Chapter 4
Solution of Differential Games with Network Structure in Marketing

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Abstract A marketing network model of goodwill accumulation with spillover effect is analysed in a differential game theory framework. Cooperative form of the game is considered under α-characteristic function. An approach is illustrated on a numerical example with particular values of the model parameters fixed.

Keywords Network differential games · Characteristic function · Differential game theory

4.1 Introduction

Modern mathematical game theory sets out to model, analyse and resolve various issues associated with conflict-controlled processes. Of particular interest are dynamic processes, the conflict processes developing over time, which could be well described in differential games terms [6].

Another essential branch of mathematical game theory covers network models. The models taking place under an assumption of some network structure among players. Differential games on networks were widely studied in [11]. Moreover, such game formulation found its place in economic and marketing issues [5].

In recent literature [8, 9], dynamic processes in marketing, which evolves over time, are often described in the framework of differential game theory. But there are only a few papers in which marketing is considered with the network structure of participants, especially in the continuous-time formulation [1].

In this paper differential game with network structure applied for a marketing model of goodwill accumulation is considered [7]. Additionally, the model includes the spillover effect [2] that accounts for the influence of other players' decisions on the total payoff of the players.

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The paper is organised as follows. Section 4.2 is dedicated to the game formulation with all the additional necessary assumptions given. The cooperative setup is proposed in Sect. 4.3. In the following section α-characteristic function of the game is calculated in the form of *maxmin* problem. In Sect. 4.4 the proposed approach is illustrated by a numerical example.

4.2 Game Formulation

Consider a differential game of three ($N = 3$) players $\Gamma(t_0, x_0)$. The game starts from the initial time instant t_0 and initial state x_0 and supposed to proceed on the infinite interval. The game is assumed to have a network structure represented by the non-oriented graph illustrated by the Fig. 4.1.

Assume that the dynamic of the common state variable for each player takes form of the following differential equation (4.1)

$$\dot{x}_i = \alpha_i u_i(t) - \delta_i x_i(t), \quad i = \{1, 2, 3\}, \quad x_i(t_0) = x_0^i. \tag{4.1}$$

The state variables $x_i(t)$ refer to the amount of stock of the player (advertising, technology, resource, capital). The control variables $u_i(t)$ are the open-loop strategies of the player i and represent the investment/extraction effort of the player (firm). In addition, both the controls and the state variables are required to be (almost everywhere) differentiable.

The player's payoff consists of components depending on his network connections with other players. If the player i is connected with the player j then the payoff component takes form (4.2)

$$h_{ij}(t) = e^{-pt}(a_i x_i(t) + c_j x_i(t) u_j(t) - \frac{1}{2} u_i^2(t)). \tag{4.2}$$

Thus, the payoff of the player is the sum of connection components

$$J_i\left(x_0^i, x_0^{K(i)}; u^i, u^{K(i)}\right) = \int\limits_{t_0}^{\infty} \sum_{j \in K(i)} h_{ij}(\tau) d\tau, \tag{4.3}$$

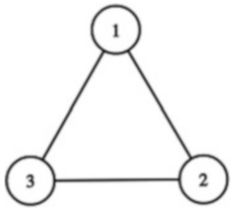

Fig. 4.1 The game's network structure

here $K(i) = \{(i, j), j \in N, (i, j) \in L\}$—the set of all connections of the player i, $L = \{i, j\}$—edges of the graph.

The payoff functional of the considered model (4.2) has a linear-quadratic form. This fact implements a number of valuable properties. In particular, Hamilton-Jacobi-Bellman approach and Maximum Principle yield the same decision if they are restricted to linear-feedback forms, see Dockner et al. (2000) [3].

Assume the following restriction which are standard for economical applications.

- Non-negative constrains on the controls' value due to its nature as an effort level, thus

$$u_i \in U_i \subseteq \mathbb{R}_+, \quad i = \{1, 2, 3\}$$

- Open-loop strategies is taken from the closed compact set

$$u_i(t) \in \hat{U}_i \subset Comp\mathbb{R}, \quad i = \{1, 2, 3\}$$

- Non-negative constrains on the common state's value due to it's nature as a stock level, thus

$$x_i \in X_i \subseteq \mathbb{R}_+ \quad i = \{1, 2, 3\}.$$

This game has a linear-quadratic structure which takes place commonly among advertising and marketing models (see Deal et al. (1979) [2] and He et al. (2007) [5]) and includes a spillover effect represented by the term $c_j x_i(t) u_j(t)$. This effect represents a specific economic behaviour in the form of positive or negative impact on the value of the economic agent i by state and investments product of the firm j. Such phenomenon is widespread for advertising and goodwill models, where the value of advertising for one firm positively depends on advertising efforts of the other firm provided they have similar products.

4.3 Differential Game in the Form of Characteristic Function

To define the cooperative game the characteristic function (ch.f.) $V(S, x_0, t_0)$ should be constructed for every coalition $S \subset N$ in the game $\Gamma(x_0, t_0)$. In the modern literature under the characteristic function in cooperative games is implied a mapping from the set of all possible coalitions to real set:

$$V(\cdot) : 2^N \to R,$$

$$V(\emptyset) = 0.$$

Note that the value of the characteristic function for the grand coalition N is equal to $V(N, x_0, t_0)$. There are several main approaches to the construction of the characteristic function which show the power of the coalition S (see, for example, [4, 14]). The most commonly used classes of characteristic functions can be indicated in the order that they appeared in literature as α-, β-, γ-, δ-, ζ-characteristic function.

The value $V(S, x_0, t_0)$ can be interpreted as a power of the coalition S. The essential property is the property of superadditivity:

$$V(S_1 \cup S_2, x_0, t_0) \geq V(S_1, x_0, t_0) + \tag{4.4}$$

$$+V(S_2, x_0, t_0), \forall S_1, S_2 \subseteq N, \;\; S_1 \cap S_2 = \emptyset.$$

However, the use of superadditive characteristic function in solving various problems in the field of cooperative game theory in static and dynamic setting, provides a number of advantages such as:

1. provides the individual rationality property for cooperative solutions,
2. encourages players to sustain large coalitions and eventually unite into a Grand coalition N,
3. delivers clear meaning to the Shapley value (a component of the division for each player is equal to its average contribution to the payoff of the Grand coalition under a certain mechanism of its formation),
4. necessary when you build a strongly dynamically stable optimality principles.

Thus, in many aspects more useful to have superadditive characteristic function.

It is rather easy to construct the characteristic function $V(S, x_0, t_0)$ in the form of α—ch.f. [12]. The characteristic function of coalition S is constructed through the classical approach of Neumann, Morgenstern, formulated in 1944 in [10]. According to this approach, under $V^{\alpha}(S, x_0, t_0)$ is understood the maximum guaranteed payoff of coalition S, and the value $V^{\alpha}(S)$ can be calculated on the basis of the auxiliary zero-sum game $\Gamma_{S,N\setminus S}(t_0, x_0)$ between the coalition S and anti-coalition $N \setminus S$.

$$V^{\alpha}(S, x_0, t_0) = \begin{cases} 0, & S = \{\emptyset\}, \\ val\Gamma_{S,N\setminus S}(x_0, t_0,), & S \subset N, \\ \max_{u_1, u_2, \ldots u_n} \sum_{i=1}^{n} J_i(x_0, t_0, u(t)), & S = N. \end{cases} \tag{4.5}$$

In this paper, without loss of generality, characteristic function calculation could be divided into three main steps: ch.f. for coalition which consists of the only one individual player, two players coalition and grand coalition.

4.3.1 One Player Coalition Characteristic Function

Calculate the value of the characteristic function for a coalition consisting only of the player $\{1\}$. Characteristic function would be calculated as a function of time moment θ, where θ is the initial moment. We assume that $\theta = 0$. The following maximisation problem is settled

$$V(\{1\}, x_0, t_0) = \max_{u_1} \min_{u_2, u_3} \int_{\theta}^{\infty} e^{-pt}(2a_1 x_1(t) + c_2 x_1(t) u_2(t) + c_3 x_1(t) u_3(t) - 2\frac{1}{2} u_1^2(t)) dt. \tag{4.6}$$

Minimisation by $u_2(t)$ and $u_3(t)$ will result in zero controls. Ultimately, we need to solve the following maximisation problem

$$\begin{cases} \int_{\theta}^{\infty} e^{-pt}(2a_1 x_1(t) - u_1^2(t)) dt \to \max_{u_1}, \\ \dot{x}_1(t) = \alpha_1 u_1(t) - \delta_1 x_1(t), \\ x_1(t_0) = x_0^1. \end{cases} \tag{4.7}$$

Using Maximum Principal [13] the following form of optimal control depending on adjoint variable is obtained

$$u_1^*(t) = 0.5 \alpha_1 \psi_1(t) e^{pt}.$$

Corresponding differential equation for adjoint variable is

$$\dot{\psi}_1(t) = \delta_1 \psi_1(t) - 2a_1 e^{-pt}.$$

Under transversality conditions

$$\lim_{t \to \infty} \psi_1(t) = 0.$$

To simplify denote the variable

$$\lambda_1(t) = e^{pt} \psi_1(t).$$

Thus, by solving the system below a final form for optimal control could be obtained

$$\begin{cases} \dot{\lambda}_1(t) = (p+\delta_1)\lambda_1(t) - 2a_1, \\ \dot{x}_1(t) = \alpha_1 u_1(t) - \delta_1 x_1(t), \\ x_1(t_0) = x_0, \\ \lim_{t\to\infty} e^{-pt}\lambda(t) = 0. \end{cases} \tag{4.8}$$

As the result

$$u_1^*(t) = \frac{\alpha_1 a_1}{p+\delta_1} = Const. \tag{4.9}$$

The optimal trajectory could be derived under the assumption that the game was started from the point $(\theta, x^*(\theta))$.

$$x^*(t) = \frac{\alpha_1^2 a_1}{\delta_1(p+\delta_1)} + x^*(\theta)e^{\delta_1(\theta-t)} - \frac{\alpha_1^2 a_1}{\delta_1(p+\delta_1)}e^{\delta_1(\theta-t)}.$$

In particular, if $\theta = t_0 = 0$ and $x(t_0) = x_0^1$

$$x^*(t) = \frac{\alpha_1^2 a_1}{\delta_1(p+\delta_1)}(1 - e^{-\delta_1 t}) + x_0^1 e^{-\delta_1 t}.$$

Characteristic function for coalition consisting of the only player one is

$$V(\{1\}, x_0, t_0) = \int_\theta^\infty e^{-pt}(2a_1 x_1^*(t) - u_1^{*^2}(t))dt =$$

$$= \int_\theta^\infty e^{-pt}(2a_1\frac{\alpha_1^2 a_1}{\delta_1(p+\delta_1)} + x^*(\theta)e^{\delta_1(\theta-t)} - \frac{\alpha_1^2 a_1}{\delta_1(p+\delta_1)}e^{\delta_1(\theta-t)} - (\frac{\alpha_1 a_1}{p+\delta_1})^2)dt =$$

$$= (\frac{2\alpha_1^2 a_1^2}{\delta_1(p+\delta_1)} - \frac{\alpha_1^2 a_1^2}{(p+\delta_1)^2})(-\frac{1}{p})e^{-pt} +$$

$$+(x^*(\theta) - \frac{\alpha_1^2 a_1}{\delta_1(p+\delta_1)})e^{\delta_1\theta}(-\frac{1}{\delta_1+p})e^{-(\delta_1+p)t}\Big|_\theta^\infty =$$

$$= -\frac{\alpha_1^2 a_1^2(2p+\delta_1)}{p\delta_1(p+\delta_1)^2}e^{-p\theta} + (x^*(\theta) - \frac{\alpha_1^2 a_1}{\delta_1(p+\delta_1)})e^{\delta_1\theta}(-\frac{1}{\delta_1+p})e^{-(\delta_1+p)\theta}.$$

4.3.2 Two Player Coalition Characteristic Function

Without loss of generality, consider the coalition of $\{1, 2\}$ as an example of two players coalition. In other cases calculation would be the same accurate to indexes.

$$V(\{1,2\}, x_0, t_0) = \max_{u_1,u_2} \min_{u_3} \int_{\theta}^{\infty} e^{-pt}(h_{12}(t) + h_{13}(t) + h_{21}(t) + h_{23}(t))dt =$$

$$= \max_{u_1,u_2} \min_{u_3} \int_{\theta}^{\infty} e^{-pt}(2a_1x_1(t) - u_1^2(t) + c_1x_2(t)u_1(t) + 2a_2x_2(t) - u_2^2(t) +$$

$$+ c_2x_1(t)u_2(t) + (x_1(t) + x_2(t))c_3u_3(t))dt.$$

Minimisation by $u_3(t)$ will result in zero controls. Ultimately, we need to solve the following maximisation problem

$$\begin{cases} \int_{\theta}^{\infty} e^{-pt}(2a_1x_1(t) - u_1^2(t) + c_1x_2(t)u_1(t) + 2a_2x_2(t) - u_2^2(t) + \\ + c_2x_1(t)u_2(t))dt \to \max_{u_1,u_2}, \\ \dot{x}_1(t) = \alpha_1u_1(t) - \delta_1x_1(t), \\ \dot{x}_2(t) = \alpha_2u_2(t) - \delta_2x_2(t), \\ x_1(t_0) = x_0^1, \\ x_2(t_0) = x_0^2. \end{cases} \quad (4.10)$$

Using Maximum principle for the first player

$$u_1^*(t) = 0.5\psi_1(t)\alpha_1e^{pt} + 0.5c_1x_2(t).$$

Due to the index symmetry among players both optimal control forms could be derived

$$\begin{cases} u_1^*(t) = 0.5\psi_1(t)\alpha_1e^{pt} + 0.5c_1x_2(t), \\ u_2^*(t) = 0.5\psi_2(t)\alpha_2e^{pt} + 0.5c_2x_1(t). \end{cases} \quad (4.11)$$

Denote again for the first player and for the second player correspondingly $\lambda_i(t) = e^{pt}\psi_i(t), \quad i = \{1, 2\}$.

$$\begin{cases} \dot{\lambda}_1(t) = (p + \delta_1)\lambda_1(t) - (2a_1 + c_2 u_2(t)), \\ \dot{\lambda}_2(t) = (p + \delta_2)\lambda_2(t) - (2a_2 + c_1 u_1(t)). \end{cases} \tag{4.12}$$

Corresponding differential equations for adjoint variables in aggregate with dynamic equations lead to the system

$$\begin{cases} \dot{\lambda}_1(t) = (p + \delta_1)\lambda_1(t) - 0.5c_2\alpha_2\lambda_2(t) - 0.5c_2^2 x_1(t) - 2a_1, \\ \dot{\lambda}_2(t) = (p + \delta_2)\lambda_2(t) - 0.5c_1\alpha_1\lambda_1(t) - 0.5c_1^2 x_2(t) - 2a_2, \\ \dot{x}_1(t) = 0.5\alpha_1^2\lambda_1(t) - \delta_1 x_1(t) + 0.5\alpha_1 c_1 x_2(t), \\ \dot{x}_2(t) = 0.5\alpha_2^2\lambda_2(t) - \delta_2 x_2(t) + 0.5\alpha_2 c_2 x_1(t). \end{cases} \tag{4.13}$$

In matrix form

$$\begin{pmatrix} \dot{\lambda}_1(t) \\ \dot{\lambda}_2(t) \\ \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} \delta_1 + p & -0.5\alpha_2 c_2 & -0.5c_2^2 & 0 \\ -0.5\alpha_1 c_1 & \delta_2 + p & 0 & -0.5c_1^2 \\ 0.5\alpha_1^2 & 0 & -\delta_1 & 0.5\alpha_1 c_1 \\ 0 & 0.5\alpha_2^2 & 0.5\alpha_2 c_2 & -\delta_2 \end{pmatrix} \times \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} -2a_1 \\ -2a_2 \\ 0 \\ 0 \end{pmatrix}.$$

To solve the system the corresponding homogeneous system.

Denote A as

$$A = \begin{pmatrix} \delta_1 + p & -0.5\alpha_2 c_2 & -0.5c_2^2 e^{pt} & 0 \\ -0.5\alpha_1 c_1 & \delta_2 + p & 0 & -0.5c_1^2 e^{pt} \\ 0.5\alpha_1^2 & 0 & -\delta_1 & 0.5\alpha_1 c_1 e^{pt} \\ 0 & 0.5\alpha_2^2 & 0.5\alpha_2 c_2 e^{pt} & -\delta_2 \end{pmatrix}.$$

To obtain the decision, eigen values and eigen vectors of the matrix A should be derived and analysed to understand if the decision is in real or complex surface. Nevertheless, the above described operation are rather computationally complex to obtain the decision in an analytical form. However, an approach could be illustrated on the simplified system in case of constant values of some of the system parameters which are denote in the way not being in contradiction with the economical meaning of the model. This result is shown in the section below.

4.3.3 Grand Coalition Characteristic Function

Calculate the characteristic function for a coalition consisting of three players (Grand coalition).

$$V(\{1,2,3\}, x_0, t_0) = \max_{u_1,u_2,u_3} \int\limits_{\theta}^{\infty} e^{-pt}(h_{12}(t) + h_{13}(t) + h_{21}(t) + h_{23}(t) + h_{31}(t) +$$

$$+h_{32}(t))dt = \max_{u_1,u_2,u_3} \int\limits_{\theta}^{\infty} e^{-pt}(2a_1x_1(t) + 2a_2x_2(t) + 2a_3x_3(t) + (x_2(t) + x_3(t))c_1u_1(t) +$$

$$+(x_1(t) + x_3(t))c_2u_2(t) + (x_1(t) + x_2(t))c_3u_3(t) - u_1^2(t) - u_2^2(t) - u_1^3(t))dt.$$

The following maximisation problem is needed to be solved

$$\begin{cases} \int\limits_{\theta}^{\infty} e^{-pt}(2a_1x_1(t) + 2a_2x_2(t) + 2a_3x_3(t) + (x_2(t) + x_3(t))c_1u_1(t) + (x_1(t) + \\ +x_3(t))c_2u_2(t) + (x_1(t) + x_2(t))c_3u_3(t) - u_1^2(t) - u_2^2(t) - u_1^3(t))dt \to \max_{u_1,u_2,u_3}, \\ \dot{x}_1(t) = \alpha_1u_1(t) - \delta_1x_1(t), \\ \dot{x}_2(t) = \alpha_2u_2(t) - \delta_2x_2(t), \\ \dot{x}_3(t) = \alpha_3u_3(t) - \delta_3x_3(t), \\ x_1(t_0) = x_0^1, \\ x_2(t_0) = x_0^2, \\ x_3(t_0) = x_0^3. \end{cases} \tag{4.14}$$

Using Maximum Principle derive the optimal controls depending on adjoint variables

$$\begin{cases} u_1^*(t) = 0.5\psi_1(t)\alpha_1e^{pt} + 0.5c_1(x_2(t) + x_3(t)), \\ u_2^*(t) = 0.5\psi_2(t)\alpha_2e^{pt} + 0.5c_2(x_1(t) + x_3(t)), \\ u_3^*(t) = 0.5\psi_3(t)\alpha_3e^{pt} + 0.5c_3(x_1(t) + x_2(t)). \end{cases} \tag{4.15}$$

Denote again for every player $\lambda_i(t) = e^{pt}\psi_i(t), \quad i \in \{1,2,3\}$.

Corresponding differential equations for adjoint variables in aggregate with dynamic equations lead to the system

$$\begin{cases} \dot\lambda_1(t) = (p+\delta_1)\lambda_1(t) - 0.5c_2\alpha_2\lambda_2(t) - 0.5c_3\alpha_3\lambda_3(t)- \\ \quad -0.5(c_2^2+c_3^2)x_1(t) - 0.5c_3^2x_2(t) - 0.5c_2^2x_3(t) - 2a_1, \\ \dot\lambda_2(t) = (p+\delta_2)\lambda_2(t) - 0.5c_1\alpha_1\lambda_1(t) - 0.5c_3\alpha_3\lambda_3(t)- \\ \quad -0.5c_1^2x_3(t) - 0.5c_3^2x_1(t) - 0.5(c_1^2+c_3^2)x_2(t) - 2a_2, \\ \dot\lambda_3(t) = (p+\delta_3)\lambda_3(t) - 0.5c_1\alpha_1\lambda_1(t) - 0.5c_2\alpha_2\lambda_2(t)- \\ \quad -0.5c_1^2x_2(t) - 0.5c_2^2x_1(t) - 0.5(c_1^2+c_2^2)x_3(t) - 2a_3, \\ \dot x_1(t) = 0.5\alpha_1^2\lambda_1(t) - \delta_1x_1(t) + 0.5\alpha_1c_1x_2(t) + 0.5\alpha_1c_1x_3(t), \\ \dot x_2(t) = 0.5\alpha_2^2\lambda_2(t) - \delta_2x_2(t) + 0.5\alpha_2c_2x_1(t) + 0.5\alpha_2c_2x_3(t), \\ \dot x_3(t) = 0.5\alpha_3^2\lambda_3(t) - \delta_3x_3(t) + 0.5\alpha_3c_3x_1(t) + 0.5\alpha_3c_3x_2(t). \end{cases} \tag{4.16}$$

Denote the system matrix $\hat A$

$$\hat A = \begin{pmatrix} \delta_1+p & -0.5\alpha_2c_2 & -0.5\alpha_3c_3 & -0.5(c_2^2+c_3^2) & -0.5c_3^2 & -0.5c_2^2 \\ -0.5\alpha_1c_1 & \delta_2+p & -0.5\alpha_3c_3 & -0.5c_3^2 & -0.5(c_1^2+c_3^2) & -0.5c_1^2 \\ -0.5\alpha_1c_1 & -0.5\alpha_2c_2 & \delta_3+p & -0.5c_2^2 & -0.5c_1^2 & -0.5(c_1^2+c_2^2) \\ 0.5\alpha_1^2 & 0 & 0 & -\delta_1 & 0.5\alpha_1c_1 & 0.5\alpha_1c_1 \\ 0 & 0.5\alpha_2^2 & 0 & 0.5\alpha_2c_2 & -\delta_2 & 0.5\alpha_2c_2 \\ 0 & 0 & 0.5\alpha_3^2 & 0.5\alpha_3c_3 & 0.5\alpha_3c_3 & -\delta_3 \end{pmatrix}.$$

As in the case of two player there is a computational complexity on the way to analytical solution which depends on all the system parameters. However, the approach is the same as for the case of two players.

4.4 Numerical Example

To show the existence of the feasible solution of such a system as (4.13), denote the parameters of the model in the following way, so the analytical form of the decision could take reasonable view

$$\alpha_1 = \alpha_2 = \alpha = 1,$$

$$c_1 = c_2 = c = 1.$$

Therefore, A matrix takes form

$$A = \begin{pmatrix} \delta_1 + p & -0.5 & -0.5 & 0 \\ -0.5 & \delta_2 + p & 0 & -0.5 \\ 0.5 & 0 & -\delta_1 & 0.5 \\ 0 & 0.5 & 0.5 & -\delta_2 \end{pmatrix}$$

Eigen values for this matrix could be simplified to the following form

$$\begin{pmatrix} 0.5\left(p - \sqrt{\left(\sqrt{(\delta_1 - \delta_2)^2 + 1} - (\delta_1 + \delta_2 + p)\right)^2 - 1}\right) \\ 0.5\left(p + \sqrt{\left(\sqrt{(\delta_1 - \delta_2)^2 + 1} - (\delta_1 + \delta_2 + p)\right)^2 - 1}\right) \\ 0.5\left(p - \sqrt{\left(\sqrt{(\delta_1 - \delta_2)^2 + 1} + (\delta_1 + \delta_2 + p)\right)^2 - 1}\right) \\ 0.5\left(p + \sqrt{\left(\sqrt{(\delta_1 - \delta_2)^2 + 1} + (\delta_1 + \delta_2 + p)\right)^2 - 1}\right) \end{pmatrix}$$

All the eigen values of A are different. However, there is an issue if they are on the real surface of they are complex ones.

If the following conditions are held then the eigen values of A are real numbers and the decision of the system exists on the real surface.

$$\begin{cases} \left(\sqrt{(\delta_1 - \delta_2)^2 + 1} - (\delta_1 + \delta_2 + p)\right)^2 \geq 1, \\ \left(\sqrt{(\delta_1 - \delta_2)^2 + 1} + (\delta_1 + \delta_2 + p)\right)^2 \geq 1. \end{cases} \tag{4.17}$$

4.5 Conclusion

We proposed an analysis of the marketing network model in the form of differential game. An approach is presented for the calculation of the α-characteristic function of the game and illustration is given for the numerical example with particular values of the number of model parameters.

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