



On the Hierarchy of Equivalence Classes Provided by Local Congruences

Roberto G. Aragón, Jesús Medina, and Eloísa Ramírez-Poussa^(✉)

Department of Mathematics, University of Cádiz, Cádiz, Spain
{roberto.aragon,jesus.medina,eloisa.ramirez}@uca.es

Abstract. In this work, we consider a special kind of equivalence relations, which are called local congruences. Specifically, local congruences are equivalence relations defined on lattices, whose equivalence classes are convex sublattices of the original lattices. In the present paper, we introduce an initial study about how the set of equivalence classes provided by a local congruence can be ordered.

Keywords: Congruence · Local congruence · Concept lattice · Ordering relation

1 Introduction

The notion of local congruence arose in an attempt to weaken the conditions imposed in the definition of a congruence relation on a lattice, with the goal of taking advantage of different properties of these relations with respect to attribute reduction in formal concept analysis [11, 17, 21].

Formal concept analysis (FCA) is a theory of data analysis that organizes the information collected in a considered dataset, by means of the algebraic structure of a complete lattice. Moreover, this theory also offers diverse mechanisms for obtaining, handling and relating (by attribute implications) information from datasets. One of the most interesting mechanisms is attribute reduction. Its main goal is the selection of the main attributes of the given dataset and detecting the unnecessary ones to preserve the structure of the complete lattice.

In [4, 5], the authors remarked that when a reduction of the set of attributes in the dataset is carried out, an equivalence relation is induced. This induced equivalence relation satisfies that the generated equivalence classes have the structure of a join-semilattice. Inspired by this fact, the original idea given in [1] was to complement these studies by proposing the use of equivalence relations

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containing the induced equivalence relation and satisfying that the generated equivalence classes be convex sublattices of the original lattice.

For example, congruence relations [6, 10, 12, 13] hold the previously exposed requirements. In addition, congruence relations have already been applied to the framework of FCA [11, 15, 18–20]. Nevertheless, in [2] was proved that congruence relations are not suitable to complement the reductions in FCA, since the constraints imposed by this kind of equivalence relation entail a great loss of information. This reason is the main justification to weaken the notion of congruence relation, appearing the definition of local congruence. These new equivalence relations are also defined on lattices and only require that the equivalence classes be convex sublattices of the original lattice. The use of local congruences considerably reduces the problem of the loss of information.

However, the appearance of local congruences uncovers new open problems that require answers. One of these open problems is to provide an ordering relation on the set of equivalence classes, that is, on the quotient set associated with the local congruence. This is the main issue addressed in this paper. First of all, we will show that the usually considered ordering relations on the set of equivalence class of a congruence relation, cannot be used for local congruences. Then, we will define a new binary relation on lattices which turns out to be a pre-order when it is used to establish a hierarchy on the equivalence classes provided by a local congruence. Finally, we will also state under what conditions this pre-order is a partial order.

The paper is organized as follows: Sect. 2 recalls some preliminary notions used throughout of the paper. Section 3 presents the study of the hierarchy among the equivalence classes provided by local congruences. The paper finishes with some conclusions and prospects for future works, which are included in Sect. 4.

2 Preliminaries

In this section, we recall basic notions used in this paper. The first notion is related to a special kind of equivalence relation on lattices, which are called congruence relations.

Definition 1 ([10]). *Given a lattice (L, \preceq) , we say that an equivalence relation θ on L is a congruence if, for all $a_0, a_1, b_0, b_1 \in L$,*

$$(a_0, b_0) \in \theta, (a_1, b_1) \in \theta \text{ imply that } (a_0 \vee a_1, b_0 \vee b_1) \in \theta, (a_0 \wedge a_1, b_0 \wedge b_1) \in \theta.$$

where \wedge and \vee are the infimum and the supremum operators.

Now, we recall the notion of quotient lattice from a congruence, based on the operations of the original lattice.

Definition 2 ([10]). *Given an equivalence relation θ on a lattice (L, \preceq) , the operators infimum and supremum, \vee_θ and \wedge_θ , can be defined on the set of equivalence classes $L/\theta = \{[a]_\theta \mid a \in L\}$ for all $a, b \in L$, as follows:*

$$[a]_\theta \vee_\theta [b]_\theta = [a \vee b]_\theta \text{ and } [a]_\theta \wedge_\theta [b]_\theta = [a \wedge b]_\theta.$$

\vee_θ and \wedge_θ are well defined on L/θ if and only if θ is a congruence.

When θ is a congruence on L , the tuple $(L/\theta, \vee_\theta, \wedge_\theta)$ is called quotient lattice of L modulo θ .

Now, let us suppose that $\{a, b, c, d\}$ is a subset of a given lattice (L, \preceq) . Then, the pairs a, b and c, d are said to be *opposite sides* of the quadrilateral $(a, b; c, d)$ if $a < b$, $c < d$ and either:

$$(a \vee d = b \text{ and } a \wedge d = c) \quad \text{or} \quad (b \vee c = d \text{ and } b \wedge c = a).$$

In addition, we say that the equivalence classes provided by an equivalence relation are *quadrilateral-closed* if whenever given two opposite sides of a quadrilateral $(a, b; c, d)$, such that $a, b \in [x]_\theta$, with $x \in L$ then there exists $y \in L$ such that $c, d \in [y]_\theta$. This notion leads us to the following result which is a characterization of the congruence notion in terms of their equivalence classes and plays a key role in the definition of local congruences as we will show later (more detailed information on the characterization and the notions involved in this result can be found in [10]).

Theorem 1 ([10]). *Let (L, \preceq) be a lattice and θ an equivalence relation on L . Then, θ is a congruence if and only if*

- (i) *each equivalence class of θ is a sublattice of L ,*
- (ii) *each equivalence class of θ is convex,*
- (iii) *the equivalence classes of θ are quadrilateral-closed.*

With the goal of obtaining a less-constraining equivalence relations than congruences, but preserving some interesting properties satisfied by this kind of equivalence relations, the notion of local congruence arose [2] in the framework of attribute reduction in FCA [7–9, 11, 16], focused on providing an optimal reduction on FCA from the application of Rough Set techniques [4, 5, 14]. This notion is recalled in the following definition and mainly consist in the elimination of a restriction (last item) in the previous theorem.

Definition 3. *Given a lattice (L, \preceq) , we say that an equivalence relation δ on L is a local congruence if the following properties hold:*

- (i) *each equivalence class of δ is a sublattice of L ,*
- (ii) *each equivalence class of δ is convex.*

Next section studies how we can define an ordering relation between the equivalence classes obtained from a local congruence.

3 Ordering Classes of Local Congruences

In this section, we are interested in studying ordering relations for local congruences. This fact is fundamental for establishing a proper hierarchy among the classes of concepts obtained after the reduction in FCA [3–5].

The set of equivalence classes of a congruence on a lattice L can be ordered by a partial order \preceq_θ which is defined, for all $a, b \in L$, by means of the operators \vee_θ and \wedge_θ presented in Definition 2, as follows:

$$[a]_\theta \preceq_\theta [b]_\theta \text{ if } [a]_\theta = [a]_\theta \wedge_\theta [b]_\theta \text{ or } [b]_\theta = [a]_\theta \vee_\theta [b]_\theta \quad (1)$$

This ordering relation cannot be used for local congruences since local congruences are not compatible with either supremum or infimum, that is, the operators \vee_θ and \wedge_θ could not be well defined when the considered relation is a local congruence due to they do not satisfy the quadrilateral-closed property unlike congruences. In the next example, we illustrate this fact.

Example 1. Let us consider the lattice (L, \preceq) shown in the left side of Fig. 1, and the local congruence δ , highlighted by means of a Venn diagram, given in the right side of Fig. 1.

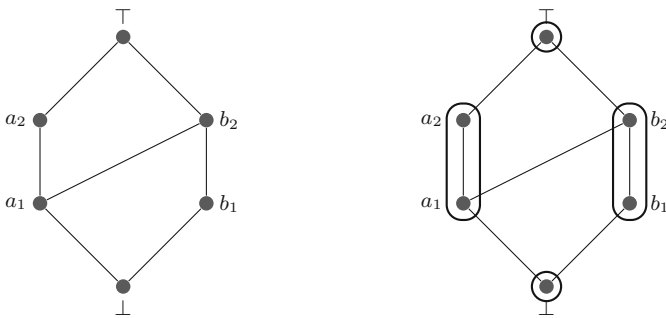


Fig. 1. Lattice (left) and local congruence (right) of Example 1.

It is easy to see that the considered local congruence δ provides four different equivalence classes which are listed below:

$$\begin{aligned} [\top]_\delta &= \{\top\} \\ [a_1]_\delta &= [a_2]_\delta = \{a_1, a_2\} \\ [b_1]_\delta &= [b_2]_\delta = \{b_1, b_2\} \\ [\perp]_\delta &= \{\perp\} \end{aligned}$$

We can observe that a_1, \perp and b_1, b_2 are opposite sides, but a_1 and \perp are not in the same equivalence class, which means that the equivalence classes of δ are not quadrilateral-closed. As a consequence, the infimum and supremum operators described in Expression (1) are not well defined. For example, we have that

$$\begin{aligned} [a_2]_\delta \wedge_\delta [b_1]_\delta &= [a_2 \wedge b_1]_\delta = [\perp]_\delta \\ [a_2]_\delta \wedge_\delta [b_1]_\delta &= [a_2 \wedge b_2]_\delta = [a_1]_\delta \end{aligned}$$

and clearly $[\perp]_\delta \neq [a_1]_\delta$. Therefore, the ordering \preceq_δ cannot be defined on local congruences. \square

A property of the ordering relation, shown in Expression (1), was shown in [10], which provides another possibility of defining an ordering on the set of local congruences.

Proposition 1 ([10]). *Let θ be a congruence on a lattice (L, \preceq) and let $[a]_\theta$ and $[b]_\theta$ be equivalence classes of L/θ . Then, the binary relation \leq defined on L/θ as: $[a]_\theta \leq [b]_\theta$, if there exist $a' \in [a]_\theta$ and $b' \in [b]_\theta$, for all $a' \preceq b'$, is an ordering relation.*

Clearly, the relation \leq is the associated ordering relation with the algebraic lattice $(L/\theta, \vee_\theta, \wedge_\theta)$. Consequently, we cannot use either this alternative definition in the equivalence classes of a local congruence. In the following example, we show a case where the application of this ordering relation for a local congruence does not satisfies the transitivity property.

Example 2. We will consider the lattice (L, \preceq) and the local congruence δ both given in Fig. 2. As we can observe, the local congruence provides five different equivalence classes:

$$\begin{aligned} [\top]_\delta &= \{\top\} \\ [a_1]_\delta &= [a_2]_\delta = \{a_1, a_2\} \\ [b_1]_\delta &= [b_2]_\delta = \{b_1, b_2\} \\ [c_1]_\delta &= [c_2]_\delta = \{c_1, c_2\} \\ [\perp]_\delta &= \{\perp\} \end{aligned}$$

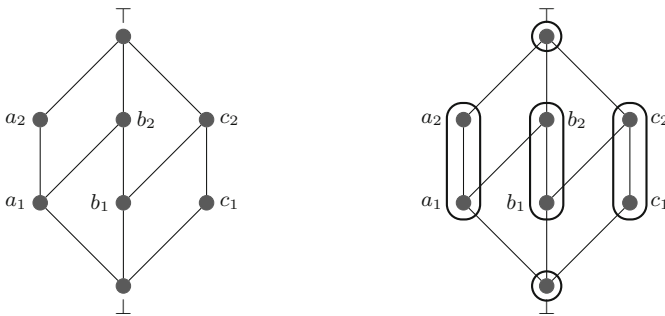


Fig. 2. Lattice (left) and local congruence (right) of Example 2.

If we try to order the equivalence classes of δ using the ordering relation described in Proposition 1, we obtain that $[a_1]_\delta \leq [b_1]_\delta$, since $a_1 \preceq b_2$, and

$[b_1]_\delta \leq [c_1]_\delta$ because $b_1 \preceq c_2$. Nevertheless, we can see that $[a_1]_\delta$ is not lesser than $[c_1]_\delta$ because neither a_1 nor a_2 are lesser than c_1 or c_2 in L . Therefore, the ordering relation defined in Proposition 1 is not transitive for local congruences in general and thus, it is not a partial order for local congruences. \square

As we have seen in the previous example, the ordering relation defined in Proposition 1 cannot be used either to order the equivalence classes obtained from local congruences. However, the underlying idea of the ordering relation of Proposition 1 can be considered to define a more suitable ordering relation for being applied on local congruences. In order to achieve this goal, we formalize some notions presented in [6], which are related to the ordering of elements in the quotient set provided from equivalence relations defined on posets. The following notion is related to the way in which two elements of the original lattice can be connected via the local congruence.

Definition 4. *Let (L, \preceq) be a lattice and a local congruence δ on L .*

- (i) *A sequence of elements of L , (p_0, p_1, \dots, p_n) with $n \geq 1$, is called a δ -sequence, denoted as $(p_0, p_n)_\delta$, if for each $i \in \{1, \dots, n\}$ either $(p_{i-1}, p_i) \in \delta$ or $p_{i-1} \preceq p_i$ holds.*
- (ii) *If a δ -sequence $(p_0, p_n)_\delta$ satisfies that $p_0 = p_n$, then it is called a δ -cycle. In addition, if the δ -cycle satisfies that $[p_0]_\delta = [p_1]_\delta = \dots = [p_n]_\delta$, then we say that the δ -cycle is closed.*

With the notions of Definition 4, we present a new binary relation on local congruences in the following definition.

Definition 5. *Given a lattice (L, \preceq) and a local congruence δ on L , we define a binary relation \preceq_δ on L/δ as follows:*

$$[x]_\delta \preceq_\delta [y]_\delta \quad \text{if there exists a } \delta\text{-sequence } (x', y')_\delta$$

for some $x' \in [x]_\delta$ and $y' \in [y]_\delta$.

Now, we go back to Example 2 in order to illustrate this relation.

Example 3. Returning to Example 2, we want to establish a hierarchy among the equivalence classes depicted in Fig. 2 by means of the relation given in Definition 5. By considering this definition, it is clear that $[a_1]_\delta \preceq_\delta [b_1]_\delta$ and $[b_1]_\delta \preceq_\delta [c_1]_\delta$. In addition, we can observe that, in this case, we also have that $[a_1]_\delta \preceq_\delta [c_1]_\delta$ since there exists a δ -sequence that connects one element of the class $[a_1]_\delta$ with another element of the class $[c_1]_\delta$, this δ -sequence is shown below:

$$(a_1, c_2)_\delta = (a_1, b_2, b_1, c_2), \quad \text{since } a_1 \preceq b_2, \quad (b_2, b_1) \in \delta \quad \text{and} \quad b_1 \preceq c_2$$

Therefore, the relationship among the elements in the quotient set L/δ given by \preceq_δ are shown in Fig. 3. \square

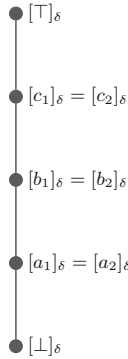


Fig. 3. Hasse diagram of the relation among the elements in L/δ of Example 3.

Observe that the binary relation \preceq_δ given in Definition 5 is a pre-order. Evidently, by definition, \preceq_δ is reflexive and transitive. Now, we present an example in which the previously defined relation \preceq_δ does not hold the antisymmetry property and, consequently, it cannot be used to establish an ordering among the equivalent classes obtained from a local congruence.

Example 4. Let us consider the lattice (L, \preceq) and the local congruence δ given in Fig. 4. The equivalence classes provided by δ are:

$$\begin{aligned}
 [\top]_\delta &= \{\top\} \\
 [a_1]_\delta &= [a_2]_\delta = [a_3]_\delta = [a_4]_\delta = \{a_1, a_2, a_3, a_4\} \\
 [b_1]_\delta &= [b_2]_\delta = \{b_1, b_2\} \\
 [c_1]_\delta &= [c_2]_\delta = \{c_1, c_2\} \\
 [\perp]_\delta &= \{\perp\}
 \end{aligned}$$

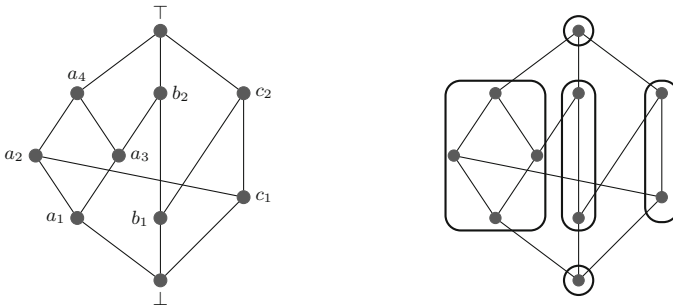


Fig. 4. Lattice and local congruence of Example 4.

If we try to establish a hierarchy among the equivalence classes using the binary relation given in Definition 5, we obtain that $[c_1]_\delta \preceq_\delta [a_1]_\delta$ since there exists another δ -sequence that connects c_1 with a_2 :

$$(c_1, a_2)_\delta = (c_1, a_2) \quad \text{since} \quad c_1 \preceq a_2$$

In addition, we also have that $[a_1]_\delta \preceq_\delta [c_1]_\delta$, because there exists a δ -sequence that connects the elements a_3 and c_2 :

$$(a_3, c_2)_\delta = (a_3, b_2, b_1, c_2) \quad \text{since} \quad a_3 \preceq b_2, (b_2, b_1) \in \delta \quad \text{and} \quad b_1 \preceq c_2$$

Therefore, we have that $[a_1]_\delta \preceq_\delta [c_1]_\delta$ and $[c_1]_\delta \preceq_\delta [a_1]_\delta$, but these classes are not equal. Thus, the antisymmetry property does not hold and, as a consequence, the obtained equivalent classes from the considered local congruence cannot be ordered by means of the considered binary relation. \square

As we have seen in the previous example, the preorder \preceq_δ is not a partial order since the antisymmetry property is not satisfied for any local congruence, in general. Therefore, it is important to study sufficient conditions to ensure that $(L/\delta, \preceq_\delta)$ is a poset. The following result states a condition under which the binary relation of Definition 5 is a partial order on local congruences.

Theorem 2. *Given a lattice (L, \preceq) and a local congruence δ on L , the preorder \preceq_δ given in Definition 5 is a partial order if and only if either no δ -cycle exists or every δ -cycle of elements in L is closed.*

Since no δ -cycle of elements in L exists with respect to the local congruences in Examples 1 and 2, we can ensure that the obtained quotient sets, together with the binary relation \preceq_δ , are posets in both examples. The following example shows a local congruence on the lattice L given in Example 4, such as $(L/\delta, \preceq_\delta)$ is a poset.

Example 5. On the lattice (L, \preceq) of Example 4, the quotient set L/δ_1 given by local congruence δ_1 depicted in the right side of Fig. 5, together with the binary relation defined in Definition 5, is a poset.

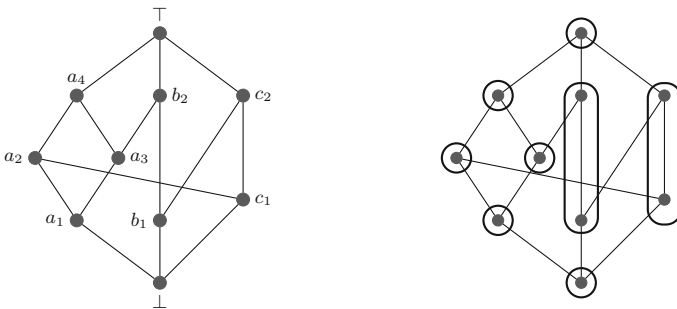


Fig. 5. Local congruence δ_1 (right) of Example 5 on the lattice of Example 4 (left).

We can ensure that because no δ_1 -cycle exists. The right side of Fig. 6 shows an equivalence relation that contains the δ -cycle of Example 4 in one equivalence class. Therefore, the least local congruence, called δ_2 , is the one that groups all the elements in a single class and, as a consequence, the δ -cycle is closed. Therefore, by Theorem 2, the pair $(L/\delta_2, \preceq_{\delta_2})$ it is also a poset. \square

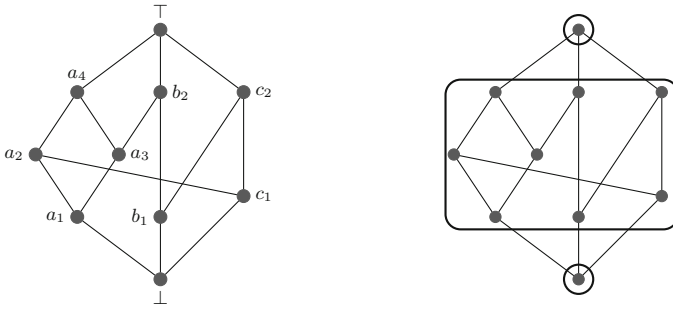


Fig. 6. Equivalence relation (right) of Example 5 on the lattice of Example 4 (left).

4 Conclusions and Future Work

In this paper, we have introduced an initial study about different ways of establish a hierarchy among the equivalence classes provided by local congruences. We have analyzed the results of applying the usually considered ordering relations on the quotient set of congruences, obtaining that these ordering relations are not suitable to be used on local congruences. Based on the underlying philosophy of one characterization of the ordering relation used for congruences, we have defined a new binary relation on the equivalence classes obtained from a local congruences. We have also proven that this binary relation is a preorder. Moreover, we have stated a sufficient condition on the lattice in which the local congruence is defined, in order to guarantee that this preorder is actually a partial order. All the ideas presented throughout this study have been illustrated by means of diverse examples.

As future work, we are interested in continuing this study and defining another binary relation, which will be a partial order on the equivalence classes of any local congruence. Furthermore, we will apply this type of equivalence relations in practical problems, such as in tasks related to the reduction of concept lattices in the framework of formal concept analysis.

References

1. Aragón, R.G., Medina, J., Ramírez-Poussa, E.: Weaken the congruence notion to reduce concept lattices. *Studies in Computational Intelligence*, pp. 1–7 (2019, in press)

2. Aragón, R.G., Medina, J., Ramírez-Poussa, E.: Weaken the congruence notion to reduce concept lattices. In: European Symposium on Computational Intelligence and Mathematics (ESCIM 2019), pp. 45–46 (2019)
3. Benítez-Caballero, M.J., Medina-Moreno, J., Ramírez-Poussa, E.: Bireducts in formal concept analysis. In: Kóczy, L.T., Medina-Moreno, J., Ramírez-Poussa, E., Šostak, A. (eds.) *Computational Intelligence and Mathematics for Tackling Complex Problems*. SCI, vol. 819, pp. 191–198. Springer, Cham (2020). https://doi.org/10.1007/978-3-030-16024-1_24
4. Benítez-Caballero, M.J., Medina, J., Ramírez-Poussa, E., Ślęzak, D.: A computational procedure for variable selection preserving different initial conditions. *Int. J. Comput. Math.* **97**, 387–404 (2020)
5. Benítez-Caballero, M.J., Medina, J., Ramírez-Poussa, E., Ślęzak, D.: Rough-set-driven approach for attribute reduction in fuzzy formal concept analysis. *Fuzzy Sets Syst.* (2020, in press)
6. Blyth, T.S.: *Lattices and Ordered Algebraic Structures*. Springer, London (2005). <https://doi.org/10.1007/b139095>
7. Cornejo, M.E., Medina, J., Ramírez-Poussa, E.: Attribute reduction in multi-adjoint concept lattices. *Inf. Sci.* **294**, 41–56 (2015)
8. Cornejo, M.E., Medina, J., Ramírez-Poussa, E.: On the use of irreducible elements for reducing multi-adjoint concept lattices. *Knowl.-Based Syst.* **89**, 192–202 (2015)
9. Cornejo, M.E., Medina, J., Ramírez-Poussa, E.: Characterizing reducts in multi-adjoint concept lattices. *Inf. Sci.* **422**, 364–376 (2018)
10. Davey, B., Priestley, H.: *Introduction to Lattices and Order*, 2nd edn. Cambridge University Press, Cambridge (2002)
11. Ganter, B., Wille, R.: *Formal Concept Analysis: Mathematical Foundation*. Springer, Heidelberg (1999). <https://doi.org/10.1007/978-3-642-59830-2>
12. Grätzer, G.: *General Lattice Theory*, 2nd edn. Birkhäuser, Basel (2007)
13. Grätzer, G.: *Universal Algebra*, 2nd edn. Springer, New York (2008)
14. Jiang, Z., Liu, K., Yang, X., Yu, H., Fujita, H., Qian, Y.: Accelerator for supervised neighborhood based attribute reduction. *Int. J. Approx. Reason.* **119**, 122–150 (2020)
15. Li, J.-Y., Wang, X., Wu, W.-Z., Xu, Y.-H.: Attribute reduction in inconsistent formal decision contexts based on congruence relations. *Int. J. Mach. Learn. Cybernet.* **8**(1), 81–94 (2016). <https://doi.org/10.1007/s13042-016-0586-z>
16. Medina, J.: Relating attribute reduction in formal, object-oriented and property-oriented concept lattices. *Comput. Math. Appl.* **64**(6), 1992–2002 (2012)
17. Medina, J., Ojeda-Aciego, M., Ruiz-Calviño, J.: Formal concept analysis via multi-adjoint concept lattices. *Fuzzy Sets Syst.* **160**(2), 130–144 (2009)
18. Viaud, J., Bertet, K., Demko, C., Missaoui, R.: The reverse doubling construction. In: 2015 7th International Joint Conference on Knowledge Discovery, Knowledge Engineering and Knowledge Management (IC3K), vol. 1, pp. 350–357, November 2015
19. Viaud, J.-F., Bertet, K., Demko, C., Missaoui, R.: Subdirect decomposition of contexts into subdirectly irreducible factors. In: International Conference on Formal Concept Analysis ICFCA2015, Nerja, Spain, June 2015
20. Viaud, J.-F., Bertet, K., Missaoui, R., Demko, C.: Using congruence relations to extract knowledge from concept lattices. *Discrete Appl. Math.* **249**, 135–150 (2018)
21. Wille, R.: Restructuring lattice theory: an approach based on hierarchies of concepts. In: Rival, I. (ed.) *Ordered Sets*, pp. 445–470. Reidel, Dordrecht (1982). https://doi.org/10.1007/978-94-009-7798-3_15