Time-Dependent Approach to Uniqueness of the Sommerfeld Solution to a Problem of Diffraction by a Half-Plane

A. Merzon, P. Zhevandrov, J. E. De la Paz Méndez, and T. J. Villalba Vega

Abstract We consider the Sommerfeld problem of diffraction by an opaque half-plane interpreting it as the limiting case as $t \to \infty$ of the corresponding non-stationary diffraction problem. We prove that the Sommerfeld formula for the solution is the limiting amplitude of the solution of this non-stationary problem which belongs to a certain functional class and is unique in it. For the proof of the uniqueness of solution to the non-stationary problem we reduce this problem, after the Fourier–Laplace transform in t, to a stationary diffraction problem with a complex wave number. This permits us to use the proof of the uniqueness in the Sobolev space H^1 as in (Castro and Kapanadze, J Math Anal Appl 421(2):1295–1314, 2015). Thus we avoid imposing the radiation condition from the beginning and instead obtain it in a natural way.

Keywords Diffraction · Uniqueness

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1 Introduction

The main goal of this paper is to prove the uniqueness of a solution to the Sommerfeld half-plane problem [23, 32, 33] with a real wave number, proceeding from the uniqueness of the corresponding time-dependent problem in a certain functional class. The existence and uniqueness of solutions to this problem was considered in many papers, for example in [8, 12, 25]. However, in our opinion, the problem of uniqueness is still not solved in a satisfactory form from the point of view of the boundary value problems (BVPs). The fact is that this problem is a homogeneous BVP boundary value problem which admits various nontrivial solutions. Usually the "correct" solutions are chosen by physical reasoning [23, 25, 32, 33], for example, using the Sommerfeld radiation conditions and regularity conditions at the edge.

The question is: from where do the radiation and regularity conditions arise, from the mathematical point of view?

Our goal is to show that they arise automatically from the *non-stationary* problem. This means the following: we prove that the Sommerfeld solution is a limiting amplitude of a solution to the corresponding non-stationary problem which is unique in an appropriate functional class. Since the Sommerfeld solution, as is well-known, satisfies the radiation and regularity conditions, our limiting amplitude also satisfies them. Of course, the limiting amplitude principle (LAP) is very well-known for the diffraction by smooth obstacles, see e.g. [28, 29], but we are unaware of its rigorous proof in the case of diffraction by a half-plane.

The literature devoted to diffraction by wedges including the Sommerfeld problem is enormous (see e.g. the review in [20]), and we will only indicate some papers where the uniqueness is treated. In paper [25] a uniqueness theorem was proven for the Helmholtz equation $(\Delta + 1)u = 0$ in two-dimensional regions D of half-plane type. These regions can have a finite number of bounded obstacles with singularities on their boundaries. In particular, the uniqueness of solution u to the Sommerfeld problem was proven by means of the decomposition of the solution into the sum $u = g+h$, where g describes the geometrical optics incoming and reflected waves and h satisfies the Sommerfeld radiation condition (clearly, u should also satisfy the regularity conditions at the edge).

In paper [8] exact conditions were found for the uniqueness in the case of complex wave number. The problem was considered in Sobolev spaces for a wide class of generalized incident waves, and for DD and NN boundary conditions. In paper [12] the same problem was considered also for the complex wave number and for DN boundary conditions. In both papers the Wiener-Hopf method has been used. Time-dependent scattering by wedges was considered in many papers although their number is not so large as the number of papers devoted to the stationary scattering by wedges. We indicate here the following papers: [1–4, 13, 14, 24, 26–31]. The detailed description of these papers is given in [19].

In [6, 7, 10, 17–20, 22], the diffraction by a wedge of magnitude ϕ (which can be a half-plane in the case $\phi = 0$ as in [20]) with real wavenumber was considered as a stationary problem which is the "limiting case" of a non-stationary one. More precisely, we seek the solutions of the classical diffraction problems as *limiting amplitudes* of solutions to corresponding non-stationary problems, which are unique in some appropriate functional class. We also, like in [25], decomposed the solution of non-stationary problem separating a "bad" incident wave, so that the other part of solution belongs to a certain appropriate functional class. Thus we avoided the a priori use of the radiation and regularity conditions and instead obtained them in a natural way. In papers [10, 17, 19] we considered the time-dependent scattering with DD, DN and NN boundary conditions and proved the uniqueness of solution in an appropriate functional class. But these results were obtained only for $\phi \neq 0$ because in the proof of uniqueness we used the Method of Complex Characteristics [15, 16, 21] which "works" only for $\phi \neq 0$.

For $\phi = 0$ we need to use other methods, namely, the reduction of the uniqueness problem for the stationary diffraction to the uniqueness problem for the corresponding non-stationary diffraction, which, in turn, is reduced to the proof of uniqueness of solution of the stationary problem but with a *complex wavenumber*, see e.g. [5].

Note that in [18] we proved the LAP for $\phi \neq 0$ and for the DD boundary conditions. Similar results for the NN and DN boundary conditions were obtained in [6, 7, 10]. A generalization of these results to the case of generalized incident wave (cf. [8]) was given in [19]. This approach (stationary diffraction as the limit of time-dependent one) permits us to justify all the classical explicit formulas [13, 14, 20, 28–31] and to prove their coincidence with the explicit formulas given in [17, 19, 22]. In other words, all the classical formulas are limiting amplitudes of solutions to non-stationary problems as $t \to \infty$. For the Sommerfeld problem, this was proven in [20], except for the proof of the uniqueness of the solution to the non-stationary problem in an appropriate class. This paper makes up for this omission.

Our plan is as follows. The non-stationary diffraction problem is reduced by means of the Fourier–Laplace transform with respect to time t to a stationary one with a complex wave number. For this problem the uniqueness theorems can be proven more easily in Sobolev classes (see an important paper [5]) and do not use the radiation conditions. Then we prove that the Fourier–Laplace transforms of solutions to non-stationary diffraction half-plane problem, whose amplitude tends to the Sommerfeld solution, also belong to a Sobolev space for a rather wide class of incident waves. This permits us to reduce the problem to the case of [5].

Let us pass to the problem setting. We consider the two-dimensional time-dependent scattering of a plane wave by the half-plane

$$W^0 := \left\{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0, x_1 \geq 0\right\}.$$

(Obviously, W^0 is a half-line in $\mathbb{R}^2$, but if one recalls that the initial problem is three-dimensional, W^0 becomes a half-plane; the third coordinate is suppressed in all what follows.) The non-stationary incident plane wave in the absence of obstacles reads

$$u_i(x,t) = e^{-i\omega_0(t-\mathbf{n}\cdot x)} f(t-\mathbf{n}\cdot x), \quad x \in \mathbb{R}^2, \quad t \in \mathbb{R}, \tag{1.1}$$

where

$$\omega_0 > 0, \quad \mathbf{n} = (n_1, n_2) = (\cos(\pi+\alpha), \sin(\pi+\alpha)), \tag{1.2}$$

and f is "a profile function", such that $f \in L^1_{loc}(\mathbb{R})$, and

$$f(s) = 0,\ s < 0,\ \sup(1+|s|)^p |f(s)| < \infty \text{ for some } p \in \mathbb{R},\ \lim_{s\to+\infty} f(s) = 1. \tag{1.3}$$

Remark 1.1 Obviously, these functions satisfy the D'Alembert equation $\Box u_i(x,t) = 0$ in the sense of distributions.

For definiteness, we assume that

$$\frac{\pi}{2} < \alpha < \pi. \tag{1.4}$$

In this case the front of the incident wave u_i reaches the half-plane W^0 for the first time at the moment $t = 0$ and at this moment the reflected wave $u_r(x,t)$ is born (see Fig. 1). Thus

$$u_r(x,t) \equiv 0, \quad t < 0.$$

Note that for $t \to \infty$ the limiting amplitude of u_i is exactly equal to the Sommerfeld incident wave [33] by (1.3), cf. also (2.1) below.

The time-dependent scattering with the Dirichlet boundary conditions is described by the mixed problem

$$\left\{ \begin{array}{r} \Box u(x,t) := (\partial_t^2 - \Delta)u(x,t) = 0,\ x \in Q \\ u(x_1, \pm 0, t) = 0,\ x_1 > 0 \end{array} \right| \ t \in \mathbb{R}, \tag{1.5}$$

where $Q := \mathbb{R}^2 \setminus W^0$. The "initial condition" reads

$$u(x,t) = u_i(x,t), \quad x \in Q, \quad t < 0, \tag{1.6}$$

where u_i is the incident plane wave (1.1).

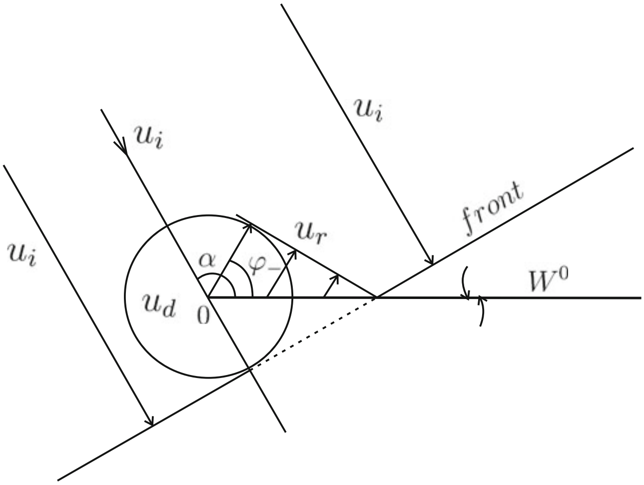


Fig. 1 Time-dependent diffraction by a half-plane

Introduce the non-stationary "scattered" wave u_s as the difference between u and u_i,

$$u_s(x,t) := u(x,t) - u_i(x,t), \quad x \in Q, \quad t \in \mathbb{R}. \tag{1.7}$$

Since $\Box u_i(x,t) = 0$, $(x,t) \in Q \times \mathbb{R}$, we get from (1.6), (1.5) that

$$\Box u_s(x,t) = 0, \quad (x,t) \in Q \times \mathbb{R}, \tag{1.8}$$

$$u_s(x,t) = 0, \quad x \in Q, \quad t < 0, \tag{1.9}$$

$$u_s(x_1, \pm 0, t) = -u_i(x_1, 0, t), \quad x_1 > 0, \quad t > 0. \tag{1.10}$$

Denote

$$\varphi_\pm := \pi \pm \alpha. \tag{1.11}$$

Everywhere below we assume that

$$x_1 = r\cos\varphi, \quad x_2 = r\sin\varphi, \quad 0 \le \varphi < 2\pi. \tag{1.12}$$

Let us define the nonstationary incident wave in the presence of the obstacle W^0, which is the opaque screen,

$$u_i^0(\rho,\varphi,t) := \begin{cases} u_i(\rho,\varphi,t), & 0 < \varphi < \varphi_+, \\ 0, & \varphi_+ < \varphi < 2\pi. \end{cases} \tag{1.13}$$

Remark 1.2 The function u_s has no physical sense, since $u_i \neq u_i^0$. The wave u_s coincides with the scattered wave $u_s^0 := u - u_i^0$ in the zone $\{(\rho, \varphi) : 0 < \varphi < \varphi_+\}$, but in the zone $\{(\rho, \varphi) : \varphi_+ < \varphi \leq 2\pi\}$ we have $u_s^0 = u_s + u_i$.

The goal of the paper is to prove that the Sommerfeld solution of half-plane diffraction problem is the limiting amplitude of the solution to time-dependent problem (1.5), (1.6) (with any f satisfying (1.3)) and this solution is unique in an appropriate functional class.

The paper is organized as follows. In Sect. 2 we recall the Sommerfeld solution. In Sect. 3 we reduce the time-dependent diffraction problem to a "stationary" one and define a functional class of solutions. In Sect. 4 we give an explicit formula for the solution of time-dependent problem and prove that the Sommerfeld solution is its limiting amplitude. In Sect. 5 we prove that the solution belongs to a certain functional class. Finally, in Sect. 6 we prove the uniqueness.

2 Sommerfeld's Diffraction

Let us recall the Sommerfeld solution [23, 33]. The stationary incident wave (rather, the incident wave limiting amplitude) in the presence of the obstacle is

$$\mathcal{A}_i^0(\rho, \varphi) = \begin{cases} e^{-i\omega_0 \rho \cos(\varphi - \alpha)}, & \varphi \in (0, \varphi_+), \\ 0, & \varphi \in (\varphi_+, 2\pi). \end{cases} \tag{2.1}$$

We denote this incident wave as $\mathcal{A}_i^0$ since it is the limiting amplitude of the non-stationary incident wave u_i^0 given by (1.13):

$$\mathcal{A}_i^0(\rho, \varphi) = \lim_{t \to \infty} e^{i\omega_0 t} u_i^0(x, t),$$

in view of formula (1.1), see Remark 1.2. The Sommerfeld half-plane diffraction problem can be formulated as follows: find a function $\mathcal{A}(x)$, $x \in \overline{Q}$, such that

$$\begin{cases} (\Delta + \omega_0^2)\mathcal{A}(x) = 0, & x \in Q, \\ \mathcal{A}(x_1, \pm 0) = 0, & x_1 > 0, \end{cases} \tag{2.2}$$

$$\mathcal{A}(x) = \mathcal{A}_i^0(x) + \mathcal{A}_r(x) + \mathcal{A}_d(x), \quad x \in Q, \tag{2.3}$$

where $\mathcal{A}_r(x)$ is the reflected wave,

$$\mathcal{A}_r(x) = \begin{cases} -e^{-i\omega_0 \rho \cos(\varphi + \alpha)}, & \varphi \in (0, \varphi_-), \\ 0, & \varphi \in (\varphi_-, 2\pi), \end{cases} \tag{2.4}$$

and $\mathcal{A}_d(x)$ is the wave diffracted by the edge,

$$\mathcal{A}_d(x) \to 0, \quad |x| \to \infty. \tag{2.5}$$

A. Sommerfeld [33] found the solution of this problem in the form

$$\mathcal{A}(\rho,\varphi) = \frac{1}{4\pi} \int\limits_{\mathcal{C}} \zeta(\gamma,\varphi) e^{-i\omega\rho\cos\gamma} d\gamma, \quad \rho \geq 0, \quad \varphi \in [0, 2\pi],$$

where

$$\zeta(\gamma,\varphi) := \left(1 - e^{i(-\gamma+\varphi-\alpha)/2}\right)^{-1} - \left(1 - e^{i(-\gamma+\varphi+\alpha)/2}\right)^{-1}, \quad \gamma \in \mathbb{C} \tag{2.6}$$

and $\mathcal{C}$ is the Sommerfeld contour (see [20, formula (1.1) and Fig. 3]).

In the rest of the paper we prove that this solution is the limiting amplitude of the solution of time-dependent problem (1.5) and is unique in an appropriate functional class.

The Sommerfeld diffraction problem can also be considered for NN and DN half-plane. The corresponding formulas for the solution can be found in [19].

Sommerfeld obtained his solution using an original method of solutions of the Helmholtz equation on a Riemann surface. Note that a similar approach was used for the diffraction by a wedge of rational angle [9], where well-posedness in suitable Sobolev space was proved.

3 Reduction to a "Stationary" Problem: Fourier–Laplace Transform

Let $\widehat{h}(\omega)$, $\omega \in \mathbb{C}^+$, denote the Fourier–Laplace transform $\mathcal{F}_{t\to\omega}$ of $h(t)$,

$$\widehat{h}(\omega) = \mathcal{F}_{t\to\omega}[h(t)] = \int\limits_0^\infty e^{i\omega t} h(t)\, dt, \quad h \in L_1(\mathbb{R}^+); \tag{3.1}$$

$\mathcal{F}_{t\to\omega}$ is extended by continuity to $S'(\overline{\mathbb{R}^+})$. Assuming that $u_s(x,t)$ belongs to $S'(\mathbb{R}^2 \times \overline{\mathbb{R}^+})$ (see (1.9) and Definition 3.1), we apply this transform to system (1.8)–(1.10), and obtain

$$\left\{ \begin{array}{ll} (\Delta + \omega^2)\widehat{u}_s(x,\omega) = 0, & x \in Q, \\ \widehat{u}_s(x_1, \pm 0, \omega) = -\widehat{u}_i(x_1, \pm 0, \omega), & x_1 > 0 \end{array} \right| \quad \omega \in \mathbb{C}^+. \tag{3.2}$$

Let us calculate $\widehat{u}_i(x,\omega)$. Changing the variable $t - \mathbf{n}\cdot x = \tau$, and using the fact that $\operatorname{supp} f \subset \overline{\mathbb{R}^+}$ we obtain from (1.1) and (1.2) that

$$\widehat{u}_i(x,\omega) = e^{i\omega\mathbf{n}\cdot x}\widehat{f}(\omega-\omega_0). \tag{3.3}$$

Hence,

$$\widehat{u}_i(x_1,0,\omega) = e^{i\omega n_1 x_1}\widehat{f}(\omega-\omega_0), \quad x_1>0,$$

and the boundary condition in (3.2) is $\widehat{u}_s(x_1,0,\omega) = -g(\omega)e^{i\omega n_1 x_1}$. Therefore we come to the following family of BVPs depending on $\omega\in\mathbb{C}^+$: find $\widehat{u}_s(x,\omega)$ such that

$$\begin{cases} (\Delta+\omega^2)\widehat{u}_s(x,\omega)=0, & x\in Q,\\ \widehat{u}_s(x_1,\pm 0,\omega) = -g(\omega)e^{i\omega n_1 x_1}, & x_1>0. \end{cases} \tag{3.4}$$

We are going to prove the existence and uniqueness of solution to problem (1.5), (1.6) such that u_s given by (1.7) belongs to the space $\mathcal{M}$, which is defined as follows:

Definition 3.1 $\mathcal{M}$ is the space of functions $u(x,t)\in S'(\mathbb{R}^2\times\overline{\mathbb{R}^+})$ such that its Fourier–Laplace transform $\widehat{u}(x,\omega)$ is a holomorphic function on $\omega\in\mathbb{C}^+$ with values in $C^2(Q)$ and

$$\widehat{u}(\cdot,\cdot,\omega)\in H^1(Q) \tag{3.5}$$

for any $\omega\in\mathbb{C}^+$.

Remark 3.2 We use the classical definition [11] of the space $H^1(Q)$ as the completion of the space of smooth functions on Q with respect to the corresponding norm. This definition does not coincide with the frequently used definition of $H^1(Q)$ as the space restrictions of distributions from $H^1(\mathbb{R})$ to Q. In our case these definitions lead to different spaces; in particular, the latter definition does not allow for functions which are discontinuous across W^0. In [34], another space allowing for the same class of functions was introduced; the proof of uniqueness of the solution to our problem in that space is an open question.

Remark 3.3 Note that $u_i(x,t)\big|_{\mathbb{R}^2\times\overline{\mathbb{R}^+}}\notin\mathcal{M}$, where for $\varphi\in D(\mathbb{R}^2)$,

$$\left\langle u_i(x,t)\big|_{\mathbb{R}^2\times\overline{\mathbb{R}^+}},\varphi\right\rangle := \int\limits_{\mathbb{R}^2\times\mathbb{R}^+} u(x,t)\varphi(x,t)\,dx\,dt.$$

In fact, $\left|e^{i\omega\mathbf{n}\cdot x}\right| = e^{\omega_2\rho\cos(\varphi-\alpha)}$ and, for $\alpha-\pi/2<\varphi<\alpha+\pi/2$, $\omega\in\mathbb{C}^+$ it grows exponentially as $\rho\to\infty$, and hence does not satisfy (3.5); because of this we use system (1.8)–(1.10) instead of (1.5) (they are equivalent by (1.6)) since (1.8)–(1.10) involves only the values of u_i on the boundary and the latter possess the Fourier–Laplace transforms which do not grow exponentially.

Remark 3.4 Since for a (weak) solution of the Helmholtz equation $u_s \in H^1(Q)$ the Dirichlet and Neumann data exist in the trace sense and in the distributional sense, respectively (see, e.g., [5]), problem (3.4) is well-posed. Hence, problem (1.8)–(1.10) is well-posed too.

4 Connection Between the Non-stationary Diffraction Problem (1.5) and (1.6) and the Sommerfeld Half-Plane Problem

In paper [20] we solved problem (1.5) and (1.6). Let us recall the corresponding construction. First we define the non-stationary reflected wave [20, formula (26)]:

$$u_r(x,t) = \left\{ \begin{array}{ll} -e^{-i\omega_0(t-\overline{\mathbf{n}}\cdot x)} f(t-\overline{\mathbf{n}}\cdot x), & \varphi \in (0,\varphi_-) \\ 0, & \varphi \in (\varphi_-, 2\pi) \end{array} \right| \quad t \geq 0, \tag{4.1}$$

where $\overline{\mathbf{n}} := (n_1, -n_2) = (-\cos\alpha, \sin\alpha)$ (see Fig. 1).

Note that its limiting amplitude coincides with (2.4) similarly to the incident wave.

Second, we define the non-stationary diffracted wave (cf. [20, formula (31) for $\phi = 0$]). Let

$$\mathcal{Z}(\beta,\varphi) := Z(\beta + 2\pi i - i\varphi), \tag{4.2}$$

and

$$u_d(\rho,\varphi,t) = \frac{i}{8\pi} \int_{\mathbb{R}} \mathcal{Z}(\beta,\varphi) F(t-\rho\cosh\beta)\, d\beta, \tag{4.3}$$

where $\varphi \in (0, 2\pi)$, $\varphi \neq \varphi_\pm$; $t \geq 0$,

$$F(s) = f(s)e^{-i\omega_0 s}, \tag{4.4}$$

$$Z(z) = -U\Big(-\frac{i\pi}{2} + z\Big) + U\Big(-\frac{5i\pi}{2} + z\Big), \tag{4.5}$$

$$U(\zeta) = \coth\Big(q(\zeta - i\frac{\pi}{2} + i\alpha)\Big) - \coth\Big(q(\zeta - i\frac{\pi}{2} - i\alpha)\Big), \quad q = \frac{1}{4} \tag{4.6}$$

for the Dirichlet boundary conditions. Below in Lemma 8.1 we give the necessary properties of the function $\mathcal{Z}$, from which the convergence of integral (4.3) follows.

Obviously, the condition supp $F \subset [0, \infty)$ (see (3.1)) implies that supp $u_d(\cdot, \cdot, t) \subset [0, +\infty)$.

Remark 4.1 The function $U(\gamma + \varphi)$ essentially coincides with the Sommerfeld kernel (2.6). This is for a reason. In paper [17] it was proven that the solution to the corresponding time-dependent diffraction problem by an arbitrary angle $\phi \in (0, \pi]$ belonging to a certain class similar to $\mathcal{M}$ necessarily has the form of the Sommerfeld type integral with the Sommerfeld type kernel.

Finally, we proved [20, Th. 3.2, Th 4.1] the following.

Theorem 4.2

(i) *For $f \in L^1_{loc}(\mathbb{R})$ the function*

$$u(\rho, \varphi, t) := u_i^0(\rho, \varphi, t) + u_r(\rho, \varphi, t) + u_d(\rho, \varphi, t), \quad \varphi \neq \varphi_\pm \tag{4.7}$$

belongs to $L^1_{loc}(Q \times \mathbb{R}^+)$. It is continuous up to $\partial Q \times \mathbb{R}$ and satisfies the boundary and initial conditions (1.5)*,* (1.6)*. The D'Alembert equation in* (1.5) *holds in the sense of distributions.*

(ii) *The LAP holds for Sommerfeld's diffraction by a half-plane:*

$$\lim_{t\to\infty} e^{i\omega_0 t} u(\rho, \varphi, t) = \mathcal{A}(\rho, \varphi), \quad \varphi \neq \varphi_\pm$$

(the limit here and everywhere else is pointwise).

Since the main object of our consideration will be the "scattered" wave $u_s(x, t)$ given by (1.7), we clarify the connection between u_s and the Sommerfeld solution $\mathcal{A}$.

Corollary 4.3 *Define $\mathcal{A}_i(x) = e^{-i\omega_0\rho\cos(\varphi+\alpha)}$, which is the limiting amplitude of $u_i(x, t)$ given by* (1.1)*. The limiting amplitude of $u_s(x, t)$ is the function*

$$\mathcal{A}_s(x) = \mathcal{A}(x) - \mathcal{A}_i(x), \tag{4.8}$$

i.e. $\lim_{t\to\infty} e^{i\omega_0 t} u_s(x, t) = \mathcal{A}_s(x)$.

Proof The statement follows from (1.7). □

Remark 4.4 The function $\mathcal{A}_s$ is the limiting amplitude of the scattered non-stationary wave $u_s(x, t)$ and $\mathcal{A}_s$ satisfies the following nonhomogeneous BVP:

$$\begin{cases} (\Delta + \omega_0^2)\mathcal{A}_s(x) = 0, & x \in Q, \\ \mathcal{A}_s(x_1, \pm 0) = -\mathcal{A}_i(x_1, 0), & x_1 > 0. \end{cases} \tag{4.9}$$

This BVP (as well as (2.2)) is ill-posed since the homogeneous problem admits many solutions (i.e., the solution is nonunique).

Remark 4.5 $\mathcal{A}_s$ can be decomposed similarly to (2.3). Namely, by (4.8) and (2.3), we have

$$\mathcal{A}_s = \mathcal{A}_i^0 + \mathcal{A}_r(x) + \mathcal{A}_d(x) - \mathcal{A}_i(x) = \mathcal{A}_r(x) + \mathcal{A}_d(x) - \mathcal{A}_i^1(x), \tag{4.10}$$

where $\mathcal{A}_i^1(x) = \mathcal{A}_i(x) - \mathcal{A}_i^0(x)$. Obviously, problems (4.9), (4.10) and (2.2), (2.3) with condition (2.5) are equivalent, but the first problem is more convenient as we will see later.

5 Solution of the "Stationary" Problem

In this section we will obtain an explicit formula for the solution of (3.4) and prove that it belongs to $H^1(Q)$ for all $\omega \in \mathbb{C}^+$.

Let $\mathcal{Z}(\beta, \varphi)$ be given by (4.2). First, we will need the Fourier–Laplace transforms of the reflected and diffracted waves (4.1), (4.3).

Lemma 5.1 *The Fourier–Laplace transforms of u_r and u_d are*

$$\widehat{u}_r(x,\omega) = \begin{cases} -\widehat{f}(\omega-\omega_0)e^{-i\omega\rho\cos(\varphi+\alpha)}, & \varphi \in (0,\varphi_-), \\ 0, & \varphi \in (\varphi_-, 2\pi), \end{cases} \tag{5.1}$$

$$\widehat{u}_d(\rho,\varphi,\omega) = \frac{i}{8\pi}\widehat{f}(\omega-\omega_0)\int_{\mathbb{R}} \mathcal{Z}(\beta,\varphi)\, e^{i\omega\rho\cosh\beta}\, d\beta, \ \omega \in \mathbb{C}^+, \ \varphi \neq \varphi_\pm. \tag{5.2}$$

Proof From (4.1) we have

$$\widehat{u}_r(x,\omega) = \begin{cases} -\mathcal{F}_{t\to\omega}\left[e^{-i\omega_0(t-\overline{\mathbf{n}}\cdot x)} f(t-\overline{\mathbf{n}}\cdot x)\right], & \varphi \in (0,\varphi_-), \\ 0, & \varphi \in (\varphi_-, 2\pi). \end{cases}$$

Further,

$$-\mathcal{F}_{t\to\omega}\left[e^{-i\omega_0(t-\overline{\mathbf{n}}\cdot x)} f(t-\overline{\mathbf{n}}\cdot x)\right] = -e^{i\omega_0(\overline{\mathbf{n}}\cdot x)}\int_0^\infty e^{i(\omega-\omega_0)t} f(t-\overline{\mathbf{n}}\cdot x)\, dt.$$

Changing the variable $t - \overline{\mathbf{n}}\cdot x = \tau$, we obtain

$$\widehat{u}_r(x,\omega) = -e^{i\omega\,\overline{\mathbf{n}}\cdot x}\int_{-\overline{\mathbf{n}}\cdot x}^\infty e^{i(\omega-\omega_0)\tau} f(\tau)\, d\tau, \quad \varphi \in (0,\varphi_-).$$

Moreover, by (4.1),

$$-\overline{\mathbf{n}} \cdot x = \rho \cos(\varphi - \alpha) \le c < 0, \quad \varphi \in (0, \varphi_-),$$

since $\pi/2 < \alpha < \varphi + \alpha < \pi$ by (1.4) and (1.11). Hence, we obtain (5.1), since $\operatorname{supp} f \subset \overline{\mathbb{R}^+}$. The second formula in (5.1) follows from definition (4.1) of u_r.

Let us prove (5.2). Everywhere below we put $\omega = \omega_1 + i\omega_2$, $\omega_{1,2} \in \mathbb{R}$, $\omega_2 > 0$, for $\omega \in \mathbb{C}^+$. By Lemma 8.1(i), (1.3) and (4.4) we have

$$\left| e^{i\omega t} \mathcal{Z}(\beta, \varphi) F(t - \rho \cosh \beta) \right| \le C e^{-\omega_2 t} e^{-\beta/2} (1+t)^{-p}, \quad \rho < 0, \varphi \neq \varphi_\pm, \beta \in \mathbb{R}.$$

Hence, by the Fubini Theorem there exists the Fourier–Laplace transform of $u_d(\cdot, \cdot, t)$ and

$$\widehat{u}_d(\rho, \varphi, \omega) = \frac{i}{8\pi} \int_{\mathbb{R}} \mathcal{Z}(\beta, \varphi) \mathcal{F}_{t\to\omega} \Big[F(t - \rho \cosh \beta) \Big] \, d\beta, \quad \varphi \neq \varphi_\pm. \tag{5.3}$$

We have

$$G(\rho, \beta, \omega) := \mathcal{F}_{t\to\omega} \Big[F(t - \rho \cosh \beta) \Big] = \int_0^\infty e^{i\omega t} F(t - \rho \cosh \beta) \, dt, \quad \omega \in \mathbb{C}^+.$$

Making the change of the variable $\tau = t - \rho \cosh \beta$ in the last integral and using the fact that supp $F \subset [0, \infty)$ and $\widehat{F}(\omega) = \widehat{f}(\omega - \omega_0)$ by (4.4), we get $G(\rho, \beta, \omega) = e^{i\omega\rho \cosh \beta} \widehat{f}(\omega - \omega_0)$. Substituting this expression into (5.3) we obtain (5.2). Lemma 5.1 is proven. □

5.1 Estimates for $\widehat{u}_r$, $\partial_\rho \widehat{u}_r$, $\partial_\varphi \widehat{u}_r$

Lemma 5.2 *For any $\omega \in \mathbb{C}$, there exist $C(\omega), c(\omega) > 0$, such that both functions $\widehat{u}_r$ and $\partial_\rho \widehat{u}_r$ admit the same estimate*

$$\left.\begin{aligned} \left|\widehat{u}_r(\rho, \varphi, \omega)\right| &\le C(\omega) e^{-c(\omega)\rho} \\ \left|\partial_\rho \widehat{u}_r(\rho, \varphi, \omega)\right| &\le C(\omega) e^{-c(\omega)\rho} \end{aligned}\right| \quad \rho > 0, \ \varphi \in (0, 2\pi), \ \varphi \neq \varphi_\pm. \tag{5.4}$$

and $\partial_\varphi \widehat{u}_r(\rho, \varphi, \omega)$ admits the estimate

$$|\partial_\varphi \widehat{u}_r(\rho, \varphi, \omega)| \le C(\omega) \rho \, e^{-c(\omega)\rho}, \quad \rho > 0. \tag{5.5}$$

Proof By (1.4) there exits $c(\omega) > 0$ such that

$$\left|e^{-i\omega\rho\cos(\varphi+\alpha)}\right| = e^{\omega_2\rho\cos(\varphi+\alpha)} \leq e^{-c(\omega)\rho}, \quad 0 < \varphi < \varphi_-$$

by (1.4). Therefore (5.4) holds for $\widehat{u}_r$. Hence, differentiating (5.1) we obtain (5.4) for $\partial_\rho \widehat{u}_r$ and (5.5) for $\partial_\varphi \widehat{u}_r$, for $\varphi \neq \varphi_-$. □

5.2 Estimates for $\widehat{u}_d$

Proposition 5.3 *There exist $C(\omega), c(\omega) > 0$ such that the function $\widehat{u}_d$, and $\partial_\rho \widehat{u}_d$, $\partial_\varphi \widehat{u}_d$ admit the estimates*

$$\begin{aligned} \left|\widehat{u}_d(\rho,\varphi,\omega)\right| &\leq C(\omega)e^{-c(\omega)\rho}, \\ \left|\partial_\rho \widehat{u}_d(\rho,\varphi,\omega)\right| &\leq C(\omega)e^{-c(\omega)\rho}(1+\rho^{-1/2}), \\ \left|\partial_\varphi \widehat{u}_d(\rho,\varphi,\omega)\right| &\leq C(\omega)e^{-c(\omega)\rho}\rho(1+\rho^{-1/2}) \end{aligned} \tag{5.6}$$

for $\rho > 0$, $\varphi \in (0, 2\pi)$, $\varphi \neq \varphi_\pm$.

Proof

(I) By (5.2), in order to prove (5.6) for $\widehat{u}_d$ it suffices to prove that

$$|A(\rho,\varphi,\omega)| \leq C(\omega)e^{-c(\omega)\rho}, \tag{5.7}$$

where

$$A(\rho,\varphi,\omega) := \int_{\mathbb{R}} \mathcal{Z}(\beta,\varphi)e^{i\omega\rho\cosh\beta}\, d\beta, \quad \varphi \neq \varphi_\pm. \tag{5.8}$$

Represent A as $A = A_1 + A_2$, where

$$\left.\begin{aligned} A_1(\rho,\varphi,\omega) &:= \int_{-1}^{1} \mathcal{Z}(\beta,\varphi)e^{i\omega\rho\cosh\beta}\, d\beta \\ A_2(\rho,\varphi,\omega) &:= \int_{|\beta|\geq 1} \mathcal{Z}(\beta,\varphi)e^{i\omega\rho\cosh\beta}\, d\beta \end{aligned}\right| \quad \varphi \in (0,2\pi), \quad \varphi \neq \varphi_\pm. \tag{5.9}$$

The estimate (5.7) for A_2 follows from (8.1) (see Appendix 1). It remains to prove the same estimate for the function A_1. Let

$$\varepsilon_\pm := \varphi_\pm - \varphi. \tag{5.10}$$

Representing A_1 as

$$A_1(\rho,\varphi,\omega) = -4\mathcal{K}_0(\rho,w,\varepsilon_+) + 4\mathcal{K}_0(\rho,w,\varepsilon_-) + \int_{-1}^{1} \check{\mathcal{Z}}(\beta,\varphi)e^{i\omega\rho\cosh\beta}\,d\beta,$$

where $\mathcal{K}_0$ is defined by (8.7), we obtain (5.7) for A_1 from Lemma 8.2 (i) and (8.3).

(II) Let us prove (5.6) for $\partial_\rho \widehat{u}_d$. By (5.2) it suffices to prove that

$$|B(\rho,\varphi,\omega)| \le C(\omega)e^{-c(\omega)\rho}(1+\rho^{1/2}), \quad \varphi \ne \varphi_\pm, \tag{5.11}$$

where

$$B(\rho,\varphi,\omega) := \int_{\mathbb{R}} \mathcal{Z}(\beta,\varphi)\cosh\beta\; e^{i\omega\rho\cosh\beta}\,d\beta.$$

Represent B as $B_1 + B_2$, where $B_{1,2}(\rho,\varphi,\omega)$ are defined similarly to (5.9),

$$B_1(\rho,\varphi,\omega) := \int_{-1}^{1} \mathcal{Z}(\beta,\varphi)\cosh\beta e^{i\omega\rho\cosh\beta}\,d\beta,$$

$$B_2(\rho,\varphi,\omega) := \int_{|\beta|\ge 1} \mathcal{Z}(\beta,\varphi)\cosh\beta e^{i\omega\rho\cosh\beta}\,d\beta, \quad \varphi \ne \varphi_\pm.$$

From (8.1) for $\mathcal{Z}$ we have

$$|B_2(\rho,\varphi,\omega)| \le C_1 \int_1^\infty e^{\beta/2}e^{-\frac{1}{2}\omega_2\rho e^\beta}\,d\beta.$$

Making the change of the variable $\xi := \rho e^\beta$, we get

$$|B_2(\rho,\varphi,\omega)| \le \begin{cases} C_1(\omega)\rho^{-1/2}, & \rho \le 1, \\ \displaystyle\int_\rho^\infty \frac{e^{-\omega_2\xi/2}}{\xi^{1/2}}d\xi, & \rho \ge 1. \end{cases}$$

Since for $\rho \geq 1$,

$$\int_{\rho}^{\infty} \frac{e^{-\omega_2 \xi/2}}{\xi^{1/2}} d\xi \leq \frac{2}{\omega_2} e^{-\omega_2 \rho/2},$$

Equation (5.11) is proved for B_2.

It remains to prove estimate (5.11) for B_1. Using (8.2) and (8.8) we write

$$B_1(\rho, \varphi, \omega) = -4\mathcal{K}_1(\rho, \omega, \varepsilon_+) + 4\mathcal{K}_1(\rho, \omega, \varepsilon_-) + \int_{-1}^{1} \check{\mathcal{Z}}(\beta, \varphi) \cdot \cos\beta \, e^{i\omega\rho\cosh\beta} \, d\beta.$$

Hence, B_1 satisfies (5.7) (and, therefore, (5.11)) by Lemma 8.2 (i) and (8.3).

(III) Let us prove (5.6) for $\partial_\varphi \widehat{u}_d$. By (5.2) it suffices to prove this estimate for $\partial_\varphi A$, where A is given by (5.8). From (9.3) we have

$$\begin{aligned} \partial_\varphi A(\rho, \varphi, \omega) &= -\omega\rho A_3(\rho, \varphi, \omega), \\ A_3(\rho, \varphi, \omega) &= \int_{\mathbb{R}} \mathcal{Z}(\beta, \varphi) \sinh\beta \, e^{i\omega\rho\cosh\beta} \, d\beta, \quad \varphi \neq \varphi_\pm. \end{aligned} \tag{5.12}$$

Similarly to the proof of estimate (5.11) for B, we obtain the same estimate for A_3, so, by (5.12), the estimate (5.6) follows. Proposition 5.3 is proven. □

Now define the function

$$u_s^0(\rho, \varphi, t) = u(\rho, \varphi, t) - u_i^0(\rho, \varphi, t), \quad \varphi \neq \varphi_+, \quad t > 0, \tag{5.13}$$

where u_i^0 is given by (1.13). Then by (4.7),

$$u_s^0(\rho, \varphi, t) = u_r(\rho, \varphi, t) + u_d(\rho, \varphi, t), \quad \varphi \neq \varphi_\pm, \quad t > 0, \tag{5.14}$$

where u_r is given by (4.1) and u_d is given by (4.3).

Corollary 5.4 *Let $\widehat{u}_s^0(\rho, \varphi, \omega)$ be the Fourier–Laplace transform of the function $u_s^0(\rho, \varphi, t)$. Then the functions $\widehat{u}_s^0$, $\partial_\rho \widehat{u}_s^0$ and $\partial_\varphi \widehat{u}_s^0$ satisfy* (5.6).

Proof From (5.14) we have

$$\widehat{u}_s^0(\rho, \varphi, \omega) = \widehat{u}_r(\rho, \varphi, \omega) + \widehat{u}_d(\rho, \varphi, \omega), \quad \varphi \neq \varphi_\pm, \quad \omega \in \mathbb{C}^+, \tag{5.15}$$

where $\widehat{u}_r$ and $\widehat{u}_d$ are defined by (5.1) and (5.2), respectively. Hence the statement follows from Lemma 5.2 and Proposition 5.3. □

5.3 Estimates for $\widehat{u}_s(x, \omega)$

To estimate $\widehat{u}_s$ it is convenient to introduce one more "part" u_i^1 of the non-stationary incident wave u_i, namely the difference between u_i and u_i^0.

From (1.7) and (5.13) it follows that

$$u_s(\rho, \varphi, t) = u_s^0(\rho, \varphi, t) - u_i^1(\rho, \varphi, t), \quad \varphi \neq \varphi_\pm \tag{5.16}$$

where $u_i^1(\rho, \varphi, t) := u_i(\rho, \varphi, t) - u_i^0(\rho, \varphi, t)$. From (1.1) and (1.13) it follows that

$$u_i^1(\rho, \varphi, t) = \begin{cases} 0, & 0 < \varphi < \varphi_+, \\ -u_i(\rho, \varphi, t), & \varphi_+ < \varphi < 2\pi. \end{cases} \tag{5.17}$$

By (3.3),

$$\widehat{u}_i^1(\rho, \varphi, \omega) = \begin{cases} 0, & 0 < \varphi < \varphi_+, \\ -\widehat{f}(\omega - \omega_0)\, e^{i\omega \mathbf{n}\cdot x}, & \varphi_+ < \varphi < 2\pi. \end{cases} \tag{5.18}$$

Lemma 5.5 *There exist $C(\omega), c(\omega) > 0$ such that $\widehat{u}_i^1$, $\partial_\rho \widehat{u}_i^1$ satisfy (5.4) and $\partial_\varphi \widehat{u}_i^1$ satisfies (5.5) for $\varphi \in (0, 2\pi)$, $\varphi \neq \varphi_\pm$.*

Proof By (3.3) it suffices to prove the statement for $e^{i\omega \mathbf{n}\cdot x}$ when $\varphi \in (\varphi_+, 2\pi)$. Since $|e^{i\omega \mathbf{n}\cdot x}| = e^{\omega_2 \rho \cos(\varphi - \alpha)}$, $\varphi \in (\varphi_+, 2\pi)$ we have

$$\begin{aligned} \partial_\rho e^{\omega_2 \rho \cos(\varphi-\alpha)} &= \omega_2 \cos(\varphi - \alpha) e^{\omega_2 \rho \cos(\varphi-\alpha)}, \\ \partial_\varphi e^{\omega_2 \rho \cos(\varphi-\alpha)} &= -\omega_2 \rho \sin(\varphi - \alpha) e^{\omega_2 \rho \cos(\varphi-\alpha)}, \end{aligned} \tag{5.19}$$

and for $\varphi \in (\varphi_+, 2\pi)$, we have $|e^{\omega_2 \rho \cos(\varphi-\alpha)}| \leq e^{-c\omega_2 \rho}$, $c > 0$, $\varphi \in (\varphi_+, 2\pi)$, because $\cos(\varphi - \alpha) \leq -c < 0$ by (1.4). Hence the statement follows from (5.19). □

Corollary 5.6 *The functions $\widehat{u}_s$, $\partial_\rho \widehat{u}_s$ and $\partial_\varphi \widehat{u}_s$ satisfy (5.6) for $\varphi \in (0, 2\pi)$, $\varphi \neq \varphi_\pm$.*

Proof From (5.16) it follows that

$$\widehat{u}_s(\rho, \varphi, \omega) = \widehat{u}_s^0(\rho, \varphi, \omega) - \widehat{u}_i^1(\rho, \varphi, \omega). \tag{5.20}$$

Thus the statement follows from Corollary 5.4 and Lemma 5.5. □

It is possible to get rid of the restriction $\varphi \neq \varphi_\pm$ in Corollary 5.6.

Let $l_\pm = \{(\rho, \varphi) : \rho > 0,\ \varphi = \varphi_\pm\}$.

Proposition 5.7 *The functions* $\widehat{u}_s(\cdot,\cdot,\omega)$, $\partial_\rho\widehat{u}_s(\cdot,\cdot,\omega)$ *and* $\partial_\varphi\widehat{u}_s(\cdot,\cdot,\omega)$ *belong to* $C^2(Q)$, *and satisfy* (5.6) *in* Q *(including* $l_+\cup l_-$*), and*

$$(\Delta+\omega^2)\widehat{u}_s(\rho,\varphi,\omega)=0,\quad (\rho,\varphi)\in Q,\quad \omega\in\mathbb{C}^+. \tag{5.21}$$

Proof The function $\widehat{u}_s(\rho,\varphi,\omega)$ satisfies (5.21) in $Q\setminus(l_+\cup l_-)$. This follows directly from the explicit formulas (5.20). In fact, (5.20) and (5.15) imply

$$\widehat{u}_s=\widehat{u}_r+\widehat{u}_d-\widehat{u}_i^1. \tag{5.22}$$

The function $\widehat{u}_r$ satisfies (5.21) for $\varphi\neq\varphi_\pm$, $\widehat{u}_i^1$ satisfies (5.21) for $\varphi\neq\varphi_\pm$ by (5.17) and (3.3) and $\widehat{u}_d$ satisfies (5.21) for $\varphi\neq\varphi_\pm$ by (5.2), see Appendix 2. It remains only to prove that $\widehat{u}_s\in C^2(Q)$, because this will mean that (5.6) holds by Corollary 5.6 (and continuity) and (5.21) holds in Q including $l_\pm$.

Let us prove this for φ close to φ_-. The case of φ close to φ_+ is analyzed similarly.

Let $h(s)$ be defined in $(\mathbb{C}\setminus\mathbb{R})\cap B(s^*)$, where $B(s^*)$ is a neighborhood of $s^*\in\mathbb{R}$. Define the jump of h at the point s^* as

$$\mathcal{J}(h,s^*):=\lim_{\varepsilon\to0+}h(s^*+i\varepsilon)-\lim_{\varepsilon\to0+}h(s^*-i\varepsilon).$$

We have $\mathcal{J}\left(\widehat{u}_r(\rho,\varphi,\omega),\varphi_-\right)=\widehat{f}(\omega-\omega_0)e^{-i\omega\rho}$ by (5.1).

Similarly,

$$\mathcal{J}\left(\partial_\varphi\widehat{u}_r(\rho,\varphi,\omega),\varphi_-\right)=0,\quad \mathcal{J}\left(\partial_{\varphi\varphi}\widehat{u}_r(\rho,\varphi,\omega),\varphi_-\right)=-\widehat{f}(\omega-\omega_0)(i\omega\rho)e^{i\omega\rho}.$$

From (5.2), (5.10), (8.2), and (8.3) we have

$$\begin{aligned}\mathcal{J}\left(\widehat{u}_d(\rho,\varphi,\omega),\varphi_-\right)&=\frac{i}{8\pi}\widehat{f}(\omega-\omega_0)\int_{-1}^{1}\frac{4}{\beta+i\varepsilon}e^{i\omega\rho\cosh\beta}\,d\beta\Big|_{\varepsilon_-=+0}^{\varepsilon_-=-0}\\&=-\mathcal{J}\left(\widehat{u}_r(\rho,\varphi,\omega),\varphi_-\right).\end{aligned} \tag{5.23}$$

Further, by (8.4),

$$\mathcal{J}\left(\partial_\varphi\widehat{u}_d(\rho,\varphi,\omega),\varphi_-\right)=0=-\mathcal{J}\left(\partial_\varphi\widehat{u}_r(\rho,\varphi,\omega),\varphi_-\right).$$

Finally, consider

$$M:=\mathcal{J}\left(\partial_{\varphi\varphi}\widehat{u}_d(\rho,\varphi,\omega),\varphi_-\right).$$

Similarly to (5.23), expanding $e^{i\omega\rho\cos\beta}$ in the Taylor series in β (at 0) and noting that all the terms $\int \frac{\beta^k\, d\beta}{(\beta+i\varepsilon_-)^3}$, $k \neq 2$, are continuous, we obtain

$$M = -\frac{i}{\pi}\widehat{f}(\omega-\omega_0)\int\limits_{-1}^{1}\frac{e^{i\omega\rho\cos\beta}}{(\beta+i\varepsilon_-)^3}d\beta\Big|_{\varepsilon_-=+0}^{\varepsilon_-=-0}$$

$$= \frac{-i\widehat{f}(\omega-\omega_0)(i\omega\rho)e^{i\omega\rho}}{2\pi}\int\limits_{-1}^{1}\frac{\beta^2}{(\beta+i\varepsilon_-)^3}d\beta\Big|_{\varepsilon_-=+0}^{\varepsilon_-=-0}.$$

Hence,

$$M = \widehat{f}(\omega-\omega_0)(i\omega\rho)e^{i\omega\rho} = -\mathcal{J}\left(\widehat{u}_r(\rho,\varphi,\omega),\varphi_-\right).$$

Since $\widehat{u}_i^i(\rho,\varphi,\omega)$ is smooth on l_- by (5.18), we obtain from (5.22) that $\widehat{u}_s \in C^2(l_-)$.

Similarly using (5.1), (5.17) and (1.1) we obtain $\widehat{u}_s \in C^2(l_+)$. So $\widehat{u}_s \in C^2(Q)$. Proposition 5.7 is proven. □

Corollary 5.8

(i) *The function $\widehat{u}_s(\cdot,\cdot,\omega)$ belongs to the space $H^1(Q)$ for any $\omega \in \mathbb{C}^+$.*
(ii) *The function $u_s(x,t) \in \mathcal{M}$.*

Proof

(i) Everywhere below $x = (\rho,\varphi) \in Q \setminus (l_1 \cup l_2)$. It suffices to prove that

$$u_s(\cdot,\cdot,\omega),\ \partial_{x_k}u_s(\cdot,\cdot,\omega) \in L_2(Q), \quad k=1,2, \quad \omega\in\mathbb{C}^+. \tag{5.24}$$

First, by Proposition 5.7, $\widehat{u}_s(x,\omega)$ satisfies (5.6). Hence, $\widehat{u}_s(\cdot,\omega) \in L_2(Q)$ for any $\omega \in \mathbb{C}^+$. Further, using (1.12), we have

$$|\partial_{x_1}u_s(\cdot,\cdot,\omega)|^2 \le |\cos\varphi|^2|\partial_\rho u_s(\cdot,\cdot,\omega)|^2 + \frac{|\sin\varphi|^2}{\rho^2}|\partial_\varphi u_s(\cdot,\cdot,\omega)|^2.$$

Hence, by Proposition 5.7,

$$|\partial_{x_1}u_s(\cdot,\cdot,\omega)|^2 \le C(\omega)e^{-2c(\omega)}\left(1+\frac{1}{\rho}\right).$$

This implies that $\partial_{x_1}u_s \in L_2(Q)$, since $c(\omega) > 0$. Similarly, $\partial_{x_2}u_s(\cdot,\cdot,\omega) \in L_2(Q)$. (5.24) is proven.

(ii) The statement follows from Definition 3.1. □

6 Uniqueness

In Sect. 5 we proved the existence of solution to (1.8)–(1.10) belonging to $\mathcal{M}$. In this section prove the uniqueness of this solution in the same space.

Recall that we understand the uniqueness of the time-dependent Sommerfeld problem (1.5)–(1.6) as the uniqueness of the solution u_s given by (1.7) of the mixed problem (1.8)–(1.10) in the space $\mathcal{M}$.

The following theorem is the main result of the paper.

Theorem 6.1

(i) *Problem* (1.8)–(1.10) *admits a solution belonging to the space* $\mathcal{M}$. *Its limiting amplitude exists and is the solution of problem* (4.9). *The connection between this limiting amplitude and the Sommerfeld solution is given by* (4.8).

(ii) *Problem* (1.8)–(1.10) *admits a unique solution in the space* $\mathcal{M}$.

Proof The statements contained in item (i) follow from Corollary 5.8, Corollary 4.3, and Remark 4.4.

(ii) Let us prove the uniqueness. We follow closely the proof of Theorem 2.1 from [5]. Suppose that there exist two solutions $u_s(x,t)$ and $v_s(x,t)$ of system (1.8)–(1.10) belonging to $\mathcal{M}$. Consider $w_s(x,t) := u_s(x,t) - v_s(x,t)$.

Then $\widehat{w}_s(\cdot,\cdot,\omega) = \widehat{u}_s(\cdot,\cdot,\omega) - \widehat{v}_s(\cdot,\cdot,\omega)$, where $\widehat{u}_s$, $\widehat{v}_s$ (and, therefore, $\widehat{w}_s$) satisfy all the conditions of Proposition 5.7 and $\widehat{w}_s|_{W^0} = 0$ by (3.4).

Let us prove that $\widehat{w}_s(\cdot,\cdot,\omega) \equiv 0$. Let R be a sufficiently large positive number and $B(R)$ be the open disk centered at the origin with radius R. Set $Q_R := Q \cap B(R)$. Note that Q_R has a piecewise smooth boundary S_R and denote by $n(x)$ the outward unit normal vector at the non-singular points $x \in S_R$.

The first Green identity for $w_s(\rho,\varphi,\cdot)$ and its complex conjugate $\overline{w}_s$ in the domain Q_R, together with zero boundary conditions on S_R, yield

$$\int\limits_{Q_R} \left[|\nabla \widehat{w}_s|^2 - \omega^2 |\widehat{w}_s|^2\right] dx = \int\limits_{\partial B(R)\cap Q} \left(\partial_n \widehat{w}_s\right)\cdot\left(\overline{w}_s\right) dS_R.$$

From the real and imaginary parts of the last identity, we obtain

$$\int\limits_{Q_R} \left[|\nabla \widehat{w}_s|^2 + \left(\operatorname{Im}\omega\right)^2 |\widehat{w}_s|^2\right] dx = \operatorname{Re} \int\limits_{\partial B(R)\cap Q} \left(\partial_n \widehat{w}_s\right)\left(\overline{\widehat{w}_s}\right) dS_R \tag{6.1}$$

for $\operatorname{Re}\omega = 0$ and

$$-2\left(\operatorname{Re}\omega\right)\left(\operatorname{Im}\omega\right) \int\limits_{Q_R} |\widehat{w}_s|^2\, dx = \operatorname{Im} \int\limits_{\partial B(R)\cap Q} \left(\partial_n w_s\right)\left(\overline{w}_s\right) dS_R \tag{6.2}$$

for Re $\omega \neq 0$. Recall that we consider the case Im $k \neq 0$. Now, note that since $\widehat{\omega}_s \in H^1(Q)$, there exist a monotonic sequence of positive numbers $\{R_j\}$ such that $R_j \to \infty$ as $j \to \infty$ and

$$\lim_{j\to\infty} \int\limits_{\partial B(R_j)\cap Q} \left[\partial_n \widehat{w}_s\right]\left[\overline{\widehat{w}_s}\right] dS_{R_j} = 0. \tag{6.3}$$

Indeed, in polar coordinates (ρ, φ), we have that the integrals

$$\int\limits_0^\infty \left(R \int\limits_0^{2\pi} |\widehat{w}_s(\rho, \varphi)|^2 d\varphi \right) dR \quad \text{and} \quad \int\limits_0^\infty \left(R \int\limits_0^{2\pi} |\partial_n \widehat{w}_s(\rho, \varphi)|^2 d\varphi \right) dR$$

are finite. This fact, in particular, implies that there exist a monotonic sequence of positive numbers R_j such that $R_j \to \infty$ as $j \to \infty$ and

$$\int\limits_0^{2\pi} |\widehat{w}_s(R_j, \varphi)|^2 d\varphi = o(R_j^{-1}), \quad \int\limits_0^{2\pi} |\partial_n \widehat{w}_s(R_j, \varphi)|^2 d\varphi = o(R_j^{-1}) \text{ as } j \to \infty.$$

Further, applying the Cauchy-Schwarz inequality for every R_j, we get

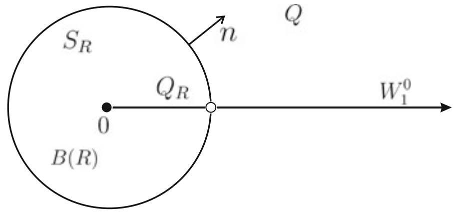
$$\begin{aligned} \left| \int\limits_0^{2\pi} \partial_n \widehat{w}_s(R_i, \varphi) \overline{\widehat{w}_s}(R_i, \varphi) d\varphi \right| &\leq \int\limits_0^{2\pi} |\partial_n \widehat{w}_s(R_i, \varphi) \widehat{w}_s(R_i, \varphi)| d\varphi \\ &\leq \left(\int\limits_0^{2\pi} |\partial_n \widehat{w}_s(R_i, \varphi)|^2 d\varphi \right)^{1/2} \left(\int\limits_0^{2\pi} |\widehat{w}_s(R_i, \varphi)|^2 d\varphi \right)^{1/2} \\ &= o(R_j^{-1}) \quad \text{as} \quad j \to \infty, \end{aligned}$$

and therefore we obtain (6.3).

Since the expressions under the integral sign in the left hand sides of equalities (6.1) and (6.2) are non-negative, we have that these integrals are monotonic with respect to R. This observation together with (6.3) implies

$$\int\limits_Q \left[|\nabla \widehat{w}_s|^2 + \left(\text{Im}\, \omega\right)^2 |\widehat{w}_s|^2 \right] d\varphi = \lim_{R\to\infty} \int\limits_{Q_R} \left[|\nabla \widehat{w}_s|^2 + \left(\text{Im}\, \omega\right)^2 |\widehat{w}_s|^2 \right] d\varphi = 0$$

Fig. 2 Uniqueness



for Re $\omega = 0$ and

$$\int_Q |\widehat{w}_s|^2 \, d\varphi = \lim_{R\to\infty} \int_{Q_R} |\widehat{w}_s|^2 \, d\varphi = 0$$

for Re $\omega \neq 0$. Thus, it follows from the last two identities that $\widehat{w}_s = 0$ in Q (Fig. 2). □

7 Conclusion

We proved that the Sommerfeld solution to the half-plane diffraction problem for a wide class of incident waves is the limiting amplitude of the solution of the corresponding time-dependent problem in a functional class of generalized solutions. The solution of the time-dependent problem is shown to be unique in this class. It is also shown that the limiting amplitude automatically satisfies the Sommerfeld radiation condition and the regularity condition at the edge.

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8 Appendix 1

Lemma 8.1

(i) *The functions* $\mathcal{Z}$ *(given by* (4.2)*) and* $\partial_\varphi \mathcal{Z}$ *admit uniform with respect to* $\varphi \in [0, 2\pi]$ *estimates*

$$|\mathcal{Z}(\beta, \varphi)| \le Ce^{-|\beta|/2}, \quad |\partial_\varphi \mathcal{Z}(\beta, \varphi)| \le Ce^{-|\beta|/2}, \quad |\beta| \ge 1. \tag{8.1}$$

(ii) *The function* $\mathcal{Z}$ *admits the representation*

$$\mathcal{Z}(\beta, \varphi) = -\frac{4}{\beta + i\varepsilon_+} + \frac{4}{\beta + i\varepsilon_-} + \check{\mathcal{Z}}(\beta, \varphi), \quad \varepsilon_\pm \neq 0 \tag{8.2}$$

with

$$\check{\mathcal{Z}}(\beta,\varphi)\in C^\infty(\mathbb{R}\times[0,2\pi]),\quad |\check{\mathcal{Z}}(\beta,\varphi)|\le C,\quad \beta\in\mathbb{R}\times[0,2\pi]. \tag{8.3}$$

(iii) *The function* $\partial_\varphi\mathcal{Z}$ *admits the representation*

$$\partial_\varphi\mathcal{Z}=-\frac{4i}{(\beta+i\varepsilon_+)^2}+\frac{4i}{(\beta+i\varepsilon_-)^2}+\check{Z}_1(\beta,\varphi),\quad \varepsilon_\pm\ne 0, \tag{8.4}$$

with

$$\check{Z}_1(\beta,\varphi)\in C^\infty(\mathbb{R}\times[0,2\pi]),\quad |\check{Z}_1(\beta,\varphi)|\le C,\quad \beta\in\mathbb{R}\times[0,2\pi]. \tag{8.5}$$

Proof

(i) For $a=im$, $b=in$, we have

$$\coth a-\coth b=\frac{-\sinh(\alpha/2)}{\sinh(b)\sinh(a)}.$$

Hence for $m=-\pi/8+a/4$ and $n=-\pi/8-a/4$ we obtain the estimate (8.1) for $U(\zeta)$ given by (4.6) with respect to ζ. So (8.1) for $\mathcal{Z}$ follows from (4.5) and (4.2).

(ii) From (4.5) and (4.6) it follows that the function $\mathcal{Z}$ admits the representation

$$\mathcal{Z}(\beta,\varphi)=Z_+(\beta,\varphi)+Z_-(\beta,\varphi)+Z^+(\beta,\varphi)+Z^-(\beta,\varphi),$$

where

$$\begin{aligned} Z_\pm(\beta,\varphi)&=\pm\coth\left(\frac{\beta+i(\varphi_\pm-\varphi)}{4}\right),\\ Z^\pm(\beta,\varphi)&=\pm\coth\left(\frac{\beta-i\big(\varphi_\pm+\varphi\big)}{4}\right).\end{aligned} \tag{8.6}$$

Further, since $|\coth z-1/z|\le C$, $|\mathrm{Im}\, z|\le\pi$, $z\ne 0$, we have

$$Z_\pm(\beta,\varphi)=\pm\frac{4}{\beta+i\varepsilon_\pm}+\check{Z}_\pm(\beta,\varphi),\quad \varphi\ne\varphi_\pm,$$

and

$$\check{Z}_\pm(\beta,\varphi)\in C^\infty(\mathbb{R}\times[0,2\pi]),\quad |\check{Z}_\pm(\beta,\varphi)|\le C,\quad (\beta,\varphi)\in\mathbb{R}\times[0,2\pi].$$

Finally, by (1.4),

$$Z^{\pm}(\beta,\varphi) \in C^{\infty}(\mathbb{R}\times[0,2\pi]), \quad |Z^{\pm}(\beta,\varphi)| \le C, \quad (\beta,\varphi)\in\mathbb{R}\times[0,2\pi].$$

Therefore, (8.2) and (8.3) are proven.

(iii) From (8.2) and (5.10) we get (8.4). Finally, by (8.6),

$$\partial_\varphi Z^{\pm}(\beta,\varphi) \in C^{\infty}(\mathbb{R}\times[0,2\pi]), \quad |\partial_\varphi Z^{\pm}(\beta,\varphi)| \le C, \quad (\beta,\varphi)\in\mathbb{R}\times[0,2\pi].$$

Moreover, since

$$\partial_\varphi Z_{\pm}(\beta,\varphi) \pm [4i/(\beta+\varepsilon_{\pm})^2] \in C^{\infty}([\mathbb{R}\times[0,2\pi]),$$

and is bounded in the same region, (8.5) holds.

□

For $\varepsilon,\beta\in\mathbb{R}$, $\varepsilon\neq 0$, $\rho>0$, $\omega\in\mathbb{C}^{+}$, let

$$K_0(\beta,\rho,\omega,\varepsilon) := \frac{e^{i\omega\rho\cosh\beta}}{\beta+i\varepsilon}, \quad \mathcal{K}_0(\rho,\omega,\varepsilon) := \int_{-1}^{1} K(\beta,\rho,\omega,\varepsilon)\,d\beta, \tag{8.7}$$

$$\begin{aligned} K_1(\beta,\rho,\omega,\varepsilon) &:= \cosh\beta\cdot e^{i\omega\rho\cosh\beta}, \quad \mathcal{K}_1(\rho,\omega,\varepsilon) := \int_{-1}^{1} K_1(\beta,\rho,\omega,\varepsilon)\,d\beta, \\ K_2(\beta,\rho,\varphi,\varepsilon) &:= \frac{e^{i\omega\rho\cosh\beta}}{(\beta+i\varepsilon)^2}, \quad \mathcal{K}_2(\rho,\omega,\varepsilon) := \int_{-1}^{1} K_2(\beta,\rho,\omega,\varepsilon)\,d\beta\,d\beta. \end{aligned} \tag{8.8}$$

Lemma 8.2 *There exist $C(\omega)>0$, $c(\omega)>0$ such that the functions $\mathcal{K}_0$, $\mathcal{K}_1$, and $\mathcal{K}_2$ satisfy the estimates*

$$|\mathcal{K}_{0,1,2}(\rho,\omega,\varepsilon)| \le C(\omega)e^{-c(\omega)\rho}, \quad \rho>0,\ \varphi\in(0,2\pi),\ \varepsilon\neq 0. \tag{8.9}$$

Proof It suffices to prove (8.9) for $0<\varepsilon<\varepsilon_0$, since the functions $\mathcal{K}_0,\mathcal{K}_1,\mathcal{K}_2$ are odd with respect to ε, and for $\varepsilon\ge\varepsilon_0>0$ they satisfy the estimate

$$\left|\mathcal{K}_{0,1,2}(\beta,\rho,\omega,\varepsilon)\right| \le C(\varepsilon_0)\int_{-1}^{1} e^{-\omega_2\rho}\,d\beta \le 2C(\varepsilon_0)e^{-\omega_2\rho}.$$

(I) Let us prove (8.9) for $\mathcal{K}_0$. Let

$$\cosh\beta := 1 + h(\beta), \quad \beta \in \mathbb{C}. \tag{8.10}$$

Define $\varepsilon_0 = \varepsilon_0(\omega)$ such that

$$|h(\beta)| < \frac{1}{4}, \quad |\omega_1||h(\beta)| \le \frac{\omega_2}{4} \quad \text{for} \quad |\beta| \le 2\varepsilon_0 := r, \tag{8.11}$$

and define the contour

$$\gamma_r := \{\beta = re^{i\theta}, \quad -\pi < \theta < 0\}. \tag{8.12}$$

Then we have by the Cauchy Theorem

$$\mathcal{K}_0(\rho,\omega,\varepsilon) = I_1(\rho,\omega,\varepsilon) + I_2(\rho,\omega,\varepsilon) - 2\pi i \ \mathrm{Res}_{\beta=-i\varepsilon} K_0(\beta,\rho,\omega,\varepsilon),$$

where

$$I_1(\rho,\omega,\varepsilon) = \int\limits_{\gamma_r} K_0(\beta,\rho,\omega,\varepsilon)\, d\beta, \quad I_2(\rho,\omega,\varepsilon) = \Big(\int\limits_{-1}^{-r} + \int\limits_{r}^{1}\Big) K_0(\beta,\rho,\omega,\varepsilon)\, d\beta$$

and $0 < \varepsilon < \varepsilon_0$. First,

$$|\,\mathrm{Res}_{\beta=-i\varepsilon} K_0(\beta,\rho,\omega,\varepsilon)| = e^{-\omega_2\rho\cos\varepsilon} \le e^{-\frac{1}{2}\omega_2\rho}, \quad 0 < \varepsilon < \varepsilon_0, \tag{8.13}$$

by (8.11). Further, from (8.10) we have

$$\begin{aligned} \big|I_1(\rho,\omega,\varepsilon)\big| &\le \int\limits_{\gamma_r} \frac{\Big|e^{-\omega_2\rho\big(1+h(\beta)\big)} e^{i\omega_1\rho\big(1+h(\beta)\big)}\Big|}{|\beta+i\varepsilon|} |d\beta| \\ &\le \frac{1}{\varepsilon_0}\, e^{-\omega_2\rho} \int\limits_{\gamma_r} \big|e^{-\omega_2\rho\, h(\beta)+i\omega_1\rho\, h(\beta)}\big|\, \big|d\beta\big|, \end{aligned} \tag{8.14}$$

since for $\beta \in \gamma_r$ we have $|\beta + i\varepsilon| \ge |\beta| - \varepsilon = 2\varepsilon_0 - \varepsilon > \varepsilon_0$, see Fig. 3.

Let $h(\beta) := h_1(\beta) + ih_2(\beta)$. Then

$$|I_1(\rho,\omega,\varepsilon)| \le \frac{1}{\varepsilon_0}\, e^{-\omega_2\rho} \int\limits_{\gamma_r} e^{\omega_2\rho\, |h_1(\beta)|}\, e^{|\omega_1|\rho\, |h_2(\beta)|}\, d\beta \le 2\pi e^{-\omega_2\rho/2}, \tag{8.15}$$

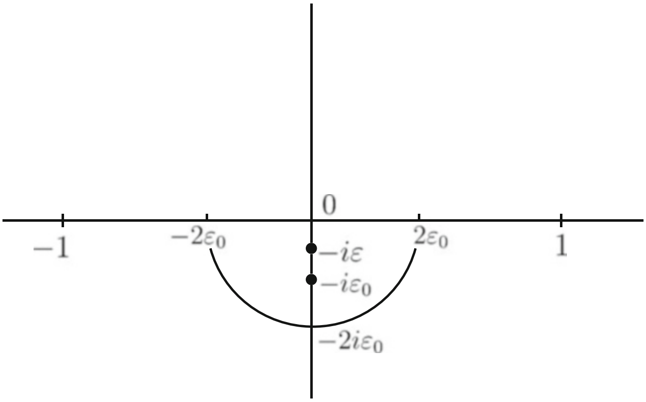

Fig. 3 Contour γ_r

by (8.11). Finally,

$$|I_2(\rho,\omega,\varepsilon)| \le \int\limits_{[-1,-r]\cup[r,1]} \left| \frac{e^{-\omega_2\rho\cosh\beta + i\omega_1\rho\cosh\beta}}{\beta + i\varepsilon} \right| d\beta \le \frac{e^{-\omega_2\rho}}{2\varepsilon_0(\omega)}, \tag{8.16}$$

since $|\beta + i\varepsilon| \ge 2\varepsilon_0,\ \beta \in [-1,-r] \cup [r,1]$. From (8.14)–(8.16), we obtain (8.9) for $\mathcal{K}_0$.

(II) Let us prove (8.9) for $\mathcal{K}_1$. Let $h(\beta)$, $\varepsilon_0(\omega)$, γ_r be defined by (8.10)–(8.12). Then we have by the Cauchy Theorem

$$\mathcal{K}_1(\rho,\omega,\varepsilon) := \int\limits_{\gamma_r\cup[-1,r]\cup[r,1]} K_1(\beta,\rho,\omega,\varepsilon)\, d\beta$$
$$- 2\pi i \ \mathrm{Res}_{\beta=-i\varepsilon}\ K_1(\beta,\rho,\omega,\varepsilon), \quad 0 < \varepsilon < \varepsilon_0. \tag{8.17}$$

First, similarly to (8.13), we obtain

$$\left| \mathrm{Res}_{\beta=-i\varepsilon}\ K_1(\beta,\rho,\omega,\varepsilon) \right| \le |\omega| e^{-\frac{\omega_2\rho}{2}},$$

by (8.11). Further, by (8.11) similarly to the proof of (8.14), (8.15), and using (8.10), we get

$$\left| \int\limits_{\gamma_r} K_1(\beta,\rho,\omega,\varepsilon)\, d\beta \right| \le \frac{|\omega|}{\varepsilon_0} \cdot \frac{5}{4}\, e^{-\omega_2\rho} \int\limits_{\gamma_r} |e^{-\omega_2\rho\, h(\beta)}\, e^{i\omega_1\rho\, h(\beta)}|\, |d\beta|$$
$$\le C(\omega) e^{-\frac{\omega_2\rho}{2}}. \tag{8.18}$$

Finally, similarly to the proof of (8.16) we get the estimate

$$\left| \int\limits_{[-1,-r]\cup[r,1]} K_1(\beta, \rho, \omega, \varepsilon)\, d\beta \right| \le C(\omega)\, e^{-\omega_2 \rho}. \tag{8.19}$$

From (8.17)–(8.19), we obtain (8.9) for $\mathcal{K}_1$.

(III) Estimate (8.9) for $\mathcal{K}_2$ is proved similarly to the same estimate for $\mathcal{K}_0$ and $\mathcal{K}_1$ with obvious changes. Lemma 8.2 is proven. □

9 Appendix 2

Lemma 9.1 *We have*

$$\left(\Delta + \omega^2\right) u_d(\rho, \varphi, \omega) = 0, \quad \varphi \ne \varphi_\pm, \quad \omega \in \mathbb{C}^+. \tag{9.1}$$

Proof By (5.2) it suffices to prove (9.1) for

$$A_d(\rho, \varphi, \omega) := \int\limits_{\mathbb{R}} \mathcal{Z}(\beta, \varphi) e^{i\omega\rho\cosh\beta}\, d\beta. \tag{9.2}$$

Since $\omega \in \mathbb{C}^+$ the integral (9.2) converges after differentiation with respect to ρ and φ. We have

$$\partial_\rho A_d(\rho, \varphi, \omega) = (i\omega) \int\limits_{\mathbb{R}} \mathcal{Z}(\beta, \varphi) \cosh\beta\; e^{i\omega\rho\cosh\beta}\, d\beta,$$

$$\partial_\rho^2 A_d(\rho, \varphi, \omega) = -\omega^2 \int\limits_{\mathbb{R}} \mathcal{Z}(\beta, \varphi) \cosh^2\beta\; e^{i\omega\rho\cosh\beta}\, d\beta.$$

Integrating by parts, we have by (4.2) and (8.1)

$$\begin{aligned} \partial_\varphi A_d(\rho, \varphi, \omega) &= \int\limits_{\mathbb{R}} \partial_\varphi \left(Z^0(\beta + 2\pi i - i\varphi) \right) e^{i\omega\rho\cosh\beta}\, d\beta \\ &= -\omega\rho \int\limits_{\mathbb{R}} \mathcal{Z}(\beta, \varphi) \sinh\beta\; e^{i\omega\rho\cosh\beta}\, d\beta, \quad \varphi \ne \varphi_\pm. \end{aligned} \tag{9.3}$$

Hence, similarly to (9.3)

$$\partial^2_{\varphi\varphi} A_d(\rho,\varphi,\omega) = -i\omega\rho \int\limits_{\mathbb{R}} \mathcal{Z}(\beta,\varphi)\Big[\cosh\beta + i\omega\rho\sinh^2\beta\Big] e^{i\omega\rho\cosh\beta}\, d\beta,$$

and

$$\begin{aligned}(\Delta+\omega^2)u_d(\rho,\varphi,\omega) =& \partial^2_\rho A_d(\rho,\varphi,\omega) + \frac{1}{\rho}\partial_\rho A_d(\rho,\varphi\omega)\\ &+ \frac{1}{\rho^2}\partial^2_\varphi A_d(\rho,\varphi,\omega) + \omega^2 A_d(\rho,\varphi,\omega) = 0. \qquad \square\end{aligned}$$

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