{m^2+hS(t)}$.

4.2 Dynamic Programming

We write the analog of (19):

$$\begin{aligned}
\tilde{V}\left(x,\hat{L},t\right) &= \max_{\pi,C}\left[h\tilde{U}_1\left(C,\hat{L},t\right)+e^{-\beta h}\int\cdots\int\tilde{V}\left(xe^{rh}\left(1\right.\right.\right. \\
&-\left.\sum_{i=1}^{n}\pi_i\right)-C\cdot h\cdot e^{rh}+xe^{rh}\sum_{i=1}^{n}\pi_i\cdot\exp\left(\left(\alpha_i(t)-r-\frac{1}{2}\sigma_{ii}(t)\right)h\right. \\
&+\left.\sqrt{h}\sum_{j=1}^{n}\sigma_{ij}\xi_j\right),\hat{L}+\left(I-\frac{\zeta^2}{2}\right)h+\zeta\rho^*\xi\sqrt{h}+\frac{\sqrt{h}S(t)}{\sqrt{m^2+hS(t)}}\psi, \\
&\left.t+h\right)\frac{\exp\left(-\frac{1}{2}\left(|\xi|^2+\psi^2\right)\right)}{(2n)^{\frac{n+1}{2}}}d\xi_1\cdots d\xi_n d\psi\Bigg], \\
\tilde{V}\left(x,\hat{L},T\right) &= \tilde{U}_2\left(x,\hat{L},T\right).
\end{aligned} \tag{51}$$

The optimal feedback $\hat{C}(x,\hat{L},t)$ is the solution of

$$\frac{\partial\tilde{U}_1}{\partial C}\left(c,\hat{L},t\right)=\frac{\partial\tilde{V}}{\partial x}(x,t), \tag{52}$$

and we will have for $\hat{\pi}$ a system analogous to (27).

For small h, we have the result

$$\hat{\pi}\left(x,\hat{L},t\right)=-\frac{(\sigma^*(t))^{-1}\left[\theta\frac{\partial\tilde{V}}{\partial x}\left(x,\hat{L},t\right)+\zeta\rho\frac{\partial^2\tilde{V}}{\partial x\partial\hat{L}}\left(x,\hat{L},t\right)\right]}{x\frac{\partial^2\tilde{V}}{\partial x^2}\left(x,\hat{L},t\right)},$$

which is similar to (30); the difference is that here we have $\hat{L}$ and $\tilde{V}$ instead of L and V in (30).

We can now state the following three-fund theorem in the case of partially observed inflation.

Theorem 2 *Under the partially observed inflation, Theorem 1 holds with a modified proportional allocations of wealth between the funds:*

$$\tilde{\mu}_2(t) = \frac{-\tilde{V}_x\left(X, \hat{L}, t\right)}{X(t)\tilde{V}_{xx}\left(X, \hat{L}, t\right)},$$

$$\tilde{\mu}_3(t) = \frac{-\zeta\tilde{V}_{Lx}\left(X, \hat{L}, t\right)}{X(t)\tilde{V}_{xx}\left(X, \hat{L}, t\right)},$$

and

$$\tilde{\mu}_1(t) = 1 - \tilde{\mu}_2(t) - \tilde{\mu}_3(t).$$

where $\tilde{\mu}_k(t)$ *is the proportional wealth invested in the* k^{th} *fund at time t.*

Theorems 1 and 2 imply that the components of the funds are arrived in the same manner under the fully observed and partially observed inflation; only the relative allocations of the wealth invested in these funds are different. Thus, in both cases the optimal portfolio is a linear combination of the risk-free fund, the growth optimum fund, and the fund that arises from the correlation between the inflation uncertainty and the market risk. The proportions of the wealth invested in these funds are different because the investor's belief on the consumption basket price is not the same under different information sets, i.e., because $\hat{L}$ is not the same as L. Thus, the noisy signals affect the optimal solution through the value function derivatives.

5 Concluding Remarks

We have formulated a discrete-time version of the optimal portfolio and consumption decision model under partially observed inflation, for the first time to our knowledge. The investor observes noisy signals on the consumption basket price over time. Based on these signals, he updates his estimates of the consumption basket and the real asset prices in any given period, and then decides on his investment portfolios and his consumption rate in that period. We show that a modified Mutual Fund Theorem consisting of three funds holds. The funds are a risk-free fund, a growth optimum fund, and a fund that arises from the correlation between the inflation uncertainty and the market risk. In general, the wealth invested in these funds depends on the investor's utility function and on his beliefs about the consumption basket price. However, the funds are robust over different information sets on the consumption basket price. That is, the investor uses the same three funds regardless of the noise in observing the consumption basket price. We show the results obtained are consistent with those obtained in the continuous-time version of the problem. Moreover, since in practice, the decisions are made in discrete time and therefore the data available on potential empirical explorations of the problem require a discrete-time formulation; this paper fills an important gap in the literature.

References

Alchian, A. A., & Klein, B. (1973). On a correct measure of inflation. *Journal of Money, Credit and Banking*, *5*, 173–191.

Basak, S., & Yan, H. (2010). Equilibrium asset prices and investor behaviour in the presence of money illusion. *The Review of Economic Studies*, *77*(3), 914–936.

Bensoussan, A., Keppo, J., & Sethi, S. P. (2009). Optimal consumption and portfolio decisions with partially observed real prices. *Mathematical Finance*, *19*(2), 215–236.

Borenstein, S., & Rose, N. L. (1994). Competition and price dispersion in the U.S. airline industry. *Journal of Political Economy*, *102*, 653–683.

Bradley, R. (2001). Finite sample effects in the estimation of substitution bias in the consumer price index. *Journal of Official Statistics*, *17*, 369–390.

Brennan, M. J. (1998). The role of learning in dynamic portfolio decisions. *European Economic Review*, *1*, 295–306.

Brennan, M. J., & Xia, Y. (2002). Dynamic asset allocation under inflation. *Journal of Finance*, *57*, 1201–1238.

Campbell, J., & Vuolteenaho, T. (2004). Inflation Illusion and stock prices. *American Economic Review Papers Proceedings*, *94*, 19–23.

Chen, S-N., & Moore, W. T. (1985). Uncertain inflation and optimal portfolio selection: A simplified approach. *The Financial Review, 20*, 343–352.

Cohen, R., Polk, C., & Vuolteenaho, T. (2005). Money illusion in the stock market: The Modigliani-Cohn hypothesis. *Quarterly Journal of Economics*, *120*, 639–668.

Karatzas, I., Lehoczky, J. P., Sethi, S. P., & Shreve, S. E. (1986). Explicit solution of a general consumption/investment problem. *Mathematics of Operations Research*, *11*(2), 261–294.

Klenow, P. J. (2003). Measuring consumption growth: The impact of new and better products. *Federal Reserve Bank Minneapolis Quarterly Review*, *27*, 10–23.

Manaster, S. (1979). Real and nominal efficient sets. *Journal of Finance*, *34*, 93–102.

Merton, R. (1971). Optimum consumption and portfolio rules in a continuous time model. *Journal of Economic Theory*, *3*, 373–413.

Munk, C., Sorensen, C., & Nygaard Vinther, T. (2004). Dynamic asset allocation under mean-reverting returns, stochastic interest rates and inflation uncertainty. *International Review of Economics & Finance*, *13*, 141–166.

Sethi, S. P. (1997). *Optimal consumption and investment with bankruptcy*. Norwell, MA: Kluwer Academic Publishers.

Shapiro, M. D., & Wilcox, D. W. (1997). Alternative strategies for aggregating prices in the CPI. *Federal Reserve Bank St. Louis Review*, *79*, 113–125.

Solnik, B. H. (1978). Inflation and optimal portfolio choice. *Journal of Financial and Quantitative Analysis*, *13*, 903–925.

Ungar, M., & Zilberhard, B.-Z. (1993). Inflation and its unpredictability-theory and empirical evidence. *Journal of Money, Credit and Banking*, *25*, 709–720.

Xia, L. (2001). Learning about predictability: The effects of parameter uncertainty on dynamic asset allocation. *Journal of Finance*, *56*, 205–246.

Zhao, Y. (2007). A dynamic model of active portfolio management with benchmark orientation. *Journal of Banking and Finance*, *31*, 3336–3356.