Chapter 3
Scalar-Tensor Gravities

3.1 General Review

In the previous chapter we demonstrated that modifications of the pure gravitational sector allow for obtaining interesting results, in particular, for a consistent explanation of the cosmic acceleration. At the same time, we noted that $f(R)$ gravities are dynamically equivalent to some gravity models whose action is given by the sum of the usual Einstein term and a new term depending on the extra scalar field [15]. This field, being related with the function of the curvature, evidently cannot be associated with the matter, hence it is natural to suggest that the complete description of gravity is given by composition of the dynamical metric tensor and this scalar field, so we have the scalar-tensor gravity model. Another motivation for a scalar-tensor gravity arises from quintessence models in cosmology which involve a very light scalar field called the quintessence field and are known to explain accelerated expansion of the Universe as well as the cosmological constant which therefore implied active application of the quintessence field within the inflationary context [36]. The advantage of the quintessence in comparison with the cosmological constant consists in the fact that the very tiny mass of the quintessence field (estimated to be about 10^{-33} eV [37]) is much more reasonable from the theoretical viewpoint than the extremal smallness of the cosmological constant giving the famous cosmological constant problem, since even the massless scalar fields are physically consistent.

While the quintessence is well discussed now (see f.e. [37] and references therein), there are other interesting manners to introduce new scalar fields in the gravity, moreover, while the quintessence field is treated as a matter, the scalar fields introduced within these approaches are interpreted as ingredients of the complete description of the gravity rather than the matter. One of these manners is the Brans–Dicke gravity where the gravitational constant whose negative dimension is responsible for a non-renormalizability of the gravity is suggested to be not a fundamental constant but a function of a some slowly varying fundamental scalar field. Another one is the four-dimensional Chern–Simons modified gravity where the pseudoscalar field

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allows to implement the CPT (and in certain cases Lorentz) symmetry breaking in the gravity context. And actually, one more model is intensively discussed in this context, that is the galileons model. Namely these theories will be considered in this chapter.

3.2 Chern–Simons Modified Gravity

3.2.1 The 4D Chern–Simons Modified Gravity Action

The three-dimensional Chern–Simons (CS) term has been originally introduced in the paper [38] within the context of electrodynamics, as an example of a term conciliating gauge invariance with a non-zero mass. It has been immediately generalized to the non-Abelian case, so, the CS Lagrangian looks like

$$\mathcal{L}^A_{CS} = \epsilon^{\mu\nu\lambda}\left(A^a_\mu \partial_\nu A^a_\lambda + \frac{2}{3} f^{abc} A^a_\mu A^b_\nu A^c_\lambda\right), \tag{3.1}$$

where $A_\mu = A^a_\mu T^a$ is the Lie-algebra valued gauge field, and f^{abc} are the structure constants. In the gravity case, the role of the gauge field is played by the connection, and the three-dimensional gravitational CS term reads as [38, 39]:

$$S_{CS} = \frac{1}{2\kappa^2\mu}\int d^3x \epsilon^{\mu\nu\lambda}\left(\Gamma^{\ b}_{\mu a}\partial_\nu \Gamma^{\ a}_{\lambda b} + \frac{2}{3}\Gamma^{\ b}_{\mu a}\Gamma^{\ c}_{\nu b}\Gamma^{\ a}_{\lambda c}\right). \tag{3.2}$$

In principle, in non-Riemannian geometries we can use an independent connection rather than the Levi–Civita one, however, this general situation is outside of the scope of our review. Here, the $\epsilon^{\mu\nu\lambda}$, which can take values $1, 0, -1$, is the usual Levi–Civita symbol, not the covariant one. Varying the CS term with respect to the metric, one finds

$$\delta S_{CS} = -\frac{1}{\kappa^2\mu}\int d^3x C^{\mu\nu}\delta g_{\mu\nu}, \tag{3.3}$$

where

$$C^{\mu\nu} = -\frac{1}{2\sqrt{|g|}}\epsilon^{\mu\alpha\beta}\nabla_\alpha R^\nu_\beta + (\mu \leftrightarrow \nu) \tag{3.4}$$

is the three-dimensional Cotton tensor. It is evidently symmetric and traceless. The μ is a some constant of the mass dimension 1. So, the modified Einstein equations look like

$$G^{\mu\nu}+\frac{1}{\mu}C^{\mu\nu}=0. \tag{3.5}$$

It is useful also to write the linearized form of the gravitational Chern–Simons action obtained from (3.2) under the replacement $g_{\mu\nu}=\eta_{\mu\nu}+\kappa h_{\mu\nu}$:

$$S^{(0)}=-\frac{1}{2\mu}\int d^3x h^{\mu\nu}\epsilon_{\alpha\mu\rho}\partial^\rho(\Box\eta_{\gamma\nu}-\partial_\gamma\partial_\nu)h^{\gamma\alpha}. \tag{3.6}$$

We see that this action is, first, explicitly gauge invariant under usual linearized gauge transformations $\delta h_{\mu\nu}=\partial_\mu\xi_\nu+\partial_\nu\xi_\mu$, second, involves higher derivatives. However, after obtaining the equations of motion for the full linearized action formed by the sum of the terms (1.11) and (3.6), one finds that the physical degrees of freedom satisfy the second-order equation [38], with their propagator behaves as $(\Box+\mu^2)^{-1}$, thus, in the $3D$ CS modified gravity there is no problems with negative-energy states discussed in the previous chapter. The similar situation occurs in the four-dimensional case as well.

The generalization of this theory to the four-dimensional case turns out to be straightforward, however, in this case, similarly to the electrodynamics, this generalization essentially involves the CPT (and in certain cases Lorentz) symmetry breaking. From the formal viewpoint such a generalization for the linearized theory is performed through replacement $\epsilon^{\mu\nu\lambda}\to b_\rho\epsilon^{\rho\mu\nu\lambda}$, with b_ρ is a constant vector, which allows to convert the CS term to the Carroll-Field-Jackiw (CFJ) term which in the Abelian case looks like

$$\mathcal{L}_{CFJ}=\epsilon^{\rho\mu\nu\lambda}b_\rho A_\mu\partial_\nu A_\lambda. \tag{3.7}$$

In principle, such a replacement of the three-dimensional Levi–Civita symbol by the four-dimensional one contracted with a vector already allows to write down the four-dimensional gravitational CS term:

$$\mathcal{L}_{CS,grav}=\int d^4x\epsilon^{\rho\mu\nu\lambda}b_\rho\left(\Gamma^{\ b}_{\mu a}\partial_\nu\Gamma^{\ a}_{\lambda b}+\frac{2}{3}\Gamma^{\ b}_{\mu a}\Gamma^{\ c}_{\nu b}\Gamma^{\ a}_{\lambda c}\right), \tag{3.8}$$

with its linearized form is

$$S^{(0)}=-\frac{1}{2}\int d^4x h^{\mu\nu}\epsilon_{\alpha\mu\rho\lambda}b^\lambda\partial^\rho(\Box\eta_{\gamma\nu}-\partial_\gamma\partial_\nu)h^{\gamma\alpha}. \tag{3.9}$$

We note that this action is invariant under the same linearized gauge transformations $\delta h_{\mu\nu}=\partial_\mu\xi_\nu+\partial_\nu\xi_\mu$. Now, it is very interesting to discuss some motivations for this term.

First of all, already in 1984, much time before the interest to Lorentz-CPT breaking strongly increased, the gravitational anomalies have been discussed in [40], where the topological current K^μ was introduced, with its explicit form is

$$K^{\rho} = 2\epsilon^{\rho\mu\nu\lambda}\left(\Gamma_{\mu a}^{\ \ b}\partial_{\nu}\Gamma_{\lambda b}^{\ \ a} + \frac{2}{3}\Gamma_{\mu a}^{\ \ b}\Gamma_{\nu b}^{\ \ c}\Gamma_{\lambda c}^{\ \ a}\right), \tag{3.10}$$

with its divergence is

$$\partial_{\rho}K^{\rho} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}R_{\mu\nu\gamma\delta}R_{\alpha\beta}^{\ \ \ \gamma\delta} \equiv {}^{*}RR. \tag{3.11}$$

We note that the $3D$ gravitational Chern–Simons term, up to overall multiplier, is equal to the K^3 component, i.e. the component of this current directed along "extra", z axis.

It is clear that the integral from (3.11) over the space-time is a surface term. To include it into the action in a consistent form, one should introduce a new field ϑ called the CS coefficient. As a result, we can add to the usual Einstein–Hilbert action the new term proportional to ϑ which we call the CS action S_{CS}:

$$S_{CS} = \frac{1}{2\kappa^2}\int d^4x\left(-\frac{1}{2}v_{\mu}K^{\mu}\right) = \frac{1}{2\kappa^2}I_{CS};$$
$$S_{EH+CS} = \frac{1}{2\kappa^2}\int d^4x\left(\sqrt{-g}R + \frac{1}{4}\vartheta^{*}RR\right). \tag{3.12}$$

Here, $v_{\mu} = \partial_{\mu}\vartheta$ is a vector. We note that in principle this vector is rather a function of space-time coordinates than the constant, hence, in general the gravitational CS term breaks the CPT symmetry. However, the ϑ can be treated as an external, but not dynamical, field, therefore one can choose v_{μ} to be the constant vector. This immediately implies the Lorentz symmetry breaking, therefore in this case the $4D$ CS modified gravity whose action is given by the second equation in (3.12) turns out to be the first example of the gravity model with the Lorentz symmetry breaking.

The equations of motion for the CS modified gravity can be easily obtained. Varying the CS term I_{CS} defined by the first equation in (3.12), we get

$$\delta I_{CS} = \int d^4x\sqrt{-g}C^{\mu\nu}\delta g_{\mu\nu}, \tag{3.13}$$

with $\varepsilon^{\alpha\beta\gamma\delta} = \frac{\epsilon^{\alpha\beta\gamma\delta}}{\sqrt{|g|}}$ is a Levi–Civita tensor (not a simple symbol!), and

$$C^{\mu\nu} = -\frac{1}{2}[v_{\sigma}(\varepsilon^{\sigma\mu\alpha\beta}\nabla_{\alpha}R^{\nu}_{\beta} + \varepsilon^{\sigma\nu\alpha\beta}\nabla_{\alpha}R^{\mu}_{\beta}) + v_{\sigma\tau}({}^{*}R^{\tau\mu\sigma\nu} + {}^{*}R^{\tau\nu\sigma\mu})], \tag{3.14}$$

is the Cotton tensor, and $v_{\sigma\tau} = \nabla_{\sigma}v_{\tau}$. One can check that the covariant divergence of the Cotton tensor is proportional to the invariant ${}^{*}RR$:

$$\nabla_{\mu}C^{\mu\nu} = \frac{1}{8}v^{\nu}\,{}^{*}RR. \tag{3.15}$$

This divergence plays the crucial role when the modified Einstein equations are considered. Their explicit form is

$$G^{\mu\nu} + C^{\mu\nu} = \kappa^2 T^{\mu\nu}, \tag{3.16}$$

so, due to the Eq. (3.15), we find that the conservation of the energy-momentum tensor requires the vanishing of the divergence of the Cotton tensor, which, according to (3.15), yields an additional consistency condition called the Pontryagin constraint:

$$^*RR = 0, \tag{3.17}$$

which must be checked for any solution. However, since in many cases, including, among others, the rotational symmetry, the curvature tensor has the structure $R_{[ab][ab]}$, i.e. its only non-zero components are R_{0101}, $R_{0202}, \ldots$, this consistency condition will be automatically satisfied in these cases.

The further extension of the Chern–Simons modified gravity (CSMG) was carried out through assuming the nontrivial dynamics for the ϑ CS coefficient. The key idea is as follows [41]: we assume that the action of CSMG includes the kinetic term for ϑ, looking like

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{|g|}\left(R + \frac{1}{2}\nabla^m\vartheta\nabla_m\vartheta - V(\vartheta) - \frac{1}{\alpha}\vartheta^* RR\right), \tag{3.18}$$

with now $^*RR \equiv \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}R_{\mu\nu\gamma\delta}R_{\alpha\beta}{}^{\gamma\delta}$, i.e. it is redefined with the Levi–Civita tensor $\varepsilon^{\mu\nu\lambda\rho} = \frac{\epsilon^{\mu\nu\lambda\rho}}{\sqrt{|g|}}$, and instead of the Pontryagin constraint (3.17), one has the equation of motion for ϑ:

$$^*RR = -\alpha\left(\Box\vartheta + \frac{\partial V}{\partial\vartheta}\right). \tag{3.19}$$

If we have a metric consistent within the non-dynamical CS framework with a specific ϑ, it is consistent in the dynamical case if the r.h.s. of this equation is zero. Then, the ϑ field generates the additional contribution to the energy-momentum tensor $T^{\mu\nu}$, and hence, to the r.h.s. of (3.16).

Now, we present, first, some classical solutions for the CS modified gravity, second, the methodology allowing the gravitational CS term as a quantum correction.

3.2.2 Classical Solutions

So, our task will consist in solving the Eq. (3.16) with the additional condition (3.17). As a first example, we consider a static spherically symmetric metric [39]:

$$ds^2 = N^2(r)dt^2 - A^2(r)dr^2 - r^2 d\Omega^2. \tag{3.20}$$

This is a very broad class of metrics including Schwarzschild, Reissner–Nordström and many other metrics. As we already said, in this case the non-zero components of the curvature tensor are $R_{[ab][ab]}$, so, the consistency condition (3.17) is automatically satisfied. For this metric, one has only non-zero components of the Ricci tensor $R^r_r = \frac{A'}{rA^2}$, $R^\theta_\theta = \frac{1}{r^2}(1-\frac{1}{A}) + \frac{A'}{rA^2}$. Then, we can consider the vacuum case $T^{\mu\nu} = 0$, and choose the vector $v_\mu = \partial_\mu\vartheta$ to be purely timelike, $v_\mu = (\frac{1}{\mu}, \mathbf{0})$, with $\mu = const$, i.e. $\vartheta = \frac{t}{\mu}$. In this case, the components C^{00} and $C^{0i} = C^{i0}$ of the Cotton tensor immediately vanish [39]. A bit more involved calculation (see details in [39]) allows to show that the C^{ij} components also vanish. As a result, we conclude that the spherically symmetric static solutions of the usual Einstein equations solve the modified Eq. (3.16) as well. It is clear that if one suggests the ϑ to be dynamical, the Eq. (3.19) for ϑ will be satisfied if the potential is zero, and $\vartheta = \frac{t}{\mu}$. We note that this choice for ϑ is a particular case of the expression $\vartheta = k_\mu x^\mu$ used within studies of the Lorentz symmetry breaking in CSMG which we will discuss further.

Moreover, it has been shown in [41] that all, even non-static ones, spherically symmetric metrics given by

$$ds^2 = g_{\mu\nu}(x^\lambda)dx^\mu dx^\nu + \Phi^2(x^\rho)d\Omega^2, \tag{3.21}$$

where $d\Omega^2$ is the 2-sphere line element, so that the coordinates on the sphere are x^i, and x^μ are two remaining coordinates (one of them is necessarily timelike), solve the modified Einstein equations (3.16) for

$$\vartheta = F(x^\mu) + \Phi(x^\mu)G(x^i), \tag{3.22}$$

where $G(x^i)$ and $F(x^\gamma)$ are the arbitrary functions of sphere coordinates and remaining coordinates respectively, and Φ is defined in (3.21). The class of spherically symmetric metrics (3.21) involves not only the static ones (3.20) but also many other metrics, including the FRW cosmological metric (the cosmological aspects of CSMG were also discussed in many papers, f.e. in [42]). Some types of metrics with cylindrical symmetry were also shown in [41] to be consistent within the CSMG.

Now, let us discuss the consistency of the Gödel-type metric (2.26) in CSMG. We consider the equations of motion (3.16) in the tetrad base, following [43].

In the non-dynamical case, with appropriate choice of units, the Eq. (3.16) imply

$$R_{AB} + C_{AB} = \kappa\left(T_{AB} - \frac{1}{2}\eta_{AB}T\right) + \Lambda\eta_{AB}; \tag{3.23}$$
$$C^{AB} = -\frac{1}{2}[\varepsilon^{CADE}(\nabla_D R^B_E)\partial_C\vartheta + {}^*R^{EAFB}\nabla_E\nabla_F\vartheta] + (A \leftrightarrow B).$$

The divergence of modified Einstein equations is

$$\nabla_A C^{AB} = \frac{1}{8}{}^*RR\partial^B\vartheta. \tag{3.24}$$

In tetrad base, the components of Ricci tensor for Gödel-type metric are constant, which is an essential advantage of this base. Actually, one has

$$R_{00} = 2\omega^2, \quad R_{11} = R_{22} = 2\omega^2 - m^2, \quad R = 2(\omega^2 - m^2). \tag{3.25}$$

Following the methodology described in [44], we consider three cases of H and D consistent with the conditions of space-time homogeneity of the metric (2.27):

(i) hyperbolic, $H = \frac{2\omega}{m^2}[\cosh mr - 1]$, $D = \frac{1}{m}\sinh mr$;
(ii) trigonometric, $H = \frac{2\omega}{\mu^2}[1 - \cos\mu r]$, $D = \frac{1}{m}\sin\mu r$; $\mu^2 = -m^2$;
(iii) linear, $H = \omega r^2$, $D = r$.

Repeating the argumentation from [44], one immediately sees that for $0 < m^2 < 4\omega^2$, there is a noncausal region with $r > r_c$, where $\sinh^2 \frac{mr_c}{2} = (\frac{4\omega^2}{m^2} - 1)$.
So, at $m^2 \geq 4\omega^2$ there is no problems with causality.

Now let us choose the matter. We have three most important its examples [43, 44]:

(i) Fluid, $T_{AB} = (\rho + p)u_A u_B + p\eta_{AB}$, $u^A = (1, 0, 0, 0)$, $T_{00} = \rho$, $T_{11,22,33} = p$.
(ii) Scalar, $\psi = s(z - z_0)$, $T_{00,33} = \frac{s^2}{2}$, $T_{11,22} = -\frac{s^2}{2}$.
(iii) Electromagnetism, $F_{03} = -F_{30} = e\sin[2\Omega(z - z_0)]$, $F_{12} = -F_{21} = -E\cos(2\Omega(z - z_0))$, $T_{00,11,22} = \frac{e^2}{2}$, $T_{33} = -\frac{e^2}{2}$.

The matter can be presented by composition of these three types. Then, the non-zero components of the Cotton tensor in this base look like

$$\begin{aligned}
C_{00} &= 2\frac{\partial\vartheta}{\partial z}\omega(4\omega^2 - m^2); \quad C_{11} = C_{22} = \frac{1}{2}C_{00};\\
C_{01} &= -\frac{1}{2}\frac{\partial^2\vartheta}{\partial z\partial t}\frac{H}{D}(4\omega^2 - m^2);\\
C_{02} &= -\frac{1}{2}\frac{\partial^2\vartheta}{\partial z\partial r}(4\omega^2 - m^2);\\
C_{03} &= -\frac{1}{2}\frac{\partial\vartheta}{\partial t}\omega(4\omega^2 - m^2);\\
C_{13} &= -\frac{1}{2}\frac{\partial^2\vartheta}{\partial t^2}\frac{H}{D}(4\omega^2 - m^2);\\
C_{23} &= \frac{1}{2}\frac{\partial^2\vartheta}{\partial r\partial t}(4\omega^2 - m^2).
\end{aligned} \tag{3.26}$$

It is clear that the Cotton tensor is traceless, $C^A_A = 0$. To cancel the off-diagonal components of C_{AB} we choose $\vartheta(z) = b(z - z_0)$ which matches the suggestion done above that the vector $v_M = \partial_M\vartheta$ is constant, which will be further used to study the Lorentz symmetry breaking. We introduce also $k = b\omega$, and require $4\omega^2 \neq m^2$.

The system of the modified Einstein equations (for 00, 11=22, 33 components respectively) looks like:

$$\begin{aligned} 2\,\omega^2 + 2\,b\omega(4\,\omega^2 - m^2) &= \frac{1}{2}\,e^2 + \frac{1}{2}\,\rho - \Lambda + \frac{3}{2}\,p, \\ 2\,\omega^2 - m^2 + b\omega(4\,\omega^2 - m^2) &= \frac{1}{2}\,e^2 - \frac{1}{2}\,p + \Lambda + \frac{1}{2}\,\rho, \\ 0 &= -\frac{1}{2}\,e^2 - \frac{1}{2}\,p + s^2 + \Lambda + \frac{1}{2}\,\rho. \end{aligned} \tag{3.27}$$

We note, that, just as in the Einstein case [44], this system is a purely algebraic one. Let us solve these equations. After some manipulations we arrive at equations for m^2 and ω^2, with $k = b\omega$ (we note that at $b = 0$, the usual GR solution is replayed since in this case, $\vartheta = 0$!):

$$(2 + 8\,k)\omega^2 - 2\,km^2 = \rho + s^2 + p, \tag{3.28}$$

$$(2 + 4\,k)\,\omega^2 - (1 + k)\,m^2 = -s^2 + e^2. \tag{3.29}$$

One of the interesting new results having no GR analogue is the vacuum noncausal solution $m^2 = \omega^2, b = -\frac{1}{3\omega}, \Lambda = 0$. Some other interesting conclusions of the above system are that, unlike the general relativity, the hyperbolic causal solutions are possible in CS modified gravity, and that trigonometric and linear solutions can arise only for a non-zero electromagnetic field [43].

If one suggests that the CS coefficient is dynamical, more new solutions having analogues neither in GR nor for the case of the non-dynamical CS coefficient are possible, see details in [43], with again the Einstein equations will be reduced to the algebraic equations involving some extra additive terms in comparison with (3.27). In particular, one can have a vacuum solution, where only cosmological constant is non-zero while density, pressure and all fields are zero.

At the same time, it is necessary to emphasize that not any solution consistent in the GR will be consistent also in CS modified gravity. The paradigmatic example is the Kerr metric which fails to solve new equations of motion [39, 45]. It has been shown then in [46] that, to satisfy the modified Einstein equations in the dynamical CS modified gravity, the Kerr metric should be also modified, by adding the ϑ-dependent terms, with the equations of motion are afterwards solved order by order in ϑ. Clearly, studies of consistency of various metrics possessing no rotational symmetry within the CS modified gravity represent an open problem.

To close the discussion of the classical solutions, it is necessary to discuss the propagation of the plane waves. Similarly to the Sect. 2.2, we introduce the transverse-traceless components h_{ij}^{TT} which are the only physical variables in the theory (so, there are only two independent components, that is, if the plane wave propagates f.e. along x_3, we have only $h_{11} = -h_{22} = T$ and $h_{12} = h_{21} = S$).

In this case, for the time-like vector $v_\mu = (\mu^{-1}, 0, 0, 0)$ the quadratic Lagrangian takes the form:

$$L_2 = -\frac{1}{4}h_{ij}^{TT}\Box h_{ij}^{TT} + \frac{1}{4\mu}\epsilon^{ijk}h_{il}^{TT}\Box\partial_k h_j^l + \cdots\,, \tag{3.30}$$

where dots are for physically irrelevant (non-propagating) degrees of freedom.

The corresponding linear equation of motion is

$$-\frac{1}{2}\Box h^{ij}_{TT}+\frac{1}{2\mu}\epsilon^{ilk}\Box\partial_k h^j_{l,TT}=0. \tag{3.31}$$

As a result, one immediately concludes that the dispersion relation is the usual one, $k_0^2=\mathbf{k}^2$, and both polarizations propagate with the speed of light.

The natural question is—what is difference of these polarizations? A more careful analysis [39] shows that, for plane waves proportional to $e^{i\omega t-ikz}$, one finds that there are two basic (circular) polarizations $T=iS$ and $T=-iS$, with their intensities proportional to $(1+\frac{k}{\mu})^{-2}$ and $(1-\frac{k}{\mu})^{-2}$ respectively. This difference of intensities can be treated as a consequence of parity breaking.

It should be noted that if we consider, instead of the CS term, the one-derivative additive term $h_{\mu\nu}\epsilon^{\lambda\alpha\mu\rho}\theta_\lambda\partial_\rho h^\nu_\alpha$, with θ^λ being a space-like vector, we will have two polarizations with physically consistent dispersion relations $E=\pm\theta+\sqrt{p^2+\theta^2}$, so, in this case the velocities differ from speed of light [47]. However, this term is not gauge invariant, which, within the gravity context, means that it breaks the general covariance.

3.2.3 Perturbative Generation

The special interest is attracted to the gravitational CS term within the context of study of the Lorentz symmetry breaking. The main reason consists in the fact that, besides of the CPT symmetry breaking, for a special choice of the CS coefficient $\vartheta=b_\mu x^\mu$, where b_μ is a constant vector (as we already noted in the previous subsection, this choice is consistent with the Gödel-type solutions), the CS term displays Lorentz symmetry breaking, taking the form (3.8), or, for the weak field, the linearized form (3.9). Therefore the natural idea consists in a generation of this term as a perturbative correction, similarly to the generation of the CFJ term in the extended QED, see f.e. [48]. This similarity is supported by a natural analogy between the gravitational anomalies [40] and the Adler–Bell–Jackiw (ABJ) anomaly [49]. Moreover, it follows from [50] that this anomaly is deeply related with the ambiguity of results, therefore, it is natural to expect the ambiguity of the gravitational CS term as well.

So, one can start with the action of spinors coupled to gravity, where the Lorentz-breaking vector b_μ is introduced:

$$S=\int d^4x e\bar{\psi}(i\partial\!\!\!/-m-b\!\!\!/\gamma_5+\omega\!\!\!/)\psi, \tag{3.32}$$

here, $b\!\!\!/=b^\mu e^a_\mu\gamma_a$, and $\omega_\mu=\frac{1}{4}\omega_{\mu bc}\sigma^{bc}$ is a (Riemannian) connection. We note that the CS term dominates in the limit $m\to 0$ while the one-derivative term discussed in [47] vanishes in this limit. The corresponding one-loop effective action is given by the following trace of the logarithm:

$$\Gamma^{(1)} = i\mathrm{Tr}\ln(i\partial\!\!\!/ - m - b\!\!\!/\gamma_5 + \not{\omega}). \tag{3.33}$$

Just the same approach was used in [48] for the Lorentz-breaking extension of QED. In the weak gravity case, we can use the approximation $e_{\mu a} \simeq \eta_{\mu a} + \frac{1}{2}h_{\mu a}$. This trace of the logarithm, however, can be calculated both in the weak field case and in the full-fledged gravity case, with use of the Feynman diagrams or of the proper-time method.

It is interesting that, similarly to the CFJ term, the $4D$ gravitational CS term is ambiguous, i.e. the results for it depend on the calculation scheme. So, within all these approaches, the linearized gravitational CS term

$$S_{CS} = C\int d^4x h_{\mu\nu}\epsilon^{\mu\rho\kappa\lambda}b_\kappa\partial_\lambda\left(\Box h_\rho{}^\nu - \partial^\nu\partial^\sigma h_{\rho\sigma}\right), \tag{3.34}$$

or its full-fledged analogue (3.8) multiplied by $2C$, was shown to arise, with the constant C depends on the method of computation. So, in [51], where the calculations were carried out in the weak gravity case with use of the Feynman diagrams constructed for the action (3.32), it was found that $C = \frac{1}{192\pi^2}$. Further, in [52], this scheme has been realized for the finite temperature case where the zero component of the internal momentum is supposed to be discrete, $k_0 = (2n+1)\pi T$, so that the result is

$$\begin{aligned} S_{CS} = \int d^4x\, h_{\mu\nu}\Bigg[&\frac{1}{192\pi^2}\epsilon^{\rho\mu\kappa\lambda}b_\kappa\partial_\lambda\left(\Box h_\rho{}^\nu - \partial^\nu\partial^\sigma h_{\rho\sigma}\right) \\ &+ \frac{T^2}{12}b_0\epsilon^{\rho\mu\kappa\lambda}u_\kappa\partial_\lambda\left(\frac{\partial_0\partial^\nu}{\Box} - u^\nu\right)\left(\frac{\partial_0\partial^\sigma}{\Box} - u^\sigma\right)h_{\rho\sigma}\Bigg], \end{aligned} \tag{3.35}$$

i.e. it looks like a sum of the zero-temperature result (3.34) and the additive term proportional to T^2.

In [53], where the proper time method has been used for the full-fledged gravity, the result was found in the form (3.8), with $C = \frac{1}{128\pi^2}$. Finally, in [54] it has been argued that due to the arbitrariness in defining of conserved currents within the functional integral approach, the constant C is actually completely ambiguous. The similar situation occurs in QED [55]. However, the ambiguity of results is known to be highly controversial, and in gravity it is even more controversial than in electrodynamics. For example, in [56] it was claimed that, if one suggests that the b_μ is the vacuum expectation value (v.e.v.) of a some dynamical field, the correct result for the $4D$ gravitational CS term is zero, as is also required by the gauge invariance of the Lagrangian (and not only the action). Nevertheless, the question whether the requirements of [56] are indeed so necessary is still open, as the presence of ambiguities in generic Lorentz-breaking theories is a strongly polemical problem.

However, there are also other interesting scalar-tensor gravity models which we will consider now.

3.3 Brans–Dicke Gravity

The Brans–Dicke (BD) gravity is one of the most known and studied scalar-tensor gravity models. Originally, it has been introduced in [57], basing on the idea that the physical space itself possesses geometrical features beyond those ones generated by matter (this is one of the forms of the so-called Mach principle), so, the action of the BD gravity was proposed in the form

$$S=\int d^4x\sqrt{|g|}\Big(\phi R+\frac{\omega}{\phi}\partial_a\phi\partial^a\phi+16\pi\mathcal{L}_{mat}\Big). \tag{3.36}$$

In this theory, the new scalar field ϕ (which does not contribute to the matter Lagrangian) plays the role of the effective gravitational constant; indeed, if one chooses $\phi=\frac{1}{2\kappa^2}$, the theory reduces to the Einstein gravity with the usual matter. One advantage of the theory consists in the fact that the coupling constant ω is dimensionless, hence the negative-dimension constants jeopardizing the renormalizability of the gravity are ruled out. Also, in this case the gravitational constant has a dynamic origin being related with an asymptotic value of the ϕ.

For this theory, one can derive equations of motion:

$$\begin{aligned}
&-\frac{2\omega}{\phi}\Box\phi+\frac{\omega}{\phi^2}\partial_\mu\phi\partial^\mu\phi+R=0; \qquad (3.37)\\
&R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=\left(\frac{8\pi}{\phi}\right)T_{\mu\nu}-\frac{\omega}{\phi^2}\left(\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}g_{\mu\nu}\partial_\rho\phi\partial^\rho\phi\right)\\
&\quad+\frac{1}{\phi}\left[\nabla_\nu(\partial_\mu\phi)-g_{\mu\nu}\Box\phi\right],
\end{aligned}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the usual matter (not including ϕ). Contracting this equation with $g^{\mu\nu}$, we find

$$R=-\left(\frac{8\pi}{\phi}\right)T-\frac{\omega}{\phi^2}\partial_\rho\phi\partial^\rho\phi+\frac{3}{\phi}\Box\phi, \tag{3.38}$$

which we can combine with the Eq. (3.37), obtaining

$$\Box\phi=\left(\frac{8\pi}{3-2\omega}\right)T. \tag{3.39}$$

Equations (3.38), (3.39) are analogues of the Einstein equations and can be solved.

As a first example, we consider the static spherically symmetric metric (3.20) which we now rewrite as

$$ds^2=e^{2\alpha(r)}dt^2-e^{2\beta(r)}(dr^2+r^2d\Omega^2) \tag{3.40}$$

In the vacuum case, $T_{\mu\nu} = 0$, this metric will be a consistent solution of equations of motion [57]. Explicitly, one finds

$$
\begin{aligned}
e^{\alpha(r)} &= e^{\alpha_0}\left[\frac{1-\frac{2B}{r}}{1+\frac{2B}{r}}\right]^{1/\lambda}; \\
e^{\beta(r)} &= e^{\beta_0}(1+\frac{2B}{r})^2\left[\frac{1-\frac{2B}{r}}{1+\frac{2B}{r}}\right]^{(\lambda-C-1)/\lambda}; \\
\phi(r) &= \phi_0 e^{\alpha_0 C}\left[\frac{1-\frac{2B}{r}}{1+\frac{2B}{r}}\right]^{C/\lambda}.
\end{aligned} \tag{3.41}
$$

The cosmological solutions also were found in [57] where they were shown, in the vacuum case, to look like

$$
\begin{aligned}
&\phi = \phi_0 t^r, \quad a = a_0 t^q; \\
&r = \frac{2}{4-3\omega}, \quad q = \frac{2-2\omega}{4-3\omega},
\end{aligned} \tag{3.42}
$$

so, accelerating solutions ($q > 1$) are possible for $\omega > 2$. Further, various papers, continuing this study, discussed cosmic acceleration in BD gravity in details, see f.e. [58].

Now, let us discuss the Gödel-type solutions (2.26) in the BD gravity. It has been shown in [59] that the nontrivial solution, i.e. that one with a non-constant scalar ϕ (otherwise the BD gravity reduces trivially to the Einstein gravity) is possible only if the action (3.36) includes the cosmological constant as well, so, one has

$$
S = \int d^4x\sqrt{|g|}\Big(\phi(R-2\Lambda) + \frac{\omega}{\phi}\partial_a\phi\partial^a\phi + 16\pi\mathcal{L}_{mat}\Big). \tag{3.43}
$$

The modified Einstein equations, in the tetrad base, look like

$$
\begin{aligned}
G^A_B - \delta^A_B\Lambda = \left(\frac{8\pi}{\phi}\right)T^A_B - \frac{\omega}{\phi^2}\left(\partial^A\phi\partial_B\phi - \frac{1}{2}\delta^A_B\partial_C\phi\partial^C\phi\right) + \\
+ \phi^{-1}\left(\nabla_B\partial^A\phi - \delta^A_B\Box\phi\right),
\end{aligned} \tag{3.44}
$$

and choosing again the matter in the form of a composition of the fluid and electromagnetic field (see Sect. 3.2.2), with the angular velocity parametrizing the Gödel-type metric (2.26) and defined within the conditions (2.27) is now denoted as Ω instead of ω, we find that the case $\phi = \phi(z)$ yields

$$
4\Omega^2 - m^2 = \left(\frac{8\pi}{\phi}\right)(\rho + E_0^2), \quad m^2 + 2\Lambda = -\frac{\phi''}{\phi}. \tag{3.45}
$$

The typical cases are:

(i) $4\Omega^2 - m^2 = 0$ (causal solution!), $\rho + E_0^2 = 0$. In this case ϕ is a trigonometric function.

(ii) $\rho = const$, $\phi = const$—trivial case reducing to GR.

For $\phi = \phi(t)$, one arrives at $\phi = const$, and this case is also trivial. In principle, more involved situations can be studied as well. As for the black hole solutions in BD gravity, we strongly recommend the classical paper [60]. In principle, many other solutions for the BD gravity have been studied, including global monopoles, wormholes etc., but the limited volume of these notes does not allow for their detailed discussion.

3.4 Galileons

One of the most important examples of the scalar-tensor gravity models is the galileons theory proposed originally in [61]. Its key idea is as follows: let us consider the most general scalar-tensor action involves no more than second derivatives of the metric tensor and no more than the first ones of the scalar field. Effectively, it was a suggestion of the Lovelock-like construction not only in the gravitational sector but also in the scalar one. So, we suggest the action to look like

$$S = \int d^4x\sqrt{-g}\mathcal{L}(g_{\mu\nu}, \partial_\lambda g_{\mu\nu}, \partial_\lambda\partial_\rho g_{\mu\nu}; \phi, \partial_\mu\phi). \tag{3.46}$$

As a result, the equations of motion involve various tensors constructed on the base of the Riemann curvature and its covariant derivatives, and various derivatives of the scalar field. In principle we can have the gravity equations of motion with Lovelock-like l.h.s. and non-canonical scalar-dependent r.h.s., and strongly nonlinear equations of motion for the scalar. We note that there is no ghost problem here since there is no higher derivatives. In principle, even on the flat background, one can have a theory of a scalar field with highly nonlinear equation of motion, the so-called K-theory (see [62] and references therein).

However, the model (3.46) was forgotten for a long time and revitalized only in 2008, in the paper [63] where the concept of galileons was formulated. Its key idea consists in invariance of the theory with respect to the combination of dilatations and conformal transformations so that the new scalar π varies as $\pi \to \pi + c + b_\mu x^\mu$, where c and b_μ are constants. These transformations look similarly to the Galilean ones, therefore the π was called the galileon. So, again, the key idea is that we have derivative couplings but no higher derivatives in the kinetic term.

There are five terms with the symmetry above. Let us introduce notations $\Pi^{\mu\nu} = \partial^\mu\partial^\nu\pi$, $[A] = A^\mu_\mu$ for trace (so, $\frac{1}{2}[\Pi]\partial\pi \cdot \partial\pi = \frac{1}{2}\Box\pi\partial^\mu\pi\partial_\mu\pi$), $[\Pi] = \Box\pi$, etc.), and use a dot for the usual scalar product like $A \cdot B \equiv A_\mu B^\mu$. So, we can write our five terms as:

$$\begin{aligned}
\mathcal{L}_1 &= \pi, \\
\mathcal{L}_2 &= -\frac{1}{2}\partial\pi\cdot\partial\pi; \\
\mathcal{L}_3 &= -\frac{1}{2}[\Pi]\partial\pi\cdot\partial\pi; \\
\mathcal{L}_4 &= -\frac{1}{4}\Big([\Pi]^2\partial\pi\cdot\partial\pi - 2[\Pi]\partial\pi\cdot\Pi\cdot\partial\pi - [\Pi^2]\partial\pi\cdot\partial\pi + 2\partial\pi\cdot\Pi^2\cdot\partial\pi\Big); \\
\mathcal{L}_5 &= -\frac{1}{5}\Big([\Pi]^3\partial\pi\cdot\partial\pi - 3[\Pi]^2\partial\pi\cdot\Pi\cdot\partial\pi - 3[\Pi][\Pi^2]\partial\pi\cdot\partial\pi + \\
&\quad + 6[\Pi]\partial\pi\cdot\Pi^2\cdot\partial\pi + 2[\Pi]^3\partial\pi\cdot\partial\pi + 3[\Pi^2]\partial\pi\cdot\Pi\cdot\partial\pi - 6\partial\pi\cdot\Pi^3\cdot\partial\pi\Big). \quad (3.47)
\end{aligned}$$

The complete Lagrangian of π is a linear combination of these terms: $\mathcal{L} = c_1\mathcal{L}_1 + c_2\mathcal{L}_2 + c_3\mathcal{L}_3 + c_4\mathcal{L}_4 + c_5\mathcal{L}_5$. Clearly, the next step consists in coupling of these Lagrangians to gravity. But let us first describe some perturbative effects of these couplings.

One of the interesting effects is that these galileon terms $\mathcal{L}_i$ are not renormalized under quantum corrections! The reasons are as follows [64]. First, the galileon is massless, so, its propagator is $1/k^2$. Then, all galileon couplings c_3, c_4, c_5 have negative mass dimensions, therefore the contributions to these terms possess quadratic and even higher divergences. After integration of subloops, the leading divergence is proportional to $\int d^4k(k^2)^n$, with $n \geq -1$, and this integral vanishes within dimensional regularization. Finally, the subleading contributions to galileon vertices vanish as well (this proof is more sophisticated being based on analysis of symmetries). In principle, such conclusions are natural for a massless theory with derivative couplings. Other divergent contributions in the galileons theory in the flat space, which do not match the form of the classical action, in particular, involve more derivatives (f.e. $\Box^2$ terms), are discussed in [65].

Clearly, the next step is the coupling of the scalar π to the gravity. One of the first ideas consists in coupling of galileons to the curvature, so we have terms like [66, 67]:

$$\delta S_4 = \int d^4x\sqrt{-g}(\pi_\mu\pi^\mu)(\pi_\nu G^{\nu\rho}\pi_\rho), \quad (3.48)$$

where $\pi_\mu \equiv \nabla_\mu\pi$, etc., or the higher terms like $\pi_\mu\pi^{\mu\nu}\pi^\rho G_{\nu\rho}$, or the simplest terms $\pi^\mu\pi^\nu G_{\mu\nu}$ (the last term is the example of the John term, see below). So, effectively we have a gravity-coupled scalar field with strongly nonlinear dynamics involving derivative depending couplings. As it has been claimed in [67], these terms are of special interest within the cosmological context, where it has been explicitly shown that the solutions with constant $H = \frac{\dot{a}}{a}$ are consistent for the presence of galileons, therefore de Sitter-like exponential expansion is possible in this case, with neither potential term for the scalar nor cosmological constant are employed, therefore the galileons theory is a sound candidate for the role of the dark energy. In [68], it was argued that only the minimal scalar-gravity couplings must be considered, as a result, there were introduced four typical galileon-gravity coupling terms called John, Paul, George and Ringo:

$$\begin{aligned}
\mathcal{L}_{John} &= V_J(\pi)G_{\mu\nu}\nabla^{\mu}\pi\nabla^{\nu}\pi;\\
\mathcal{L}_{Paul} &= V_P(\pi)P_{\mu\nu\rho\sigma}\nabla^{\mu}\pi\nabla^{\nu}\pi\nabla^{\rho}\nabla^{\sigma}\pi;\\
\mathcal{L}_{George} &= V_G(\pi)R;\\
\mathcal{L}_{Ringo} &= V_R(\pi)\mathcal{G}.
\end{aligned} \tag{3.49}$$

where $P^{\mu\nu\alpha\beta} = -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\delta}R_{\rho\sigma\gamma\delta}$ is the double dual of the Riemann curvature. In [68], the cosmological aspects of the theory involving these terms were studied, especially, it was argued how the known cosmological self-tuning problem is solved in this theory. Various issues related to the cosmic acceleration in this context are studied numerically also in [69]. Many other papers are also devoted to galileon cosmology. However, up to now the galileons are mostly considered namely within the cosmological context, there are only a few papers on other solutions such as f.e. black holes (see f.e. [70]). An interesting review of galileons is presented in [71]. To close this section, we note that many aspects of galileons still must be studied.

3.5 Conclusions

We formulated several examples of scalar-tensor gravity models whose form does not match the standard quintessence-gravity Lagrangian $\mathcal{L} = \sqrt{|g|}(\frac{1}{16\pi G}R - \frac{1}{2}g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi - V(\phi))$ which is well studied, both within the cosmological and QFT contexts. Explicitly, we considered the $4D$ CS modified gravity, the Brans–Dicke gravity and the galileons theory. These theories display new interesting features.

First of all, the CSMG allows for the CPT symmetry breaking, and, for a certain form of the CS coefficient, also for the Lorentz symmetry breaking, opening thus a way for intensive studies of Lorentz-breaking modifications of gravity. Some of these studies will be discussed in the next chapter. Besides, in the presence of the gravitational CS term new solutions impossible within the usual GR arise.

Second, the Brans–Dicke gravity represents itself as a theory allowing to rule out the gravitational constant possessing negative mass dimension and hence implying in problems with quantum description of the gravity. Moreover, it turns to be that some new solutions which are not consistent within the GR, are also possible.

Third, the galileons theory turns out to be a sound candidate for a description of the dark energy allowing for accelerated solutions. Besides of this, the galileons contributions to the action arise within applying the Stuckelberg approach for the massive gravity. Essentially, at the first step one introduces the new vector field to construct the gauge invariant extension for the mass term of the gravity, and at the second step, to achieve the gauge symmetry for this vector field, one introduces the scalar field whose action matches the galileon form [72].

To conclude, for the scalar-tensor gravity models, one has essentially new results. One of the most interesting conclusions is the possibility to introduce the Lorentz symmetry breaking within the gravitational context, for a special form of the CS coefficient. However, it is clear that in this context, an extension of gravity through introduction of vector fields seems to be more promising since the vacuum expectations of vector fields can yield constant vectors necessary to introduce privileged space-time directions breaking thus the Lorentz symmetry.