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Dumitru I. Stamate
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Combinatorial Structures in Algebra and Geometry

NSA 26, Constanța, Romania, August 26–September 1, 2018

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Preface

The summer school *Combinatorial structures in algebra and geometry* is the 26th event organized in Romania within the series of National Schools on Algebra. It took place between the 26th of August and the 1st of September 2018 at the Faculty of Mathematics and Computer Science, Ovidius University, on the sunny shore of the Black See in Constanţa, Romania. The School was organized by the Faculty of Mathematics and Computer Science, Ovidius University in partnership with Osnabrück University, Germany, the Faculty of Mathematics and Computer Science of the University of Bucharest, the Simion Stoilow Institute of Mathematics of the Romanian Academy, and the Romanian Mathematical Society. These institutions were represented in the Organizing Committee by Alexandru Bobe, Cristodor Ionescu, Anca Măcinic, Anda Olteanu and Tim Römer. The Committee was led by Viviana Ene. The School was generously supported by the European Mathematical Society.

The Scientific Committee: Jürgen Herzog (Essen University, Germany), Takayuki Hibi (Osaka University, Japan), Dorin Popescu (Institute of Mathematics, Bucharest, Romania), and Tim Römer (Osnabrück University, Germany) admitted 21 talks to be presented during the school. The speakers were carefully chosen among researchers in different phases of the career, ranging from graduate students, through postdocs, early faculty, and senior professors. The core of the school were the afternoon discussion sessions. Some outcomes of these sessions are reflected in the present proceedings. More details about the program of the school are available at https://math.univ-ovidius.ro/sna/edition.aspx?cat=GeneralInfoitemID=11.

These proceedings contain material from three fields of mathematics: algebra, geometry, and discrete mathematics. Whereas interactions between algebra and geometry go back at least to Hilbert, the ties to combinatorics are much more recent and they are subject of immense interest at the forefront of contemporary research in mathematics. Transplanting methods between different branches of mathematics has proved to be very fruitful in the past. For example, the application of fixed point theorems in topology to solving nonlinear differential equations in analysis. Similarly, combinatorial structures, e.g., Newton–Okounkov bodies, have led to significant progress in understanding asymptotic properties of line bundles in geometry and multiplier ideals in algebra. These proceedings are addressed to emerging scientists and established researchers bringing in one volume up to date developments in the theory of ideals, graphs, and local positivity of line bundles. It is our hope that the publication will be useful for all those wishing to study ideas or problems at the exciting crossroads of three research areas.

We owe many thanks to Viviana Ene and Tim Römer for planning the 2018 edition of the School, for taking care that everything runs smoothly, and for suggesting our involvement in editing this volume. We thank the contributors for their enthusiasm and for sharing their research. We thank the Springer Verlag for publishing these proceedings and for the constant support during their preparation.

Bucharest, Romania — Dumitru I. Stamate
Kraków, Poland — Tomasz Szemberg
April 2020

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Nearly Normally Torsionfree Ideals

Claudia Andrei-Ciobanu

Abstract We describe all connected graphs whose edge ideals are nearly normally torsionfree. We also prove that the facet ideal of a special odd cycle is nearly normally torsionfree. Finally, we give a necessary condition for a t-spread principal Borel ideal generated in degree 3 to be nearly normally torsionfree.

Keywords Associated prime ideals · Connected graphs · Simplicial complexes · t-Spread principal Borel ideals

2010 Mathematics Subject Classification Primary 13A15, 13A30, 05C25 · Secondary 05E45

1 Introduction

Combinatorial algebra is closely related to monomial ideals. A special class of monomial ideals are squarefree monomial ideals. They can be associated with simplicial complexes or graphs. In this paper, we study squarefree monomial ideals which are very near to normally torsionfree ideals.

Let $\mathbb{K}$ be a field and $S = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring in n variables over $\mathbb{K}$. If I is a monomial ideal in S, then it is known that the associated prime ideals of I are generated by variables; see [7, Sect. 1.3]. In [10, Lemma 2.3] and [5, Lemma 2.11], it was given a method for checking whether a prime ideal P is an associated prime ideal of I. More precisely, $P \in \mathrm{Ass}(I)$ if and only if $\mathrm{depth}(S_P/I_P) = 0$, where S_P is the polynomial ring $\mathbb{K}[\{x_i : x_i \in P\}]$ and I_P is the localization of I with respect to P.

An important result of Brodmann from [2] shows in particular, that the set of associated prime ideals of a monomial ideal $I \subset S$ stabilizes which means that there exists an integer k_0 such that $\mathrm{Ass}(I^k) = \mathrm{Ass}(I^{k_0})$ for all $k \geq k_0$. A monomial ideal

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I of the polynomial ring S satisfies the persistence property if

$$\text{Ass}(I) \subset \text{Ass}(I^2) \subset \ldots \subset \text{Ass}(I^k) \subset \text{Ass}(I^{k+1}) \subset \ldots.$$

Classes of monomial ideals which have the persistence property are studied in [1, 5, 10, 12, 13]. For example, in [1, Corollary 2.6], it was shown that a t-spread principal Borel ideal has the persistence property, while in [12, Lemma 2.12], it was proved that the edge ideal of a simple graph has the persistence property.

In [7, Sect. 1.4], normally torsionfree ideals are studied. A monomial ideal in S is called normally torsionfree if $\text{Ass}(I^k) = \text{Min}(I)$ for all $k \geq 1$. There it was shown that the normally torsionfree squarefree monomial ideals have the property that their symbolic powers coincide with the ordinary powers. Moreover, in [7, Theorem 10.3.16], a combinatorial characterization of normally torsionfree facet ideals of simplicial complexes was given.

In this paper, we generalize normally torsionfree ideals and we introduce nearly normally torsionfree ideals. A monomial ideal $I \subset S$ is called nearly normally torsionfree if there exist a positive integer k and a monomial prime ideal P such that $\text{Ass}(I^m) = \text{Min}(I)$ for $m \leq k$ and $\text{Ass}(I^m) \subseteq \text{Min}(I) \cup \{P\}$ for $m \geq k+1$. This definition is slightly more general than that considered in [9] for almost normally torsionfree ideals, where the monomial prime ideal P is only given by the graded maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n) \subset S$. Notice that any normally torsionfree ideal is nearly normally torsionfree.

In the first section, we refer to edge ideals. According to [14, Theorem 5.9], it is known that the edge ideal of a simple graph G is normally torsionfree if and only if G is a bipartite graph. The aim of the main result of this section is to classify all connected graphs whose edge ideals are nearly normally torsionfree; see Theorem 2.3.

The second section is dedicated to facet ideals of special odd cycles. We have seen before that [7, Theorem 10.3.16] gives a large class of normally torsionfree facet ideals by using special odd cycles. In Theorem 3.1, we prove that the facet ideal of a special odd cycle is nearly normally torsionfree.

In the last section, we study t-spread principal Borel ideals generated in degree 3. They have been recently introduced in [4]. The main result of this section is Proposition 4.5, where we give a necessary condition for a t-spread principal Borel ideal generated in degree 3 to be nearly normally torsionfree. We notice that for degree 2, the behaviour of the set of associated prime ideals of the powers of a t-spread principal Borel ideal was given in [11, Theorem 1.1]. We recover here this result by a different proof; see Proposition 4.4.

2 Nearly Normally Torsionfree Edge Ideals

In this section, we present the behaviour of the set of associated prime ideals of the powers of an edge ideal.

Let G be a finite simple graph on the vertex set $V(G) = [n]$. Recall that a *walk of length r* in G is a sequence of edges

$$\gamma = (\{i_0, i_1\}, \{i_1, i_2\}, \dots, \{i_{r-1}, i_r\})$$

with each $i_k \in [n]$ for $k \in \{0, 1, \dots, r\}$. An *even walk* (respectively *odd walk*) is a walk of even (respectively odd) length. A *closed walk* is a walk with $i_0 = i_r$ and a *cycle* is a closed walk of the form

$$C = (\{i_0, i_1\}, \{i_1, i_2\}, \dots, \{i_{r-1}, i_0\})$$

with $i_j \neq i_k$ for all $0 \leq j < k \leq r - 1$.

The graph G is a *bipartite graph* if it does not contain any odd cycle.

Let $S = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring over the field K. The *edge ideal* of G is the monomial ideal

$$I(G) = (x_i x_j : \{i, j\} \text{ is an edge in } G) \subset S.$$

By [7, Lemma 9.1.4], every minimal prime ideal of $I(G)$ is given by a minimal vertex cover of G. A *vertex cover* of G is a subset $A \subset [n]$ such that $\{i, j\} \cap A \neq \emptyset$ for all edges $\{i, j\}$ of G. A vertex cover A is called minimal if no proper subset of A is a vertex cover of G. In [3, Theorem 3.3], a method to construct associated prime ideals of the powers of $I(G)$ by considering the odd cycles of G was given.

Let $I \subset S$ be a monomial ideal and $P \subset S$ be a monomial prime ideal containing I. Then P is a *monomial persistent prime ideal* of I, if whenever $P \in \mathrm{Ass}(I^k)$ for some k, then $P \in \mathrm{Ass}(I^{k+1})$. The set of monomial persistent prime ideals is denoted by $\mathrm{Ass}^{\infty}(I)$.

According to [12, Lemma 2.12], $I(G)$ has the *persistence property* which means that the sets of associated primes of powers of $I(G)$ form an ascending chain. In other words, all monomial prime ideals $P \in \bigcup_{k>0} \mathrm{Ass}(I(G)^k)$ are monomial persistent prime ideals.

In [14, Theorem 5.9], it was shown that $I(G)$ is normally torsionfree if and only if G is a bipartite graph. Recall that a monomial ideal $I \subset S$ is *normally torsionfree* if $\mathrm{Min}(S/I) = \mathrm{Ass}(S/I^k)$ for all $k \geq 1$.

Definition 2.1 A monomial ideal $I \subset S$ is called *nearly normally torsionfree* if there exist a positive integer k and a monomial prime ideal P such that $\mathrm{Ass}(I^m) = \mathrm{Min}(I)$ for $m \leq k$ and $\mathrm{Ass}(I^m) \subseteq \mathrm{Min}(I) \cup \{P\}$ for $m \geq k + 1$.

In particular, any normally torsionfree ideal is nearly normally torsionfree.

In this section we study edge ideals which are nearly normally torsionfree. The following lemma will allow us to restrict our attention to connected graphs. It generalizes [6, Lemma 3.4] and its proof is the same with that given in [6].

Let I be a monomial ideal of S. From now on, $G(I)$ denotes the minimal set of monomial generators of I.

Lemma 2.2 *Let $I_1 \subset S_1 = \mathbb{K}[x_1, \dots, x_n]$, $I_2 \subset S_2 = \mathbb{K}[y_1, \dots, y_m]$ be two monomial ideals in disjoint sets of variables. Let*

$$I = I_1S + I_2S \subset S = \mathbb{K}[x_1, \dots, x_n, y_1, \dots, y_m].$$

Then $P \in \mathrm{Ass}_S(I^k)$ *if and only if* $P = P_1S + P_2S$ *where* $P_1 \in \mathrm{Ass}_{S_1}(I_1^{k_1})$ *and* $P_2 \in \mathrm{Ass}_{S_2}(I_2^{k_2})$ *for some positive integers* k_1, k_2 *with* $k_1 + k_2 = k + 1$.

Let G be a graph on the vertex set $[n]$. For a subset $A \subset [n]$, one denotes $N(A)$ the set of all the vertices of G which are adjacent to some vertices in A. In other words, $N(A)$ is the set of all the neighbors of the vertices in A. In addition, if $A \subset [n]$, $G \setminus A$ denotes the induced subgraph of G on the vertex set $[n] \setminus A$. If $A = \{j\}$, we write $G \setminus j$ instead of $G \setminus \{j\}$.

Theorem 2.3 *Let G be a connected graph on the vertex set* $[n]$ *and* $I = I(G)$ *the edge ideal of G. Suppose that G contains an odd cycle and let*

$$k = \min\{j : G \text{ has an odd cycle } C_{2j+1}\}.$$

Then $\mathrm{Ass}^\infty(I) = \mathrm{Min}(I) \cup \{\mathfrak{m}\}$ *if and only if, for every odd cycle* C *of* G, *we have* $V(C) \cup N(C) = [n]$. *Moreover, if the above condition holds, then* $\mathrm{Ass}(I^m) = \mathrm{Min}(I)$ *if* $m \le k$ *and* $\mathrm{Ass}(I^m) = \mathrm{Min}(I) \cup \{\mathfrak{m}\}$ *if* $m \ge k + 1$.

Proof Let $\mathrm{Ass}^\infty(I) = \mathrm{Min}(I) \cup \{\mathfrak{m}\}$ and assume that there exists an odd cycle $C = C_{2j+1}$ for some $j \ge 1$ such that $V(C) \cup N(C) \subsetneq [n]$. By [3, Theorem 3.3], it follows that

$$P = (x_i : i \in V(C) \cup N(C) \cup W)$$

is an associated prime ideal of I^m for $m \ge j + 1$, where W is any minimal subset of $[n]$ for which $V(C) \cup N(C) \cup W$ is a vertex cover of G. If $V(C) \cup N(C)$ is a vertex cover of G, then $P = (x_i : i \in V(C) \cup N(C))$ is an associated prime of I^m for $m \ge j + 1$ with $P \notin \mathrm{Min}(I)$ and $P \ne \mathfrak{m}$. This is a contradiction to our hypothesis. If $V(C) \cup N(C)$ is not a vertex cover of G, then there exists at least two minimal subsets W_1, W_2 of $[n]$ for which $V(C) \cup N(C) \cup W_1$ and $V(C) \cup N(C) \cup W_2$ are vertex covers of G. Therefore, we get at least two minimal prime ideals

$$P_1 = (x_i : i \in V(C) \cup N(C) \cup W_1) \text{ and } P_2 = (x_i : i \in V(C) \cup N(C) \cup W_2)$$

which are not minimal primes of I and are associated with I^m for $m \ge j + 1$, a contradiction.

Let $V(C) \cup N(C) = [n]$ for every odd cycle of G. In particular, this equality holds for the cycle $C = C_{2k+1}$ with the smallest number of vertices. First we show that $\mathfrak{m} \in \mathrm{Ass}(I^m)$ for $m \ge k + 1$. Since, by [12, Theorem 2.15], $I = I(G)$ has the persistence property, it is enough to show that $\mathfrak{m} \in \mathrm{Ass}(I^{k+1})$. Let $u = \prod_{i \in V(C)} x_i$ be the product of all the variables of cycle C. Clearly, $u \in I^k$ and since $\deg(u) = 2k + 1$, we have $u \notin I^{k+1}$ which implies that $I^{k+1} : u \subseteq \mathfrak{m}$. It is also easily seen that for all $j \in V(C) \cup N(C)$, $x_j u \in I^{k+1}$. Therefore, we have $\mathfrak{m} \subseteq I^{k+1} : u$ since $V(C) \cup N(C) = [n]$.

Further on, we show that $\mathfrak{m} \notin \operatorname{Ass}(I^m)$ for $m \leq k$. We proceed by induction on the number of vertices of G. If $G = C$, then the claim follows by [3, Lemma 3.1]. Now let $V(C) \subsetneq [n]$ and assume that there exists $m \leq k$ such that $\mathfrak{m} \in \operatorname{Ass}(I^m)$. By the persistence property of edge ideals, we may assume that $\mathfrak{m} \in \operatorname{Ass}(I^k)$. Thus, there exists a monomial $v \notin I^k$ such that $I^k : v = \mathfrak{m}$. There exists some vertex $j \in V(C)$ such that x_j does not divide v, since, otherwise, $v \in I^k$. Let J be the edge ideal of $H = G \setminus j$ and $Q = (x_i : i \neq j)$. Then, by [3, Lemma 2.6], it follows that $Q \in \operatorname{Ass}(J^k)$. Note that the graph H is either bipartite or it has an odd cycle. If H is bipartite, then $Q \in \operatorname{Min}(J)$. If H is not bipartite, the inductive hypothesis implies that $Q \in \operatorname{Min}(J)$. But this is not possible since the set $[n] \setminus \{j\}$ is not a minimal vertex cover of H.

What is left is to show that, in our hypothesis, if $P \in \operatorname{Ass}(I^m)$ for some integer $m \geq 1$ and $P \neq \mathfrak{m}$, then $P \in \operatorname{Min}(I)$. Let $P \in \operatorname{Ass}(I^m)$ for some integer $m \geq 1$ with $P \neq \mathfrak{m}$. We proceed again by induction on the number of vertices of G. The claim is obvious if $G = C$ by [3, Lemma 3.1]. Let $V(C) \subsetneq [n]$. Then there exists a vertex $j \in V(G)$ such that $x_j \notin P$. We localize at the prime ideal $Q = (x_i : i \neq j)$. Then $I_Q = (I_1, I_2)$ where I_1 is generated by all the variables x_i with $i \in N(j)$ and $I_2 = I(G \setminus N[j])$. Here $N[j]$ denotes the set $N(j) \cup \{j\}$. By [3, Corollary 2.3], it follows that $P_Q = (I_1, P_2)$ where $P_2 \in \operatorname{Ass}(I_2^{m'})$ for some $m' \leq m$. If $G \setminus N[j]$ is bipartite, then P_2 gives a minimal vertex cover of $G \setminus N[j]$ and, thus, P gives a minimal vertex cover of I, equivalently, $P \in \operatorname{Min}(I)$. Let $G \setminus N[j]$ be not bipartite. If it is connected, by induction it follows that P_2 is a minimal prime ideal of I_2 and, consequently, $P \in \operatorname{Min}(I)$. If $G \setminus N[j]$ is not connected, say it has the connected components $G_1, \ldots, G_c$ for some $c \geq 2$, then P_2 is a sum of minimal prime ideals of $I(G_1), \ldots, I(G_c)$, and again, it follows that $P \in \operatorname{Min}(I)$. □

Remark 2.4 Let G be a graph on the vertex set $[n]$ and $I = I(G) \subset S$ be its edge ideal. Condition $V(C) \cup N(C) = [n]$, for every odd cycle C of G, in the hypothesis of previous theorem needs to be checked for every odd cycle C of G. Indeed, if G is the graph of Fig. 1, then

$$|\operatorname{Ass}(I(G)^2)| = 13 > 8 = |\operatorname{Min}(I(G))|.$$

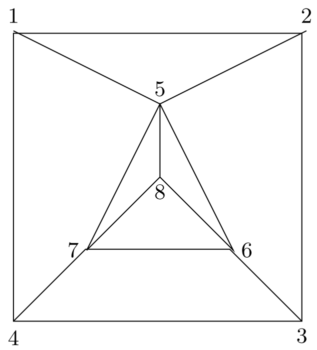
Notice that $V(C_1) \cup N(C_1) = [8]$ for $C_1 = (\{5, 6\}, \{6, 7\}, \{7, 5\})$, while for $C_2 = (\{6, 7\}, \{7, 8\}, \{8, 6\})$, we have $V(C_2) \cup N(C_2) = \{3, 4, 5, 6, 7, 8\} \neq [8]$.

Let G be a graph on the vertex set $[n]$ and $I = I(G) \subset S$ its edge ideal. The symbolic Rees algebra of I is

$$\mathcal{R}_s(I) = \oplus_{k \geq 0} I^{(k)} t^k,$$

where $I^{(k)}$ is the k-th symbolic power of I. This algebra is actually the vertex cover algebra of G which is generated as an S-algebra by all the monomials $\mathbf{x}^{\mathbf{a}} t^b$ where $\mathbf{a}$ is an indecomposable vertex cover of G of order b. Recall from [8] that a vector $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ is a vertex cover of G of order b if $a_i + a_j \geq b$ for all i, j with $\{i, j\} \in E(G)$. A vertex cover $\mathbf{a}$ of order b is decomposable if there exists a

Fig. 1 G



vertex cover $\mathbf{a}_1$ of order b_1 and a vertex cover $\mathbf{a}_2$ of order b_2 such that $\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2$ and $b = b_1 + b_2$. The vector $\mathbf{a}$ is called indecomposable if $\mathbf{a}$ is not decomposable.

In [8, Proposition 5.3], it was shown that the graded S-algebra $R_s(I)$ is generated by the monomial $x_1x_2\cdots x_nt^2$ and the monomials $\mathbf{x}^{\mathbf{a}}t$ such that $\mathbf{a}$ is a minimal vertex cover of G if and only if for every odd cycle C of G and for every vertex $i \in [n]$, there exists a vertex $j \in C$ such that $\{i, j\} \in E(G)$. This latter condition characterizes the nearly normally torsionfree edge ideals. Therefore, we can derive the characterization of nearly normally torsionfree edge ideals in terms of their associated symbolic Rees algebras.

Corollary 2.5 *Let G be a graph on the vertex set $[n]$. Then the following statements are equivalent:*

(i) The edge ideal $I(G)$ is nearly normally torsionfree;
(ii) The symbolic Rees algebra $\mathcal{R}_s(I(G))$ is generated as a graded S-algebra by the monomial $x_1x_2\cdots x_nt^2$ and the monomials $\mathbf{x}^{\mathbf{a}}t$ such that $\mathbf{a}$ is a minimal vertex cover of G.

3 Special Odd Cycles and the Associated Primes of Their Powers

In this section, we study the associated prime ideals of the powers of a facet ideal of a special odd cycle.

Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field $\mathbb{K}$ and Δ be a simplicial complex on the vertex set $[n]$ with the facet set $\mathcal{F}(\Delta)$. For every subset $F \subset [n]$, we set $\mathbf{x}_F = \prod_{j\in F} x_j$. Let $I(\Delta) = (\mathbf{x}_F : F \in \mathcal{F}(\Delta)) \subset S$ be the facet ideal of Δ.

A *cycle of length* $r \geq 2$ in Δ is an alternate sequence of distinct vertices and facets

$$\mathcal{C} = (v_1, F_1, v_2, F_2, v_3, \ldots v_r, F_r, v_{r+1} = v_1)$$

such that $v_i, v_{i+1} \in F_i$ for $1 \leq i \leq r$. A cycle $\mathcal{C}$ is called *special* if no facet of it contains more than two vertices of the cycle.

By [7, Theorem 10.3.16], if Δ has no special odd cycle, then the facet ideal $I(\Delta)$ is normally torsionfree, that is, $\mathrm{Ass}(I(\Delta)^m) = \mathrm{Min}(I(\Delta))$ for $m \geq 1$.

In what follows, we study $\mathrm{Ass}^{\infty}(I)$ when $I = I(\Delta)$ and Δ is a special odd cycle.

Theorem 3.1 *Let* $\Delta = (v_1, F_1, v_2, F_2, v_3, \dots v_{2s+1}, F_{2s+1}, v_{2s+2} = v_1)$ *be a special odd cycle with* $s \geq 1$. *Then* $\mathrm{Ass}(I(\Delta)^m) = \mathrm{Min}(I(\Delta))$ *for* $m \leq s$ *and* $\mathrm{Ass}(I(\Delta)^m) = \mathrm{Min}(I(\Delta)) \cup \{(x_{v_1}, \dots, x_{v_{2s+1}})\}$ *for* $m \geq s+1$. *In particular, the facet ideal of a special odd cycle is nearly normally torsionfree.*

Proof Without loss of generality, we may assume that the vertices $v_1, \dots, v_{2s+1}$ of Δ are labeled as $1, \dots, 2s+1$. We set $I = I(\Delta)$ and $P = (x_1, \dots, x_{2s+1})$. We split the proof in three steps.

Step 1. We show that $P \in \mathrm{Ass}(I^m)$ for $m \geq s+1$. First, we show that $P \in \mathrm{Ass}(I^{s+1})$. Let

$$u = x_1 \mathbf{x}_{F_1 \setminus \{1,2\}} x_2 \mathbf{x}_{F_2 \setminus \{2,3\}} x_3 \cdots x_{2s} \mathbf{x}_{F_{2s} \setminus \{2s, 2s+1\}} x_{2s+1} \mathbf{x}_{F_{2s+1} \setminus \{2s+1, 1\}}.$$

We notice that u is the product of the variables x_v, where v is a vertex of Δ. We claim that $u \notin I^{s+1}$. Indeed, if $u \in I^{s+1}$, then there exists $\mathbf{x}_{F_{j_1}}, \dots, \mathbf{x}_{F_{j_{s+1}}} \in I$ with $1 \leq j_1 \leq \cdots \leq j_{s+1} \leq 2s+1$, such that $\prod_{k=1}^{s+1} \mathbf{x}_{F_{j_k}}$ divides u.

If there exists some $1 \leq j \leq 2s+1$ such that $\mathbf{x}_{F_j}^2 \mid u$, then

$$\mathbf{x}_{F_j} \mid x_1 \mathbf{x}_{F_1 \setminus \{1,2\}} x_2 \cdots x_{j-1} \mathbf{x}_{F_{j-1} \setminus \{j-1, j\}} \mathbf{x}_{F_{j+1} \setminus \{j+1, j+2\}} x_{j+2} \cdots x_{2s+1} \mathbf{x}_{F_{2s+1} \setminus \{2s+1, 1\}}.$$

Clearly, x_j does not divide $\mathbf{x}_{F_{j-1} \setminus \{j-1, j\}}$, thus there exists $\ell < j-1$ or $\ell \geq j+1$ such that $F_\ell \ni j$. But this is a contradiction, since Δ is a special cycle, thus no facet may contain more than two vertices of the cycle. Therefore, we may assume that $1 \leq j_1 < \cdots < j_{s+1} \leq 2s+1$, and $\prod_{k=1}^{s+1} \mathbf{x}_{F_{j_k}}$ divides u. If $j_k \geq j_{k-1} + 2$ for all $2 \leq k \leq s+1$, we get $j_1 = 1, j_2 = 3, \dots, j_{s+1} = 2s+1$ and $\prod_{k=1}^{s+1} \mathbf{x}_{F_{2k-1}} \mid u$. Then it follows that $x_1^2 \mid u$, which is false. Therefore, there must be an index j such that $\mathbf{x}_{F_j} \mathbf{x}_{F_{j+1}} \mid u$. This yields

$$\mathbf{x}_{F_{j+1}} \mid x_1 \mathbf{x}_{F_1 \setminus \{1,2\}} x_2 \cdots x_{j-1} \mathbf{x}_{F_{j-1} \setminus \{j-1, j\}} \mathbf{x}_{F_{j+1} \setminus \{j+1, j+2\}} x_{j+2} \cdots x_{2s+1} \mathbf{x}_{F_{2s+1} \setminus \{2s+1, 1\}}$$

which implies that x_{j+1} divides the right side monomial, a contradiction since Δ is a special cycle.

Thus, we have proved that $u \notin I^{s+1}$.

We show that $x_j u \in I^{s+1}$ for $1 \leq j \leq 2s+1$.

We observe that we may write

$$x_1 u = \mathbf{x}_{F_1} \mathbf{x}_{F_3} \cdots \mathbf{x}_{F_{2s-1}} \mathbf{x}_{F_{2s+1}} w$$

for some monomial w. This shows that $x_1u \in I^{s+1}$. Let $j \geq 2$, j even. Then we see that

$$\mathbf{x}_{F_1}\mathbf{x}_{F_3}\cdots\mathbf{x}_{F_{j-1}}\mathbf{x}_{F_j}\mathbf{x}_{F_{j+2}}\cdots\mathbf{x}_{F_{2s}}$$

divides x_ju, which shows that $x_ju \in I^{s+1}$. If $j \geq 2$, j odd, then

$$\mathbf{x}_{F_1}\mathbf{x}_{F_3}\cdots\mathbf{x}_{F_j}\mathbf{x}_{F_{j+1}}\mathbf{x}_{F_{j+3}}\cdots\mathbf{x}_{F_{2s}}$$

divides x_ju, thus $x_ju \in I^{s+1}$. Consequently, we obtained $P \subset I^{s+1} : u$.

For the inclusion $I^{s+1} : u \subset P$, let us consider some monomial $v \in I^{s+1} : u$ and assume that $v \notin P$. This means that none of the variables $x_1, \ldots, x_{2s+1}$ divides v. As $vu \in I^{s+1}$, there exists $G_1, \ldots, G_{s+1} \in \mathcal{F}(\Delta)$ such that $w = \mathbf{x}_{G_1}\cdots\mathbf{x}_{G_{s+1}}$ divides vu. Then

$$\sum_{k=1}^{2s+1} \deg_{x_k}(w) = 2s+2 > 2s+1 = \sum_{k=1}^{2s+1} \deg_{x_k}(vu),$$

contradiction. Therefore, $I^{s+1} : u = P$, which shows that $P \in \operatorname{Ass}(I^{s+1})$.

In order to show that $P \in \operatorname{Ass}(I^m)$ for $m > s+1$, we may argue in a similar way as above, but using the monomial $w = u(\mathbf{x}_{F_1})^{m-s-1}$ instead of u.

Step 2. We show that if $m \leq s$, then $P \notin \operatorname{Ass}(I^m)$. Assume there exists $u \notin I^m$ for some $m \leq s$ such that $P = I^m : u$. Then there exists some vertex j of Δ with $1 \leq j \leq 2s+1$, such that x_j does not divide u. Indeed, let us assume that $x_1x_2\cdots x_{2s+1}$ divides u. But u cannot be divisible by the product of all the variables since, otherwise, we would get $u \in I^s \subseteq I^m$. Then there exists some vertices in $\Delta \setminus \{1, 2, \ldots, 2s+1\}$, say $k_1, \ldots, k_a$, such that the product $x_{k_1}\cdots x_{k_a}$ does not divide u. Then $x_{k_1}\cdots x_{k_a}u$ is divisible by the product of all the variables, thus $x_{k_1}\cdots x_{k_a}u \in I^s \subseteq I^m$ which implies that $x_{k_1}\cdots x_{k_a} \in P$, contradiction. Consequently, we have proved that there exists some vertex j of Δ with $1 \leq j \leq 2s+1$, such that x_j does not divide u.

Let Δ' the simplicial complex obtained from Δ by removing the facets F_{j-1}, F_j which contain the vertex j and $I' = I(\Delta')$. Then Δ' has no special odd cycle, thus $\operatorname{Ass}((I')^m) = \operatorname{Min}(I')$ for all $m \geq 1$. We will show that $(I')^m : u = P'$ where P' is the prime ideal obtained from P by removing the generator x_j. This leads to a contradiction since it implies that $P' \in \operatorname{Ass}((I')^m)$, but, on the other hand, $\{1, 2, \ldots, j-1, j+1, \ldots, 2s+1\}$ is not a minimal vertex cover of Δ'.

Let $i \neq j$, $1 \leq i \leq 2s+1$. Then $x_iu \in I^m$. Since x_iu is not divisible by x_j, it follows that $x_iu \in (I')^m$, hence $P' \subset (I')^m : u$. Let now assume that $P' \subsetneq (I')^m : u$. Then there exists a minimal monomial generator $v \in (I')^m : u$ such that $v \notin P'$. As $vu \in (I')^m \subset I^m$, it follows that $v \in I^m : u = P$. Thus we may write $v = x_j^t w$ where t is a positive integer and w is a monomial which is not divisible by x_j. Then we get $x_j^t wu = uv \in (I')^m$, thus $wu \in (I')^m : x_j^t = (I')^m$. This implies that $w \in (I')^m : u$, a contradiction with the choice of v. Consequently, $P' = (I')^m : u$ and the proof of Step 2 is completed.

Step 3. It remains to show that, for every $m \geq 1$, if Q is an associated prime of I^m different from P, then Q is a minimal prime ideal of I. Since $Q \neq P$, there exists

$1 \leq j \leq 2s+1$ such that $x_j \notin Q$. Let us consider the localization of I with respect to the prime ideal $\mathfrak{p} = (x_i : i \neq j, 1 \leq i \leq n)$. Then $I_\mathfrak{p} = I' S_\mathfrak{p}$ where $I' = I(\Delta')$ and Δ' is generated by the facets

$$F_1, \ldots, F_{j-2}, F_{j-1} \setminus \{j\}, F_j \setminus \{j\}, F_{j+1}, \ldots, F_{2s+1}$$

if $|F_{j-1}|, |F_j| \geq 3$. If $|F_{j-1}| = 2, |F_j| \geq 3$, then Δ' has two connected components, one of them trivial consisting of the facet $\{j-1\}$, and the other one generated by the facets F_i of Δ with $i \neq j-2, j-1, j$ and the face $F_j \setminus \{j\} \in \Delta$. If $|F_{j-1}| \geq 3, |F_j| = 2$, then Δ' consists of the trivial component $\{j+1\}$ and the component generated by the facets F_i of Δ with $i \neq j-1, j, j+1$ and the face $F_{j-1} \setminus \{j\} \in \Delta$. Finally, if $|F_{j-1}| = 2, |F_j| = 2$, then in Δ' we have two trivial components, namely $\{j-1\}, \{j+1\}$ and a non-trivial component determined by the facets F_i of Δ with $i \neq j-2, j-1, j, j+1$.

In the first case, namely if $|F_{j-1}|, |F_j| \geq 3$, Δ' has no special odd cycle, thus I' is normally torsionfree, which implies that Q is a minimal prime of I' and, clearly, of I as well. In all the other cases, by applying Lemma 2.2, we derive that Q is a minimal prime of I which does not contain the vertex j. □

4 Associated Prime Ideals of t-Spread Principal Borel Ideals Generated in Degree 3

The t-spread strongly stable ideals with $t \geq 1$ have been recently introduced in [4] and they represent a special class of square-free monomial ideals.

Fix a field $\mathbb{K}$ and a polynomial ring $S = \mathbb{K}[x_1, \ldots, x_n]$. Let t be a positive integer. A monomial $x_{i_1} x_{i_2} \cdots x_{i_d} \in S$ with $i_1 \leq i_2 \leq \ldots \leq i_d$ is called *t-spread* if $i_j - i_{j-1} \geq t$ for $2 \leq j \leq d$.

A monomial ideal in S is called a *t-spread monomial ideal* if it is generated by t-spread monomials. For example,

$$I = (x_1x_4x_9, x_1x_5x_8, x_1x_5x_9, x_2x_6x_9, x_4x_8) \subset \mathbb{K}[x_1, \ldots, x_9]$$

is a 3-spread monomial ideal, but not 4-spread, because $x_1x_5x_8$ is not a 4-spread monomial.

For a monomial $u \in S$, we set

$$\operatorname{supp}(u) = \{i : x_i \mid u\}.$$

Definition 4.1 [4, Definition 1.1] Let I be a t-spread monomial ideal. Then I is called a *t-spread strongly stable ideal* if for all t-spread monomials $u \in G(I)$, all $j \in \operatorname{supp}(u)$ and all $1 \leq i < j$ such that $x_i(u/x_j)$ is a t-spread monomial, it follows that $x_i(u/x_j) \in I$.

Let t be a positive integer. A monomial ideal $I \subset S = \mathbb{K}[x_1, \ldots, x_n]$ is called *t-spread principal Borel* if there exists a monomial $u \in G(I)$ such that I is the smallest t-spread strongly stable ideal which contains u. According to [4], we denote $I = B_t(u)$. For example, for an integer $d \geq 2$, if $u = x_{n-(d-1)t} \cdots x_{n-t}x_n$, then $B_t(u)$ is minimally generated by all the t-spread monomials of degree d in S.

Let $u = x_{i_1}x_{i_2} \cdots x_{i_d} \in S = \mathbb{K}[x_1, \ldots, x_n]$ be a t-spread monomial. Notice that

$$x_{j_1} \cdots x_{j_d} \in G(B_t(u))$$

if and only if

$$j_1 \leq i_1, \ldots, j_d \leq i_d \text{ and } j_k - j_{k-1} \geq t \text{ for } k \in \{2, \ldots, d\}.$$

In what follows, we study some classes of t-spread principal Borel ideals which are nearly normally torsionfree.

The minimal prime ideals of a t-spread principal Borel ideal were determined implicitly in the proof of [1, Theorem 1.1]. Therefore, in the following result, we reformulate the statement of [1, Theorem 1.1].

Theorem 4.2 *Let $t \geq 1$ be an integer and $I = B_t(u)$, where $u = x_{i_1}x_{i_2} \cdots x_{i_d} \in S$ is a t-spread monomial. We assume that $i_d = n$. Then every associated prime ideal of $B_t(u)$ is of one of the following forms:*

$$(x_1, \ldots, x_{j_1-1}, x_{j_1+t}, x_{j_1+t+1}, \ldots, x_{j_2-1}, x_{j_2+t}, \ldots, x_{j_{d-1}-1}, x_{j_{d-1}+t}, x_{j_{d-1}+t+1}, \ldots, x_n) \quad (1)$$

with $j_l \leq i_l$ for $1 \leq l \leq d-1$ and $j_l - j_{l-1} \geq t$ for $2 \leq l \leq d-1$.

$$(x_1, x_2, \ldots, x_{i_1}) \quad (2)$$

$$(x_1, \ldots, x_{j_1-1}, x_{j_1+t}, x_{j_1+t+1}, \ldots, x_{j_2-1}, x_{j_2+t}, \ldots, x_{j_{s-1}-1}, x_{j_{s-1}+t}, x_{j_{s-1}+t+1}, \ldots, x_{i_s}) \quad (3)$$

with $2 \leq s \leq d-1$, $j_l \leq i_l$ for $1 \leq l \leq s-1$ and $j_l - j_{l-1} \geq t$ for $2 \leq l \leq s-1$.

Example 4.3 Let $u = x_3x_7x_{10} \in \mathbb{K}[x_1, \ldots, x_{10}]$ and $I = B_3(u)$. Then we have

$$\begin{aligned}\text{Ass}(I) = \{&P_1 = (x_7, x_8, x_9, x_{10}), P_2 = (x_4, x_8, x_9, x_{10}), P_3 = (x_4, x_5, x_9, x_{10}),\\ &P_4 = (x_4, x_5, x_6, x_{10}), P_5 = (x_1, x_8, x_9, x_{10}), P_6 = (x_1, x_5, x_9, x_{10}),\\ &P_7 = (x_1, x_5, x_6, x_{10}), P_8 = (x_1, x_2, x_9, x_{10}), P_9 = (x_1, x_2, x_6, x_{10}),\\ &P_{10} = (x_1, x_2, x_3),\\ &P_{11} = (x_4, x_5, x_6, x_7), P_{12} = (x_1, x_5, x_6, x_7), P_{13} = (x_1, x_2, x_6, x_7)\}.\end{aligned}$$

In the same paper, in [1, Corollary 2.6], it was also proved that every t-spread principal Borel ideal satisfies the persistence property.

Let $u = x_ix_n$ be a t spread monomial in S, where t is a positive integer. In [11, Theorem 1.1], it was proved that $\text{Ass}^{\infty}(B_t(u)) = \text{Min}(B_t(u)) \cup (x_1, x_2, \ldots, x_n)$ if

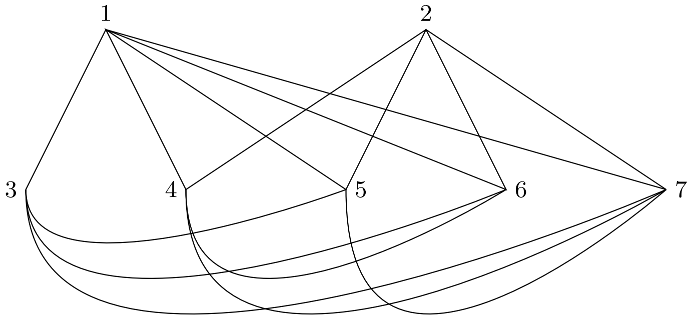


Fig. 2 G

$i \geq t + 1$ and $B_t(u)$ is normally torsionfree if $i = t$. In the following proposition, we give another proof for [11, Theorem 1.1] by using the main result of the first section, Theorem 2.3.

Proposition 4.4 *Let $u = x_i x_n$ be a t-spread monomial in S with $i \geq t$. Then $I = B_t(u)$ is nearly normally torsionfree. Moreover, I is normally torsionfree if and only if $i = t$.*

Proof If $i < t$, then $\cup_{v \in G(I)} \operatorname{supp}(v) = [n] \setminus \{i+1, i+2, \ldots, t\}$ and I is in fact an i-spread principal Borel ideal in the polynomial ring $\mathbb{K}[x_j : j \notin \{i+1, i+2, \ldots, t\}]$. Therefore, we may consider $i \geq t$.

If $i = t$, then

$$I = (x_1 x_{t+1}, x_1 x_{t+2}, \ldots, x_1 x_n, x_2 x_{t+2}, \ldots, x_2 x_n, \ldots, x_t x_{2t}, x_t x_{2t+1}, \ldots, x_t x_n)$$

is the edge ideal of a bipartite graph on the vertex set

$$\{1, 2, \ldots, t\} \cup \{t+1, t+2, \ldots, n\}.$$

Thus, I is normally torsionfree.

If $i > t$, then I is the edge ideal of a connected graph G which contains an odd cycle. For example, if $I = B_2(x_5 x_7)$, then I is the edge ideal of the graph G of Fig. 2.

Let

$$C = (\{i_0, i_1\}, \{i_1, i_2\}, \ldots, \{i_{r-1}, i_0\})$$

be an odd cycle of G. We consider $i_r = i_0$ and $i_{-1} = i_{r-1}$.

In the case that there exists $k \in \{0, 1, \ldots, r-1\}$ such that

$$t + 1 \leq i_{k-1} \leq i, \, i_{k-1} + t \leq i_k \leq i \text{ and } i_k + t \leq i_{k+1}, \tag{4}$$

I is nearly normally torsionfree and $\operatorname{Ass}^{\infty}(I) = \operatorname{Min}(I) \cup \{(x_1, \ldots, x_n)\}$. Indeed, we have

$$x_1x_{i_k}, x_2x_{i_k}, \ldots, x_tx_{i_k}, x_{t+1}x_{i_k}, \ldots, x_{i_{k-1}}x_{i_k} \in I \text{ and}$$

$$1, 2, \ldots, t, t+1, \ldots, i_{k-1} \in N(\{i_k\}) \subset V(C) \cup N(C)$$

because $i_k \geq i_{k-1} + t$ and $t + 1 \leq i_{k-1} \leq i$.

Since $i_{k+1} \geq i_k + t \geq i_{k-1} + 2t$ and $i_{k-1} + t \leq i_k \leq i$, we also have

$$x_{i_{k-1}+1}x_{i_{k+1}}, x_{i_{k-1}+2}x_{i_{k+1}}, \ldots, x_{i_k}x_{i_{k+1}}, x_{i_{k-1}}x_{i_k+1}, x_{i_{k-1}}x_{i_k+2}, \ldots, x_{i_{k-1}}x_n \in I \text{ and}$$

$$i_{k-1}+1, i_{k-1}+2, \ldots, i_k, i_k+1, i_k+2, \ldots, n \in N(\{i_{k+1}\}) \cup N(\{i_{k-1}\}) \subset V(C) \cup N(C).$$

Therefore, $V(C) \cup N(C) = [n]$ and by Theorem 2.3, the proof of this case is completed.

In the case that for any $k \in \{0, 1, \ldots, r-1\}$, the inequalities (4) are not fulfilled, there exists $\ell \in \{0, 1, \ldots, r-1\}$ such that $1 \leq i_\ell \leq t$. Since C is an odd cycle, we may choose ℓ such that

$$t < i_{\ell+1} \leq i \text{ and } i_{\ell+2} \geq i_{\ell+1} + t.$$

In this case,

$$1, 2, \ldots, t, t+1, \ldots, i_{\ell+1} \in N(\{i_{\ell+2}\}) \subset V(C) \cup N(C)$$

because $i_{\ell+2} \geq i_{\ell+1} + t$ and $i_{l+1} \leq i$. We also obtain

$$i_{\ell+1} + t, i_{\ell+1} + t + 1, \ldots, n \in N(\{i_{\ell+1}\}) \subset V(C) \cup N(C).$$

Since $i_{\ell+1} \geq t + i_\ell$,

$$i_{\ell+1} + 1, i_{\ell+1} + 2, \ldots, i_{\ell+1} + t - 1 \in N(\{i_\ell\}) \subset V(C) \cup N(C).$$

Thus, $V(C) \cup N(C) = [n]$ and by Theorem 2.3, I is again nearly normally torsionfree with $\mathrm{Ass}^\infty(I) = \mathrm{Min}(I) \cup \{(x_1, \ldots, x_n)\}$. □

The following result gives a necessary condition for a t-spread principal Borel ideal generated in degree 3 to be nearly normally torsionfree.

Proposition 4.5 *Let t be a positive integer and $I = B_t(u)$, where $u = x_{i_1}x_{i_2}x_n$ is a t-spread monomial with $i_2 \geq 2t + 1$. Then I is not nearly normally torsionfree.*

Proof Since $i_2 \geq 2t + 1$,

$$P_1 = (x_{t+1}, x_{t+2}, \ldots, x_n) \text{ and } P_2 = (x_1, x_2, \ldots, x_{t-1}, x_{2t}, x_{2t+1}, \ldots, x_n)$$

belong to $\mathrm{Ass}(I^2) \setminus \mathrm{Ass}(I)$. Indeed, we have

$$I_{P_1} = B_t(x_{i_2}x_n) \subset S_{P_1} \text{ and } I_{P_2} = B_t(x_{i_2}x_n) \subset S_{P_2},$$

where I_{P_j} is the localization of I with respect to P_j for $j \in \{1, 2\}$. According to [1, Theorem 3.1], we obtain $\operatorname{depth}(S_{P_1}/I_{P_1}^k) = 0$ and $\operatorname{depth}(S_{P_2}/I_{P_2}^k) = 0$ for every $k \geq 2$. It implies that $P_1, P_2 \in \operatorname{Ass}(I^2)$. By Theorem 4.2, $P_1, P_2 \notin \operatorname{Ass}(I) = \operatorname{Min}(I)$. In other words, I is not nearly normally torsionfree. $\square$

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Gröbner–Nice Pairs of Ideals

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Abstract We introduce the concept of a Gröbner nice pair of ideals in a polynomial ring and we present some applications.

Keywords Gröbner basis · Polynomial ring · S-polynomial · Regular sequence

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1 Introduction

One feature of a Gröbner basis is that it extends a system of generators for an ideal in a polynomial ring so that several invariants or algebraic properties are easier to read. It is a natural question to ask how to obtain a Gröbner basis for an ideal obtained by performing basic algebraic operations. A first situation we discuss in this note is when a Gröbner basis for the sum of the ideals I and J is obtained by taking the union of two Gröbner bases for the respective ideals. In that case we say that (I, J) is a Gröbner nice pair (G-nice pair, for short). In Theorem 2.1 we prove that for any given monomial order, (I, J) is a G-nice pair if and only if $\text{in}(I + J) = \text{in}(I) + \text{in}(J)$, which is also equivalent to having $\text{in}(I \cap J) = \text{in}(I) \cap \text{in}(J)$.

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Given the ideals I and J in a polynomial ring, they could be a G-nice pair for some, for any, or for no monomial order. Situations of G-nice pairs of ideals have naturally occurred in the literature, especially related to ideals of minors, e.g. [1, 6, 7].

One application of the new concept is in Corollary 2.7: assume J is any ideal in the polynomial ring S and let $f_1, \dots, f_r$ in S be a regular sequence on S/J. Then the sequence $\mathrm{in}(f_1), \dots, \mathrm{in}(f_r)$ is regular on $S/\mathrm{in}(J)$ if and only if $\mathrm{in}(J, f_1, \dots, f_r) = \mathrm{in}(J) + (\mathrm{in}(f_1), \dots, \mathrm{in}(f_r))$.

We introduced the notion of a Gröbner nice pair in an attempt to unify some cases of distributivity in the lattice of ideals in a polynomial ring S. For instance, one consequence of Proposition 2.8 is that if (J, E), (J, E') and $(J, E \cap E')$ are G-nice pairs, then $(J + E) \cap (J + E') = J + (E \cap E')$ if and only if $\mathrm{in}((J + E) \cap (J + E')) = \mathrm{in}(J) + \mathrm{in}(E \cap E')$. Moreover, in Proposition 2.11 we show that when $(E_i)_{i\in\Lambda}$ is a family of monomial ideals such that (J, E_i) is G-nice for all $i \in \Lambda$, then $(J, \bigcap_{i\in\Lambda} E_i)$ is a G-nice pair and $\bigcap_{i\in\Lambda}(J + E_i) = J + (\bigcap_{i\in\Lambda} E_i)$.

One problem we raise in Sect. 3 is how to efficiently transform a pair of ideals (J, E) into a G-nice pair (J, F) so that $F \supseteq E$. We show that in general there is no minimal such ideal F so that also $J + F = J + E$, see Example 3.4. However, if E is a monomial ideal then we may consider $\widehat{E}$ the smallest monomial ideal in S containing E and so that $(J, \widehat{E})$ is G-nice. Furthermore, when J is a binomial ideal (i.e. it is generated by binomials) then $J + E = J + \widehat{E}$, see Corollary 3.9.

Buchberger's criterion [3–5] asserts that a set of polynomials form a Gröbner basis for the ideal they generate if and only if for any two elements in the set their S-polynomial reduces to zero with respect to the given set. Having this fact in mind, a special class of Gröbner nice pairs is introduced in Sect. 4. Namely, if $\mathscr{G}_J$ is a Gröbner basis for J, then we say that the ideal E is S-nice with respect to $\mathscr{G}_J$ if for all $f \in \mathscr{G}_J$ and $g \in E$ their S-polynomial $S(f, g) \in E$. To check that property it is enough to verify it for g in some Gröbner basis for E, see Proposition 4.4. A good property is that if E_i is S-nice with respect to $\mathscr{G}_J$ for all $i \in \Lambda$, then so is $\bigcap_{i\in\Lambda} E_i$.

This way, given $\mathscr{G}_J$, for any ideal (resp. monomial ideal E) we can define the S-nice (monomial) closure: $\widetilde{E}$ (resp. $E^\sharp$) is the smallest (monomial) ideal which is S-nice w.r.t. $\mathscr{G}_J$ and we show how to compute it in Proposition 4.10. While $J + \widetilde{E} = J + E$, it is not always the case that $J + E^\sharp = J + E$. In Proposition 4.12 we show that if $\mathscr{G}_J$ consists of binomials and E is a monomial ideal, then $\widehat{E} = \widetilde{E}$.

We provide many examples for the notions that we introduce.

2 Gröbner-Nice Pairs of Ideals

Let K be any field. Throughout this paper, we usually denote by S the polynomial ring $K[x_1, \dots, x_n]$, unless it is stated otherwise. For an ideal I in S and a monomial order $\leq$ on S, the set of all Gröbner bases of I with respect to the given monomial order will be denoted $\mathrm{Gröb}_{\leq}(I)$, or simply $\mathrm{Gröb}(I)$ when there is no risk of confusion. As a piece of notation, with respect to a fixed monomial order, for f in S its leading term is denoted $\mathrm{LT}(f)$, and its leading monomial $\mathrm{in}(f)$.

The following result is at the core of our work.

Theorem 2.1 *We fix a monomial order in the polynomial ring S. Let J and E be ideals in S. The following conditions are equivalent:*

(a) $\mathrm{in}(J+E) = \mathrm{in}(J) + \mathrm{in}(E)$*;*
(b) *for any* $\mathscr{G}_J \in$ *Gröb*(J) *and* $\mathscr{G}_E \in$ *Gröb*(E)*, we have* $\mathscr{G}_J \cup \mathscr{G}_E \in$ *Gröb*$(J+E)$*;*
(c) *there exist* $\mathscr{G}_J \in$ *Gröb*(J) *and* $\mathscr{G}_E \in$ *Gröb*(E)*, such that* $\mathscr{G}_J \cup \mathscr{G}_E \in$ *Gröb*$(J+E)$*;*
(d) $\mathrm{in}(J \cap E) = \mathrm{in}(J) \cap \mathrm{in}(E)$*;*
(e) *for any* $f \in J$ *and* $g \in E$*, there exists* $h \in J \cap E$ *such that* $\mathrm{in}(h) = \mathrm{lcm}(\mathrm{in}\, f, \mathrm{in}\, g)$*;*
(f) *for any* $0 \neq h \in J+E$*, there exist* $f \in J$ *and* $g \in E$ *with* $h = f - g$ *and* $\mathrm{in}(f) \neq \mathrm{in}(g)$.

Proof We set $I = J + E$.

(a) $\Rightarrow$ (b): Let $\mathscr{G}_J = \{f_1, \dots, f_r\} \in \mathrm{Gröb}(J)$ and $\mathscr{G}_E = \{g_1, \dots, g_p\} \in \mathrm{Gröb}(E)$. Clearly, $\mathscr{G}_J \cup \mathscr{G}_E$ generates the ideal I. Since $\mathrm{in}(J) = (\mathrm{in}(f_1), \dots, \mathrm{in}(f_r))$ and $\mathrm{in}(E) = (\mathrm{in}(g_1), \dots, \mathrm{in}(g_p))$, by (a), it follows that $\mathrm{in}(I) = (\mathrm{in}(f_1), \dots, \mathrm{in}(f_r), \mathrm{in}(g_1), \dots, \mathrm{in}(g_p))$ and therefore, $\mathscr{G}_J \cup \mathscr{G}_E$ is a Gröbner basis of I.

(b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (a): Let $\mathscr{G}_J = \{f_1, \dots, f_r\} \in \mathrm{Gröb}(J)$ and $\mathscr{G}_E = \{g_1, \dots, g_p\} \in \mathrm{Gröb}(E)$, such that $\mathscr{G}_J \cup \mathscr{G}_E \in \mathrm{Gröb}(J+E)$. It follows that $\mathrm{in}(J+E) = \mathrm{in}(J) + \mathrm{in}(E)$, as required.

(a) $\Rightarrow$ (d): Since always $\mathrm{in}(J \cap E) \subseteq \mathrm{in}(J) \cap \mathrm{in}(E)$, it remains to prove the other inclusion. Let m be any monomial in $\mathrm{in}(J) \cap \mathrm{in}(E)$. Hence there exist $f \in J$ and $g \in E$ such that $m = \mathrm{in}(f) = \mathrm{in}(g)$. If $f = g$ then $m \in \mathrm{in}(J \cap E)$ and we are done.

Otherwise, since $f - g \in J + E$, by (a), $\mathrm{in}(f-g) \in \mathrm{in}(J) + \mathrm{in}(E)$. If $\mathrm{in}(f - g) \in \mathrm{in}(J)$, then there exists $f_1 \in J$ with $\mathrm{LT}(f-g) = \mathrm{LT}(f_1)$, and we set $g_1 = 0$. If $\mathrm{in}(f-g) \in \mathrm{in}(E)$, then there exists $g_1 \in E$ with $\mathrm{LT}(f-g) = \mathrm{LT}(g_1)$, and we set $f_1 = 0$. In either case, we have $\mathrm{in}(f - f_1) = \mathrm{in}(g - g_1) = m$. If $f - f_1 = g - g_1$ we are done. Otherwise, since $\mathrm{in}((f - f_1) - (g - g_1)) < \mathrm{in}(f - g)$, we repeat the above procedure for $f - f_1$ and $g - g_1$. By Dickson's Lemma [4, Theorem 1.9] this process eventually stops, hence $m \in \mathrm{in}(J \cap E)$.

Condition (e) is a restatement of (d).

(e) $\Rightarrow$ (a): It is enough to prove that $\mathrm{in}(J+E) \subseteq \mathrm{in}(J) + \mathrm{in}(E)$. Let $p \in J + E$ and write $p = f - g$, with $f \in J$ and $g \in E$. If $\mathrm{LT}(f) \neq \mathrm{LT}(g)$ then $\mathrm{in}(p) = \mathrm{in}(f)$ or $\mathrm{in}(p) = \mathrm{in}(g)$ and we are done. Assume $\mathrm{LT}(f) = \mathrm{LT}(g)$. By (e), there exists $h \in J \cap E$ with $\mathrm{LT}(h) = \mathrm{LT}(f) = \mathrm{LT}(g)$. We let $f_1 = f - h$ and $g_1 = g - h$. We note that $p = f - g = f_1 - g_1$, $\mathrm{in}(f_1) < \mathrm{in}(f)$ and $\mathrm{in}(g_1) < \mathrm{in}(g)$. If $\mathrm{LT}(f_1) \neq \mathrm{LT}(g_1)$ then $\mathrm{in}(p) \in \mathrm{in}(J) + \mathrm{in}(E)$, arguing as above. Otherwise, we repeat the same procedure for f_1 and g_1 until it stops.

(f) $\Rightarrow$ (a) is obvious.

(d) $\Rightarrow$ (f): Let $h \in I$. We write $h = f - g$ with $f \in J$ and $g \in E$. If $h \in J$ or $h \in E$ then there is nothing to prove.

If $\text{in}(f) \neq \text{in}(g)$, we are done. Otherwise, assume $\text{in}(f) = \text{in}(g)$. We distinguish two cases, depending whether $\text{in}(f) = \text{in}(h)$ or $\text{in}(f) > \text{in}(h)$.

If $\text{in}(h) = \text{in}(f) = \text{in}(g)$, according to (d) we have that $\text{in}(h) \in \text{in}(J) \cap \text{in}(E) = \text{in}(J \cap E)$. Therefore, there exists $w \in J \cap E$ with $\text{LT}(w) = \text{LT}(g)$. We write $h = (f - w) - (g - w)$. Note that $f - w \in J$, $g - w \in E$ and $\text{in}(g - w) < \text{in}(g) = \text{in}(h)$. It follows that $\text{in}(h) = \text{in}(f - w) > \text{in}(g - w)$ so we are done. Note that $g - w \neq 0$, otherwise h would be in J, a contradiction.

If $\text{in}(h) < \text{in}(f) = \text{in}(g) =: m$, since $\text{in}(h) = \text{in}(f - g) < m$, it follows that $\text{LT}(f) = \text{LT}(g)$. Also, $m \in \text{in}(J) \cap \text{in}(E) = \text{in}(J \cap E)$. Let $w \in J \cap E$ with $\text{in}(w) = m$. We can assume $\text{LT}(w) = \text{LT}(f) = \text{LT}(g)$. We write $h = f - g = f_1 - g_1$, where $f_1 = f - w$ and $g_1 = g - w$. Arguing as before, $0 \neq f_1 \in J$, $0 \neq g_1 \in E$, $\text{in}(f_1) < m$ and $\text{in}(g_1) < m$. If $\text{in}(f_1) \neq \text{in}(g_1)$ we are done. If $\text{in}(f_1) = \text{in}(g_1) = \text{in}(h)$ we are done by the discussion of the former case. Otherwise, we apply the same procedure for f_1 and g_1, until it eventually stops. □

Remark 2.2 The equivalence of the conditions (a), (d) and (e) in Theorem 2.1 was proved in a different way by A. Conca in [1, Lemma 1.3] under the extra hypothesis that the ideals J and E are homogeneous.

Definition 2.3 Let S be a polynomial ring with a fixed monomial order. If the ideals J and E of S fulfill one of the equivalent conditions of Theorem 2.1, we say that (J, E) is a *Gröbner nice (G-nice) pair* of ideals.

Remark 2.4 The chosen monomial order is essential. For example, let $J = (x^2 + y^2)$, $E = (x^2)$ and $I = J + E = (x^2, y^2)$ as ideals in $S = K[x, y]$. If $x > y$, then $\text{in}(J) = \text{in}(E) = (x^2)$, but $\text{in}(I) = (x^2, y^2)$. Thus the pair (J, E) is not G-nice. On the other hand, if $y > x$, then $\text{in}(I) = \text{in}(J) + \text{in}(E) = (x^2, y^2)$, hence the pair (J, E) is G-nice.

On the other hand, some pairs of ideals are never G-nice, regardless of the monomial order which is used. Indeed, let $J = (x^2 + y^2)$ and $E = (xy)$ in $S = K[x, y]$. It is easy to see that $S(x^2 + y^2, xy)$ equals either y^3 (if $x^2 > y^2$), or x^3 (if $x^2 < y^2$), and in either case $\text{in}(S(x^2 + y^2, xy)) \notin (\text{in}(J), \text{in}(E))$.

Here are some examples of classes of Gröbner nice pairs of ideals.

Example 2.5 1. If $J \subseteq E \subseteq S$, then the pair (J, E) is G-nice.
2. If J and E are monomial ideals in S, then the pair (J, E) is G-nice.
3. If J and E are ideals in S whose generators involve disjoint sets of variables, then the pair (J, E) is G-nice.
4. (Conca [1]) Let $X = (x_{ij})$ be an $n \times m$ matrix of indeterminates and let Z be a set of consecutive rows (or columns) of X. For t an integer with $1 \leq t \leq \min\{n, m\}$ we let $J = I_t(X)$ be the ideal in $S = K[X]$ generated by the t-minors of X. Also, let $E = I_{t-1}(Z) \subset S$. Then (J, E) is a G-nice pair of ideals, according to [1, Proposition 3.2].
   More generally, [1, Proposition 3.3] states that when Y is a *ladder*, $J = I_t(X)$ and $E = I_{t-1}(Y \cap Z)$, then the pair (J, E) is G-nice. We refer to [1] for the unexplained terminology and further details on this topic.

5. Ideals generated by various minors in a generic matrix are a source of G-nice pairs of ideals, see [6, Lemma 4.2], [7, Lemma 2.10].

The following results characterize situations when one of the ideals of the G-pair is generated by a regular sequence.

Proposition 2.6 *Let J be an ideal in the polynomial ring S and let $f \in S$ which is regular on S/J. The following conditions are equivalent:*

(a) $\operatorname{in}(J, f) = \operatorname{in}(J) + (\operatorname{in}(f))$*;*
(b) $\operatorname{in}(f)$ *is regular on* $S/\operatorname{in}(J)$*.*

Proof We note that since f is regular on S/J we get that

$$\operatorname{in}(J \cap (f)) = \operatorname{in}(fJ) = \operatorname{in}(f)\operatorname{in}(J).$$

By Theorem 2.1(d), property (a) is equivalent to $\operatorname{in}(J \cap (f)) = \operatorname{in}(J) \cap (\operatorname{in}(f))$. That in turn is equivalent to $\operatorname{in}(f)\operatorname{in}(J) = \operatorname{in}(J) \cap (\operatorname{in}(f))$, which is a restatement of the condition (b), since S is a domain. □

Corollary 2.7 *Let J be any ideal in the polynomial ring S and let $f_1, \ldots, f_r$ in S be a regular sequence on S/J. Then the sequence $\operatorname{in}(f_1), \ldots, \operatorname{in}(f_r)$ is regular on $S/\operatorname{in}(J)$ if and only if*

$$\operatorname{in}(J, f_1, \ldots, f_r) = \operatorname{in}(J) + (\operatorname{in}(f_1), \ldots, \operatorname{in}(f_r)).$$

In particular, if $f_1, \ldots, f_r$ is a regular sequence on S, then $\operatorname{in}(f_1), \ldots, \operatorname{in}(f_r)$ is regular on S if and only if $\{f_1, \ldots, f_r\}$ is a Gröbner basis for $(f_1, \ldots, f_r)$.

Proof This follows from Proposition 2.6 by induction on r. □

The G-nice condition is also connected to the distributivity property in the lattice of ideals of S, as the following result shows.

Proposition 2.8 *Let J, E and E' be ideals in the polynomial ring S such that (J, E) and (J, E') are G-nice pairs. The following conditions are equivalent:*

(a) $(J + E) \cap (J + E') = J + (E \cap E')$ *and* $(J, E \cap E')$ *is a G-nice pair of ideals;*
(b) $\operatorname{in}((J + E) \cap (J + E')) = \operatorname{in}(J) + \operatorname{in}(E \cap E')$*.*

Proof (a) $\Rightarrow$ (b) is straightforward.

(b) $\Rightarrow$ (a): We denote $I = J + E$ and $I' = J + E'$. We have that

$$\operatorname{in}(J) + \operatorname{in}(E \cap E') \subseteq \operatorname{in}(J + (E \cap E')) \subseteq \operatorname{in}(I \cap I')$$

and thus, by (b), these inclusions are in fact equalities. In particular, $\operatorname{in}(J) + \operatorname{in}(E \cap E') = \operatorname{in}(J + (E \cap E'))$, hence the pair $(J, E \cap E')$ is G-nice.

On the other hand, since $J + (E \cap E') \subseteq I \cap I'$ and $\operatorname{in}(J + (E \cap E')) = \operatorname{in}(I \cap I')$, it follows that $I \cap I' = J + (E \cap E')$. □

The two parts of condition (a) in Proposition 2.8 are independent, as the following example shows.

Example 2.9 1. Let $J = (x^2 + y^2 + z^2)$, $E = (xy, y^3 + yz^2)$ and $E' = (xy, y^3 + yz^2 + x^2 + y^2 + z^2)$ be ideals in $S = K[x, y, z]$. Then $I = J + E = J + E' = (J, xy)$.
On S we consider the reverse lexicographic monomial order (or revlex, for short) with $x > y > z$. We have $\mathrm{in}(I) = (x^2, xy, y^3)$, $\mathrm{in}(E) = \mathrm{in}(E') = (xy, y^3)$ and one can check with Singular [2] that $\mathrm{in}(E \cap E') = (xy, y^4)$.
Therefore, (J, E) and (J, E') are G-nice pairs of ideals, $(J + E) \cap (J + E') = J + (E \cap E') = I$, but the pair $(J, E \cap E')$ is not G-nice.

2. In $S = K[x, y, z, t]$ let $J = (x^4 + y^3 + z^2, xy^3 - t^2)$, $E = (-y^2zt^2 + xz^2 + t^2, x^2yz^2 - zt^4 + xyt^2, x^2zt^4 + y^4z^2 - x^3yt^2 + yz^4, yzt^6 - xy^2t^4 - x^3z^3 - x^2zt^2, -zt^8 - y^5z^3 + xyt^6 - y^2z^5 - y^3z^2 + x^3t^2 - z^4)$ and $E' = (xz^5 - yzt^2, y^4zt^2 - z^5t^2, y^3z^5 + z^7 + x^3yzt^2, y^2z^5t^2 + y^3z^3t^2 + x^3zt^4, -z^9t^2 - yz^7t^2 - x^3y^2zt^4)$.
We set $I = J + E$ and $I' = J + E'$.
We claim that the inclusion $J + (E \cap E') \subset I \cap I'$ is strict. Indeed, considering the reverse lexicographic order on S with $x > y > z > t$, one can check with Singular [2] that $y^2z^6t^2 \in \mathrm{in}(I \cap I')$, but $y^2z^6t^2 \notin \mathrm{in}(J + (E \cap E'))$. However, one can verify that the pair $(J, E \cap E')$ is G-nice.

The following result is a dual form of Proposition 2.8.

Proposition 2.10 *Let J, E and E' be ideals in the polynomial ring S such that the pairs (J, E) and (J, E') are G-nice. The following conditions are equivalent:*

(a) $J \cap E + J \cap E' = J \cap (E + E')$ *and the pair* $(J, E + E')$ *is* G*-nice.*
(b) $\mathrm{in}(J \cap E + J \cap E') = \mathrm{in}(J) \cap \mathrm{in}(E + E')$.

Proof (a) $\Rightarrow$ (b): We have that $\mathrm{in}(J \cap E + J \cap E') = \mathrm{in}(J \cap (E + E')) = \mathrm{in}(J) \cap \mathrm{in}(E + E')$, since the pair $(J, E + E')$ is G-nice.
(b) $\Rightarrow$ (a): We have that

$$\mathrm{in}(J) \cap \mathrm{in}(E + E') = \mathrm{in}(J \cap E + J \cap E') \subseteq \mathrm{in}(J \cap (E + E')) \subseteq \mathrm{in}(J) \cap \mathrm{in}(E + E'),$$

hence the inequalities in this chain become equalities. It follows that $\mathrm{in}(J \cap (E + E')) = \mathrm{in}(J) \cap \mathrm{in}(E + E')$ and thus, by Theorem 2.1(d), the pair $(J, E + E')$ is G-nice. Also, $\mathrm{in}(J \cap E + J \cap E') = \mathrm{in}(J \cap (E + E'))$, therefore $J \cap E + J \cap E' = J \cap (E + E')$. □

Given E a monomial ideal, we denote by $G(E)$ its unique minimal set of monomial generators. Clearly, $G(E) \in \mathrm{Gröb}(E)$ for any monomial order.

Proposition 2.11 *Let J be any ideal in the polynomial ring S and let $(E_i)_{i \in \Lambda}$ be a family of monomial ideals in S such that the pair (J, E_i) is G-nice for all $i \in \Lambda$. We set $I_i = J + E_i$ for all $i \in \Lambda$, $I = \bigcap_{i \in \Lambda} I_i$ and $E = \bigcap_{i \in \Lambda} E_i$. Then (J, E) is a G-nice pair and $I = J + E$.*
Also, if $\mathscr{G}_J \in \mathrm{Gröb}(J)$ then $\mathscr{G}_J \cup G(E)$ is a Gröbner basis of I.

Proof Since the E_i's are monomial ideals then E is a monomial ideal, too. We have $\mathrm{in}(E_i) = E_i$ for all $i \in \Lambda$ and $\mathrm{in}(E) = E$. Obviously, $\mathrm{in}(J) + E \subseteq \mathrm{in}(J + E)$. On the other hand, $\mathrm{in}(J + E) \subseteq \bigcap_{i\in\Lambda} \mathrm{in}(J + E_i) = \bigcap_{i\in\Lambda}(\mathrm{in}(J) + E_i) = \mathrm{in}(J) + \bigcap_{i\in\Lambda} E_i = \mathrm{in}(J) + E$. Thus, the pair (J, E) is G-nice.

Since $J + E \subseteq I$ and $\mathrm{in}(J + E) = \mathrm{in}(I)$, it follows that $I = J + E$. The last assertion follows immediately. □

The following proposition shows that the G-nice property behaves well with respect to taking sums of ideals. For any positive integer m we denote $[m] = \{1, \ldots, m\}$.

Proposition 2.12 *Let $E_1, \ldots, E_m$ be ideals in S such that (E_i, E_j) is a G-nice pair for all $1 \le i, j \le m$. Let $X \subset [m]$. We denote $E_X = \sum_{i\in X} E_i$ and $E_{X^c} = \sum_{j\in[m]\setminus X} E_j$. Then (E_X, E_{X^c}) is a G-nice pair of ideals.*

Proof For all i we pick a Gröbner basis $\mathcal{G}_i \in \mathrm{Gröb}(E_i)$. We claim that $\mathcal{G}_Y = \bigcup_{i\in Y} \mathcal{G}_i$ is a Gröbner basis of E_Y, for any $Y \subseteq [m]$.

If $|Y| = 1$, there is nothing to prove. Assume $|Y| \ge 2$ and let $f, g \in \mathcal{G}_Y$. If $f, g \in G_i$, for some $i \in Y$, then $S(f, g) \to_{G_i} 0$ and therefore $S(f, g) \to_{G_Y} 0$. If $f \in G_i$ and $g \in G_j$, with $i \ne j$ in Y, then $S(f, g) \to_{G_i\cup G_j} 0$, since $G_i \cup G_j$ is a Gröbner basis of $E_i + E_j$. It follows that $S(f, g) \to_{G_Y} 0$. Thus, G_Y is a Gröbner basis of E_Y, which proves our claim.

In follows that $\mathcal{G}_{[m]} = \mathcal{G}_X \cup \mathcal{G}_{X^c}$ is a Gröbner basis for $E_{[m]} = E_X + E_{X^c}$, and therefore the pair (E_X, E_{X^c}) is G-nice. □

Corollary 2.13 *If J is any ideal and $(E_i)_{i\in\Lambda}$ is a family of monomial ideals, such that the pair (J, E_i) is G-nice for all $i \in \Lambda$, then the pair $(J, \sum_{i\in\Lambda} E_i)$ is G-nice.*

Proof Note that $\sum_{i\in\Lambda} E_i$ can be written as the sum of finitely many terms in the sum. On the other hand, any two monomial ideals form a G-nice pair, so the conclusion follows by Proposition 2.12. □

3 Creating Gröbner-Nice Pairs

Let J be an ideal in S. Given any ideal $E \subset S$ such that (J, E) is not a G-nice pair, we are interested in finding ideals F in S, "close" to E, such that

$$J + F = J + E \text{ and } (J, F) \text{ is a } G\text{-nice pair.} \tag{1}$$

Lemma 3.1 *If (J, F) is a G-nice pair with $E \supseteq F$ so that $J + E = J + F$, then (J, E) is a G-nice pair.*

Proof We have $\mathrm{in}(J + E) = \mathrm{in}(J + F) = \mathrm{in}(J) + \mathrm{in}(F) \subseteq \mathrm{in}(J) + \mathrm{in}(E)$, hence $\mathrm{in}(J + E) = \mathrm{in}(J) + \mathrm{in}(E)$ and (J, E) is a G-nice pair. □

Based on Lemma 3.1, for (1) we should look at ideals $F \supseteq E$. In general, $(J, J + E)$ is a G-nice pair, so in the worst case we may take $F = J + E$. But sometimes, it is also the best choice, as the following example shows.

Example 3.2 Let $J = (x^2 - y^2)$ and $E = (x^2)$. We consider on $S = K[x, y]$ the revlex order with $x > y$. Let $I = J + E$. Then $\mathrm{in}(I) = I = (x^2, y^2)$. Let $F \supseteq E$ be an ideal with $J + F = I$ and $\mathrm{in}(I) = \mathrm{in}(J) + \mathrm{in}(F)$. We claim that $F = I$.

Indeed, since $y^2 \in \mathrm{in}(F)$, there exists $f \in F$ with $\mathrm{LT}(f) = y^2$. It follows that $f = y^2 + ax + by + c$, where $a, b, c \in K$. Since $y^2, f \in I$, we get that $ax + by + c \in I$, and therefore $a = b = c = 0$. Thus $y^2 \in F = I$.

Remark 3.3 For J and E ideals in S, assume the ideal F satisfies (1) and $F \supseteq E$. In order to find an ideal $E' \subseteq S$ such that $E \subseteq E' \subseteq F$ and (J, E') is a G-nice pair, where E' is as small as possible, a natural approach is the following. Set $I = J + E$. We write

$$\mathrm{in}(I) = \mathrm{in}(J) + \mathrm{in}(E) + (m_1, \ldots, m_s),$$

where $m_1, \ldots, m_s$ are the monomials in $G(\mathrm{in}(I)) \setminus (\mathrm{in}(J) + \mathrm{in}(E))$. Since $\mathrm{in}(I) = \mathrm{in}(J + F) = \mathrm{in}(J) + \mathrm{in}(F)$, it follows that $m_1, \ldots, m_s \in \mathrm{in}(F)$. We choose $g_1, \ldots, g_s \in F$ such that $\mathrm{in}(g_i) = m_i$ for $1 \leq i \leq s$. Let $E_1 = E + (g_1, \ldots, g_s)$. Then $J + E_1 = J + E = I$ and

$$\mathrm{in}(I) = \mathrm{in}(J) + \mathrm{in}(E) + (m_1, \ldots, m_s) \subseteq \mathrm{in}(J) + \mathrm{in}(E_1) \subseteq \mathrm{in}(J + E_1) = \mathrm{in}(I).$$

Thus (J, E_1) is a G-nice pair of ideals.

The following example shows that given the ideal I there does not always exist a minimal ideal E' (eventually containing E) such that (J, E') is a G-nice pair with $I = J + E'$.

Example 3.4 (1) We consider the lexicographic order on $S = K[x, y]$ induced by $x > y$. Let $J = (y^3) \subset I = (y^3, x^2 - y^2, xy)$. Then $\mathrm{in}(I) = (y^3, x^2, xy)$. We define the sequence $(\alpha_k)_{k \geq 1}$ by $\alpha_1 = 5$ and $\alpha_{k+1} = 3\alpha_k - 4$ for $k \geq 1$. Let $g_k = xy - y^{\alpha_k}$ and $E_k = (x^2 - y^2, xy - y^{\alpha_k})$. One can easily check that $J + E_k = I$, and $\mathrm{in}(E_k) = (x^2, xy, y^{2\alpha_k - 1})$. Therefore, (J, E_k) is a G-nice pair of ideals for all k.

Since $xg_k - y(x^2 - y^2) = y^3 - xy^{\alpha_k} = y^3 - xyg_k - y^{2\alpha_k - 1}$, it follows that $y^3 - y^{2\alpha_k - 1} \in E_k$. Then $g_{k+1} = g_k + y^{\alpha_k} - y^{3\alpha_k - 4} = g_k + y^{\alpha_k - 3}(y^3 - y^{2\alpha_k - 1})$. Hence $E_k \subsetneq E_{k+1}$ for all k. We also note that $\cap_{k \geq 1} E_k = (x^2 - y^2)$, $(J, (x^2 - y^2))$ is a G-nice pair of ideals, and $J + (x^2 - y^2) \subsetneq I$.

(2) In $S = K[x, y, z]$ ordered lexicographically (with $x > y > x$) we let $J = (x^2 - y^2, z^2)$, $E = (xy)$, $F = (xy, y^3, z^2)$. Denote $I = J + E = J + F = (x^2 - y^2, z^2, xy)$. We note that $\mathrm{in}(I) = (x^2, xy, y^3, z^2) = \mathrm{in}(J) + \mathrm{in}(F)$. Therefore, (J, E) is not a G-nice pair, while (J, F) is one. We set $E_k = (xy, y^3 + z^{k+1} + z^k + \cdots + z^2, z^{k+2})$. Then $E \subsetneq E_{k+1} \subsetneq E_k \subsetneq F$, while (J, E_k) is a G-nice pair for all $k \geq 1$.

We recall the definition of a normal form, see [5].

Definition 3.5 Consider the ideal $J \subset S$ and $\mathcal{G}_J \in \text{Gröb}(J)$. A *normal form* with respect to $\mathcal{G}_J$ is a map $NF(-|\mathcal{G}_J) : S \to S$, which satisfies the following conditions:

(i) $NF(0|\mathcal{G}_J) = 0$;
(ii) if $NF(g|\mathcal{G}_J) \neq 0$ then $\text{in}(NF(g|\mathcal{G}_J)) \notin \text{in}(J)$;
(iii) $g - NF(g|\mathcal{G}_J) = \sum_{f \in \mathcal{G}_J} c_f f$, where $c_f \in S$ and $\text{in}(g) \geq \text{in}(c_f f)$, for all $f \in \mathcal{G}_J$ with $c_f \neq 0$.

Moreover, NF is called a *reduced normal form*, if for any $f \in S$ no monomial of $NF(f|\mathcal{G}_J)$ is contained in $\text{in}(\mathcal{G}_J)$.

Given the ideals J, E in S, $\mathcal{G}_J \in \text{Gröb}(J)$ and $NF(-|\mathcal{G}_J)$ a normal form with respect to $\mathcal{G}_J$, we denote $NF(E|\mathcal{G}_J) = (NF(g|\mathcal{G}_J) : \ g \in E)$.

Proposition 3.6 *With notation as above, we have:*

(i) $J + E = J + NF(E|\mathcal{G}_J)$;
(ii) $(J, NF(E|\mathcal{G}_J))$ *is a G-nice pair of ideals;*
(iii) if NF *is a reduced normal form and* $E = NF(E|\mathcal{G}_J)$*, then any* $h \in J + E$ *can be written as* $h = f + g$*, such that* $f \in J$, $g \in E$ *and no monomial of* g *is contained in* $\text{in}(J)$;
(iv) if $(E_i)_{i \in \Lambda}$ *is a family of ideals with* $E_i = NF(E_i|\mathcal{G}_J)$ *for all* $i \in \Lambda$*, then*

$$NF\left(\bigcap_{i \in \Lambda} E_i|\mathcal{G}_J\right) \subseteq \bigcap_{i \in \Lambda} E_i.$$

Proof (i) : If $h \in J + E$, then $h = NF(h|\mathcal{G}_J) + \sum_{f \in \mathcal{G}_J} c_f f$, and therefore $NF(h|\mathcal{G}_J) \in J + E$.

(ii) : Since, by (i), $J + E = J + NF(E|\mathcal{G}_J)$, it is enough to consider the case $E = NF(E|\mathcal{G}_J)$ and to prove that $\text{in}(J + E) \subseteq \text{in}(J) + \text{in}(E)$. Let $h \in J + E$. Then $h = j + g$ where $j \in J$ and $g \in E$. We also write $g = j_1 + g_1$, where we let $g_1 = NF(g|\mathcal{G}_J) \in NF(E|\mathcal{G}_J) = E$. Thus $h = (j + j_1) + g_1$. If $g_1 = 0$, then $\text{in}(h) \in \text{in}\, J$. Otherwise, by the definition of the normal form, $\text{in}(g) \notin \text{in}(J)$ which implies $\text{in}(h) \in \text{in}(J) + \text{in}(E)$.

(iii) : For any $h \in J + E$, the decomposition $h = (h - NF(h|\mathcal{G}_J)) + NF(h|\mathcal{G}_J)$ satisfies the required condition.

(iv) is straightforward.

Given the ideals $J, E \subseteq S$ we introduce the sets

$$\mathcal{E}_{J,E} = \{F \subseteq S : E \subseteq F, J + E = J + F, (J, F) \text{ is a } G\text{-nice pair of ideals}\},$$
$$\mathcal{E}^m_{J,E} = \{F \in \mathcal{E}_{J,E} : F \text{ is a monomial ideal}\}$$

The previous discussion shows that the set $\mathcal{E}_{J,E}$ may not have a minimal element. However, when E is a monomial ideal and $\mathcal{E}^m_{J,E} \neq \emptyset$, then the latter set has a minimum.

Definition 3.7 Let J be an ideal in S, and E a monomial ideal in S. The G-nice monomial closure of E with respect to J is the (monomial) ideal

$$\widehat{E} = \bigcap_{E \subseteq F,\ F \text{ monomial ideal},\ (J,F) \text{ is } G\text{-nice}} F.$$

The ideal $\widehat{E}$ naturally depends on the ideal J, although this is not reflected in the notation. We prefer not to complicate the notation since it will be clear from the context what J is.

Proposition 3.8 *Let J be any ideal in S, and E a monomial ideal in S. Then $(J, \widehat{E})$ is a G-nice pair. Moreover, if $\mathscr{E}^m_{J,E} \neq \emptyset$, then $\widehat{E}$ is the smallest element in $\mathscr{E}^m_{J,E}$ with respect to inclusion.*

Proof The first part follows from Proposition 2.11. For the second assertion, note that if $F \in \mathscr{E}^m_{J,E}$ then $J + E \subseteq J + \widehat{E} \subseteq J + F$, hence $J + E = J + \widehat{E}$, $\widehat{E} \in \mathscr{E}^m_{J,E}$ and it is its smallest element. □

Corollary 3.9 *With notation as in Proposition 3.8, if J is a binomial ideal, then $J + E = J + \widehat{E}$.*

Proof Note that if b is any binomial in S and m is any monomial in S, then their S-polynomial $S(b, m)$ is a monomial. Also, since J is a binomial ideal, it has a Gröbner basis $\mathscr{G}_J$ consisting of binomials. Therefore, we can define the monomial ideal F which extends E by adding the monomials $S(b, m)$ where $m \in G(E)$ and $b \in \mathscr{G}_J$. Then $F \in \mathscr{E}^m_{J,E}$, and we apply Proposition 3.8. □

The ideal $\widehat{E}$ can be computed as follows.

Remark 3.10 Let $J \subset S$ be an ideal, and let $E \subset S$ be a monomial ideal. Let $F \subset S$ be any monomial ideal such that (J, F) is a G-nice pair and $E \subseteq F$. Let G_0 be the minimal monomial generators of $\text{in}(J + E)$ which are not in $\text{in}(J)$ nor in $\text{in}(E)$. Clearly, $G_0 \subset F$. We let $E_1 = E + (G_0)$. If (J, E_1) is G-nice, then $\widehat{E} = E_1$. Else, we argue as above and we get a chain of monomial ideals $E_1 \subseteq E_2 \subseteq \cdots \subseteq F$. By notherianity, this chain stabilizes at some point $E_i = E_{i+1} = \cdots$ and we get $\widehat{E} = E_i$.

Proposition 3.11 *Let J be a binomial ideal and let $(E_i)_{i\in\Lambda}$ be a family of monomial ideals. Assume $F_i \supseteq E_i$ are monomial ideals such that (J, F_i) is a G-nice pair and $J + E_i = J + F_i$, for all $i \in \Lambda$. Then*

$$\bigcap_{i\in\Lambda}(J + E_i) = J + \bigcap_{i\in\Lambda} \widehat{E}_i = J + \bigcap_{i\in\Lambda} F_i.$$

Proof Using the Corollary 3.9 we have that $J + E_i = J + \widehat{E}_i = F + F_i$ for all $i \in \Lambda$, hence

$$\bigcap_{i\in\Lambda}(J + E_i) = \bigcap_{i\in\Lambda}(J + \widehat{E}_i) = \bigcap_{i\in\Lambda}(J + F_i).$$

On the other hand, by Proposition 3.8, $(J, \widehat{E_i})$ is a G-nice pair for all i. Now using Proposition 2.11 we get that $\bigcap_{i\in\Lambda}(J+\widehat{E_i}) = J + \bigcap_{i\in\Lambda}\widehat{E_i}$ and $\bigcap_{i\in\Lambda}(J+F_i) = J+\bigcap_{i\in\Lambda}F_i$. □

Example 3.12 We consider the revlex order with $x > y > z$ on $S = K[x, y, z]$. Let $J = (x^2 + y^2 + z^2)$ and $E = (xy)$ be ideals in S. Note that $\mathscr{G} = \{x^2 + y^2 + z^2, xy, y^3 + yz^2\}$ is a Gröbner basis of $I = J + E$. Therefore, $\mathrm{in}(I) = (x^2, xy, y^3)$ strictly includes $\mathrm{in}(J) + \mathrm{in}(E) = (x^2, xy)$, and the pair (J, E) is not G-nice.

Let $F \subset S$ be any monomial ideal such that the pair (J, F) is G-nice and $E \subseteq F$. Since $\mathrm{in}(J + E) \subseteq \mathrm{in}(J + F) = (x^2) + F$, it follows that $(xy, y^3) \subset F$. Let $E_1 = (xy, y^3)$. We have $(x^2, xy, y^3, yz^2) = \mathrm{in}(J + E') \subseteq (x^2) + F$. Thus $yz^2 \in F$. Clearly, $E_2 = (xy, y^3, yz^2) \subseteq F$. Since (J, E_2) is a G-nice pair, we conclude that $E_2 = \widehat{E}$.

4 A Special Class of Gröbner-Nice Pairs of Ideals

To verify if a set is a Gröbner basis implies computing the S-polynomial of any two elements in the set and testing if it reduces to zero with respect to the given set, see [3, 4]. Inspired by this, we propose the following.

Definition 4.1 Let J, E be ideals in S and $\mathscr{G}_J \in \mathrm{Gröb}(J)$. We say that E is S-nice with respect to $\mathscr{G}_J$ if for any $f \in \mathscr{G}_J$ and $g \in E$ we have $S(f, g) \in E$.

Example 4.2 If $J \subseteq E$, then E is S-nice with respect to $\mathscr{G}_J$ for any $\mathscr{G}_J \in \mathrm{Gröb}(J)$.

Proposition 4.3 *Assume the ideal E is S-nice with respect to $\mathscr{G}_J \in \mathrm{Gröb}(J)$. Then (J, E) is a G-nice pair of ideals.*

Proof Let $\mathscr{G}_E$ be any Gröbner basis for E. We claim that $\mathscr{G}_E \cup \mathscr{G}_J$ is a Gröbner basis for $E + J$. Indeed, we only need to consider S-polynomials $S(f, g)$ where $f \in \mathscr{G}_J$ and $g \in \mathscr{G}_E$. Since $S(f, g) \in E$ we infer that the former reduces to 0 w.r.t. $\mathscr{G}_J \cup \mathscr{G}_E$. Applying Theorem 2.1(c) finishes the proof. □

Proposition 4.4 *Let J, E be ideals in S and $\mathscr{G}_J \in \mathrm{Gröb}(J)$. The following statements are equivalent:*

(a) the ideal E is S-nice with respect to $\mathscr{G}_J$;
(b) for any $\mathscr{G}_E \in \mathrm{Gröb}(E)$, for any $f \in \mathscr{G}_J$ and $g \in \mathscr{G}_E$ one has that $S(f, g) \in E$;
(c) there exists $\mathscr{G}_E \in \mathrm{Gröb}(E)$ such that for any $f \in \mathscr{G}_J$ and $g \in \mathscr{G}_E$ one has that $S(f, g) \in E$.

Proof The implications (a) ⇒ (b) ⇒ (c) are clear. We suppose (c) holds. Without loss of generality, we may also assume that the polynomials in $\mathscr{G}_J$ and $\mathscr{G}_E$ are monic. Let $f \in \mathscr{G}_J$ and $g \in \mathscr{G}_E$. We can write $g = \sum_{i=1}^{p} u_i g_i$ where $g_i \in \mathscr{G}_E$ and $\mathrm{in}(g) \geq \mathrm{in}(u_i g_i)$ for $i = 1, \dots, p$ and such that $\mathrm{LT}(g) = \mathrm{LT}(u_1 g_1)$. We set $h = g - u_1 g_1$. Then

$$\begin{aligned} S(f,g) &= \frac{\mathrm{lcm}(\mathrm{in}(f),\mathrm{in}(g))}{\mathrm{in}(f)} f - \frac{\mathrm{lcm}(\mathrm{in}(f),\mathrm{in}(g))}{\mathrm{in}(g)} g \\ &= \frac{\mathrm{lcm}(\mathrm{in}(f),\mathrm{in}(g))}{\mathrm{in}(f)} (f-\mathrm{LT}(f)) - \frac{\mathrm{lcm}(\mathrm{in}(f),\mathrm{in}(g))}{\mathrm{in}(g)} (g-\mathrm{LT}(u_1g_1)) \\ &= \frac{\mathrm{lcm}(\mathrm{in}(f),\mathrm{in}(g))}{\mathrm{in}(f)} (f-\mathrm{LT}(f)) - \frac{\mathrm{lcm}(\mathrm{in}(f),\mathrm{in}(g))}{\mathrm{in}(g)} (u_1g_1-\mathrm{LT}(u_1g_1)) \\ &\quad - \frac{\mathrm{lcm}(\mathrm{in}(f),\mathrm{in}(g))}{\mathrm{in}(g)} h. \end{aligned}$$

Note that $u_1g_1 - \mathrm{LT}(u_1g_1) = LT(u_1)(g_1 - \mathrm{LT}(g_1)) + (u_1 - LT(u_1))g_1$. Thus,

$$\begin{aligned} S(f,g) &= \frac{\mathrm{lcm}(\mathrm{in}(f),\mathrm{in}(g))}{\mathrm{in}(f)} (f-\mathrm{LT}(f)) - \frac{\mathrm{lcm}(\mathrm{in}(f),\mathrm{in}(g))}{\mathrm{in}(g_1)} (g_1-\mathrm{LT}(g_1)) \\ &\quad - \frac{\mathrm{lcm}(\mathrm{in}(f),\mathrm{in}(g))}{\mathrm{in}(g)} ((u_1 - LT(u_1))g_1 + h) \\ &= \frac{\mathrm{lcm}(\mathrm{in}(f),\mathrm{in}(g))}{\mathrm{lcm}(\mathrm{in}(f),\mathrm{in}(g_1))} S(f,g_1) - \frac{\mathrm{lcm}(\mathrm{in}(f),\mathrm{in}(g))}{\mathrm{in}(g)} ((u_1 - LT(u_1))g_1 + h). \end{aligned}$$

Since $S(f, g_1), g_1, h \in E$ we obtain that $S(f, g) \in E$, too. This proves statement (a). □

The S-nice property is stable when taking intersections.

Proposition 4.5 *Let J be an ideal in S and $\mathcal{G}_J \in$ Gröb(J). Assume that in the family of ideals $(E_i)_{i\in\Lambda}$ each is S-nice with respect to $\mathcal{G}_J$. Then*

(a) the ideal $\bigcap_{i\in\Lambda} E_i$ is S-nice with respect to $\mathcal{G}_J$;
(b) if (E_i, E_j) is a G-nice pair for all $i, j \in \Lambda$, then $\sum_{i\in\Lambda} E_i$ is S-nice with respect to $\mathcal{G}_J$.

Proof (a): Let $f \in \mathcal{G}_J$ and $g \in \bigcap_{i\in\Lambda} E_i$ Since E_i is S-nice w.r.t. $\mathcal{G}_J$ we get that $S(f, g) \in E_i$ for all $i \in \Lambda$. This proves (a).

(b): For all $i \in \Lambda$ we pick $\mathcal{G}_i \in$ Gröb(E_i). Arguing as in the proof of Proposition 2.12 we get that $\mathcal{G} = \bigcup_{i\in\Lambda} \mathcal{G}_i$ is a Gröbner basis for $\sum_{i\in\Lambda} E_i$. Then for any $f \in \mathcal{G}_J$ and any $g \in \mathcal{G}$ we have that $S(f, g) \in \sum_{i\in\Lambda} E_i$. Conclusion follows by Proposition 4.4. □

An immediate consequence of the previous result is the following related form of Proposition 2.8.

Corollary 4.6 *Let J, E and E' be ideals in S and $\mathcal{G}_J \in$ Gröb(J). Assume that E and E' are S-nice with respect to $\mathcal{G}_J$. Then the following conditions are equivalent:*

(a) $(J + E) \cap (J + E') = J + (E \cap E')$;
(b) $\mathrm{in}((J + E) \cap (J + E')) = \mathrm{in}(J) + \mathrm{in}(E \cap E')$.

In view of Proposition 4.5, for any ideal we can define its S-nice (monomial) closure.

Definition 4.7 Let J be any ideal in S and let $\mathscr{G}_J \in \text{Gröb}(J)$. For any ideal $E \subset S$ we set

$$\widetilde{E} = \bigcap_{E \subseteq F,\ F \text{ is } S\text{-nice w.r.t. } \mathscr{G}_J} F$$

and we call it the S-nice closure of E with respect to $\mathscr{G}_J$.

Moreover, if E is a monomial ideal, we set

$$E^\sharp = \bigcap_{E \subseteq F,\ F \text{ monomial ideal is } S\text{-nice w.r.t. } \mathscr{G}_J} F$$

and we call it the S-nice monomial closure of E with respect to $\mathscr{G}_J$.

As with Definition 3.7, we prefer not to complicate notation and include $\mathscr{G}_J$ in it, as it will be clear from the context the Gröbner basis which is used.

Remark 4.8 By Proposition 4.5, $\widetilde{E}$ (resp. $E^\sharp$) is indeed the smallest ideal (resp. the smallest monomial ideal) in S which is S-nice with respect to $\mathscr{G}_J$. Clearly, $E \subseteq \widetilde{E} \subseteq E^\sharp$, and also $\widehat{E} \subseteq E^\sharp$. By Example 4.2, the ideal $J + E$ is S-nice w.r.t. $\mathscr{G}_J$, hence $J + E \subseteq J + \widetilde{E} \subseteq J + E$. The latter implies that

$$J + E = J + \widetilde{E}.$$

Example 4.9 In $S = K[x, y]$ we consider $J = (x^2 + y^2)$ with $\mathscr{G}_J = \{x^2 + y^2\}$ and $E = (x^2)$. If $x > y$ then $S(x^2 + y^2, x^2) = y^2$, and so $\widetilde{E} = (x^2, y^2) = J + E$. On the other hand, if $y > x$ then $S(x^2 + y^2, x^2) = x^4 \in E$ and thus $\widetilde{E} = E = (x^2)$.

In general, the following proposition is useful for computing $\widetilde{E}$ and $E^\sharp$.

Proposition 4.10 *Let J, E be ideals in S and $\mathscr{G}_J \in \text{Gröb}(J)$. Then*

(a) $\widetilde{E} = \sum_{i \geq 0} E_i$*, where the ideal E_i is defined inductively as $E_0 = E$ and $E_{i+1} = E_i + (S(f, g) : f \in \mathscr{G}_J, g \in E_i)$ for all $i > 0$.*
(b) If E is a monomial ideal, then $E^\sharp = \sum_{i \geq 0} F_i$, where $F_0 = E$ and F_{i+i} is the ideal generated by the monomial terms of the polynomials in $\widetilde{F_i}$, for all $i > 0$.

Proof (a): If $f \in \mathscr{G}_J$ and $g \in G_i$, then $S(f, g) \in E_{i+1}$. Therefore, $\sum_{i \geq 0} E_i$ is S-nice with respect to $\mathscr{G}_J$. Conversely, since $E_0 \subset \widetilde{E}$ and $\widetilde{E}$ is S-nice w.r.t. $\mathscr{G}_J$, it follows that $S(f, g) \in \widetilde{E}$ for any $f \in \mathscr{G}_J$ and $g \in E$. Therefore, $E_1 \subseteq \widetilde{E}$. Inductively, we get that $E_i \subseteq \widetilde{E}$ for any $i \geq 0$. This completes the proof.

(b): If $m \in F_i$ is a monomial and $f \in \mathscr{G}_J$, then $S(f, m) \in \widetilde{F_i} \subseteq F_{i+1}$. Therefore, $\sum_{i \geq 0} F_i$ is S-nice with respect to $\mathscr{G}_J$. Conversely, if $m \in E$ is a monomial and $f \in \mathscr{G}_J$, then $S(f, m) \in E^\sharp$. Since $E^\sharp$ is a monomial ideal,, any monomial which is in the support of $S(f, m)$ is in $E^\sharp$. Therefore, $F_1 \subseteq E^\sharp$. Inductively, we get $F_i \subseteq E^\sharp$ for all $i \geq 0$. Thus $\sum_{i \geq 0} F_i = E^\sharp$. □

Example 4.11 In $S = K[x, y, z]$ we consider the revlex order with $x > y > z$ and the ideals $J = (x^2 + y^2 + z^2)$, $E = (xy)$, $E_1 = (xy, y^3 + yz^2)$ and $E_2 = (xy, y^3 + yz^2 + x^2 + y^2 + z^2)$. From Example 2.9 we have that $J + E_1 = J + E_2 = J + E$, and that (J, E_1), (J, E_2) are G-nice pairs of ideals. Let $\mathscr{G}_J = \{x^2 + y^2 + z^2\} \in$ Gröb(J). We claim that E_1 is S-nice with respect to $\mathscr{G}_J$, but E_2 is not.

We note that $\mathscr{G}_1 = \{xy, y^3 + yz^2\} \in$ Gröb(E_1) and $\mathscr{G}_2 = \{xy, y^3 + yz^2 + x^2 + y^2 + z^2, x^3 + xz^2\} \in$ Gröb(E_2). Also, $S(x^2 + y^2 + z^2, xy) = y^3 + yz^2 \in E_1$ and $S(x^2 + y^2 + z^2, y^3 + yz^2) = y^3(y^2 + z^2) - x^2(y^3 + yz^2) = y^2(y^3 + yz^2) - xy(xy^2 + xz^2) \in E_1$. This proves the claim. Moreover, since $S(x^2 + y^2 + z^2, xy) = y^3 + yz^2$ we infer that, when computed with respect to $\mathscr{G}_J$, $\widetilde{E} = E_1$.

Let U be a monomial ideal in S which is S-nice with respect to $\mathscr{G}_J$, and $E \subseteq U$. Since $S(x^2 + y^2 + z^2, xy) = y^3 + yz^2 \in U$ one has that $y^3, yz^2 \in U$. Let $L = (xy, y^3, yz^2)$. As $S(x^2 + y^2 + z^2, y^3) = y^5 + y^3z^2 \in L$ and $S(x^2 + y^2 + z^2, yz^2) = y^3z^2 + yz^4 \in L$, it follows that L is S-nice with respect to $\mathscr{G}_J$ and moreover, $E^\sharp = L$. We also remark that $J + E \subsetneq J + E^\sharp$.

Proposition 4.12 *Let J be an ideal in S, $\mathscr{G}_J \in$ Gröb(J) and E a monomial ideal in S. Then*

(a) if there exists a monomial ideal F in S which is S-nice with respect to $\mathscr{G}_J$ and $J + E = J + F$, then $J + E^\sharp = J + E$;
(b) if J is a binomial ideal and $\mathscr{G}_J \in$ Gröb(J) consists of binomials, then $\widetilde{E} = E^\sharp$.

Proof Part (a) is clear. For (b) we let $E_0 = E$ and we note that $S(b, m)$ is a monomial, for any binomial b and monomial m. Therefore, using the notation from Proposition 4.10 we obtain an ascending chain of monomial ideals $E_{i+1} = E_i + (S(f, g) : f \in \mathscr{G}_J, g \in E_i)$. This shows that $\widetilde{E} = \sum_{i \geq 0} E_i$ is a monomial ideal and we are done. □

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Veneroni Maps

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Abstract Veneroni maps are a class of birational transformations of projective spaces. This class contains the classical Cremona transformation of the plane, the cubo-cubic transformation of the space and the quarto-quartic transformation of $\mathbb{P}^4$. Their common feature is that they are determined by linear systems of forms of degree n vanishing along $n+1$ general flats of codimension 2 in $\mathbb{P}^n$. They have appeared recently in a work devoted to the so called unexpected hypersurfaces. The purpose of this work is to refresh the collective memory of the mathematical community about these somewhat forgotten transformations and to provide an elementary description of their basic properties given from a modern point of view.

Keywords Cremona transformation · Birational transformation

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1 Introduction

The aim of this note is to give a detailed description of Veneroni's Cremona transformations in $\mathbb{P}^n$. They were first described by Veneroni in [13], and then discussed for $n = 4$ by Todd in [12] and by Blanch in [2] and for $n \geq 3$ by Snyder and Rusk in [11] (with a focus on $n = 5$) and by Blanch again in [1]. The base loci of the Veneroni transformations involve certain varieties swept by lines that were considered for $n = 4$ by Segre in [10] and for $n \geq 3$ by Eisland in [6]. Evolution in terminology and rigor can make it a challenge to study classical papers. Our purpose here is to bring this work together in one place, in a form accessible to a modern audience. In order to use Bertini's Theorem, we assume the ground field $\mathbb{K}$ has characteristic 0.

Consider $n + 1$ distinct linear subspaces $\Pi_0, \ldots, \Pi_n \subset \mathbb{P}^n$ of codimension 2. Let $\mathscr{L}_n$ be the linear system of hypersurfaces in $\mathbb{P}^n$ of degree n containing $\Pi_0 \cup \cdots \cup \Pi_n$ and let $N + 1$ be the vector space dimension of $\mathscr{L}_n$ (we will see that $N = n$ when the Π_j are general, hence by semi-continuity we have $N \geq n$). We denote by $v_n : \mathbb{P}^n \dashrightarrow \mathbb{P}^N$ the rational map given by $\mathscr{L}_n$. If $N = n$ and if in addition v_n is birational, we refer to v_n as a Veneroni transformation. (Here we raise an interesting question: is v_n birational to its image if and only if $N = n$? When $n = 2$, it is not hard to check that $N = n$ always holds and that v_n is always birational.)

When the Π_j are general, we will see that v_n is a Veneroni transformation whose inverse is also given by a linear system of forms of degree n vanishing on $n + 1$ codimension 2 linear subspaces of $\mathbb{P}^n$. In this situation, v_2 is the standard quadratic Cremona transformation of $\mathbb{P}^2$, v_3 is a cubo-cubic Cremona transformation of $\mathbb{P}^3$ (see [4, Example 3.4.3]) and v_4 is a quarto-quartic Cremona transformation of $\mathbb{P}^4$ (see [12]). In [8] the quarto-quartic Cremona transformation was used to produce some unexpected hypersurfaces.

The paper is organized as follows: we start in Sect. 2 with characterizing degree $n - 1$ hypersurfaces in $\mathbb{P}^n$, containing n general linear subspaces of codimension 2.

In Sect. 3 we investigate the linear system giving the Veneroni transformation. When the spaces Π_i are general, we prove that the dimension of $\mathscr{L}_n$ is $n + 1$, we describe the base locus of this system, and prove that v_n is birational.

In Sect. 4 we give the inverse u_n of v_n explicitly and show that u_n is given by a possibly linear subsystem of the linear system of forms of degree n vanishing on a certain set of $n + 1$ codimension 2 linear subspaces.

The last section, Sect. 5, is devoted to the additional description of the intersection of two hypersurfaces of the type described in Sect. 2.

2 Codimension 2 Linear Subspaces

Given linear subspaces $\Lambda_1, \ldots, \Lambda_s$ of $\mathbb{P}^n$, a line intersecting them all is called a *transversal* (for $\Lambda_1, \ldots, \Lambda_s$).

Proposition 2.1 *Let $\Pi_1, \dots, \Pi_{n-1}$ be general codimension 2 linear subspaces of $\mathbb{P}^n$. For every point $p \in \mathbb{P}^n$, there is a transversal for $\Pi_1, \dots, \Pi_{n-1}$ through p. If p is general, then there is a unique transversal, which we denote t_p, and it meets $\Pi_1 \cup \dots \cup \Pi_{n-1}$ in $n-1$ distinct points. If however there are at least two transversals through p, then p lies on a subspace T_p (of dimension $d_p > 1$) intersecting each Π_j along a subspace of dimension $d_p - 1$, $j = 1, \dots, n-1$, and T_p is the union of all transversals for $\Pi_1, \dots, \Pi_{n-1}$ through p.*

Proof Let H be a general hyperplane in $\mathbb{P}^n$ and consider the projection $\pi_p : \mathbb{P}^n \dashrightarrow H$ from $p \in \mathbb{P}^n$. If $p \notin \Pi_j$, let $\Pi'_j = \pi_p(\Pi_j)$ and define

$$\Pi' = \bigcap_{\substack{1 \le j < n \\ p \notin \Pi_j}} \Pi'_j.$$

The intersection Π' is not empty, since each Π'_j is a hyperplane in H and Π' is the intersection of at most $n-1$ hyperplanes in H. Let $q \in \Pi'$. Then the line L_{pq} is transversal to all Π_i (because either $q \in \Pi'_i$, and hence L_{pq} intersects Π_i, or $p \in \Pi_i$). Conversely, a transversal from p intersects Π'. Observe that for a general p, the points $\pi^{-1}(q)|_{\Pi_j}$ are different, so the transversal meets Π_j in different points.

Consequently, for a general p there is a unique transversal. If $\dim \Pi' = k > 0$, then we have a subspace T_p of the transversals of dimension $k+1$. This subspace is a cone over Π' and over $\Pi_j \cap T_p$ as well, hence $\dim \Pi_j \cap T_p = k$. □

Example 2.2 For 3 general codimension 2 linear subspaces Π_1, Π_2, Π_3 of $\mathbb{P}^4$, the pairwise intersections $\Pi_{ij} = \Pi_i \cap \Pi_j$, $i \neq j$, are points. These three points span a plane T which intersects each Π_i in a line. (For Π_1 this line is the line L_{23} through Π_{12} and Π_{13}, and similarly for Π_2 and Π_3.) The lines L_{12}, L_{13}, L_{23} all lie in T, hence every point $p \in T$ has a pencil of transversals, namely the lines in T through p.

Remark 2.3 In the preceding example, not every transversal is in T; this follows from Proposition 2.1. What is more, even if a point p has a pencil of transversals, it need not be true that $p \in T$. Take, for example, a general point $p \in \Pi_1$. The cone on Π_2 with vertex p intersects Π_3 in a line L. Every line through p in the plane spanned by p and L is a transversal, so the general point $p \in \Pi_1$ has a pencil of transversals.

Remark 2.4 We will eventually be interested in $n+1$ general codimension 2 subspaces $\Pi_0, \dots, \Pi_n$ of $\mathbb{P}^n$. They are defined by $2(n+1)$ general linear forms f_{j1}, f_{j2}, $j = 0, \dots, n$, where $I_{\Pi_j} = (f_{j1}, f_{j_2})$. After a change of coordinates we may assume that $f_{j1} = x_j$ and that $f_{j2} = a_{j0}x_0 + \dots + a_{jn}x_n$ with $a_{ji} = 0$ if and only if $i = j$. Here the homogeneous coordinate ring R of $\mathbb{P}^n$ is the polynomial ring $R = \mathbb{K}[\mathbb{P}^n] = \mathbb{K}[x_0, \dots, x_n]$.

Now, we establish existence and uniqueness of a hypersurface Q of degree $n-1$ containing n general codimension 2 linear subspaces in $\mathbb{P}^n$ for $n \ge 2$.

Proposition 2.5 *Let $\Pi_1, \ldots, \Pi_n$ be general codimension 2 linear subspaces of $\mathbb{P}^n$. Then there exists a unique hypersurface Q of degree $n-1$ containing Π_j for $j = 1, \ldots, n$. Moreover, Q is reduced and irreducible, it is the union of the transversals for $\Pi_1, \ldots, \Pi_n$, and for each point $q \in Q$ we have $\operatorname{mult}_q Q \geq r$, where r is the number of indices i such that $q \in \Pi_i$. If q is a general point of Q, then there is a unique transversal for $\Pi_1, \ldots, \Pi_n$ through q.*

Proof Let Δ be the determinantal variety in $(\mathbb{P}^n)^{n+1}$ of all $(n+1) \times (n+1)$ matrices M of rank at most 2 whose entries are the variables x_{ij}. It is known that Δ is reduced and irreducible of dimension $3n-1$, see [9]. It consists of the locus of points $(p_1, \ldots, p_{n+1})$ whose span in $\mathbb{P}^n$ is contained in a line.

Let $\pi_i : (\mathbb{P}^n)^{n+1} \to \mathbb{P}^n$ be projection to the ith factor (so $1 \leq i \leq n+1$). Now, for $1 \leq i \leq n$, let $\Pi_i' = \pi_i^{-1}(\Pi_i)$. Then $D = \Delta \cap \bigcap_{1 \leq i \leq n} \Pi_i'$ has dimension $(3n-1) - 2n = n-1$. Indeed, Δ is irreducible, thus intersection with a divisor (preimage of a form by π_j) drops the dimension by one (Δ does not lie in one summand, hence cannot lie in the preimage). By Bertini we can do this again and again ($2n$ times, the dimension drops by 2 for every Π_j). We see that D is reduced and irreducible. Since $\Pi_1 \cap \cdots \cap \Pi_n = \varnothing$, we see that D is the locus of all points $(p_1, \ldots, p_{n+1})$ such that the span $\langle p_1, \ldots, p_n \rangle$ is a line with $p_i \in \Pi_i$ for $1 \leq i \leq n$ and p_{n+1} being on that line. Thus $\overline{D} = \pi_{n+1}(D)$ is irreducible, properly contains $\Pi_1 \cup \cdots \cup \Pi_n$ and is the union of all transversals for $\Pi_1, \ldots, \Pi_n$. (To get Π_j in the image of the last projection, take a point p in Π_j, take a general line ℓ through p, and $(\ell \cap \Pi_1, \ell \cap \Pi_2, \ldots, \ell \cap \Pi_j = p, \ell \cap \Pi_{j+1}, \ldots, \ell \cap \Pi_n, p)$ lies in D and projects to $p \in \overline{D}$).

In particular, $\overline{D}$ has dimension $n-1$, and since by Proposition 2.1 there is a line through a general point meeting $n-1$ of the spaces Π_i in distinct points, we see that $\deg \overline{D} \geq n-1$.

Below we will check that there is a hypersurface Q of degree $n-1$ containing $\Pi_1 \cup \cdots \cup \Pi_n$. Since any such hypersurface must by Bezout contain all transversals for $\Pi_1, \ldots, \Pi_n$, we see that $\deg \overline{D} = n-1$ and $Q = \overline{D}$ and thus that there is a unique hypersurface of degree $n-1$ containing $\Pi_1 \cup \cdots \cup \Pi_n$, and it is irreducible.

To show existence of Q we follow [2]. As mentioned in Remark 2.4, we may assume that the ideal of Π_k is

$$I_k = (x_k, f_k = a_{k,0}x_0 + \cdots + a_{k,k-1}x_{k-1} + a_{k,k+1}x_{k+1} + \cdots + a_{k,n}x_n),$$

where we write f_k instead of $f_{k,2}$. By generality we may assume that $a_{i,j} \neq 0$ for all $i \neq j$.

Now consider the $n \times n$ matrix

$$A = \begin{pmatrix} -f_1 & \ldots & a_{1,k}x_k & \ldots & a_{1,n}x_n \\ \vdots & & \vdots & & \vdots \\ a_{k,1}x_1 & \ldots & -f_k & \ldots & a_{k,n}x_n \\ \vdots & & \vdots & & \vdots \\ a_{n,1}x_1 & \ldots & a_{n,k}x_k & \ldots & -f_n \end{pmatrix}$$

and let $F = \det(A)$. Note that F is not identically 0 (since its value at the point $(1, 0, \dots, 0)$ is not 0) so $\deg F = n$. It is clear, developing $\det(A)$ with respect to the k-th column, that $F \in I_k$ for every $k = 1, \dots, n$. For each k, adding to the kth column of A all of the other columns of A gives a matrix A_k whose entries in the kth column are nonzero scalar multiples of x_0; in particular,

$$A_k = \begin{pmatrix} -f_1 & \dots & a_{1,k-1}x_{k-1} & -a_{1,0}x_0 & a_{1,k+1}x_{k+1} & \dots & a_{1,n}x_n \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{k-1,1}x_1 & \dots & -f_{k-1} & -a_{k-1,0}x_0 & a_{k-1,k+1}x_{k+1} & \dots & a_{k-1,n}x_n \\ a_{k,1}x_1 & \dots & a_{k,k-1}x_{k-1} & -a_{k,0}x_0 & a_{k,k+1}x_{k+1} & \dots & a_{k,n}x_n \\ a_{k+1,1}x_1 & \dots & a_{k+1,k-1}x_{k-1} & -a_{k+1,0}x_0 & -f_{k+1} & \dots & a_{k+1,n}x_n \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n,1}x_1 & \dots & a_{n,k-1}x_{k-1} & -a_{n,0}x_0 & a_{n,k+1}x_{k+1} & \dots & -f_n \end{pmatrix}.$$

so

$$A_1 = \begin{pmatrix} -a_{1,0}x_0 & a_{1,2}x_2 & \dots & a_{1,k}x_k & \dots & a_{1,n}x_n \\ -a_{2,0}x_0 & -f_2 & \dots & a_{2,k}x_k & \dots & a_{2,n}x_n \\ \vdots & \vdots & & \vdots & & \vdots \\ -a_{k,0}x_0 & a_{k,2}x_2 & \dots & -f_k & \dots & a_{k,n}x_n \\ \vdots & \vdots & & \vdots & & \vdots \\ -a_{n,0}x_0 & a_{n,2}x_2 & \dots & a_{n,k}x_k & \dots & -f_n \end{pmatrix}.$$

Thus $F = \det(A) = \det(A_k) = x_0 \cdot G$ for some polynomial G. Since x_0 is not an element of any I_k, it follows that $G \in I_k$ for $k = 1, \dots, n$, hence G vanishes on each of $\Pi_1, \dots, \Pi_n$. Since $\deg F = n$, we have $\deg(G) = n - 1$. Thus G defines a hypersurface Q of degree $n - 1$ containing each Π_i.

Now consider a point $q \in Q$. The matrix A_k will have r columns which vanish at q, where r is the number of indices i such that $q \in \Pi_i$. In particular, each entry in each such column is in the ideal I_q. Thus $G = \det(A_k)/x_0 \in I_q^r$ so $\mathrm{mult}_q\, Q \geq r$.

Finally assume p is a general point of Π_n. Since Π_n is general, p is a general point of $\mathbb{P}^n$, hence by Proposition 2.1 there is a unique transversal t_p for $\Pi_1, \dots, \Pi_{n-1}$ through p, hence t_p is also the unique transversal for $\Pi_1, \dots, \Pi_n$ through p. Thus there is an open neighborhood U of p of points q through each of which there is a unique transversal t_q for $\Pi_1, \dots, \Pi_{n-1}$, and for those points q of $U \cap Q$, t_q also meets Π_n, hence for a general point $q \in Q$ there is a unique transversal t_q for $\Pi_1, \dots, \Pi_n$. □

Remark 2.6 Let $p_0, \dots, p_n$ be the coordinate vertices of $\mathbb{P}^n$ with respect to the variables $x_0, \dots, x_n$, so $p_0 = (1, 0, \dots, 0), \dots, p_n = (0, \dots, 0, 1)$. We saw in the proof of Proposition 2.5 that $p_0 \notin Q$ (since $F \neq 0$ at p_0). Let A'_k be the matrix from the proof of Proposition 2.5 arising after dividing x_0 from column k of A_k. Then Q is defined by $\det(A'_k) = 0$ but A'_k at p_k is a matrix which, except for column k, is a diagonal matrix with nonzero entries on the diagonal, and whose kth column has no zero entries. Thus $\det(A'_k) \neq 0$ at p_k so $p_k \notin Q$. In particular, none of the coordinate vertices is on Q.

3 The System $\mathscr{L}_n$

Let us start with some notation. Assume $\Pi_0, \ldots, \Pi_n \subset \mathbb{P}^n$ are general linear subspaces of codimension 2. From the previous section it follows that for each subset $\Pi_0, \ldots, \Pi_{j-1}, \Pi_{j+1}, \ldots, \Pi_n$ of n of them there is a unique hypersurface Q_j of degree $n-1$ containing them. Depending on the context, we may also denote by Q_j the form defining this hypersurface. We may assume $I_{\Pi_i} = (x_i, f_i)$ where f_j is as given in Remark 2.4. In this case we have the $(n+1) \times (n+1)$ matrix

$$B = \begin{pmatrix} -f_0 & a_{0,1}x_1 & \ldots & a_{0,k}x_k & \ldots & a_{0,n}x_n \\ a_{1,0}x_0 & -f_1 & \ldots & a_{1,k}x_k & \ldots & a_{1,n}x_n \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{k,0}x_0 & a_{k,1}x_1 & \ldots & -f_k & \ldots & a_{k,n}x_n \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n,0}x_0 & a_{n,1}x_1 & \ldots & a_{n,k}x_k & \ldots & -f_n \end{pmatrix}.$$

Let B_i be the $n \times n$ submatrix obtained by deleting row i and column i of B (where we have i run from 0 to n). The matrix A in the proof of Proposition 2.5 is thus B_0, and we have $\det(B_i) = x_i Q_i$. The next result shows that v_n is the map given by $(x_0, \ldots, x_n) \mapsto (x_0 Q_0, \ldots, x_n Q_n)$.

Proposition 3.1 *The polynomials $x_i Q_i$, $i = 0, \ldots, n$, give a basis for $\mathscr{L}_n$, hence* $\dim \mathscr{L}_n = n+1$*, so v_n is a rational map to $\mathbb{P}^n$ whose image is not contained in a hyperplane.*

Proof By Remark 2.6, no coordinate vertex p_j is in Q_i for any i. But $x_i Q_i \in \mathscr{L}_n$ for every i, and $(x_i Q_i)(p_j) \neq 0$ if and only if $i = j$. Thus the polynomials $x_i Q_i$ span a vector space of dimension at least $n+1$.

To show that these sections in fact give a basis, we show that $\dim \mathscr{L}_n = n+1$. We proceed by induction (the proof that $\mathscr{L}_2$ has three independent sections is clear, since three general points impose independent conditions on forms of degree 2 on $\mathbb{P}^2$). Let A be a fixed hyperplane that contains Π_1. There is, by Proposition 2.5, a unique section of $\mathscr{L}_n$ containing A, namely AQ_1. Moreover, the restrictions to A of sections s_n of $\mathscr{L}_n$ which do not contain A give divisors $s_n \cap A$ of degree n, containing Π_1, and containing $A \cap \Pi_j$, $j > 1$. So on A, the linear system of restrictions residual to Π_1 has degree $n-1$ and contains the n general subspaces $\Pi_i \cap A$, $i > 1$, of codimension 2. From the inductive assumption this has dimension n, so $\dim \mathscr{L}_n = n+1$.

We may also see the result from the exact sequence

$$0 \to \mathscr{L}_n(-A) \to \mathscr{L}_n \to \mathscr{L}_n|_A \to 0,$$

where A is as above and $\mathscr{L}_n(-A)$ is the linear subsystem of all elements of $\mathscr{L}_n$ containing A. Then, from the inductive assumption, the dimension of $\mathscr{L}_n|_A$ is n, and from Proposition 2.5 the dimension of $\mathscr{L}_n(-A)$ (which is of degree $n-1$ passing through n codimension 2 subspaces in A) is 1. □

Remark 3.2 If the hyperplanes $H_j \supset \Pi_j$, $j = 0, \ldots, n$ are such that any n of them intersect in a point outside all Q_i and $\bigcap_j H_j = \emptyset$, then $H_j Q_j$ are linearly independent.

Proof If this is not the case, then one of them is linearly dependent of others, let it be $H_0 Q_0$. Thus, if $H_j Q_j$ vanish in some point p for $j = 1, \ldots, n$, then $H_0 Q_0$ also does. Let then $p = \bigcap_{j=1}^n H_j$. Thus, $H_0 Q_0$ vanishes on p, but $p \notin H_0$, so $p \in Q_0$, a contradiction. □

Remark 3.3 Observe also, that up to an isomorphism of (the target) $\mathbb{P}^n$, the map v_n may be defined by any set of $n+1$ linearly independent elements of $\mathscr{L}_n$.

Let T_n be the closure of the union of all lines transversal to $\Pi_0, \ldots, \Pi_n$, and let $R_n = Q_0 \cap \cdots \cap Q_n$ and let B_n be the base locus of $\mathscr{L}_n$ (i.e., the locus where $v_n : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is not defined). We note that $T_n \subseteq R_n$, by Proposition 2.5.

Proposition 3.4 *We have* $B_n = \Pi_0 \cup \cdots \cup \Pi_n \cup R_n$.

Proof Since v_n is given by $(x_0, \ldots, x_n) \mapsto (x_0 Q_0, \ldots, x_n Q_n)$, the base locus consists of the common zeros of the $x_i Q_i$. Clearly each Q_i (and hence each $x_i Q_i$) vanishes on R_n (as R_n is the intersection of all Q_i.) But Q_i vanishes on Π_j for $j \neq i$ and x_i vanishes on Π_i, so each $x_i Q_i$ vanishes on $\Pi_0 \cup \cdots \cup \Pi_n$. Thus $\Pi_0 \cup \cdots \cup \Pi_n \cup R_n \subseteq B_n$.

Conversely, let p be a point in B_n not in $\Pi_0 \cup \cdots \cup \Pi_n$. By Remarks 3.2 and 3.3, v_n may be defined by the forms $H_i Q_i$ for sufficiently general H_i. Since H_i does not vanish on p, Q_i does for all i. Thus $p \in R_n$, so $B_n \subseteq \Pi_0 \cup \cdots \cup \Pi_n \cup R_n$. □

Proposition 3.5 *We have* $\dim T_n = n-2$ *for* $n \geq 3$*, and* T_n *is irreducible for* $n > 3$.

Proof Consider the Grassmannian V of lines in $\mathbb{P}^n$ and the incidence variety $W = \{(v, p) \in V \times \mathbb{P}^n : p \in L_v\}$, where L_v is the line corresponding to a point $v \in V$. We also have the two projections $\pi_1 : W \to V$ and $\pi_2 : W \to \mathbb{P}^n$. Then V is an irreducible variety of dimension $2(n-1)$ and degree $\frac{(2(n-1))!}{n!(n-1)!}$ embedded in $\mathbb{P}^N$, $N = \binom{n+1}{2} - 1$, see [7], Chap. 1, Sect. 5. The condition of being incident to a codimension 2 linear space is given by a hyperplane in $\mathbb{P}^N$ (see p. 128 in [3]), so the intersections of V with $n+1$ general hyperplanes gives the locus ρ_n in V parametrizing the lines comprising T_n; notice that $\pi_2(\pi_1^{-1}(\rho_n)) = T_n$. Thus $\dim \rho_n = 2(n-1) - (n+1) = n-3$, so $\dim \pi_1^{-1}(\rho_n) = n-2$, and by Proposition 2.1 the projection π_2 is generically injective on $\pi_1^{-1}(\rho_n)$ so we have $\dim T_n = n-2$. Moreover, by Bertini's Theorem, ρ_n (and hence T_n) is irreducible when $\dim \rho_n > 0$. □

Proposition 3.6 *With the notation as above we have* $T_n = R_n$ *in* $\mathbb{P}^n$.

Proof Let us start with the following fact. Let $L_0, \dots, L_k, L$ be lines through a common point p. Let L belong to the space spanned by $L_0, \dots, L_k$, let $\mathscr{P}$ be a linear subspace, such that p does not lie on $\mathscr{P}$. Let L_j intersect $\mathscr{P}$ at a point l_j, $j = 0, \dots, k$. Then L intersects $\mathscr{P}$, as the linear combination of a projection of some vectors is a projection of the combination.

Now we can show that the intersection of all Q_j lies in T_n, the union of all transversals. Observe, that the opposite inclusion is obvious.

Take a point p in all Q_j, but not in any Π_j. So for each j, there is L_j through p, transversal to all Q_i except Q_j. We have $n + 1$ such lines, but they must span a space of dimension less than $n + 1$ (being in $\mathbb{P}^n$).

Without loss of generality, let L_0 belong to the space spanned by the others. Then using the fact we started with, for $\mathscr{P} = \Pi_0$, we get that L_0 intersects Π_0 (since $L_1, \dots, L_n$ intersect Π_0), which finishes the proof.

If $p \in \Pi_j$ for some j, the proof is trivial. □

Proposition 3.7 *The Veneroni transformation $v_n : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is injective off $Q_0 \cup \dots \cup Q_n$, hence it is a Cremona transformation.*

Proof Let p, q be two different points off $Q_0 \cup \dots \cup Q_n$. Let H_j denote the unique hyperplane through p and Π_j. Then $\bigcap_{j=0}^n H_j = \{p\}$ as if the intersection of all such H_i is not exactly p, then the intersection $H_0 \cap \dots \cap H_n$ is a positive dimensional linear space, and any line through p in this space intersects each Π_i and hence is a transversal for $\Pi_0, \dots, \Pi_n$, and so p, being on a transversal, is in $T_n \subseteq R_n \subseteq B_n$. Take j_0 such that $q \notin H_{j_0}$. Then $H_{j_0} Q_{j_0}$ is a non-zero section of $\mathscr{L}_n$ and may be extended to a basis of $\mathscr{L}_n$. Then v_n defined by the sections of this basis separates p and q. Thus v_n is injective off $Q_0 \cup \dots \cup Q_n$. □

4 An Inverse for v_n

It is of interest to determine an inverse for v_n, and to observe that the inverse is again given by forms of degree n vanishing on $n + 1$ codimension 2 linear subspaces. We explicitly define such a map u_n and then check that it is an inverse for $v_n : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. If we regard $x_0, \dots, x_n$ as homogeneous coordinates on the source $\mathbb{P}^n$ and $y_0, \dots, y_n$ as homogeneous coordinates on the target $\mathbb{P}^n$, then v_n is defined by the homomorphism h on homogeneous coordinate rings given by $h(x_0, \dots, x_n) = (y_0, \dots, y_n)$, where $y_i = x_i Q_i = \det(B_i)$, as we saw in Sect. 3.

To define u_n, we slightly modify matrix B from Sect. 3 by replacing the diagonal entries $-f_i$ in B by $-g_i$ (defined below) and by replacing each entry $a_{i,j} x_j$ in B by $a_{i,j} y_j$ to obtain a new matrix

$$C = \begin{pmatrix} -g_0 & a_{0,1}y_1 & \dots & a_{0,k}y_k & \dots & a_{0,n}y_n \\ a_{1,0}y_0 & -g_1 & \dots & a_{1,k}y_k & \dots & a_{1,n}y_n \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{k,0}y_0 & a_{k,1}y_1 & \dots & -g_k & \dots & a_{k,n}y_n \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n,0}y_0 & a_{n,1}y_1 & \dots & a_{n,k}y_k & \dots & -g_n \end{pmatrix}.$$

To define g_i, recall that since $f_i Q_i \in \mathscr{L}_n$ for each i and the forms $x_j Q_j$ give a basis for $\mathscr{L}_n$, we can for each i and appropriate scalars $b_{i,j}$ write

$$f_i Q_i = b_{i,0}x_0 Q_0 + \cdots + b_{i,n}x_n Q_n.$$

We define g_i to be $g_i = b_{i,0}y_0 + \cdots + b_{i,n}y_n$, so we see that $h(g_i) = f_i Q_i$.

As an aside we also note that $b_{i,j} = 0$ if and only if $i = j$. (To see this, recall by Remark 2.6 that no Q_j vanishes at any coordinate vertex p_k, but f_i vanishes at the coordinate vertex p_j if and only if $i = j$. Thus, evaluating $f_i Q_i = b_{i,0}x_0 Q_0 + \cdots + b_{i,n}x_n Q_n$ at p_i gives $0 = b_{i,i} Q_i$, hence $b_{i,i} = 0$, while evaluating at p_j for $j \neq i$ gives $0 \neq b_{i,j} Q_j$, hence $b_{i,j} \neq 0$.)

Let C_i be the matrix obtained from C by deleting row i and column i. Define a homomorphism $\lambda : \mathbb{K}[x_0, \dots, x_n] \to \mathbb{K}[y_0, \dots, y_n]$ by $\lambda(x_i) = \det(C_i)$.

The next result gives an inverse for v_n.

Proposition 4.1 *The homomorphism λ defines a birational map $u_n : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ which is inverse to v_n.*

Proof Note that applying h to the entries of C gives the matrix obtained from BD, where D is the diagonal matrix whose diagonal entries are $Q_0, \dots, Q_n$, from which it is easy to see that $h(\det(C_i)) = \det(B_i) Q_0 \cdots Q_{i-1} Q_{i+1} \cdots Q_n = x_i Q_0 \cdots Q_n$.

We now have $h(\lambda(x_i)) = h(\det(C_i)) = x_i Q_0 \cdots Q_n$, so $u_n v_n = id_U$, where U is the complement of $Q_0 \cdots Q_n = 0$. Since v_n is a Cremona transformation, so is u_n and thus u_n is the inverse of v_n. □

Remark 4.2 We now confirm that the forms $\det(C_i)$ defining u_n have degree n and vanish on $n + 1$ codimension 2 linear subspaces $\Pi_i^* \subset \mathbb{P}^n$. That $\deg(\det(C_i)) = n$ is clear, since C_i is an $n \times n$ matrix of linear forms.

Consider the codimension two linear spaces defined by the ideals $J_k = (y_k, g_k) = b_{k,0}y_0 + \cdots + b_{k,n}y_n$. Since the entries of column k of C are in the ideal J_k, it follows that $\det(C_i)$ vanishes on Π_j^* for $j \neq i$. It remains to check that $\det(C_i)$ vanishes on Π_i^*. But let $q \in Q_i$ be a point where v_n is defined. Note that $y_i(v_n(q)) = h(y_i)(q) = x_i Q_i(q) = 0$ and that $g_i(v_n(q)) = h(g_i)(q) = f_i Q_i(q) = 0$. Thus $v_n|_{Q_i}$ gives a rational map to Π_i^* whose image is in the zero locus of $\det(C_i)$ since $\det(C_i)(v_n(q)) = (h(\det(C_i)))(q) = h\lambda(x_i)(q) = (x_i Q_0 \cdots Q_n)(q) = 0$. Thus $\det(C_i)$ vanishing on Π_i^* will follow if we show that $v_n|_{Q_i}$ gives a dominant rational map to Π_i^*. This in turn will follow if we show for a general $q \in Q_i$ that the fiber over $v_n(q)$ has dimension 1 (since Q_i as dimension $n - 1$ and Π_i^* has dimension $n - 2$). But the space of forms in $\mathscr{L}_n$ vanishing on q is spanned by forms of the form $H_j Q_j$ where H_j is a hyperplane containing q and Π_j. For a general point q, since the Π_j are general, the intersection of any $n-1$ of the H_j with $j \neq i$ has dimension 1. Since the Π_j are general, the same is true for a general point $q \in Q_i$ except now, since there is a transversal t_q through q for Π_j, $j \neq i$, we see that $\cap_{j\neq i} H_j$ still has dimension 1 and is thus exactly t_q. Hence the locus of points on which the forms in $\mathscr{L}_n$ vanishing at q vanish is exactly t_q. Thus the fiber over $v_n(q)$ has dimension 1, as we wanted to show.

It is still unclear to us whether u_n is itself a Veneroni transformation whenever v_n is. If we denote by $\mathscr{L}_n^*$ the forms in $\mathbb{K}[y_0, \dots, y_n]$ of degree n vanishing on $\Pi_0^* \cup \cdots \cup \Pi_n^*$, what we saw above is that u_n is defined by an $n+1$ dimensional linear system contained in $\mathscr{L}_n^*$; the issue is whether the linear system is all of $\mathscr{L}_n^*$ (i.e., whether $\dim \mathscr{L}_n^* = n+1$).

In any case, when $\Pi_0, \dots, \Pi_n$ are general, we now see that v_n gives a birational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ whose restriction to Q_i gives a rational map to Π_i^* for $i = 0, \dots, n$ and the fiber of Q_i over Π_i^* generically has dimension 1. It is convenient to denote the linear system of divisors of degree n vanishing on $\Pi_0 \cup \cdots \cup \Pi_n$ by $nH - \Pi_0 - \cdots - \Pi_n$. Similarly, the linear system of divisors of degree $n-1$ vanishing on Π_j for $j \neq i$ is represented by $(n-1)H - \Pi_0 - \cdots - \Pi_n + \Pi_i$. Thus, if H^* is the linear system of divisors of degree 1 on the target $\mathbb{P}^n$ for v_n, then v_n pulls H^* back to $nH - \Pi_0 - \cdots - \Pi_n$, and it pulls Π_i^* back to Q_i, represented by $(n-1)H - \Pi_0 - \cdots - \Pi_n + \Pi_i$. We can represent the pullback by a matrix map $M_n : \mathbb{Z}^{n+1} \to \mathbb{Z}^{n+1}$ where

$$M_n = \begin{pmatrix} n & n-1 & n-1 & \dots & n-1 \\ -1 & 0 & -1 & \dots & -1 \\ -1 & -1 & 0 & \dots & -1 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & -1 & -1 & \dots & 0 \end{pmatrix}.$$

If in fact the spaces Π_i^* can be taken to be sufficiently general, then $\dim \mathscr{L}_n^* = n+1$, and u_n pulls H back to $nH^* - \Pi_0^* - \cdots - \Pi_n^*$, and it pulls Π_i back to $(n-1)H^* - \Pi_0^* - \cdots - \Pi_n^* + \Pi_i^*$, and hence is represented by the same matrix M_n. Since M_n^2 corresponds to the pullback map for $u_n v_n$ and $u_n v_n$ is the identity (where defined), we would expect that $M_n^2 = I_n$, which is indeed the case.

5 Intersection of Q_i and Q_j

This section is devoted to investigating the intersections of Q_i and Q_j, assuming that $\Pi_0, \dots, \Pi_n$ are general linear subspaces of codimension 2. These intersections were already treated in [11] and in more detail than here, but here we use more modern language.

Without loss of generality assume that $i = 0$, $j = 1$, so take $Q_0 \cap Q_1$. From the considerations above (Proposition 3.4) we may write

$$Q_0 \cap Q_1 = T_n \cup \Pi_2 \cup \cdots \cup \Pi_n \cup M_n$$

where M_n is the closure of the complement of $T_n \cup \Pi_2 \cup \cdots \cup \Pi_n$ in $Q_0 \cap Q_1$.

Proposition 5.1 *The complement of $T_n \cup \Pi_2 \cup \cdots \cup \Pi_n$ in $Q_0 \cap Q_1$ is the set of all points $q \in Q_0 \cap Q_1$ through which there is no transversal for $\Pi_0, \ldots, \Pi_n$, (in which case there is more than one transversal through q for $\Pi_2, \ldots, \Pi_n$).*

Proof For $n = 2$ it is easy to check that $Q_0 \cap Q_1 = \Pi_2$ and that $T_n = M_n = \emptyset$. For $n = 3$, keeping in mind that $Q_0 = \mathbb{P}^1 \times \mathbb{P}^1$, $Q_0 \cap Q_1$ is a divisor on Q_0 of multi-degree $(2, 2)$, consisting of the lines Π_2 and Π_3 together with the two transversals for $\Pi_0, \ldots, \Pi_3$ (these two transversals give T_n); again M_n is empty. (See, for example, the description of the cubo-cubic Cremona transformation from [4] or [5].)

So now assume that $n \geq 4$. Take a point q from $Q_0 \cap Q_1$. Suppose q is not in $\Pi_2 \cup \cdots \cup \Pi_n$. Since $q \in Q_0$, by Proposition 2.5 there is at least one transversal through q for $\Pi_1, \ldots, \Pi_n$ and since $q \in Q_1$ there is similarly at least one transversal through q for $\Pi_0, \Pi_2 \ldots, \Pi_n$. If one of the transversals coming from $q \in Q_0$ is also a transversal coming from $q \in Q_1$, then it follows that the transversal goes through all Π_j, so the transversal (and hence q) is contained in T_n. Otherwise, $q \notin T_n$, hence there are two lines through q transversal for $\Pi_2, \ldots, \Pi_n$. □

Example 5.2 We close by showing for $n = 4$ that the complement of $T_4 \cup \Pi_2 \cup \cdots \cup \Pi_4$ in $Q_0 \cap Q_1$ is nonempty.

Take three points p_{ij}, where $p_{ij} = \Pi_i \cap \Pi_j$, for $j = 2, 3, 4$, $i \neq j$. Let π be the plane spanned by the three points. Take a general point q on π. From the fact that all Π_j are general, we have that q, $p_0 := \pi \cap \Pi_0$ and $p_1 := \pi \cap \Pi_1$ are not on a line. Then the line through q and p_0 is a transversal to $\Pi_0, \Pi_2, \Pi_3, \Pi_4$, so it is in Q_0 (and in π of course). In the same way, the line through q and p_1 is a transversal to $\Pi_1, \Pi_2, \Pi_3, \Pi_4$, so it is in Q_1, thus q is in $Q_0 \cap Q_1$.

To prove that $M_4 \not\subset T_4$, take a point r not in Π_2, Π_3, Π_4, and consider a projection from r to a general hyperplane. Then the intersection of the images of Π_2, Π_3, Π_4 is either a point—and then there is only one transversal to Π_2, Π_3, Π_4 through this point—or this intersection is a line, and then we have a plane of transversals from our point r. From this construction it follows that we may have at most a plane of transversals to Π_2, Π_3, Π_4. As Π_1, Π_0 are general, the generic transversal on π is not transversal to Π_1, Π_0.

Remark 5.3 Snyder and Rusk in [11] assert that $\deg(R_n) = \frac{(n+1)(n-2)}{2}$ and that $\deg(M_n) = \frac{(n-2)(n-3)}{2}$. We plan a future paper explaining these results and showing also precisely that the inverse of a Veneroni transformation is always a Veneroni.

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On the Symbolic Powers of Binomial Edge Ideals

Viviana Ene and Jürgen Herzog

Abstract We show that under some conditions, if the initial ideal $\mathrm{in}_<(I)$ of an ideal I in a polynomial ring has the property that its symbolic and ordinary powers coincide, then the ideal I shares the same property. We apply this result to prove the equality between symbolic and ordinary powers for binomial edge ideals with quadratic Gröbner basis.

Keywords Symbolic power · Binomial edge ideal · Chordal graphs

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1 Introduction

Binomial edge ideals were introduced in [11] and, independently, in [12]. Let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be the polynomial ring in $2n$ variables over a field K and G a simple graph on the vertex set $[n]$ with edge set $E(G)$. The binomial edge ideal of G is generated by the set of 2-minors of the generic matrix $X = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$ indexed by the edges of G. In other words,

$$J_G = (x_i y_j - x_j y_i : i < j \text{ and } \{i, j\} \in E(G)).$$

We will often use the notation $[i, j]$ for the maximal minor $x_i y_j - x_j y_i$ of X.

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In the last decade, several properties of binomial edge ideals have been studied. In [11], it was shown that, for every graph G, the ideal J_G is a radical ideal and the minimal prime ideals are characterized in terms of the combinatorics of the graph. Several articles treated the Cohen-Macaulay property of binomial edge ideals—see, for example, [1, 8, 14–16]. A significant effort has been done while studying the resolution of binomial edge ideals. For relevant results on this topic we refer to the following recent survey [17] and references therein.

In this paper, we consider symbolic powers of binomial edge ideals. The study and use of symbolic powers have been a reach topic of research in commutative algebra for more than 40 years. Symbolic powers and ordinary powers do not coincide in general. However, there are classes of homogeneous ideals in polynomial rings for which the symbolic and ordinary powers coincide. For example, if I is the edge ideal of a graph, then $I^k = I^{(k)}$ for all $k \geq 1$ if and only if the graph is bipartite. More general, the facet ideal $I(\Delta)$ of a simplicial complex Δ has the property that $I(\Delta)^k = I(\Delta)^{(k)}$ for all $k \geq 1$ (equivalently, $I(\Delta)$ is normally torsion free) if and only if Δ is a Mengerian complex; see [10, Sect. 10.3.4]. The ideal of the maximal minors of a generic matrix shares the same property, that is, the symbolic and ordinary powers coincide [6].

To the best of our knowledge, the comparison between symbolic and ordinary powers for binomial edge ideals was considered so far only in [13]—in Sect. 4 therein, Ohtani proved that if G is a complete multipartite graph, then $J_G^k = J_G^{(k)}$ for all integers $k \geq 1$.

In our paper we prove that for any binomial edge ideal with quadratic Gröbner basis the symbolic and ordinary powers of J_G coincide. Our proof is based on the transfer of the equality for symbolic and ordinary powers from the initial ideal to the ideal itself.

The structure of the paper is the following. In Sect. 2, we give an outline on basic results needed in the next section, namely on symbolic powers of ideals in Noetherian rings and on binomial edge ideals and their primary decompositions.

In Sect. 3, we discuss symbolic powers in connection to initial ideals. Under some specific conditions on a homogeneous ideal I in a polynomial ring over a field, one may derive that if $\text{in}_<(I)^k = \text{in}_<(I)^{(k)}$ for some integer $k \geq 1$, then $I^k = I^{(k)}$; see Lemma 3.1. By using this lemma and some properties of binomial edge ideals, we show in Theorem 3.3 that if $\text{in}_<(J_G)$ is a normally torsion-free ideal, then the symbolic and ordinary powers of J_G coincide. This is the case, for example, if G is a closed graph (Corollary 3.4) or cycle C_4. However, in general, $\text{in}_<(J_G)$ is not a normally torsion-free ideal. For example, for the binomial edge ideal of C_5, we have $J_{C_5}^2 = J_{C_5}^{(2)}$, but $(\text{in}_<(J_{C_5}))^2 \subsetneq (\text{in}_<(J_{C_5}))^{(2)}$.

2 Preliminaries

In this section we summarize basic facts about symbolic powers of ideals and binomial edge ideals.

2.1 Symbolic Powers of Ideals

Let $I \subset R$ be an ideal in a Noetherian ring R, and let $\mathrm{Min}(I)$ be the set of the minimal prime ideals of I. For an integer $k \geq 1$, one defines the kth *symbolic power* of I as follows:

$$I^{(k)} = \bigcap_{\mathfrak{p} \in \mathrm{Min}(I)} (I^k R_{\mathfrak{p}} \cap R) = \bigcap_{\mathfrak{p} \in \mathrm{Min}(I)} \ker(R \to (R/I^k)_{\mathfrak{p}}) =$$

$$= \{a \in R : \text{ for every } \mathfrak{p} \in \mathrm{Min}(I), \text{ there exists } w_{\mathfrak{p}} \notin \mathfrak{p} \text{ with } w_{\mathfrak{p}} a \in I^k\} =$$

$$= \{a \in R : \text{ there exists } w \notin \bigcup_{\mathfrak{p} \in \mathrm{Min}(I)} \mathfrak{p} \text{ with } wa \in I^k\}.$$

By the definition of the symbolic power, we have $I^k \subseteq I^{(k)}$ for $k \geq 1$. Symbolic powers do not, in general, coincide with the ordinary powers. However, if I is a complete intersection or it is the determinantal ideal generated by the maximal minors of a generic matrix, then it is known that $I^k = I^{(k)}$ for $k \geq 1$; see [6] or [2, Corollary 2.3].

Let $I = Q_1 \cap \cdots \cap Q_m$ be an irredundant primary decomposition of I with $\sqrt{Q_i} = \mathfrak{p}_i$ for all i. If the minimal prime ideals of I are $\mathfrak{p}_1, \ldots \mathfrak{p}_s$, then

$$I^{(k)} = Q_1^{(k)} \cap \cdots \cap Q_s^{(k)}.$$

In particular, if $I \subset R = K[x_1, \ldots, x_n]$ is a square-free monomial ideal in a polynomial ring over a field K, then

$$I^{(k)} = \bigcap_{\mathfrak{p} \in \mathrm{Min}(I)} \mathfrak{p}^k.$$

Moreover, I is normally torsion-free (i.e. $\mathrm{Ass}(I^m) \subseteq \mathrm{Ass}(I)$ for $m \geq 1$) if and only if $I^k = I^{(k)}$ for all $k \geq 1$, if and only if I is the Stanley-Reisner ideal of a Mengerian simplicial complex; see [10, Theorem 1.4.6, Corollary 10.3.15]. In particular, if G is a bipartite graph, then its monomial edge ideal $I(G)$ is normally torsion-free [10, Corollary 10.3.17].

In what follows, we will often use the binomial expansion of symbolic powers [9]. Let $I \subset R$ and $J \subset R'$ be two homogeneous ideals in the polynomial algebras R, R' in disjoint sets of variables over the same field K. We write I, J for the extensions of these two ideals in $R \otimes_K R'$. Then, the following binomial expansion holds.

Theorem 2.1 ([9, Theorem 3.4]) *In the above settings,*

$$(I + J)^{(n)} = \sum_{i+j=n} I^{(i)} J^{(j)}.$$

Moreover, we have the following criterion for the equality of the symbolic and ordinary powers.

Corollary 2.2 ([9, Corollary 3.5]) *In the above settings, assume that $I^t \neq I^{t+1}$ and $J^t \neq J^{t+1}$ for $t \leq n-1$. Then $(I+J)^{(n)} = (I+J)^n$ if and only if $I^{(t)} = I^t$ and $J^{(t)} = J^t$ for every $t \leq n$.*

2.2 Binomial Edge Ideals

Let G be a simple graph on the vertex set $[n]$ with edge set $E(G)$, and let S be the polynomial ring $K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ in $2n$ variables over a field K. The binomial edge ideal $J_G \subset S$ associated with G is

$$J_G = (f_{ij} : i < j, \{i, j\} \in E(G)),$$

where $f_{ij} = x_i y_j - x_j y_i$ for $1 \leq i < j \leq n$. Note that f_{ij} are exactly the maximal minors of the $2 \times n$ generic matrix $X = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$. We will use the notation $[i, j]$ for the 2- minor of X determined by the columns i and j.

We consider the polynomial ring S endowed with the lexicographic order induced by the natural order of the variables, and $\operatorname{in}_<(J_G)$ denotes the initial ideal of J_G with respect to this monomial order. By [11, Corollary 2.2], J_G is a radical ideal. Its minimal prime ideals may be characterized in terms of the combinatorics of the graph G. We introduce the following notation. Let $\mathcal{S} \subset [n]$ be a (possible empty) subset of $[n]$, and let $G_1, \ldots, G_{c(\mathcal{S})}$ be the connected components of $G_{[n]\setminus \mathcal{S}}$ where $G_{[n]\setminus \mathcal{S}}$ is the induced subgraph of G on the vertex set $[n] \setminus \mathcal{S}$. For $1 \leq i \leq c(\mathcal{S})$, let $\tilde{G}_i$ be the complete graph on the vertex set $V(G_i)$. Let

$$P_{\mathcal{S}}(G) = (\{x_i, y_i\}_{i \in \mathcal{S}}) + J_{\tilde{G}_1} + \cdots + J_{\tilde{G}_{c(\mathcal{S})}}.$$

Then $P_{\mathcal{S}}(G)$ is a prime ideal. Since the symbolic powers of an ideal of maximal minors of a generic matrix coincide with the ordinary powers, and using Corollary 2.2, we get

$$P_{\mathcal{S}}(G)^{(k)} = P_{\mathcal{S}}(G)^k \text{ for } k \geq 1. \tag{1}$$

By [11, Theorem 3.2], $J_G = \bigcap_{\mathcal{S} \subset [n]} P_{\mathcal{S}}(G)$. In particular, the minimal primes of J_G are among the prime ideals $P_{\mathcal{S}}(G)$ with $\mathcal{S} \subset [n]$. The following proposition characterizes the sets $\mathcal{S}$ for which the prime ideal $P_{\mathcal{S}}(G)$ is minimal.

Proposition 2.3 ([11, Corollary 3.9]) *$P_{\mathcal{S}}(G)$ is a minimal prime of J_G if and only if either $\mathcal{S} = \emptyset$ or $\mathcal{S}$ is non-empty and for each $i \in \mathcal{S}$, $c(\mathcal{S} \setminus \{i\}) < c(\mathcal{S})$.*

In combinatorial terminology, for a connected graph G, $P_{\mathcal{S}}(G)$ is a minimal prime ideal of J_G if and only if $\mathcal{S}$ is empty or $\mathcal{S}$ is non-empty and is a *cut-point set* of G, that is, i is a cut point of the restriction $G_{([n]\setminus\mathcal{S})\cup\{i\}}$ for every $i \in \mathcal{S}$. Let $\mathcal{C}(G)$ be the set of all sets $\mathcal{S} \subset [n]$ such that $P_{\mathcal{S}}(G) \in \operatorname{Min}(J_G)$.

Let us also mention that, by [4, Theorem 3.1] and [4, Corollary 2.12], we have

$$\operatorname{in}_<(J_G) = \bigcap_{\mathcal{S}\in\mathcal{C}(G)} \operatorname{in}_< P_{\mathcal{S}}(G). \tag{2}$$

Remark 2.4 The cited results of [4] require that K is algebraically closed. However, in our case, we may remove this condition on the field K. Indeed, neither the Gröbner basis of J_G nor the primary decomposition of J_G depend on the field K, thus we may extend the field K to its algebraic closure $\bar{K}$.

When we study symbolic powers of binomial edge ideals, we may reduce to connected graphs. Let $G = G_1 \cup \cdots \cup G_c$, where $G_1, \ldots, G_c$ are the connected components of G and let $J_G \subset S$ be the binomial edge ideal of G. Then we may write

$$J_G = J_{G_1} + \cdots + J_{G_c},$$

where $J_{G_i} \subset S_i = K[x_j, y_j : j \in V(G_i)]$ for $1 \leq i \leq c$. In the above equality, we used the notation J_{G_i} for the extension of J_{G_i} in S as well.

Proposition 2.5 *In the above settings, we have $J_G^k = J_G^{(k)}$ for every $k \geq 1$ if and only if $J_{G_i}^k = J_{G_i}^{(k)}$ for every $k \geq 1$.*

Proof The equivalence is a direct consequence of Corollary 2.2. □

3 Symbolic Powers and Initial Ideals

In this section, we discuss the transfer of the equality between symbolic and ordinary powers from the initial ideal to the ideal itself.

Let $R = K[x_1, \ldots, x_n]$ be the polynomial ring over the field K and $I \subset R$ be a homogeneous ideal. We assume that there exists a monomial order $<$ on R such that $\operatorname{in}_<(I)$ is a square-free monomial ideal. In particular, it follows that I is a radical ideal. Let $\operatorname{Min}(I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$. Then $I = \bigcap_{i=1}^s \mathfrak{p}_i$.

Lemma 3.1 *In the above settings, we assume that the following conditions are fulfilled:*

(i) $\operatorname{in}_<(I) = \bigcap_{i=1}^s \operatorname{in}_<(\mathfrak{p}_i)$;
(ii) For an integer $t \geq 1$ we have:

- *(a)* $\mathfrak{p}_i^{(t)} = \mathfrak{p}_i^t$ *for* $1 \leq i \leq s$;
- *(b)* $\operatorname{in}_<(\mathfrak{p}_i^t) = (\operatorname{in}_<(\mathfrak{p}_i))^t$ *for* $1 \leq i \leq s$;
- *(c)* $(\operatorname{in}_<(I))^{(t)} = (\operatorname{in}_<(I))^t$.

Then $I^{(t)} = I^t$.

Proof In our hypothesis, we obtain:

$$\mathrm{in}_<(I^t) \supseteq (\mathrm{in}_<(I))^t = (\mathrm{in}_<(I))^{(t)} = \bigcap_{i=1}^{s}(\mathrm{in}_<(\mathfrak{p}_i))^{(t)} \supseteq \bigcap_{i=1}^{s}(\mathrm{in}_<(\mathfrak{p}_i))^t = \bigcap_{i=1}^{s}\mathrm{in}_<(\mathfrak{p}_i^t) \supseteq$$

$$\supseteq \mathrm{in}_<(\bigcap_{i=1}^{s}\mathfrak{p}_i^t) = \mathrm{in}_<(\bigcap_{i=1}^{s}\mathfrak{p}_i^{(t)}) = \mathrm{in}_<(I^{(t)}) \supseteq \mathrm{in}_<(I^t).$$

Therefore, it follows that $\mathrm{in}_<(I^{(t)}) = \mathrm{in}_<(I^t)$. Since $I^t \subseteq I^{(t)}$, we get $I^t = I^{(t)}$. □

Now we investigate whether one may use the above lemma for studying symbolic powers of binomial edge ideals. Note that, by (2), the first condition in Lemma 3.1 holds for any binomial edge ideal J_G. In addition, as we have seen in (1), condition (a) in Lemma 3.1 holds for any prime ideal $P_{\mathcal{S}}(G)$ and any integer $t \geq 1$.

Lemma 3.2 *Let* $\mathcal{S} \subset [n]$. *Then* $\mathrm{in}_<(P_{\mathcal{S}}(G)^t) = (\mathrm{in}_<(P_{\mathcal{S}}(G)))^t$, *for every* $t \geq 1$.

Proof To shorten the notation, we write P instead of $P_{\mathcal{S}}(G)$, c instead of $c(\mathcal{S})$, and J_i instead of $J_{\tilde{G}_i}$ for $1 \leq i \leq c$. Let $\mathcal{R}(P)$, respectively $\mathcal{R}(\mathrm{in}_<(P))$ be the Rees algebras of P, respectively $\mathrm{in}_<(P)$. Then, as the sets of variables $\{x_j, y_j : j \in V(\tilde{G}_i)\}$ are pairwise disjoint, we get

$$\mathcal{R}(P) = \mathcal{R}((\{x_i, y_i\}_{i\in\mathcal{S}})) \otimes_K (\otimes_{i=1}^{c}\mathcal{R}(J_i)). \tag{3}$$

On the other hand, since $\mathrm{in}_<(P) = (\{x_i, y_i\}_{i\in\mathcal{S}}) + \mathrm{in}_<(J_1) + \cdots + \mathrm{in}_<(J_c)$, due to the fact that $J_1, \ldots, J_c$ are ideals in disjoint sets of variables different from $\{x_i, y_i\}_{i\in\mathcal{S}}$ (see [11]), we obtain

$$\begin{aligned}\mathcal{R}(\mathrm{in}_< P) &= \mathcal{R}((\{x_i, y_i\}_{i\in\mathcal{S}})) \otimes_K (\otimes_{i=1}^{c}\mathcal{R}(\mathrm{in}_< J_i)) = \\ &= \mathcal{R}((\{x_i, y_i\}_{i\in\mathcal{S}})) \otimes_K (\otimes_{i=1}^{c}\mathrm{in}_< \mathcal{R}(J_i)).\end{aligned} \tag{4}$$

For the last equality we used the equality $\mathrm{in}_<(J_i^t) = (\mathrm{in}_< J_i)^t$ for all $t \geq 1$ which is a particular case of [3, Theorem 2.1] and the equality $\mathcal{R}(\mathrm{in}_< J_i) = \mathrm{in}_< \mathcal{R}(J_i)$ due to [5, Theorem 2.7]. We know that $\mathcal{R}(P)$ and $\mathrm{in}_<(\mathcal{R}(P))$ have the same Hilbert function. On the other hand, equalities (3) and (4) show that $\mathcal{R}(P)$ and $\mathcal{R}(\mathrm{in}_< P)$ have the same Hilbert function since $\mathcal{R}(J_i)$ and $\mathrm{in}_< \mathcal{R}(J_i)$ have the same Hilbert function for every $1 \leq i \leq s$. Therefore, $\mathcal{R}(\mathrm{in}_< P)$ and $\mathrm{in}_< \mathcal{R}(P)$ have the same Hilbert function. As $\mathcal{R}(\mathrm{in}_< P) \subseteq \mathrm{in}_<(\mathcal{R}(P))$, we have $\mathcal{R}(\mathrm{in}_< P) = \mathrm{in}_<(\mathcal{R}(P))$, which implies by [5, Theorem 2.7] that $\mathrm{in}_<(P^t) = (\mathrm{in}_< P)^t$ for all t. □

Theorem 3.3 *Let* G *be a connected graph on the vertex set* $[n]$. *If* $\mathrm{in}_<(J_G)$ *is a normally torsion-free ideal, then* $J_G^{(k)} = J_G^k$ *for* $k \geq 1$.

Proof The proof is a consequence of Lemma 3.2 combined with relations (2) and (1). □

There are binomial edge ideals whose initial ideal with respect to the lexicographic order are normally torsion-free. For example, the binomial edge ideals which have a quadratic Gröbner basis have normally torsion-free initial ideals. They were characterized in [11, Theorem 1.1] and correspond to the so-called closed graphs. The graph G is *closed* if there exists a labeling of its vertices such that for any edge $\{i, k\}$ with $i < k$ and for every $i < j < k$, we have $\{i, j\}, \{j, k\} \in E(G)$. If G is closed with respect to its labeling, then, with respect to the lexicographic order $<$ on S induced by the natural ordering of the indeterminates, the initial ideal of J_G is $\text{in}_<(J_G) = (x_i y_j : i < j \text{ and } \{i, j\} \in E(G))$. This implies that $\text{in}_<(J_G)$ is the edge ideal of a bipartite graph, hence it is normally torsion-free. Therefore we get the following.

Corollary 3.4 *Let G be a closed graph on the vertex set $[n]$. Then $J_G^{(k)} = J_G^k$ for $k \geq 1$.*

Let C_4 be the 4-cycle with edges $\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}$. Let $<$ be the lexicographic order on $K[x_1, \ldots, x_4, y_1, \ldots, y_4]$ induced by $x_1 > x_2 > x_3 > x_4 > y_1 > y_2 > y_3 > y_4$. With respect to this monomial order, we have

$$\text{in}_<(J_{C_4}) = (x_1x_4y_3, x_1y_2, x_1y_4, x_2y_1y_4, x_2y_3, x_3y_4).$$

Let Δ be the simplicial complex whose facet ideal $I(\Delta) = \text{in}_<(J_{C_4})$. It is easily seen that Δ has no special odd cycle, therefore, by [10, Theorem 10.3.16], it follows that $I(\Delta)$ is normally torsion-free. Note that the 4-cycle is a complete bipartite graph, thus the equality $J_{C_4}^k = J_{C_4}^{(k)}$ for all $k \geq 1$ follows also from [13].

In view of this result, one would expect that initial ideals of binomial edge ideals of cycles are normally torsion-free. But this is not the case.

Indeed, let C_5 be the 5-cycle with edges $\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{1, 5\}$ and $I = \text{in}_<(J_{C_5})$ the initial ideal of J_{C_5} with respect to the lexicographic order on $K[x_1 \ldots, x_5, y_1, \ldots, y_5]$. By using SINGULAR [7], we checked that $I^2 \subsetneq I^{(2)}$. Indeed, the monomial $x_1^2x_4x_5y_3y_5 \in I^2$ is a minimal generator of I^2. On the other hand, the monomial $x_1x_4x_5y_3y_5 \in I^{(2)}$, thus $I^2 \neq I^{(2)}$, and I is not normally torsion-free. On the other hand, again with SINGULAR, we have checked that $J_{C_5}^2 = J_{C_5}^{(2)}$.

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Multigraded Betti Numbers of Some Path Ideals

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Abstract We provide formulas for multigraded Betti numbers of path ideals of lines and star graphs.

Keywords Path ideals · Betti numbers · Graphs

2010 Mathematics Subject Classification Primary: 05E40 · Secondary: 05E45, 13P99

1 Introduction

The path ideal of a directed graph was introduced by Conca and De Negri [8] and, since then these ideals and their generalizations have been studied by many authors, see [1–3, 5, 7, 14, 16–18].

Let $S = \Bbbk[x_1, \ldots, x_n]$ be the polynomial ring over a field $\Bbbk$. If G is a graph with vertex set $V = \{x_1, \ldots, x_n\}$, then its **path ideal** $I_t(G)$ is the monomial ideal of S defined by

$$I_t(G) = (x_{i_1} \ldots x_{i_t} \mid G \text{ has a line subgraph with vertices } x_{i_1}, \ldots, x_{i_t}).$$

Note that when $t = 2$ the path ideal $I_t(G)$ is the same as the well-known edge ideal of G. Therefore path ideals generalize edge ideals. Formulas for Betti numbers of edge ideals of lines, cycles and stars were given by Jacques in [15] using Hochster's formula. Alilooee and Faridi [1, 2] generalized the techniques of Jacques to find Betti numbers of path ideals of lines and cycles. Also, path ideals of trees were studied in [5, 6, 16].

In this paper, we follow a different approach and compute the multigraded Betti numbers of path ideals of lines and stars using Theorem 2.4 which requires the study

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of certain subcomplexes of the Taylor simplex of the ideal. Our results[1] provide more detailed information about the resolutions and reduce the problem of computing graded Betti numbers to combinatorial problems since such numbers can be obtained by summing over suitable multigraded Betti numbers.

2 Background

2.1 Simplicial Complexes and Homology

An **abstract simplicial complex** Δ on a set of **vertices** $V(\Delta) = \{v_1, \ldots, v_n\}$ is a collection of subsets of $V(\Delta)$ such that $\{v_i\} \in \Delta$ for all i, and $F \in \Delta$ implies that all subsets of F are also in Δ. The elements of Δ are called **faces** and the maximal faces under inclusion are called **facets**. If the facets $F_1, \ldots, F_q$ generate Δ, we write $\Delta = \langle F_1, \ldots, F_q \rangle$ or $\text{Facets}(\Delta) = \{F_1, \ldots, F_q\}$.

The **dimension** of a face F is equal to $|F| - 1$. The dimension of Δ is the maximum of the dimensions of its faces.

A face $\{v_1, v_2, \ldots, v_n\} \backslash \{v_{i_1}, \ldots, v_{i_s}\}$ will be denoted by $\{v_1, \ldots, \widehat{v_{i_1}}, \ldots, \widehat{v_{i_s}}, \ldots, v_n\}$ for $i_1 < i_2 < \cdots < i_s$.

Two simplicial complexes Δ and Γ are **isomorphic** if there is a bijection $\varphi : V(\Delta) \to V(\Gamma)$ between their vertex sets such that F is a face of Δ if and only if $\varphi(F)$ is a face of Γ.

Let Δ and Γ be simplicial complexes which have no common vertices. Then the **join** of Δ and Γ is the simplicial complex given by

$$\Delta * \Gamma = \{\delta \cup \gamma : \delta \in \Delta, \gamma \in \Gamma\}.$$

A **cone** with **apex** v is a special join obtained by joining a simplicial complex Δ with $\{\emptyset, v\}$ where v is not in the vertex set of Δ. Equivalently, a simplicial complex is a cone with apex v if v is a member of every facet.

If σ is a face of Δ, then the **deletion** of Δ with respect to σ is the subcomplex

$$\text{del}_\Delta(\sigma) = \{\tau \in \Delta \mid \tau \cap \sigma = \emptyset\}.$$

For further definitions from combinatorial topology the reader can refer to [4].

For each integer i, the $\mathbb{k}$ -vector space $\tilde{H}_i(\Delta, \mathbb{k})$ is the ith **reduced homology** of Δ over $\mathbb{k}$. For the sake of simplicity, we drop $\mathbb{k}$ and write $\tilde{H}_i(\Delta)$ whenever we work on a fixed ground field $\mathbb{k}$.

A **simplex** is a simplicial complex that contains all subsets of its nonempty vertex set. The **boundary** Σ of a simplex $\Delta = \langle \{v_1, \ldots, v_n\} \rangle$ is obtained from Δ by removing the maximal face of Δ. The homology groups of Σ are given by

[1] The contents of this article appeared in the Ph.D. dissertation of the author [9].

$$\tilde{H}_p(\Sigma, \Bbbk) \cong \begin{cases} \Bbbk, & \text{if } p = n-2 \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

The **irrelevant complex** $\{\emptyset\}$ has the homology groups

$$\tilde{H}_p(\{\emptyset\}, \Bbbk) \cong \begin{cases} \Bbbk, & \text{if } p = -1 \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

whereas the **void complex** $\{\}$ has trivial reduced homology in all degrees.

A simplicial complex Δ is **acyclic** (over $\Bbbk$) if $\tilde{H}_i(\Delta, \Bbbk)$ is trivial for all i. Examples of acyclic complexes include cones and simplices, see page 1853 of [4].

The homology of two simplicial complexes is related to homology of their union and intersection by the **Mayer-Vietoris long exact sequence**.

Theorem 2.1 ([13, p. 149]) *Let Δ_1 and Δ_2 be two simplicial complexes. Then there is a long exact sequence*

$$\cdots \to \tilde{H}_p(\Delta_1) \oplus \tilde{H}_p(\Delta_2) \to \tilde{H}_p(\Delta_1 \cup \Delta_2) \to \tilde{H}_{p-1}(\Delta_1 \cap \Delta_2) \to \tilde{H}_{p-1}(\Delta_1) \oplus \tilde{H}_{p-1}(\Delta_2) \to \cdots \tag{3}$$

provided that $\Delta_1 \cap \Delta_2 \neq \{\}$.

A particular case of Theorem 2.1 occurs when a simplicial complex $\Delta = \Delta_1 \cup \Delta_2$ is a union of two acyclic subcomplexes Δ_1 and Δ_2. In that case, the sequence (3) becomes

$$\cdots \to 0 \to \tilde{H}_p(\Delta_1 \cup \Delta_2) \to \tilde{H}_{p-1}(\Delta_1 \cap \Delta_2) \to 0 \to \cdots$$

whence $\tilde{H}_p(\Delta_1 \cup \Delta_2)$ and $\tilde{H}_{p-1}(\Delta_1 \cap \Delta_2)$ are isomorphic for all p. Since we will make frequent use of this specific case, we state it separately as an immediate corollary.

Corollary 2.2 *If Δ_1 and Δ_2 are acyclic simplicial complexes over $\Bbbk$, then*

$$\tilde{H}_p(\Delta_1 \cup \Delta_2, \Bbbk) \cong \tilde{H}_{p-1}(\Delta_1 \cap \Delta_2, \Bbbk)$$

for every p, provided that $\Delta_1 \cap \Delta_2 \neq \{\}$.

2.2 Graphs and Resolutions

Let $S = \Bbbk[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field $\Bbbk$. Given a minimal multigraded free resolution

$$0 \longrightarrow \bigoplus_{\mathbf{m} \in \mathbb{N}^n} S(-\mathbf{m})^{b_{r,\mathbf{m}}(I)} \xrightarrow{\partial_r} \ldots \longrightarrow \bigoplus_{\mathbf{m} \in \mathbb{N}^n} S(-\mathbf{m})^{b_{1,\mathbf{m}}(I)} \xrightarrow{\partial_1} \bigoplus_{\mathbf{m} \in \mathbb{N}^n} S(-\mathbf{m})^{b_{0,\mathbf{m}}(I)} \xrightarrow{\partial_0} I \longrightarrow 0$$

Fig. 1 P_4

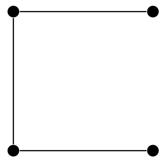


Fig. 2 C_4

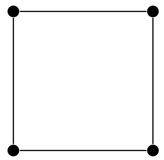
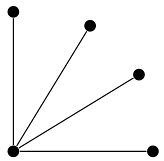


Fig. 3 $\mathscr{S}_4$



of I, its **multigraded Betti numbers** are denoted by $b_{i,\mathbf{m}}(I)$. Graded and multigraded Betti numbers are related by the equation

$$b_{i,j}(I) = \sum_{\deg(\mathbf{m})=j} b_{i,\mathbf{m}}(I) \tag{4}$$

where $\deg(\mathbf{m})$ stands for the standard degree of $\mathbf{m}$.

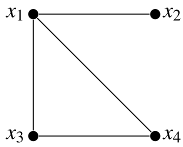
Given a **(finite simple) graph** G, the vertex and edge sets are denoted by $V(G)$ and $E(G)$ respectively. Two vertices u and v are **adjacent** to one another if $\{u, v\}$ is an edge of G. A vertex u of G is called an **isolated vertex** if it is not adjacent to any vertex of G. We will say that G is of **size** e and of **order** n if it has e edges and n vertices. For two vertices u and v of G, a **path** of length $t-1$ from u to v is a sequence of $t \geq 2$ distinct vertices $u = z_1, \dots, z_t = v$ such that $\{z_i, z_{i+1}\} \in E(G)$ for all $i = 1, \dots, t-1$. We will denote by P_n a **line** graph of order n. Also, C_n and $\mathscr{S}_n$ will be respectively the **cycle** and the **star** graphs of size n (Figs. 1, 2 and 3).

If G is a graph with vertex set $V = \{x_1, \dots, x_n\}$ then its **path ideal** $I_t(G)$ is the monomial ideal of $S = \Bbbk[x_1, \dots, x_n]$ defined by

$$I_t(G) = (x_{i_1} \dots x_{i_t} \mid x_{i_1}, \dots, x_{i_t} \text{ is a path of length } t-1 \text{ in } G).$$

When $t = 2$, the path ideal $I_t(G)$ is also known as the edge ideal and, it is denoted by $I(G)$.

Example 2.3 A graph G of order 4 which has the path ideals $I_4(G) = (x_2x_1x_4x_3)$, $I_3(G) = (x_2x_1x_4, x_2x_1x_3, x_1x_3x_4)$, $I_2(G) = (x_1x_2, x_1x_3, x_1x_4, x_3x_4)$.



We say H is a **subgraph** of G if the vertex and edge sets of H are respectively contained in those of G. A subgraph H of G is called an **induced subgraph** if for every pair of vertices u and v of H, $\{u, v\}$ is an edge of H whenever $\{u, v\}$ is an edge of G. For a square-free monomial $\mathbf{m}$ we denote by $G_{\mathbf{m}}$ the **induced subgraph** of G on the set of vertices that divide $\mathbf{m}$.

Let $I = (m_1, \dots, m_s)$ be a monomial ideal of S which is minimally generated by the set of monomials $M = \{m_1, \dots, m_s\}$. The **Taylor simplex** Taylor(I) of I is a simplex on s vertices which are labelled with the minimal generators of I. If $\tau = \{m_{i_1}, \dots, m_{i_r}\}$ is a face of Taylor(I), then by lcm(τ) we mean lcm$(m_{i_1}, \dots, m_{i_r})$. For any monomial $\mathbf{m}$ in S

$$\text{Taylor}(I)_{\leq \mathbf{m}} = \{\tau \in \text{Taylor}(I) \mid \text{lcm}(\tau) \text{ divides } \mathbf{m}\}$$

and

$$\text{Taylor}(I)_{< \mathbf{m}} = \{\tau \in \text{Taylor}(I) \mid \text{lcm}(\tau) \text{ strictly divides } \mathbf{m}\}$$

are subcomplexes of Taylor(I). Then we have

$$\text{Taylor}(I)_{<\mathbf{m}} = \bigcup_{x_i \text{ divides } \mathbf{m}} \text{Taylor}(I)_{\leq \frac{\mathbf{m}}{x_i}} \tag{5}$$

The following theorem will be our main tool to calculate Betti numbers.

Theorem 2.4 ([19, Theorem 57.9]) *Let I be a monomial ideal of S which is minimally generated by the monomials $m_1, \dots, m_s$. For $i \geq 1$, the multigraded Betti numbers of S/I are given by*

$$b_{i,\mathbf{m}}(S/I) = \begin{cases} \dim_{\Bbbk} \widetilde{H}_{i-2}(\text{Taylor}(I)_{<\mathbf{m}}; \Bbbk), & \textit{if } \mathbf{m} \textit{ divides } \text{lcm}(m_1, \dots, m_s) \\ 0, & \textit{otherwise.} \end{cases} \tag{6}$$

Remark 2.5 If $I = (m_1, \dots, m_s)$ and $q = \deg \text{lcm}(m_1, \dots, m_s)$ then for any $r > q$ we have $b_{i,r}(I) = 0$ for all i. Therefore we call the numbers $b_{i,q}(I)$, $i \in \mathbb{Z}$ as the **top degree Betti numbers**.

Remark 2.6 Suppose that Δ is the Taylor simplex of $I(G)$ for some graph G. If the induced graph $G_{\mathbf{m}}$ contains an isolated vertex, then $\Delta_{<\mathbf{m}} = \Delta$ is a simplex. So $b_{i,\mathbf{m}}(S/I(G)) = 0$ for all i by Theorem 2.4.

Lemma 2.7 ([11, Lemma 2]) *Let $I_1, I_2, \dots, I_N$ be squarefree monomial ideals whose minimal generators contain no common variable. For each $k = 1, \dots, N$ let q_k be the degree of the least common multiple of the entire minimal monomial generating set of I_k. Then*

$$b_{i,q_1+\cdots+q_N}(S/(I_1+I_2+\cdots+I_N)) = \sum_{u_1+\cdots+u_N=i} b_{u_1,q_1}(S/I_1)\dots b_{u_N,q_N}(S/I_N).$$

Lemma 2.8 *If* $\mathbf{m}$ *is a squarefree monomial of degree* j *and* $t \geq 2$*, then* $b_{i,\mathbf{m}}(S/I_t(G)) = b_{i,j}(S/I_t(G_{\mathbf{m}}))$.

Proof Proof is similar to Lemma 3.1 in [10]. □

3 The Simplicial Complex Ω_t^n

For any $n \geq t \geq 1$ we define a simplicial complex Ω_t^n on the set of vertices $\{1, \dots, n\}$ by

$$\Omega_t^n = \langle \{1, \dots, \hat{i}, \widehat{i+1}, \dots, \widehat{i+t-1}, i+t, \dots, n\} |\, i = 1, \dots, n-t+1 \rangle.$$

Example 3.1 For $n = 5$ and $t = 2$ the simplicial complex Ω_2^5 has facets $\{\hat{1}, \hat{2}, 3, 4, 5\}$, $\{1, \hat{2}, \hat{3}, 4, 5\}$, $\{1, 2, \hat{3}, \hat{4}, 5\}$ and $\{1, 2, 3, \hat{4}, \hat{5}\}$.

Remark 3.2 For $n = t$ the simplicial complex Ω_t^n is the irrelevant complex $\{\emptyset\}$. If $t = 1$, then Ω_1^n coincides with the boundary of an $n-1$ dimensional simplex.

In this section, we compute the homology of Ω_t^n because this simplicial complex will arise when we study subcomplexes of Taylor simplex of path ideals in Sect. 4.2.

The following lemma relates the pth homology of Ω_t^n to $(p-2)$nd homology of Ω_t^{n-t-1} which will allow us to apply induction on the number of vertices of the simplicial complex to find a closed formula for the homology.

Lemma 3.3 *For* $n \geq 2t+1$ *we have* $\tilde{H}_p(\Omega_t^n) \cong \tilde{H}_{p-2}(\Omega_t^{n-t-1})$ *for each integer* p*. Otherwise,*

$$\tilde{H}_p(\Omega_t^n) \cong \begin{cases} \tilde{H}_p(\{\emptyset\}), & \text{if } n = t \\ \tilde{H}_{p-1}(\{\emptyset\}), & \text{if } n = t+1 \\ 0, & \text{if } t+2 \leq n \leq 2t. \end{cases} \tag{1}$$

Proof The case $n = t$ is clear as $\Omega_t^t = \{\emptyset\}$. So we assume that $n > t$ and fix an index p. Note that

$$\Omega_t^n = \langle \sigma_1, \sigma_2, \dots, \sigma_{n-t+1} \rangle$$

where for each i, $\sigma_i = \{1, \dots, n\} \setminus \{i, i+1, \dots, i+t-1\}$. Then $\Omega_t^n = S \cup C$ where $S = \langle \sigma_1 \rangle$ is the simplex on vertices $\{t+1, \dots, n\}$ and $C = \langle \sigma_2, \dots, \sigma_{n-t+1} \rangle$ is the cone generated by the facets of Ω_t^n that contain the vertex 1. Note that by Corollary 2.2 we have

$$\tilde{H}_p(\Omega_t^n) \cong \tilde{H}_{p-1}(S \cap C).$$

We consider the three possible cases left.

Case 1: Suppose that $n = t + 1$. Then $S = \langle\{t + 1\}\rangle$ and $C = \langle\{1\}\rangle$. Thus $S \cap C$ is the irrelevant complex and we are done.

Case 2: Suppose that $t + 2 \le n \le 2t$. Observe that since $n \le 2t$ we have $\sigma_1 \cap \sigma_i \subseteq \sigma_1 \cap \sigma_2$ for all $i = 2, 3, \dots, n - t + 1$. Therefore $S \cap C = \langle\sigma_1 \cap \sigma_2\rangle$ is a simplex whose maximal face is $\{t + 2, \dots, n\}$ as $t + 2 \le n$.

Case 3: Suppose that $n \ge 2t + 1$. In this case, we have $\sigma_1 \cap \sigma_i \subseteq \sigma_1 \cap \sigma_2$ for all $i = 2, 3, \dots, t + 1$. Thus we obtain

$$S \cap C = \langle\sigma_1 \cap \sigma_2\rangle \bigcup \langle\sigma_1 \cap \sigma_{t+2}, \dots, \sigma_1 \cap \sigma_{n-t+1}\rangle.$$

Let us set $S_1 := \langle\sigma_1 \cap \sigma_2\rangle$ and $C_1 := \langle\sigma_1 \cap \sigma_{t+2}, \dots, \sigma_1 \cap \sigma_n\rangle$. Then S_1 is a simplex with vertex set $\{t + 2, \dots, n\}$, and C_1 is a cone with apex $t + 1$ such that

$$\text{Facets}(C_1) = \big\{\{t + 1, \dots, n\} \setminus \{i, i + 1, \dots, i + t - 1\} \mid i = t + 2, \dots, n - t + 1\big\}.$$

Hence we get $S_1 \cap C_1 \cong \Omega_t^{n-t-1}$. Since both S_1 and C_1 are acyclic, by Corollary 2.2 we get $\tilde{H}_{p-1}(S \cap C) \cong \tilde{H}_{p-2}(S_1 \cap C_1) \cong \tilde{H}_{p-2}(\Omega_t^{n-t-1})$ which completes the proof. □

Notation 3.4 For two integers i, j the symbol $\delta_{i,j}$ is 1 if $i = j$, and is 0 otherwise.

Corollary 3.5 *The dimensions of reduced homology modules of Ω_t^n are independent of the ground field. And they are given by*

$$\dim \tilde{H}_p(\Omega_t^n) = \begin{cases} \delta_{p+2,\frac{2n}{t+1}}, & \text{if } n \equiv 0 \bmod t + 1 \\ \delta_{p+3,\frac{2(n+1)}{t+1}}, & \text{if } n \equiv t \bmod t + 1 \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

Proof Follows from a straightforward induction using Lemma 3.3 and Eq. (2). □

4 Path Ideals of Lines

4.1 The Taylor Simplex of $I_t(P_n)$

Throughout this section let Δ be the Taylor simplex of $I_t(P_n)$ where P_n is a line on vertices $x_1, \dots, x_n$. If $n < t$, then there is no path on P_n of order t, and therefore $I_t(P_n) = 0$. Let us assume $n \ge t$ then we have

$$\Delta = \langle\{x_i x_{i+1} \dots x_{i+t-1} \mid i = 1, \dots, n - t + 1\}\rangle.$$

For simplicity, we replace the label of a vertex $x_i x_{i+1} \dots x_{i+t-1}$ of Δ with τ_i for all $i = 1, \dots, n - t + 1$. Hence Δ is the simplex given by

$$\Delta = \langle \{\tau_1, \tau_2, \ldots, \tau_{n-t+1}\} \rangle.$$

Now, in the light of Theorem 2.4 we would like to determine the facets of $\Delta_{<\mathbf{m}}$ where $\mathbf{m} = x_1 \ldots x_n$. Observe that we have

- $\Delta_{\leq \frac{\mathbf{m}}{x_i}} \subseteq \Delta_{\leq \frac{\mathbf{m}}{x_1}}$ for all $1 \leq i \leq t$,
- $\Delta_{\leq \frac{\mathbf{m}}{x_i}} = \langle \{\tau_1, \ldots, \tau_{n-t+1}\} \setminus \{\tau_{(i-t)+1}, \ldots, \tau_{(i-t)+t}\} \rangle$ for all $t+1 \leq i \leq n-t$,
- $\Delta_{\leq \frac{\mathbf{m}}{x_i}} \subseteq \Delta_{\leq \frac{\mathbf{m}}{x_n}}$ for all $n-t+1 \leq i \leq n$.

Therefore,

$$\text{Facets}(\Delta_{<\mathbf{m}}) = \text{Facets}(\Delta_{\leq \frac{\mathbf{m}}{x_1}}) \bigcup \text{Facets}(\Delta_{\leq \frac{\mathbf{m}}{x_n}}) \bigcup \bigcup_{i=t+1}^{n-t} \text{Facets}(\Delta_{\leq \frac{\mathbf{m}}{x_i}}).$$

For future reference, we explicitly list the facets of $\Delta_{<\mathbf{m}}$.

$$\begin{aligned} \text{Facets}(\Delta_{<\mathbf{m}}) = &\Big\{\{\widehat{\tau_1}, \tau_2, \ldots, \tau_{n-t+1}\}, \{\tau_1, \ldots, \tau_{n-t}, \widehat{\tau_{n-t+1}}\}\Big\} \\ &\bigcup \Big\{\{\tau_1, \ldots, \tau_{n-t+1}\} \setminus \{\tau_i, \ldots, \tau_{i+t-1}\} \mid i = 2, \ldots, n-2t+1\Big\}. \end{aligned} \tag{1}$$

Note that when $n < 2t+1$ there is no i such that $t+1 \leq i \leq n-t$. Therefore the list of items above become

- $\Delta_{\leq \frac{\mathbf{m}}{x_i}} \subseteq \Delta_{\leq \frac{\mathbf{m}}{x_1}}$ for all $1 \leq i \leq t$,
- $\Delta_{\leq \frac{\mathbf{m}}{x_i}} \subseteq \Delta_{\leq \frac{\mathbf{m}}{x_n}}$ for all $n-t+1 \leq i \leq n$

for $n < 2t+1$. Thus we get

$$\Delta_{<\mathbf{m}} = \langle \{\widehat{\tau_1}, \tau_2, \ldots, \tau_{n-t+1}\}, \{\tau_1, \ldots, \tau_{n-t}, \widehat{\tau_{n-t+1}}\} \rangle. \tag{2}$$

for $n < 2t+1$.

4.2 Multigraded Betti Numbers

We will first determine the top degree Betti numbers of path ideals of lines. Such numbers were also obtained in [2, Theorem 4.13] using a different method.

Theorem 4.1 (Top degree Betti numbers of path ideals of lines) *For all $i \geq 1$, $n \geq t$ and $n \geq 1$, we have*

$$b_{i,n}(S/I_t(P_n)) = \begin{cases} \delta_{i,\frac{2n}{t+1}}, & \textit{if } n \equiv 0 \bmod t+1 \\ \delta_{i+1,\frac{2n+2}{t+1}}, & \textit{if } n \equiv t \bmod t+1 \\ 0, & \textit{otherwise.} \end{cases} \tag{3}$$

Proof Let $\mathbf{m}$ be the product of vertices of P_n. By Eq. (4) and Theorem 2.4 we have

$$b_{i,n}(S/I_t(P_n)) = b_{i,\mathbf{m}}(S/I_t(P_n)) = \dim_{\Bbbk} \tilde{H}_{i-2}(\Delta_{<\mathbf{m}}, \Bbbk).$$

We consider two cases:

Case 1: Suppose that $n < 2t + 1$. By Eq. (2) we have

$$\Delta_{<\mathbf{m}} = \mathrm{del}_{\Delta}(\{\tau_1\}) \cup \mathrm{del}_{\Delta}(\{\tau_{n-t+1}\}).$$

Observe that if $n = t$, then $\Delta_{<\mathbf{m}} = \{\emptyset\}$ and so that

$$\dim_{\Bbbk} \tilde{H}_{i-2}(\Delta_{<\mathbf{m}}, \Bbbk) = \dim_{\Bbbk} \tilde{H}_{i-2}(\{\emptyset\}, \Bbbk) = \delta_{i-2,-1}.$$

But for $n = t$ we have $\delta_{i-2,-1} = \delta_{i+1,\frac{2n+2}{t+1}}$ which proves the formula given in (3) for this case. So, we may assume that $n > t$. Note that by Corollary 2.2 we have

$$\dim_{\Bbbk} \tilde{H}_{i-2}(\Delta_{<\mathbf{m}}, \Bbbk) \cong \dim_{\Bbbk} \tilde{H}_{i-3}(\mathrm{del}_{\Delta}(\{\tau_1\}) \cap \mathrm{del}_{\Delta}(\{\tau_{n-t+1}\}), \Bbbk)$$

as both $\mathrm{del}_{\Delta}(\{\tau_1\})$ and $\mathrm{del}_{\Delta}(\{\tau_{n-t+1}\})$ are simplices. We consider two cases.

Case 1.1: Suppose that $n = t + 1$. Then $\Delta = \langle\{\tau_1, \tau_2\}\rangle$ and $\mathrm{del}_{\Delta}(\{\tau_1\}) \cap \mathrm{del}_{\Delta}(\{\tau_{n-t+1}\})$ is the irrelevant complex. Therefore we have

$$\dim_{\Bbbk} \tilde{H}_{i-3}(\mathrm{del}_{\Delta}(\{\tau_1\}) \cap \mathrm{del}_{\Delta}(\{\tau_{n-t+1}\}), \Bbbk) \cong \dim_{\Bbbk} \tilde{H}_{i-3}(\{\emptyset\}, \Bbbk) = \delta_{i-3,-1}.$$

Check that if $n = t + 1$, then $\delta_{i-3,-1} = \delta_{i,\frac{2n}{t+1}}$ and the proof is complete for this case.

Case 1.2: Next, suppose that $n > t + 1$. Then $\mathrm{del}_{\Delta}(\{\tau_1\}) \cap \mathrm{del}_{\Delta}(\{\tau_{n-t+1}\}) = \mathrm{del}_{\Delta}(\{\tau_1, \tau_{n-t+1}\})$ is a simplex and has trivial reduced homology in all degrees.

Case 2: Suppose that $n \geq 2t + 1$. Then by Eq. (1) we have

$$\Delta_{<\mathbf{m}} = \mathrm{del}_{\Delta}(\{\tau_1\}) \cup \mathrm{del}_{\Delta}(\{\tau_{n-t+1}\}) \cup \Upsilon$$

where $\Upsilon = \langle\{\tau_1, \ldots, \tau_{n-t+1}\} \setminus \{\tau_i, \ldots, \tau_{i+t-1}\} \mid i = 2, \ldots, n - 2t + 1\rangle$. Now we can see that $\Delta_{<\mathbf{m}}$ is a union of $\mathrm{del}_{\Delta}(\{\tau_1\})$ and $\mathrm{del}_{\Delta}(\{\tau_{n-t+1}\}) \cup \Upsilon$. But since $\mathrm{del}_{\Delta}(\{\tau_1\})$ is a simplex and $\mathrm{del}_{\Delta}(\{\tau_{n-t+1}\}) \cup \Upsilon$ is a cone with apex τ_1, by virtue of Corollary 2.2 we have

$$\dim_{\Bbbk} \tilde{H}_{i-2}(\Delta_{<\mathbf{m}}, \Bbbk) = \dim_{\Bbbk} \tilde{H}_{i-3}(\mathrm{del}_{\Delta}(\{\tau_1\}) \cap \left(\mathrm{del}_{\Delta}(\{\tau_{n-t+1}\}) \cup \Upsilon\right), \Bbbk).$$

Now observe that $\mathrm{del}_{\Delta}(\{\tau_1\}) \cap \left(\mathrm{del}_{\Delta}(\{\tau_{n-t+1}\}) \cup \Upsilon\right) = C \cup \mathrm{del}_{\Delta}(\{\tau_1, \tau_{n-t+1}\})$ where C is the cone generated by the facets of $\mathrm{del}_{\Delta}(\{\tau_1\}) \cap \left(\mathrm{del}_{\Delta}(\{\tau_{n-t+1}\}) \cup \Upsilon\right)$ that contain the vertex τ_{n-t+1}. Again, as $\mathrm{del}_{\Delta}(\{\tau_1, \tau_{n-t+1}\})$ is a simplex, by Corollary 2.2 we get

$$\dim_{\Bbbk} \tilde{H}_{i-3}(\mathrm{del}_{\Delta}(\{\tau_1\}) \cap \left(\mathrm{del}_{\Delta}(\{\tau_{n-t+1}\}) \cup \Upsilon\right), \Bbbk) = \dim_{\Bbbk} \tilde{H}_{i-4}(C \cap \mathrm{del}_{\Delta}(\{\tau_1, \tau_{n-t+1}\}), \Bbbk).$$

Finally, observe that $C \cap \mathrm{del}_{\Delta}(\{\tau_1, \tau_{n-t+1}\})$ is isomorphic to the simplicial complex Ω_t^{n-t-1} which was defined in Sect. 3 Therefore, Corollary 3.5 yields

$$\dim \tilde{H}_{i-4}(\Omega_t^{n-t-1}) = \begin{cases} \delta_{i-2, \frac{2(n-t-1)}{t+1}}, & \text{if } n \equiv 0 \bmod t+1 \\ \delta_{i-1, \frac{2(n-t)}{t+1}}, & \text{if } n \equiv t \bmod t+1 \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

which agrees with the formula given in Eq. (3). □

Remark 4.2 In [14, Corollary 2.9] He and Van Tuyl proved that the facet complex of the path ideal of a rooted tree is a simplicial tree. Their result implies that the facet complex of the path ideal of a line is a simplicial tree. Therefore all results in this section can be stated in terms of facet ideals of simplicial forests.

Theorem 4.3 (Multigraded Betti numbers of path ideals of lines) *Let* $t \geq 2$ *and* $\mathbf{m}$ *be a squarefree monomial of degree* j*. Then the multigraded Betti number* $b_{i,\mathbf{m}}(S/I_t(P_n)) = 1$ *if the induced graph* $(P_n)_{\mathbf{m}}$ *consists of a collection of disjoint lines that satisfy the following conditions:*

(1) Each line is of order 0 *or* $t \bmod t+1$
(2) The number of lines of order $t \bmod t+1$ *is equal to* $\frac{i(t+1)-2j}{1-t}$.

Otherwise, $b_{i,\mathbf{m}}(S/I_t(P_n)) = 0$.

Proof Let $Q_1, \ldots, Q_p$ be the connected components of $(P_n)_{\mathbf{m}}$ where each Q_ℓ is a line of order v_ℓ for $1 \leq \ell \leq p$. We have

$$\begin{aligned} b_{i,\mathbf{m}}(S/I_t(P_n)) &= b_{i,j}(S/I_t((P_n)_{\mathbf{m}})) \text{ by Lemma 2.8} \\ &= \sum_{u_1+\cdots+u_p=i} b_{u_1,v_1}(S/I_t(Q_1)) \ldots b_{u_p,v_p}(S/I_t(Q_p)) \text{ by Lemma 2.7} \end{aligned}$$

By Theorem 4.1 if one of Q_ℓ is not of order 0 or $t \bmod t+1$, then the sum above is 0. So without loss of generality let us assume that $Q_1, \ldots, Q_z$ are of order $0 \bmod t+1$ and $Q_{z+1}, \ldots, Q_p$ are of order $t \bmod t+1$ for some $0 \leq z \leq p$. Again by Theorem 4.1, the sum above is equal to 1 if

$$\sum_{\ell=1}^{z} \frac{2v_\ell}{t+1} + \sum_{\ell=z+1}^{p} \left(\frac{2v_\ell+2}{t+1} - 1\right) = i \tag{5}$$

and 0 otherwise. Observe that (5) holds if and only if $p - z = \frac{i(t+1)-2j}{1-t}$ since $v_1 + \cdots + v_p = j$. Hence the result follows. □

Suppose that C_n is a cycle and $\mathbf{m}$ is the product of all vertices of C_n. For any $\mathbf{u}$ that strictly divides $\mathbf{m}$, the induced subgraph $(C_n)_{\mathbf{u}}$ can be seen as an induced subgraph of a line. Therefore, we can get a formula for multigraded Betti numbers of path ideals of cycles in all multidegrees except the top degree one.

Corollary 4.4 (Multigraded Betti numbers of path ideals of cycles) *Let* $t \geq 2$ *and* $\mathbf{m}$ *be a squarefree monomial of degree* $j < n$. *Then the multigraded Betti number* $b_{i,\mathbf{m}}(S/I_t(C_n)) = 1$ *if the induced graph* $(C_n)_{\mathbf{m}}$ *consists of a collection of disjoint lines that satisfy the following conditions:*

(1) Each line is of order 0 *or* $t \mod t+1$
(2) The number of lines of order $t \mod t+1$ *is equal to* $\frac{i(t+1)-2j}{1-t}$.

Otherwise, $b_{i,\mathbf{m}}(S/I_t(C_n)) = 0$.

Proof By Lemma 2.8 we have $b_{i,\mathbf{m}}(S/I_t(C_n)) = b_{i,j}(S/I_t((C_n)_{\mathbf{m}}))$. Since $(C_n)_{\mathbf{m}}$ is a disjoint union of lines the proof follows from Theorem 4.3. □

5 Path Ideals of Stars

Throughout this section let $\mathscr{S}_n$ denote a star graph on $n+1$ vertices. We say a vertex v of a graph G is a **free vertex** if v belongs to exactly one edge of G.

Lemma 5.1 *Let* G *be a connected graph with no isolated vertices. Then* G *is a star graph if and only if every edge of* G *contains a free vertex.*

Proof If G is a star graph, then every edge contains a free vertex by definition. So, suppose that every edge of G contains a free vertex. Then G has no line subgraph of order at least 4. Indeed, if e_1, e_2, e_3 are distinct edges of G, then $e_1 \nsubseteq e_2 \cup e_3$ because e_1 has a free vertex. This implies that the longest path of G has 2 edges. In particular, since G is connected every pair of edges intersect. □

Lemma 5.2 *Let* G *be a connected graph with no isolated vertices. If* $\mathbf{m}$ *is the product of the vertices of* G, *then the simplicial complex* $\text{Taylor}(I(G))_{<\mathbf{m}}$ *is the boundary of* $\text{Taylor}(I(G))$ *if and only if* G *is a star.*

Proof Suppose that $e_1, \ldots, e_q$ are the edges of the graph G. Then, $\text{Taylor}(I(G))_{<\mathbf{m}}$ is the boundary of $\text{Taylor}(I(G))$ if and only if $F_i := \{e_1, \ldots, e_q\} \setminus \{e_i\}$ is a facet of $\text{Taylor}(I)_{<\mathbf{m}}$ for each $i = 1, \ldots, q$. But observe that

$$\begin{aligned} F_i \text{ is a facet of } \text{Taylor}(I(G))_{<\mathbf{m}} &\Leftrightarrow F_i \text{ is a face of } \text{Taylor}(I(G))_{<\mathbf{m}} \\ &\Leftrightarrow \text{lcm}(e_1, \ldots, \hat{e_i}, \ldots, e_q) \neq \mathbf{m}, \text{ for each } i = 1, \ldots, q \\ &\Leftrightarrow e_i \text{ contains a free vertex, for each } i = 1, \ldots, q. \end{aligned}$$

Since G is a connected graph, our claim follows from Lemma 5.1. □

Using the lemma above, we will determine the graded Betti numbers of edge ideals of stars recovering results of Jacques [15], Hà and Van Tuyl [12].

Corollary 5.3 (Graded Betti numbers of edge ideals of stars) [15, Theorem 5.4.11], [12, Theorem 2.7] *Let* $\mathscr{S}_n$ *be a star on* $n+1$ *vertices. Then*

$$b_{i,j}(S/I(\mathscr{S}_n)) = \begin{cases} \binom{n}{j-1}, & \text{if } i = j-1 \\ 0, & \text{otherwise.} \end{cases}$$

Proof First, we find the top degree Betti numbers. Combining Lemma 5.2, Theorem 2.4 and Eq. (1) we get

$$b_{i,n+1}(S/I(\mathscr{S}_n)) = \delta_{i,n} \text{ for all } i. \tag{1}$$

Fix j and recall Eq. (4) and Lemma 2.8. Any induced subgraph of $\mathscr{S}_n$ is either a star or contains an isolated vertex. If it contains an isolated vertex, then by Remark 2.6 the multigraded Betti number for such an induced subgraph is 0. Hence by (1) we see that $b_{i,j}(S/I(\mathscr{S}_n))$ is the number of induced star subgraphs of $\mathscr{S}_n$ of order j if $i = j - 1$ and is 0 otherwise. □

We will extend the result above to path ideals. To this end, we will need the following proposition.

Proposition 5.4 *Let Γ be a simplicial complex which is not a cone. Suppose that $\langle F_1, \dots, F_q \rangle = \Gamma$ and there exists a sequence of distinct vertices $v_1, \dots, v_q$ of Γ such that $v_i \notin F_j$ if and only if $i = j$. Then $\tilde{H}_p(\Gamma, \Bbbk) \cong \tilde{H}_{p-q+1}(\{\emptyset\}, \Bbbk)$ for any field $\Bbbk$.*

Proof We use induction on q, the number of facets. Since there is no simplex which satisfies the assumptions of the given proposition, the base case starts at $q = 2$.

Suppose that $\Gamma = \langle F_1, F_2 \rangle$ is not a cone and it has two vertices v_1, v_2 such that $v_i \notin F_j \Leftrightarrow i = j$. Since Γ is not a cone, $\langle F_1 \rangle \cap \langle F_2 \rangle = \{\emptyset\}$. Now $\Gamma = \langle F_1 \rangle \cup \langle F_2 \rangle$, and the simplicial complexes $\langle F_1 \rangle, \langle F_2 \rangle$ are acyclic. By virtue of Corollary 2.2 we obtain

$$\tilde{H}_p(\Gamma) \cong \tilde{H}_{p-1}(\langle F_1 \rangle \cap \langle F_2 \rangle) = \tilde{H}_{p-1}(\{\emptyset\})$$

as desired.

Now let $\Gamma = \langle F_1, \dots, F_q \rangle, q \geq 3$ be a simplicial complex as in the statement of the Proposition. We write

$$\Gamma = \langle F_1, \dots, F_{q-1} \rangle \cup \langle F_q \rangle$$

where $\langle F_q \rangle$ is a simplex and $\langle F_1, \dots, F_{q-1} \rangle$ is a cone with apex v_q. By Corollary 2.2 we have $\tilde{H}_p(\Gamma) \cong \tilde{H}_{p-1}(\langle F_1, \dots, F_{q-1} \rangle \cap \langle F_q \rangle)$. But observe that

$$\langle F_1, \dots, F_{q-1} \rangle \cap \langle F_q \rangle = \langle F_1 \cap F_q, \dots, F_{q-1} \cap F_q \rangle$$

as for all $1 \leq i \neq j \leq q-1$, $v_j \in F_i \cap F_q$, $v_j \notin F_j \cap F_q$ so that $F_i \cap F_q \nsubseteq F_j \cap F_q$. Observe that the simplicial complex $\langle F_1 \cap F_q, \dots, F_{q-1} \cap F_q \rangle$ is not a cone since Γ is not a cone. Moreover for all $1 \leq i, j \leq q-1$

$$v_i \notin F_j \cap F_q \Longleftrightarrow i = j.$$

Hence it satisfies the inductive hypothesis and we get

$$\tilde{H}_{p-1}(\langle F_1 \cap F_q, \dots, F_{q-1} \cap F_q \rangle) \cong \tilde{H}_{p-q+1}(\{\emptyset\})$$

which completes the proof. □

Theorem 5.5 *Let $\mathscr{S}_n$ be a star graph of size $n \geq 2$. Then for all $i \geq 1$*

$$b_{i,n+1}(S/I_3(\mathscr{S}_n)) = \begin{cases} i, & \text{if } i = n-1 \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Proof Let $\mathscr{S}_n$ be a star graph of size n with the edge set $E(\mathscr{S}_n) = \{\{x_0, x_i\} \mid i = 1, \dots, n\}$. Suppose that Θ is the Taylor simplex of $I_3(\mathscr{S}_n)$. We prove the given statement by induction on n. We will use Theorem 2.4 so that for all $i \geq 1$ we have

$$b_{i,n+1}(S/I_3(\mathscr{S}_n)) = \dim_{\Bbbk} \tilde{H}_{i-2}(\Theta_{<\mathbf{m}}, \Bbbk)$$

where $\mathbf{m} = x_0 \dots x_n$ is the product of vertices of $\mathscr{S}_n$. For $n = 2$, we have $\Theta_{<\mathbf{m}} = \{\emptyset\}$ so the base case is settled by (2).

Now suppose that $n \geq 3$ is fixed. Recall that by (5) we have a decomposition

$$\Theta_{<\mathbf{m}} = \Theta_{\leq \frac{\mathbf{m}}{x_n}} \bigcup \Big(\bigcup_{i=1}^{n-1} \Theta_{\leq \frac{\mathbf{m}}{x_i}} \Big) \tag{3}$$

since $\Theta_{\leq \frac{\mathbf{m}}{x_0}} = \{\emptyset\}$. For $i \geq 1$ we set

$$F_i := \{x_0 x_j x_k \mid j, k \in \{1, \dots, n\} \setminus \{i\} \text{ and } j < k\}$$

Because of the symmetry of star graphs, every element of $\{F_1, \dots, F_n\}$ is maximal with respect to inclusion. Consequently, we get $\Theta_{<\mathbf{m}} = \langle F_1, \dots, F_n \rangle$ and (3) becomes

$$\Theta_{<\mathbf{m}} = \langle F_n \rangle \cup \langle F_1, \dots, F_{n-1} \rangle \tag{4}$$

Let $H = \mathscr{S}_n - \{\{x_0, x_n\}\}$ be the star obtained by removing the edge $\{x_0, x_n\}$ from $\mathscr{S}_n$. We claim the following:

Claim 1: $\langle F_n \rangle \cap \langle F_1, \dots, F_{n-1} \rangle \cong \text{Taylor}(I_3(H))_{<x_0 \dots x_{n-1}}$.

Claim 2: $\tilde{H}_p(\langle F_1, \dots, F_{n-1} \rangle) \cong \tilde{H}_{p-n+2}(\{\emptyset\})$.

Observe that K is a facet of $\langle F_n \rangle \cap \langle F_1, \dots, F_{n-1} \rangle$ if and only if $K = F_n \cap F_i$ for some $1 \leq i \leq n-1$. But the latter means that K consists of all monomials corresponding to paths of the form $x_0 x_j x_k$ where $j, k \in \{x_1, \dots, x_n\} \setminus \{x_i, x_n\}$ and $j \neq k$. And this proves Claim 1.

For Claim 2 we show that Proposition 5.4 applies to the simplicial complex $\langle F_1, \dots, F_{n-1} \rangle$. To this end, we first check that $\langle F_1, \dots, F_{n-1} \rangle$ is not a cone. Assume for a contradiction it is a cone with apex $x_0 x_i x_j$. Then $x_0 x_i x_j \in F_1 \cap \dots \cap F_{n-1}$

which is only possible if $i = j = n$ which is a contradiction. Now consider $v_1 := x_0x_1x_n, \dots, v_{n-1} := x_0x_{n-1}x_n$, a sequence of vertices of $\langle F_1, \dots, F_{n-1}\rangle$. By definition of $F_1, \dots, F_n$ we have $v_i \notin F_j \Leftrightarrow i = j$, and this proves Claim 2.

Therefore (2) yields

$$\dim \tilde{H}_p(\langle F_1, \dots, F_{n-1}\rangle) = \begin{cases} 1, & \text{if } p = n-3 \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Also by our induction hypothesis and Claim 1 we have

$$\dim \tilde{H}_p(\langle F_n\rangle \cap \langle F_1, \dots, F_{n-1}\rangle) = \begin{cases} p+2, & \text{if } p = n-4 \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Therefore the Mayer-Vietoris sequence for (4) is

$$\cdots \to 0 \to 0 \to \tilde{H}_n(\Theta_{<m}) \to 0 \to 0 \to \tilde{H}_{n-1}(\Theta_{<m}) \to 0 \to 0 \to \tilde{H}_{n-2}(\Theta_{<m}) \to 0$$
$$\to \tilde{H}_{n-3}(\langle F_1, \dots, F_{n-1}\rangle) \to \tilde{H}_{n-3}(\Theta_{<m}) \to \tilde{H}_{n-4}(\langle F_n\rangle \cap \langle F_1, \dots, F_{n-1}\rangle) \to 0 \to$$
$$\tilde{H}_{n-4}(\Theta_{<m}) \to 0 \to 0 \to \tilde{H}_{n-5}(\Theta_{<m}) \to 0 \to 0 \to \tilde{H}_{n-6}(\Theta_{<m}) \to 0 \to 0 \to \cdots.$$

It follows $\tilde{H}_i(\Theta_{<m}) = 0$ for $i \geq n-2$ and $i \leq n-4$. Hence we have

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \tilde{H}_{n-3}(\langle F_1, \dots, F_{n-1}\rangle) \longrightarrow \tilde{H}_{n-3}(\Theta_{<x_0 \dots x_n})$$
$$\longrightarrow \tilde{H}_{n-4}(\langle F_n\rangle \cap \langle F_1, \dots, F_{n-1}\rangle) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

which gives that

$$\begin{aligned} \dim \tilde{H}_{n-3}(\Theta_{<m}) &= \dim \tilde{H}_{n-3}(\langle F_1, \dots, F_{n-1}\rangle) + \dim \tilde{H}_{n-4}(\langle F_n\rangle \cap \langle F_1, \dots, F_{n-1}\rangle) \\ &= 1 + (n-2) \text{ by (5) and (6)} \\ &= n-1 \end{aligned}$$

and the proof is completed. □

Finally, we extend Corollary 5.3 to path ideals.

Theorem 5.6 (Graded Betti numbers of path ideals of stars) *Let $\mathcal{S}_n$ be a star graph of size $n \geq 2$. For all $i \geq 1$ and $j \leq n+1$*

$$b_{i,j}(S/I_3(\mathcal{S}_n)) = \begin{cases} i\binom{n}{j-1}, & \text{if } i = j-2 \\ 0, & \text{otherwise.} \end{cases}$$

Proof Similar to proof of Corollary 5.3. □

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Depth of an Initial Ideal

Takayuki Hibi and Akiyoshi Tsuchiya

Abstract Given an arbitrary integer $d > 0$, we construct a homogeneous ideal I of the polynomial ring $S = K[x_1, \ldots, x_{3d}]$ in $3d$ variables over a field K for which S/I is a Cohen–Macaulay ring of dimension d with the property that, for each of the integers $0 \leq r \leq d$, there exists a monomial order $<_r$ on S with $\text{depth}(S/\text{in}_{<_r}(I)) = r$, where $\text{in}_{<_r}(I)$ is the initial ideal of I with respect to $<_r$.

Keywords Initial ideal · Depth · Gröbner basis

2010 Mathematics Subject Classification 13P10

1 Background

In order to answer a question suggested in [2, p. 38], the first author [6, p. 285] discovered a graded Gorenstein Hodge algebra whose corresponding discrete Hodge algebra is not a Cohen–Macaulay ring. In the modern language of Gröbner bases and initial ideals, the work guarantees the existence of a homogeneous ideal I of the polynomial ring $S = K[x_1, \ldots, x_n]$ over a field K for which S/I is Cohen–Macaulay with the property that there is an initial ideal $\text{in}_<(I)$ of I for which $S/(\text{in}_<(I))$ is not Cohen–Macaulay. On the other hand, in [1, Corollary 3.9], it is shown that if A is an ASL (algebra with straightening laws [3]) and A_0 is its discrete ASL, then depth A = depth A_0. In particular the discrete ASL of a Cohen–Macaulay ASL is again Cohen–Macaulay.

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Take the above background into consideration, one cannot escape the temptation to study the question as follows:

Question 1.1 Given an arbitrary integer $d > 0$, does there exist a homogeneous ideal I of the polynomial ring S over a field K for which S/I is a Cohen–Macaulay ring of dimension d with the property that, for each of the integers $0 \leq r \leq d$, there is a monomial order $<_r$ on S with $\mathrm{depth}(S/\mathrm{in}_{<_r}(I)) = r$, where $\mathrm{in}_{<_r}(I)$ is the initial ideal of I with respect to $<_r$?

The purpose of the present paper is to solve Question 1.1 and, in addition, to supply related questions.

2 Result

We refer the reader to [4, Chap. 2] for fundamental materials and standard notation on Gröbner bases.

Let $S = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over a field K. Given a vector $\mathbf{w} = (w_1, \ldots, w_d) \in \mathbb{Z}^n$ and a monomial $u = x_1^{a_1} \cdots x_n^{a_n} \in S$, the *weight* of u with respect to $\mathbf{w}$ is defined to be $a_1 w_1 + \cdots + a_n w_n$.

Theorem 2.1 *Given an arbitrary integer $d > 0$, there exists a homogeneous ideal I of the polynomial ring $S = K[x_1, \ldots, x_{3d}]$ in $3d$ variables over a field K for which S/I is a Cohen–Macaulay ring of dimension d with the property that, for each of the integers $0 \leq r \leq d$, there is a monomial order $<_r$ on S with* $\mathrm{depth}(S/\mathrm{in}_{<_r}(I)) = r$.

Proof **(First Step)** Let $d = 1$. Let $S = K[x_1, x_2, x_3]$ and

$$I = (x_1^2 - x_2x_3, x_1x_2 - x_3^2, x_1x_3 - x_2^2) \subset S$$

([4, Example 3.3.6]). Then S/I is a one-dimensional Cohen–Macaulay ring. Let $\prec$ be the lexicographic order on S with $x_3 \prec x_2 \prec x_1$. Let, in addition, $\mathbf{w}_0 = (1, 1, 1)$ and $\mathbf{w}_1 = (1, 2, 2)$. For each $r \in \{0, 1\}$, we introduce the monomial order $<_r$ on S as follows: One has $u <_r v$ if and only if one of the following holds:

- The weight of u is less than that of v with respect to $\mathbf{w}_r$;
- The weight of u is equal to that of v with respect to $\mathbf{w}_r$ and $u \prec v$.

Then

$$\{x_1^2 - x_2x_3, x_1x_2 - x_3^2, x_1x_3 - x_2^2, x_2^3 - x_3^3\}$$

is a Gröbner basis of I with respect to $<_0$ and $\mathrm{depth}(S/\mathrm{in}_{<_0}(I)) = 0$. On the other hand,

$$\{x_2x_3 - x_1^2, x_3^2 - x_1x_2, x_2^2 - x_1x_3\}$$

is a Gröbner basis of I with respect to $<_1$ and $\mathrm{depth}(S/\mathrm{in}_{<_1}(I)) = 1$.

(Second Step) Let $d > 1$. Let $S = K[x_1, \ldots, x_{3d}]$ and

$$I = (\{ x_{3i-2}^2 - x_{3i-1}x_{3i},\ x_{3i-2}x_{3i-1} - x_{3i}^2,\ x_{3i-2}x_{3i} - x_{3i-1}^2 \ :\ 1 \leq i \leq d \}) \subset S.$$

Let $S_i = K[x_{3_i-2}, x_{3i-1}, x_{3i}]$ and

$$I_i = (x_{3i-2}^2 - x_{3i-1}x_{3i}, x_{3i-2}x_{3i-1} - x_{3i}^2, x_{3i-2}x_{3i} - x_{3i-1}^2) \subset S_i,$$

where $1 \leq i \leq d$. Thus

$$S/I \cong S_1/I_1 \otimes_K \cdots \otimes_K S_d/I_d$$

and S/I is a Cohen–Macaulay ring of dimension d.

Now, we employ the lexicographic order $\prec$ on S with

$$x_{3d} \prec x_{3d-1} \prec \cdots \prec x_1$$

on S and the vectors

$$\mathbf{w}_r = (\underbrace{1, 2, 2, \ldots, 1, 2, 2}_{3r}, \underbrace{1, 1, 1, \ldots, 1, 1, 1}_{3(d-r)}) \in \mathbb{Z}^{3d}, \quad 0 \leq r \leq d.$$

For each $0 \leq r \leq d$, we introduce the monomial order $<_r$ on S as follows: One has $u <_r v$ if and only if one of the following holds:

- The weight of u is less than that of v with respect to $\mathbf{w}_r$;
- The weight of u is equal to that of v with respect to $\mathbf{w}_r$ and $u \prec v$.

It then follows that the set $A \cup B$, where

$$A = \{ x_{3i-1}x_{3i} - x_{3i-2}^2, x_{3i-1}^2 - x_{3i-2}x_{3i}, x_{3i}^2 - x_{3i-2}x_{3i-1} \ :\ 1 \leq i \leq r \}$$

and

$$B = \{ x_{3i-2}^2 - x_{3i-1}x_{3i}, x_{3i-2}x_{3i-1} - x_{3i}^2, x_{3i-1}x_{3i} - x_{3i}^2, x_{3i-1}^3 - x_{3i}^3 \ :\ r+1 \leq i \leq d \},$$

is a Gröbner basis of I with respect to $<_r$. Since

$$\begin{aligned} S/\mathrm{in}_{<_r}(I) \cong\ & \frac{K[x_1, x_2, x_3]}{(x_2x_3, x_3^2, x_2^2)} \otimes_K \cdots \otimes_K \frac{K[x_{3r-1}, x_{3r-1}, x_{3r}]}{(x_{3r-1}x_{3r}, x_{3r}^2, x_{3r-1}^2)} \\ & \otimes_K \frac{K[x_{3r+1}, x_{3r+2}, x_{3r+3}]}{(x_{3r+1}^2, x_{3r+1}x_{3r+2}, x_{3r+1}x_{3r+3}, x_{3r+2}^3)} \\ & \otimes_K \cdots \\ & \otimes_K \frac{K[x_{3d-2}, x_{3d-1}, x_{3d}]}{(x_{3d-2}^2, x_{3d-2}x_{3d-1}, x_{3d-2}x_{3d}, x_{3d-1}^3)}, \end{aligned}$$

one has $\text{depth}(S/\text{in}_{<_r}(I)) = r$, as desired. □

Remark 2.2 Let S/I be the quotient ring studied in the proof of Theorem 2.1 and $\text{reg}(S/I)$ its regularity. On has $\text{reg}(S/I) = d$. Furthermore, it follows that

$$\text{depth}(S/\text{in}_{<_r}(I)) + \text{reg}(S/\text{in}_{<_r}(I)) = 2d$$

for each of $0 \leq r \leq d$.

3 Questions

We conclude the present paper with related questions.

Question 3.1 In Question 1.1, one may ask if S/I can be a Gorenstein ring.

Question 3.2 In Question 1.1, one may ask if I can be a prime ideal.

Question 3.3 Let $I \subset S = K[x_1, \ldots, x_n]$ be a homogeneous ideal with $\text{reg}(S/I) = r$ and $\text{depth}(S/I) = e$. Suppose that there is a monomial order $\prec$ on S with $\text{reg}(S/\text{in}_\prec(I)) = r'$ and $\text{depth}(S/\text{in}_\prec(I)) = e'$. Then, for each $r \leq r'' \leq r'$ and for each $e' \leq e'' \leq e$, does there exist a monomial order $\prec'$ on S with $\text{reg}(S/\text{in}_{\prec'}(I)) = r''$ and $\text{depth}(S/\text{in}_{\prec'}(I)) = e''$?

Let $I \subset S = K[x_1, \ldots, x_n]$ be the toric ideal of a *unimodular* convex polytope [5, p. 107]. Thus, for *any* monomial order $\prec$ on S, its initial ideal $\text{in}_\prec(I)$ is generated by squarefree monomials. It then follows from [1] that $\text{reg}(S/\text{in}_\prec(I)) = \text{reg}(S/I)$ and $\text{depth}(S/I) = \text{depth}(S/\text{in}_\prec(I))$.

Question 3.4 Is there a nice class of toric ideals $I \subset S$ for which $\text{reg}(S/\text{in}_\prec(I)) = \text{reg}(S/I)$ for *any* monomial order $\prec$ on S and for which there is a monomial order $\prec$ on S with $\text{depth}(S/\text{in}_\prec(I)) < \text{depth}(S/I)$?

Let D be a finite distributive lattice and $\mathscr{R}_K[D]$ the ASL on D over a field K ([7, p. 98]). It is known that $\mathscr{R}_K[D]$ is normal and Cohen–Macaulay. Its discrete ASL is Cohen–Macaulay. Let $S = K[\{x_\alpha : \alpha \in D\}]$ the polynomial ring in $|D|$ variables over K. The defining ideal of $\mathscr{R}_K[D]$ is

$$I_D = (\{\, x_\alpha x_\beta - x_{\alpha\wedge\beta} x_{\alpha\vee\beta} \;:\; \alpha \not\leq \beta,\ \beta \not\leq \alpha \,\}) \subset S.$$

Question 3.5 For which finite distributive lattices D, does there exist a monomial order $\prec$ on $S = K[\{x_\alpha : \alpha \in D\}]$ with $\min \text{depth}(S/\text{in}_\prec(I_D)) = 0$?

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Asymptotic Behavior of Symmetric Ideals: A Brief Survey

Martina Juhnke-Kubitzke, Dinh Van Le, and Tim Römer

Abstract Recently, chains of increasing symmetric ideals have attracted considerable attention. In this note, we summarize some results and open problems concerning the asymptotic behavior of several algebraic and homological invariants along such chains, including codimension, projective dimension, Castelnuovo-Mumford regularity, and Betti tables.

Keywords Invariant ideal · Monoid · Polynomial ring · Symmetric group

2010 Mathematics Subject Classification 13A50 · 13C15 · 13D02 · 13F20 · 16P70 · 16W22

1 Introduction

This note provides a brief survey on recent developments in the study of asymptotic properties of chains of ideals in increasingly larger polynomial rings that are invariant under the action of symmetric groups. Such chains have received much attention in the last decades as they arise naturally in various areas of mathematics, including algebraic chemistry [1, 10], algebraic statistics [2, 11–13, 17–19, 33], group theory [6], and representation theory [5, 16, 27, 29, 31, 32].

As we will see, chains of ascending symmetric ideals are intimate related to their limits in a polynomial ring in infinitely many variables. Let us begin by fixing some notation. Throughout the paper, $\mathbb{N}$ denotes the set of positive integers, c an element of $\mathbb{N}$, and K an arbitrary field. Let

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$$R = K[x_{k,j} \mid 1 \le k \le c,\ j \ge 1]$$

be the polynomial ring in "$c \times \mathbb{N}$" variables over K, and for each $n \in \mathbb{N}$ consider the following Noetherian subring of R:

$$R_n = K[x_{k,j} \mid 1 \le k \le c, 1 \le j \le n].$$

It is useful to view R as the limit of the following ascending chain of polynomial rings

$$R_1 \subseteq R_2 \subseteq \cdots \subseteq R_n \subseteq \cdots .$$

Let $\mathrm{Sym}(\infty) := \bigcup_{n \ge 1} \mathrm{Sym}(n)$ denote the infinite symmetric group, where $\mathrm{Sym}(n)$ is the symmetric group on $\{1, \ldots, n\}$ regarded as stabilizer of $n + 1$ in $\mathrm{Sym}(n + 1)$. One can also view $\mathrm{Sym}(\infty)$ as the limit of the following ascending chain of finite subgroups

$$\mathrm{Sym}(1) \subseteq \mathrm{Sym}(2) \subseteq \cdots \subseteq \mathrm{Sym}(n) \subseteq \cdots .$$

Consider the action of $\mathrm{Sym}(\infty)$ on R by acting on the second index of the variables, i.e.,

$$\sigma \cdot x_{k,j} = x_{k,\sigma(j)} \quad \text{for } \sigma \in \mathrm{Sym}(\infty),\ 1 \le k \le c,\ \ j \ge 1.$$

Observe that this action restricts to an action of $\mathrm{Sym}(n)$ on R_n.

We say that an ideal $I \subseteq R$ is $\mathrm{Sym}(\infty)$-*invariant* (or Sym-*invariant* for short) if $\sigma(f) \in I$ for every $f \in I$ and $\sigma \in \mathrm{Sym}(\infty)$. In order to investigate such an ideal, a natural approach is to consider the truncated ideals $I_n = I \cap R_n$ for $n \ge 1$. Note that the sequence of truncated ideals $(I_n)_{n \ge 1}$ forms an ascending chain, and furthermore, it is Sym-*invariant* in the sense that

$$\mathrm{Sym}(m)(I_n) \subseteq I_m \quad \text{for all } \ m \ge n \ge 1.$$

Conversely, when $(I_n)_{n \ge 1}$ is a Sym-invariant chain, its limit in R, i.e. the union $\bigcup_{n \ge 1} I_n$, forms a Sym-invariant ideal in R.

Although R is not a Noetherian ring, a number of useful finiteness results have been established for this ring. For instance, it is known that Sym-invariant ideals in R satisfy the ascending chain condition, or in other words, R is Sym-*Noetherian*. This celebrated result was first discovered by Cohen in his investigation of the variety of metabelian groups [6, 7]. The result was later rediscovered by Aschenbrenner and Hillar [1] and Hillar and Sullivant [18] with motivations from finiteness questions in chemistry and algebraic statistics. Generalizations of the Sym-Noetherianity of R were obtained by Nagel and Römer [27] in the context of FI-modules.

Based on the Sym-Noetherianity of R, Nagel and Römer [26] introduced Hilbert series for Sym-invariant chains and showed that they are rational functions (see also [21] for another proof using formal languages and [15] for some explicit results in a special case). As a consequence, they determined the asymptotic behaviors of the codimension and multiplicity along Sym-invariant chains: the codimension grows eventually linearly, whereas the multiplicity grows eventually exponentially. This result leads to the following general problem (see [23, Problem 1.1]):

Problem 1.1 Study the asymptotic behavior of invariants along Sym-invariant chains of ideals.

The aim of this note is to briefly summarize some recent results and open problems arising from the study of the previous problem. Apart from the aforementioned results of Nagel and Römer, we will discuss further results on the asymptotic behaviors of the codimension [23], the Castelnuovo-Mumford regularity [22, 25, 30], the projective dimension [23, 25], and the Betti table [25, 27] along Sym-invariant chains of ideals. For the sake of simplicity, some results will not be stated in their most general form. Moreover, for the reader's convenience, a large number of examples will be provided.

The note is divided into seven sections. In Sect. 2 we recall some basic notions and facts on invariant chains of ideals. Section 3 contains Nagel-Römer's result on rationality of Hilbert series and its consequences on the asymptotic behaviors of codimension and multiplicity, together with an improvement on the codimension obtained in [23]. The asymptotic behaviors of the Castelnuovo-Mumford regularity, the projective dimension, and the Betti table are discussed in Sects. 4, 5, and 6, respectively. Finally, some open problems are proposed in Sect. 7.

2 Preliminaries

We use the notation and definitions from the introduction. In particular, c is a fixed positive integer, R is the polynomial ring $K[x_{k,j} \mid 1 \leq k \leq c,\ j \geq 1]$, and for each $n \geq 1$, R_n is the subring of R generated by the first $c \times n$ variables. Moreover, for any monomial order $\leq$ on R, we will use the same notation to denote its restriction to R_n.

Let $I \subseteq R$ be a Sym-invariant ideal and consider the chain $(I_n)_{n\geq 1}$ of truncations of I. A useful technique for investigating the asymptotic properties of the chain $(I_n)_{n\geq 1}$ is to pass to the chain $(\mathrm{in}_{\leq}(I_n))_{n\geq 1}$ of initial ideals. But one issue then arises: unfortunately, the chain $(\mathrm{in}_{\leq}(I_n))_{n\geq 1}$ is typically not Sym-invariant (see Example 2.4). The reason is that the action of the symmetric group $\mathrm{Sym}(\infty)$ on R is not compatible with monomial orders: for any monomial order $\leq$ on R, there exist monomials $u, v \in R$ with $u < v$ and some $\sigma \in \mathrm{Sym}(\infty)$ such that $\sigma(u) > \sigma(v)$; see [2, Remark 2.1]. To deal with this issue, one introduces another action on R with a larger class of invariant ideals which behaves better with respect to monomial orders.

Consider the following monoid of strictly increasing functions on $\mathbb{N}$:

$$\mathrm{Inc} = \{\pi : \mathbb{N} \to \mathbb{N} \mid \pi(j) < \pi(j+1) \text{ for all } j \geq 1\}.$$

Analogously to $\mathrm{Sym}(\infty)$, one defines the action of Inc on R as follows[1]:

$$\pi \cdot x_{k,j} = x_{k,\pi(j)} \quad \text{for } \pi \in \mathrm{Inc},\ 1 \le k \le c,\ j \ge 1.$$

Definition 2.1 An ideal $I \subseteq R$ is called Inc-*invariant* if

$$\pi(I) := \{\pi(f) \mid f \in I\} \subseteq I \quad \text{for all } \pi \in \mathrm{Inc}\,.$$

A chain of ideals $(I_n)_{n\ge1}$ with $I_n \subseteq R_n$ is Inc-*invariant* if

$$\mathrm{Inc}_{m,n}(I_m) := \{\pi(I_m) \mid \pi \in \mathrm{Inc}_{m,n}\} \subseteq I_n \quad \text{for all } m \le n,$$

where

$$\mathrm{Inc}_{m,n} := \{\pi \in \mathrm{Inc} \mid \pi(m) \le n\}.$$

It is evident that if $I \subseteq R$ is an Inc-invariant ideal, then its truncations $I \cap R_n$ form an Inc-invariant chain. Conversely, if $(I_n)_{n\ge1}$ is an Inc-invariant chain, then $I := \bigcup_{n\ge1} I_n$ is an Inc-invariant ideal in R.

Although Inc is not a submonoid of $\mathrm{Sym}(\infty)$, it turns out that the class of Inc-invariant ideals contains the one of Sym-invariant ideals (see, e.g., [26, Lemma 7.6]):

Proposition 2.2 *For any $f \in R_m$ and $\pi \in \mathrm{Inc}_{m,n}$ with $m \le n$, there exists $\sigma \in \mathrm{Sym}(n)$ such that $\pi(f) = \sigma(f)$. Thus, it holds that $\mathrm{Inc}_{m,n}(f) \subseteq \mathrm{Sym}(n)(f)$. In particular:*

(i) every Sym-*invariant ideal $I \subseteq R$ is also an* Inc-*invariant ideal;*
(ii) every Sym-*invariant chain $(I_n)_{n\ge1}$ is also an* Inc-*invariant chain.*

In addition, the class of Inc-invariant ideals is closed under the action of certain monomial orders on R, which is not true for the class of Sym-invariant ideals (see Lemma 2.3 and Example 2.4 below). We say that a monomial order $\le$ *respects* Inc if $\pi(u) \le \pi(v)$ whenever $\pi \in \mathrm{Inc}$ and u, v are monomials of R with $u \le v$. This condition implies that

$$\mathrm{in}_\le(\pi(f)) = \pi(\mathrm{in}_\le(f)) \quad \text{for all } f \in R \ \text{ and } \pi \in \mathrm{Inc}\,.$$

Examples of monomial orders respecting Inc include the lexicographic order and the reverse-lexicographic order on R induced by the following ordering of the variables:

[1]Let $i \ge 0$ be an integer and consider the following submonoid of Inc that fixes the first i natural numbers:

$$\mathrm{Inc}^i = \{\pi \in \mathrm{Inc} \mid \pi(j) = j \ \text{ for all } \ j \le i\}.$$

Thus, in particular, $\mathrm{Inc} = \mathrm{Inc}^0$. We note that the notions and results presented in this survey can be extended to a more general setting, where the monoid Inc is replaced by Inc^i for any $i \ge 0$. However, for the sake of simplicity, we will only concentrate on the monoid Inc, referring the reader to [22, 23, 26, 27] for the general case.

$$x_{k,j} \le x_{k',j'} \quad \text{if either } k < k' \text{ or } k = k' \text{ and } j < j'. \tag{1}$$

The next simple lemma makes it possible to use the Gröbner bases method in studying Inc-invariant ideals, and hence, Sym-invariant ideals as well:

Lemma 2.3 *Let $(I_n)_{n\ge 1}$ be an* Inc*-invariant chain of ideals. Then for any monomial order $\le$ respecting* Inc*, the chain $(\text{in}_\le(I_n))_{n\ge 1}$ is also* Inc*-invariant. In particular, if $I \subseteq R$ is an* Inc*-invariant ideal, then $\text{in}_\le(I) := \bigcup_{n\ge 1} \text{in}_\le(I \cap R_n)$ is also an* Inc*-invariant ideal in R.*

Proof For the first assertion see, e.g., [26, Lemma 7.1]. The second assertion follows from the first one and the observation after Definition 2.1. □

Example 2.4 *(see also* [23, Example 2.2]*)* Let $R = K[x_j \mid j \in \mathbb{N}]$ (i.e., $c = 1$) with the field K having characteristic 0. Consider the Sym-invariant ideal I generated by $x_1^2 + x_2x_3$. Let $(I_n)_{n\ge 1}$ be the chain of truncations of I. Then $I_1 = I_2 = \langle 0\rangle$ and

$$\begin{aligned}
I_3 &= \text{Sym}(3)(x_1^2 + x_2x_3) = \langle x_1^2 + x_2x_3,\ x_2^2 + x_1x_3,\ x_3^2 + x_1x_2\rangle,\\
I_4 &= \text{Sym}(4)(x_1^2 + x_2x_3)\\
&= I_3 + \langle x_1^2 + x_2x_4,\ x_1^2 + x_3x_4,\ x_2^2 + x_1x_4,\ x_2^2 + x_3x_4,\\
&\qquad x_3^2 + x_1x_4,\ x_3^2 + x_2x_4,\ x_4^2 + x_1x_2,\ x_4^2 + x_1x_3,\ x_4^2 + x_2x_3\rangle.
\end{aligned}$$

Computations with Macaulay2 [14] using the reverse-lexicographic order on R induced by $x_1 < x_2 < x_3 < \cdots$ give

$$\begin{aligned}
\text{in}(I_3) &= \langle x_2^2,\ x_3x_2,\ x_3^2,\ x_2x_1^2,\ x_3x_1^2,\ x_1^4\rangle,\\
\text{in}(I_4) &= \langle x_2x_1,\ x_3x_1,\ x_4x_1,\ x_2^2,\ x_3x_2,\ x_4x_2,\ x_3^2,\ x_4x_3,\ x_4^2,\ x_1^3\rangle.
\end{aligned}$$

Since $x_2^2 \in \text{in}(I_3)$ but $x_1^2 \notin \text{in}(I_3)$, we see that $\text{Sym}(3)(\text{in}(I_3)) \nsubseteq \text{in}(I_3)$. Thus, the chain $(\text{in}(I_n))_{n\ge 1}$ is not Sym-invariant. Moreover, the ideal $\text{in}(I) = \bigcup_{n\ge 1} \text{in}(I_n)$ is also not Sym-invariant: otherwise, x_1^2 would belong to $\text{in}(I)$, which implies $x_1^2 \in I$ (since x_1^2 is the smallest element of R of degree 2), contradicting the fact that $I_1 = 0$.

On the other hand, the chain $(\text{in}(I_n))_{n\ge 1}$ and the ideal $\text{in}(I)$ are both Inc-invariant by Proposition 2.2 and Lemma 2.3. For instance, one can check that

$$\text{Inc}_{3,4}(\text{in}(I_3)) = \text{in}(I_3) + \langle x_4x_2,\ x_4x_3,\ x_4^2,\ x_4x_1^2\rangle \subseteq \text{in}(I_4).$$

Proposition 2.2, Lemma 2.3, and Example 2.4 suggest that even if one is primarily interested in Sym-invariant chains of ideals it is worthwhile and often more convenient to study the larger class of Inc-invariant chains.

To conclude this section let us recall the following fundamental finiteness result (see [18, Theorems 3.1, 3.6, Corollaries 3.5, 3.7] and also [26, Corollaries 3.6, 5.4, Lemma 5.2]):

Theorem 2.5 *The following statements hold:*

(i) The ring R is Inc*-Noetherian, i.e., for any* Inc*-invariant ideal $I \subseteq R$ there exist finitely many elements $f_1, \ldots, f_m \in R$ such that*

$$I = \langle \operatorname{Inc}(f_1), \ldots, \operatorname{Inc}(f_m) \rangle.$$

(ii) Every Inc*-invariant chain $\mathscr{I} = (I_n)_{n \geq 1}$ stabilizes, i.e., there exists an integer $r \geq 1$ such that for all $n \geq m \geq r$ one has*

$$I_n = \langle \operatorname{Inc}_{m,n}(I_m) \rangle$$

as ideals in R_n. The least integer r with this property is called the stability index *of $\mathscr{I}$, denoted by* $\operatorname{ind}(\mathscr{I})$.

In particular, analogous statements hold if Inc *is replaced by* $\operatorname{Sym}(\infty)$.

Example 2.6 Let $R = K[x_j \mid j \in \mathbb{N}]$ with char $K = 0$ and n_0 a natural number. Consider the ideal $I \subseteq R$ generated by the Sym-orbits of the elements $x_1^n + \cdots + x_n^n$ for all $n \geq n_0$, i.e.,

$$I = \langle \operatorname{Sym}(\infty)(x_1^n + \cdots + x_n^n) \mid n \geq n_0 \rangle.$$

It is evident that I is a Sym-invariant ideal. So by the previous theorem, I has a finite set of generators up to the action of $\operatorname{Sym}(\infty)$. How many elements do we need to generate I?[2] Quite surprisingly, one polynomial is enough, and in fact, I is a monomial ideal. We claim that

$$I = \langle \operatorname{Sym}(\infty)(x_1^{n_0}) \rangle.$$

Let J denote the ideal on the right hand side. Then clearly $I \subseteq J$. To show the reverse inclusion observe that for $j > 1$ one has

$$\begin{aligned} x_j^{n_0} - x_1^{n_0} &= (x_j^{n_0} + x_{j+1}^{n_0} + \cdots + x_{j+n_0-1}^{n_0}) - (x_1^{n_0} + x_{j+1}^{n_0} + \cdots + x_{j+n_0-1}^{n_0}) \\ &\in \langle \operatorname{Sym}(\infty)(x_1^{n_0} + \cdots + x_{n_0}^{n_0}) \rangle \subseteq I. \end{aligned}$$

It follows that

$$n_0 x_1^{n_0} = (x_1^{n_0} + \cdots + x_{n_0}^{n_0}) - (x_2^{n_0} - x_1^{n_0}) - \cdots - (x_{n_0}^{n_0} - x_1^{n_0}) \in I.$$

This gives $x_1^{n_0} \in I$ since char $K = 0$. Hence, $I = J$.

3 Hilbert Series, Multiplicity, and Codimension

The celebrated Hilbert theorem says that the Hilbert series of any finitely generated graded module over a standard graded K-algebra S is a rational function, and furthermore, this rational function encodes the dimension as well as the multiplicity of

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[2] We thank H. Brenner for asking this question and Hop D. Nguyen for suggesting the answer.

the module. In particular, given a proper graded ideal $J \subset S$, the Hilbert series of the quotient ring S/J,

$$H_{S/J}(t) = \sum_{u \geq 0} \dim_K (S/J)_u t^u,$$

can be uniquely expressed in the form

$$H_{S/J}(t) = \frac{Q(t)}{(1-t)^d}$$

with $Q(t) \in \mathbb{Z}[t]$ and $Q(1) > 0$. Moreover, one has that $d = \dim(S/J)$ and $Q(1) = e(S/J)$, the multiplicity of S/J. Note that $e(S/J)$ is also called the degree of J, denoted by $\deg(J)$.

The above result was extended to Inc-invariant chains of ideals by Nagel and Römer [26]. They defined the Hilbert series for such a chain, showed its rationality, and obtained from that the asymptotic behaviors of the codimension and the degree of the ideals in the chain. Let us now summarize their results.

Let $\mathscr{I} = (I_n)_{n \geq 1}$ be an Inc-invariant chain of graded ideals. Thus, each I_n is a graded ideal in the polynomial ring $R_n = K[x_{k,j} \mid 1 \leq k \leq c, 1 \leq j \leq n]$. The *(equivariant) Hilbert series* of $\mathscr{I}$ is defined as the following bivariate formal power series

$$H_{\mathscr{I}}(s,t) = \sum_{n \geq 0} H_{R_n/I_n}(t) s^n = \sum_{n \geq 0,\ u \geq 0} \dim_K (R_n/I_n)_u s^n t^u.$$

This series was introduced in [26] as a way to encode all the Hilbert series of the quotient rings R_n/I_n simultaneously. See also [31, 34] for related notions. The following crucial fact is given in [26, Proposition 7.2].

Theorem 3.1 *Let $\mathscr{I} = (I_n)_{n \geq 1}$ be an* Inc-*invariant chain of graded ideals. Then the Hilbert series of $\mathscr{I}$ is a rational function of the form*

$$H_{\mathscr{I}}(s,t) = \frac{g(s,t)}{(1-t)^a \prod_{l=1}^{b} [(1-t)^{c_l} - s \cdot f_l(t)]},$$

where $a,\ b,\ c_l \in \mathbb{Z}_{\geq 0}$ with $c_l \leq c$, $g(s,t) \in \mathbb{Z}[s,t]$, $f_l(t) \in \mathbb{Z}[t]$ and $f_l(1) > 0$ for every $l = 1, \ldots, b$.

The proof of this result is rather long and involved. Nevertheless, the proof technique is very useful, as it has been employed in [22, 23] to study the asymptotic behavior of the Castelnuovo-Mumford regularity, codimension and projective dimension along Inc-invariant chains, the main results of which are summarized below. A shorter proof of the rationality of $H_{\mathscr{I}}(s,t)$ using formal languages is given by Krone, Leykin and Snowden [21]. This approach, however, does not seem to yield information on the denominator of $H_{\mathscr{I}}(s,t)$ as stated in Theorem 3.1. In the case that the chain $\mathscr{I}$ is generated by one monomial, the rational form of $H_{\mathscr{I}}(s,t)$ can be determined explicitly, as carried out in [15]. Recently, inspired by the independent set theorem [18, Theorem 4.7], Maraj and Nagel [24] define multigraded Hilbert series of chains of ideals that are invariant under the action of a product of symmetric groups. They found a sufficient condition for the rationality of such series; see [24, Theorem 3.5].

Example 3.2 Let $c = 2$ and consider the ideal

$$I = \langle x_{1,j}x_{2,k} \mid j,k \in \mathbb{N}\rangle \subseteq R.$$

It is clear that I is an Inc-invariant ideal. Let $\mathscr{I} = (I_n)_{n\geq 1}$ denote the chain of truncations of I. Thus,

$$I_n = I \cap R_n = \langle x_{1,j}x_{2,k} \mid 1 \leq j,k \leq n\rangle.$$

One may view I_n as the edge ideal of the complete bipartite graph $K_{n,n}$ with vertex set $\{x_{1,1},\dots,x_{1,n}\} \cup \{x_{2,1},\dots,x_{2,n}\}$. So it is well-known that (see, e.g., [36, Exercise 7.6.11] or [8, Theorem 2.1])

$$H_{R_n/I_n}(t) = \frac{2}{(1-t)^n} - 1.$$

Hence, we get

$$\begin{aligned} H_{\mathscr{I}}(s,t) &= 1 + \sum_{n\geq 1}\Big(\frac{2}{(1-t)^n} - 1\Big)s^n = \sum_{n\geq 0}\frac{2s^n}{(1-t)^n} - \sum_{n\geq 0} s^n \\ &= \frac{2(1-t)}{1-t-s} - \frac{1}{1-s} = \frac{2(1-s)(1-t)-1}{(1-t-s)(1-s)}. \end{aligned}$$

Theorem 3.1 has implications for the asymptotic behaviors of the codimension and degree of ideals in Inc-invariant chains, which might be regarded as analogous to the fact mentioned above that the Hilbert series of an ideal encodes the codimension and degree of the ideal. The next result follows from [26, Theorem 7.10].

Corollary 3.3 *Let $\mathscr{I} = (I_n)_{n\geq 1}$ be an* Inc-*invariant chain of proper graded ideals. Then the following statements hold:*

(i) The codimension of I_n is eventually a linear function on n. More precisely, there are integers A, B with $0 \leq A \leq c$ such that

$$\operatorname{codim}(I_n) = An + B \quad \textit{for all} \quad n \gg 0.$$

(ii) The degree of I_n grows eventually exponentially. More precisely, there are integers $M > 0$, $L \geq 0$ and a rational number $Q > 0$ such that

$$\lim_{n\to\infty} \frac{\deg(I_n)}{M^n n^L} = Q.$$

The linear function $\operatorname{codim}(I_n)$ (for $n \gg 0$) is further investigated in [23], where its leading coefficient is (somewhat combinatorially) determined. We will discuss this result in the remaining part of this section.

In view of Lemma 2.3, we may assume that $\mathscr{I} = (I_n)_{n\geq 1}$ is an Inc-invariant chain of monomial ideals. Let $G(I_n)$ denote the minimal set of monomial generators of I_n. The following notion is essential for determining the growth of the function $\operatorname{codim}(I_n)$.

Definition 3.4 Let C be a subset of $[c] = \{1, \dots, c\}$. Assume that I_n is a proper monomial ideal in R_n. We say that C is a *cover* of I_n if for every $u \in G(I_n)$ there exist $k \in C$ and $j \geq 1$ such that $x_{k,j}$ divides u. Set

$$\gamma(I_n) = \min\{\#C \mid C \text{ is a cover of } I_n\}.$$

Example 3.5 Let $\mathscr{I} = (I_n)_{n\geq 1}$ be the Inc-invariant chain in Example 3.2. It is clear that each I_n has 2 minimal covers, namely, $C_1 = \{1\}$ and $C_2 = \{2\}$. Thus,

$$\gamma(I_n) = 1 \quad \text{for all } \ n \geq 1.$$

There is an alternative way to compute $\gamma(I_n)$ using minimal primes of I_n. Let φ be the map defined on the variables of R by

$$\varphi(x_{k,j}) = k \quad \text{for every } \ j \geq 1.$$

Then φ clearly induces a map, still denoted by φ, from the set $\operatorname{Min}(I_n)$ of minimal primes of I_n to the set of covers of I_n. Moreover, it can be shown that any minimal cover of I_n is of the form $\varphi(P)$ for some $P \in \operatorname{Min}(I_n)$; see [23, Proposition 3.3]. Therefore,

$$\gamma(I_n) = \min\{\#\varphi(P) \mid P \in \operatorname{Min}(I_n)\}.$$

Example 3.6 Assume $c \geq 4$ and consider the ideal

$$I_3 = \langle x_{1,2}^2 x_{2,3}^3,\ x_{1,1}x_{3,2}^2,\ x_{4,2}^2\rangle \subset R_3.$$

We have that

$$\operatorname{Min}(I_3) = \{\langle x_{1,1},\ x_{1,2},\ x_{4,2}\rangle,\ \langle x_{1,1},\ x_{2,3},\ x_{4,2}\rangle,\ \langle x_{1,2},\ x_{3,2},\ x_{4,2}\rangle,\ \langle x_{2,3},\ x_{3,2},\ x_{4,2}\rangle\}.$$

Thus,

$$\varphi(\operatorname{Min}(I_3)) = \{\{1, 4\},\ \{1, 2, 4\},\ \{1, 3, 4\},\ \{2, 3, 4\}\},$$

and so

$$\gamma(I_3) = \min\{\#\varphi(P) \mid P \in \operatorname{Min}(I_3)\} = 2.$$

One can also compute $\gamma(I_3)$ by observing that I_3 has two minimal covers: $C_1 = \{1, 4\}$ and $C_2 = \{2, 3, 4\}$.

Given an Inc-invariant chain of proper monomial ideals $\mathscr{I} = (I_n)_{n\geq 1}$, it follows readily from Definition 3.4 that γ is a non-decreasing function along this chain, in the sense that $\gamma(I_n) \leq \gamma(I_{n+1})$ for all $n \geq 1$ (see [23, Lemma 3.5]). Since $0 \leq \gamma(I_n) \leq c$ by definition, γ must stabilize, i.e., $\gamma(I_n) = \gamma(I_{n+1})$ for all $n \gg 0$. In fact, one has $\gamma(I_n) = \gamma(I_r)$ for all $n \geq r$, where $r = \operatorname{ind}(\mathscr{I})$ denotes the stability index of $\mathscr{I}$. This is because the set of covers of I_n is stable for $n \geq r$, which follows from the fact that the action of Inc on R keeps the first index of the variables unchanged (see [23, Lemma 3.6]). So we may define

$$\gamma(\mathscr{I}) = \gamma(I_n) \quad \text{for some } \ n \geq \operatorname{ind}(\mathscr{I}).$$

This number is exactly the leading coefficient of the function $\operatorname{codim}(I_n)$ when $n \gg 0$ (see [23, Theorem 3.8]):

Theorem 3.7 *Let $\mathscr{I} = (I_n)_{n\geq 1}$ be an* Inc*-invariant chain of proper monomial ideals. Then there exists an integer B such that*

$$\operatorname{codim}(I_n) = \gamma(\mathscr{I})n + B \quad \textit{for all } \ n \gg 0.$$

As a consequence, we obtain the following more explicit and slightly more general version of Corollary 3.3(i) (see [23, Corollary 3.12]). Note that this result is applicable also to non-graded ideals.

Corollary 3.8 *Let $\mathscr{I} = (I_n)_{n\geq 1}$ be an* Inc*-invariant chain of proper ideals, and let $\leq$ be any monomial order on R respecting* Inc. *Then there exists an integer B such that*

$$\operatorname{codim}(I_n) = \gamma(\operatorname{in}_{\leq}(\mathscr{I}))n + B \quad \textit{for all } \ n \gg 0.$$

In particular, the coefficient $\gamma(\operatorname{in}_{\leq}(\mathscr{I}))$ does not depend on the order $\leq$ and therefore will be simply denoted by $\gamma(\mathscr{I})$.

Example 3.9 Let p be a positive integer with $p \leq c$. For $n \geq 1$ we view the variables of R_n as a matrix $X_{c\times n}$ of size $c \times n$ and we denote by I_n the ideal generated by all p-minors of this matrix. Then the chain $\mathscr{I} = (I_n)_{n\geq 1}$ is evidently Sym-invariant, and therefore, it is also Inc-invariant. Let $\leq$ be a *diagonal term order* on R that respects Inc (e.g., the lexicographic order extending the order of the variables as given in (1)). Then by [35, Theorem 1] (see also [3, Theorem 5.3]), the p-minors of $X_{c\times n}$ form a Gröbner basis for I_n with respect to $\leq$ for all $n \geq p$. It follows that

$$\operatorname{in}_{\leq}(I_n) = \langle x_{k_1,j_1}\cdots x_{k_p,j_p} \mid 1 \leq k_1 < \cdots < k_p \leq c,\ 1 \leq j_1 < \cdots < j_p \leq n\rangle \ \text{ for } \ n \geq p.$$

Now one can easily check that the minimal covers of $\operatorname{in}_{\leq}(I_n)$ for $n \geq p$ are exactly subsets of $[c]$ of cardinality $c - p + 1$. Thus, $\gamma(\operatorname{in}_{\leq}(\mathscr{I})) = c - p + 1$, and so Corollary 3.8 gives

Fig. 1 Betti table of J

	0	1	J 2	…
0	$\beta_{0,0}(J)$	$\beta_{1,1}(J)$	$\beta_{2,2}(J)$	…
1	$\beta_{0,1}(J)$	$\beta_{1,2}(J)$	$\beta_{2,3}(J)$	…
2	$\beta_{0,2}(J)$	$\beta_{1,3}(J)$	$\beta_{2,4}(J)$	…
⋮	⋮	⋮	⋮	

$$\operatorname{codim}(I_n) = (c - p + 1)n + B \quad \text{for all} \ \ n \gg 0,$$

where B is a constant integer. Note that an exact formula for $\operatorname{codim}(I_n)$ is known, namely,

$$\operatorname{codim}(I_n) = (c - p + 1)(n - p + 1) \quad \text{for all} \ \ n \geq p$$

(see, e.g., [3, Theorem 6.8]).

Theorem 3.7 and Corollary 3.8 provide a convenient way to compute the constant A in Corollary 3.3. It is therefore desirable to ask for similar results for the other constants.

Problem 3.10 Determine the constants B, M, L, Q in Corollary 3.3.

In the following we will concentrate on some homological invariants, such as Betti numbers, the Castelnuovo-Mumford regularity or the projective dimension. Let us briefly recall these notions.

Let $S = K[y_1, \ldots, y_m]$ be a standard graded polynomial ring and $J \subsetneq S$ a nonzero graded ideal. Assume that the minimal graded free resolution of J is given by

$$0 \to \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{p,j}(J)} \xrightarrow{\partial_p} \cdots \xrightarrow{\partial_2} \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{1,j}(J)} \xrightarrow{\partial_1} \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{0,j}(J)} \xrightarrow{\partial_0} J \xrightarrow{\partial_{-1}} 0,$$

where ∂_{-1} is the zero map. Then the numbers $\beta_{i,j}(J)$ are called the *graded Betti numbers* of J. They can be expressed as

$$\beta_{i,j}(J) = \dim_K \operatorname{Tor}_i^S(J, K)_j.$$

The graded Betti numbers of J are usually given in a table, called the *Betti table* of J, in which the entry in the i-th column and j-th row is $\beta_{i,i+j}(J)$; see Fig. 1.

For $i \geq 0$ the module

$$\operatorname{Im}(\partial_i) = \operatorname{Ker}(\partial_{i-1})$$

is called the *i-th syzygy module* of J. The *projective dimension* and the *Castelnuovo-Mumford regularity* (or *regularity* for short) of J are defined as

$$\operatorname{pd}(J) = \max\{i \mid \beta_{i,j}(J) \neq 0 \ \text{ for some } \ j\},$$
$$\operatorname{reg}(J) = \max\{j \mid \beta_{i,i+j}(J) \neq 0 \ \text{ for some } \ i\}.$$

Thus, $\operatorname{pd}(J)$ and $\operatorname{reg}(J)$, respectively, are the index of the last nonzero column and last nonzero row of the Betti table of J. Note that $\operatorname{pd}(J)$ and $\operatorname{reg}(J)$ can also be interpreted as

$$\operatorname{pd}(J) = \max\{i \mid \operatorname{Ext}^i_S(J, S)_j \neq 0 \text{ for some } j\},$$
$$\operatorname{reg}(J) = \max\{j \mid \operatorname{Ext}^i_S(J, S)_{-i-j} \neq 0 \text{ for some } i\}.$$

Moreover, by Hilbert's Syzygy Theorem (see, e.g., [28, Theorem 15.2]) one always has

$$\operatorname{pd}(J) \leq \dim(S) - 1 = m - 1.$$

4 Castelnuovo-Mumford Regularity

The regularity of powers of a graded ideal in a polynomial ring is eventually a linear function. This beautiful result was independently proven by Cutkosky-Herzog-Trung [9] and Kodiyalam [20]. A similar behavior is expected for the regularity of ideals in an Inc-invariant chain, as proposed in [22, Conjecture 1.1]:

Conjecture 4.1 *Let $\mathscr{I} = (I_n)_{n\geq 1}$ be a nonzero* Inc*-invariant chain of graded ideals. Then* $\operatorname{reg}(I_n)$ *is eventually a linear function, that is, there exist integers C and D such that*

$$\operatorname{reg}(I_n) = Cn + D \quad \textit{whenever } n \gg 0.$$

As evidence for this conjecture, a rather sharp upper linear bound for $\operatorname{reg}(I_n)$ is obtained in [22]. Additionally, some special cases of the conjecture are verified in [22, 25, 30]. This section is devoted to discussing these results.

Let us begin with an upper linear bound for $\operatorname{reg}(I_n)$. We first assume that $\mathscr{I} = (I_n)_{n\geq 1}$ is a nonzero Inc-invariant chain of monomial ideals. As before, let $G(I_n)$ denote the minimal set of monomial generators of I_n. In order to bound $\operatorname{reg}(I_n)$, the following weights are introduced in [22, Definition 3.2]:

Definition 4.2 Let $k \in [c]$. For a nonzero monomial $u \in R_n$ set

$$w_k(u) = \max\{e \mid x^e_{k,j} \text{ divides } u \text{ for some } j \geq 1\},$$
$$w(u) = \max\{w_k(u) \mid k \in [c]\}.$$

Define the following weights for I_n:

$$w_k(I_n) = \max\{w_k(u) \mid u \in G(I_n)\},$$
$$\omega(I_n) = \min\{w(u) \mid u \in G(I_n)\}.$$

Example 4.3 Let $c = 3$ and consider the ideal $I_4 = \langle u_1, u_2, u_3, u_4\rangle \subset R_4$ with

$$u_1 = x_{1,1}^2 x_{2,1}^3 x_{2,2}, \quad u_2 = x_{1,3}^3 x_{2,2}^4 x_{3,2}^5, \quad u_3 = x_{3,1}, \quad u_4 = x_{1,4}^2.$$

Then we have that

$$\begin{aligned} w_1(u_1) &= 2, \quad w_2(u_1) = 3, \quad w_3(u_1) = 0, \quad w(u_1) = 3, \\ w_1(u_2) &= 3, \quad w_2(u_2) = 4, \quad w_3(u_2) = 5, \quad w(u_2) = 5, \\ w_1(u_3) &= 0, \quad w_2(u_3) = 0, \quad w_3(u_3) = 1, \quad w(u_3) = 1, \\ w_1(u_4) &= 2, \quad w_2(u_4) = 0, \quad w_3(u_4) = 0, \quad w(u_4) = 2. \end{aligned}$$

Thus,

$$w_1(I_4) = 3, \quad w_2(I_4) = 4, \quad w_3(I_4) = 5, \quad \omega(I_4) = 1.$$

Similarly to the function γ in the previous section, the weights defined in Definition 4.2 have the following stabilization property:

$$w_k(I_{n+1}) = w_k(I_n) \quad \text{and} \quad \omega(I_{n+1}) = \omega(I_n) \quad \text{for all} \;\; n \gg 0.$$

Indeed, let $r = \operatorname{ind}(\mathscr{I})$ be the stability index of $\mathscr{I}$. Then for $n \geq r$ one has that

$$\langle G(I_{n+1})\rangle = I_{n+1} = \langle \operatorname{Inc}_{n,n+1}(I_n)\rangle = \langle \operatorname{Inc}_{n,n+1}(G(I_n))\rangle,$$

giving

$$G(I_{n+1}) \subseteq \operatorname{Inc}_{n,n+1}(G(I_n)).$$

Using the latter inclusion one can show that (see [22, Lemma 3.5, Remark 3.6])

$$w_k(I_{n+1}) \leq w_k(I_n) \quad \text{and} \quad \omega(I_{n+1}) = \omega(I_n)$$

for every $k \in [c]$ and $n \geq r$. Now since $w_k(I_n)$ is a nonnegative integer, it must hold that $w_k(I_{n+1}) = w_k(I_n)$ for $n \gg 0$.

The stabilization of the weights w_k and ω allows us to extend them to the chain $\mathscr{I}$ by setting

$$\begin{aligned} w_k(\mathscr{I}) &= w_k(I_n) \quad \text{for} \;\; n \gg 0, \\ \omega(\mathscr{I}) &= \omega(I_n) \quad \text{for} \;\; n \geq r. \end{aligned}$$

As the next example demonstrates, it can happen that $w_k(\mathscr{I}) \neq w_k(I_r)$.

Example 4.4 Consider again the ideal I_4 in Example 4.3 and let $\mathscr{I} = (I_n)_{n\geq 1}$ be an Inc-invariant chain with $I_n = \langle \operatorname{Inc}_{4,n}(I_4)\rangle$ for all $n \geq 4$. Then $\operatorname{ind}(\mathscr{I}) = 4$. Using induction on n one can easily show that

$$I_n = \langle x_{1,j}^2 x_{2,j}^3 x_{2,k} \mid 1 \leq j < k \leq n-2, \; j < 4\rangle + \langle x_{3,j} \mid 1 \leq j \leq n-3\rangle + \langle x_{1,j}^2 \mid 4 \leq j \leq n\rangle$$

for all $n \geq 5$. This yields

$$w_1(\mathscr{I}) = w_1(I_n) = 2,\ w_2(\mathscr{I}) = w_2(I_n) = 3,\ w_3(\mathscr{I}) = w_3(I_n) = 1,\ \omega(\mathscr{I}) = \omega(I_n) = 1$$

for all $n \geq 5$. So from Example 4.3 we see that $w_k(\mathscr{I}) \neq w_k(I_4)$ for $k = 1, 2, 3$.

Now we are ready to state an upper linear bound for $\operatorname{reg}(I_n)$ (see [22, Theorem 4.1]):

Theorem 4.5 *Let* $\mathscr{I} = (I_n)_{n\geq 1}$ *be a nonzero* Inc*-invariant chain of monomial ideals. Set*

$$C(\mathscr{I}) = \max\{\omega(\mathscr{I}) - 1, 0\} + \max\Big\{\sum_{k\neq l} w_k(\mathscr{I}) \mid l \in [c]\Big\}.$$

Then there exists a constant $D(\mathscr{I})$ *such that*

$$\operatorname{reg}(I_n) \leq C(\mathscr{I})n + D(\mathscr{I}) \quad \textit{for all} \quad n \gg 0.$$

As illustrated by the following example, the bound provided in the previous theorem is rather sharp. Furthermore, it is predicted that the bound is tight in the case $c = 1$; see the discussion below.

Example 4.6 Let m be a positive integer. Consider the Inc-invariant chain $\mathscr{I} = (I_n)_{n\geq 1}$ given by $I_n = \langle \operatorname{Inc}_{1,n}(I_1)\rangle$ if $n \geq 1$ and

$$I_1 = \langle x_{1,1}^m \cdots x_{c,1}^m\rangle.$$

Then one has

$$I_n = \langle x_{1,1}^m \cdots x_{c,1}^m, \ldots, x_{1,n}^m \cdots x_{c,n}^m\rangle \quad \text{for} \quad n \geq 1.$$

Thus, it is evident that

$$w_1(\mathscr{I}) = \cdots = w_c(\mathscr{I}) = \omega(\mathscr{I}) = m.$$

Hence,

$$C(\mathscr{I}) = \omega(\mathscr{I}) - 1 + \max\Big\{\sum_{k\neq l} w_k(\mathscr{I}) \mid l \in [c]\Big\} = cm - 1.$$

Note that

$$\operatorname{reg}(I_n) = (cm-1)n + 1 \quad \text{for all} \quad n \geq 1,$$

as the generators of I_n form a regular sequence; see, e.g., [28, Theorem 20.2].

Theorem 4.5 can be immediately extended to any Sym- and Inc-invariant chain of graded ideals by virtue of Lemma 2.3 (see [22, Corollaries 4.6, 4.7]).

Corollary 4.7 *Let* $\mathscr{I} = (I_n)_{n\geq 1}$ *be a nonzero* Inc*-invariant chain of graded ideals. Let* $\leq$ *be a monomial order on* R *respecting* Inc*. Then there exists a constant* $D(\mathscr{I})$ *such that*

$$\operatorname{reg}(I_n) \le C(\operatorname{in}_{\le}(\mathscr{I}))n + D(\mathscr{I}) \quad \textit{for all } n \gg 0.$$

In particular, the conclusion is true if $\mathscr{I}$ *is a* Sym-*invariant chain of graded ideals.*

Next, we discuss some special cases of Conjecture 4.1. The following result summarizes several instances where Conjecture 4.1 holds true (see [22, Propositions 4.14, 4.16, Corollary 6.5]).

Theorem 4.8 *Let* $\mathscr{I} = (I_n)_{n\ge 1}$ *be a nonzero* Inc-*invariant chain of graded ideals. Then Conjecture 4.1 is true in the following cases:*

(i) R_n/I_n *is an Artinian ring for* $n \gg 0$.
(ii) There exists an Inc-*invariant ideal* $I \subseteq R$ *such that* $I_n = I \cap R_n$ *for* $n \gg 0$, *and*

  a. either I *is generated by the* Inc-*orbit of one monomial, i.e., there is a monomial* $u \in R$ *such that* $I = \langle \operatorname{Inc}(u)\rangle$, *or*
  b. $c = 1$ *and* I *is a squarefree monomial ideal.*

In the remainder of this section we consider the case $c = 1$, i.e., R has only one row of variables. For simplicity, we will write the variables of R as $x_i,\ i \in \mathbb{N}$. It is apparent that if $\mathscr{I} = (I_n)_{n\ge 1}$ is a nonzero Inc-invariant chain of proper monomial ideals, then $C(\mathscr{I}) = \omega(\mathscr{I}) - 1$. So Theorem 4.5 yields the following bound:

$$\operatorname{reg}(I_n) \le (\omega(\mathscr{I}) - 1)n + D(\mathscr{I}) \quad \text{for } n \gg 0,$$

where $D(\mathscr{I})$ is a suitable constant (see [22, Corollary 4.8]). Based on computational experiments it is conjectured in [22, Conjecture 4.12] that this bound is tight:

Conjecture 4.9 *Let* $c = 1$ *and let* $\mathscr{I} = (I_n)_{n\ge 1}$ *be a nonzero* Inc-*invariant chain of proper monomial ideals. Then there exists a constant* $D(\mathscr{I})$ *such that*

$$\operatorname{reg}(I_n) = (\omega(\mathscr{I}) - 1)n + D(\mathscr{I}) \quad \textit{for all } n \gg 0.$$

This conjecture is a special case of Conjecture 4.1 with a precise description of the slope of the linear function. Recently, using different approaches Murai [25] and Raicu [30] have independently verified this conjecture for Sym-invariant chains of monomial ideals. Note that if $\mathscr{I} = (I_n)_{n\ge 1}$ is a Sym-invariant chain of monomial ideals, then by Theorem 2.5 there exist monomials $u_1, \dots, u_m \in R$ such that

$$I_n = \langle \sigma(u_i) \mid 1 \le i \le m,\ \sigma \in \operatorname{Sym}(n)\rangle \quad \text{for } n \gg 0.$$

Evidently, we can choose each u_i of the form $u_i = x_1^{a_{i,1}} x_2^{a_{i,2}} \cdots x_{k_i}^{a_{i,k_i}}$ with

$$\mathbf{a}_i = (a_{i,1}, \dots, a_{i,k_i}) \in \mathbb{N}^{k_i} \quad \text{and} \quad a_{i,1} \ge a_{i,2} \ge \cdots \ge a_{i,k_i}.$$

Such an $\mathbf{a}_i$ is called a *partition of length* k_i. Setting $r = \max\{k_i \mid 1 \le i \le m\}$ we see that I_n is generated by the $\operatorname{Sym}(n)$-orbits of $u_1, \dots, u_m$ for all $n \ge r$. For brevity, we also say that the chain $\mathscr{I}$ is generated by the partitions $\mathbf{a}_1, \dots, \mathbf{a}_m$. Notice that $w(u_i) = a_{i,1}$ and

$$\omega(\mathscr{I}) = \min\{w(u_i) \mid 1 \le i \le m\} = \min\{a_{i,1} \mid 1 \le i \le m\}.$$

Using the above notation, the result of Murai [25, Proposition 3.9] and Raicu [30, Theorem 6.1] can be stated as follows:

Theorem 4.10 *Let $c = 1$ and let $\mathscr{I} = (I_n)_{n\ge1}$ be a nonzero* Sym*-invariant chain of proper monomial ideals. Assume that $\mathscr{I}$ is generated by the partitions $\mathbf{a}_1, \dots, \mathbf{a}_m$ of length at most r. Set $\alpha = (x_1 \cdots x_r)^{\omega(\mathscr{I})-1}$. Then one has*

$$\operatorname{reg}(I_n) = (\omega(\mathscr{I}) - 1)n + \operatorname{reg}(I_r : \alpha) \quad \textit{for} \ \ n \gg 0.$$

It should be noted that Murai and Raicu in fact prove stronger results: Murai [25, Theorem 1.1] obtains the asymptotic behavior of the Betti table of I_n (see Sect. 6), whereas Raicu [30, Theorem 3.1] provides a description of the graded components of the Ext modules $\operatorname{Ext}^j_{R_n}(R_n/I_n, R_n)$ for $j \ge 0$.

Example 4.11 Let $\mathscr{I} = (I_n)_{n\ge1}$ be the Sym-invariant chain of ideals generated by the two partitions (4, 1) and (3, 3). Then $I_1 = \langle 0\rangle$ and I_n is generated by the Sym(n)-orbits of $x_1^4x_2,\ x_1^3x_2^3$ for $n \ge 2$. For instance,

$$I_2 = \langle x_1^4x_2,\ x_1x_2^4,\ x_1^3x_2^3\rangle,$$
$$I_3 = \langle x_1^4x_2,\ x_1^4x_3,\ x_1x_2^4,\ x_1x_3^4,\ x_2^4x_3,\ x_2x_3^4,\ x_1^3x_2^3,\ x_1^3x_3^3,\ x_2^3x_3^3\rangle.$$

We have that $\omega(\mathscr{I}) = \min\{4, 3\} = 3$ and $\alpha = (x_1x_2)^{\omega(\mathscr{I})-1} = x_1^2x_2^2$. Thus,

$$I_2 : \alpha = \langle x_1^2,\ x_2^2,\ x_1x_2\rangle$$

with regularity $\operatorname{reg}(I_2 : \alpha) = 2$. So by Theorem 4.10,

$$\operatorname{reg}(I_n) = 2n + 2 \quad \text{for} \ \ n \gg 0.$$

5 Projective Dimension

Analogously to the codimension and Castelnuovo-Mumford regularity, the projective dimension is also expected to grow eventually linearly along Inc-invariant chains of ideals, as proposed in [23, Conjecture 1.1]:

Conjecture 5.1 *Let $(I_n)_{n\ge1}$ be an* Inc*-invariant chain of ideals. Then* $\operatorname{pd}(I_n)$ *is eventually a linear function, that is, there exist integers E and F such that*

$$\operatorname{pd}(I_n) = En + F \quad \textit{whenever} \ \ n \gg 0.$$

This section is devoted to discussing some evidence for this conjecture. First, recall that one has

$$\operatorname{pd}(I_n) \geq \operatorname{codim}(I_n) - 1, \tag{2}$$

with equality if I_n is a Cohen-Macaulay ideal. So Corollary 3.8 immediately gives the following (see [23, Proposition 4.1]):

Proposition 5.2 *Let $\mathscr{I} = (I_n)_{n\geq 1}$ be an* Inc*-invariant chain of ideals such that I_n is Cohen-Macaulay for all $n \gg 0$. Then Conjecture 5.1 is true for $\mathscr{I}$.*

Note that the ideal I_n generated by the p-minors of a $c \times n$ generic matrix is a Cohen-Macaulay ideal; see, e.g., [4, Theorem 7.3.1]. So Proposition 5.2 is applicable to the chain $\mathscr{I} = (I_n)_{n\geq 1}$ considered in Example 3.9. It is also applicable to any chain that is generated by one monomial orbit; see [15, Corollary 2.2].

Further evidence for Conjecture 5.1 is given by the next result, which provides upper and lower linear bounds for $\operatorname{pd}(I_n)$ (see [23, Propositions 4.3, 4.13]):

Proposition 5.3 *Let $\mathscr{I} = (I_n)_{n\geq 1}$ be an* Inc*-invariant chain of proper ideals. Then there exists an integer B such that*

$$\gamma(\mathscr{I})n + B \leq \operatorname{pd}(I_n) \leq cn - 1 \quad \textit{for} \quad n \gg 0.$$

In particular, if $c = 1$, then there is a positive integer B' such that

$$n - B' \leq \operatorname{pd}(I_n) \leq n - 1 \quad \textit{for} \quad n \gg 0.$$

Note that the upper bound for $\operatorname{pd}(I_n)$ in this proposition follows from Hilbert's Syzygy Theorem, while the lower bound is a consequence of Corollary 3.8 and Inequality (2). In the case that $\mathscr{I} = (I_n)_{n\geq 1}$ is a chain of monomial ideals, the lower bound for $\operatorname{pd}(I_n)$ can be considerably improved; see [23, Theorems 4.6, 4.10].

We conclude this section with the following result of Murai [25, Corollary 3.7] which verifies Conjecture 5.1 for Sym-invariant chains of monomial ideals in the case $c = 1$. Together with Theorem 4.10 this is another consequence of Murai's investigation of the asymptotic behavior of Betti tables along such chains, to be discussed in the next section.

Theorem 5.4 *Let $c = 1$ and let $\mathscr{I} = (I_n)_{n\geq 1}$ be a nonzero* Sym*-invariant chain of proper monomial ideals. Assume that $\mathscr{I}$ is generated by the partitions $\mathbf{a}_1, \ldots, \mathbf{a}_m$ of length at most r. Then there is a positive integer B' such that*

$$\operatorname{pd}(I_n) = n - B' \quad \textit{for} \quad n \geq r.$$

It is worth noting that the integer B' in the previous theorem can be combinatorially determined; see [30, Theorem on regularity and projective dimension].

6 Syzygies and Betti Tables

Given an Inc-invariant chain $\mathscr{I} = (I_n)_{n\geq 1}$ of graded ideals, what can be said about the asymptotic behavior of the syzygies of I_n? A first answer to this question was provided by Nagel and Römer [27], who established a stabilization result for the syzygy modules of I_n when $n \gg 0$. However, a full description of this interesting result would require the theory of FI- and OI-modules with varying coefficients that is beyond the scope of this survey. Therefore, in this section, we will only discuss a rather informal version of Nagel-Römer's result and its consequence on the Betti table of I_n. We close the section with a recent result of Murai [25], which provides more information on the Betti table of I_n in the case $c = 1$ and $\mathscr{I} = (I_n)_{n\geq 1}$ is a Sym-invariant chain of monomial ideals.

Let us begin with a simple example.

Example 6.1 Let $c = 1$ and consider the Inc-invariant chain $\mathscr{I} = (I_n)_{n\geq 1}$ with

$$I_n = \langle x_j x_k \mid 1 \leq j < k \leq n\rangle \subseteq R_n.$$

Then the first syzygies of I_n are given by the following equations:

$$\begin{aligned} x_{k+1}(x_j x_k) - x_k(x_j x_{k+1}) &= 0 \quad \text{for} \quad j < k, \\ x_{j+1}(x_j x_k) - x_j(x_{j+1} x_k) &= 0 \quad \text{for} \quad j+1 < k. \end{aligned}$$

We see that these syzygies are determined through the Inc-action by two first syzygies of I_3, namely,

$$\begin{aligned} x_3(x_1x_2) - x_2(x_1x_3) &= 0, \\ x_2(x_1x_3) - x_1(x_2x_3) &= 0. \end{aligned}$$

This example illustrates a much more general result due to Nagel and Römer [27, Theorem 7.1]. Informally, it implies that for any Inc-invariant chain $\mathscr{I} = (I_n)_{n\geq 1}$ of graded ideals and any integer $p \geq 0$, the p-syzygies of the ideals I_n "look alike" eventually. Slightly more precisely, this means that there exists a positive integer n_p such that the p-syzygies of I_n are determined through the Inc-action by the p-syzygies of I_{n_p} for all $n \geq n_p$. Note that [27, Theorem 7.1] is actually stated in a more general context of FI- and OI-modules. See also [34, Theorem A] for a related result.

The above result of Nagel and Römer leads to the following stabilization of the Betti table (see [27, Theorem 7.7] for a more general version):

Theorem 6.2 *Let $\mathscr{I} = (I_n)_{n\geq 1}$ be an* Inc*-invariant chain of graded ideals. Then for any integer $p \geq 0$ the set*

$$\{j \in \mathbb{Z} \mid \beta_{p,j}(I_n) \neq 0\}$$

I_2	0	1
5	2	.
6	1	2

I_3	0	1	2
5	6	3	.
6	3	6	.
7	.	.	.
8	.	2	3

I_4	0	1	2	3
5	12	12	4	.
6	6	12	.	.
7	.	.	.	.
8	.	8	12	.
9	.	.	.	.
10	.	.	3	4

I_5	0	1	2	3	4
5	20	30	20	5	.
6	10	20	.	.	.
7	.	.	.	.	.
8	.	20	30	.	.
9	.	.	.	.	.
10	.	.	15	20	.
11	.	.	.	.	.
12	.	.	.	4	5

I_6	0	1	2	3	4	5
5	30	60	60	30	6	.
6	15	30	.	.	.	.
7	.	.	.	.	.	.
8	.	40	60	.	.	.
9	.	.	.	.	.	.
10	.	.	45	60	.	.
11	.	.	.	.	.	.
12	.	.	.	24	30	.
13	.	.	.	.	.	.
14	.	.	.	.	5	6

I_6	0	1	2	3	4	5
5	①	①	①	①	①	.
6	②	③	.	.	.	.
7	.	.	.	.	.	.
8	.	②	③	.	.	.
9	.	.	.	.	.	.
10	.	.	②	③	.	.
11	.	.	.	.	.	.
12	.	.	.	②	③	.
13	.	.	.	.	.	.
14	.	.	.	.	②	③

Fig. 2 Betti tables of some I_n. The circled numbers in the last table represent 3 line segments

stabilizes for $n \gg 0$. More precisely, there exist integers $j_0 < \cdots < j_t$ depending on p and $\mathscr{I}$ such that for $n \gg 0$,

$$\beta_{p,j}(I_n) \neq 0 \quad \textit{if and only if} \quad j \in \{j_0, \ldots, j_t\}.$$

This result implies that for any $p \geq 0$, the p-th column of the Betti table of I_n has a stable shape whenever $n \gg 0$. In the special case when $c = 1$ and $\mathscr{I} = (I_n)_{n\geq 1}$ is a Sym-invariant chain of monomial ideals, Murai [25] recently obtained a much stronger result, yielding the stabilization of the whole Betti table of I_n when $n \gg 0$. Before stating his result, let us consider an example.

Example 6.3 As in Example 4.11, let $\mathscr{I} = (I_n)_{n\geq 1}$ be the Sym-invariant chain generated by the two partitions (4, 1) and (3, 3). The Betti tables of some ideals in this chain, computed by Macaulay2, are shown in Fig. 2.

As one might have noticed, these tables suggest that the set

$$\{(i, j) \in \mathbb{Z}^2 \mid \beta_{i,i+j}(I_n) \neq 0\}$$

of nonzero positions in the Betti table of I_n is a union of *line segments* of length $n - 2$. Here, for integers $i, j, s, \ell \geq 0$, a line segments of length ℓ with starting point (i, j) and slope s is the following subset of $\mathbb{Z}^2$:

$$\mathscr{L}((i, j), s, \ell) = \{(i + k, j + sk) \in \mathbb{Z}^2 \mid k = 0, 1, \ldots, \ell\}.$$

In our example, it holds for $n \geq 2$ that (see [25, Proposition 2.7]):

$$\{(i, j) \mid \beta_{i,i+j}(I_n) \neq 0\} = \mathscr{L}((0,5), 0, n-2) \cup \mathscr{L}((0,6), 2, n-2) \cup \mathscr{L}((1,6), 2, n-2).$$

The main result of Murai [25, Theorem 1.1] asserts that the phenomenon observed in the previous example is true asymptotically for any Sym-invariant chain of monomial ideals:

Theorem 6.4 *Let $c = 1$ and let $\mathscr{I} = (I_n)_{n\geq 1}$ be a* Sym*-invariant chain of monomial ideals. Assume that $\mathscr{I}$ is generated by the partitions $\mathbf{a}_1, \ldots, \mathbf{a}_m$ of length at most r. Then there exist finite sets $M \subseteq \mathbb{Z}^2_{\geq 0}$ and $N \subseteq \{0, 1, \ldots, r-1\}$ such that for every $n \geq r$ one has*

$$\{(i, j) \mid \beta_{i,i+j}(I_n) \neq 0\} = \bigcup_{(i,j)\in M,\ s\in N} \mathscr{L}((i, j), s, n-r) \cup \{(i, j) \mid \beta_{i,i+j}(I_{r-1}) \neq 0\}.$$

This theorem implies the stabilization of the shape of the Betti table of I_n for $n \geq r$: the nonzero positions in the Betti table of I_{n+1} is obtained from the ones of I_n by just extending each line segment $\mathscr{L}((i, j), s, n-r)$ one point further. Thus, in particular, the shape of the Betti table of I_r completely determines that of I_n for all $n \geq r$. Interesting consequences of this fact are the asymptotic behavior of $\operatorname{reg}(I_n)$ and $\operatorname{pd}(I_n)$ mentioned earlier (see Theorems 4.10 and 5.4)

It should be noted that Murai proved Theorem 6.4 by studying the multigraded components of the Tor modules $\operatorname{Tor}_i(I_n, K)$ through homology modules of simplicial complexes. In fact, he was able to determine the (non-)vanishing of the multigraded Betti numbers of I_n (see [25, Theorem 3.2]).

7 Open Problems

The study of asymptotic behavior of Inc-invariant chains is in its early stage and many interesting problems are still open. Some of them are scattered throughout the previous sections. In this section we provide some further problems.

First of all, the following problem arises naturally from Proposition 5.2.

Problem 7.1 Characterize those Sym- or Inc-invariant chains of ideals $\mathscr{I} = (I_n)_{n\geq 1}$ for which I_n is Cohen-Macaulay whenever $n \gg 0$.

In the case $c = 1$ and $\mathscr{I} = (I_n)_{n\geq 1}$ is a Sym-invariant chain of monomial ideals, this problem has been resolved recently by Raicu [30, Theorem on injectivity of maps from Ext to local cohomology]:

Theorem 7.2 *Let $c = 1$ and let $\mathscr{I} = (I_n)_{n\geq 1}$ be a* Sym*-invariant chain of monomial ideals. Assume that $\mathscr{I}$ is generated by the partitions $\mathbf{a}_1, \ldots, \mathbf{a}_m$ of length at most r. Write $\mathbf{a}_i = (a_{i,1}, \ldots, a_{i,k_i})$. Then the following are equivalent for $n \geq r$:*

(i) I_n is Cohen-Macaulay.

(ii) I_n *is unmixed.*
(iii) $a_{i,1} = \cdots = a_{i,p}$ *for every* $i = 1, \ldots, m$, *where* $p = \dim(R_n/I_n) + 1$.

Under the assumption of the preceding theorem note that $\dim(R_n/I_n) = p - 1$ for all $n \geq r$, where p is the minimal length of a partition $\mathbf{a}_i$; see [25, Corollary 3.8].

The next problem is closely related to Problem 7.1.

Problem 7.3 Let the chain $\mathscr{I} = (I_n)_{n \geq 1}$ be Sym- or Inc-invariant. Study the primary decomposition of I_n.

Finally, in view of Theorems 6.2 and 6.4 the following problem is of great interest.

Problem 7.4 Study the asymptotic behavior of Betti tables of ideals of Sym- or Inc-invariant chains.

It should be mentioned that two questions related to this problem are proposed by Murai in [25, Questions 5.3, 5.5], where he asks for a generalization of Theorem 6.4 to Inc-invariant chains and for a way to determine the Betti numbers of ideals in Sym-invariant chains.

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On Piecewise-Linear Homeomorphisms Between Distributive and Anti-blocking Polyhedra

Christoph Pegel and Raman Sanyal

Abstract Stanley (1986) introduced the order polytope and chain polytope of a partially ordered set and showed that they are related by a piecewise-linear homeomorphism. In this paper we view order and chain polytopes as instances of distributive and anti-blocking polytopes, respectively. Both these classes of polytopes are defined in terms of the componentwise partial order on $\mathbb{R}^n$. We generalize Stanley's PL-homeomorphism to a large class of distributive polyhedra using infinite walks in marked networks.

Keywords Order polytopes · Chain polytopes · Distributive polyhedra · Anti-blocking polyhedra · Piecewise-linear maps · Marked networks

2010 Mathematics Subject Classification 52B12 · 05C20 · 52A41

1 Introduction

Let $(P, \preceq)$ be a finite partially ordered set (**poset**, for short). Stanley [22] introduced two convex polytopes associated to P, the **order polytope**

$$\mathscr{O}(P) := \left\{ f \in \mathbb{R}^P : \begin{array}{rcll} 0 & \le & f(a) \le 1 & \text{for all } a \in P \\ f(a) & \le & f(b) & \text{for all } a \prec b \end{array} \right\} \qquad (1)$$

and the **chain polytope**

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$$\mathscr{C}(P) := \left\{ g \in \mathbb{R}^P : \begin{array}{rcll} g(a) & \geq & 0 & \text{for all } a \in P \\ g(a_1) + g(a_2) + \cdots + g(a_k) & \leq & 1 & \text{for all } a_1 \prec a_2 \prec \cdots \prec a_k \end{array} \right\}. \quad (2)$$

The poset can be completely recovered from $\mathscr{O}(P)$ and many geometric properties of $\mathscr{O}(P)$ can be translated into combinatorial properties of P. In particular, the Ehrhart polynomial of $\mathscr{O}(P)$ is the order polynomial of P and the normalized volume $(|P|)! \cdot \mathrm{vol}(\mathscr{O}(P))$ is the number of linear extensions of P. We refer the reader to Stanley's original paper and [3, Ch. 6] for more details. So it is fair to say that the order polytope $\mathscr{O}(P)$ gives a geometric representation of P. The chain polytope, on the other hand, is defined in terms of the **comparability graph** $G_P = (P, E)$ of P. Two elements $a, b \in P$ share an edge in G if and only if $a \prec b$ or $b \prec a$. Chains in P correspond to cliques in G. The comparability graph can be recovered from $\mathscr{C}(P)$ but P is in general not determined by G_P. Stanley defines a piecewise-linear (PL) homeomorphism $\phi \colon \mathbb{R}^P \to \mathbb{R}^P$ called the **transfer map** that is volume- and lattice preserving and that maps $\mathscr{O}(P)$ to $\mathscr{C}(P)$. This shows, quite unexpectedly, that both polytopes have the same Ehrhart polynomial and normalized volume and, consequently, that order polynomial and number of linear extensions only depend on the comparability graph. Order and chain polytopes have many applications in combinatorics as well as in geometry and, together with their connecting PL-homeomorphism, have been generalized to marked posets [1, 15, 21], to marked chain-order polytopes [9, 10], and to double posets [8], to name a few. The aim of this paper is to give a generalization of Stanley's transfer map to a larger class of geometric objects that we now define.

Let V be some finite set and $\mathbb{R}^V$ equipped with the usual componentwise partial order $\leq$. A convex polyhedron $Q \subseteq \mathbb{R}^V_{\geq 0}$ is called **anti-blocking** [14] or a **convex corner** [4] if for $y \in Q$ and $x \in \mathbb{R}^V_{\geq 0}$

$$x \leq y \quad \Longrightarrow \quad x \in Q\,. \quad (3)$$

The chain polytope is easily seen to be anti-blocking. An **order ideal** in a poset is a subset that is down-closed with respect to the partial order. Condition (3) thus states that anti-blocking polyhedra can be viewed as **convex order ideals** in $(\mathbb{R}^n_{\geq 0}, \leq)$.

For $x, y \in \mathbb{R}^V$, let us write $x \wedge y$ and $x \vee y$ for the coordinate-wise minimum and maximum, respectively. In particular, $(\mathbb{R}^V, \wedge, \vee)$ is an (infinite) distributive lattice with meet $\wedge$ and join $\vee$. It is straightforward to verify that $\mathscr{O}(P)$ is closed under meets and joins. Thus $\mathscr{O}(P)$ is a polyhedron as well as a sublattice of $\mathbb{R}^P$. Such polyhedra were introduced by Felsner and Knauer [12] under the name **distributive polyhedra**. Felsner and Knauer noted that order polytopes and, more generally, alcoved polytopes [19] are distributive. Since marked order polytopes are coordinate sections of dilated order polytopes, they are automatically distributive. There are many other polyhedra in combinatorics that turn out to be distributive. For example, the t-Cayley and t-Gayley polytopes of Konvalinka and Pak [17], the s-lecture hall polytopes and cones of Bousquet-Mélou and Eriksson [5, 6], and their poset generalizations due to Brändén–Leander [7]. See Sect. 5 for more on these classes of examples.

Stanley's piecewise-linear homeomorphism connects the distributive polytopes $\mathscr{O}(P)$ to the anti-blocking polytope $\mathscr{C}(P)$ with phenomenal combinatorial consequences. Similar PL-maps have been constructed in other contexts. For example, the polytope $P_n(x)$ studied by Pitman–Stanley is an anti-blocking polytope and a linear isomorphism to a distributive polytope is constructed in [24, Sect. 4]. Beck, Braun, and Le [2] introduced **Cayley polytopes** C_n (denoted by $\mathbf{A}_n$ in [18]) as

$$C_n \;=\; \{x \in \mathbb{R}^n : 1 \le x_i \le 2x_{i-1} \text{ for all } 1 \le i \le n\}\,,$$

where $x_0 := 1$. Cayley polytopes are distributive. In [18], Konvalinka and Pak define the polytope $\mathbf{Y}_n$, that turns out to be anti-blocking, as the set of all $y \in \mathbb{R}^n$ with $y \ge 0$ and for all $1 \le h \le n$

$$\sum_{j=1}^{h} 2^{h-j} y_j \le 2^h - 1$$

and a linear lattice-preserving map $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ with $\phi(\mathbf{Y}_n) = C_n$ to give a simple proof of a conjecture of Braun on partitions [2].

In this paper, we study the relation between distributive and anti-blocking polyhedra more closely and we construct PL-homeomorphisms for a large class of distributive polyhedra that subsumes marked order polyhedra. Our PL-maps generalize Stanley's original construction as well as the mentioned examples and depends on the convergence of series given by infinite walks in directed networks. Most of the work presented here also appeared in the first authors PhD thesis [20].

2 Distributive Polyhedra and Marked Networks

A **marked network** is a tuple $\Gamma = (V \uplus A, E, \alpha, c, \lambda)$. It consists of a finite loop-free directed multigraph $(V \uplus A, E)$ on nodes $V \uplus A$ with edges E. We refer to the nodes in A as **marked nodes** with **marking** $\lambda \in \mathbb{R}^A$. To every directed edge $v \xrightarrow{e} w$ there are two associated weights $\alpha_e, c_e \in \mathbb{R}$ with $\alpha_e > 0$. In drawings of a marked network, we will depict an edge $v \xrightarrow{e} w$ with weights α_e and c_e as

$$v \bullet\!\xrightarrow{\alpha_e, c_e}\!\bullet\, w\,,$$

where blue labels are node names. Marked nodes are drawn as squares with red labels and when edge weights are omitted, we always assume $\alpha_e = 1$ and $c_e = 0$. See Figs. 1 and 2a.

To a marked network, we associate the polyhedron $\mathscr{O}(\Gamma) \subseteq \mathbb{R}^V$ consisting of all points $x \in \mathbb{R}^V$ such that

$$\alpha_e x_w + c_e \;\le\; x_v \quad \text{for all edges } v \xrightarrow{e} w\,, \tag{4}$$

where we set $x_v := \lambda_v$ for $v \in A$.

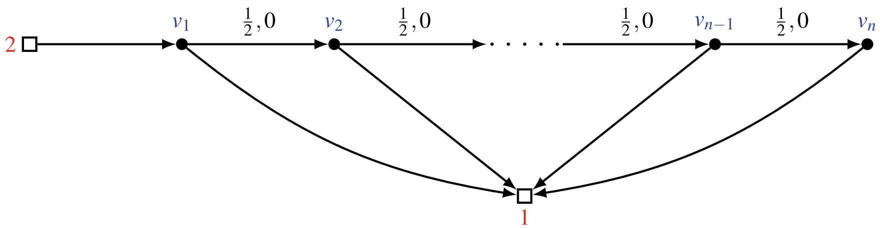


Fig. 1 The marked network defining the Cayley polytope C_n

Example 2.1 (*Marked order polyhedra*) For a poset $(P, \preceq)$ let $\widehat{P} = P \uplus \{\hat{0}, \hat{1}\}$ be the poset with minimum $\hat{0}$ and maximum $\hat{1}$. A marked network is obtained from the Hasse diagram of $\widehat{P}$ with $A := \{\hat{0}, \hat{1}\}$, $V := P$, and E consisting of edges $v \to w$ for w covered by v. Setting $\alpha \equiv 1$, $c \equiv 0$ and $(\lambda_{\hat{0}}, \lambda_{\hat{1}}) = (0, 1)$, we obtain the order polytope $\mathcal{O}(P)$. By allowing more general A, this yields the marked order polyhedra [1, 21].

In a similar fashion one sees that the Cayley polytope C_n is also of the form $\mathcal{O}(\Gamma)$ for the simple network given in Fig. 1. It is straightforward to verify that $\mathcal{O}(\Gamma)$ is a distributive polyhedron. The main result in [12] is a characterization of distributive polyhedra in terms of marked networks.

Theorem 2.2 ([12, Thm. 4]) *Every distributive polyhedron is of the form $\mathcal{O}(\Gamma)$ for some marked network Γ.*

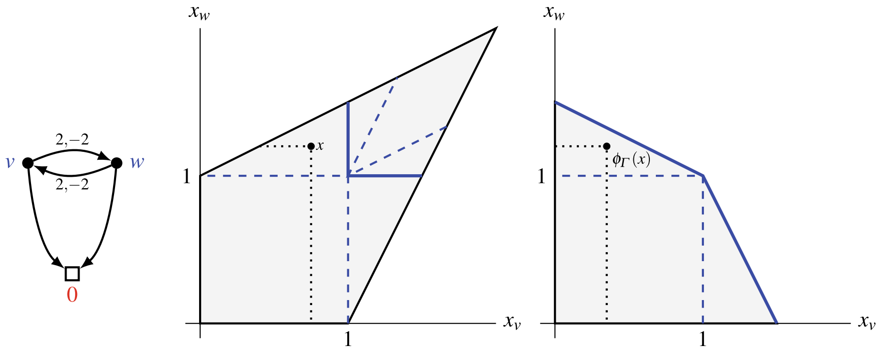
To a marked network with at least all sinks marked, we associate the **transfer map** $\phi_\Gamma : \mathbb{R}^V \to \mathbb{R}^V$ defined as

$$\phi_\Gamma(x)_v \; := \; x_v - \max_{v \xrightarrow{e} w} (\alpha_e x_w + c_e) \, . \tag{5}$$

Let us point out again, that $x_v = \lambda_v$ for $v \in A$.

Remark 2.3 Our definition of marked networks is slightly different from that employed in [12]. Instead of loops, we use markings and require $\alpha_e > 0$. Moreover, our edge weights (α_e, c_a) translate to $(\lambda, c) = (\frac{1}{\alpha_e}, \frac{-c_e}{\alpha_e})$ in the notation of [12]. The reason for this change of notation is that it allows for a simpler definition of transfer maps (avoiding fractions and a separate treatment of loops) which is in line with the transfer maps defined for (marked) order polyhedra in [1, 8–10, 15, 21]. Weights as used in [12] will naturally appear in the dual transfer map in Sect. 4.1.

If Γ is derived from a poset P as in Example 2.1, the map ϕ_Γ is the original transfer map from [22]. If Γ is **acyclic**, that is, the underlying directed graph has no directed cycles, then we will see in Theorem 3.5 that ϕ_Γ is bijective. In the non-acyclic situation, this need not be true. In order to illustrate, let us give a geometric reformulation of the transfer map. We denote the standard basis of $\mathbb{R}^V$ by $\{e_v\}_{v \in V}$.



(a) Network Γ (b) the polytope $\mathscr{O}(\Gamma)$ (c) the image $\phi_\Gamma(\mathscr{O}(\Gamma))$

Fig. 2 The marked network Γ of Example 2.4 with the associated distributive polytope and its "folded" image under the non-injective transfer map

For a polyhedron $Q \subseteq \mathbb{R}^V$ that does not contain any ray of the form $x - \mathbb{R}_{\geq 0} e_v$ for $x \in Q$ and $v \in V$, define the map $\phi_Q \colon Q \to \mathbb{R}^V$ by

$$\phi_Q(x)_v \;:=\; \max(\mu \geq 0 : x - \mu e_v \in Q) \qquad (6)$$

for all $v \in V$. If Q is defined by linear inequalities of the form $\ell_i(x) \leq b_i$, then

$$\phi_Q(x)_v \;=\; \min\left(\tfrac{b_i - \ell_i(x)}{-\ell_i(e_v)} \;:\; \text{for } i \text{ with } \ell_i(e_v) < 0\right) .$$

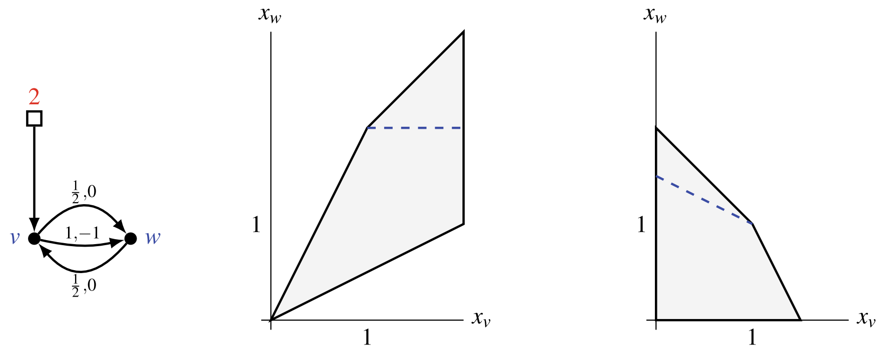
From (4), we conclude that $\phi_{\mathscr{O}(\Gamma)}$ coincides with ϕ_Γ.

Example 2.4 Let Γ be the marked network depicted in Fig. 2a. The distributive polyhedron $\mathscr{O}(\Gamma)$ is a "kite" given by the inequalities $0 \leq x_v$, $0 \leq x_w$, $2x_v - 2 \leq x_w$ and $2x_w - 2 \leq x_v$ as shown in Fig. 2b. The transfer map for this network is given by

$$\phi_\Gamma \begin{pmatrix} x_v \\ x_w \end{pmatrix} \;=\; \begin{pmatrix} x_v - \max\{0, 2x_w - 2\} \\ x_w - \max\{0, 2x_v - 2\} \end{pmatrix} .$$

The transfer map is not injective on $\mathscr{O}(\Gamma)$. For example the vertices (0, 0) and (2, 2) both get mapped to the origin. In fact, the map is 2-to-1 and "folds" the polytope along the thick blue line in Fig. 2b. The dashed lines in the lower left part stay fixed under the transfer map and have the same image as the dashed lines in the upper right part. The geometric behavior of the transfer map given above is shown for some $x \in \mathscr{O}(\Gamma)$ using dotted lines.

Example 2.5 Let Γ be the marked network depicted in Fig. 3a. The distributive polyhedron $\mathscr{O}(\Gamma)$ is a quadrilateral given by the inequalities $\frac{1}{2}x_v \leq x_w$, $\frac{1}{2}x_w \leq x_v$,



(a) a network Γ (b) the polytope $\mathcal{O}(\Gamma)$ (c) the image $\phi_\Gamma(\mathcal{O}(\Gamma))$

Fig. 3 The marked network Γ of Example 2.5 with the associated distributive polytope and its bijective image under the transfer map

$x_w - 1 \leq x_v$ and $x_v \leq 2$ as shown in Fig. 3b. The transfer map for this network is given by

$$\phi_\Gamma \begin{pmatrix} x_v \\ x_w \end{pmatrix} = \begin{pmatrix} x_v - \max\{\frac{1}{2}x_w, x_w - 1\} \\ x_w - \frac{1}{2}x_v \end{pmatrix}.$$

In this example, the transfer map is bijective and maps $\mathcal{O}(\Gamma)$ to the anti-blocking polytope depicted in Fig. 3c. The dashed line divides $\mathcal{O}(\Gamma)$ into the two linearity regions of the transfer map. We will come back to this example in Sect. 4 after constructing inverse transfer maps and describing the inequalities for $\phi_\Gamma(\mathcal{O}(\Gamma))$.

As we have seen in Examples 2.4 and 2.5, some cyclic networks lead to bijective transfer maps while others do not. The important difference in the two examples is the product of weights along the cycles. This motivates the following definition.

Definition 2.6 Let $\Gamma = (V \uplus A, E, \alpha, c, \lambda)$ be a marked network. A finite (resp. infinite) **walk** W in Γ is a finite (resp. infinite) sequence

$$W = v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} \cdots \xrightarrow{e_r} v_{r+1} \left(\xrightarrow{e_{r+1}} \cdots\right).$$

If W is finite, then its **length** $|W| := r$ is the number of edges. In this case, the **weight** of W is

$$\alpha(W) := \prod_{i=1}^{|W|} \alpha_{e_i}.$$

If all nodes are distinct, then W is called a **path**. If all nodes are distinct except for $v_{r+1} = v_1$, then we call W a **directed cycle**. In accordance with [12], a directed cycle C is called **gainy** if $\alpha(C) < 1$, **lossy** if $\alpha(C) > 1$ and **breakeven** if $\alpha(C) = 1$.

Finally, we call a marked network **gainy/lossy/breakeven** if all directed cycles are gainy/lossy/breakeven.

Note that in contrast to [12] we only consider and define weights for *directed* cycles.

In the following section, we will show that the observation made in Examples 2.4 and 2.5 is true in general: when all directed cycles in Γ are gainy, the transfer map is bijective.

3 Gainy Networks and Infinite Walks

Throughout this section we assume that $\Gamma = (V \uplus A, E, \alpha, c, \lambda)$ is a gainy marked network such that every sink is marked. Our goal is to construct an inverse to the transfer map ϕ_Γ and show that the image $\phi_\Gamma(\mathscr{O}(\Gamma))$ is an anti-blocking polyhedron by giving explicit inequalities determined by walks in Γ. Note that if the underlying directed graph of Γ is acyclic, then Γ is gainy and the following results apply. This, in particular, includes the case of (marked) order polyhedra and their transfer maps.

Definition 3.1 To Γ associate the set $\mathscr{W}$ consisting of finite walks

$$v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} \cdots \xrightarrow{e_r} v_{r+1} \quad \text{with } v_i \in V \text{ for } i \leq r \text{ and } v_{r+1} \in A, \tag{7}$$

as well as infinite walks

$$v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} v_3 \xrightarrow{e_3} \cdots \quad \text{with all } v_i \in V. \tag{8}$$

In particular, $A \subseteq \mathscr{W}$, since walks of length 0 are allowed.

Given a walk $W \in \mathscr{W}$ starting in w and an edge $v \xrightarrow{e} w$ from an unmarked node $v \in V$, denote by $v \xrightarrow{e} W$ the walk in $\mathscr{W}$ obtained by prepending the edge e.

In order to define the inverse transfer map, we want to associate to each walk $W \in \mathscr{W}$ an affine-linear form $\Sigma(W)$ on $\mathbb{R}^V$ satisfying $\Sigma(a)(x) := \lambda_a$ for all trivial walks at a marked element $a \in A$ and for all walks $W = v \to W'$ of positive length, the recursion

$$\Sigma(W)(x) \ = \ \alpha_e \Sigma(W')(x) + (x_v + c_e) \ . \tag{9}$$

In order to see that Σ is well-defined on infinite walks, we need the following statement on convergence of infinite series.

Proposition 3.2 *Let $W \in \mathscr{W}$ be an infinite walk as in* (8). *The infinite series*

$$\sum_{k=1}^{\infty} \left(\prod_{j=1}^{k-1} \alpha_{e_j} \right) \left(x_{v_k} + c_{e_k} \right)$$

absolutely converges for all $x \in \mathbb{R}^V$.

Proof Since Γ has only finitely many nodes and edges, we have $\left|x_{v_k} + c_{e_k}\right| \leq M$ for some M. It is therefore enough to show absolute convergence of $\sum_{k=1}^{\infty} \prod_{j=1}^{k-1} \alpha_{e_j}$. Using the root test, it is sufficient to show that

$$\limsup_{k\to\infty} \left(\prod_{j=1}^{k} \alpha_{e_j}\right)^{\frac{1}{k}} < 1.$$

Since Γ is finite, there are only finitely many paths and directed cycles and we can define

$$a := \max\left\{\alpha(C)^{\frac{1}{|C|}} : C \text{ directed cycle}\right\} \quad \text{and} \quad b := \max\left\{\alpha(P)^{\frac{1}{|P|}} : P \text{ path}\right\}.$$

Now fix some $k \in \mathbb{N}$ and consider the truncated walk

$$W^{(k)} = v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} \cdots \xrightarrow{e_k} v_{k+1}$$

We may decompose $W^{(k)}$ into a path from v_1 to v_{k+1} and finitely many directed cycles as depicted in Fig. 4. If the path has $t \leq k$ edges, the total number of edges in the cycles is $k - t$ and we obtain

$$\alpha_{W^{(k)}} = \prod_{j=1}^{k} \alpha_{e_j} \leq a^{k-t} b^t = \left(\frac{b}{a}\right)^t a^k.$$

Let $\ell \geq 0$ be the maximal length of a path in Γ and set $c = \max(b/a, 1)$ to obtain

$$\left(\prod_{j=1}^{k} \alpha_{e_j}\right)^{\frac{1}{k}} \leq \left(c^{\ell} a^k\right)^{\frac{1}{k}} = c^{\frac{\ell}{k}} a \xrightarrow{k\to\infty} a.$$

Since all directed cycles in Γ are gainy by assumption, we have $a < 1$, finishing the proof. □

Using Proposition 3.2, we can define the desired linear forms.

Definition 3.3 For $W \in \mathscr{W}$ define an affine-linear form $\Sigma(W)\colon \mathbb{R}^V \to \mathbb{R}$ as follows. If W is a finite walk as in (7), set

$$\Sigma(W)(x) := \sum_{k=1}^{r} \left(\prod_{j=1}^{k-1} \alpha_{e_j}\right) \left(x_{v_k} + c_{e_k}\right) + \left(\prod_{j=1}^{r} \alpha_{e_j}\right) x_{v_{r+1}}.$$

If W is an infinite walk as in (8), set

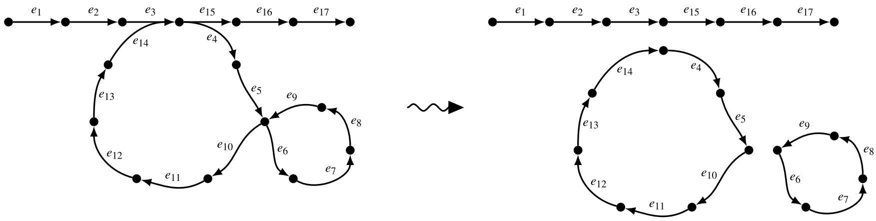


Fig. 4 The decomposition of a finite walk into a path and directed cycles as used in the proof of Proposition 3.2

$$\Sigma(W)(x) \;:=\; \sum_{k=1}^{\infty}\left(\prod_{j=1}^{k-1}\alpha_{e_j}\right)\left(x_{v_k}+c_{e_k}\right).$$

By construction, the defined linear forms satisfy the recursion (9). Indeed (9) together with $\Sigma(a)(x) := \lambda_a$ uniquely determines the linear forms $\Sigma(W)$ given the convergence in Proposition 3.2.

Proposition 3.4 *For any* $x \in \mathbb{R}^V$ *we have* $\sup_{W\in\mathscr{W}} \Sigma(W)(x) < \infty$.

Proof Let the constants M, a, b, c, ℓ be given as in the proof of Proposition 3.2. For finite walks $W \in \mathscr{W}$ as in (7), we have

$$\Sigma(W)(x) \;=\; \sum_{k=1}^{r}\left(\prod_{j=1}^{k-1}\alpha_{e_j}\right)\left(x_{v_k}+c_{e_k}\right) + \left(\prod_{j=1}^{r}\alpha_{e_j}\right) x_{v_{r+1}} \;\le\; M\sum_{k=1}^{r+1} c^{\ell} a^{k-1} \;\le\; \frac{Mc^{\ell}}{1-a}.$$

Likewise, for infinite walks as in (8), we have

$$\Sigma(W)(x) \;=\; \sum_{k=1}^{\infty}\left(\prod_{j=1}^{k-1}\alpha_{e_j}\right)\left(x_{v_k}+c_{e_k}\right) \;\le\; M\sum_{k=1}^{\infty} c^{\ell} a^{k-1} \;=\; \frac{Mc^{\ell}}{1-a}.$$

□

3.1 The Inverse Transfer Map

We are now ready to construct an inverse to the transfer map ϕ_Γ. For $v \in V$ denote by $\mathscr{W}_v$ the set of all walks $\gamma \in \mathscr{W}$ starting in v.

Theorem 3.5 *Let* $\Gamma = (V \uplus A, E, \alpha, c, \lambda)$ *be a gainy marked network with all sinks marked. The transfer map* $\phi_\Gamma : \mathbb{R}^V \to \mathbb{R}^V$ *is a piecewise-linear bijection with inverse* $\psi_\Gamma : \mathbb{R}^V \to \mathbb{R}^V$ *given by*

$$\psi_\Gamma(y)_v \;:=\; \sup_{W\in\mathscr{W}_v} \Sigma(W)(y).$$

Since part of the proof of Theorem 3.5 will be relevant when we give a description of $\phi_\Gamma(\mathscr{O}(\Gamma))$ below, we provide the following lemma first.

Lemma 3.6 *For any $x \in \mathbb{R}^V$ and $v \in V \uplus A$ we have*

$$\sup_{W \in \mathscr{W}_v} \Sigma(W)(\phi_\Gamma(x)) \leq x_v.$$

Proof Let $y = \phi_\Gamma(x)$ for $x \in \mathbb{R}^V$. For a finite walk $W \in \mathscr{W}$ as in (7) starting in $v_1 = v$, we have

$$\begin{aligned}
\Sigma(W)(y) &= \sum_{k=1}^{r} \left(\prod_{j=1}^{k-1} \alpha_{e_j} \right) \left(y_{v_k} + c_{e_k} \right) + \left(\prod_{j=1}^{r} \alpha_{e_j} \right) y_{v_{r+1}} \\
&= \sum_{k=1}^{r} \left(\prod_{j=1}^{k-1} \alpha_{e_j} \right) \left(x_{v_k} - \max_{v_k \xrightarrow{e} w} (\alpha_e x_w + c_e) + c_{e_k} \right) + \left(\prod_{j=1}^{r} \alpha_{e_j} \right) x_{v_{r+1}} \\
&\leq \sum_{k=1}^{r} \left(\prod_{j=1}^{k-1} \alpha_{e_j} \right) \left(x_{v_k} - \alpha_{e_k} x_{v_{k+1}} \right) + \left(\prod_{j=1}^{r} \alpha_{e_j} \right) x_{v_{r+1}} \\
&= x_{v_1} = x_v.
\end{aligned}$$

For an infinite walk as in (8) starting in $v_1 = v$, we have

$$\begin{aligned}
\Sigma(W)(y) &= \sum_{k=1}^{\infty} \left(\prod_{j=1}^{k-1} \alpha_{e_j} \right) \left(y_{v_k} + c_{e_k} \right) \\
&\leq \sum_{k=1}^{\infty} \left(\prod_{j=1}^{k-1} \alpha_{e_j} \right) \left(x_{v_k} - \alpha_{e_k} x_{v_{k+1}} \right) = x_{v_1} = x_v.
\end{aligned}$$

□

Proof (***Theorem*** 3.5) For $v \in V$, all walks $W \in \mathscr{W}_v$ are of the form $v \xrightarrow{e} W'$ for an edge $v \xrightarrow{e} w$ and $W' \in \mathscr{W}_w$. Hence, by the recursive property (9) we have

$$\Sigma(W)(y) = \alpha_e \Sigma(W')(y) + (y_v + c_e).$$

We conclude that ψ satisfies the recursion

$$\psi_\Gamma(y)_v = y_v + \max_{v \xrightarrow{e} w} (\alpha_e \psi_\Gamma(y)_w + c_e) \quad \text{for all } v \in V.$$

Comparing this to the definition of ϕ_Γ, we see that $\phi_\Gamma \circ \psi_\Gamma$ is the identity on $\mathbb{R}^V$.

Regarding the composition $\psi_\Gamma \circ \phi_\Gamma$, first note that $\psi_\Gamma(\phi_\Gamma(x))_v \leq x_v$ for all $v \in V$ by Lemma 3.6. Hence, to show that $\psi_\Gamma(\phi_\Gamma(x))_v = x_v$ for $v \in V$, it is enough to construct a walk $W \in \mathscr{W}_v$ such that $\Sigma(W)(\phi_\Gamma(x)) \geq x_v$. Let $v_1 = v$ and successively pick an edge $v_k \xrightarrow{e_k} v_{k+1}$ such that

$$\alpha_{e_k} x_{v_{k+1}} + c_{e_k} = \max_{v_k \xrightarrow{e} w} (\alpha_e x_w + c_e),$$

until either v_{k+1} is marked or v_{k+1} already appeared in $\{v_1, \ldots, v_k\}$.

In the first case we constructed a finite walk $W \in \mathscr{W}_v$ as in (7) satisfying

$$\begin{aligned}\Sigma(W)(\phi_\Gamma(x)) &= \sum_{k=1}^{r} \left(\prod_{j=1}^{k-1} \alpha_{e_j} \right) \left(\phi_\Gamma(x)_{v_k} + c_{e_k} \right) + \left(\prod_{j=1}^{r} \alpha_{e_j} \right) \phi_\Gamma(x)_{v_{r+1}} \\ &= \sum_{k=1}^{r} \left(\prod_{j=1}^{k-1} \alpha_{e_j} \right) \left(x_{v_k} - \alpha_{e_k} x_{v_{k+1}} \right) + \left(\prod_{j=1}^{r} \alpha_{e_j} \right) x_{v_{r+1}} = x_{v_1} = x_v.\end{aligned}$$

In the second case, we ended at an unmarked element $v_{r+1} = v_s$ for $s \leq r$. This yields an infinite walk $W \in \mathscr{W}_v$ of the form

$$v_1 \xrightarrow{e_1} \cdots \xrightarrow{e_{s-1}} v_s \xrightarrow{e_s} \cdots \xrightarrow{v_{r-1}} e_r \xrightarrow{e_r} v_s \xrightarrow{e_s} \cdots \xrightarrow{v_{r-1}} e_r \xrightarrow{e_r} v_s \xrightarrow{e_s} \cdots \tag{10}$$

That is, W walks from v_1 to v_s and then infinitely often runs through the cycle

$$v_s \xrightarrow{e_s} v_{s+1} \xrightarrow{e_{s+1}} \cdots \xrightarrow{v_{r-1}} e_r \xrightarrow{e_r} v_s.$$

Treating indices $k > r$ accordingly, we obtain

$$\begin{aligned}\Sigma(W)(\phi_\Gamma(x)) &= \sum_{k=1}^{\infty} \left(\prod_{j=1}^{k-1} \alpha_{e_j} \right) \left(\phi_\Gamma(x)_{v_k} + c_{e_k} \right) \\ &= \sum_{k=1}^{\infty} \left(\prod_{j=1}^{k-1} \alpha_{e_j} \right) \left(x_{v_k} - \alpha_{e_k} x_{v_{k+1}} \right) = x_{v_1} = x_v.\end{aligned}$$

In both cases $\Sigma(W)(\phi_\Gamma(x)) = x_v$ and we obtain $\psi_\Gamma(\phi_\Gamma(x))_v = x_v$ as desired. We conclude that ϕ_Γ and ψ_Γ are mutually inverse piecewise-linear self-maps of $\mathbb{R}^V$. □

Inspecting the proof of Theorem 3.5, we see that only a finite subset of $\mathscr{W}$ is necessary to define ψ_Γ. Namely, the paths with only the last node marked and the infinite walks that keep repeating a cycle after a finite number of steps as in (10). We will refer to walks of the latter kind as **monocycles** and denote them by $W = P * C$, where P is the path and C the cycle. Note that only the end node of P is shared with C. A visual representation of a monocycle can be found in Fig. 5.

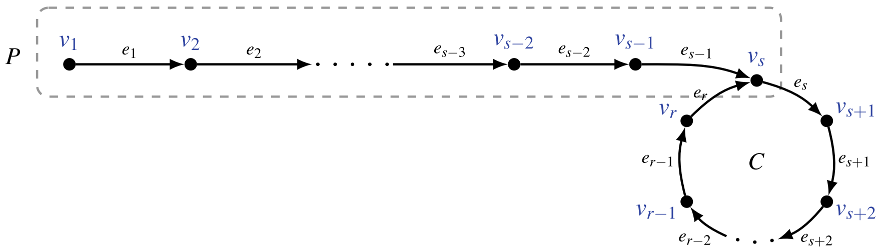


Fig. 5 A monocycle. Note that the visible nodes are pairwise distinct

Definition 3.7 Let $\widehat{\mathscr{W}} \subseteq \mathscr{W}$ be the subset of walks $W \in \mathscr{W}$ such that W is either a path or a monocycle as in (10) with pairwise distinct $v_1, \ldots, v_r$. For $v \in V$, denote by $\widehat{\mathscr{W}_v} = \widehat{\mathscr{W}} \cap \mathscr{W}_v$ the set of walks in $\widehat{\mathscr{W}}$ starting in v.

Corollary 3.8 *The inverse transfer map $\psi_\Gamma : \mathbb{R}^V \to \mathbb{R}^V$ is given by*

$$\psi_\Gamma(y)_v = \max_{W \in \widehat{\mathscr{W}_v}} \Sigma(W)(y).$$

Since some of the $W \in \widehat{\mathscr{W}_v}$ appearing in this description of the inverse transfer map might be monocycles, we want to give a finite expression for the linear form $\Sigma(W)$.

Proposition 3.9 *Let $W = P * C \in \widehat{\mathscr{W}}$ be a monocycle with*

$$\begin{aligned} P &= v_1 \xrightarrow{e_1} \cdots \xrightarrow{e_{s-1}} v_s, \\ C &= v_s \xrightarrow{e_s} \cdots \xrightarrow{e_{r-1}} v_r \xrightarrow{e_r} v_s. \end{aligned}$$

Then for all $x \in \mathbb{R}^V$ we have

$$\Sigma(W)(x) = \sum_{k=1}^{s-1} \left(\prod_{j=1}^{k-1} \alpha_{e_j} \right) \left(x_{v_k} + c_{e_k} \right) + \frac{\alpha(P)}{1 - \alpha(C)} \sum_{k=s}^{r} \left(\prod_{j=s}^{k-1} \alpha_{e_j} \right) \left(x_{v_k} + c_{e_k} \right).$$

Proof The infinite series in Definition 3.3 yields that $\Sigma(W)(x)$ is equal to

$$\sum_{k=1}^{s-1} \left(\prod_{j=1}^{k-1} \alpha_{e_j} \right) \left(x_{v_k} + c_{e_k} \right) + \alpha(P) \sum_{l=0}^{\infty} \left[\alpha(C)^l \sum_{k=s}^{r} \left(\prod_{j=s}^{k-1} \alpha_{e_j} \right) \left(x_{v_k} + c_{e_k} \right) \right]. \quad (11)$$

Since all directed cycles in Γ are gainy, we have $\alpha(C) < 1$ and the geometric series $\sum_{l=0}^{\infty} \alpha(C)^l$ converges to $(1 - \alpha(C))^{-1}$. □

Let $x \in \mathbb{R}^V$. Now for every $v \in V$, select an edge $v \to w$ which attains the maximum in the definition of $\phi_\Gamma(x)_v$ in (5). It becomes clear from our discussion that this yields a marked subnetwork Γ_x composed of paths and monocycles. More precisely, deleting the directed cycles, leaves a *rooted* forest, that is, an acyclic digraph in which every node has one edge pointing out. This network realizes ϕ_Γ as an affine-linear function at x. The matrix $B = B(\Gamma_x) \in \mathbb{R}^{V \times V}$ with $B_{ww} = 1$ and

$$B_{wv} \ = \ -\alpha_e \quad \text{if } w \xrightarrow{e} v$$

and 0 otherwise determines the linear part of ϕ_Γ at x. The determinant of B is $\prod_C (1 - \alpha(C))$ where C ranges over the directed cycles in Γ_x.

Corollary 3.10 *Let $\Gamma = (V \uplus A, E, \alpha, c, \lambda)$ be a gainy marked network with all sinks marked. If $\alpha(C) = 2$ for all directed cycles C, then ϕ_Γ is volume preserving. If the weights α and c are integral, then ϕ_Γ is lattice-preserving if and only if for every directed cycle there is a unique edge e' with weight $\alpha_{e'} = 2$ and all other edges have weight $\alpha_e = 1$.*

4 Anti-blocking Images

In the previous section, we showed that distributive polyhedra given by gainy marked networks with at least all sinks marked admit a piecewise-linear bijective transfer map $\phi_\Gamma : \mathbb{R}^V \to \mathbb{R}^V$ analogous to the transfer map for (marked) order polytopes. In this section we keep the same premise and focus on the image $\phi_\Gamma(\mathscr{O}(\Gamma))$. We show that it is an anti-blocking polyhedron with describing inequalities given by the walks in $\widehat{\mathscr{W}}$, similar to the chain polytope being described by inequalities given by chains in the poset.

Definition 4.1 Let $\Gamma = (V \uplus A, E, \alpha, c, \lambda)$ be a gainy marked network with at least all sinks marked. The polyhedron $\mathscr{C}(\Gamma)$ is the set of all $y \in \mathbb{R}^V$ with $y \geq 0$ and

$$\alpha_e \Sigma(W)(y) + c_e \ \leq \ \lambda_a \tag{12}$$

for each walk $a \xrightarrow{e} W$ with $a \in A$ and $W \in \widehat{\mathscr{W}}$.

Theorem 4.2 *Let $\Gamma = (V \uplus A, E, \alpha, c, \lambda)$ be a gainy marked network with at least all sinks marked. The transfer map ϕ_Γ restricts to a piecewise-linear homeomorphism $\mathscr{O}(\Gamma) \to \mathscr{C}(\Gamma)$.*

Proof To show that $\phi_\Gamma(\mathscr{O}(\Gamma)) \subseteq \mathscr{C}(\Gamma)$, let $y = \phi_\Gamma(x)$ for $x \in \mathscr{O}(\Gamma)$. By definition of ϕ_Γ we have $y_v \geq 0$ for $v \in V$. Now let $a \xrightarrow{e} W$ be a walk with $a \in A$ and $W \in \widehat{\mathscr{W}}$. It follows from Lemma 3.6 that

$$\alpha_e \Sigma(W)(y) + c_e \leq \alpha_e x_v + c_e \leq x_a = \lambda_a.$$

Now let y be any point in $\mathscr{C}(\Gamma)$ and let $x = \psi_\Gamma(y)$. For any edge $v \xrightarrow{e} w$ we have to show that $\alpha_e x_w + c_e \leq x_v$. Let $W \in \widehat{\mathscr{W}}_w$ be a walk starting in w constructed as in the proof of Theorem 3.5 such that $\Sigma(W)(y) = x_w$. If $v \in V$, we can again appeal to Lemma 3.6 together with $y \geq 0$ to obtain

$$\alpha_e x_w + c_e = \alpha_e \Sigma(W)(y) + c_e = \Sigma(v \xrightarrow{e} W)(y) - x_v \leq \Sigma(v \xrightarrow{e} W)(y) \leq x_v.$$

Otherwise, if $v \in A$, the walk $v \xrightarrow{e} W$ appears in Definition 4.1, so that

$$\alpha_e x_w + c_e = \alpha_e \Sigma(W)(y) + c_e \leq \lambda_v = x_v.$$

Fulkerson [14] introduced anti-blocking polyhedra and gave the following characterization.

Proposition 4.3 ([14]) *A polyhedron $Q \subseteq \mathbb{R}^d_{\geq 0}$ is anti-blocking if and only if there are $a_1, \ldots, a_m \in \mathbb{R}^d_{\geq 0}$ and $b_1, \ldots, b_m \in \mathbb{R}_{\geq 0}$ such that*

$$Q = \{x \in \mathbb{R}^d_{\geq 0} : a_i^t x \leq b_i \text{ for } i = 1, \ldots, m\}.$$

This description allows us to prove the following.

Corollary 4.4 *The polyhedron $\mathscr{C}(\Gamma)$ is anti-blocking.*

Proof By definition $\mathscr{C}(\Gamma) \subseteq \mathbb{R}^V_{\geq 0}$. Furthermore, the coefficients in an inequality $\alpha_e \Sigma(\gamma)(y) + c_e \leq \lambda_a$ are all non-negative: for finite walks they are just finite products of edge weights $\alpha_{e'}$ while for monocycles some of them are multiplied by the positive factor $\alpha(P)/(1 - \alpha(C))$ as described in Proposition 3.9. □

Example 4.5 (*continuation of Example* 2.5) Recall the marked network Γ with two unmarked nodes depicted in Fig. 3 together with the distributive polytope $\mathscr{O}(\Gamma)$ and its anti-blocking image now denoted by $\mathscr{C}(\Gamma)$. We label the three edges between v and w from top to bottom by e, f, g.

Since Γ does not have marked nodes with incoming edges and all directed cycles contain only unmarked nodes, the set of monocycles $\widehat{\mathscr{W}}$ is given by the directed cycles with trivial acyclic beginning:

$$W_1 = v \xrightarrow{e} w \xrightarrow{g} v \xrightarrow{e} w \xrightarrow{g} \cdots \qquad W_2 = v \xrightarrow{f} w \xrightarrow{g} v \xrightarrow{f} w \xrightarrow{g} \cdots$$
$$W_3 = w \xrightarrow{g} v \xrightarrow{e} w \xrightarrow{g} v \xrightarrow{e} \cdots \qquad W_4 = w \xrightarrow{g} v \xrightarrow{f} w \xrightarrow{g} v \xrightarrow{f} \cdots$$

From Proposition 3.9 with trivial acyclic beginning ($s = 1$) we obtain

$$\Sigma(W_1)(y) = \tfrac{4}{3}x_v + \tfrac{2}{3}x_w \qquad \Sigma(W_2)(y) = 2x_v + 2x_w - 2$$
$$\Sigma(W_3)(y) = \tfrac{2}{3}x_v + \tfrac{4}{3}x_w \qquad \Sigma(W_4)(y) = x_v + 2x_w - 1$$

Hence, the inverse transfer map on $\mathbb{R}^V$ is given by

$$\psi_\Gamma \begin{pmatrix} y_v \\ y_w \end{pmatrix} = \begin{pmatrix} \max\{\frac{4}{3}x_v + \frac{2}{3}x_w, 2x_v + 2x_w - 2\} \\ \max\{\frac{2}{3}x_v + \frac{4}{3}x_w, x_v + 2x_w - 1\} \end{pmatrix}.$$

Note that the linearity regions are the two half-spaces given by the hyperplane $\frac{1}{3}x_v + \frac{2}{3}x_w = 1$ containing the dashed line in Fig. 3c. For the anti-blocking image $\mathscr{C}(\Gamma)$ the only walks appearing in Definition 4.1 are $2 \to W_1$ and $2 \to W_2$ giving inequalities

$$\tfrac{4}{3}x_v + \tfrac{2}{3}x_w \le 2 \quad \text{and} \quad 2x_v + 2x_w \le 4.$$

These correspond to the two non-trivial facets in Fig. 3c.

In Example 2.4, where we have a lossy directed cycle and the transfer map is not injective, the image was still an anti-blocking polytope. However, this is not true in general: in the following example we have a lossy directed cycle, an injective transfer map nevertheless, but the image $\phi_\Gamma(\mathscr{O}(\Gamma))$ is not anti-blocking.

Example 4.6 Let Γ be the marked network shown in Fig. 6a. The distributive polyhedron $\mathscr{O}(\Gamma)$ is the unbounded polyhedron in Fig. 6b given by the inequalities $2x_v - 4 \le x_w$, $2x_w - 4 \le x_v$, $x_v \le 3$ and $x_w \le 3$. The transfer map is given by

$$\phi_\Gamma \begin{pmatrix} x_v \\ x_w \end{pmatrix} = \begin{pmatrix} x_v - 2x_w + 4 \\ x_w - 2x_v + 4 \end{pmatrix}.$$

Thus, the image $\phi_\Gamma(\mathscr{O}(\Gamma))$ is the polyhedron given by inequalities $0 \le y_v$, $0 \le y_w$, $y_v + 2y_w \ge 3$ and $2y_v + y_w \ge 3$. It is depicted in Fig. 6c and is not an anti-blocking polyhedron. In fact it is what is called a **blocking polyhedron** in [14]: it is given given by inequalities $x_i \ge 0$ for all coordinates together with inequalities of the form $a^t x \ge 1$ with $a \in \mathbb{R}^n_{\ge 0}$.

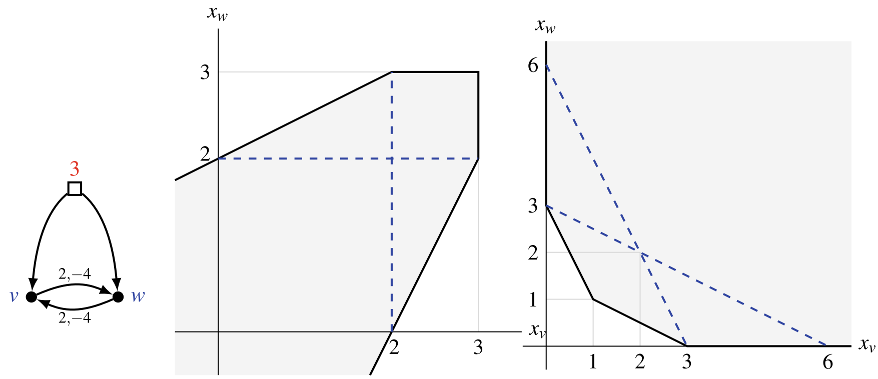
4.1 Duality

Let $Q \subseteq \mathbb{R}^n$ be a distributive polyhedron. It follows from the definition that $-Q$ is distributive as well. If $Q = \mathscr{O}(\Gamma)$, then $-Q = \mathscr{O}(\Gamma^{\text{op}})$, where $\Gamma^{\text{op}} = (V \uplus A, E', \alpha', c', \lambda')$ is the **opposite network** with edges $w \xrightarrow{e'} v$ for each edge $v \xrightarrow{e} w$ in Γ, weights $\alpha_{e'} = \frac{1}{\alpha_e}$, $c_{e'} = \frac{c_e}{\alpha_e}$, and $\lambda' = -\lambda$. If $(P, \preceq)$ is a poset, then $-\mathscr{O}(P)$ is, up to a translation, the order polytope $\mathscr{O}(P^{\text{op}})$, where P^{op} is the opposite poset.

If Γ is a network with all *sources* marked, then $\phi_\Gamma^{\text{op}} : \mathbb{R}^V \to \mathbb{R}^V$ given by $\phi_\Gamma^{\text{op}}(x) := \phi_{\Gamma^{\text{op}}}(-x)$ is a piecewise-linear map. More precisely, it is given by

$$\phi_\Gamma^{\text{op}}(x)_v \;=\; -x_v + \min_{w \xrightarrow{e} v} \left(\tfrac{1}{\alpha_e} x_w - \tfrac{c_e}{\alpha_e} \right).$$

If Γ has only lossy directed cycles, then ϕ_Γ^{op} is bijective and restricts to a homeomorphism $\mathscr{O}(\Gamma) \to \mathscr{C}(\Gamma^{\text{op}})$.



(a) a network Γ (b) the polyhedron $\mathscr{O}(\Gamma)$ (c) the image $\phi_\Gamma(\mathscr{O}(\Gamma))$

Fig. 6 The marked network Γ of Example 4.6 with the associated distributive polyhedron and its non-anti-blocking image under the transfer map

When Γ is acyclic and both all sinks and all sources are marked, we can compare the anti-blocking polyhedra $\mathscr{C}(\Gamma)$ and $\mathscr{C}(\Gamma^{\mathrm{op}})$. If Γ is the Hasse diagram of a poset, we have $\mathscr{C}(\Gamma) = \mathscr{C}(\Gamma^{\mathrm{op}})$ as a consequence of the opposite poset having the same comparability graph. By comparing the defining inequalities of the two polyhedra in the general case, we can see that this observation still holds for arbitrary acyclic marked networks with all sinks and sources marked.

5 Applications and Questions

5.1 Cayley Polytopes

Recall that the Cayley polytope C_n is the distributive polytope $\mathscr{O}(\Gamma)$ associated to the marked network in Fig. 1. The geometric bijection in [18] is a linear transformation $\phi^{-1}: C_n \to \mathbf{Y}_n$, where $\mathbf{Y}_n$ is an anti-blocking polytope defined in the introduction. This map is exactly the transfer map $\psi_\Gamma^{\mathrm{op}}: \mathscr{C}(\Gamma) \to \mathscr{O}(\Gamma)$.

5.2 Lecture Hall Order Cones and Polytopes

The s-lecture hall cones and polytopes of Bousquet-Mélou and Eriksson [5, 6] and Stanley's P-partitions [23, Sect. 3.15] were elegantly combined in [7] to *lecture hall order cones/polytopes*. Here we briefly sketch a generalization to a marked version. Let $(P, \preceq, \lambda)$ be a marked poset with $\lambda \in \mathbb{R}^A$ for $A \subseteq P$. For any $s \in \mathbb{R}^P_{>0}$, define the **marked lecture hall order polyhedron** $\mathscr{O}(P, \lambda, s)$ as the set of points $x \in \mathbb{R}^{P \setminus A}$

$$\frac{x_p}{s_p} \le \frac{x_q}{s_q} \quad \text{for } p \prec q,$$

where we set $x_a = s_a \lambda_a$ for $a \in A$. If $s \equiv 1$, then $\mathcal{O}(P, \lambda, s)$ is the marked order polyhedron $\mathcal{O}(P, \lambda)$. When P is the linear poset $\hat{0} \prec p_1 \prec \cdots \prec p_n$ and $\lambda_{\hat{0}} = 0$, we recover the s-lecture hall cones and adding a maximal element $\hat{1}$ with marking $\lambda_{\hat{1}} = 1$ we get the s-lecture hall polytopes.

Note that $\mathcal{O}(P, \lambda, s) = \mathcal{O}(\Gamma)$ for the marked network given by the Hasse diagram of P with edge weights $c \equiv 0$ and $\alpha_e = \frac{s_q}{s_p}$ for an edge e given by a covering relation $p \prec q$. We may also express $\mathcal{O}(P, \lambda, s)$ as a linear transformation $T_s(\mathcal{O}(P, \lambda))$ of the usual marked order polyhedron, where $T_s(x)_p = s_p x_p$. This transformation is compatible with the transfer maps associated to $\mathcal{O}(P, \lambda)$ and $\mathcal{O}(P, \lambda, s) = \mathcal{O}(\Gamma)$ in the sense that $T_s \circ \phi_{(P,\lambda)} = \phi_\Gamma \circ T_s$. If λ and s are integral, then the marked lecture hall order polytopes are lattice polytopes. Furthermore, if s satisfies $s_p \mid s_q$ for $p \prec q$, then the transfer map is lattice preserving.

5.3 Coordinates in Polytopes

The geometric reformulation (6) admits the following generalization: Given a polyhedron $Q \subseteq \mathbb{R}^d$ and vectors $U = (u_1, \ldots, u_m) \in \mathbb{R}^{d \times m}$. Define $\phi_{Q,U} \colon Q \to \mathbb{R}^m$ by

$$\phi_{Q,U}(x)_i \;:=\; \max(\mu \ge 0 : x - \mu v_i \in Q)\,.$$

Question 5.1 For which (Q, U) is $\phi_{Q,U}$ injective? When is the image convex?

If $\phi_{Q,U}$ is injective and convex, then its image gives a representation of Q up to translation, akin to its *slack representation*; see [16, Sect. 3.2]. Our results show that ϕ_Γ yields a class of examples for gainy networks. However, Example 4.6 shows that even for some distributive polyhedra associated to marked networks with non-gainy directed cycles the transfer map can still be injective.

5.4 Continuous Families

In [9], Fang and Fourier generalized the marked poset polytopes $\mathcal{O}(P, \lambda)$ and $\mathcal{C}(P, \lambda)$ to a discrete family of marked chain-order polytopes $\mathcal{O}_{C,O}(P, \lambda)$. It is parametrized by partitions $P \setminus A = C \uplus O$ such that $C = \varnothing$ yields the order polytope, which is distributive, and $O = \varnothing$ yields the chain polytope, which is anti-blocking. When both C and O are non-empty, $\mathcal{O}_{C,O}(P, \lambda)$ is neither distributive nor anti-blocking in general.

Definition 5.2 Let D and A be finite sets. A polyhedron $Q \subseteq \mathbb{R}^D \times \mathbb{R}^A_{\ge 0}$ is called **mixed distributive anti-blocking** if it satisfies the following properties:

(i) given $(x, z) \in Q$ and $(y, z) \in Q$, we have $(x \wedge y, z) \in Q$ and $(x \vee y, z) \in Q$,
(ii) when $(x, z) \in Q$ and $0 \leq y \leq z$, then $(x, y) \in Q$.

When $A = \varnothing$ or $D = \varnothing$, this recovers the notions of distributive and anti-blocking polyhedra, respectively. The marked chain-order polytopes are then mixed distributive anti-blocking with respect to the decomposition $\mathbb{R}^{P\setminus A} = \mathbb{R}^O \times \mathbb{R}^C$.

This discrete family of marked chain-order polytopes has been embedded into a continuous family of polytopes $\mathscr{O}_t(P, \lambda)$ parametrized by $t \in [0, 1]^{P\setminus A}$ in [10]. The marked chain-order polytopes are obtained for characteristic functions $t = \chi_C$. These polytopes are all obtained as images of the marked order polytope $\mathscr{O}(P, \lambda)$ under parametrized transfer maps

$$\phi_t(x)_p \;:=\; x_p - t_p \cdot \max_{q \prec p} x_q .$$

Hence, it is natural to ask whether we can obtain an analogous continuous family of polyhedra associated to marked networks.

Question 5.3 Does introducing a parameter $t \in [0, 1]^V$ in the transfer map of distributive polyhedra associated to gainy marked networks with marked sinks yield a continuous family of polyhedra such that

(i) the combinatorial type of the images is constant along relative interiors of the parametrizing hypercube and
(ii) the polyhedra at the vertices of the hypercube are mixed distributive anti-blocking?

5.5 *Domains of Linearity, Faces, Minkowski Summands*

At the end of Sect. 3.1, we gave an idea of the domains of linearity of ϕ_Γ. They are related to rooted forests with cycles attached to some leafs. Stanley [22] considered a refined subdivision of $\mathscr{O}(P)$ that had the property of being unimodular. For marked order polytopes a corresponding subdivision was described in [15] in terms of products of dilated unimodular simplices. In the general case with arbitrary weights it is not clear if such fine subdivisions exist.

Question 5.4 Do distributive polyhedra admit a natural subdivision into products of simplices on which the transfer map is linear?

The face structure of marked order polyhedra can be described by so-called face partitions [15, 21, 22]. The question of describing the vertices of $\mathscr{O}(\Gamma)$ was also raised in [12].

Question 5.5 Give a combinatorial description of the faces of $\mathscr{O}(\Gamma)$ in terms of the underlying network.

In [13, 21], a marked poset is called **regular** if the inequalities derived from the cover relations are irredundant (or facet-defining).

Question 5.6 When is a marked network *regular*?

Finally, polyhedra may be decomposed into Minkowski summands. For marked order polyhedra this was done in [15, 21], for marked chain-order polyhedra in [9, 11].

Question 5.7 Is there a Minkowski sum decomposition of distributive polyhedra similar to the one for marked order polyhedra and marked chain-order polyhedra?

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The Bass-Quillen Conjecture and Swan's Question

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Abstract We present a question which implies a complete positive answer for the Bass-Quillen Conjecture.

Keywords Regular rings · Smooth morphisms · Projective modules

2010 Mathematics Subject Classification Primary 13C10 · Secondary 19A13, 13H05, 13B40

1 Introduction

The theory of projective modules over polynomial algebras over a regular ring R was an important subject in Commutative Algebra starting with Serre's Conjecture (see [2]) and its extension considered by Bass and Quillen.

Conjecture 1.1 (Bass-Quillen Conjecture, [1, Problem IX], [9]) *Let R be a regular ring. Every finitely generated projective module P over a polynomial R-algebra, $R[T]$, $T = (T_1, \ldots, T_n)$ is extended from R, i.e. $P \cong R[T] \otimes_R (P/(T)P)$.*

Important positive answers were given by Quillen (see [9]) and Suslin (see [13]) in dimension ≤ 1 and Murthy [5] in dimension 2. Later Lindel [3] and Swan [6] gave, in particular, positive answers for many regular local rings, essentially of finite type $\mathbf{Z}$-algebras.

Theorem 1.2 (Lindel, Swan) *Let $(R, \mathfrak{m}, k)$ be a regular local ring, essentially of finite type over $\mathbf{Z}$ and $p = \text{char}\, k$. The following statements hold.*

1. *If $p = 0$ then R is essentially smooth over its prime field.*
2. *If $p \notin \mathfrak{m}^2$ then R is essentially smooth over $\mathbf{Z}$.*

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3. *If $p = 0$ or $p \notin \mathfrak{m}^2$ then the BQ Conjecture holds for R.*

Then Swan noticed that it is useful to have a positive answer to the following question.

Question 1.3 (Swan [6]) Is it a regular local ring a filtered inductive limit of regular local rings, essentially of finite type over $\mathbf{Z}$?

The Bass-Quillen Conjecture (shortly BQ Conjecture) is connected with the following one.

Conjecture 1.4 (Bass-Quillen-Suslin Conjecture) *Let R be a local ring, Assume either that R is regular local ring or that $1/r! \in R$. Let v be a unimodular vector over $R[T]$ of length $(r+1)$. Then v can be completed to an invertible matrix over $R[T]$.*

This conjecture holds in many cases given for example by Rao [10, 11] who also proved the BQ Conjecture for regular local rings of dimension 3 with residue characteristic $\neq 2, 3$.

For a regular local ring $(R, \mathfrak{m}, k)$ containing a field, or with $p := \text{char } k \notin \mathfrak{m}^2$, Swan's question has a positive answer in [7]. Using this partial positive answer and (iii) from Theorem 1.2 we got the following corollary (see [7, Theorem 4.1] and also [14, Theorems 2.1, 2.2]).

Corollary 1.5 *The Bass-Quillen Conjecture holds for R if $p = 0$ or $p \notin \mathfrak{m}^2$.*

Recently, we gave a complete positive answer to Swan's Question in [8, Theorem 17] (see here Theorem 2.1).

The purpose of this paper is to show that a positive answer to the following question gives a complete positive answer to the BQ Conjecture (see Theorem 2.6).

Question 1.6 Let $(R, \mathfrak{m})$ be a regular local ring, which is essentially smooth over $\mathbf{Z}_{(p)}$ and $b \in \mathfrak{m}^2$. Is it true the BQ Conjecture for the regular local ring $R/(p-b)$?

We owe thanks to Ravi Rao for some useful comments.

2 The Bass-Quillen Conjecture

We start reminding some definitions concerning smooth morphisms after [4], or [14]. A ring morphism $R \to R'$ of Noetherian rings has *regular fibers* if for all prime ideals $\mathfrak{p} \in \text{Spec } R$ the ring $R'/\mathfrak{p}R'$ is a regular ring. It has *geometrically regular fibers* if for all prime ideals $\mathfrak{p} \in \text{Spec } R$ and all finite field extensions K of the fraction field of $R/\mathfrak{p}$ the ring $K \otimes_{R/\mathfrak{p}} R'/\mathfrak{p}R'$ is regular. A flat morphism of Noetherian rings is *regular* if its fibers are geometrically regular. If it is regular of finite type, or essentially of finite type then it is called *smooth*, resp. *essentially smooth*.

The proof of our positive answer to Swan's Question says actually a little more (see [8, Theorem 17]).

Theorem 2.1 *Every regular local ring* $(R, \mathfrak{m}, k)$ *with* $0 \neq p = char\ k \in \mathfrak{m}^2$ *is a filtered inductive limit of regular local rings* R_i*, essentially smooth over a regular local* $\mathbf{Z}$*-algebra* $A_i/(p - b_i)$*, where* $(A_i, \mathfrak{a}_i)$ *is a regular local ring, essentially smooth over* $\mathbf{Z}_{(p)}$*, and* $b_i \in \mathfrak{a}_i^2$.

In fact, this theorem gives also the structure of regular local rings essentially of finite type over $\mathbf{Z}$ in the case $0 \neq \text{char}\ k \in \mathfrak{m}^2$, which is not covered by (i), (ii) from Theorem 1.2.

Corollary 2.2 *Let* B *be a* $\mathbf{Z}$*-algebra regular local, essentially of finite type. Then* B *has the form* $B = A/(p - b)$*, where* $(A, \mathfrak{a})$ *is a regular local ring essentially smooth over* $\mathbf{Z}$ *and* $b \in \mathfrak{a}^2$.

Proof Applying Theorem 2.1 to B we see that there exists a regular local ring D, essentially smooth over a regular local $\mathbf{Z}$-algebra $A/(p - b)$, where $(A, \mathfrak{a})$ is a regular local ring, essentially smooth over $\mathbf{Z}_{(p)}$, and $b \in \mathfrak{a}^2$ such that the identity of B factors through D. Then $B \cong D/q$ for some prime ideal $q \subset D$. Unfortunately, we cannot conclude that B is among these D (see the next remark).

Note that D is a factor of an essentially smooth $\mathbf{Z}_{(p)}$-algebra D' by $(p - b')$, where b' is a lifting of b to D'. Let q' be the prime ideal of D' containing $p - b'$ and such that $q'/(p - b') = q$. Changing D by D' and q by q' we may assume that D is essentially smooth over $\mathbf{Z}_{(p)}$. Then D is an etale neighborhood of a localization of a polynomial algebra in t variables Y over $\mathbf{Z}_{(p)}$ (see e.g. [14, Theorem 2.5]) and $(p - b, Y)$ generates the maximal ideal of D. Since D, B are regular local we see that q is generated by a part of a regular system of parameters of D, let us say $p - b, z$, where $z = (z_1, \ldots, z_r)$, $r \leq t$. After some linear transformations on Y we may assume that $z_i = Y_i$, $1 \leq i \leq r$. Then $B = D/q$ is an etale neighborhood of a localization of $\mathbf{Z}_{(p)}[Y_{r+1}, \ldots, Y_t]/(p - b'')$, b'' being induced by b. □

Remark 2.3 Let B be a regular local ring essentially of finite type over $\mathbf{Z}$, $D_n = B[X_n]$, $n \in \mathbf{N}$ and $\phi_{n,n+1} : D_n \to D_{n+1}$ be the B-morphism given by $X_n \mapsto 0$. Then B is the limit of $(D_n, \phi_{n,n+1})$, the inclusion $B \subset D_n$ has a retraction and $B \ncong D_n$ for any $n \in \mathbf{N}$.

Next we will need the following two lemmas, the first one is elementary (see e. g. [7, Theorem 4.2]).

Lemma 2.4 *Let* R *be a regular local ring, which is a filtered inductive limit of some regular local rings* $(R_i)_{i \in I}$*. If the BQ Conjecture holds for all* R_i*,* $i \in I$*, then it holds for* R *too.*

Proof Let M be a finitely generated projective module over $R[T]$, $T = (T_1, \ldots, T_n)$. Then $M \cong R \otimes_{R_i} M_i$ for some finitely generated projective $R_i[T]$-module M_i. Indeed, if M is defined by an idempotent ϕ from $\text{End}(L)$ for some $L = R^t$ then we may find i such that ϕ is extended from an endomorphism ϕ_i of R_i^t. Also we may find i such that ϕ_i is idempotent, and defines the wanted M_i. As BQ Conjecture holds for R_i we get M_i free and so M is free too. □

The following lemma follows easily from [3]. However, we give here a proof in sketch.

Lemma 2.5 *Let $R \to R'$ be an essentially smooth morphism between regular local rings. If the BQ Conjecture holds for R then it holds for R' too.*

Proof R' is an etale neighborhood of a localization of a polynomial algebra A over R (see e.g. [14, Theorem 2.5]). If the BQ Conjecture holds for R then it holds for A too by [12]. Now it is enough to apply the Corollary from [3]. □

Theorem 2.6 *If Question 1.6 has a positive answer then the BQ Conjecture holds for all regular rings.*

Proof By Quillen's Patching Theorem [9, Theorem 1] we may prove the conjecture only for regular local rings. Let $(R, \mathfrak{m}, k)$ be a regular local ring. Using [7, Theorem 3.1] we may suppose that $0 \neq p := \text{char}\, k \in \mathfrak{m}^2$.

By Theorem 2.1, R is a filtered inductive limit of some regular local rings D, essentially smooth over a regular local ring of the form $A/(p-b)$, where $(A, \mathfrak{a})$ is a regular local ring, essentially smooth over $\mathbf{Z}$ and $b \in \mathfrak{a}^2$.

Then the BQ Conjecture holds for A by Lemma 2.5 and for $A/(p-b)$ by Question 1.6. Applying Lemma 2.5 it follows that BQ holds for D. The final result is a consequence of Lemma 2.4. □

We end the section with a special form of Theorem 2.1 in the frame of the discrete valuation rings (DVRs for short) in the idea of [8, Theorem 8]. Actually, [8, Theorem 8] has a complicated proof given to illustrate [8, Theorem 17] in the DVR case. The proof below is easier and does not use Néron Desingularization.

Theorem 2.7 *Let $(A, \mathfrak{m}, k)$ be a DVR with $0 \neq p = \text{char}\, k \in \mathfrak{m}^2$. Suppose that k is separably generated over $\mathbf{F}_p$. Then A is a filtered inductive union of DVRs essentially of finite type over $\mathbf{Z}$.*

Proof As in [8, Theorem 8], let $y = (y_i)_{i \in I}$ be a system of elements of A inducing a separable transcendence base $(\bar{y})$ of k over $\mathbf{F}_p$. Then $C_0 = \mathbf{Z}[Y]_{p\mathbf{Z}[Y]}$ for some variables $Y = (Y_i)_{i \in I}$ is a DVR and the map $C_0 \to A$, $Y \to y$ defines a ramified extension inducing an algebraic separable residue field extension $\mathbf{F}_p(\bar{y}) \subset k$.

Note that A is a filtered inductive union of DVRs $A_L = A \cap L$ with $L \subset \text{Fr}(A)$ a finite type field extension of $\mathbf{Q}(y)$. We claim that L must be finite over $\mathbf{Q}(y)$. Indeed, assume that $z \in L$ is transcendental over $\mathbf{Q}(y)$ and $\mathfrak{m}^s \subset A$ for some $s \in \mathbf{N}$. Choose r such that $p^r > s$ and consider the DVR extension $A_{L''} \subset A_{L'}$ for $L' = \mathbf{Q}(y, z)$ and $L'' = \mathbf{Q}(y, z^{p^r})$. The residue field extension induced by $A_{L''} \subset A_{L'}$ is pure inseparable and also separable by assumption. Then it is trivial and so the ramification index of $A_{L''} \subset A_{L'}$ is $p^r > s$. Contradiction!

Since L is finite over $\mathbf{Q}(y)$ it follows that A_L is a localization of the integral closure of C_0 in L and so essentially finite over C_0, which is enough. □

3 Swan's Question in the Non-reduced Case

We start this section reminding the first part of [8, Theorem 17].

Theorem 3.1 *Let* $(A, \mathfrak{m}, k)$ *be a Noetherian local ring,* $s = (s_1, \ldots, s_m)$ *some positive integers and* $\gamma = (\gamma_1, \ldots, \gamma_m)$ *a system of nilpotents of* A*. Suppose that* $0 \neq p \in \mathfrak{m}^2$*,* $R = A/(\gamma)$ *is a regular local ring and* A *is a flat* $\Phi = \mathbf{Z}_{(p)}[\Gamma]/(\Gamma^s)$*-algebra,* $\Gamma \mapsto \gamma$ *with* $\Gamma = (\Gamma_1, \ldots, \Gamma_m)$ *some variables, and* (Γ^s) *denotes the ideal* $(\Gamma_1^{s_1} \cdots \Gamma_m^{s_m})$*. Then* A *is a filtered inductive limit of some Noetherian local* Φ*-algebras* $(F_i)_i$ *essentially of finite type with* $F_i/\Gamma F_i$ *regular local rings.*

Remark 3.2 The above theorem holds also when $p = 0$ or $p \notin \mathfrak{m}^2$ as in [7, Theorem 4.1] because the map $\Phi \to A$ is regular in this case.

Lemma 3.3 *In the notation and hypothesis of the above theorem, let* $(B, \mathbf{b})$ *be a Noetherian local ring, and* $z = (z_1, \ldots, z_m)$ *a system of elements of* B*. Suppose that*

1. *for all* $i \in [m]$ $s_i > 1$, $z_i^{s_i} = 0$,
2. *for all* $i \in [m]$ *and* $j \in [s_i - 1]$, $(z_1, \ldots, z_{i-1}) : z_i^j = (z_1, \ldots, z_{i-1}, z_i^{s_i - j})$,
3. $B/(z)$ *is a regular local ring.*

Then B *is a flat* Φ*-algebra by the map* $\phi : \Phi \to B$*,* $\Gamma \mapsto z$*.*

Proof Suppose $m = 1$. Note that the minimal free resolution of $\Phi/(\Gamma)$,

$$\ldots \to \Phi/(\Gamma^s) \xrightarrow{\Gamma} \Phi/(\Gamma^s) \xrightarrow{\Gamma^{s-1}} \Phi/(\Gamma^s) \xrightarrow{\Gamma} \Phi/(\Gamma^s) \to \Phi/(\Gamma) \to 0$$

gives after tensorizing with B a minimal free resolution of $B/(z)$. Thus

$\mathrm{Tor}_1^{\Phi}(\Phi/(\Gamma), B) = 0$. As $B/(z)$ is a flat $\mathbf{Z}$-algebra we get ϕ flat using the Local Criteria of Flatness [4, Theorem 49, (20.C)].

Induct on m and assume $m > 1$. Then $B/(z_m)$ is a flat $\Phi/(\Gamma_m)$-algebra by induction hypothesis. As in the case $m = 1$ we see that $\mathrm{Tor}_1^{\Phi}(\Phi/(\Gamma_m), B) = 0$ and so ϕ is flat using the Local Criteria of Flatness. □

Remark 3.4 Let R be a regular local ring, $s = (s_1, \ldots, s_m) \in \mathbf{N}^m$ and $z = (z_1, \ldots, z_m)$ be a part of a regular system of parameters of R. Then $B = R/(z_1^{s_1}, \ldots, z_m^{s_m})$ satisfies the above lemma.

Using the above lemma and Theorem 3.1 we get the following result.

Theorem 3.5 *In the notation and hypothesis of the above lemma,* B *is a filtered inductive limit of some Noetherian local* $\mathbf{Z}$*-algebras* $(F_i)_i$ *essentially of finite type with* $F_i/\sqrt{(0)}$ *regular local rings.*

As in Corollary 2.2 we get the following result.

Corollary 3.6 *In the notation and hypothesis of Lemma 3.3, suppose that $B/(z)$ is essentially of finite type over* $\mathbf{Z}$. *Then B is essentially smooth over a local Φ-algebra D of type $C/(p-b)$, where $(C,\mathfrak{q})$ is a local ring, esentially smooth over Φ with $b \in \mathfrak{q}^2$ (so $D/\Gamma D$ and $C/\Gamma C$ are regular).*

Theorem 3.7 *Let $(A,\mathfrak{m},k)$ be a Noetherian local ring, $T=(T_1,\dots,T_n)$ some variables and* $\mathbf{b}$ *a nilpotent ideal of A. Suppose that $R=A/\mathbf{b}$ is a regular local ring and the BQ Conjecture holds for R (for example $p := \text{char}\, k = 0$, or $p \notin \mathfrak{m}^2$). Then any finitely generated projective $A[T]$-module is free.*

Proof Let M be a finitely generated projective $A[T]$-module. Then $\bar{M} = M/\mathbf{b}M$ is finitely generated projective over $R[T]$ and so it is free. Let $x=(x_1,\dots,x_r)$ be a system of elements from M inducing a basis in $\bar{M}$. Then $M = <x> + \mathbf{b}M = <x> + \mathbf{b}^t M = <x>$ for $t \in \mathbf{N}$ with $\mathbf{b}^t = 0$. Thus the map $\phi : A[T]^n \to M$ given by $(a_1,\dots,a_n) \mapsto \sum_i a_i x_i$ is surjective. Set $N = \text{Ker}\,\phi$. Tensorizing with $A[T]/(\mathbf{b})$ the exact sequence

$$0 \to N \to A[T]^n \to M \to 0$$

we get the following exact sequence

$$\text{Tor}_1^{A[T]}(M, A[T]/(\mathbf{b})) \to N/\mathbf{b}N \to R[T]^n \to \bar{M} \to 0,$$

where the first module is zero because M is a flat $A[T]$-module. As the last map is injective we see that $N = \mathbf{b}N = \mathbf{b}^t N = 0$, that is ϕ is an isomorphism. □

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Licci Level Stanley-Reisner Ideals with Height Three and with Type Two

Giancarlo Rinaldo, Naoki Terai, and Ken-Ichi Yoshida

Abstract Via computer-aided classification we show that the following three conditions are equivalent for level* squarefree monomial ideals I with codimension 3, with Cohen-Macaulay type 2 and with $\dim S/I \leq 4$: (1) $IS_{\mathfrak{m}}$ is licci, (2) the twisted conormal module of I is Cohen-Macaulay, (3) $S/I^{(2)}$ is Cohen-Macaulay, where S is a polynomial ring over a field of characteristic 0 and $\mathfrak{m}$ is its graded maximal ideal.

Keywords Stanley-Reisner ideal · Level ring · Licci · Twisted conormal module linkage

2010 Mathematics Subject Classification Primary 13F55 · Secondary 13H10

1 Introduction

Let k be a fixed field of characteristic 0 throughout the article. Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring with $\deg x_i = 1$ for all $i \in [n] = \{1, 2, \ldots, n\}$, and put $\mathfrak{m} = (x_1, \ldots, x_n)$ and $R = S_{\mathfrak{m}}$.

In liaison theory the following theorems are classical:

Theorem 1.1 ([10, 16]) *Let $I \subseteq R$ be an ideal.*

1. *If I is a Cohen-Macaulay ideal of height 2, then it is licci.*

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2. *If I is a Gorenstein ideal of height* 3, *then it is licci.*

Then what can be said about a Cohen-Macaulay ideal of height 3 or a Gorenstein ideal of height 4? They are very interesting but very difficult questions. As for a Gorenstein squarefree monomial ideal with height 4, the following holds:

Theorem 1.2 ([11]) *Let I be a Gorenstein squarefree monomial ideal with height* 4 *and with* $\dim S/I \leq 4$. *Then the following conditions are equivalent:*

1. *IR is licci.*
2. *S/I^2 is Cohen-Macaulay.*

We would like to give an analogous condition for a licci Cohen-Macaulay squarefree monomial ideal with height 3. In this article we consider a level squarefree monomial ideal with height 3 and with Cohen-Macaulay type 2 as the first step. We show the following as a main result:

Theorem 1.3 *Let $I \subseteq S$ be a level* squarefree monomial ideal with height* 3, *with Cohen-Macaulay type 2 and with* $\dim S/I \leq 4$. *Then the following conditions are equivalent:*

1. *IR is licci.*
2. *The twisted conormal module of I is Cohen-Macaulay.*
3. *$S/I^{(2)}$ is Cohen-Macaulay.*

We give a computer-aided proof via the classification of level squarefree monomial ideals with height 3, with Cohen-Macaulay type 2 and with $\dim S/I \leq 4$.

Without the restriction of the dimension does these equivalence hold? Namely:

Problem 1.4 Let I be a level* squarefree monomial ideal with height 3 and with Cohen-Macaulay type 2. Then are the following conditions equivalent?

1. IR is licci.
2. The twisted conormal module of I is Cohen-Macaulay.
3. $S/I^{(2)}$ is Cohen-Macaulay.

After introducing necessary notation and lemmas in Sect. 1, in Sect. 2 we treat the cases of Krull dimensions 2 and 3. In Sect. 3 we treat the case of Krull dimension 4. In Sect. 4 we give an example of licci level* squarefree monomial ideal with Cohen-Macaulay type 2 for an arbitrary high dimension such that its second symbolic power is Cohen-Macaulay.

2 Preliminaries

Let k be a fixed field of characteristic 0. Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring with $\deg x_i = 1$ for all $i \in [n] = \{1, 2, \ldots, n\}$. Let $\mathfrak{m} = (x_1, \ldots, x_n)S$ and put $R = S_{\mathfrak{m}}$.

Let $I \subseteq S$ be a homogeneous ideal such that S/I is a d-dimensional Cohen-Macaulay ring. Suppose S/I has the following graded minimal free resolution:

$$0 \to \bigoplus_j S(-j)^{\beta_{n-d,j}} \to \cdots \to \bigoplus_j S(-j)^{\beta_{1,j}} \to S \to S/I \to 0,$$

where $\beta_{i,j} = \dim_k[\mathrm{Tor}_i^S(S/\mathfrak{m}, S/I)]_j$ is the (i, j)-th *graded Betti number* of S/I. Under this situation S/I is called *level* if $\bigoplus_j S(-j)^{\beta_{n-d,j}} = S(-m)^{\beta_{n-d,m}}$ for some m. When I is a squarefree monomial ideal, we have $j \le n$ if $\beta_{i,j} \ne 0$ by Lemma 2.3 below. We call S/I *level** if $\bigoplus_j S(-j)^{\beta_{n-d,j}} = S(-n)^{\beta_{n-d,n}}$. We say that S/I has Cohen-Macaulay type t if $t = \sum_j \beta_{n-d,j}$.

We introduce notion on licci ideals according to [15]. Refer to it for further information.

Put $\mathfrak{m} = (x_1, \ldots, x_n)$ and $R = S_{\mathfrak{m}}$. For two ideals I and J of R, I and J are said to be *directly linked*, denoted by $I \sim J$, if there exists a regular sequence $\underline{z} = z_1, \ldots, z_g$ in $I \cap J$ such that $I = (\underline{z}) : J$ and $J = (\underline{z}) : I$. Furthermore, I is *linked* to J if there exists a sequence of ideals $I_0, I_1, \ldots, I_k$ such that $I = I_0 \sim I_1 \sim I_2 \sim \cdots \sim I_k = J$. Then we say that I and J are in the same linkage class.

Definition 2.1 *(Licci)* An ideal $I \subseteq R$ is said to be *in the linkage class of a complete intersection* if I is linked to some complete intersection ideal (that is, the ideal generated by a regular sequence). We call such an ideal *licci*.

Lemma 2.2 ([6, Remark 0.2], [10, Proposition 1.3]) *Let $I \subseteq R$ be an ideal.*

1. *Suppose that I is unmixed of height g and $\underline{z} = z_1, \ldots, z_g$ is a regular sequence in I. If $J = (\underline{z}) : I$, then I is directly linked to J.*
2. *If I and J are in the same linkage class, then R/I is Cohen-Macaulay if and only if R/J is Cohen-Macaulay. In particular, any licci ideal is Cohen-Macaulay.*

The *twisted conormal module* of R/I (or I) is $I \otimes_R \omega_{R/I}$, where $\omega_{R/I}$ is the canonical module of R/I. And R/I (or I) is called *strongly unobstructed* if the *twisted conormal module* of R/I (or I) is Cohen-Macaulay. It is known that I is strongly unobstructed if I is licci [3].

We recall some notation on simplicial complexes and Stanley-Reisner rings. We refer the reader to [1, 14] for the detailed information about combinatorial and algebraic background.

Set $V = [n]$. A nonempty subset Δ of the power set 2^V of V is called a *simplicial complex* on V if $\{v\} \in \Delta$ for all $v \in V$, and $F \in \Delta, H \subseteq F$ imply $H \in \Delta$. An element $F \in \Delta$ is called a *i-face* of Δ if $\sharp(F) = i + 1$. The dimension of Δ is defined by $\dim \Delta = \max\{\sharp(F) - 1 \ : \ F$ is a face of $\Delta\}$. Set $d = \dim \Delta + 1$. A maximal face of Δ is called a *facet* of Δ. Let $\mathscr{F}(\Delta)$ denote the set of facets of Δ.

The *Stanley–Reisner ideal* of Δ, denoted by I_Δ, is the squarefree monomial ideal generated by

$$\{x_{i_1} x_{i_2} \cdots x_{i_p} \mid 1 \le i_1 < \cdots < i_p \le n, \ \{x_{i_1}, \ldots, x_{i_p}\} \notin \Delta\},$$

and $K[\Delta] = K[x_1, \dots, x_n]/I_\Delta$ is called the *Stanley–Reisner ring* of Δ.

We say that a simplicial complex Δ is Cohen-Macaulay (or level*) if $K[\Delta]$ is so. A 1-dimensional simplicial complex Δ is Cohen-Macaulay if and only if Δ is connected. The Betti number $\beta_{i,j}(k[\Delta])$ can be expressed in term of the reduced homology groups of subcomplexes of Δ as follows:

Lemma 2.3 (Hochster's formula)

$$\beta_{i,j}(k[\Delta]) = \sum_{W \subset V,\, \#(W)=j} \dim_k \widetilde{H}_{j-i-1}(\Delta_W; k),$$

where $\Delta_W = \{F \in \Delta : F \subset W\}$ *is the restriction of* Δ *to* W.

We say that a simplicial complex Δ is *licci* if $I_\Delta S_\mathfrak{m}$ is licci for the Stanley-Reisner ideal I_Δ.

In [7] squarefree monomial ideals generated in degree 2 or with deviation 2 are considered. In particular, the following fact is shown:

Lemma 2.4 ([7, Corollary 7.2]) *Let* $R = S_\mathfrak{m}$ *and* $I \subseteq R$ *be a Cohen-Macaulay squarefree monomial ideal. If* I *is generated by 5 or less elements, then it is licci.*

We introduce the notion of a 1-vertex inflation according to [2]. Let $I = I_\Delta$ be the Stanley-Reisner ideal of a simplicial complex Δ on the vertex set $[n]$. Set $I = (m_1, m_2, \dots, m_\mu)$ where $m_1, m_2, \dots, m_k$ are monomials divisible by x_n and $m_{k+1}, m_{k+2}, \dots, m_\mu$ are monomials indivisible by x_n. Let J be the squarefree monomial ideal in $S' = K[x_1, x_2 \dots, x_{n+1}]$ generated by

$$m_1x_{n+1}, m_2x_{n+1}, \dots, m_kx_{n+1}, m_{k+1}, m_{k+2}, \dots, m_\mu.$$

Let Δ' be the simplicial complex on $[n+1]$ such that $J = I_{\Delta'}$. The simplicial complex Δ' is called a *1-vertex inflation* of Δ. Note that $\dim \Delta' = \dim \Delta + 1$. It is easy to see that Δ' is level* with Cohen Macaulay type t (resp. licci) if so is Δ.

Let $I = I_\Delta$ is the Stanley-Reisner ideal of Δ. Set $P_F = (x \mid x \in [n] \setminus F)$ for each facet F, then we have the minimal prime decomposition

$$I_\Delta = \bigcap_{F \in \mathscr{F}(\Delta)} P_F.$$

For any integer $\ell \geq 1$, we define the ℓ-th symbolic power of I_Δ by

$$I_\Delta^{(\ell)} = \bigcap_{F \in \mathscr{F}(\Delta)} P_F^\ell.$$

3 2 and 3-Dimensional Cases

First note that a 0-dimensional simplicial complex Δ with height$I_\Delta = 3$ is the simplicial complex which consists of 4 vertices, which is level*. But its h-vector is $(1, 3)$ and then its Cohen-Macaulay type is 3. Hence there is no 0-dimensional level* simplicial complex Δ with height$I_\Delta = 3$ and with Cohen-Mcacaulay type 2.

Now we classify all 1-dimensional level* simplicial complexes Δ with 5 vertices and with Cohen-Macaulay type 2, which shows our main theorem (Theorem 1.3) is true in the case of $\dim S/I = 2$.

Proposition 3.1 *Let Δ be a 1-dimensional simplicial complex on $V = \{a, b, c, d, e\}$. Set $S = k[a, b, c, d, e]$. Then Δ is level* with Cohen-Macaulay type 2 if and only if I_Δ is one of the following ideals up to permutation of variables:*

(1) (ac, ad, bd, ce, abe).
(2) (ac, ad, be, cd).

When this is the case, Δ is licci and $S/I_\Delta^{(2)}$ is Cohen-Macaulay.

Proof Suppose Δ is level* with Cohen-Macaulay type 2. By Euler-Poincaré relation and Hochster's formula, we have

$$f_2 - f_1 + 1 = \dim \widetilde{H}_1(\Delta) - \dim \widetilde{H}_0(\Delta) = \dim \widetilde{H}_1(\Delta) = \beta_{3,5}(k[\Delta]) = 2.$$

Hence we have $f_2 = 6$. Since Δ is level*, by Hochster's formula

$$\sum_{\sharp(W)=4} \dim \widetilde{H}_0(\Delta_W) = \beta_{3,4}(k[\Delta]) = 0.$$

Hence graph Δ has no cut vertex. Hence Δ has 5 vertices and 6 edges without cut vertex. From the list of graphs (e.g., The Appendix 1 in [5]) we see that I_Δ is isomorphic to (1) or (2).

It is easy to see that for both cases Δ is level* with Cohen-Macaulay type 2. They are licci by Lemma 2.4. By [9, Theorem 2.3] $S/I_\Delta^{(2)}$ is Cohen-Macaulay. □

Next we list up all 2-dimensional level* simplicial complexes Δ with Cohen-Macaulay type 2 with 6 vertices using NAUTY [8] and Macaulay2 [4]. See [13] for the concrete algorithm used.

Theorem 3.2 *Let Δ be a 2-dimensional simplicial complex on $V = \{a, b, c, d, e, f\}$ with f_2 2-faces. Then Δ is level* with Cohen-Macaulay type 2 if and only if I_Δ is one of the following ideals up to permutation of variables:*

1. *If $f_2 = 9$, then*
   - *a.* (df, cd, cf, abe).
   - *b.* $(be, bf, cf, acd, adef)$.

2. *If* $f_2 = 10$*, then*

   a. $(be, bf, cde, acd, adf, acef)$.
   b. (bf, cd, aef, abe).
   c. (cf, df, abe, acd, bde).
   d. $(af, be, bcd, cdf, acde)$.

3. *If* $f_2 = 11$*, then*

   a. $(cd, abe, abf, aef, bdf, cef)$.
   b. $(bf, abe, acd, ace, bcd, def)$.

4. *If* $f_2 = 12$*, then* $(abf, ace, acf, ade, bcd, bde, bef, cdf)$.

Based upon the above classification, we prove Theorem 1.3 in the case of $\dim S/I = 3$ using Macaulay2 [4]. See [13] for the concrete algorithm used.

Theorem 3.3 *Let* Δ *be a* 2*-dimensional simplicial complex on* $V = \{a, b, c, d, e, f\}$. *Set* $S = k[a, b, c, d, e, f]$. *Suppose* $k[\Delta] = S/I_\Delta$ *is level* with Cohen-Macaulay type* 2*. Then the following conditions are equivalent*:

1. Δ *is licci.*
2. $k[\Delta]$ *is strongly unobstructed.*
3. $S/I_\Delta^{(2)}$ *is Cohen-Macaulay.*
4. $f_2 \leq 11$ *and* I_Δ *is one of the ideals in Theorem 3.2 up to permutation of variables.*

Proof (1) $\Longrightarrow$ (2) is known.

(2) $\Longrightarrow$ (4). It is enough to show that in the case (4) in Theorem 3.2 $k[\Delta]$ is not strongly unobstructed. It can be checked by a computer.

(4) $\Longrightarrow$ (1). It is known that a Cohen-Macaulay squarefree monomial ideal generated by 5 or less elements is licci by Lemma 2.4. Hence it is enough to show that the ideals in (2)(a), (3)(a) and (3)(b) are licci in $R = S_\mathfrak{m}$.

Case (2)(a).
Set

$$I = (be, bf, acd, adf, cde, acef) \subseteq R.$$

Set

$$J = (be + bf, acd + adf + cde, acef),$$

which is a complete intersection. Then

$$\begin{aligned} K &:= J : I \\ &= (be + bf, acd + adf + cde, ace + ce^2 + cef, acf + aef + af^2, ce^2 f + cef^2). \end{aligned}$$

Set

$$M = (be + bf, acd + adf + cde, ace + ce^2 + cef),$$

which is a complete intersection. Then

$$N := M : K = (b, ad - cd, cd + de + df, ace + ce^2 + cef).$$

Since $(ad - cd, cd + de + df, ace + ce^2 + cef)$ is Cohen-Macaulay with height 2, hence N is licci.

Case (3)(a).
Set

$$I = (cd, abe, abf, aef, bdf, cef) \subseteq R.$$

Set

$$J = (cd, abe + abf + aef, bdf + cef),$$

which is a complete intersection. Then

$$\begin{aligned} K &:= J : I \\ &= (cd, bd + ce, ce + cf - df, abe + abf + aef). \end{aligned}$$

Set

$$M = (cd, bd + ce, abe + abf + aef),$$

which is a complete intersection. Then

$$N := M : K = (ab, ae, bd, cd, ce).$$

Since N is Gorenstein with height 3, hence it is licci.

Case (3)(b).
Set

$$I = (bf, abe, acd, ace, bcd, def) \subseteq R.$$

Set

$$J = (bf, abe + acd + ace, bcd + def),$$

which is a complete intersection. Then

$$\begin{aligned} K &:= J : I \\ &= (b^2 + bc - df, bf, ef, abe + acd + ace, bcd - e^2 f). \end{aligned}$$

Set

$$M = (b^2 + bc - df, bf, abe + acd + ace),$$

which is a complete intersection. Then

$$N := M : K = (ac, b^2 + bc, bf, df).$$

Set

$$P = (ac, b^2 + bc, df),$$

which is a complete intersection. Then

$$Q = P : N = (b + c, ac, ad, df).$$

Since $(b + c, ac, ad)$ is Cohen-Macaulay with height 2, hence Q is licci.

(3)$\Longrightarrow$(4). It is enough to show that in the case (4) in Theorem 3.2 $S/I_\Delta^{(2)}$ is not Cohen-Macaulay. It can be checked by a computer.

(4)$\Longrightarrow$(3). It is enough to show that $S/I_\Delta^{(2)}$ is Cohen-Macaulay for each ideal I_Δ in (4). It can be checked by a computer. □

4 4-Dimensional Case

First we list up all 3-dimensional level* simplicial complexes with Cohen-Macaulay type 2 with 7 vertices using NAUTY [8] and Macaulay2 [4]. See [13] for the concrete algorithm used.

Theorem 4.1 (3-dimensional level* complex) *Let Δ be a 3-dimensional simplicial complex on $V = \{a, b, c, d, e, f, g\}$ with f_3 3-faces. Then Δ is level* with Cohen-Macaulay type 2 if and only if I_Δ is one of the following ideals up to permutation of variables:*

1. *If $f_3 = 12$, then*
   a. $(cf, cg, fg, abde)$.
   b. $(bg, cf, cg, adef, abcde)$.
2. *If $f_3 = 13$, then*
   a. $(cg, dg, bcf, ade, abefg)$.
3. *If $f_3 = 14$, then*
   a. $(cf, de, abdg, abeg)$.
   b. $(cg, dg, cdf, abef, abde)$.
   c. $(cg, dg, cef, abef, abde, abcdf)$.
   d. $(cf, cg, aeg, abde, bdfg)$.
   e. $(bg, cf, acde, adeg, abdef)$.
4. *If $f_3 = 15$, then*
   a. $(cg, acf, bde, deg, abfg)$.
   b. $(ef, acg, bdg, acf, abcde)$.

c. $(cf, aeg, ceg, bdf, abdg, abcde)$.
d. $(de, dg, bcfg, abcf, acef, abeg)$.
e. (bg, bcf, ade, cfg).

5. *If* $f_3 = 16$*, then*

a. $(de, acf, bdg, abeg, bcfg)$.
b. $(cf, bdg, cdg, aefg, abde, abeg)$.
c. $(acf, bcf, deg, bde, aeg, bdf, abcdg)$.
d. $(bg, acf, cfg, acde, adeg, bdef)$.
e. $(cf, acg, abf, bdeg, bcde, adeg)$.
f. $(bg, dfg, bcf, adef, acef, acde)$.
g. $(cg, ade, bef, abdf)$.

6. *If* $f_3 = 17$*, then*

a. $(acf, beg, bde, bdg, ceg, adef)$.
b. $(acf, ade, def, bcg, bfg)$.
c. $(adg, bdf, bdg, bef, cef, abcg, acde)$.
d. $(acf, acg, bdf, ceg, efg, abde)$.
e. $(acf, bcf, bcg, bdg, deg, abde, abef)$.
f. $(cf, aeg, abde, abdg, bcdg, bdef)$.
g. $(bg, bcf, acde, acdf, adef, aefg, cdeg)$.
h. $(acf, acg, bde, bdf, beg, defg)$.

7. *If* $f_3 = 18$*, then*

a. $(ade, bcf, bcg, bfg, aceg, adfg, cdef)$
b. $(ade, acf, bcg, bfg, bdef, cdeg)$.
c. $(acg, bde, bef, cdg, abcf, adef, adfg)$.
d. $(adf, adg, aeg, bcf, bcde, bdeg, cefg)$.
e. $(acg, ade, adg, cde, abcf, bcef, bdfg, befg)$.
f. $(ade, beg, cdg, deg, abcf, abdf, acfg, bcef)$.
g. $(bde, bdf, beg, ceg, abcf, acdg, acef, adfg)$.

Based upon the classification, we prove Theorem 1.3 in the case of dim $S/I = 4$ using Macaulay 2 [4]. See [13] for the concrete algorithm used.

Theorem 4.2 (3-dimensional licci level* complex) *Let* Δ *be a 3-dimensional simplicial complex on* $V = \{a, b, c, d, e, f, g\}$*. Set* $S = k[a, b, c, d, e, f, g]$*. Suppose* $k[\Delta] = S/I_\Delta$ *is level* with Cohen-Macaulay type 2. Then the following conditions are equivalent:*

1. Δ *is licci.*
2. $k[\Delta]$ *is strongly unobstructed.*
3. $S/I_\Delta^{(2)}$ *is Cohen-Macaulay.*
4. I_Δ *is one of the ideals in Theorem 4.1 up to permutation of variables except* (e)*,* (f)*,* (g) *of* (7).

Proof (1) $\Longrightarrow$ (2) is known.

(2) $\Longrightarrow$ (4). It is enough to show that $k[\Delta]$ is not strongly unobstructed in the case (e), (f), (g) of (7) in Theorem 4.1. It can be checked by a computer.

(4) $\Longrightarrow$ (1). It is known that a Cohen-Macaulay squarefree monomial ideal generated by 5 or less elements is licci by Lemma 2.4. Hence we only have to consider the ideals generated by 6 and 7 monomials in Theorem 4.1.

Case (3)(c).

Since a and b appear simultaneously in the generators in the ideal, this ideal corresponds to a 1-vertex inflation of the simplicial complex of (2)(a) in Theorem 3.2. Hence it is licci.

Case (4)(c).

Since b and d appear simultaneously in the generators in the ideal, this ideal corresponds to a 1-vertex inflation of the simplicial complex of (2)(a) in Theorem 3.2. Hence it is licci.

Case (4)(d).

Since c and f appear simultaneously in the generators in the ideal, this ideal corresponds to a 1-vertex inflation of the simplicial complex of (2)(a) in Theorem 3.2. Hence it is licci.

Case (5)(b).

Since a and e appear simultaneously in the generators in the ideal, this ideal corresponds to a 1-vertex inflation of the simplicial complex of (3)(a) in Theorem 3.2. Hence it is licci.

Case (5)(c).

Set

$$I = (acf, bcf, deg, bde, aeg, bdf, abcdg) \subseteq R,$$

and

$$J = (acf + bcf, deg + bde, aeg + bdf + abcdg),$$

which is a complete intersection. Then

$$\begin{aligned} K &:= J : I \\ &= (acf + bcf, abcd + b^2cd + acdg + bcdg, ae + be - df - acdg + eg, \\ &\quad bdf - b^2cdg - beg + dfg - bcdg^2 - eg^2, bde + deg). \end{aligned}$$

Set

$$M = (acf + bcf, bde + deg, ae + be - df - acdg + eg),$$

which is a complete intersection. Then

$$N := M : K = (cf, ae + be + eg, de, acdg, df).$$

Set $b' = a + b + g$. Then

$$N = (cf, b'e, de, acdg, df).$$

Since N is a Cohen-Macaulay squarefree monomial ideal generated by 5 elements, N is licci.

Case (5)(d).

Since d and e appear simultaneously in the generators in the ideal, this ideal corresponds to a 1-vertex inflation of the simplicial complex of (3)(b) in Theorem 3.2. Hence it is licci.

Case (5)(e).

Since d and e appear simultaneously in the generators in the ideal, this ideal corresponds to a 1-vertex inflation of the simplicial complex of (3)(b) in Theorem 3.2. Hence it is licci.

Case (5)(f).

Since a and e appear simultaneously in the generators in the ideal, this ideal corresponds to a 1-vertex inflation of the simplicial complex of (3)(a) in Theorem 3.2. Hence it is licci.

Case (6)(a).

Since a and f appear simultaneously in the generators in the ideal, this ideal corresponds to a 1-vertex inflation of the simplicial complex of (3)(a) in Theorem 3.2. Hence it is licci.

Case (6)(c).

Set

$$I = (adg, bdf, bdg, bef, cef, abcg, acde) \subseteq R,$$

and

$$J = (adg + bdg, bdf + bef + cef, abcg + acde),$$

which is a complete intersection. Then

$$\begin{aligned}K &:= J : I\\ &= (adg+bdg, bdf+bef+cef, abg+b^2g+acg+bcg, ad^2+bd^2+ade+bde+cde-acg,\\ &\quad acde-b^2cg-ac^2g-bcg).\end{aligned}$$

Set

$$M=(adg+bdg, bdf+bef+cef, abg+b^2g+bcg+ad^2+bd^2+ade+bde+cde),$$

which is a complete intersection. Then

$$N := M : K = (df, dg, ef, abg+b^2g+bcg, ad^2+bd^2+ade+bde+cde).$$

Set

$$P=(dg, ef, abg+b^2g+bcg+ad^2+bd^2+ade+bde+cde),$$

which is a complete intersection. Then

$$Q := P : N = (g, de, ef, ad^2+bd^2).$$

Since (de, ef, ad^2+bd^2) is a Cohen-Macaulay ideal with height 2, Q is licci.

Case (6)(d).

Since b and d appear simultaneously in the generators in the ideal, this ideal corresponds to a 1-vertex inflation of the simplicial complex of (3)(b) in Theorem 3.2. Hence it is licci.

Case (6)(e).

Set

$$I=(acf, , bcf, bcg, bdg, deg, abde, abef) \subseteq R,$$

and

$$J=(acf, +bcf, bcg+bdg+deg, abde+abef),$$

which is a complete intersection. Then

$$\begin{aligned}K &:= J : I\\ &= (acf, +bcf, bcg+bdg+deg, ade+aef, abc+b^2c+abd+b^2d+bde+abf+b^2f+bef).\end{aligned}$$

Set

$$M=(acf, +bcf, bcg+bdg+deg, ade+aef),$$

which is a complete intersection. Then

$$N := M : K = (cf, cg, dg, .ade, aef)$$

Since N is a Gorenstein ideal with height 3, N is licci.

Case (6)(f).

Since b and d appear simultaneously in the generators in the ideal, this ideal corresponds to a 1-vertex inflation of the simplicial complex of (3)(a) in Theorem 3.2. Hence it is licci.

Case (6)(g).

Set

$$I = (bg, bcf, acde, acdf, adef, aefg, cdeg) \subseteq R,$$

and

$$J = (bg + bcf, acde + acdf + adef, aefg + cdeg),$$

which is a complete intersection. Then

$$\begin{aligned} K &:= J : I \\ &= (bg + bcf, acde + acdf + adef, aefg + cdeg, acef + acf^2 + aef^2 + aeg - deg + afg, \\ &\quad c^2de + c^2df + cdef + cdg - aeg + deg). \end{aligned}$$

Set

$$M = (bg + bcf, acde + acdf + adef, acef + acf^2 + aef^2 + aeg - deg + afg),$$

which is a complete intersection. Then

$$N := M : K = (bf, ade, adf, bg, deg, acef + acf^2 + aef^2 + aeg + afg).$$

Set

$$P = (bf, ade + adf + bg, deg + acef + acf^2 + aef^2 + aeg + afg),$$

which is a complete intersection. Then

$$Q := P : N = (bf, ae + de + af + df, d^2e + d^2 f - bg, def + df^2, af^3 + dfg).$$

Set

$$H = (bf + ae + de + af + df, d^2e + d^2f - bg + def + df^2, af^3 + dfg),$$

which is a complete intersection. Then

$$\begin{aligned} T &:= H : Q \\ &= (bf + ae + de + af + df, d^2f - af^2 + df^2 + ag, af^2 + dg, d^2e + def - ag - bg - dg, \\ &\quad df^3 + f^4 + f^2g - g^2). \end{aligned}$$

Set

$$U = (bf + ae + de + af + df, d^2f - af^2 + df^2 + ag, af^2 + dg, df^3 + f^4 + f^2g - g^2),$$

which is a complete intersection. Then

$$V := U : T = (df + f^2, ae + de + af + bf - f^2, ag + dg, fg, g^2).$$

Set

$$W = (df + f^2, ae + de + af + bf - f^2, g^2),$$

which is a complete intersection. Then

$$X := W : V = (g, df + f^2, ae + af + bf - ef - f^2, de + df + ef + f^2).$$

Since $(df + f^2, ae + af + bf - ef - f^2, de + df + ef + f^2)$ is a Cohen-Macaulay ideal with height 2, X is licci.

Case (6)(h).

Since a and c appear simultaneously in the generators in the ideal, this ideal corresponds to a 1-vertex inflation of the simplicial complex of (3)(b) in Theorem 3.2. Hence it is licci.

Case (7)(a).

Set

$$I = (ade, bcf, bcg, bfg, aceg, adfg, cdef) \subseteq R,$$

and

$$J = (ade, bcf + bcg + bfg, aceg + adfg + cdef),$$

which is a complete intersection. Then

$$\begin{aligned} K &:= J : I \\ &= (ade, bcf + bcg + bfg, ace - cde - adg + aeg, adf - def + adg - aeg, cdef + cdeg + defg). \end{aligned}$$

Set

$$M = (ade, bcf + bcg + bfg, ace - cde + adf - def, cdef + cdeg + defg),$$

which is a complete intersection. Then

$$N := M : K = (bc, bf, ace, ade, adf, cde + def).$$

Set

$$P = (bc + ace, bf + adf, ade + cde + def),$$

which is a complete intersection. Then

$$Q := P : N = (b + ad + cd + ae + ef, cdf - cef, ade + cde + def, aef + cef + ef^2, acd + c^2d + cef).$$

Set

$$H = (b + ad + cd + ae + ef, cdf - cef, ade + cde + def),$$

which is a complete intersection. Then

$$\begin{aligned} T &:= H : Q \\ &= (df, de, ce, b + ad + cd + ae + ef). \end{aligned}$$

Set

$$U = (df, ce, b + ad + cd + ae + ef),$$

which is a complete intersection. Then

$$V := U : T = (c, f, b + ad + ae),$$

which is a complete intersection. Hence it is licci.

Case (7)(b).

Since d and e appear simultaneously in the generators in the ideal, this ideal corresponds to a 1-vertex inflation of the simplicial complex of (3)(b) in Theorem 3.2. Hence it is licci.

Case (7)(c).

Set

$$I = (acg, bde, bef, cdg, abcf, adef, adfg) \subseteq R,$$

and

$$J = (acg + cdg, bde + bef, abcf + adef + adfg).$$

which is a complete intersection. Then

$$\begin{aligned} K &:= J : I \\ &= (abc + bcd + bcf - aef - afg, abc + bcd + ade + bcf + def + adg + d^2g + dfg, acg+cdg, \\ &\quad bde + bef, abcf + adef + adfgbcdf + d^2ef + bcf^2 + def^2 + d^2fg + df^2g). \end{aligned}$$

Set

$$M = (abc + bcd + bcf - aef - afg, bde + bef, abcf + adef + adfg),$$

which is a complete intersection. Then

$$N := M : K = (be, cg, aef, afg, abc + bcd + bcf).$$

Set

$$P = (be, cg, aef + afg + abc + bcd + bcf),$$

which is a complete intersection. Then

$$Q := P : N = (e + g, bc, bg, cg).$$

Set $e' = e + g.$. Then $Q = (e', bc, bg, cg)$. Since (bc, bg, cg) is is a Cohen-Macaulay ideal with height 2, Q is licci.

Case (7)(d).

Set

$$I = (adf, adg, aeg, bcf, bcde, bdeg, cefg) \subseteq R,$$

and

$$J = (adf + bcf, adg + aeg, bcde + bdeg + cefg),$$

which is a complete intersection. Then

$$\begin{aligned} K &:= J : I \\ &= (adf + bcf, adg + aeg, bcd + ad^2 + ade + bdg + cfg, bce + beg - cfg, cdfg + cefg). \end{aligned}$$

Set

$$M = (adf + bcf, adg + aeg, bce + beg - cfg),$$

which is a complete intersection. Then

$$N := M : K = (af, ag, bce, bcf, beg, cfg).$$

Set

$$P = (af + ag, bce + bcf + beg, cfg),$$

which is a complete intersection. Then

$$Q := P : N = (af + ag, cg + fg + g^2, bce + bcf + beg, cef + cf^2, f^2g + fg^2).$$

Set

$$H = (af + ag, cg + fg + g^2, bce + bcf + beg),$$

which is a complete intersection. Then

$$\begin{aligned} T &:= H : Q \\ &= (a, cg + fg + g^2, bc + bf + bg, bc - be). \end{aligned}$$

Since $(cg + fg + g^2, bc + bf + bg, bc - be)$ is a Cohen-Macaulay ideal with height 2, T is licci.

(3)$\Longrightarrow$(4). It is enough to show that in the case (e), (f), (g) of (7) in Theorem 4.1 $S/I_\Delta^{(2)}$ is not Cohen-Macaulay. It can be checked by a computer.

(4)$\Longrightarrow$(3). It is enough to show that $S/I_\Delta^{(2)}$ is Cohen-Macaulay for each ideal I_Δ in (4). It can be checked by a computer.

We finished the proof of Theorem 4.2 and thus Theorem 1.3. □

5 High Dimensional Case

In this section we give an example of a licci level Stanley-Reisner ideal with Cohen-Macaulay type 2 with arbitrary dimension, which is similar to the Stanley-Reisner ideal of the boundary complex of a stellar subdivision of a cross polytope (see [12]).

Theorem 5.1 *Set*

$$I = (x_1y_1, x_2y_2, \ldots, x_dy_d, y_1z, y_2z, \ldots, y_dz, x_1x_2\cdots x_dz)$$

in $S = k[x_1, x_2 \ldots, x_d, y_1, y_2, \ldots, y_d, z]$. *Then*

1. *I is a level* Stanley-Reisner ideal with Cohen-Macaulay type* 2 *and* $\dim S/I = d$.
2. *IR is licci, where* $\mathfrak{m} = (x_1, x_2 \ldots, x_d, y_1, y_2, \ldots, y_d, z)S$ *and* $R = S_\mathfrak{m}$.
3. *$S/I^{(2)}$ is Cohen-Macaulay.*
4. *$I^{(2)} \neq I^2$.*

Proof (1) Set

$$I_\Gamma = (x_1y_1, x_2y_2, \ldots, x_dy_d).$$

Then Γ is the boundary complex of the d-dimensional cross polytope. Set

$$I_\Pi = (x_1x_2\cdots x_dz).$$

Then Π is the boundary complex of a pyramid on $\{x_1, x_2, \dots, x_d\}$. Set $\Delta = \Gamma \cup \Pi$. Then it is easy to see that $I = I_\Delta$ is a level* Stanley-Reisner ideal with Cohen-Macaulay type 2.

(2) Now we show that IR is licci. If we put $t_i = x_i + z$ for $i = 1, 2, \dots, d$ and

$$\begin{aligned} J &= (x_1y_1 + y_1z, x_2y_2 + y_2z, \dots, x_dy_d + y_dz, x_1x_2\cdots x_dz) \\ &= (t_1y_1, t_2y_2, \dots, t_dy_d, x_1\cdots x_dz), \end{aligned}$$

Since J is an ideal generated by $(d+1)$ elements with height $(d+1)$, J is a complete intersection. Moreover, since $I = J + (y_1, \dots, y_d)z$, we have

$$\begin{aligned} J : I &= \bigcap_{i=1}^{d} (J : z) : y_i \\ &= \bigcap_{i=1}^{d} (t_1y_1, t_2y_2, \dots, t_dy_d,\ x_1x_2\cdots x_d) : y_i \\ &= \bigcap_{i=1}^{d} (t_1y_1, \dots, \widehat{t_iy_i}, \dots, t_dy_d,\ t_i,\ x_1x_2\cdots x_d) \\ &= (t_1y_1, t_2y_2, \dots, t_dy_d,\ t_1t_2\cdots t_d,\ x_1x_2\cdots x_d) \end{aligned}$$

Set $K = J : I$ and

$$M = (t_1y_1, t_2y_2, \dots, t_dy_d,\ x_1x_2\cdots x_d).$$

Then M is a complete intersection ideal and

$$M : K = M : t_1t_2\cdots t_d = (y_1, \dots, y_d, x_1x_2\cdots x_d)$$

is also a complete intersection ideal. Hence IR is licci.

(3) In order to prove that $S/I^{(2)}$ is Cohen-Macaulay, if we put

$$I_0 = (x_1y_1, x_2y_2, \dots, x_dy_d, y_1z, y_2z, \dots, y_dz,\ x_1x_2\cdots x_d)S \quad \text{and} \quad P = (y_1, \dots, y_d, z)S,$$

then $I = I_0 \cap P$. By [12, Example 5.6], S/I_0 is a d-dimensional Gorenstein ring and $I_0^{(2)} = I_0^2$ and S/I_0^2 is Cohen-Macaulay. Then $I^{(2)} = I_0^{(2)} \cap P^{(2)} = I_0^2 + P^2$ and we have the following exact sequence:

$$0 \to S/I^{(2)} \to S/I_0^2 \oplus S/P^2 \to S/(I_0^2 + P^2) \to 0. \qquad (1)$$

One can easily see that

$$L := I_0^2 + P^2 = (y_1, y_2, \dots, y_d, z)^2 + x_1x_2\cdots x_d(x_1y_1, x_2y_2, \dots, x_dy_d, x_1x_2\cdots x_d).$$

Moreover, if we put $P_i = (x_i, y_2, \dots, y_d, z)$ $(i = 1, 2, \dots, d)$, then

$$\{P \in \operatorname{Spec} S \mid \dim S/P = \dim S/L\} = \{P_i \mid i = 1, 2, \dots, d\}.$$

Hence the multiplicity of S/L is given by

$$e(S/L) = \sum_{i=1}^{d} e(S/P_i) \cdot \ell_{S_{P_i}}(S_{P_i}/LS_{P_i}) = d(d+5).$$

On the other hand, since

$$L + (x_2 - x_1, \dots, x_d - x_1) = (y_1, \dots, y_d, z)^2 + (x_1^{d+1}y_1, x_1^{d+1}y_2, \dots, x_1^{d+1}y_d, x_1^{2d}),$$

we get $\ell_S(S/L + (x_2 - x_1, \dots, x_d - x_1)) = d(d+5)$, which implies S/L is Cohen-Macaulay with $\dim S/L = d - 1$.

From the exact sequence (1), we conclude that $S/I^{(2)}$ is Cohen-Macaulay.

(4) One can check that $x_1 \cdots x_d y_1 z \in I^{(2)} \setminus I^2$. □

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Homological and Combinatorial Properties of Powers of Cover Ideals of Graphs

S. A. Seyed Fakhari

Abstract Over the last 25 years the study of algebraic, homological and combinatorial properties of powers of ideals has been one of the major topics in Commutative Algebra. In this article, we survey the recent results concerning the associated primes, regularity, depth and Stanley depth of (symbolic) powers of cover ideals of graphs.

Keywords Associated prime · Cover ideal · Depth · Regularity · Stanley depth · Symbolic power

2010 Mathematics Subject Classification Primary: 13C15, 13D02, 05E99 · Secondary: 13C13

1 Introduction

The study of homological and combinatorial properties of monomial ideals is an active research area of mathematics which employs methods of abstract algebra in combinatorial contexts and vice versa. We refer the reader to the books by Stanley [61], Bruns and Herzog [10], Miller and Sturmfels [46], Villarreal [67], as well as Herzog and Hibi [35] as general references in the subject.

One of the connections between combinatorics and commutative algebra is via rings constructed from the combinatorial objects. Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field $\mathbb{K}$. There is a natural correspondence between quadratic squarefree monomial ideals of S and finite simple graphs with n vertices. To every simple graph G with vertex set $V(G) = \{x_1, \ldots, x_n\}$ and edge set $E(G)$, we associate its *edge ideal* $I = I(G)$ defined by

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$$I(G) = \big(x_i x_j : \{x_i, x_j\} \in E(G)\big) \subseteq S.$$

This ideal was introduced by Villarreal [66] and has been studied by several authors (see e.g., [4–6, 22, 26, 37, 39, 47, 57–59, 64]).

The focus of this article is on the Alexander dual of edge ideals. Namely, the ideal

$$J(G) = \bigcap_{\{x_i, x_j\} \in E(G)} (x_i, x_j),$$

which is called the *cover ideal* of G.

Let G be a graph. A subset C of $V(G)$ is called a *vertex cover* of G if every edge of G is incident to at least one vertex of C. A vertex cover C is called a *minimal vertex cover* of G if no proper subset of C is a vertex cover of G. On can easily check that

$$J(G) = \big(\prod_{x_i \in C} x_i \mid C \text{ is a minimal vertex cover of } G\big),$$

and this equality clarifies the reason of naming the cover ideals.

Example 1.1 Let G be the 5-cycle graph with vertices $V(G) = \{x_1, x_2, x_3, x_4, x_5\}$ and edge set

$$E(G) = \big\{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\}, \{x_1, x_5\}\big\}.$$

Then

$$\begin{aligned} J(G) &= (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4) \cap (x_4, x_5) \cap (x_1, x_5) \\ &= (x_1x_2x_4, x_1x_3x_5, x_1x_3x_4, x_2x_4x_5, x_2x_3x_5). \end{aligned}$$

It is clear that $\{x_1, x_2, x_4\}$, $\{x_1, x_3, x_5\}$, $\{x_1, x_3, x_4\}$, $\{x_2, x_4, x_5\}$ and $\{x_2, x_3, x_5\}$ are the minimal vertex covers of G.

In this article, we survey the recent results concerning the homological and combinatorial properties of ordinary and symbolic powers of cover ideals of graphs.

2 Associated Prime Ideals

In this section, we investigate the set of associated prime ideals of powers of cover ideals. It is clear from the definition of cover ideals that for every graph G, we have

$$\mathrm{Ass}(S/J(G)) = \big\{(x_i, x_j) \mid \{x_1, x_j\} \in E(G)\big\}.$$

But determining the associated primes of powers of cover ideals is a challenging problem. However, the situation is nice, if G is a bipartite graph. In order to state the relevant results, we first recall the definition of symbolic powers.

Definition 2.1 Assume that I is an ideal of S and $\mathrm{Min}(I)$ is the set of minimal primes of I. For every integer $k \geq 0$, the k-th *symbolic power* of I, denoted by $I^{(k)}$, is defined to be

$$I^{(k)} = \bigcap_{\mathfrak{p} \in \mathrm{Min}(I)} \mathrm{Ker}(S \to (S/I^k)_{\mathfrak{p}}).$$

Let I be a squarefree monomial ideal with irredundant primary decomposition

$$I = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r,$$

where every $\mathfrak{p}_i$ is a prime ideal generated by a subset of the variables. It follows from [35, Proposition 1.4.4] that for every integer $k \geq 0$,

$$I^{(k)} = \mathfrak{p}_1^k \cap \ldots \cap \mathfrak{p}_r^k.$$

In particular, for every graph G and every integer $k \geq 0$, we have

$$J(G)^{(k)} = \bigcap_{\{x_i, x_j\} \in E(G)} (x_i, x_j)^k.$$

It is natural to ask when the ordinary and symbolic powers of a cover ideal coincide. In order to answer this question, Herzog, Hibi and Trung [36] studied the degree of the minimal generators of the so-called *vertex cover algebra* of a graph.

Theorem 2.2 ([36, Theorem 5.1]) *For any graph G, set*

$$A(G) := \bigoplus_{k \geq 0} J(G)^{(k)} t^k \subset S[t].$$

Then

(i) The graded S-algebra $A(G)$ is generated in degree at most 2.
(ii) The graded S-algebra $A(G)$ is a standard graded S-algebra if and only if G is a bipartite graph.

It follows from Theorem 2.2(ii) that for every bipartite graph G and every integer $k \geq 0$, we have $J(G)^k = J(G)^{(k)}$. This fact was also proved by Gitler et al. [28, Corollary 2.6]. In particular, one obtains the following corollary.

Corollary 2.3 *Let G be a bipartite graph. Then for any integer $k \geq 1$,*

$$\mathrm{Ass}(S/J(G)^k) = \big\{(x_i, x_j) \mid \{x_1, x_j\} \in E(G)\big\}.$$

Computing the associated primes of powers of cover ideals of non-bipartite graphs is more complicated. However, Francisco, Hà and Van Tuyl [25] provided a combinatorial description for them. To state their result, we need to recall some definitions from graph theory.

Let G be a graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$. For every subset $A \subset V(G)$, the graph $G \setminus A$ has vertex set $V(G \setminus A) = V(G) \setminus A$ and edge set $E(G \setminus A) = \{e \in E(G) \mid e \cap A = \emptyset\}$. If $A = \{x\}$ is a singleton, we write $G \setminus x$ instead of $G \setminus \{x\}$. For every subset $T \subseteq V(G)$ the *induced subgraph* of G on T, denoted by G_T, is the graph with $V(G_T) = T$ and $E(G_T) = \{e \in E(G) \mid e \subseteq T\}$.

Suppose $k \geq 1$ is a positive integer. A *k-coloring* of G is a partition of $V(G)$, say $V(G) = C_1 \cup \cdots \cup C_k$, with $|e \cap C_i| \leq 1$, for every integer $1 \leq i \leq k$ and any edge $e \in E(G)$. The *chromatic number* of G, denoted by $\chi(G)$ is the smallest integer k such that G admits a k-coloring. We say G is *critically k-chromatic* if $\chi(G) = k$ but $\chi(G \setminus x_i) = k - 1$ for every integer i with $1 \leq i \leq n$.

To state the next result, we also need to introduce the following notation.

As before, let G be a graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$. For every integer $s \geq 1$, the graph G^s is defined to be the graph with vertex set

$$V(G^s) = \{x_{i,k} \mid 1 \leq i \leq n, 1 \leq k \leq s\},$$

and edge set

$$\begin{aligned} E(G^s) = &\Big\{\{x_{i,k}, x_{j,l}\} \mid \{x_i, x_j\} \in E(G), 1 \leq k, l \leq s\Big\} \\ &\cup \Big\{\{x_{i,k}, x_{i,l}\} \mid 1 \leq i \leq n, 1 \leq k, l \leq s, k \neq l\Big\}. \end{aligned}$$

Theorem 2.4 ([25, Corollary 4.5]) *Let G be a graph and let $s \geq 1$ be a positive integer. A prime ideal $\mathfrak{p} = (x_{i_1}, \ldots, x_{i_m})$ belongs to $\mathrm{Ass}(S/J(G)^s)$ if and only if there exists a subset $T \subseteq V(G^s)$ with*

$$\{x_{i_1,1}, \ldots, x_{i_m,1}\} \subseteq T \subseteq \{x_{i_1,1}, \ldots, x_{i_1,s} \ldots, x_{i_m,1}, \ldots, x_{i_m,s}\}$$

such that the induced graph G^s_T is critically $(s+1)$-chromatic.

Remark 2.5 Francisco, Hà and Van Tuyl [25] indeed proved that the conclusion of Theorem 2.4 is true if one replaces the graph G with any arbitrary *simple hypergraph*. Since the study of the cover ideals of hypergraphs is not the goal of this paper, we only state a special case of their result.

It is easy to see that any critically 3-chromatic graph is an odd cycle. Thus, using Theorem 2.4, together with a simple graph theoretic computation, one can explicitly determine the associated primes of the second powers of cover ideals, as follows.

Theorem 2.6 ([23, Theorem 1.1]) *Let G be a graph. A prime ideal $\mathfrak{p} = (x_{i_1}, \ldots, x_{i_s})$ belongs to $\mathrm{Ass}(S/J(G)^2)$ if and only if*

(i) $s = 2$ and $\{x_{i_1}, x_{i_2}\}$ is and edge of G, or

Fig. 1 The graph G in Example 2.7

(ii) s is odd and the induced subgraph of G on $\{x_{i_1}, \ldots, x_{i_s}\}$ is a cycle.

Example 2.7 Let G be the graph with vertices $V(G) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and edge set

$$E(G) = \big\{\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\}, \{x_4, x_6\}, \{x_5, x_6\}\big\}.$$

The graph G is shown in Fig. 1.

We have

$$\begin{aligned} J(G) =&(x_1x_3x_4x_5, x_1x_3x_4x_6, x_1x_3x_5x_6, x_1x_2x_4x_5,\\ &x_1x_2x_4x_6, x_2x_3x_4x_5, x_2x_3x_4x_6, x_2x_3x_5x_6). \end{aligned}$$

One can check that

$$\begin{aligned} J(G)^2 = &(x_1^2, x_2) \cap (x_1, x_2^2) \cap (x_1^2, x_3) \cap (x_1, x_3^2)\\ &\cap (x_2^2, x_3) \cap (x_2, x_3^2) \cap (x_3^2, x_4) \cap (x_3, x_4^2)\\ &\cap (x_4^2, x_5) \cap (x_4, x_5^2) \cap (x_4^2, x_6) \cap (x_4, x_6^2)\\ &\cap (x_5^2, x_6) \cap (x_5, x_6^2) \cap (x_1^2, x_2^2, x_3^2)\\ &\cap (x_4^2, x_5^2, x_6^2). \end{aligned}$$

In particular,

$$\begin{aligned} \mathrm{Ass}(S/J(G)^2) = \big\{&(x_1, x_2), (x_1, x_3), (x_2, x_3), (x_3, x_4), (x_4, x_5), (x_4, x_6), (x_5, x_6),\\ &(x_1, x_2, x_3), (x_4, x_5, x_6)\big\}, \end{aligned}$$

as claimed by Theorem 2.6.

For any graph G, its *complementary graph* $\overline{G}$ is the graph with $V(\overline{G}) = V(G)$ and $E(\overline{G})$ consists of those 2-element subsets $\{x_i, x_j\}$ of $V(G)$ for which $\{x_i, x_j\} \notin E(G)$. A subset W of $V(G)$ is a *clique* if any pair of vertices of W are adjacent in G. The cardinality of the largest clique of G is the *clique number* of G and is denoted by $w(G)$. It is clear that for any graph G, we have $w(G) \leq \chi(G)$. A graph G is called *perfect* if for any induced subgraph H of G, the equality $w(H) = \chi(H)$ holds. A well-known result in graph theory, namely, the *Strong Perfect Graph Theorem* [14] states that a graph G is perfect if and only if neither G nor its complementary graph $\overline{G}$

has an odd induced cycle of length at least five. Hence, as a consequence of Theorem 2.6, one obtains the following corollary.

Corollary 2.8 ([23, Corollary 3.5]) *A graph G is perfect if and only if every prime ideal belonging to* $\mathrm{Ass}(S/J(G)^2)$ *or* $\mathrm{Ass}(S/J(\overline{G})^2)$ *has height at most three.*

Let G be a graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$. For any vertex $x_i \in V(G)$, the expansion of G at x_i is the graph $G[\{x_i\}]$ with

$$V(G[\{x_i\}]) = (V(G) \setminus \{x_i\}) \cup \{x_{i,1}, x_{i,2}\}$$

(where $x_{i,1}$ and $x_{i,2}$ are new vertices) and

$$E(G[\{x_i\}]) = E(G \setminus \{x_i\}) \cup \big\{\{x_j, x_{i,1}\}, \{x_j, x_{i,2}\} \mid \{x_j, x_i\} \in E(G)\big\} \cup \{x_{i,1}, x_{i,2}\}.$$

For every subset $W \subseteq V(G)$, the expansion of G at W, denoted by $G[W]$, is obtained by successively expanding the vertices of W. Francisco, Hà and Van Tuyl [24], conjectured that for any critically s-chromatic graph G, there exists a subset $W \subseteq V(G)$ such that the expansion $G[W]$ is critically $(s+1)$-chromatic. They also proved that this conjecture is equivalent to say that for every graph G,

$$\mathrm{Ass}(S/J(G)^{s-1}) \subseteq \mathrm{Ass}(S/J(G)^s).$$

This conjecture has been disproved in [44]. This means there exists a graph G and an integer $s \geq 1$ such that

$$\mathrm{Ass}(S/J(G)^{s-1}) \nsubseteq \mathrm{Ass}(S/J(G)^s).$$

However, G can not be a perfect graph, as the following theorem shows.

Theorem 2.9 ([25, Theorem 5.9 and Corollary 5.11]) *Let G be graph. Then the following conditions are equivalent.*

(i) *G is a perfect graph.*
(ii) *For every integer $s \geq 1$, we have $\mathfrak{p} = (x_{i_1}, \ldots, x_{i_r}) \in \mathrm{Ass}(S/J(G)^s)$ if and only if the induced graph on the vertices $\{x_{i_1}, \ldots, x_{i_r}\}$ is a clique of size $2 \leq r \leq s+1$ in G.*

In particular, for every perfect graph G and for any integer $s \geq 1$, we have

$$\mathrm{Ass}(S/J(G)^s) \subseteq \mathrm{Ass}(S/J(G)^{s+1}).$$

Example 2.10 Let G be the graph which was considered in Example 2.7. Then it is easy to see that G is a perfect graph. Since any clique of G has cardinality at most three, it follows from Theorem 2.9 that for every integer $s \geq 2$,

$$\mathrm{Ass}(S/J(G)^s) = \mathrm{Ass}(S/J(G)^2) = \big\{(x_1,x_2),(x_1,x_3),(x_2,x_3),(x_3,x_4),(x_4,x_5),\\ (x_4,x_6),(x_5,x_6),(x_1,x_2,x_3),(x_4,x_5,x_6)\big\}.$$

We remind that the associated primes of $S/J(G)^2$ are already computed in Example 2.7.

Let $I \subset S$ be a monomial ideal. By Brodmann [9], there exists an integer $k \geq 1$ such that for any integer $m \geq k$, the equality

$$\mathrm{Ass}(S/I^m) = \mathrm{Ass}(S/I^k)$$

holds. The smallest integer k with this property is called the *index of Ass-stability* of I and is denoted by $\mathrm{astab}(I)$. Francisco, Hà and Van Tuyl [25] determined a lower bound for the index of Ass-stability of squarefree monomial ideals. Here, we mention a special case of their result.

Theorem 2.11 ([25, Corollaries 4.9 and 5.11]) *Let G be a graph and assume that $a \geq 1$ is an integer with*

$$\bigcup_{k=1}^{\infty} \mathrm{Ass}(S/J(G)^k) = \bigcup_{k=1}^{a} \mathrm{Ass}(S/J(G)^k).$$

Then $a \geq \chi(G) - 1$ and the equality holds if G is a perfect graph.

3 Regularity

Let M be a finitely generated graded S-module. Suppose that the minimal graded free resolution of M is given by

$$0 \to \cdots \to \bigoplus_j S(-j)^{\beta_{1,j}(M)} \to \bigoplus_j S(-j)^{\beta_{0,j}(M)} \to M \to 0.$$

The Castelnuovo-Mumford regularity (or simply, regularity) of M is defined as

$$\mathrm{reg}(M) = \max\{j - i \mid \beta_{i,j}(M) \neq 0\},$$

and it is an important invariant in commutative algebra and algebraic geometry.

Computing and finding bounds for the regularity of powers of a monomial ideal have been studied by a number of researchers (see for example [1, 2, 4, 6, 16, 20, 30, 43, 47]). Cutkosky, Herzog, Trung, [20], and independently Kodiyalam [45], proved that for a homogenous ideal I in a polynomial ring, $\mathrm{reg}(I^s)$ is a linear function for $s \gg 0$, i.e., there exist integers a, b, and s_0 such that

$$\text{reg}(I^s) = as + b \quad \text{for all } s \geq s_0.$$

It is known that a is bounded above by the maximum degree of elements in a minimal generating set of I. Also, there have been several studies on integers b and s_0 (see e.g., [7, 12, 13]). But we still have no general combinatorial description for these invariants, even in the case of squarefree monomial ideals.

The asymptotic behavior of regularity of symbolic powers of squarefree monomial ideals is also known. Indeed, let I be a squarefree monomial of S. By [36, Theorem 3.2], we know that the *symbolic Rees ring* $\mathscr{R}_s(I) = \bigoplus_{k=0}^{\infty} I^{(k)}t^k$ is a finitely generated $\mathbb{K}$-algebra. It follows that the regularity of symbolic powers of I is asymptotically quasi-linear, i.e., there exists an integer $d \geq 1$ and linear polynomials $p_1(x), \ldots, p_d(x) \in \mathbb{Q}[x]$, such that

$$\text{reg}(I^{(s)}) = p_i(s), \qquad \text{for } s \gg 0, \text{ where } s \equiv i \pmod{d}.$$

In this section, we review the recent results about the regularity of ordinary and symbolic powers of cover ideals.

Hang and Trung [31] studied the regularity of powers of cover ideals of the so-called *unimodular hypergraphs*. Here, we state a special case of their result. We recall that for any monomial ideal I, the maximum degree of its minimal monomial generators is denoted by $\deg(I)$.

Theorem 3.1 ([31, Corollary 3.4]) *Let G be a bipartite graph with n vertices. Then there is a non-negative integer*

$$e \leq n - \deg(J(G)) - 1$$

such that

$$\text{reg}(J(G)^k) = k\deg(J(G)) + e,$$

for every integer $k \geq n + 2$.

As an immediate consequence of Theorem 3.1, for every bipartite graph G and any integer $k \geq |V(G)| + 2$, we have

$$\text{reg}(J(G)^k) \leq (k-1)\deg(J(G)) + |V(G)| - 1.$$

It is natural to ask whether the above inequality is valid for every non-negative integer k. In [56], we gave a positive answer to this question. Indeed, we proved a stronger result. Before stating this result, we recall that for every graph G and for any vertex $x_i \in V(G)$, the *neighbor set* of x_i is $N_G(x_i) = \big\{x_j \mid \{x_i, x_j\} \in E(G)\big\}$ and we set $N_G[x_i] = N_G(x_i) \cup \{x_i\}$.

Theorem 3.2 ([56, Theorem 3.2]) *Let $\mathscr{H}$ be a family of graphs which satisfies the following conditions.*

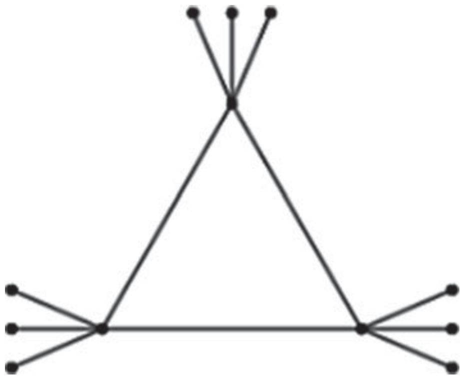

Fig. 2 The graph $G_{3,3}$

(i) For every graph $G \in \mathscr{H}$ *and every vertex* $x \in V(G)$*, the graph* $G \setminus N_G[x]$ *belongs to* $\mathscr{H}$*.*
(ii) If $G \in \mathscr{H}$ *has no isolated vertex, then it admits a minimal vertex cover with cardinality at least* $\frac{|V(G)|}{2}$*.*

Then for every graph $G \in \mathscr{H}$ *and every integer* $k \geq 1$*, we have*

$$k\deg(J(G)) \leq \operatorname{reg}(J(G)^{(k)}) \leq (k-1)\deg(J(G)) + |V(G)| - 1.$$

Remark 3.3 It can be easily checked that for every squarefree monomial ideal I and every integer $k \geq 1$, we have $k\deg(I) \leq \deg(I^{(k)})$. In particular,

$$k\deg(I) \leq \operatorname{reg}(I^{(k)}).$$

This shows that the first inequality of Theorem 3.2 is true for any arbitrary graph G.

The following example from [29] shows that not every graph satisfies condition (ii) of Theorem 3.2. We first mention that an edge of a graph is a *pendant edge*, if it is incident to a leaf (i.e., a vertex of degree one). Also, the complete graph with n vertices will be denoted by K_n.

Example 3.4 For every pair of integers $n \geq 3$ and $s \geq 2$, let $G_{n,s}$ be the graph obtained by attaching s pendant edges at each vertex of K_n. The graph $G_{3,3}$ is shown in Fig. 2. It is easy to check that the largest minimal vertex cover of $G_{n,s}$ has $n + s - 1 < \frac{|V(G_{n,s})|}{2}$ vertices. (Note that every vertex cover of G contains at least $n - 1$ vertices of $V(K_n)$.)

Let G be a bipartite graph with bipartition $V(G) = A \cup B$. If every vertex of A is adjacent to every vertex of B, then we say that G is a *complete bipartite* graph and denote it by $K_{a,b}$, where $a = |A|$ and $b = |B|$. The graph $K_{1,3}$ is called a *claw* and the graph G is said to be *claw–free* if it has no claw as an induced subgraph. A graph G is *unmixed* if the minimal vertex covers of G have the same size.

Let G be a bipartite graph with bipartition $V(G) = A \cup B$ and assume that G has no isolated vertex. Then either $|A| \geq \frac{|V(G)|}{2}$ or $|B| \geq \frac{|V(G)|}{2}$. Since A and B are both vertex covers of G, it follows that every bipartite graph satisfies condition (ii) of Theorem 3.2. We also know from [29] (see also [19, Theorem 0.1]) that every unmixed graph satisfies condition (ii) of Theorem 3.2. Moreover, it is shown in the proof of [56, Theorem 3.7] that every claw-free graph satisfies condition (ii) of Theorem 3.2 too. As a consequence, we obtain the following result.

Corollary 3.5 *Let G be a graph with n vertices and assume that*

(i) G is a bipartite graph, or
(ii) G is an unmixed graph, or
(iii) G is a claw-free graph.

Then for every integer $k \geq 1$, we have

$$k\deg(J(G)) \leq \operatorname{reg}(J(G)^{(k)}) \leq (k-1)\deg(J(G)) + n - 1.$$

Remark 3.6 Let G be a bipartite graph with n vertices. We remind that by Theorem 2.2(ii), for every integer $k \geq 1$, we have $J(G)^k = J(G)^{(k)}$. Thus, Corollary 3.5 implies that

$$k\deg(J(G)) \leq \operatorname{reg}(J(G)^k) \leq (k-1)\deg(J(G)) + n - 1,$$

for every integer $k \geq 1$.

Example 3.7 Let G be the complete bipartite graph $K_{1,m}$. Then $\deg(J(G)) = m$. Thus, the upper and the lower bounds provided in Corollary 3.5 coincide. Therefore,

$$\operatorname{reg}(J(K_{1,m})^k) = km,$$

for every integer $k \geq 1$. In particular, the bounds of Corollary 3.5 are sharp.

Definition 3.8 Let M be a finitely generated graded S-module. M is said to have a *linear resolution*, if for some integer d, $\beta_{i,i+t}(M) = 0$ for all i and every integer $t \neq d$.

It is clear from the definition that if a module M has a linear resolution, then M is generated in a single degree.

There are many attempts to characterize the monomial ideals with a linear resolution. One of the most important results in this direction is due to Fröberg [27, Theorem 1], who characterized all edge ideals which have a linear resolution. He proved that the edge ideal of a graph G has a linear resolution if and only if the complementary graph $\overline{G}$ is *chordal*, i.e., it has no induced cycle of length at least four. Herzog, Hibi and Zheng [38] proved that $I(G)$ has a linear resolution if and only if every power of $I(G)$ has a linear resolution. It is also known [40] that polymatroidal ideals have linear resolution and that powers of polymatroidal ideals are again polymatroidal (see [35]). In particular, they have again linear resolution. However, in general, the powers of ideals with a linear resolution need not to have linear resolution [62].

A graph G with n vertices is said to be *very well-covered* if n is an even integer and every minimal vertex cover of G has cardinality $n/2$. It is clear that every unmixed bipartite graph (without isolated vertices) is very well-covered. Let G be a bipartite graph such that $J(G)$ has a linear resolution. Mohammadi and Moradi [48, Theorem 2.2] proved that every power of $J(G)$ has a linear resolution too. Using Theorem 2.2(ii), we conclude that $J(G)^{(k)}$ has a linear resolution, for every integer $k \geq 1$. This result has been generalized in [54], as follows.

Theorem 3.9 ([54, Theorem 3.6]) *Let G be a very well-covered graph and suppose that its cover ideal $J(G)$ has a linear resolution. Then $J(G)^{(k)}$ has a linear resolution, for every integer $k \geq 1$.*

It is natural to ask whether the converse of the above theorem is true. More precisely, we propose the following question.

Question 3.10 Let G be a very well-covered graph and suppose that $J(G)^{(k)}$ has a linear resolution, for some integer $k \geq 2$. Is it true that $J(G)$ has a linear resolution?

In general, we do not know the answer of Question 3.10. However, the following proposition gives a positive answer to this question for bipartite graphs.

Proposition 3.11 ([54, Corollary 3.7]) *Let G be a bipartite graph and $k \geq 1$ be an integer. If $J(G)^{(k)} = J(G)^k$ has a linear resolution, then $J(G)$ has a linear resolution too.*

4 Depth and Stanley Depth

In this section, we collect the results about the depth and the Stanley depth of powers cover ideals of graphs. We start with depth and first recall some general facts about the depth of powers of monomial ideals.

Let $I \subset S$ be a monomial ideal. The *analytic spread* of I, denoted by $\ell(I)$, is defined as the Krull dimension of $\mathscr{R}(I)/\mathfrak{m}\mathscr{R}(I)$, where $\mathscr{R}(I) = \bigoplus_{k=0}^{\infty} I^k t^k$ is the *Rees ring* of I and $\mathfrak{m} = (x_1, \ldots, x_n)$ is the maximal ideal of S. A classical result by Burch [11] says that

$$\min_k \operatorname{depth}(S/I^k) \leq n - \ell(I).$$

By a theorem of Brodmann [8], $\operatorname{depth}(S/I^k)$ is constant for large k. We call this constant value the *limit depth* of I, and denote it by $\lim_{k\to\infty} \operatorname{depth}(S/I^k)$. Brodmann improved the Burch inequality by showing that

$$\lim_{k\to\infty} \operatorname{depth}(S/I^k) \leq n - \ell(I).$$

The smallest integer $t \geq 1$ such that $\mathrm{depth}(S/I^m) = \lim_{k\to\infty} \mathrm{depth}(S/I^k)$ for all $m \geq t$ is called the *index of depth stability of powers* of I and is denoted by $\mathrm{dstab}(I)$. It is of great interest to compute the limit of the sequence $\{\mathrm{depth}(S/I^k)\}_{k=1}^{\infty}$ and to determine or bound its index of stability. The most general results in this direction were obtained in [33, 34, 63]. In [34], Herzog and Hibi proved that if the associated graded ring $\mathrm{gr}_I(S) = \bigoplus_{k=0}^{\infty} I^k/I^{k+1}$ is Cohen-Macaulay, then

$$\lim_{t\to\infty} \mathrm{depth}(S/I^t) = n - \ell(I).$$

Herzog and Qureshi [33] showed that for every polymatroidal ideal I, we have $\mathrm{dstab}(I) \leq \ell(I)$. Furthermore, they asked whether it is true that for every squarefree monomial ideal I, the inequality $\mathrm{dstab}(I) < n$ holds. Trung [63] investigated the case of edge ideals and proved that for any edge ideal $I(G) \subset S$, the limit $\lim_{k\to\infty} \mathrm{depth}(S/I(G)^k)$ is $n - \ell(I(G))$ which is equal to the number of bipartite connected components of G. Moreover, in the same paper, it is shown that for any graph G with n vertices, we have $\mathrm{dstab}(I(G)) < n$. This gives a positive answer to the above mentioned question of Herzog and Qureshi, in the case of edge ideals.

It is also of interest to consider the similar problems for symbolic powers of monomial ideals. Let $I \subset S$ be a squarefree monomial ideal. It immediately follows from [42, Theorem 4.7] that the sequence $\{\mathrm{depth}(S/I^{(k)})\}_{k=1}^{\infty}$ is convergent. As before, let $\mathscr{R}_s(I)$ denotes the symbolic Rees ring of I. The Krull dimension of $\mathscr{R}_s(I)/\mathfrak{m}\mathscr{R}(I)$ is called the *symbolic analytic spread* of I and is denoted by $\ell_s(I)$. Varbaro [65, Proposition 2.4] proved that

$$\min_k \mathrm{depth}(S/I^{(k)}) = n - \ell_s(I).$$

Hoa, Kimura, Terai and Trung [41, Theorem 2.4] improved this equality by showing that

$$\min_k \mathrm{depth}(S/I^{(k)}) = \lim_{k\to\infty} \mathrm{depth}(S/I^{(k)}) = n - \ell_s(I).$$

Constantinescu and Varbaro [18] provided a combinatorial description for the symbolic analytic spread of cover ideals. Before stating their result, we need to recall some notions from graph theory.

Let G be a graph. A *matching* in G is a set of edges such that no two different edges share a common vertex. A subset W of $V(G)$ is called an *independent subset* of G if there are no edges among the vertices of W. Let $M = \{\{a_i, b_i\} \mid 1 \leq i \leq r\}$ be a nonempty matching of G. We say that M is an *ordered matching* of G if the following conditions hold.

(1) $A := \{a_1, \ldots, a_r\} \subseteq V(G)$ is an independent subset vertices of G; and
(2) $\{a_i, b_j\} \in E(G)$ implies that $i \leq j$.

The *ordered matching number* of G, denoted by $\nu_o(G)$, is defined to be

$$\nu_o(G) = \max\{|M| \mid M \subseteq E(G) \text{ is an ordered matching of } G\}.$$

Theorem 4.1 ([18, Theorem 2.8]) *For any graph G,*

$$\ell_s(J(G)) = \nu_o(G) + 1.$$

As a consequence of Theorem 4.1 and the above discussion, one obtain the following result concerning the depth of high powers of cover ideals.

Corollary 4.2 *For every graph G with n vertices, we have*

$$\lim_{k\to\infty} \operatorname{depth}(S/J(G)^{(k)}) = n - \nu_o(G) - 1.$$

Let I be a squarefree monomial ideal and let $\operatorname{dstab}_s(I)$ denote the *index of depth stability of symbolic powers* of I which is the smallest integer $t \geq 1$ with $\operatorname{depth}(S/I^{(m)}) = \lim_{k\to\infty} \operatorname{depth}(S/I^{(k)})$ for all $m \geq t$. In [41, Theorem 2.4], it was proven that

$$\operatorname{dstab}_s(I) \leq n(n+1)\operatorname{bight}(I)^{n/2},$$

where $\operatorname{bight}(I)$ is the maximum height of associated primes of I. However, the situation is much better for cover ideals. Indeed, we have a linear upper bound for the index of depth stability of symbolic powers of cover ideals, as the following theorem shows.

Theorem 4.3 ([41, Theorem 3.4] and [52, Theorem 3.1]) *For any graph G with n vertices, we have*

$$\operatorname{dstab}_s(J(G)) \leq 2\nu_o(G) - 1.$$

Let M be a finitely generated $\mathbb{Z}^n$-graded S-module. Let $u \in M$ be a homogeneous element and $Z \subseteq \{x_1, \ldots, x_n\}$. The $\mathbb{K}$-subspace $u\mathbb{K}[Z]$ generated by all elements uv with $v \in \mathbb{K}[Z]$ is called a *Stanley space* of dimension $|Z|$, if it is a free $\mathbb{K}[Z]$-module. Here, as usual, $|Z|$ denotes the number of elements of Z. A decomposition $\mathscr{D}$ of M as a finite direct sum of Stanley spaces is called a *Stanley decomposition* of M. The minimum dimension of a Stanley space in $\mathscr{D}$ is called the *Stanley depth* of $\mathscr{D}$ and is denoted by $\operatorname{sdepth}(\mathscr{D})$. The quantity

$$\operatorname{sdepth}(M) := \max\left\{\operatorname{sdepth}(\mathscr{D}) \mid \mathscr{D} \text{ is a Stanley decomposition of } M\right\}$$

is called the *Stanley depth* of M. For a reader friendly introduction to Stanley depth, we refer to [49] and for surveys on this topic, we refer to [32, 55].

We say that a $\mathbb{Z}^n$-graded S-module M satisfies *Stanley's inequality* if

$$\operatorname{depth}(M) \leq \operatorname{sdepth}(M).$$

In fact, Stanley [60] conjectured that every $\mathbb{Z}^n$-graded S-module satisfies Stanley's inequality. This conjecture has been recently disproved in [21]. The counterexample presented in [21] lives in the category of squarefree monomial ideals. Thus, one can ask whether Stanley's inequality holds for non-squarefree monomial ideals. Of particular interest are the high powers of monomial ideals. In fact, in [52], we proposed the following question.

Question 4.4 ([52, Question 1.1]) Let I be a monomial ideal. Is it true that I^k and S/I^k satisfy the Stanley's inequality for every integer $k \gg 0$?

Question 4.4 has been investigated for several classes of ideals, including complete intersections [15], polymatroidal ideals [50] and edge ideals [3, 50, 53].

An analogue of Question 4.4 for symbolic powers was also asked in [52].

Question 4.5 ([52, Question 1.1]) Let I be a squarefree monomial ideal. Is it true that $I^{(k)}$ and $S/I^{(k)}$ satisfy the Stanley's inequality for every integer $k \gg 0$?

As we mentioned above, for any graph G and for any integer $k \gg 0$, we have

$$\mathrm{depth}(S/J(G)^{(k)}) = n - \nu_o(G) - 1.$$

Hence, in order to give a positive answer to Question 4.5 for cover ideals, one needs to prove

$$\mathrm{sdepth}(S/J(G)^{(k)}) \geq n - \nu_o(G) - 1,$$

and

$$\mathrm{sdepth}(J(G)^{(k)}) \geq n - \nu_o(G),$$

for every $k \gg 0$. The above inequalities were proved for bipartite graphs in [51, Theorem 3.3] and for any arbitrary graph in [52].

Theorem 4.6 ([52, Theorem 3.5 and Corollary 3.6]) *Let G be a graph with n vertices. Then for every integer $k \geq 1$, we have*

$$\mathrm{sdepth}(J(G)^{(k)}) \geq n - \nu_o(G) \quad \text{and} \quad \mathrm{sdepth}(S/J(G)^{(k)}) \geq n - \nu_o(G) - 1.$$

In particular, $J(G)^{(k)}$ and $S/J(G)^{(k)}$ satisfy Stanley's inequality, for every integer $k \geq 2\nu_o(G) - 1$.

In [54], we studied the Stanley depth of symbolic powers of cover ideals of very well-covered graphs and showed that any power of cover ideals of a certain class of very well-covered graphs satisfies Stanley's inequality. More precisely, we proved the following stronger result.

Proposition 4.7 ([54, Corollary 3.8]) *Let G be a very well-covered graph and suppose that its cover ideal $J(G)$ has a linear resolution. Then $J(G)^{(k)}$ and $S/J(G)^{(k)}$ satisfy Stanley's inequality, for every integer $k \geq 1$.*

In [34], Herzog and Hibi asked whether for a squarefree monomial ideal, the sequence $\{\mathrm{depth}(S/I^k)\}_{k=1}^{\infty}$ is non-increasing. We know from [44] that the answer of this question is in general negative and the counterexample given in [44] is the cover ideal of a graph. However, it was shown in [17, Theorem 3.2] that the sequence $\{\mathrm{depth}(S/J(G)^k)\}_{k=1}^{\infty}$ is non-increasing, if G is a bipartite graph. This result was extended by Hoa, et al., as follows.

Theorem 4.8 ([41, Theorem 3.2]) *Let G be a graph. Then for every integer $k \geq 1$, we have*

$$\mathrm{depth}(S/J(G)^{(k)}) \geq \mathrm{depth}(S/J(G)^{(k+1)}).$$

The conclusion of Theorem 4.8 remains true, if one replaces depth by sdepth.

Theorem 4.9 ([52, Theorem 3.3]) *Let G be a graph. Then for every integer $k \geq 1$, we have*

(i) $\mathrm{sdepth}(S/J(G)^{(k)}) \geq \mathrm{sdepth}(S/J(G)^{(k+1)})$*, and*
(ii) $\mathrm{sdepth}(J(G)^{(k)}) \geq \mathrm{sdepth}(J(G)^{(k+1)})$.

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Fermat-Type Arrangements

Justyna Szpond

Abstract The purpose of this work is to collect in one place available information on line arrangements known in the literature as braid, monomial, Ceva or Fermat arrangement. They have been studied for a long time and appeared recently in connection with highly interesting problems, namely: the containment problem between symbolic and ordinary powers of ideals and the existence of unexpected hypersurfaces. We also study also derived configurations of points (or more general: linear flats) which arise by intersecting hyperplanes in Fermat arrangements or by taking duals of these hyperplanes. Furthermore we discuss briefly higher dimensional generalizations and present results arising by applying this approach to problems mentioned above. Some of our results are original and appear for the first time in print.

Keywords Fermat-type arrangement · Containment problem · Unexpected hypersurfaces · Arrangements · Freeness

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1 Introduction

Fermat arrangements of lines, as reflection arrangements, appear (under the name of Ceva arrangements) in Hirzebruch's work [18]. Hirzebruch's interest in them was motivated by seeking ways to construct surfaces of general type which are ball quotients. This treatment has been considerably extended in the book by Barthel, Hirzebruch and Höfer [3]. For a recent update on relations between ball quotients and line arrangements we refer to Tretkoff's book [29].

It seems that the name Fermat arrangements first appears in PhD thesis of Urzúa [30], which is devoted to the geography problem of surfaces of general type. They

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came into focus in connection with examples of the non-containment between the third symbolic and the second ordinary power of an ideal of a set of points in $\mathbb{P}^2$, which appeared first in the work of Dumnicki, Szemberg and Tutaj-Gasińska [14] and were considerable generalized by Harbourne and Seceleanu [17]. Subsequently asymptotic invariants of associated ideals of points have been computed in [12]. Nagel and Seceleanu provided in [22] detailed description of Rees algebras of ideals of points derived from Fermat arrangements of lines.

The idea of studying generalizations of Fermat line arrangements to Fermat-type arrangements of hyperplanes in higher dimensional projective spaces appeared during the workshop "Ordinary and Symbolic Powers of Ideals" held in Oaxaca in May 2017, particularly during discussions with Juan Migliore and Uwe Nagel which I enjoyed so much. My joint papers with Malara [19, 20] show how configurations of codimension two flats derived from Fermat-type arrangements provide additional non-containment results.

Additional interest in examples studied here comes from yet another direction. The Bounded Negativity Conjecture predicts that on any complex algebraic surface the self-intersection numbers of reduced and irreducible curves are bounded from below. This Conjecture is one of the central and most difficult problems in the theory of algebraic surfaces. Seminal work of Bauer, Harbourne, Knutsen, Küronya, Müller-Stach, Roulleau, and Szemberg [5] renewed interest in this conjecture. The subsequent article by Bauer, Di Rocco, Harbourne, Huizenga, Lundman, Pokora, and Szemberg [4] revealed a link between line arrangements and the Bounded Negativity Conjecture. Fermat-type arrangements served there as examples with extremal Harbourne constants.

My initial idea was to treat here all this interesting developments. I realized soon that this would exceed by far the scope of a conference proceedings article. Therefor I decided to focus on the latest theory where Fermat-type arrangements and derived configurations seem to play a prominent role. I mean here the theory of unexpected hypersurfaces initiated in the edge-cutting work of Cook II, Harbourne, Migliore, and Nagel [8] developing very rapidly. Thus after collecting some general facts about reflection arrangements I pass directly to Fermat-type arrangements and unexpected hypersurfaces. Some of results presented there are new and hopefully so surprising, that they will ignite new pathes of research.

I failed in writing a comprehensive survey on Fermat-type arrangements and their appearances in commutative algebra and algebraic geometry. Maybe some other time.

1.1 Notation

We adopt the combinatorial convention and write

$$[t] = \{1, \ldots, t\}\,.$$

Even though many arguments are valid over an arbitrary field containing enough roots of unity, in order to avoid additional assumptions at some places and to streamline the discussion, we make a general assumption of working solely over the field of complex numbers.

2 Reflection Groups and Arrangements

In this section we recall briefly some general facts about arrangements associated to finite reflection groups. This will serve as a motivation for one of possible generalizations of Fermat arrangements of lines.

Let V be the unitary space $\mathbb{C}^{N+1}$ with standard basis $e_0, \ldots, e_N$ and unitary product $< \cdot, \cdot >$. Let $R = \mathbb{C}[x_0, \ldots, x_N]$ be the ring of polynomials. Let G be a finite group of linear automorphisms of V. Then G acts on R by

$$(g \cdot f)(x) = f(g^{-1}(x))$$

for all $x \in V$, $f \in R$ and $g \in G$.

Definition 2.1 (*Imprimitive group of automorphisms*) A group G of unitary automorphisms of V is called *imprimitive*, if V is a direct sum

$$V = V_1 \oplus \ldots \oplus V_t \tag{1}$$

of non-trivial proper linear subspaces V_i of V such that the set $\{V_i, i \in [t]\}$ is invariant under G. This set is called a *system of imprimitivity* for G.

We are mainly interested here in reflection groups.

Definition 2.2 (*Reflections and reflection groups*) A linear automorphism s of V of finite order is a *reflection* in V if it has exactly N eigenvalues equal to 1.
A *reflection group* in V is a group generated by reflections in V.

Equivalently, a reflection is a non-trivial linear automorphism of finite order that fixes a hyperplane. We call this hyperplane a *reflection hyperplane*.

Remark 2.3 Any reflection s in V of order $d \geq 2$ has the form $s = s_{a,\varepsilon}$ with

$$s_{a,\varepsilon}(x) = x - (1 - \varepsilon)\frac{< x, a >}{< a, a >}a$$

for some vector $a \in V$ and a primitive root of unity ε of order d.

Remark 2.4 If V does not admit any decomposition as in (1), then G is called primitive.

Definition 2.5 (*Irreducible group*) A group $G \subset \mathrm{GL(V)}$ is called *irreducible* if there is no non-trivial invariant proper subspace of V invariant under G.

The following example is important for our considerations.

Example 2.6 (*Monomial groups*) Let $\Pi_{N+1} \subset \mathrm{GL(V)}$ be the group of all $(N+1) \times (N+1)$ permutation matrices. It is of course isomorphic with the permutation group S_{N+1} of $(N+1)$ elements. Let $n \geq 2$ and $p \geq 1$ be integers with $p|n$ and let $A(n, p, N+1)$ be the group of $(N+1) \times (N+1)$ diagonal matrices $A = \left(a_{ij}\right)_{i,j \in [N+1]}$ with $a_{ij} = \varepsilon^{\alpha_i}\delta_{ij}$, where ε is a primitive root of unity of order n, $\alpha_i \in [n]$ and such the product

$$\det(A) = \prod_{i \in [N+1]} a_{ii}$$

is a power of ε^p. Let $G(n, p, N+1)$ be the semi-direct product of $A(n, p, N+1)$ and Π_{N+1}. Then $G(n, p, N+1)$ is an irreducible unitary reflection group. The generators are matrices in $A(n, p, N+1)$, whose all, but exactly one, diagonal entries are 1 and products of these matrices with permutation matrices.

Expressed somewhat simpler, $G(n, p, N+1)$ consists of all monomial $(N+1) \times (N+1)$ matrices whose non-zero elements are roots of unity of order n and the product of these entries is a root of unity of order n/p. Recall that a matrix is called *monomial* if in any row and any column there is exactly one non-zero element. In particular permutation matrices are monomial.
For $n = 1$ we could identify Π_{N+1} with the group $G(1, 1, N+1)$. Note that this group is reducible and the following inclusions holds:

$$G(1, 1, N+1) \subset G(n, p, N+1) \subset G(n, 1, N+1).$$

The group $G(n, 1, N+1)$ is called the *full monomial group*, see [23, Example 6.29] or [11, Example 2.23].

Remark 2.7 The reflection hyperplanes for $G(n, p, N+1)$ with $p < n$ are of the form

$$x_i = \varepsilon^{\alpha} x_j$$

for $0 \leq i < j \leq N$ and $\alpha \in [n]$ or the coordinate hyperplanes

$$x_i = 0$$

for $i \in \{0, \ldots, N\}$.
For $G(n, n, N+1)$ only the hyperplanes of the first kind are reflection hyperplanes.

Finite complex reflection groups were classified by Shephard and Todd [24]. In particular they showed the following result [24, Sect. 2], see also [7, Theorem 2.4] for an alternative proof.

Theorem 2.8 (Imprimitive reflection groups) *The only (up to conjugation) irreducible imprimitive unitary groups generated by reflections are*

$$G(n, p, N + 1)$$

for $n \geq 2$, $N \geq 1$ and $p|n$ with exception of $G(2, 2, 2)$ which is reducible.

Remark 2.9 The groups denoted here by $G(n, p, N + 1)$ are usually in the literature denoted by $G(m, p, n)$. We thought it less confusing to use right away the notation which applies in other parts of this article.

Definition 2.10 (*Semi-invariant polynomials*) A polynomial $f \in R$ is called *semi-invariant* with respect to G, if there exists a linear character $\eta : G \to \mathbb{C}^*$ such that

$$g \cdot f = \eta(g) \cdot f$$

for all $g \in G$.

Remark 2.11 It is well known (see e.g. [7, Proposition 2.2] that if G is an imprimitive and irreducible finite reflection group and N is at least 1 (i.e. $\dim(V) \geq 2$), then $\dim(V_i) = 1$ for all $i \in [t]$, hence $t = N + 1$ and there are mutually distinct homogeneous linear polynomials $\ell_1, \ldots, \ell_t$ such that their product $\ell_1 \cdot \ldots \cdot \ell_t$ is a homogeneous, semi-invariant with respect to G, polynomial of degree $N + 1$ in R.

Definition 2.12 (*Reflection arrangement*) A *reflection arrangement* is the hyperplane arrangement $\mathscr{H}(G)$ which consists of reflection hyperplanes defined by elements of a finite reflection group G.

Example 2.13 (*Braid arrangement*) The group $G(1, 1, N + 1)$ is just a representation of the symmetry group S_{N+1} acting on $V = \mathbb{C}^{N+1}$ by permuting coordinates. The reflecting hyperplanes are given by equations

$$x_i = x_j$$

for $0 \leq i < j \leq N$. This arrangement can be studied also projectively. The hyperplanes in $\mathbb{P}^N$ are defined by linear factors of the semi-invariant polynomial

$$F_{N,1} = \prod_{0 \leq i < j \leq N} (x_i - x_j).$$

Let us consider the case of $N = 2$ in detail. The group $G(1, 1, 3)$ consists of 6 matrices:

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The reflection hyperplanes are eigenspaces of 1 of matrices A_4, A_5 and A_6. These are thus three lines defined by linear factors of $(x_0 - x_1)(x_0 - x_2)(x_1 - x_2)$. The resulting arrangement consists of 3 lines passing through the point $(1 : 1 : 1)$. We refer to [25, Example 1.3] for more details on these arrangements.

In the next section we define hyperplane arrangements in general and we recall also some of their fundamental properties.

3 Arrangements and Their Basic Properties

The theory of hyperplane arrangements is a classical subject of study in mathematics. It seems that the first non-trivial line arrangement studied by Greek geometers is that associated to the Theorem of Pappus. Nowadays hyperplane arrangements constitute an area of intensive study with far reaching connections to algebra, analysis, combinatorics, geometry and topology. In this section we establish the basic terminology and properties.

Definition 3.1 (*Hyperplane arrangement*) A *hyperplane arrangement* $\mathscr{H}$ in the projective space $\mathbb{P}^N$ is a finite collection of mutually distinct hyperplanes.

Remark 3.2 Projective arrangements of hyperplanes correspond to *central* affine arrangements, that is arrangements where all hyperplanes pass through the origin.

A hyperplane H in $\mathbb{P}^N$ is defined by a linear polynomial f_H, which is determined uniquely up to a non-zero multiplicative scalar. To any arrangement $\mathscr{H}$ one can thus associate its *defining polynomial*

$$Q(\mathscr{H}) = \prod_{H \in \mathscr{H}} f_H,$$

which again is defined up to a scalar. It defines a unique principal ideal $I(\mathscr{H}) = \langle Q(\mathscr{H}) \rangle$ in R, which we call the *arrangement ideal*.

A fundamental combinatorial object associated to an arrangement is its intersection lattice.

Definition 3.3 (*Intersection lattice*) Let $\mathscr{H}$ be an arrangement. The set $L(\mathscr{H})$ of all non-empty intersections of hyperplanes in $\mathscr{H}$ is the *intersection lattice* of $\mathscr{H}$. This set has a natural structure of a poset defined by reversed inclusion relation.

We shall now introduce the arrangements which are in our focus.

Example 3.4 (*Fermat arrangements*) It is natural to extend the arrangements in Example 2.13 replacing linear factors $(x_i - x_j)$ by factors containing powers of variables. More precisely, for a positive integer n, let

$$F_{N,n} = \prod_{0 \le i < j \le N} (x_i^n - x_j^n).$$

This polynomial splits over complex numbers into linear factors of the type

$$x_i - \varepsilon^k x_j, \tag{2}$$

where ε is a primitive root of unity of degree n and $k \in [n]$. The Fermat arrangement in $\mathbb{P}^N$ consists of zeros of all linear factors of $F_{N,n}$. Following Orlik and Terao [23, Example 6.29] we denote this arrangement by $\mathscr{A}^0_{N+1}(n)$. Of course, for $n = 1$ we recover a braid arrangement.

It is well known that Fermat arrangements are reflection arrangements. Indeed, the group $G(n, n, N + 1)$ contains reflections in all hyperplanes defined in (2). Thus the example generalizes readily as follows.

Example 3.5 (*Extended Fermat arrangements*) For the groups $G = G(n, p, N + 1)$ with $p < n$, we obtain reflection arrangement with

$$Q(\mathscr{H}(G)) = x_0 \cdot \ldots \cdot x_N \cdot \prod_{0 \le i < j \le N} (x_i^n - x_j^n).$$

Thus the reflection hyperplanes are all those of the corresponding Fermat arrangement with the addition of coordinate hyperplanes. We call the resulting arrangement $\mathscr{H}(G)$ the extended Fermat arrangement. In the literature it can be also encountered under the name of Ceva arrangement.

Again, following [23] we denote this arrangement by $\mathscr{A}^{N+1}_{N+1}(n)$. In turn, following Hirzebruch [18], we introduce intermediate Fermat arrangements $\mathscr{A}^{k+1}_{N+1}(n)$ as defined by linear factors of polynomials

$$F_{N,n,k} = x_0 \cdot \ldots \cdot x_k \cdot \prod_{0 \le i < j \le N} (x_i^n - x_j^n),$$

for $k = 0, \ldots, N$.

We introduce now briefly some useful properties of arrangements.

We denote by $\mathrm{Der(R)}$ the R-module of $\mathbb{C}$-linear derivation of R. It is a free R-module with basis $D_0, \ldots, D_N$, where D_i stands as usual for the partial derivation $\partial/\partial x_i$ for $i = 0, \ldots, N$. We say that a derivation $\theta \in \mathrm{Der(R)}$ is homogeneous of polynomial degree d if

$$\theta = \sum_{i=0}^{N} f_i D_i,$$

with f_i a homogeneous polynomial in R of degree d. In this way $\mathrm{Der(R)}$ becomes a $\mathbb{Z}$-graded R-module with

$$\mathrm{Der(R)} = \bigoplus_{\mathrm{d} \in \mathbb{Z}} \mathrm{Der(R)_d},$$

where $\mathrm{Der(R)_d}$ consists of all homogeneous derivations of polynomial degree d.

The *Euler derivation*

$$D_E = \sum_{i=0}^{N} x_i D_i$$

is a distinguished element of polynomial degree 1. For each homogeneous polynomial $f \in R$ we have

$$D_E(f) = \deg(f) f.$$

Thus the Euler derivation keeps any homogeneous principal ideal invariant. We want now to distinguish these derivations which keep an arrangement ideal invariant.

Definition 3.6 (*Module of derivations*) Let $\mathscr{H}$ be an arrangement of hyperplanes in V. The *module of $\mathscr{H}$-derivations of $\mathscr{H}$* is defined by

$$\mathrm{Der}(\mathscr{H}) := \{\theta \in \mathrm{Der(R)} : \ \theta(\mathrm{Q}(\mathscr{H})) \in \mathrm{I}(\mathscr{H})\}.$$

As mentioned above, the Euler derivation is an element in $\mathrm{Der}(\mathscr{H})$ for any $\mathscr{H}$. Thus, it is more interesting to study the quotient module

$$\mathrm{Der}(\mathscr{H})_0 = \mathrm{Der}(\mathscr{H})/\mathrm{D_E R},$$

where $D_E R$ denotes the rank 1 submodule of $\mathrm{Der}(\mathscr{H})$ generated by D_E. The module $\mathrm{Der}(\mathscr{H})_0$ consists hence of derivatives θ, which annihilate $Q(\mathscr{H})$, i.e., $\theta(Q(\mathscr{H})) = 0$. If we write

$$\theta = \sum_{i=0}^{N} s_i D_i$$

with homogeneous polynomials $s_0, \ldots, s_N$, then the condition

$$\left(\sum_{i=0}^{N} s_i D_i\right)(Q(\mathscr{H})) = 0$$

can be restated as

$$\sum_{i=0}^{N} s_i (D_i(Q(\mathscr{H}))) = 0, \tag{3}$$

which is clearly a syzygy relation between partial derivatives of the polynomial $Q(\mathscr{H})$ defining the arrangement $\mathscr{H}$.

All this leads to the following important notion.

Definition 3.7 (*Free arrangements*) We say that an arrangement $\mathscr{H}$ is *free* if the module $\mathrm{Der}(\mathscr{H})$ is a free R-module.

If $\mathscr{H}$ is free then there exist integers $a_0, \ldots, a_N$ such that $\mathrm{Der}(\mathscr{H}) \simeq \oplus_{\mathrm{i=0}}^{\mathrm{N}} \mathrm{R}(-\mathrm{a_i})$. These numbers are called the *exponents* of the arrangement $\mathscr{H}$. Customarily, we assume that the exponents are ordered with $a_0 \leq a_1 \leq \ldots \leq a_N$.

Remark 3.8 Since

$$\mathrm{Der}(\mathscr{H}) = \mathrm{D_E R} \oplus \mathrm{Der}(\mathscr{H})_0,$$

it is clear that the freeness of an arrangement is equivalent to the freeness of the module $\mathrm{Der}(\mathscr{H})_0$. It follows also that $a_0 = 1$ and this exponent is typically omitted. For example, a free line arrangement $\mathscr{H}$ has exponents (sometimes described also as the splitting type) (a, b) with $a + b = d - 1$, where d is the number of lines in $\mathscr{H}$.

The relations in (3) show that the freeness of $\mathscr{H}$ is also equivalent to the freeness of the module of syzygies of the Jacobian ideal of $Q(\mathscr{H})$, i.e., the ideal $\mathrm{Jac}(\mathscr{H}) = \mathrm{Jac}(\mathrm{Q}(\mathscr{H}))$ generated by partial derivatives of $Q(\mathscr{H})$. The minimal free resolution of this ideal is in this case of the form

$$0 \to \mathrm{Der}(\mathscr{H})_0 \to \mathrm{R}(-\mathrm{d}+1)^{\oplus(\mathrm{N}+1)} \to \mathrm{Jac}(\mathscr{H}) \to 0.$$

It follows from Remark 3.8 that one of the exponents of a free arrangement is 1 (it corresponds to the submodule generated by the Euler derivation). We omit it listing in the sequel exponents of free arrangements.

Remark 3.9 The sheafification of $\mathrm{Der}(\mathscr{H})_0$ is the sheaf $T_{\mathbb{P}^N}(-log\mathscr{H})$ of logarithmic vector fields.

It is difficult to decide in general if an arrangement is free. However for some classes of arrangements it is known. In particular we have the following result, see [28].

Theorem 3.10 (Freeness of reflection arrangements) *Any reflection arrangement is free.*

Example 3.11 (*Fermat line arrangements*) The arrangements $F_{2,n}$ are free for all $n \geq 1$ and for $n \geq 3$ they have the splitting type $(n+1, 2n-2)$, see [8, Proposition 6.12]. See [23, Corollary 3.86] for the explicit formula for exponents of $F_{N,n}$ arrangements.

Example 3.12 (*Extended Fermat line arrangements*) The arrangements $\mathscr{A}_3^3(n)$ are free for all $n \geq 1$ with the splitting type $(n+1, 2n+1)$.

4 Unexpected Curves

The concept of *unexpected curves* has been introduced in the ground breaking article [8] of Cook II, Harbourne, Migliore and Nagel. Initially it has been defined only for curves in $\mathbb{P}^2$ with strong constraints on the relation between the degree and the multiplicity of the unexpected curve. More precisely we had (see [8, Definition 2.1]).

Definition 4.1 (*Unexpected plane curves*) We say that a finite set Z of reduced points in $\mathbb{P}^2$ *admits an unexpected curve* of degree $m+1$, if for a general point P, the fat point scheme mP (i.e. defined by the ideal $I(P)^m$) fails to impose independent conditions on the linear system of curves of degree m vanishing at all points of Z. In other words, Z admits an unexpected curve of degree $m+1$ if

$$h^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(d) \otimes I(Z+mP)) \;>\; \max\left\{h^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(d) \otimes I(Z)) - \binom{m+1}{2},\ 0\right\}.$$

Note, that it follows immediately from the definition (taking a projection from the point P), that an unexpected curve is rational. Moreover, it is irrelevant if the points in Z impose independent conditions on curves of any degree or not. In particular, they can be arranged in a special position. In fact, in all example discovered so far the points in Z exhibit a lot of symmetries.

Research in [8] has been motivated by the article [10] by Di Gennaro, Illardi and Valles, where the existence of unexpected curves has been firstly observed. Incidentally, in the example studied in [10] the set Z is dual to the B_3 arrangement of lines. It is the arrangement associated to the Weyl group of a B_3 root system.

Definition 4.2 (B_3 *arrangement of lines*) The B_3 arrangement is the reflection arrangement defined by the the group $G(2, 1, 3)$.

Thus, according to Remark 2.7, the lines in the B_3 arrangement are described by linear factors of the polynomial

$$x_0x_1x_2(x_0^2 - x_1^2)(x_1^2 - x_2^2)(x_2^2 - x_0^2).$$

In the notation of Example 3.4 this is the arrangement $\mathscr{A}_3^3(2)$.

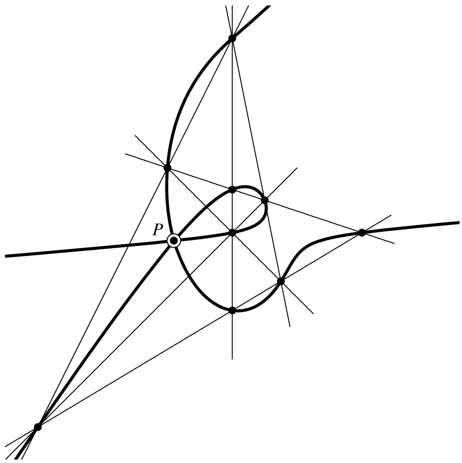
Dually, we obtain a set Z of 9 points with the following coordinates

$$\begin{array}{lll} P_1 = (1:0:0), & P_2 = (0:1:0), & P_3 = (0:0:1), \\ P_4 = (1:1:0), & P_5 = (1:-1:0), & P_6 = (1:0:1), \\ P_7 = (1:0:-1), & P_8 = (0:1:1), & P_9 = (0:1:-1). \end{array}$$

Figure 1 shows an unexpected curve admitted by a B_3 arrangement. The coordinate system in this Figure has been so chosen that the set Z is completely contained in the affine part of the plane.

It has been realized in [6] that one can actually write explicitly the equation of an unexpected quartic Q_P in this case. If $P = (a:b:c)$ is general, then

Fig. 1 A visualization of an unexpected quartic admitted for B_3



$$Q_P(x:y:z) = 3a(b^2-c^2)\cdot x^2yz + 3b(c^2-a^2)\cdot xy^2z + 3c(a^2-b^2)\cdot xyz^2 \\ +a^3\cdot y^3z - a^3\cdot yz^3 + b^3\cdot xz^3 - b^3\cdot x^3z + c^3\cdot x^3y - c^3\cdot xy^3. \quad (4)$$

It is natural to wonder if the set of points dual to the Fermat arrangement $\mathscr{A}_3^0(2)$ also admits an unexpected curve. It turns out that this does not happen for degree 2, but allowing high enough degree we obtain in this way additional examples of unexpected curves. More precisely, we have the following result, see [8, Theorem 6.12].

Theorem 4.3 (Cook II, Harbourne, Migliore, Nagel) *For $m \geq 5$ let Z be the set of points dual to lines in the Fermat arrangement $\mathscr{A}_3^0(m)$, i.e., given by linear factors of*

$$F_{2,m} = (x_0^m - x_1^m)(x_1^m - x_2^m)(x_2^m - x_0^m).$$

Then Z admits an unexpected curve of degree $m+2$ with a point of multiplicity $m+1$. Moreover the unexpected curve is unique and irreducible.

It is natural to wonder if for lower values of m intermediate Fermat arrangements $\mathscr{A}_3^k(m)$ for $k = 1, 2$ might lead to unexpected curves. We show that this is indeed the case.

4.1 Fermat-Type Arrangement for $m = 3$

Theorem 4.4 *Let Z be the set of points dual to lines in the Fermat arrangement $\mathscr{A}_3^2(3)$. Then Z admits a unique and irreducible unexpected quintic with a point of multiplicity 4 at a general point $R = (a:b:c)$.*

Proof The lines in $\mathscr{A}_3^2(3)$ are given by linear factors of

$$x_0x_1(x_0^3 - x_1^3)(x_1^3 - x_2^3)(x_2^3 - x_0^3),$$

so that the 11 points in Z are the coordinate points $(1 : 0 : 0)$ and $(0 : 1 : 0)$ together with the following points

$$\begin{aligned} P_1 &= (1 : -1 : 0),\ P_2 = (1 : -\varepsilon : 0),\ P_3 = (1 : -\varepsilon^2 : 0),\\ P_4 &= (1 : 0 : -1),\ P_5 = (1 : 0 : -\varepsilon),\ P_6 = (1 : 0 : -\varepsilon^2),\\ P_7 &= (0 : 1 : -1),\ P_8 = (0 : 1 : -\varepsilon),\ P_9 = (0 : 1 : -\varepsilon^2). \end{aligned}$$

The points $P_1, \ldots, P_9$ form a complete intersection given by the coordinate axes and the lines in the corresponding Fermat arrangement $\mathscr{A}_3^0(3)$, i.e., their ideal is generated by $x_0x_1x_2$ and $x_0^3 + x_1^3 + x_2^3$. Intersecting with the ideals of the remaining 2 points we obtain

$$I(Z) = \langle x_0x_1x_2,\ x_0^3x_2 + x_1^3x_2 + x_2^4,\ x_1^4x_2 + x_1x_2^4 \rangle.$$

Since the regularity of $I(Z)$ is 5, the linear system of quintics vanishing in all points of Z has (projective) dimension 9. Thus it is not expected that for a general point R, it contains a member which vanishes at R to order 4. However, there exists such an unexpected curve given by the equation

$$\begin{aligned} Q_R(x_0 : x_1 : x_2) = {} & a^4 \cdot x_1x_2 \cdot (x_1^3 + x_2^3) + b^4 \cdot x_0x_2 \cdot (x_0^3 + x_2^3) + c^4 \cdot x_0x_1 \cdot (x_0^3 + x_1^3)\\ & - 4a(b^3 + c^3) \cdot x_0^3x_1x_2 - 4b(a^3 + c^3) \cdot x_0x_1^3x_2\\ & - 4c(a^3 + b^3) \cdot x_0x_1x_2^3\\ & + 6a^2b^2 \cdot x_0^2x_1^2x_2 + 6a^2c^2 \cdot x_0^2x_1x_2^2 + 6b^2c^2 \cdot x_0x_1^2x_2^2. \end{aligned}$$

The latter claim can be easily verified by a direct computation. □

Passing to the next degree of the Fermat-type arrangement we can drop another coordinate line.

4.2 Fermat-Type Arrangement for $m = 4$

Theorem 4.5 *Let Z be the set of points dual to lines in the Fermat arrangement $\mathscr{A}_3^1(4)$. Then Z admits a unique and irreducible unexpected sextic with a point of multiplicity 5 at a general point $R = (a : b : c)$.* □

Proof The lines in $\mathscr{A}_3^1(4)$ are given by linear factors of

$$x_0(x_0^4 - x_1^4)(x_1^4 - x_2^4)(x_2^4 - x_0^4),$$

so that the 13 points in Z are the coordinate point $(1 : 0 : 0)$ and the points with coordinates

$$\begin{aligned}
&P_1 = (1:1:0),\ P_2 = (1:i:0),\ \ P_3 = (1:-1:0),\ \ P_4 = (1:-i:0),\\
&P_5 = (1:0:1),\ P_6 = (1:0:i),\ \ P_7 = (1:0:-1),\ \ P_8 = (1:0:-i),\\
&P_9 = (0:1:1),\ P_{10} = (0:1:i),\ P_{11} = (0:1:-1),\ P_{12} = (0:1:-i).
\end{aligned}$$

The ideal of Z is generated by

$$x_0x_1x_2,\ x_0^4x_2 + x_1^4x_2 - x_2^5,\ x_0^4x_1 - x_1^5 + x_1x_2^4,$$

so Z is an almost complete intersection ideal (that means that the number of generators is one higher than the hight of an ideal). Such ideals have an easy minimal free resolution and either writing it explicitly down or using a symbolic algebra program (we used Singular [9]) we get reg(I(Z)) = 6, so that Z imposes independent conditions on curves of degree 6. Thus $\dim(I)_{[6]} = 15$ and we do not expect that for a general point R there exists an element vanishing at R to order 5.

However, such an element can be written down explicitly as follows:

$$\begin{aligned}
S_R(x_0:x_1:x_2) = {}& a^5 \cdot x_1x_2 \cdot (x_1^4 - x_2^4) + b^5 \cdot x_0x_2 \cdot (x_2^4 - x_0^4) + c^5 \cdot x_0x_1 \cdot (x_0^4 - x_1^4)\\
&+ 10a^3x_0^2x_1x_2 \cdot (b^2x_1^2 - c^2x_2^2)\\
&+ 10b^3x_0x_1^2x_2 \cdot (c^2x_2^2 - a^2x_0^2)\\
&+ 10c^3x_0x_1x_2^2 \cdot (a^2x_0^2 - b^2x_1^2)\\
&+ 5a(b^4 - c^4) \cdot x_0^4x_1x_2 + 5b(c^4 - a^4) \cdot x_0x_1^4x_2 + 5c(a^4 - b^4) \cdot x_0x_1x_2^4.
\end{aligned} \tag{5}$$

Vanishing order in R can be checked by a direct (but tedious) computation. □

Theorems 4.4 and 4.5 fill thus a gap between the B_3 example and a general Theorem 4.3, showing that all these examples belong in fact to the same family. We summarize this section by the following statement.

Theorem 4.6 *Let Z be the set of points dual to lines in the Fermat arrangement $\mathscr{A}_3^k(m)$. Then for $m \geq 2$ and $0 \leq k \leq 3$ such that $k + m \geq 5$, the set Z admits a unique and irreducible unexpected curve C_R of degree $m + 2$ with a point of multiplicity $m + 1$ at a general point $R = (a:b:c)$.*
Moreover, the curve C_R does not depend on k. Thus, for example an unexpected curve for $\mathscr{A}_3^0(m)$ automatically passes through all three coordinate points. □

Proof We provide general formulas for curves C_R depending on the parity of m. We omit lengthy and not very instructive computational arguments showing that these formulas indeed define curves satisfying conditions required in our statement.

For an even $m \geq 2$ we have the following formula.

$$C_R(x_0:x_1:x_2) =$$
$$\sum_{k=1}^{\frac{m}{2}+1} \binom{m+1}{2k-1} a^{2k-1} \cdot \Big[\left(b^{m-(2k-2)}x_1^{2k-2} - c^{m-(2k-2)}x_2^{2k-2}\right) \cdot x_0^{m-(2k-2)}x_1x_2$$

$$+\left(c^{m-(2k-2)}x_2^{2k-2}-a^{m-(2k-2)}x_0^{2k-2}\right)\cdot x_0x_1^{m-(2k-2)}x_2$$
$$+\left(a^{m-(2k-2)}x_0^{2k-2}-b^{m-(2k-2)}x_1^{2k-2}\right)\cdot x_0x_1x_2^{m-(2k-2)}\Big].$$

For an odd $m \geq 3$ we have in turn

$$\begin{aligned}
&C_R(x_0 : x_1 : x_2) = \\
&a^{m+1}x_1x_2(x_1^m + x_2^m) + b^{m+1}x_0x_2(x_0^m + x_2^m) + c^{m+1}x_0x_1(x_0^m + x_1^m) \\
&- (m+1)\left[a(b^m + c^m)x_0^m x_1x_2 + b(a^m + c^m)x_0x_1^m x_2 + c(a^m + b^m)x_0x_1x_2^m\right] \\
&+ \sum_{k=2}^{\frac{m-1}{2}}(-1)^k\binom{m+1}{k}\Big[a^{m+1-k}x_0^k x_1x_2(b^k x_1^{m-k} + c^k x_2^{m-k}) \\
&+ b^{m+1-k}x_0x_1^k x_2(a^k x_0^{m-k} + c^k x_2^{m-k}) + c^{m+1-k}x_0x_1x_2^k(a^k x_0^{m-k} + b^k x_1^{m-k})\Big] \\
&+ \binom{m+1}{\frac{m+1}{2}}\cdot(-1)^{\frac{m+1}{2}}\Big[a^{\frac{m+1}{2}}b^{\frac{m+1}{2}}x_0^{\frac{m+1}{2}}x_1^{\frac{m+1}{2}}x_2 + b^{\frac{m+1}{2}}c^{\frac{m+1}{2}}x_0x_1^{\frac{m+1}{2}}x_2^{\frac{m+1}{2}} \\
&+ a^{\frac{m+1}{2}}c^{\frac{m+1}{2}}x_0^{\frac{m+1}{2}}x_1x_2^{\frac{m+1}{2}}\Big].
\end{aligned}$$

□

5 Unexpected Hypersurfaces

It has been quickly realized that Definition 4.1 is too restrictive. First, exactly the same phenomena can be studied in projective spaces of arbitrary dimension. More importantly, there is no need to couple the degree of the unexpected hypersurface and its multiplicity in a general point. In the subsequent to [8] article [16] Harbourne, Migliore, Nagel and Teitler generalize Definition 4.1 in the following way.

Definition 5.1 (*Unexpected hypersurface*) Let $Q_1, \ldots, Q_s$ be mutually distinct points in $\mathbb{P}^N$ and let $n_1, \ldots, n_s$ be positive integers. Let $Z = n_1Q_1 + \cdots + n_sQ_s$ be a scheme of fat points, i.e.,

$$I(Z) = I(Q_1)^{n_1} \cap \ldots \cap I(Q_s)^{n_s}.$$

Let m be a positive integer and let R be a general point in $\mathbb{P}^N$. We say that Z admits an *unexpected hypersurface* with respect to $X = mR$ of degree d, if

$$h^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d) \otimes I(Z+X)) > \max\left\{h^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d) \otimes I(Z)) - \binom{N+m-1}{N},\ 0\right\}.$$

Remark 5.2 Note that in fact one can pose the same definition replacing a fat points scheme Z by an arbitrary subscheme of $\mathbb{P}^N$. □

Interestingly, the first example of an unexpected surface in $\mathbb{P}^3$ has been announced by Bauer, Malara, Szemberg and the author in [6, Theorem 1] and it is related to a Fermat-type arrangement of planes in $\mathbb{P}^3$.

Let us begin with a general definition of a Fermat-derived configuration of flats.

Definition 5.3 (*Fermat-derived configurations*) Let $\mathscr{H} = \mathscr{A}_{N+1}^{k+1}(n)$ be a Fermat-type arrangement of hyperplanes in $\mathbb{P}^N$. Taking intersections of hyperplanes in the arrangement we obtain a number of related objects. For an integer $0 \leq t \leq N-1$ we denote by $\mathscr{H}(t)$ the set theoretical union of all t–dimensional flats in the intersection lattice. We call these sets *Fermat-derived configurations* of flats. □

In particular $\mathscr{H}(N-1)$ is the union of all arrangement hyperplanes, while $\mathscr{H}(0)$ is the union of all points in $L(\mathscr{H})$. These configurations have been introduced in [19] and investigated further in [20] in the context of the containment problem between symbolic and ordinary powers of homogeneous ideals, see [26] for an introduction to this circle of ideas.

We are now in a position to recall the main result from [6].

Theorem 5.4 (Unexpected quartic surface) *Let Z be the subset of points $(\mathscr{A}_4^4(3))(0)$ derived from the Fermat-type arrangement $(\mathscr{A}_4^4(3))$ and defined by the following binomial ideal:*

$$x_0(x_1^3 - x_2^3),\ x_0(x_2^3 - x_3^3),\ x_1(x_0^3 - x_2^3),\ x_1(x_2^3 - x_3^3)\,,$$

$$x_2(x_0^3 - x_1^3),\ x_2(x_1^3 - x_3^3),\ x_3(x_0^3 - x_1^3),\ x_3(x_1^3 - x_2^3)\,.$$

Then Z admits a unique and irreducible unexpected quartic surface Q_R, which vanishes at a general point $R = (a : b : c : d)$ to order 3. □

Proof Also in this case the statement is effective in the sense that we can write down explicitly the equation for Q_R, namely

$$\begin{aligned} Q_R(x_0 : x_1 : x_2 : x_3) = {} & b^2(c^3 - d^3) \cdot x_0^3 x_1 + a^2(d^3 - c^3) \cdot x_0 x_1^3 + c^2(d^3 - b^3) \cdot x_0^3 x_2 \\ & + c^2(a^3 - d^3) \cdot x_1^3 x_2 + a^2(b^3 - d^3) \cdot x_0 x_2^3 + b^2(d^3 - a^3) \cdot x_1 x_2^3 \\ & + d^2(b^3 - c^3) \cdot x_0^3 x_3 + d^2(c^3 - a^3) \cdot x_1^3 x_3 + d^2(a^3 - b^3) \cdot x_2^3 x_3 \\ & + a^2(c^3 - b^3) \cdot x_0 x_3^3 + b^2(a^3 - c^3) \cdot x_1 x_3^3 + c^2(b^3 - a^3) \cdot x_2 x_3^3. \end{aligned}$$

□

Let ε be a primitive root of the unity of degree 3. Then the set Z in Theorem 5.4 can be written more explicitly as all points of the form

$$(1 : \varepsilon^\alpha : \varepsilon^\beta : \varepsilon^\gamma), \quad \text{where} \quad \alpha, \beta, \gamma = 1, 2, 3$$

together with the three coordinate points in $\mathbb{P}^3$. Note that, for example, the point

$$(0:0:1:1)$$

is contained in the $L(\mathscr{A}_4^4(3))$ since it is the intersection point of arrangement hyperplanes

$$x_0 - x_1 = 0,\ x_0 - \varepsilon x_1 = 0,\ x_0 - \varepsilon^2 x_1 = 0 \text{ and } x_2 - x_3 = 0,$$

but is not an element of Z.

5.1 Unexpected Hypersurfaces with Multiple General Fat Points

After passing from Definitions 4.1–5.1, it has been realized that one can allow more general fat points to appear.

Definition 5.5 (*Unexpected hypersurfaces with multiple points*) We say that a subscheme $Z \subset \mathbb{P}^N$ *admits an unexpected hypersurface* of degree d with respect to $X = m_1P_1 + \cdots + m_sP_s$, where $m_1, \ldots, m_s$ are integers and $P_1, \ldots, P_s$ are **general** points, if the fat points scheme $X = m_1P_1 + \cdots + m_sP_s$ fails to impose independent conditions on forms of degree d vanishing along Z, i.e.,

$$h^0(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(d) \otimes I(Z + \sum j = 1^s m_jP_j)) >$$
$$\max\left\{h^0(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(d) \otimes I(Z)) - \sum_{j=1}^{s} \binom{m_j+1}{2}, 0\right\}.$$

Remark 5.6 It is relatively easy to construct examples of this kind of behaviour. For instance, let Z be an empty set and let $P_1, \ldots, P_7$ be general points in $\mathbb{P}^4$. Then it is well known (one of special cases in the Alexander-Hirschowitz classification of special linear systems with double base points [2]) that the scheme $X = 2P_1 + \cdots + 2P_7$ fails to impose independent conditions on forms of degree 3. There exists a threefold T of degree 3 singular in these 7 points. However, T is singular along the rational quartic curve passing through $P_1, \ldots, P_7$. It is much harder to find examples where the points in X are isolated in the singular locus. □

Until recently it was not clear that if there exist unexpected hypersurfaces with *isolated* multiple general fat points. The first example of this kind has been announced in [27]. Expectedly, it was constructed with a Fermat-derived configuration of points. More precisely, let Z be the union of the six coordinate points in $\mathbb{P}^5$ with the set of all points of the form

$$(1 : \varepsilon^{s_1} : \varepsilon^{s_2} : \varepsilon^{s_3} : \varepsilon^{s_4} : \varepsilon^{s_5}) \text{ with } s_1, \ldots, s_5 = 1, 2, 3,$$

where as usually ε is a primitive root of unity of order 3.

Theorem 5.7 *Let $R = (a_0 : a_1 : \ldots : a_5)$ and $S = (b_0 : b_1 : \ldots : b_5)$ be general points in $\mathbb{P}^5$. The set Z as above admits a unique expected quartic 4–fold $Q_{R,S} \subset \mathbb{P}^5$. More precisely, $Q_{R,S}$ passes through*

- *all points in Z,*
- *has at R a singularity of order 3,*
- *has at S a singularity of order 2.*

□

6 Unexpected Curves and Fermat-Derived Point Configurations

In Sect. 4 we have seen how configurations of points dual to lines in a Fermat-type arrangement lead to unexpected curves. In the present part we come back to Fermat line arrangements and investigate if the configurations of points derived from them give rise to unexpected curves. We show that this is indeed the case and somewhat surprisingly we discover a new phenomena: no matter how big degree of the Fermat arrangement we consider, the multiplicity of an unexpected curve in a general point is always 4. This is in clear opposition to Theorem 4.3 where the multiplicity of the unexpected curve in a general point grows with m. We have no conceptual explanation for this fact at the moment.

Theorem 6.1 (Unexpected curves with a point of multiplicity 4) *Let Z be the configuration of points in $\mathbb{P}^2$ derived from the Fermat arrangement $\mathscr{A}_3^0(n)$ for $n \geq 3$. Let $P = (a : b : c)$ be a general point in $\mathbb{P}^2$. We define the following numbers:*

$$u = \binom{n}{2} - 1, \quad v = \binom{n-1}{2}, \quad w = \binom{n+1}{2}.$$

Then the polynomial

$$\begin{aligned} Q_P(x:y:z) = &- cxy((ub^n + vc^n)(z^n - x^n) + (ua^n + vc^n)(y^n - z^n)) \\ &- bxz((ua^n + vb^n)(y^n - z^n) + (uc^n + vb^n)(x^n - y^n)) \\ &- ayz((ub^n + va^n)(z^n - x^n) + (uc^n + va^n)(x^n - y^n)) \\ &+ wa^{n-1}bcx^2(y^n - z^n) + wab^{n-1}cy^2(z^n - x^n) \\ &+ wabc^{n-1}z^2(x^n - y^n) \end{aligned} \tag{6}$$

- *vanishes at all points of Z,*
- *vanishes to order 4 at P,*
- *defines an unexpected curve of degree $n + 2$ for Z with respect to P.*

□

Proof The points in Z are of the form

$$P_{(\alpha,\beta)} = (1 : \varepsilon^{\alpha} : \varepsilon^{\beta})$$

where ε is a primitive root of unity of order n and $1 \le \alpha, \beta \le n$; and three coordinate points $P_1 = (1:0:0)$, $P_2 = (0:1:0)$, $P_3 = (0:0:1)$.

It is clear that Q_P vanishes at the coordinate points, since every summands of Q_P vanishes at these points. Similarly, vanishing at points $P_{(\alpha,\beta)}$ is guaranteed by vanishing of all summands in (6) in these points.

The ideal $I(P)$ is generated by

$$f_1 = cx - az, \quad \text{and } f_2 = cy - bz,$$

and then the ideal $I(4P)$ is generated by

$$g_1 = f_2^4,\ g_2 = f_1 \cdot f_2^3,\ g_3 = f_1^2 \cdot f_2^2,\ g_4 = f_1^3 \cdot f_2,\ g_5 = f_1^4.$$

These generators form in fact a Gröbner basis of the ideal $I(4P)$. In this basis the polynomial Q_P is presented as follows

$$\begin{aligned} c^4 Q_P = &\big((a^4 + 2ac^3)z - (2a^3c + c^4)\big) \cdot g_1 + \big(6a^2bcx - (4a^3b + 2bc^3)z\big) \cdot g_2 + \\ &+ \big((4ab^3 + 2ac^3)z - 6ab^2cy\big) \cdot g_4 + \big((2b^3c + c^4)y - (b^4 + 2bc^3)z\big) \cdot g_5. \end{aligned}$$

Interestingly, the third, “most symmetric”, generator of $I(4P)$ is not necessary to define the unexpected polynomial Q_P. □

7 Unexpected Surfaces and Flats

By a flat we mean here a linear subspace of a projective space, i.e., a (reduced) subscheme defined by linear equations. Linear systems with base loci imposed along higher dimensional flats have been studied recently by Guardo, Harbourne and Van Tuyl in [15]. Their study has been motivated by the containment problem between symbolic and ordinary powers of the associated homogeneous ideals. Dumnicki, Harbourne, Szemberg and Tutaj-Gasińska in [13] studied linear systems of this kind from the positivity point of view. Very recently Migliore, Nagel and Schenck in [21] initiated a systematic investigation of singular loci of hyperplane arrangements. The schemes they consider generalize our concept of Fermat-derived configurations. In this spirit we generalize Definition 5.1 once again.

Definition 7.1 (*Unexpected hypersurface for flats*) Let Z be a subscheme of $\mathbb{P}^N$, let X be a general flat in $\mathbb{P}^N$ and let m be a positive integer. We say that Z admits an *unexpected hypersurface* of degree d with respect to mX if

$$h^0(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(d) \otimes I(Z + mX)) \ > \ h^0(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(d) \otimes I(Z)) - \mathrm{HF_X(d)},$$

where HF_X denotes the Hilbert function of the scheme X. □

In other words, there is an unexpected hypersurface if X fails to impose the expected number of conditions on linear series.

Let $c(N, r, m, d)$ be the number of conditions imposed on forms of degree d in $\mathbb{P}^N$ by vanishing along an r–dimensional flat to multiplicity at least m. These numbers have been computed in [13, Lemma 2.1] and we have the following formula

$$c(N, r, m, d) = \sum_{0 \le i < m} \binom{d - i + r}{r} \binom{N - r - 1 + i}{i}.$$

Specializing to a line in $\mathbb{P}^N$ and rearranging terms we obtain

$$c(N, 1, m, d) = \frac{m(Nd + 2N + m - mN - 1)}{N(N-1)} \binom{N + m - 2}{m}.$$

And finally specializing to a line in $\mathbb{P}^3$ we get

$$c(m, d) = c(3, 1, m, d) = \binom{m+1}{2}(t+1) - 2\binom{m+1}{3}.$$

It is well known that a single line with arbitrary multiplicity imposes independent conditions on forms of any degree. This is in parallel with conditions imposed by a single point. A ground breaking discovery of Cook II, Harbourne, Migliore and Nagel in [8] was that it may happen that a fat point imposes *dependent* conditions in a *noncomplete* linear system (of hypersurfaces vanishing along Z). It is clear that a general point must be taken with multiplicity at least 2 in order to exhibit this kind of behavior. In fact Akesseh noticed in [1] that a general point of multiplicity 2 can impose dependent conditions on a noncomplete linear series only if the characteristic of the ground field is 2. Farnik, Galuppi, Sodomaco and Trok extended this picture showing that the only unexpected curve with a point of multiplicity 3 is the B_3 quartic given in (4).

We show now that, surprisingly, it may happen that a single general line fails to impose independent conditions on forms with a base locus consisting of a Fermat-derived configuration of lines in $\mathbb{P}^3$. Our observation is experimental based on Singular computations.

Let $Z \subset \mathbb{P}^3$ be the union of those lines derived from Fermat-type arrangement $\mathscr{A}_4^0(3)$, where at least three of arrangement planes intersect. It has been computed in [19, Sect. 3.1] that there are 42 mutually distinct (but not disjoint) lines in Z. By [19, Lemma 4.1] the ideal $I(Z) \subset R = \mathbb{C}[x, y, z, w]$ is generated by

$$(x^3 - y^3)(z^3 - w^3)xy, \ (x^3 - y^3)(z^3 - w^3)zw,$$

$$(x^3 - z^3)(y^3 - w^3)xz,\ (x^3 - z^3)(y^3 - w^3)yw,$$

$$(x^3 - w^3)(y^3 - z^3)xw,\ (x^3 - w^3)(y^3 - z^3)yz.$$

It is easy to check that the minimal free resolution of $I(Z)$ has the form

$$I(Z) \leftarrow R(-8)^6 \leftarrow R(-9)^4 \oplus R(-12) \leftarrow 0.$$

In particular the dimension of the space of forms of degree 9 in $I(Z)$ is exactly 20. Since a reduced line imposes $c(1, d) = d + 1$ conditions on forms of degree 10, we expect that for a general line $L \subset \mathbb{P}^3$ the space of forms of degree 9 in $I(Z + L)$ will be 10. However, according to Singular, this space has dimension 12, which is unexpected. To be more precise we run computations with a *random* line rather than a general line, but nevertheless the indications that 12 is indeed the dimension are rather strong. We were not able to verify the computations for a general line because the process has never stopped. It would be, of course, desirable to have a theoretical explanation for this phenomena and a more general phenomena, which we state now.

Remark 7.2 Let Z be the set of lines as above. Let m be a positive integer and let L be a random line in $\mathbb{P}^3$. Then Z admits unexpected surfaces of degree $m + 8$ with respect to the scheme mL for all $m \geq 1$.

Remark 7.3 Unfortunately, in opposition to results presented so far, we were not able to write down explicitly equations of the unexpected surfaces. In fact, Singular computations break, if the line is supposed to be defined by two general linear equations of the form

$$ax + by + cz + dw = 0.$$

We hope to come back to this problem in the very near future.

8 Concluding Remarks

In this note I made an attempt to introduce in a systematic way Fermat-type arrangements of hyperplanes in projective spaces and configurations of flats (including points) derived as intersections among those hyperplanes. I argued that these arrangements are always free since they are reflection arrangements. I also explained how they constitute an ample group of examples in the theory of unexpected hypersurfaces. Since they play also an important role in the containment problem for powers of ideals in commutative algebra, it can be expected that there might more areas of algebra, geometry and combinatorics where Fermat-type arrangements appear as interesting source of examples. This work can be regarded as an invitation to study these interesting objects in greater detail and to consider them as a rich testing field for various statements and ideas.

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