A Note on Group Representations, Determinantal Hypersurfaces and Their Quantizations

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Abstract Recently, there have been exciting developments on the interplay between representation theory of finite groups and determinantal hypersurfaces. For example, a finite Coxeter group is determined by the determinantal hypersurface described by its natural generators under the regular representation. This short note solves three problems about extending this result in the negative. On the affirmative side, it is shown that a quantization of a determinantal hypersurface, the so-called free locus, correlates well with representation theory. If $A_1, \dots, A_\ell \in \mathrm{GL}_d(\mathbb{C})$ generate a finite group G, then the family of hypersurfaces $\{X \in \mathrm{M}_n(\mathbb{C})^d : \det(I + A_1 \otimes X_1 + \cdots + A_\ell \otimes X_\ell) = 0\}$ for $n \in \mathbb{N}$ determines G up to isomorphism.

Keywords Linear pencil · Group representation · Determinantal hypersurface · Free locus

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1 Introduction

To $A_0, \dots, A_\ell \in \mathrm{M}_d(\mathbb{C})$ one assigns the **determinantal hypersurface**

$$\{[\xi_0 : \cdots : \xi_\ell] \in \mathbb{CP}^\ell :\ \det(\xi_0 A_0 + \cdots + \xi_\ell A_\ell) = 0\}. \tag{1.1}$$

This is a classical object in algebraic geometry [1, 6, 10, 11], where a key question asks which hypersurfaces admit determinantal representations. When A_j are real symmetric matrices, determinantal hypersurfaces pertain to hyperbolic and stable polynomials [2, 15, 18, 23, 24]. The geometry of the hypersurface (1.1) is also explored in multivariate operator theory [3, 4, 26]. If A_j are bounded operators on a Hilbert space and the determinant in (1.1) is replaced with the condition that $\xi_0 A_0 + \cdots + \xi_\ell A_\ell$ is not invertible, then (1.1) is known as the **projective joint spectrum** of $A_0, \dots, A_\ell$ (cf. Taylor spectrum [22] for ensembles of commuting operators).

Through the work of Frobenius [13] and Dedekind [7] on group determinants (see also [9]), determinantal hypersurfaces also pertain to representation theory. Several fascinating developments in this direction [5, 14, 21] have been recently made. This note addresses certain limitations for extensions of these results.

Let G be a finitely generated group. If $T = (g_1, \dots, g_\ell)$ is a finite sequence of generators for G and $\rho : G \to \mathrm{GL}_d(\mathbb{C})$ is a representation of G, then denote

$$\mathscr{Z}_1(T, \rho) = \left\{\xi \in \mathbb{C}^\ell :\ \det\left(I_d + \xi_1 \rho(g_1) + \cdots + \xi_\ell \rho(g_\ell)\right) = 0\right\}. \tag{1.2}$$

It is natural to ask what kind of information the affine hypersurface $\mathscr{Z}_1(T, \rho)$ carries about ρ and G. For example, if G_1, G_2 are finite groups with left regular representations λ_1, λ_2, then $\mathscr{Z}_1(G_1 \backslash \{1\}, \lambda_1) = \mathscr{Z}_1(G_2 \backslash \{1\}, \lambda_2)$ implies that G_1, G_2 are isomorphic [12]. However, one is typically interested in smaller generating sets or in finitely generated groups which are not necessarily finite. In [14], the authors computed the joint spectrum for the infinite dihedral group

$$D_\infty = \langle a, t \mid a^2 = t^2 = 1 \rangle$$

with respect to the generating set $(1, a, t)$, and analyzed its properties through the representation theory of D_∞. Determinantal hypersurfaces also have a strong connection with representation theory in the case of finite Coxeter groups [5]. A Coxeter group is a finitely generated group on generators $g_1, \dots, g_\ell$ satisfying

$$(g_i g_j)^{m_{ij}} = 1$$

where $m_{ii} = 1$ and $m_{ij} \geq 2$ for $i \neq j$. In [5] the authors first showed that if G is a finite Coxeter group, λ is its left regular representation, and $T = (g_1, \dots, g_\ell)$ are the generators as above, then $\mathscr{Z}_1(T, \lambda)$ determines G up to isomorphism. Furthermore, if G is not of exceptional type (in the Coxeter diagram sense) and ρ is an arbitrary finite-dimensional representation of G, then $\mathscr{Z}_1(T,\rho)$ determines ρ.

These theorems were presented during the Multivariable Spectral Theory and Representation Theory workshop at the Banff International Research Station in April 2019. Several problems about extending these results beyond Coxeter groups were posed by the speakers; among them were the following.

Questions 1.1 Let G be a finite group, T a fixed generating set for G, and ρ_1,ρ_2 irreducible complex representations of G.

(1) Is $\mathscr{Z}_1(T,\rho_1)$ a reduced and irreducible hypersurface?
(2) If $\mathscr{Z}_1(T,\rho_1)=\mathscr{Z}_1(T,\rho_2)$, are ρ_1 and ρ_2 equivalent?

As usual, $\rho_1 : G \to \mathrm{GL}_{d_1}(\mathbb{C})$ and $\rho_2 : G \to \mathrm{GL}_{d_2}(\mathbb{C})$ are equivalent if $d_1 = d_2$ and $\rho_2 = P\rho_2 P^{-1}$ for some $P \in \mathrm{GL}_{d_1}(\mathbb{C})$. A representation ρ_1 is irreducible if its image does not admit a nontrivial common invariant subspace; equivalently, it generates $\mathrm{M}_{d_1}(\mathbb{C})$ as a $\mathbb{C}$-algebra by Burnside's theorem [17, Corollary 1.17]. The hypersurface $\mathscr{Z}_1(T,\rho_1)$ is reduced and irreducible (in the scheme-theoretic sense) if its defining determinant in (1.2) is an irreducible polynomial. The main result of this note is the following.

Theorem 1.2 *Questions* 1.1(1) and (2) *have negative answers in general.*

See Sects. 2.1 and 2.2 for concrete examples. On a more positive side, in Sect. 3 we show that representation theory aligns well with a quantization of the determinantal hypersurface, the free locus; see Theorem 3.1. Furthermore, Proposition 3.4 determines whether a free locus arises from a representation of a finite group, and Proposition 3.7 characterizes finite abelian groups from the perspective of determinantal hypersurfaces. We conclude this note with an open question.

2 Representations Versus Determinants

In this section we give negative answers to Questions 1.1. The representations were found with the help of the computer algebra system GAP and the online repository ATLAS of Finite Group Representations. Verifying equivalence and irreducibility of representations was sometimes done symbolically with the computing system Mathematica.

2.1 Irreducible Representation with Reducible Determinant

The alternating group $G = A_6$ admits a presentation

$$G = \left\langle g_1, g_2 \mid g_1^2, g_2^4, (g_1g_2)^5, (g_1g_2^2)^5 \right\rangle.$$

Let

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $\rho(g_1) = A_1$ and $\rho(g_2) = A_2$ determines a faithful and irreducible representation $\rho : G \to \mathrm{GL}_9(\mathbb{C})$. Indeed, we can directly check that

$$A_1^2 = A_2^4 = (A_1 A_2)^5 = (A_1 A_2^2)^5 = I,$$

so ρ is a representation of G, and is moreover faithful since it is nontrivial and G is simple. Furthermore, all the possible products of A_1 and A_2 with at most 8 factors span the whole $\mathrm{M}_9(\mathbb{C})$, so ρ is irreducible. However, we claim that the curve $\mathscr{Z}_1((g_1, g_2), \rho)$ in $\mathbb{C}^2$ is not irreducible. We can compute the determinant of $I + x_1\rho(g_1) + x_2\rho(g_2)$,

$$\det \begin{pmatrix} 1+x_1 & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & x_1 & x_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_1 & 1 & 0 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & 1 & 0 & x_2 & 0 & 0 \\ x_2 & 0 & 0 & 0 & 0 & 1 & 0 & x_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1+x_1 & 0 & x_2 \\ 0 & 0 & 0 & 0 & 0 & x_1 & 0 & 1+x_2 & 0 \\ -x_1 & -x_1 & x_2-x_1 & -x_1 & -x_1 & -x_1 & -x_1 & -x_1 & 1-x_1 \end{pmatrix}$$

by cofactor expansion along the rows. The reader will have no difficulty verifying that $\det(I + x_1\rho(g_1) + x_2\rho(g_2))$ equals

$$1 + x_1 - 4x_1^2 - 4x_1^3 + 6x_1^4 + 6x_1^5 - 4x_1^6 - 4x_1^7 + x_1^8 + x_1^9 + x_2 + 2x_1x_2 - 2x_1^2x_2$$
$$- 6x_1^3x_2 + 6x_1^5x_2 + 2x_1^6x_2 - 2x_1^7x_2 - x_1^8x_2 + x_1^2x_2^2 + x_1^3x_2^2 - 2x_1^4x_2^2 - 2x_1^5x_2^2$$
$$+ x_1^6x_2^2 + x_1^7x_2^2 - x_1^2x_2^3 + 2x_1^4x_2^3 - x_1^6x_2^3 - 2x_2^4 + x_1^2x_2^4 - x_1^3x_2^4 + x_1^4x_2^4 + x_1^5x_2^4$$
$$- 2x_2^5 - 2x_1x_2^5 - x_1^2x_2^5 + x_1^4x_2^5 - x_1^2x_2^6 - x_1^3x_2^6 + x_1^2x_2^7 + x_2^8 - x_1x_2^8 + x_2^9$$

which is the product of the following two irreducible polynomials:

$$1 + 2x_1 - 2x_1^3 - x_1^4 + x_1x_2 + 2x_1^2x_2 + x_1^3x_2 - x_1x_2^2 - x_1^2x_2^2 + x_1x_2^3 - x_2^4,$$
$$1 - x_1 - 2x_1^2 + 2x_1^3 + x_1^4 - x_1^5 + x_2 - x_1x_2 - x_1^2x_2 + x_1^3x_2 - x_2^4 - x_2^5.$$

Some of the subsequent examples are presented in a more terse way to maintain the focus on their intent.

Note that the above irreducible representation of A_6 has dimension 9, which is not the minimum among nontrivial complex representations of A_6; namely, A_6 admits a representation σ of minimal dimension 5, and $\mathscr{Z}_1((g_1, g_2), \sigma)$ is irreducible. One might thus be tempted to suggest that for a group G generated by a finite set T and its (irreducible) representation σ of minimal dimension, $\mathscr{Z}_1(T, \sigma)$ is irreducible. However, even this weaker conjecture fails. The counterexample is given by the Janko group J_2,

$$J_2 = \left\langle g_1, g_2 \mid g_1^2, g_2^3, (g_1g_2)^7, (g_1g_2g_1^{-1}g_2^{-1})^{12}, (g_1g_2(g_1g_2g_1g_2^{-1})^2)^6 \right\rangle.$$

This sporadic simple group of order 604800 admits two non-isomorphic complex representations σ_1, σ_2 of minimal dimension 14, courtesy of ATLAS of Finite Group Representations. As in the previous example (albeit with slightly longer calculations), one can explicitly check that the curve $\mathscr{Z}_1((g_1, g_2), \sigma_1) = \mathscr{Z}_1((g_1, g_2), \sigma_2)$ has two irreducible components.

2.2 Non-equivalent Representations with the Same Determinant

The classical group $G = \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})$ admits the presentation

$$G = \left\langle g_1, g_2, g_3 \mid g_1^2, (g_1g_2^{-1})^2, (g_1g_3^{-1})^2, g_2^2g_3g_2^{-1}g_3, g_2g_3^2g_2g_3^{-1} \right\rangle.$$

Let A_1, A_2, A_3 be the matrices

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{2}-\frac{i}{2} \\ -\frac{1}{2}+\frac{i}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{2}+\frac{i}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2}-\frac{i}{2} \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{2}-\frac{i}{2} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{2}+\frac{i}{2} \end{pmatrix}.$$

There are faithful irreducible unitary representations $\rho_+, \rho_- : G \to \mathrm{GL}_2(\mathbb{C})$ given by

$$\rho_\pm(g_1) = \pm A_1, \quad \rho_\pm(g_2) = A_2, \quad \rho_\pm(g_3) = A_3.$$

It is easy to check that ρ_+ and ρ_- are not equivalent. On the other hand,

$$\mathcal{Z}_1((g_1, g_2, g_3), \rho_\pm) = \{(\xi_1, \xi_2, \xi_3) \colon 1 - \xi_1^2 + \xi_2 + \xi_2^2 + \xi_3 + \xi_3^2 = 0\}.$$

3 Free Locus Perspective

In this section we will see how representations of a finitely generated group are determined by a noncommutative relaxation of (1.2). To $A \in \mathrm{M}_d(\mathbb{C})^\ell$ we associate the monic matrix pencil $L_A = I + A_1x_1 + \cdots + A_\ell x_\ell$ of size d in freely noncommuting variables $x = (x_1, \dots, x_\ell)$. Thus L is an affine matrix over the free algebra $\mathbb{C}\langle x \rangle$. At a matrix point $X \in \mathrm{M}_n(\mathbb{C})^\ell$ it evaluates as

$$L_A(X) = I_{dn} + A_1 \otimes X_1 + \cdots + A_\ell \otimes X_\ell \in \mathrm{M}_{dn}(\mathbb{C}).$$

The **free locus** [19] of L_A is the disjoint union of determinantal hypersurfaces

$$\mathscr{Z}(L_A) = \bigsqcup_{n \in \mathbb{N}} \mathscr{Z}_n(L_A), \quad \mathscr{Z}_n(L_A) = \left\{X \in \mathrm{M}_n(\mathbb{C})^\ell \colon \det L_A(X) = 0\right\}.$$

Given a group G generated by $T = (g_1, \dots, g_n)$ and a complex representation $\rho : G \to \mathrm{GL}_d(\mathbb{C})$, we write

$$\mathscr{Z}(T, \rho) = \mathscr{Z}\left(L_{\rho(g_1),\dots,\rho(g_\ell)}\right). \tag{3.1}$$

By the definition of the free locus we see that (3.1) is indeed a quantization of (1.2). The existing results on free loci [16, 19] readily apply to group representations.

Theorem 3.1 *For $i = 1, 2$ let G_i be a group generated by a finite sequence T_i and let ρ_i be a complex representation of G_i. Assume $|T_1| = |T_2|$.*

(1) If ρ_i is irreducible, then there exists $n_0 \in \mathbb{N}$ such that $\mathscr{Z}_n(T_1, \rho_1)$ is a reduced and irreducible hypersurface for all $n \geq n_0$.

(2) If ρ_1 and ρ_2 are irreducible, then $\mathscr{Z}(T_1, \rho_1) = \mathscr{Z}(T_2, \rho_2)$ if and only if $G_1/\ker\rho_1 \cong G_2/\ker\rho_2$ and ρ_1, ρ_2 are equivalent.
(3) For $i = 1, 2$ assume that G_i is finite and ρ_i is a faithful representation. Then $\mathscr{Z}(T_1, \rho_1) = \mathscr{Z}(T_2, \rho_2)$ if and only if $G_1 \cong G_2$ via an isomorphism mapping T_1 to T_2.

Proof

(1) A consequence of [16, Theorem 3.4].
(2) A consequence of [19, Theorem 3.11].
(3) Let $\mathcal{T}_i$ be the $\mathbb{C}$-algebra generated by T_i. Since G_i is finite, its group algebra $\mathbb{C}G_i$ is semisimple by Maschke's theorem [17, Theorem 1.9]. Since $\mathcal{T}_i$ is a quotient of $\mathbb{C}G_i$, it is also semisimple. Then $\mathscr{Z}(T_1, \rho_1) = \mathscr{Z}(T_2, \rho_2)$ if and only if $T_1 \mapsto T_2$ induces an algebra isomorphism $\mathcal{T}_1 \to \mathcal{T}_2$ by [19, Corollary 3.8]. This isomorphism then restricts to the group isomorphism $G_1 \to G_2$. □

Remark 3.2 There is a deterministic bound on n_0 in Theorem 3.1(1) that is exponential in $|T_1|$ and the dimension of ρ_1 by [16, Remark 3.5] (the bound is likely not optimal). Similarly, to verify $\mathscr{Z}(T_1, \rho_1) = \mathscr{Z}(T_2, \rho_2)$ of Theorem 3.1(2,3), it suffices to check $\mathscr{Z}_n(T_1, \rho_1) = \mathscr{Z}_n(T_2, \rho_2)$ for a fixed large enough n, exponential in $|T_i|$ and the dimension of ρ_i by [19, Remark 3.7].

Free loci are defined for monic pencils with arbitrary matrix coefficients; we now describe how the geometry of the free locus $\mathscr{Z}(L_A)$ detects whether the coefficients $A_1, \dots, A_\ell$ generate a finite group. See also [8] for an efficient algorithm that determines finiteness of a finitely generated linear group.

Definition 3.3 Let $\ell, n \in \mathbb{N}$. Let $C \in \mathrm{GL}_n(\mathbb{Z})$ be the permutation matrix corresponding to the cycle $(1\,2\,\cdots\,n)$. If $\{1, \dots, n\} = S_1 \sqcup \cdots \sqcup S_\ell$ and P_j is the orthogonal projection onto $\mathrm{span}\{e_k : k \in S_j\}$, then the matrix point

$$X = (P_1 C, \cdots, P_\ell C) \in \mathrm{M}_n(\mathbb{Z})^\ell$$

is called a **cycle partition**. For given ℓ, n we thus have ℓ^n cycle partitions.

Let $\mu_\infty \subset \mathbb{C} \setminus \{0\}$ be the group of all roots of unity. The next proposition shows that if $A_1, \dots, A_\ell$ generate a finite group, then $\mathscr{Z}(L_A)$ intersects complex lines through cycle partitions only in points from μ_∞.

Proposition 3.4 *Let $A \in \mathrm{M}_d(\mathbb{C})^\ell$. Then $A_1, \dots, A_\ell$ generate a finite group if and only if the following hold:*

(i) there is a positive definite $P \in \mathrm{M}_d(\mathbb{C})$ such that $A_j^* P A_j = P$ for all j;
(ii) for every cycle partition X and $t \in \mathbb{C}$,

$$tX \in \mathscr{Z}(L_A) \implies t \in \mu_\infty.$$

Proof ($\Leftarrow$) Every A_j is invertible by (i). Let G be a group generated by $A_1, \dots, A_\ell$. Also by (i), G is a subgroup of the unitary group in $\mathrm{GL}_d(\mathbb{C})$ with respect to the inner product $\langle u, v\rangle = u^* P v$. Hence every element of G is diagonalizable. By [25, Corollary 4.9], a finitely generated subgroup of $\mathrm{GL}_d(\mathbb{C})$ is finite if and only if it is periodic (or torsion; i.e., every element has finite order). Since a diagonalizable matrix has a finite order if and only if all its eigenvalues lie in μ_∞, it suffices to verify that eigenvalues of every element of G lie in μ_∞.

To $(i_1, \dots, i_n) \in \{1, \dots, \ell\}^n$ we associate the cycle partition $X \in \mathrm{M}_n(\mathbb{Z})^\ell$ by choosing $S_j = \{e_k : i_k = j\}$. We claim that $tX \in \mathscr{Z}(L_A)$ if and only if $(-t)^n$ is an eigenvalue of $A_{i_1} \cdots A_{i_n}$. Indeed, using Schur complements it is easy to check that

$$\begin{aligned}
\det(I - (-1)^n t^n A_{i_1} \cdots A_{i_n}) &= \det \begin{pmatrix} I & tA_{i_1} \\ (-1)^n t^{n-1} A_{i_2} \cdots A_{i_n} & I \end{pmatrix} \\
&= \det \begin{pmatrix} I & tA_{i_1} & 0 \\ 0 & I & tA_{i_2} \\ -(-1)^n t^{n-2} A_{i_3} \cdots A_{i_n} & 0 & I \end{pmatrix} \\
&= \cdots \\
&= \det \begin{pmatrix} I & tA_{i_1} & & \\ & \ddots & \ddots & \\ & & I & tA_{i_{n-1}} \\ tA_{i_n} & & & I \end{pmatrix} \\
&= \det L_A(tX).
\end{aligned}$$

Thus the matrix $A_{i_1} \cdots A_{i_n}$ has finite order if and only if $tX \in \mathscr{Z}(L_A)$ implies $t \in \mu_\infty$, which holds by (ii).

($\Rightarrow$) If $A_1, \dots, A_\ell$ generate a finite group G, then $\mathbb{C}^d$ admits a G-invariant inner product

$$\langle u, v\rangle = \sum_{g \in G} (gu)^*(gv).$$

If P is the positive definite matrix satisfying $\langle u, v\rangle = u^* P v$, then (i) holds. Furthermore, the proof of (ii) is already given in the previous paragraph. □

Remark 3.5 If additional information about $A_1, \dots, A_\ell$ is given, say that their entries generate a number field (finite extension of $\mathbb{Q}$), then the size of the cycle partitions, which have to be tested in Proposition 3.4, can be bounded using Schur's theorem on orders of finite matrix groups [17, Theorem 14.19].

Remark 3.6 Let $p \in \mathbb{N}$ be prime. If μ_∞ in Proposition 3.4 is replaced by the group of power-of-p roots of unity, one obtains a free locus characterization of matrix tuples that generate a finite p-group.

We also show how the free locus certifies whether its defining coefficients generate a finite abelian group. The degree of an affine variety of codimension m is the number of intersection points of the variety with m hyperplanes in general position; in the case of a hypersurface, it is simply the degree of its square-free defining polynomial.

Proposition 3.7 *Let G be a finite group generated by $A_1, \dots, A_\ell \in \mathrm{M}_d(\mathbb{C})$. Then G is abelian if and only if the irreducible components of $\mathscr{Z}_n(L_A)$ have degree n for all $n \in \mathbb{N}$.*

Proof Let $\mathcal{A}$ be the $\mathbb{C}$-algebra generated by $A_1, \dots, A_\ell$. As in the proof of Theorem 3.1(3) we see that $\mathcal{A}$ is semisimple. After a basis change (which does not affect the structure of G or $\mathscr{Z}(L_A)$) we can thus assume that

$$A_j = A_j^{(1)} \oplus \cdots \oplus A_j^{(s)}$$

where $A_1^{(k)}, \dots, A_\ell^{(k)} \in \mathrm{M}_{d_k}(\mathbb{C})$ determine an irreducible representation of G for every $k = 1, \dots, s$. For $X \in \mathrm{M}_n(\mathbb{C})^d$ let us view $\det L_{A^{(k)}}(X)$ as a polynomial in the entries of X. If $d_k = 1$, then $\det L_{A^{(k)}}(X)$ is up to an affine change of coordinates equal to the determinant of a generic $n \times n$ matrix, and hence an irreducible polynomial of degree n. On the other hand, if $d_k > 1$, then $\det L_{A^{(k)}}(X)$ is a polynomial of degree $d_k n > n$ for all n, and irreducible for all large enough n by [16, Theorem 3.4]. Since G is abelian if and only if $d_1 = \cdots = d_s = 1$, and

$$\mathscr{Z}_n(L_A) = \mathscr{Z}_n(L_A^{(1)}) \cup \cdots \cup \mathscr{Z}_n(L_A^{(s)}),$$

it follows that G is abelian if and only if the irreducible components of $\mathscr{Z}_n(L_A)$ are hypersurfaces of degree n. □

Remark 3.8 If $\ell = 2$ and A_1, A_2 are hermitian, then $\mathscr{Z}_1(L_A)$ alone determines whether G is abelian, cf. [20].

The last two propositions offer some directions for future research. Theorem 3.1 implies that the linear group G generated by a tuple A is determined by $\mathscr{Z}(L_A)$. It would be interesting to know which properties of G can be deduced from the geometry of $\mathscr{Z}(L_A)$. For example, intersections of $\mathscr{Z}(L_A)$ with certain lines and hyperplanes determine whether G is finite or abelian. An open problem is how to decide whether a finite group G is nilpotent/solvable/simple (or any other group-theoretic property) by considering the geometry of the hypersurfaces $\mathscr{Z}_n(L_A)$.

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