Veneroni Maps

Marcin Dumnicki, Łucja Farnik, Brian Harbourne, Tomasz Szemberg, and Halszka Tutaj-Gasińska

Abstract Veneroni maps are a class of birational transformations of projective spaces. This class contains the classical Cremona transformation of the plane, the cubo-cubic transformation of the space and the quarto-quartic transformation of $\mathbb{P}^4$. Their common feature is that they are determined by linear systems of forms of degree n vanishing along $n+1$ general flats of codimension 2 in $\mathbb{P}^n$. They have appeared recently in a work devoted to the so called unexpected hypersurfaces. The purpose of this work is to refresh the collective memory of the mathematical community about these somewhat forgotten transformations and to provide an elementary description of their basic properties given from a modern point of view.

Keywords Cremona transformation · Birational transformation

2010 Mathematics Subject Classification 14E07

M. Dumnicki · H. Tutaj-Gasińska
Faculty of Mathematics and Computer Science, Jagiellonian University, Łojasiewicza 6, 30-348 Kraków, Poland
e-mail: Marcin.Dumnicki@im.uj.edu.pl

H. Tutaj-Gasińska
e-mail: Halszka.Tutaj@im.uj.edu.pl

Ł. Farnik · T. Szemberg (✉)
Pedagogical University of Cracow, Podchorążych 2, 30-084 Cracow, Poland
e-mail: tomasz.szemberg@gmail.com

Ł. Farnik
e-mail: lucja.farnik@gmail.com

B. Harbourne
Department of Mathematics, University of Nebraska, Lincoln, NE 68588-0130, USA
e-mail: bharbourne1@unl.edu

D. I. Stamate and T. Szemberg (eds.), *Combinatorial Structures in Algebra and Geometry*, Springer Proceedings in Mathematics & Statistics 331,
https://doi.org/10.1007/978-3-030-52111-0_3

1 Introduction

The aim of this note is to give a detailed description of Veneroni's Cremona transformations in $\mathbb{P}^n$. They were first described by Veneroni in [13], and then discussed for $n = 4$ by Todd in [12] and by Blanch in [2] and for $n \geq 3$ by Snyder and Rusk in [11] (with a focus on $n = 5$) and by Blanch again in [1]. The base loci of the Veneroni transformations involve certain varieties swept by lines that were considered for $n = 4$ by Segre in [10] and for $n \geq 3$ by Eisland in [6]. Evolution in terminology and rigor can make it a challenge to study classical papers. Our purpose here is to bring this work together in one place, in a form accessible to a modern audience. In order to use Bertini's Theorem, we assume the ground field $\mathbb{K}$ has characteristic 0.

Consider $n + 1$ distinct linear subspaces $\Pi_0, \ldots, \Pi_n \subset \mathbb{P}^n$ of codimension 2. Let $\mathcal{L}_n$ be the linear system of hypersurfaces in $\mathbb{P}^n$ of degree n containing $\Pi_0 \cup \cdots \cup \Pi_n$ and let $N + 1$ be the vector space dimension of $\mathcal{L}_n$ (we will see that $N = n$ when the Π_j are general, hence by semi-continuity we have $N \geq n$). We denote by $v_n : \mathbb{P}^n \dashrightarrow \mathbb{P}^N$ the rational map given by $\mathcal{L}_n$. If $N = n$ and if in addition v_n is birational, we refer to v_n as a Veneroni transformation. (Here we raise an interesting question: is v_n birational to its image if and only if $N = n$? When $n = 2$, it is not hard to check that $N = n$ always holds and that v_n is always birational.)

When the Π_j are general, we will see that v_n is a Veneroni transformation whose inverse is also given by a linear system of forms of degree n vanishing on $n + 1$ codimension 2 linear subspaces of $\mathbb{P}^n$. In this situation, v_2 is the standard quadratic Cremona transformation of $\mathbb{P}^2$, v_3 is a cubo-cubic Cremona transformation of $\mathbb{P}^3$ (see [4, Example 3.4.3]) and v_4 is a quarto-quartic Cremona transformation of $\mathbb{P}^4$ (see [12]). In [8] the quarto-quartic Cremona transformation was used to produce some unexpected hypersurfaces.

The paper is organized as follows: we start in Sect. 2 with characterizing degree $n - 1$ hypersurfaces in $\mathbb{P}^n$, containing n general linear subspaces of codimension 2.

In Sect. 3 we investigate the linear system giving the Veneroni transformation. When the spaces Π_i are general, we prove that the dimension of $\mathcal{L}_n$ is $n + 1$, we describe the base locus of this system, and prove that v_n is birational.

In Sect. 4 we give the inverse u_n of v_n explicitly and show that u_n is given by a possibly linear subsystem of the linear system of forms of degree n vanishing on a certain set of $n + 1$ codimension 2 linear subspaces.

The last section, Sect. 5, is devoted to the additional description of the intersection of two hypersurfaces of the type described in Sect. 2.

2 Codimension 2 Linear Subspaces

Given linear subspaces $\Lambda_1, \ldots, \Lambda_s$ of $\mathbb{P}^n$, a line intersecting them all is called a *transversal* (for $\Lambda_1, \ldots, \Lambda_s$).

Proposition 2.1 *Let $\Pi_1, \dots, \Pi_{n-1}$ be general codimension 2 linear subspaces of $\mathbb{P}^n$. For every point $p \in \mathbb{P}^n$, there is a transversal for $\Pi_1, \dots, \Pi_{n-1}$ through p. If p is general, then there is a unique transversal, which we denote t_p, and it meets $\Pi_1 \cup \dots \cup \Pi_{n-1}$ in $n-1$ distinct points. If however there are at least two transversals through p, then p lies on a subspace T_p (of dimension $d_p > 1$) intersecting each Π_j along a subspace of dimension $d_p - 1$, $j = 1, \dots, n-1$, and T_p is the union of all transversals for $\Pi_1, \dots, \Pi_{n-1}$ through p.*

Proof Let H be a general hyperplane in $\mathbb{P}^n$ and consider the projection $\pi_p : \mathbb{P}^n \dashrightarrow H$ from $p \in \mathbb{P}^n$. If $p \notin \Pi_j$, let $\Pi'_j = \pi_p(\Pi_j)$ and define

$$\Pi' = \bigcap_{\substack{1 \le j < n \\ p \notin \Pi_j}} \Pi'_j.$$

The intersection Π' is not empty, since each Π'_j is a hyperplane in H and Π' is the intersection of at most $n-1$ hyperplanes in H. Let $q \in \Pi'$. Then the line L_{pq} is transversal to all Π_i (because either $q \in \Pi'_i$, and hence L_{pq} intersects Π_i, or $p \in \Pi_i$). Conversely, a transversal from p intersects Π'. Observe that for a general p, the points $\pi^{-1}(q)|_{\Pi_j}$ are different, so the transversal meets Π_j in different points.

Consequently, for a general p there is a unique transversal. If $\dim \Pi' = k > 0$, then we have a subspace T_p of the transversals of dimension $k+1$. This subspace is a cone over Π' and over $\Pi_j \cap T_p$ as well, hence $\dim \Pi_j \cap T_p = k$. □

Example 2.2 For 3 general codimension 2 linear subspaces Π_1, Π_2, Π_3 of $\mathbb{P}^4$, the pairwise intersections $\Pi_{ij} = \Pi_i \cap \Pi_j$, $i \neq j$, are points. These three points span a plane T which intersects each Π_i in a line. (For Π_1 this line is the line L_{23} through Π_{12} and Π_{13}, and similarly for Π_2 and Π_3.) The lines L_{12}, L_{13}, L_{23} all lie in T, hence every point $p \in T$ has a pencil of transversals, namely the lines in T through p.

Remark 2.3 In the preceding example, not every transversal is in T; this follows from Proposition 2.1. What is more, even if a point p has a pencil of transversals, it need not be true that $p \in T$. Take, for example, a general point $p \in \Pi_1$. The cone on Π_2 with vertex p intersects Π_3 in a line L. Every line through p in the plane spanned by p and L is a transversal, so the general point $p \in \Pi_1$ has a pencil of transversals.

Remark 2.4 We will eventually be interested in $n+1$ general codimension 2 subspaces $\Pi_0, \dots, \Pi_n$ of $\mathbb{P}^n$. They are defined by $2(n+1)$ general linear forms f_{j1}, f_{j2}, $j = 0, \dots, n$, where $I_{\Pi_j} = (f_{j1}, f_{j_2})$. After a change of coordinates we may assume that $f_{j1} = x_j$ and that $f_{j2} = a_{j0}x_0 + \dots + a_{jn}x_n$ with $a_{ji} = 0$ if and only if $i = j$. Here the homogeneous coordinate ring R of $\mathbb{P}^n$ is the polynomial ring $R = \mathbb{K}[\mathbb{P}^n] = \mathbb{K}[x_0, \dots, x_n]$.

Now, we establish existence and uniqueness of a hypersurface Q of degree $n-1$ containing n general codimension 2 linear subspaces in $\mathbb{P}^n$ for $n \geq 2$.

Proposition 2.5 *Let $\Pi_1, \dots, \Pi_n$ be general codimension 2 linear subspaces of $\mathbb{P}^n$. Then there exists a unique hypersurface Q of degree $n-1$ containing Π_j for $j = 1, \dots, n$. Moreover, Q is reduced and irreducible, it is the union of the transversals for $\Pi_1, \dots, \Pi_n$, and for each point $q \in Q$ we have $\operatorname{mult}_q Q \geq r$, where r is the number of indices i such that $q \in \Pi_i$. If q is a general point of Q, then there is a unique transversal for $\Pi_1, \dots, \Pi_n$ through q.*

Proof Let Δ be the determinantal variety in $(\mathbb{P}^n)^{n+1}$ of all $(n+1) \times (n+1)$ matrices M of rank at most 2 whose entries are the variables x_{ij}. It is known that Δ is reduced and irreducible of dimension $3n-1$, see [9]. It consists of the locus of points $(p_1, \dots, p_{n+1})$ whose span in $\mathbb{P}^n$ is contained in a line.

Let $\pi_i : (\mathbb{P}^n)^{n+1} \to \mathbb{P}^n$ be projection to the ith factor (so $1 \leq i \leq n+1$). Now, for $1 \leq i \leq n$, let $\Pi_i' = \pi_i^{-1}(\Pi_i)$. Then $D = \Delta \cap \bigcap_{1 \leq i \leq n} \Pi_i'$ has dimension $(3n-1) - 2n = n-1$. Indeed, Δ is irreducible, thus intersection with a divisor (preimage of a form by π_j) drops the dimension by one (Δ does not lie in one summand, hence cannot lie in the preimage). By Bertini we can do this again and again ($2n$ times, the dimension drops by 2 for every Π_j). We see that D is reduced and irreducible. Since $\Pi_1 \cap \cdots \cap \Pi_n = \varnothing$, we see that D is the locus of all points $(p_1, \dots, p_{n+1})$ such that the span $\langle p_1, \dots, p_n \rangle$ is a line with $p_i \in \Pi_i$ for $1 \leq i \leq n$ and p_{n+1} being on that line. Thus $\overline{D} = \pi_{n+1}(D)$ is irreducible, properly contains $\Pi_1 \cup \cdots \cup \Pi_n$ and is the union of all transversals for $\Pi_1, \dots, \Pi_n$. (To get Π_j in the image of the last projection, take a point p in Π_j, take a general line ℓ through p, and $(\ell \cap \Pi_1, \ell \cap \Pi_2, \dots, \ell \cap \Pi_j = p, \ell \cap \Pi_{j+1}, \dots, \ell \cap \Pi_n, p)$ lies in D and projects to $p \in \overline{D}$).

In particular, $\overline{D}$ has dimension $n-1$, and since by Proposition 2.1 there is a line through a general point meeting $n-1$ of the spaces Π_i in distinct points, we see that $\deg \overline{D} \geq n-1$.

Below we will check that there is a hypersurface Q of degree $n-1$ containing $\Pi_1 \cup \cdots \cup \Pi_n$. Since any such hypersurface must by Bezout contain all transversals for $\Pi_1, \dots, \Pi_n$, we see that $\deg \overline{D} = n-1$ and $Q = \overline{D}$ and thus that there is a unique hypersurface of degree $n-1$ containing $\Pi_1 \cup \cdots \cup \Pi_n$, and it is irreducible.

To show existence of Q we follow [2]. As mentioned in Remark 2.4, we may assume that the ideal of Π_k is

$$I_k = (x_k, f_k = a_{k,0}x_0 + \cdots + a_{k,k-1}x_{k-1} + a_{k,k+1}x_{k+1} + \cdots + a_{k,n}x_n),$$

where we write f_k instead of $f_{k,2}$. By generality we may assume that $a_{i,j} \neq 0$ for all $i \neq j$.

Now consider the $n \times n$ matrix

$$A = \begin{pmatrix} -f_1 & \dots & a_{1,k}x_k & \dots & a_{1,n}x_n \\ \vdots & & \vdots & & \vdots \\ a_{k,1}x_1 & \dots & -f_k & \dots & a_{k,n}x_n \\ \vdots & & \vdots & & \vdots \\ a_{n,1}x_1 & \dots & a_{n,k}x_k & \dots & -f_n \end{pmatrix}$$

and let $F = \det(A)$. Note that F is not identically 0 (since its value at the point $(1, 0, \dots, 0)$ is not 0) so $\deg F = n$. It is clear, developing $\det(A)$ with respect to the k-th column, that $F \in I_k$ for every $k = 1, \dots, n$. For each k, adding to the kth column of A all of the other columns of A gives a matrix A_k whose entries in the kth column are nonzero scalar multiples of x_0; in particular,

$$A_k = \begin{pmatrix} -f_1 & \dots & a_{1,k-1}x_{k-1} & -a_{1,0}x_0 & a_{1,k+1}x_{k+1} & \dots & a_{1,n}x_n \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{k-1,1}x_1 & \dots & -f_{k-1} & -a_{k-1,0}x_0 & a_{k-1,k+1}x_{k+1} & \dots & a_{k-1,n}x_n \\ a_{k,1}x_1 & \dots & a_{k,k-1}x_{k-1} & -a_{k,0}x_0 & a_{k,k+1}x_{k+1} & \dots & a_{k,n}x_n \\ a_{k+1,1}x_1 & \dots & a_{k+1,k-1}x_{k-1} & -a_{k+1,0}x_0 & -f_{k+1} & \dots & a_{k+1,n}x_n \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n,1}x_1 & \dots & a_{n,k-1}x_{k-1} & -a_{n,0}x_0 & a_{n,k+1}x_{k+1} & \dots & -f_n \end{pmatrix}.$$

so

$$A_1 = \begin{pmatrix} -a_{1,0}x_0 & a_{1,2}x_2 & \dots & a_{1,k}x_k & \dots & a_{1,n}x_n \\ -a_{2,0}x_0 & -f_2 & \dots & a_{2,k}x_k & \dots & a_{2,n}x_n \\ \vdots & \vdots & & \vdots & & \vdots \\ -a_{k,0}x_0 & a_{k,2}x_2 & \dots & -f_k & \dots & a_{k,n}x_n \\ \vdots & \vdots & & \vdots & & \vdots \\ -a_{n,0}x_0 & a_{n,2}x_2 & \dots & a_{n,k}x_k & \dots & -f_n \end{pmatrix}.$$

Thus $F = \det(A) = \det(A_k) = x_0 \cdot G$ for some polynomial G. Since x_0 is not an element of any I_k, it follows that $G \in I_k$ for $k = 1, \dots, n$, hence G vanishes on each of $\Pi_1, \dots, \Pi_n$. Since $\deg F = n$, we have $\deg(G) = n - 1$. Thus G defines a hypersurface Q of degree $n - 1$ containing each Π_i.

Now consider a point $q \in Q$. The matrix A_k will have r columns which vanish at q, where r is the number of indices i such that $q \in \Pi_i$. In particular, each entry in each such column is in the ideal I_q. Thus $G = \det(A_k)/x_0 \in I_q^r$ so $\text{mult}_q\, Q \geq r$.

Finally assume p is a general point of Π_n. Since Π_n is general, p is a general point of $\mathbb{P}^n$, hence by Proposition 2.1 there is a unique transversal t_p for $\Pi_1, \dots, \Pi_{n-1}$ through p, hence t_p is also the unique transversal for $\Pi_1, \dots, \Pi_n$ through p. Thus there is an open neighborhood U of p of points q through each of which there is a unique transversal t_q for $\Pi_1, \dots, \Pi_{n-1}$, and for those points q of $U \cap Q$, t_q also meets Π_n, hence for a general point $q \in Q$ there is a unique transversal t_q for $\Pi_1, \dots, \Pi_n$. □

Remark 2.6 Let $p_0, \dots, p_n$ be the coordinate vertices of $\mathbb{P}^n$ with respect to the variables $x_0, \dots, x_n$, so $p_0 = (1, 0, \dots, 0), \dots, p_n = (0, \dots, 0, 1)$. We saw in the proof of Proposition 2.5 that $p_0 \notin Q$ (since $F \neq 0$ at p_0). Let A'_k be the matrix from the proof of Proposition 2.5 arising after dividing x_0 from column k of A_k. Then Q is defined by $\det(A'_k) = 0$ but A'_k at p_k is a matrix which, except for column k, is a diagonal matrix with nonzero entries on the diagonal, and whose kth column has no zero entries. Thus $\det(A'_k) \neq 0$ at p_k so $p_k \notin Q$. In particular, none of the coordinate vertices is on Q.

3 The System $\mathscr{L}_n$

Let us start with some notation. Assume $\Pi_0, \ldots, \Pi_n \subset \mathbb{P}^n$ are general linear subspaces of codimension 2. From the previous section it follows that for each subset $\Pi_0, \ldots, \Pi_{j-1}, \Pi_{j+1}, \ldots, \Pi_n$ of n of them there is a unique hypersurface Q_j of degree $n-1$ containing them. Depending on the context, we may also denote by Q_j the form defining this hypersurface. We may assume $I_{\Pi_i} = (x_i, f_i)$ where f_j is as given in Remark 2.4. In this case we have the $(n+1) \times (n+1)$ matrix

$$B = \begin{pmatrix} -f_0 & a_{0,1}x_1 & \ldots & a_{0,k}x_k & \ldots & a_{0,n}x_n \\ a_{1,0}x_0 & -f_1 & \ldots & a_{1,k}x_k & \ldots & a_{1,n}x_n \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{k,0}x_0 & a_{k,1}x_1 & \ldots & -f_k & \ldots & a_{k,n}x_n \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n,0}x_0 & a_{n,1}x_1 & \ldots & a_{n,k}x_k & \ldots & -f_n \end{pmatrix}.$$

Let B_i be the $n \times n$ submatrix obtained by deleting row i and column i of B (where we have i run from 0 to n). The matrix A in the proof of Proposition 2.5 is thus B_0, and we have $\det(B_i) = x_i Q_i$. The next result shows that v_n is the map given by $(x_0, \ldots, x_n) \mapsto (x_0 Q_0, \ldots, x_n Q_n)$.

Proposition 3.1 *The polynomials $x_i Q_i$, $i = 0, \ldots, n$, give a basis for $\mathscr{L}_n$, hence* $\dim \mathscr{L}_n = n+1$*, so v_n is a rational map to $\mathbb{P}^n$ whose image is not contained in a hyperplane.*

Proof By Remark 2.6, no coordinate vertex p_j is in Q_i for any i. But $x_i Q_i \in \mathscr{L}_n$ for every i, and $(x_i Q_i)(p_j) \neq 0$ if and only if $i = j$. Thus the polynomials $x_i Q_i$ span a vector space of dimension at least $n+1$.

To show that these sections in fact give a basis, we show that $\dim \mathscr{L}_n = n+1$. We proceed by induction (the proof that $\mathscr{L}_2$ has three independent sections is clear, since three general points impose independent conditions on forms of degree 2 on $\mathbb{P}^2$). Let A be a fixed hyperplane that contains Π_1. There is, by Proposition 2.5, a unique section of $\mathscr{L}_n$ containing A, namely AQ_1. Moreover, the restrictions to A of sections s_n of $\mathscr{L}_n$ which do not contain A give divisors $s_n \cap A$ of degree n, containing Π_1, and containing $A \cap \Pi_j$, $j > 1$. So on A, the linear system of restrictions residual to Π_1 has degree $n-1$ and contains the n general subspaces $\Pi_i \cap A$, $i > 1$, of codimension 2. From the inductive assumption this has dimension n, so $\dim \mathscr{L}_n = n+1$.

We may also see the result from the exact sequence

$$0 \to \mathscr{L}_n(-A) \to \mathscr{L}_n \to \mathscr{L}_n|_A \to 0,$$

where A is as above and $\mathscr{L}_n(-A)$ is the linear subsystem of all elements of $\mathscr{L}_n$ containing A. Then, from the inductive assumption, the dimension of $\mathscr{L}_n|_A$ is n, and from Proposition 2.5 the dimension of $\mathscr{L}_n(-A)$ (which is of degree $n-1$ passing through n codimension 2 subspaces in A) is 1. □

Remark 3.2 If the hyperplanes $H_j \supset \Pi_j$, $j = 0, \dots, n$ are such that any n of them intersect in a point outside all Q_i and $\bigcap_j H_j = \emptyset$, then $H_j Q_j$ are linearly independent.

Proof If this is not the case, then one of them is linearly dependent of others, let it be $H_0 Q_0$. Thus, if $H_j Q_j$ vanish in some point p for $j = 1, \dots, n$, then $H_0 Q_0$ also does. Let then $p = \bigcap_{j=1}^n H_j$. Thus, $H_0 Q_0$ vanishes on p, but $p \notin H_0$, so $p \in Q_0$, a contradiction. □

Remark 3.3 Observe also, that up to an isomorphism of (the target) $\mathbb{P}^n$, the map v_n may be defined by any set of $n+1$ linearly independent elements of $\mathscr{L}_n$.

Let T_n be the closure of the union of all lines transversal to $\Pi_0, \dots, \Pi_n$, and let $R_n = Q_0 \cap \dots \cap Q_n$ and let B_n be the base locus of $\mathscr{L}_n$ (i.e., the locus where $v_n : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is not defined). We note that $T_n \subseteq R_n$, by Proposition 2.5.

Proposition 3.4 *We have* $B_n = \Pi_0 \cup \dots \cup \Pi_n \cup R_n$.

Proof Since v_n is given by $(x_0, \dots, x_n) \mapsto (x_0 Q_0, \dots, x_n Q_n)$, the base locus consists of the common zeros of the $x_i Q_i$. Clearly each Q_i (and hence each $x_i Q_i$) vanishes on R_n (as R_n is the intersection of all Q_i.) But Q_i vanishes on Π_j for $j \neq i$ and x_i vanishes on Π_i, so each $x_i Q_i$ vanishes on $\Pi_0 \cup \dots \cup \Pi_n$. Thus $\Pi_0 \cup \dots \cup \Pi_n \cup R_n \subseteq B_n$.

Conversely, let p be a point in B_n not in $\Pi_0 \cup \dots \cup \Pi_n$. By Remarks 3.2 and 3.3, v_n may be defined by the forms $H_i Q_i$ for sufficiently general H_i. Since H_i does not vanish on p, Q_i does for all i. Thus $p \in R_n$, so $B_n \subseteq \Pi_0 \cup \dots \cup \Pi_n \cup R_n$. □

Proposition 3.5 *We have* $\dim T_n = n-2$ *for* $n \geq 3$, *and* T_n *is irreducible for* $n > 3$.

Proof Consider the Grassmannian V of lines in $\mathbb{P}^n$ and the incidence variety $W = \{(v, p) \in V \times \mathbb{P}^n : p \in L_v\}$, where L_v is the line corresponding to a point $v \in V$. We also have the two projections $\pi_1 : W \to V$ and $\pi_2 : W \to \mathbb{P}^n$. Then V is an irreducible variety of dimension $2(n-1)$ and degree $\frac{(2(n-1))!}{n!(n-1)!}$ embedded in $\mathbb{P}^N$, $N = \binom{n+1}{2} - 1$, see [7], Chap. 1, Sect. 5. The condition of being incident to a codimension 2 linear space is given by a hyperplane in $\mathbb{P}^N$ (see p. 128 in [3]), so the intersections of V with $n+1$ general hyperplanes gives the locus ρ_n in V parametrizing the lines comprising T_n; notice that $\pi_2(\pi_1^{-1}(\rho_n)) = T_n$. Thus $\dim \rho_n = 2(n-1) - (n+1) = n-3$, so $\dim \pi_1^{-1}(\rho_n) = n-2$, and by Proposition 2.1 the projection π_2 is generically injective on $\pi_1^{-1}(\rho_n)$ so we have $\dim T_n = n-2$. Moreover, by Bertini's Theorem, ρ_n (and hence T_n) is irreducible when $\dim \rho_n > 0$. □

Proposition 3.6 *With the notation as above we have* $T_n = R_n$ *in* $\mathbb{P}^n$.

Proof Let us start with the following fact. Let $L_0, \ldots, L_k, L$ be lines through a common point p. Let L belong to the space spanned by $L_0, \ldots, L_k$, let $\mathscr{P}$ be a linear subspace, such that p does not lie on $\mathscr{P}$. Let L_j intersect $\mathscr{P}$ at a point l_j, $j = 0, \ldots, k$. Then L intersects $\mathscr{P}$, as the linear combination of a projection of some vectors is a projection of the combination.

Now we can show that the intersection of all Q_j lies in T_n, the union of all transversals. Observe, that the opposite inclusion is obvious.

Take a point p in all Q_j, but not in any Π_j. So for each j, there is L_j through p, transversal to all Q_i except Q_j. We have $n + 1$ such lines, but they must span a space of dimension less than $n + 1$ (being in $\mathbb{P}^n$).

Without loss of generality, let L_0 belong to the space spanned by the others. Then using the fact we started with, for $\mathscr{P} = \Pi_0$, we get that L_0 intersects Π_0 (since $L_1, \ldots, L_n$ intersect Π_0), which finishes the proof.

If $p \in \Pi_j$ for some j, the proof is trivial. □

Proposition 3.7 *The Veneroni transformation $v_n : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is injective off $Q_0 \cup \cdots \cup Q_n$, hence it is a Cremona transformation.*

Proof Let p, q be two different points off $Q_0 \cup \cdots \cup Q_n$. Let H_j denote the unique hyperplane through p and Π_j. Then $\bigcap_{j=0}^{n} H_j = \{p\}$ as if the intersection of all such H_i is not exactly p, then the intersection $H_0 \cap \cdots \cap H_n$ is a positive dimensional linear space, and any line through p in this space intersects each Π_i and hence is a transversal for $\Pi_0, \ldots, \Pi_n$, and so p, being on a transversal, is in $T_n \subseteq R_n \subseteq B_n$. Take j_0 such that $q \notin H_{j_0}$. Then $H_{j_0} Q_{j_0}$ is a non-zero section of $\mathscr{L}_n$ and may be extended to a basis of $\mathscr{L}_n$. Then v_n defined by the sections of this basis separates p and q. Thus v_n is injective off $Q_0 \cup \cdots \cup Q_n$. □

4 An Inverse for v_n

It is of interest to determine an inverse for v_n, and to observe that the inverse is again given by forms of degree n vanishing on $n + 1$ codimension 2 linear subspaces. We explicitly define such a map u_n and then check that it is an inverse for $v_n : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. If we regard $x_0, \ldots, x_n$ as homogeneous coordinates on the source $\mathbb{P}^n$ and $y_0, \ldots, y_n$ as homogeneous coordinates on the target $\mathbb{P}^n$, then v_n is defined by the homomorphism h on homogeneous coordinate rings given by $h(x_0, \ldots, x_n) = (y_0, \ldots, y_n)$, where $y_i = x_i Q_i = \det(B_i)$, as we saw in Sect. 3.

To define u_n, we slightly modify matrix B from Sect. 3 by replacing the diagonal entries $-f_i$ in B by $-g_i$ (defined below) and by replacing each entry $a_{i,j}x_j$ in B by $a_{i,j}y_j$ to obtain a new matrix

$$C = \begin{pmatrix} -g_0 & a_{0,1}y_1 & \dots & a_{0,k}y_k & \dots & a_{0,n}y_n \\ a_{1,0}y_0 & -g_1 & \dots & a_{1,k}y_k & \dots & a_{1,n}y_n \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{k,0}y_0 & a_{k,1}y_1 & \dots & -g_k & \dots & a_{k,n}y_n \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n,0}y_0 & a_{n,1}y_1 & \dots & a_{n,k}y_k & \dots & -g_n \end{pmatrix}.$$

To define g_i, recall that since $f_i Q_i \in \mathscr{L}_n$ for each i and the forms $x_j Q_j$ give a basis for $\mathscr{L}_n$, we can for each i and appropriate scalars $b_{i,j}$ write

$$f_i Q_i = b_{i,0} x_0 Q_0 + \cdots + b_{i,n} x_n Q_n.$$

We define g_i to be $g_i = b_{i,0} y_0 + \cdots + b_{i,n} y_n$, so we see that $h(g_i) = f_i Q_i$.

As an aside we also note that $b_{i,j} = 0$ if and only if $i = j$. (To see this, recall by Remark 2.6 that no Q_j vanishes at any coordinate vertex p_k, but f_i vanishes at the coordinate vertex p_j if and only if $i = j$. Thus, evaluating $f_i Q_i = b_{i,0} x_0 Q_0 + \cdots + b_{i,n} x_n Q_n$ at p_i gives $0 = b_{i,i} Q_i$, hence $b_{i,i} = 0$, while evaluating at p_j for $j \neq i$ gives $0 \neq b_{i,j} Q_j$, hence $b_{i,j} \neq 0$.)

Let C_i be the matrix obtained from C by deleting row i and column i. Define a homomorphism $\lambda : \mathbb{K}[x_0, \dots, x_n] \to \mathbb{K}[y_0, \dots, y_n]$ by $\lambda(x_i) = \det(C_i)$.

The next result gives an inverse for v_n.

Proposition 4.1 *The homomorphism λ defines a birational map $u_n : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ which is inverse to v_n.*

Proof Note that applying h to the entries of C gives the matrix obtained from BD, where D is the diagonal matrix whose diagonal entries are $Q_0, \dots, Q_n$, from which it is easy to see that $h(\det(C_i)) = \det(B_i) Q_0 \cdots Q_{i-1} Q_{i+1} \cdots Q_n = x_i Q_0 \cdots Q_n$.

We now have $h(\lambda(x_i)) = h(\det(C_i)) = x_i Q_0 \cdots Q_n$, so $u_n v_n = id_U$, where U is the complement of $Q_0 \cdots Q_n = 0$. Since v_n is a Cremona transformation, so is u_n and thus u_n is the inverse of v_n. □

Remark 4.2 We now confirm that the forms $\det(C_i)$ defining u_n have degree n and vanish on $n + 1$ codimension 2 linear subspaces $\Pi_i^* \subset \mathbb{P}^n$. That $\deg(\det(C_i)) = n$ is clear, since C_i is an $n \times n$ matrix of linear forms.

Consider the codimension two linear spaces defined by the ideals $J_k = (y_k, g_k) = b_{k,0} y_0 + \cdots + b_{k,n} y_n$. Since the entries of column k of C are in the ideal J_k, it follows that $\det(C_i)$ vanishes on Π_j^* for $j \neq i$. It remains to check that $\det(C_i)$ vanishes on Π_i^*. But let $q \in Q_i$ be a point where v_n is defined. Note that $y_i(v_n(q)) = h(y_i)(q) = x_i Q_i(q) = 0$ and that $g_i(v_n(q)) = h(g_i)(q) = f_i Q_i(q) = 0$. Thus $v_n|_{Q_i}$ gives a rational map to Π_i^* whose image is in the zero locus of $\det(C_i)$ since $\det(C_i)(v_n(q)) = (h(\det(C_i)))(q) = h\lambda(x_i)(q) = (x_i Q_0 \cdots Q_n)(q) = 0$. Thus $\det(C_i)$ vanishing on Π_i^* will follow if we show that $v_n|_{Q_i}$ gives a dominant rational map to Π_i^*. This in turn will follow if we show for a general $q \in Q_i$ that the fiber over $v_n(q)$ has dimension 1 (since Q_i as dimension $n - 1$ and Π_i^* has dimension $n - 2$). But the space of forms in $\mathscr{L}_n$ vanishing on q is spanned by forms of the form $H_j Q_j$ where H_j is a hyperplane containing q and Π_j. For a general point q, since the Π_j are general, the intersection of any $n-1$ of the H_j with $j \neq i$ has dimension 1. Since the Π_j are general, the same is true for a general point $q \in Q_i$ except now, since there is a transversal t_q through q for Π_j, $j \neq i$, we see that $\cap_{j \neq i} H_j$ still has dimension 1 and is thus exactly t_q. Hence the locus of points on which the forms in $\mathscr{L}_n$ vanishing at q vanish is exactly t_q. Thus the fiber over $v_n(q)$ has dimension 1, as we wanted to show.

It is still unclear to us whether u_n is itself a Veneroni transformation whenever v_n is. If we denote by $\mathscr{L}_n^*$ the forms in $\mathbb{K}[y_0, \dots, y_n]$ of degree n vanishing on $\Pi_0^* \cup \cdots \cup \Pi_n^*$, what we saw above is that u_n is defined by an $n+1$ dimensional linear system contained in $\mathscr{L}_n^*$; the issue is whether the linear system is all of $\mathscr{L}_n^*$ (i.e., whether $\dim \mathscr{L}_n^* = n+1$).

In any case, when $\Pi_0, \dots, \Pi_n$ are general, we now see that v_n gives a birational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ whose restriction to Q_i gives a rational map to Π_i^* for $i = 0, \dots, n$ and the fiber of Q_i over Π_i^* generically has dimension 1. It is convenient to denote the linear system of divisors of degree n vanishing on $\Pi_0 \cup \cdots \cup \Pi_n$ by $nH - \Pi_0 - \cdots - \Pi_n$. Similarly, the linear system of divisors of degree $n-1$ vanishing on Π_j for $j \neq i$ is represented by $(n-1)H - \Pi_0 - \cdots - \Pi_n + \Pi_i$. Thus, if H^* is the linear system of divisors of degree 1 on the target $\mathbb{P}^n$ for v_n, then v_n pulls H^* back to $nH - \Pi_0 - \cdots - \Pi_n$, and it pulls Π_i^* back to Q_i, represented by $(n-1)H - \Pi_0 - \cdots - \Pi_n + \Pi_i$. We can represent the pullback by a matrix map $M_n : \mathbb{Z}^{n+1} \to \mathbb{Z}^{n+1}$ where

$$M_n = \begin{pmatrix} n & n-1 & n-1 & \dots & n-1 \\ -1 & 0 & -1 & \dots & -1 \\ -1 & -1 & 0 & \dots & -1 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & -1 & -1 & \dots & 0 \end{pmatrix}.$$

If in fact the spaces Π_i^* can be taken to be sufficiently general, then $\dim \mathscr{L}_n^* = n+1$, and u_n pulls H back to $nH^* - \Pi_0^* - \cdots - \Pi_n^*$, and it pulls Π_i back to $(n-1)H^* - \Pi_0^* - \cdots - \Pi_n^* + \Pi_i^*$, and hence is represented by the same matrix M_n. Since M_n^2 corresponds to the pullback map for $u_n v_n$ and $u_n v_n$ is the identity (where defined), we would expect that $M_n^2 = I_n$, which is indeed the case.

5 Intersection of Q_i and Q_j

This section is devoted to investigating the intersections of Q_i and Q_j, assuming that $\Pi_0, \dots, \Pi_n$ are general linear subspaces of codimension 2. These intersections were already treated in [11] and in more detail than here, but here we use more modern language.

Without loss of generality assume that $i = 0$, $j = 1$, so take $Q_0 \cap Q_1$. From the considerations above (Proposition 3.4) we may write

$$Q_0 \cap Q_1 = T_n \cup \Pi_2 \cup \cdots \cup \Pi_n \cup M_n$$

where M_n is the closure of the complement of $T_n \cup \Pi_2 \cup \cdots \cup \Pi_n$ in $Q_0 \cap Q_1$.

Proposition 5.1 *The complement of $T_n \cup \Pi_2 \cup \cdots \cup \Pi_n$ in $Q_0 \cap Q_1$ is the set of all points $q \in Q_0 \cap Q_1$ through which there is no transversal for $\Pi_0, \ldots, \Pi_n$, (in which case there is more than one transversal through q for $\Pi_2, \ldots, \Pi_n$).*

Proof For $n = 2$ it is easy to check that $Q_0 \cap Q_1 = \Pi_2$ and that $T_n = M_n = \varnothing$. For $n = 3$, keeping in mind that $Q_0 = \mathbb{P}^1 \times \mathbb{P}^1$, $Q_0 \cap Q_1$ is a divisor on Q_0 of multi-degree $(2, 2)$, consisting of the lines Π_2 and Π_3 together with the two transversals for $\Pi_0, \ldots, \Pi_3$ (these two transversals give T_n); again M_n is empty. (See, for example, the description of the cubo-cubic Cremona transformation from [4] or [5].)

So now assume that $n \geq 4$. Take a point q from $Q_0 \cap Q_1$. Suppose q is not in $\Pi_2 \cup \cdots \cup \Pi_n$. Since $q \in Q_0$, by Proposition 2.5 there is at least one transversal through q for $\Pi_1, \ldots, \Pi_n$ and since $q \in Q_1$ there is similarly at least one transversal through q for $\Pi_0, \Pi_2 \ldots, \Pi_n$. If one of the transversals coming from $q \in Q_0$ is also a transversal coming from $q \in Q_1$, then it follows that the transversal goes through all Π_j, so the transversal (and hence q) is contained in T_n. Otherwise, $q \notin T_n$, hence there are two lines through q transversal for $\Pi_2, \ldots, \Pi_n$. □

Example 5.2 We close by showing for $n = 4$ that the complement of $T_4 \cup \Pi_2 \cup \cdots \cup \Pi_4$ in $Q_0 \cap Q_1$ is nonempty.

Take three points p_{ij}, where $p_{ij} = \Pi_i \cap \Pi_j$, for $j = 2, 3, 4$, $i \neq j$. Let π be the plane spanned by the three points. Take a general point q on π. From the fact that all Π_j are general, we have that q, $p_0 := \pi \cap \Pi_0$ and $p_1 := \pi \cap \Pi_1$ are not on a line. Then the line through q and p_0 is a transversal to $\Pi_0, \Pi_2, \Pi_3, \Pi_4$, so it is in Q_0 (and in π of course). In the same way, the line through q and p_1 is a transversal to $\Pi_1, \Pi_2, \Pi_3, \Pi_4$, so it is in Q_1, thus q is in $Q_0 \cap Q_1$.

To prove that $M_4 \not\subset T_4$, take a point r not in Π_2, Π_3, Π_4, and consider a projection from r to a general hyperplane. Then the intersection of the images of Π_2, Π_3, Π_4 is either a point—and then there is only one transversal to Π_2, Π_3, Π_4 through this point—or this intersection is a line, and then we have a plane of transversals from our point r. From this construction it follows that we may have at most a plane of transversals to Π_2, Π_3, Π_4. As Π_1, Π_0 are general, the generic transversal on π is not transversal to Π_1, Π_0.

Remark 5.3 Snyder and Rusk in [11] assert that $\deg(R_n) = \frac{(n+1)(n-2)}{2}$ and that $\deg(M_n) = \frac{(n-2)(n-3)}{2}$. We plan a future paper explaining these results and showing also precisely that the inverse of a Veneroni transformation is always a Veneroni.

Acknowledgements Farnik was partially supported by National Science Centre, Poland, grant 2018/28/C/ST1/00339, Harbourne was partially supported by Simons Foundation grant #524858. Szemberg was partially supported by National Science Centre grant 2018/30/M/ST1/00148. Harbourne and Tutaj-Gasińska were partially supported by National Science Centre grant 2017/26/M/ST1/00707. Harbourne and Tutaj-Gasińska thank the Pedagogical University of Cracow, the Jagiellonian University and the University of Nebraska for hosting reciprocal visits by Harbourne and Tutaj-Gasińska when some of the work on this paper was done. The paper is in final form and no similar paper has been or is being submitted elsewhere.

References

1. G. Blanch, The Veneroni transformation in S_n. Am. J. Math. **59**(4), 783–786 (1937)
2. G.K. Blanch, Properties of the Veneroni transformation in S_4. Am. J. Math. **58**(3), 639–645 (1936)
3. R. Cid-Muñoz, M. Pedreira, Classification of incidence scrolls. I. Manuscripta Math. **105**(1), 125–138 (2001)
4. I. Dolgachev, Lectures on Cremona transformations, p. 121, http://www.math.lsa.umich.edu/~idolga/cremonalect.pdf. Access date: March 2019
5. M. Dumnicki, B. Harbourne, J. Roé, T. Szemberg, H. Tutaj-Gasińska, Unexpected surfaces singular on lines in $\mathbb{P}^3$, arXiv:1901.03725
6. J. Eiesland, On a class of ruled $(n-1)$-spreads in S_n the V_4^4 in S_5. Rend. Circ. Mat. Palermo **54**, 335–365 (1929)
7. P. Griffiths, J. Harris, *Principles of Algebraic Geometry* (Wiley-Interscience [Wiley], New York, 1978). Pure and Applied Mathematics
8. B. Harbourne, J. Migliore, H. Tutaj-Gasińska, New constructions of unexpected hypersurfaces in $\mathbb{P}^N$. Rev Mat Complut (2020). https://doi.org/10.1007/s13163-019-00343-w
9. M. Hochster, J.A. Eagon, Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci. Am. J. Math. **93**, 1020–1058 (1971)
10. C. Segre, Sulla varieta cubica con dieci punti doppi dello spazio a quattro dimensioni. Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **XXII**, 791–801 (1887)
11. V. Snyder, E. Carroll-Rusk, The Veneroni transformation in S_n. Bull. Am. Math. Soc. **42**(8), 585–592 (1936)
12. J.A. Todd, The quarto-quartic transformation of four-dimensional space associated, with certain projectively generated loci. Proc. Camb. Philos. Soc. **26**, 323–333 (1930)
13. E. Veneroni, Sopra una trasformazione birazionale fra due S_n. Istit. Lombardo Accad. Sci. Lett. Rend. A **34**(2Serie 2, Fascicolo 11-12), 1901