Chapter 6
Nonlocal Gravity

6.1 Motivations

As we have noted several times along this review, the main problem of various gravity models is the development of a consistent quantum description. Indeed, the Einstein gravity is non-renormalizable, and introduction of higher-derivative additive terms implies in arising of ghosts. We have argued in the previous chapter that the Horava–Lifshitz gravity seems to be a good solution since it is power-counting renormalizable, and ghosts ate absent since the action involves only second time derivatives. However, the HL gravity, first, is very complicated, second, breaks the Lorentz symmetry strongly, third, displays a problem of extra degrees of freedom whose solving, as we noted, requires special efforts. At the same time, the concept of nonlocality developed originally within phenomenological context in order to describe finite-size effects (see f.e. [107]), began to attract the interest. Besides of this, the nonlocality enjoys also a stringy motivation since the factors like $e^{\Box}$ emerge naturally within the string context [108]. The key idea of nonlocal field theories looks like follows. Let us consider for example the free scalar field whose Lagrangian is

$$\mathcal{L} = \frac{1}{2}\phi f(\Box/\Lambda^2)\phi, \tag{6.1}$$

where $f(z)$ is a some non-polynomial function (with Λ is the characteristic nonlocality scale) which we choose to satisfy the following requirements.

First, at small arguments this function should behave as $f(z) = a + z$, in order to provide the correct $\Box + m^2$ IR asymptotic behavior. Second, this function must decay rapidly at $|z| \to \infty$ (in principle, we can consider only Euclidean space, so, z is essentially positive), so that integrals like $\int_0^\infty f(z)z^n dz$ are finite for any finite non-negative n, to guarantee finiteness of the theory (in principle in some case this requirement is weakened, if the theory is required to be not finite but only renormalizable). Third, the $f(z)$ is required to be so-called entire function, i.e. it cannot be presented in the form of a product of primitive multipliers like $(z - a_1)(z - a_2)\ldots$,

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so, its propagator has no different poles (as we noted in the Chap. 2, namely presence of such a set of poles implies in existence of ghost modes). The simplest example of such a function is the exponential, $f(z) = e^{-z}$.

Another motivations for nonlocality are the loop quantum gravity dealing with finite-size objects, and the noncommutativity, where the Moyal product is essentially nonlocal by construction. At the same time, it is interesting to note that although the so-called coherent states approach [109] has been motivated by quantum mechanics, by its essence it represents itself as a natural manner to implement nonlocality, so that all propagators carry the factor $e^{-\theta k^2}$, with θ is the noncommutativity parameter. Within the gravity context, use of the nonlocal methodology appears to be especially promising since it is expected that the nonlocality, being implemented in a proper manner, can allow to achieve renormalizability without paying the price of arising the ghosts. The first step in this study has been done in the seminal paper [110].

6.2 Some Results in Non-gravitational Nonlocal Theories

Before embarking to studies of gravity, let us first discuss the most interesting results in non-gravitational nonlocal theories, especially within the context of quantum corrections.

As we already noted, effectively the nonlocal methodology has been applied to perturbative studies for the first time within the coherent states approach [109] which includes Gaussian propagator guaranteeing convergence of quantum corrections. Further, various other studies have been performed. An important role was played by the paper [111] where the effective potential in a nonlocal theory has been calculated for the first time. In that paper, the following theory has been introduced:

$$\mathcal{L} = -\frac{1}{2}\phi(\exp(\Box/\Lambda^2)\Box + m^2)\phi - V(\phi). \tag{6.2}$$

Here, Λ is a characteristic nonlocality scale. For this theory, one can calculate the one-loop effective potential given by the following integral:

$$V^{(1)} = \frac{1}{2}\int \frac{d^4k_E}{(2\pi)^4} \ln\left(\exp\left(-\frac{k_E^2}{\Lambda^2}\right)k_E^2 + m^2 + V''\right). \tag{6.3}$$

It is clear that at $k^2 \ll \Lambda^2$, the theory is reduced to usual one. The exponential factors guarantee finiteness. It is easy to see that there is no ghosts in the theory since there is no different denominators $\Box + m_i^2$ in the propagator of the theory. However, the integral (6.3) can be calculated only approximately for various limits, and it is easy to see that it diverges as $\Lambda \to \infty$ (in [111], a some procedure to isolate this divergence has been adopted). Further, this study has been generalized for the superfield theories representing themselves as various nonlocal extensions of Wess-Zumino model and super-QED, in [112]. It is clear that when, in these theories, one consider the limit of an infinite nonlocality scale $\Lambda \to \infty$, the theory returns to the local limit and becomes to be divergent, i.e. the nonlocality acts as a kind of the higher-derivative regularization, so, the quantum contributions are singular in this limit growing as Λ^2 if the local counterpart of the theory involves quadratic divergences, or as $\ln \Lambda^2$, if it involves the logarithmic ones. From a formal viewpoint, the existence of this singularity can be exemplified by the fact that the typical integral in nonlocal (Euclidean) theory grows quadratically with Λ scale since $\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} e^{-k^2/\Lambda^2} \propto \Lambda^2$. Effectively, the problem of the singularity of the result at $\Lambda \to \infty$ is nothing more that the problem of large quantum corrections arising also in higher-derivative and noncommutative field theories.

At the same time, the problems of unitarity and causality in nonlocal theories require special attention since the nonlocality is commonly associated with an instant propagation of a signal. These problems were discussed in details in various papers. So, it has been claimed in [113] that the problems of unitarity and causality can be solved at least for certain forms of nonlocal functions. Further this result was corroborated and discussed in more details in [114]. However, the complete discussion of unitarity and causality in nonlocal field theories is still to be done. Otherwise, the nonlocal theories must be treated only as effective ones.

So, to begin with studies of gravity, we can formulate some preliminary conclusions: (i) there is a mechanism allowing to avoid UV divergences: (ii) this mechanism is Lorentz covariant and ghost free: (iii) the unitarity and causality still are to be studied.

6.3 Classical Solutions in Nonlocal Gravity Models

So, let us introduce examples of nonlocal gravity models. The paradigmatic form has been proposed in [115], where the Lagrangian $\mathcal{L} = \frac{1}{G}\sqrt{|g|}F(R)$ was studied, with

$$F(R) = R - \frac{R}{6}\left(\frac{e^{-\Box/M^2} - 1}{\Box}\right)R. \tag{6.4}$$

Here the d'Alembertian operator $\Box$ is covariant: $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$. This is the nonlocal extension of R^2-gravity.

First of all, it is easy to show that this theory is ghost-free. Indeed, we can expand

$$F(R) = R + \sum_{n=0}^{\infty} \frac{c_n}{M^{2n+2}} R\Box^n R, \tag{6.5}$$

with $c_n = -\frac{1}{6}\frac{(-1)^{n+1}}{(n+1)!}$. We can rewrite this Lagrangian with auxiliary field Φ and scalar ψ (we can eliminate first Φ, and then ψ, through their equations of motion):

$$\mathcal{L} = \frac{1}{G}\sqrt{|g|}\left(\Phi R + \psi \sum_{n=1}^{\infty} \frac{c_n}{M^{2n+2}} \Box^n \psi - \left[\psi(\Phi - 1) - \frac{c_0}{M^2}\psi^2\right]\right). \tag{6.6}$$

Then we do conformal transformations $g_{mn} \to \Phi g_{mn}$, with $\Phi \simeq 1 + \phi$, to absorb Φ in curvature term. As a result, we arrive at the Lagrangian

$$\mathcal{L} = \frac{1}{G}\sqrt{|g|}\left(R + \psi \sum_{n=0}^{\infty} \frac{c_n}{M^{2n+2}} \Box^n \psi - \psi\phi + \frac{3}{2}\phi\Box\phi\right). \tag{6.7}$$

with the equations of motion are

$$\psi = 3\Box\phi; \quad \phi = 2\sum_{n=0}^{\infty} \frac{c_n}{M^{2n+2}} \Box^n \psi. \tag{6.8}$$

From here we have equation of motion for ϕ:

$$\left(1 - 6\sum_{n=1}^{\infty} c_n \frac{\Box^{n+1}}{M^{2n+2}}\right)\phi = \left[1 + \frac{e^{\Box/M^2} - 1}{\Box/M^2}\right]\phi = 0, \tag{6.9}$$

The l.h.s. is evidently entire, so we have no ghosts.

We conclude that the nonlocality in gravity sector can be transferred to matter sector! This is valid for various models. In a certain sense, this fact is analogous to the observation made in the Sect. 2.3 where it was argued that the $f(R)$ gravity, representing itself as an example of higher-derivative theory, can be mapped to a some scalar-tensor gravity with no higher derivatives in the gravity sector.

The Lagrangian (6.4) can be rewritten as [116]:

$$\mathcal{L} = \sqrt{|g|}\Big(\frac{1}{G}R + \frac{\lambda}{2}RF(\Box)R - \Lambda + \mathcal{L}_M\Big). \tag{6.10}$$

The function $F(\Box)$ is assumed to be analytic, as it is motivated by string theory, and, moreover, in the analytic case the theory does not display problems in IR limit. The Gaussian case, which is especially convenient from the viewpoint of the UV finiteness, is the perfect example. The equations of motion, for $M_P^2 = G^{-1}$, take the form

$$\begin{aligned}
&[\frac{M_P^2}{2} + 2\lambda F(\Box)R]G^\mu_\nu = T^\mu_\nu + \Lambda\delta^\mu_\nu + \lambda K^\mu_\nu - \frac{\lambda}{2}(K^\alpha_\alpha + K_1) - \\
&\quad - \frac{\lambda}{2}RF(\Box)R\delta^\mu_\nu + 2\lambda(g^{\mu\alpha}\nabla_\alpha\nabla_\nu - \delta^\mu_\nu\Box)F(\Box)R, \qquad (6.11)\\
&K^\mu_\nu = g^{\mu\rho}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \partial_\rho\Box^l R\,\partial_\mu\Box^{n-l-1}R;
\end{aligned}$$

$$K_1 = \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \Box^l R \, \Box^{n-l} R; \quad F(\Box) = \sum_{n=0}^{\infty} f_n \Box^n.$$

It is important to note that in two last lines $\Box^l$ acts **only** to the adjacent R.

Now, the natural problem is finding some solutions of these equations. In [116], the following ansatz has been proposed, with r_1, r_2 are some real numbers:

$$\Box R - r_1 R - r_2 = 0 \tag{6.12}$$

which implies (here f_0 is zeroth order in expansion of $F(\Box)$ in series)

$$F(\Box)R = F(r_1)R + \frac{r_2}{r_1}(F(r_1) - f_0). \tag{6.13}$$

This allows to reduce the order of equations to at maximum second. It is clear that constant curvature makes the equation trivial, just this situation occurs for Gödel-type solutions.

One can find nontrivial cosmological solutions for this theory. In particular, bouncing solutions, for $r_1 > 0$, are possible:

$$a(t) = a_0 \cosh\left(\sqrt{\frac{r_1}{2}} t\right). \tag{6.14}$$

Let us give more details for cosmology. Indeed, if we substitute the FRW metric (1.6) to (6.11), and suggest that, as usual in cosmology, $\rho = \rho_0(\frac{a_0}{a})^4$, we have from (6.12), with $r_1 \neq 0$:

$$\frac{d^3 H}{dt^3} + 7H\ddot{H} + 4\dot{H}^2 - 12H^2\dot{H} = -2r_1 H^2 - r_1 \dot{H} - \frac{r_2}{6}, \tag{6.15}$$

whose solution is $H = \sqrt{\frac{r_1}{2}} \tanh(\sqrt{\frac{r_1}{2}} t)$ which just implies hyperbolic dependence of $a(t)$ (6.14). It is well known that namely such a scenario (decreasing of scale factor changing then to increasing) is called bouncing scenario. We also introduce $h_1 = \ddot{H}/M^3$.

The density can be found as well: if we use $G = M_P^{-2}$, and redefine $F(\Box) \to F(\Box/M^2)$, with M is the characteristic nonlocality scale, we find

$$\rho_0 = \frac{3(M_P^2 r_1 - 2\lambda f_0 r_2)(r_2 - 12h_1 M^4)}{12r_1^2 - 4r_2}. \tag{6.16}$$

Let us discuss possible implications of the Eq. (6.15). The cosmological constant turns out to be equal to $\Lambda = -\frac{r_2 M_P^2}{4r_1}$, and there are three scenarios for evolution of the Universe:

1. $\Lambda < 0$, $r_1 > 0$, $r_2 > 0$—cyclic Universe (in particular one can have cyclic inflation).

2. $\Lambda > 0$, $r_1 < 0$, $r_2 > 0$—first contraction, then very rapid inflation (super-inflation) $a(t) \propto \exp(kt^2)$.

3. $\Lambda > 0$, $r_1 > 0$, $r_2 < 0$—constant curvature $R = 4\frac{\Lambda}{M_P^2}$, i.e. de Sitter solution.

So we find that accelerating solutions are possible within all these scenarios. Again, we note that in the constant scalar curvature case, we have drastic reducing of equations.

Moreover, it has been shown in [117] that for $\mathcal{L} = \sqrt{|g|}\sqrt{R - 2\Lambda}F(\Box)\sqrt{R - 2\Lambda}$, with $F(\Box)$ being an arbitrary analytic function, there are hyper-exponentially accelerating cosmological solutions $a(t) \propto e^{kt^2}$.

The next step in study of nonlocal theories consists in introducing non-analytic functions of the d'Alembertian operator. The simplest case is $F(\Box) = \frac{1}{\Box}$. Actually it means that we must consider terms like $R\Box^{-1}R$. It is clear that the gravity extension with such a term is non-renormalizable since the propagator behaves as only $\frac{1}{k^2}$, so we gain nothing in comparison with the usual Einstein-Hlbert gravity [118]. However, theories with negative degrees of the d'Alembertian operator can display new tree-level effects, especially within the cosmological context where an important class of nonlocal gravity models has been introduced in [119]. The action of this class of theories is

$$S = \int d^4x\sqrt{|g|}\Big(\frac{1}{2G}\Big(R + Rf(\Box^{-1}R) - 2\Lambda\Big) + \mathcal{L}_m\Big). \tag{6.17}$$

We note that the presence of the factor $\Box^{-1}$ actually implies in "retarded" solutions behaving similarly to the potential of a moving charge in electrodynamics. Further, this action has been considered in [120], and below, we review the discussion given in that paper.

It is convenient to rewrite the action (6.17) with use of two extra scalar fields ξ and η:

$$S = \int d^4x\sqrt{|g|}\Big[\frac{1}{2G}[R(1 + f(\eta) - \xi) + \xi\Box\eta - 2\Lambda] + \mathcal{L}_m\Big]. \tag{6.18}$$

Varying this action with respect to ξ and expressing $\eta = \Box^{-1}R$, we return to (6.17). This corroborates the already mentioned idea that the modified gravity is in many cases equivalent to a some scalar-tensor gravity.

Then, one varies (6.18) with respect to the metric and η respectively:

$$\begin{aligned}
&\Box\xi + f_\eta(\eta)R = 0;\\
&\frac{1}{2}g_{\mu\nu}[R(1 + f(\eta) - \xi) - \partial_\alpha\xi\partial^\alpha\eta - 2\Lambda] - R_{\mu\nu}(1 + f(\eta) - \xi) +\\
&+ \frac{1}{2}(\partial_\mu\xi\partial_\nu\eta + \partial_\mu\eta\partial_\nu\xi) - (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)(f(\eta) - \xi) = -GT_{\mu\nu}.
\end{aligned} \tag{6.19}$$

We consider the FRW cosmological metric (1.6) with $k = 0$. As usual, the Hubble parameter is $H = \frac{\dot{a}}{a}$. The evolution equation for matter is usual:

$$\dot{\rho} = -3H(\rho + p). \tag{6.20}$$

For scale factor and scalars, we have

$$\begin{aligned}
&2\dot{H}(1 + f(\eta) - \xi) + \dot{\xi}\dot{\eta} + \left(\frac{d^2}{dt^2} - H\frac{d}{dt}\right)(f(\eta) - \xi) + G(\rho + p) = 0;\\
&\ddot{\eta} + 3H\dot{\eta} = -6(\dot{H} + 2H^2);\\
&\ddot{\xi} + 3H\dot{\xi} = -6(\dot{H} + 2H^2) f_\eta(\eta).
\end{aligned} \tag{6.21}$$

We start with the de Sitter space corresponding to $H = H_0 = const$, with the scalar curvature is $R = 12H_0^2$. The equation of state is $p = \omega\rho$, as usual, so, we have the following solutions for the scalar η and the density:

$$\begin{aligned}
\eta(t) &= -4H_0(t - t_0) - \eta_0 e^{-H_0(t-t_0)};\\
\rho(t) &= \rho_0 e^{3(1+\omega)H_0 t}.
\end{aligned} \tag{6.22}$$

Then we introduce the new variable $\Psi = f(\eta) - \xi$, and its equation of evolution is

$$\ddot{\Psi} + 5H_0\dot{\Psi} + 6H_0^2(1 + \Psi) - 2\Lambda + G(\omega - 1)\rho = 0. \tag{6.23}$$

For η we have

$$\dot{\eta}^2 f_{\eta\eta} + (\ddot{\eta} + 3H_0\dot{\eta} - 12H_0^2) f_\eta = \ddot{\Psi} + 3H_0\dot{\Psi}. \tag{6.24}$$

This equation is a necessary condition for existence of the de Sitter solution.

Let us consider the particular case $\eta_0 = 0$ in (6.22). So, (6.24) reduces to

$$16H_0^2 f_{\eta\eta} - 24H_0^2 f_\eta = \ddot{\Psi} + 3H_0\dot{\Psi}. \tag{6.25}$$

So, knowing Ψ, one can find $f(\eta)$. It remains to solve (6.23). Some characteristic cases are:

- $\rho_0 = 0$: $\Psi = C_1 e^{-3H_0 t} + C_2 e^{-2H_0 t} - 1 + \frac{\Lambda}{3H_0^2}$;
- $w = 0$: $\Psi = C_1 e^{-3H_0 t} + C_2 e^{-2H_0 t} - 1 + \frac{\Lambda}{3H_0^2} - \frac{G\rho_0}{H_0} e^{-3H_0 t} t$.
- $w = -1/3$: $\Psi = C_1 e^{-3H_0 t} + C_2 e^{-2H_0 t} - 1 + \frac{\Lambda}{3H_0^2} + \frac{4G\rho_0}{3H_0} e^{-2H_0 t} t$.

As for the function $f(\eta)$, in all cases it will be proportional to $e^{\eta/\beta}$, with $\beta > 0$ (or, at most, linear combination of such functions with various values of β). Effectively we demonstrated arising of the exponential potential widely used in cosmology.

An important particular case is $\eta_0 = 0$. It follows from (6.22) that we have for $\beta \neq 4/3$:

$$\xi = -\frac{3f_0\beta}{3\beta-4}e^{-H_0(t-t_0)/\beta} + \frac{c_0}{3H_0}e^{-3H_0(t-t_0)} - \xi_0;$$
$$\eta = -4H_0(t-t_0); \quad \omega = \frac{4}{3\beta} - 1, \quad \Lambda = 3H_0^2(1+\xi_0);$$
$$\rho_0 = \frac{6(\beta-2)H_0^2 f_0}{\beta G}, \tag{6.26}$$

so we can have exotic matter for $0 < \beta < 2$. And at $\beta = 2$ we have vacuum. If $\beta = 4/3$, we have $\omega = 0$, and $\rho < 0$ (ghost-like dust).

However, we note that the nonlocal modifications of gravity are used mostly in cosmology. One of a few discussions of other metrics within the nonlocal gravity has been presented in [121] where not only cosmological but also (anti) de Sitter-like solutions were discussed for theories involving, besides of.already mentioned term $RF(\Box)R$, also the terms $R_{\mu\nu}F_1(\Box)R^{\mu\nu}$ and $R_{\mu\nu\alpha\beta}F_2(\Box)R^{\mu\nu\alpha\beta}$, with F, F_1, F_2 are some functions of the covariant d'Alembertian operator.

Let us say a few words about other non-analytic nonlocal extensions of gravity. In [122], the additive term $\mu^2 R\Box^{-2}R$ was introduced and shown to be consistent with cosmological observations. However, this theory turns out to be problematic from the causality viewpoint [123]. Also, in [124], the first-order correction in μ^2 to the Schwarzschild solution in a theory with this term has been obtained explicitly.

To close the discussion, it is important to note that the nonlocal gravity can arise as an effective theory as a result of integration over some matter fields. Namely in this manner, the term $R\Box^{-1}R$ contributes to the trace anomaly, at least in two dimensions, in [16]. Therefore, the presence of nonlocal terms can be apparently treated as a consequence of some hidden couplings with matter.

6.4 Conclusions

We discussed various nonlocal extensions of gravity. The key property of nonlocal theories is the possibility to achieve UV finiteness for an appropriate choice for non-local form factor(s). However, apparently explicit quantum calculations in nonlocal gravity models would be extremely complicated from the technical viewpoint, therefore, up to now, all studies of such theories are completely classical ones. Moreover, most papers on nonlocal gravity models are devoted to cosmological aspects of these theories, and the results demonstrated along this chapter allow to conclude that non-local extensions of gravity can be treated as acceptable solutions for the dark energy problem. At the same time, nonlocal theories, including gravitational ones, display certain difficulties. The main problem is that one of unitarity and causality which still requires special attention.

To conclude this chapter, let us emphasize the main directions for studies of nonlocal gravity models. First, clearly, it will be very important to check consistency of different known GR solutions, especially, various black holes (including f.e. nonsingular and rotating ones). Second, various nonlocal form factors, not only Gaussian ones, are to be introduced, and their impact must be tested within the gravity context. Third, study of quantum effects in nonlocal gravity models is of special importance since namely at the perturbative level the main advantages of these theories such as the expected UV finiteness are crucial. It is natural to hope that these studies will be performed in next years.