# Chapter 9
# Anticipative, Incursive and Hyperincursive Discrete Equations for Simulation-Based Cyber-Physical System Studies

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**Abstract** This chapter will present algorithms for simulation of discrete space-time partial differential equations in classical physics and relativistic quantum mechanics. In simulation-based cyber-physical system studies, the main properties of the algorithms must meet the following conditions. The algorithms must be numerically stable and must be as compact as possible to be embedded in cyber-physical systems. Moreover the algorithms must be executed in real-time as quickly as possible without too much access to the memory. The presented algorithms in this paper meet these constraints. As a first example, we present the second-order hyperincursive discrete harmonic oscillator that shows the conservation of energy. This recursive discrete harmonic oscillator is separable to two incursive discrete oscillators with the conservation of the constant of motion. The incursive discrete oscillators are related to forward and backward time derivatives and show anticipative properties. The incursive discrete oscillators are not recursive but time inverse of each other and are executed in series without the need of a work memory. Then, we present the second-order hyperincursive discrete Klein–Gordon equation given by space-time second-order partial differential equations for the simulation of the quantum Majorana real 4-spinors equations and of the relativistic quantum Dirac complex 4-spinors equations. One very important characteristic of these algorithms is the fact that they are space-time symmetric, so the algorithms are fully invertible (reversible) in time and space. The development of simulation-based cyber-physical systems indeed evolves to quantum computing. So the presented computing tools are well adapted to these future requirements.

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## 9.1 Introduction

This chapter begins with a presentation step by step of the second-order hyperincursive discrete equation of the position of the harmonic oscillator. We show that the second-order hyperincursive discrete harmonic oscillator is represented by the equations of the position and velocity of the hyperincursive discrete harmonic oscillator that is separable into two incursive discrete harmonic oscillators. We demonstrate that these incursive discrete equations of the position and velocity of the harmonic oscillators can be described by a constant of motion. After that, we give a numerical simulation of the two incursive discrete harmonic oscillators. The numerical values correspond exactly to the analytical solutions. Then we present the hyperincursive discrete harmonic oscillator. And we give also a numerical simulation of the hyperincursive discrete harmonic oscillator. The numerical values correspond also to the analytical solutions. After that, we demonstrate that a rotation on the position and velocity variables transforms the incursive discrete harmonic oscillators to recursive discrete harmonic oscillators.

Then, this chapter presents the second-order hyperincursive discrete Klein–Gordon equation.

This discrete Klein–Gordon equation bifurcates to the hyperincursive discrete Majorana equations which tend to the real 4-spinors Majorana first-order partial differential equations for the intervals of time and space tending to zero.

After that, we demonstrate that the Majorana equations bifurcate to the 8 real Dirac first-order partial differential equations that are transformed to the original Dirac 4-spinors equations. The 4 hyperincursive discrete Dirac 4-spinors equations are then presented.

Finally, we show that there are 16 discrete functions associated with the space and time symmetric discrete Klein–Gordon equation. This is in agreement with the Proca thesis on the 16 components of the Dirac wave function in 4 groups of 4 equations.

In this chapter, we restricted our derivation of the Majorana and Dirac equations to the first group of 4 equations depending on 4 functions.

This chapter is based on my papers in this field.

The paper [1] concerns the hyperincursive algorithms of classical harmonic oscillator applied to quantum harmonic oscillator separable into incursive oscillators. The paper [2] deals with a unified discrete mechanics given by the bifurcation of the hyperincursive discrete harmonic oscillator, the hyperincursive discrete Schrödinger quantum equation, the hyperincursive discrete Klein–Gordon equation and the Dirac quantum relativist equations. In this paper [2], I have demonstrated that the second-order hyperincursive discrete Klein–Gordon equation bifurcates to the 4 Dirac first-order equations, in one space dimension.

An introduction to incursion and hyperincursion is given in the following series of papers on the total incursive control of linear, non-linear and chaotic systems [3], on computing anticipatory systems with incursion and hyperincursion [4], on the computational derivation of quantum and relativist systems with forward–backward space-time shifts [5], on a review of incursive, hyperincursive and anticipatory systems, with the foundation of anticipation in electromagnetism [6], then, on the precision and stability analysis of Euler, Runge–Kutta and incursive algorithms for the harmonic oscillator [7], and finally, on the new concept of deterministic anticipation in natural and artificial systems [8].

I wrote a series of theoretical papers on the discrete physics with Adel Antippa on the harmonic oscillator via the discrete path approach [9], on anticipation, orbital stability, and energy conservation in discrete harmonic oscillators [10], on the dual incursive system of the discrete harmonic oscillator [11], on the superposed hyperincursive system of the discrete harmonic oscillator [12], on the incursive discretization, system bifurcation, and energy conservation [13], on the hyperincursive discrete harmonic oscillator [14], on the synchronous discrete harmonic oscillator [15], on the discrete harmonic oscillator, a short compendium of formulas [16], on the time-symmetric discretization of the harmonic oscillator [17], and finally, on the discrete harmonic oscillator, evolution of notation and cumulative erratum [18]. This discrete physics is based on the fundamental mathematical development of the hyperincursive and incursive discrete harmonic oscillator.

An important purpose of this chapter deals with the bifurcation of the second-order hyperincursive discrete Klein–Gordon equation firstly to the 4 hyperincursive discrete real 4-spinors Majorana equations, secondly to the 8 hyperincursive discrete real 8-spinors Dirac equations that can be rewritten as the 4 hyperincursive discrete complex 4-spinors Dirac equations.

In 1926, Klein [19] and Gordon [20] presented independently what is called the Klein–Gordon equation. In 1928, Dirac [21] introduced the relativist quantum mechanics based on this Klein–Gordon equation. His fundamental equation is based on 4-spinors and is given by 4 first-order complex partial differential equations. All the work of Dirac is well explained in his book [22].

In 1930 and 1932, Proca [23, 24] proposed a generalization of the Dirac theory with the introduction of 4 groups of 4-spinors, and with 16 first-order complex partial differential equations.

In 1937, Majorana [25] proposed a real 4-spinors Dirac equation, given by 4 first-order real partial differential equations. Ettore Majorana disappears just after having written this fundamental paper. Pessa [26] presented a very interesting paper on the Majorana oscillator based on the 4 first-order real partial differential equations.

An excellent introduction to quantum mechanics is given in the books of Messiah [27].

This chapter is essentially based on my following recent papers.

The paper [28] deals with deduction of the Majorana real 4-Spinors generic Dirac equation from the computable hyperincursive discrete Klein–Gordon equation. Then the paper [29] shows that the hyperincursive discrete Klein–Gordon Equation is the algorithm for computing the Majorana real 4-spinors equation and the real 8-spinors Dirac equation. In fact, this corresponds to bifurcation of the hyperincursive discrete Klein–Gordon equation to real 4-spinors Dirac equation related to the Majorana Equation [30].

Then the next paper is a continuation of the paper on the unified discrete mechanics [2], dealing with the bifurcation of hyperincursive discrete harmonic oscillator, Schrödinger's quantum oscillator, Klein–Gordon's equation and Dirac's quantum relativist equations. Indeed this next paper on the unified discrete mechanics II [31] deals with the space and time-symmetric hyperincursive discrete Klein–Gordon equation that bifurcates to the 4 incursive discrete Majorana real 4-spinors equations. Then the paper on the unified discrete mechanics III [32] deals with the hyperincursive discrete Klein–Gordon equation that bifurcates to the 4 incursive discrete Majorana and Dirac equations and to the 16 Proca equations.

The review paper [33], as an update of my paper [3], deals with the time-symmetric hyperincursive discrete harmonic oscillator separable into two incursive harmonic oscillators with the conservation of the constant of motion. As a novelty, we present the transformation of the incursive discrete equations to recursive discrete equations by a rotation of the position and velocity variables of the harmonic oscillator [33], as described in this chapter.

More developments are given in the paper [34] on the rotation of the two incursive discrete harmonic oscillators to recursive discrete harmonic oscillators with the Hadamard matrix. Then, a continuation deals with the rotation of the relativistic quantum Majorana equation with the Hadamard matrix and Unitary matrix U [35]. Finally in this chapter, we give the analytical solution of the quantum Dirac equation for a particle at rest following our last paper on the relations between the Majorana and Dirac quantum equations [36].

This chapter is organized as follows.

Section 9.2 deals with a presentation step by step of the second-order hyperincursive discrete harmonic oscillator.

Section 9.3 develops the 4 incursive discrete equations of the hyperincursive discrete harmonic oscillator. Then Sect. 9.4 presents the constants of motion of the two incursive discrete harmonic oscillators. In Sect. 9.5, we give numerical simulations of the two incursive discrete harmonic oscillators.

Section 9.6 presents the hyperincursive discrete harmonic oscillator. Section 9.7 gives numerical simulations of the hyperincursive discrete harmonic oscillator.

Section 9.8 deals with a rotation of the position and velocity variables of the incursive discrete equations of the harmonic oscillator which are transformed to recursive discrete equations. This result is fundamental because it gives an explanation of the anticipative effect of the discretization of the time in discrete physics. The information obtained from the hyperincursive discrete equations is richer than obtained by continuous physics.

In Sect. 9.9, we present the Klein–Gordon partial differential equation and the space and time-symmetric second-order hyperincursive discrete Klein–Gordon equation that bifurcates to the relativistic quantum Majorana and Dirac equations.

Then, in Sect. 9.10, we present the hyperincursive discrete relativistic quantum Majorana equations. For intervals of time and space tending to zero, these discrete equations tend to the 4 first-order partial differential Majorana equations.

In Sect. 9.11, next, in defining the Majorana functions by 2-spinors real functions, after some mathematical manipulations, we demonstrate that the Majorana real 4-spinors equations bifurcate into the 8 real equations. These 8 real first-order partial differential equations represent the Dirac real 8-spinors equations that are transformed to the original Dirac complex 4-spinors equations.

Then Sect. 9.12 presents the 4 hyperincursive discrete Dirac 4-spinors equations depending on 4 complex discrete Dirac wave functions.

In Sect. 9.13, we show that there are 16 complex functions associated with this second-order hyperincursive discrete Klein–Gordon equation. This is in agreement with the Proca thesis, for which the Dirac function has 16 components and divided into 4 groups of 4 functions with 4 equations. In this chapter, we restricted our derivation of the Majorana and Dirac equations to the first group of 4 equations depending to 4 functions.

Finally Sect. 9.14 deals with numerical simulations of the hyperincursive discrete Majorana and Dirac wave equations depending on time and one spatial dimension (1D) and with a null mass.

## 9.2 Presentation Step by Step of the Second-Order Hyperincursive Discrete Harmonic Oscillator

The harmonic oscillator is represented by the second-order temporal ordinary differential equations

$$d^2x(t)/dt^2 = -\omega^2 x(t) \quad (9.2.1a)$$

with the velocity given by

$$v(t) = dx(t)/dt \quad (9.2.1b)$$

where $x(t)$ is the position and $v(t)$ the velocity as functions of the time $t$ and where the pulsation $\omega$ is related to the spring constant $k$ and the oscillating $m$ by

$$\omega^2 = k/m \quad (9.2.1c)$$

The harmonic oscillator can be represented by the two ordinary differential equations:

$$\begin{aligned} dx(t)/dt &= v(t) \\ dv(t)/dt &= -\omega^2 x(t) \end{aligned} \quad (9.2.2a, b)$$

The solution is given by

$$\begin{aligned} x(t) &= x(0)\cos(\omega t) + [v(0)/\omega]\sin(\omega t) \\ v(t) &= -\omega x(0)\sin(\omega t) + v(0)\cos(\omega t) \end{aligned} \quad (9.2.2c, d)$$

with the initial conditions $x(0)$ and $v(0)$.

In the phase space, given by $x(t)$, $v(t)$, the solutions are given by closed curves (orbital stability).

The period of oscillations is given by $T = 2\pi/\omega$.

The energy $e(t)$ of the harmonic oscillator is constant and is given by

$$e(t) = kx^2(t)/2 + mv^2(t)/2 = kx^2(0)/2 + mv^2(0)/2 = e(0) = e_0 \qquad (9.2.3)$$

The harmonic oscillator is computable by recursive functions from the discretization of the differential equations. The differential equations of the harmonic oscillator depend on the current time.

In the discrete form, there are the discrete current time $t$ and the interval of time $\Delta t = h$.

The discrete time is defined as $t_k = t_0 + kh,\ \ k = 0, 1, 2, \ldots,$

where $t_0$ is the initial value of the time and $k$ is the counter of the number of interval of time $h$.

The discrete position and velocity variables are defined as $x(k) = x(t_k)$ and $v(k) = v(t_k)$.

The discrete equations consists in computing firstly the first equation to obtain, $x(k + 1)$, and then compute the second equation in using the just computed, $x(k + 1)$, as follows

$$\begin{aligned} x(k+1) &= x(k) + hv(k) \\ v(k+1) &= v(k) - h\omega^2 x(k+1) \end{aligned} \qquad (9.2.4\text{a, b})$$

In fact, the first equation used the forward derivative and the second equation used the backward derivative,

$$\begin{aligned} [x(k+1) - x(k)]/h &= v(k) \\ [v(k) - v(k-1)]/h &= -h\omega^2 x(k) \end{aligned} \qquad (9.2.4\text{c, d})$$

The position, $x(k + 1)$, and the velocity, $v(k)$, are not computed at the same time step.

I called such a system, an incursive system, for inclusive or implicit recursive system, e.g., [4].

A second possibility occurs if the second equation is firstly computed, and then the first equation is computed in using the just computed, $v(k + 1)$, as follows

$$\begin{aligned} v(k+1) &= v(k) - h\omega^2 x(k) \\ x(k+1) &= x(k) + hv(k+1) \end{aligned} \qquad (9.2.5\text{a, b})$$

In fact, the first equation used the forward derivative and the second equation used the backward derivative,

$$[v(k+1) - v(k)]/h = -\omega^2 x(k)$$
$$[x(k) - x(k-1)]/h = v(k) \quad (9.2.5c, d)$$

The position, $x(k)$, and the velocity, $v(k+1)$, are not computed at the same time step.

But in using the two incursive systems, we see that the position in the first incursion, $x(k+1)$, corresponds to the velocity in the second incursion, $v(k+1)$, at the same time step, $(k+1)$. And similarly, we see that the velocity in the first incursion, $v(k)$, corresponds to the position in the second incursion, $x(k)$, at the same time step $k$. So, both incursions give two successive positions and velocities at two successive time steps, $k$, and, $k+1$.

An important difference between the incursive and the recursive discrete systems is the fact that in the incursive system, the order in which the computations are made is important: this is a sequential computation of equations. In the recursive systems, the order in which the computations are made is without importance: this is a parallel computation of equations.

The two incursive harmonic oscillators are numerically stable, contrary to the classical recursive algorithms like the Euler and Runge–Kutta algorithms [7].

In the following paragraphs, it will be given a generalized equation that integrates both incursions to form a hyperincursive system.

In my paper [3], I defined a generalized forward-backward discrete derivative

$$D_w = wD_f + (1-w)D_b \quad (9.2.6)$$

where $w$ is a weight taking the values between 0 and 1, and where the discrete forward and backward derivatives on a function $f$ are defined by

$$D_f(f) = \Delta^+ f/\Delta t = \left[f(k+1) - f(k)\right]/h$$
$$D_b(f) = \Delta^- f/\Delta t = \left[f(k) - f(k-1)\right]/h \quad (9.2.7a, b)$$

The generalized incursive discrete harmonic oscillator is given by Dubois [3] and reprinted in the review paper [33]:

$$(1-w)x(k+1) + (2w-1)x(k) - wx(k-1) = hv(k)$$
$$wv(k+1) + (1-2w)v(k) + (w-1)v(k-1) = -h\omega^2 x(k) \quad (9.2.8a, b)$$

When $w = 0$, $D_0 = D_b$, this gives the first incursive equations:

$$x(k+1) - x(k) = hv(k)$$
$$v(k) - v(k-1) = -h\omega^2 x(k) \quad (9.2.9a, b)$$

When $w = 1$, $D_1 = D_f$, this gives the second incursive equations:

$$x(k) - x(k-1) = hv(k)$$
$$v(k+1) - v(k) = -h\omega^2 x(k) \quad (9.2.10a, b)$$

When $w = 1/2$, $D_{1/2} = [D_f + D_b]/2$, this gives the hyperincursive equations:

$$x(k+1) - x(k-1) = +2hv(k)$$
$$v(k+1) - v(k-1) = -2h\omega^2 x(k) \quad (9.2.11a, b)$$

where the discrete derivative is given by

$$D_s = D_{1/2} = [D_f + D_b]/2$$
$$D_s(f) = D_{1/2}(f) = [f(k+1) - f(k-1)]/2h \quad (9.2.7c)$$

that defines a time-symmetric derivative noted, $D_s$.

NB: the time-symmetric derivative $D_s$ in hyperincursive discrete equations

$$D_s(f) = [f(k+1) - f(k-1)]/2h$$

is not the same as the classical central derivative $D_c$ given in classical difference equations theory

$$D_c(f) = [f(k+1/2) - f(k-1/2)]/h.$$

These Eqs. (9.2.11a, b) integrate the two incursive equations [4–6].

Let us remark that this first hyperincursive Eq. (9.2.11a) can be also obtained by adding the Eq. (9.2.9a) to the Eq. (9.2.10a), and the second hyperincursive Eq. (9.2.11b) by adding the Eq. (9.2.9b) to the Eq. (9.2.10b).

In putting the velocity, $v(k)$, of the first Eq. (9.2.11b)

$$v(k) = [x(k+1) - x(k-1)]/2h \quad (9.2.12a)$$

to the second Eq. (9.2.11b),

$$x(k+2) - 2x(k) + x(k-2) = -4h^2\omega^2 x(k) \quad (9.2.12b)$$

one obtains what I called "the second-order hyperincursive discrete harmonic oscillator", corresponding to the second-order differential equations of the harmonic oscillator given by Eq. (9.2.1a), with the velocity given by the Eq. (9.2.1b).

In this section, we have presented the second-order hyperincursive discrete harmonic oscillator given by the Eq. (9.2.12b) that is separable into 4 first-order incursive discrete equations of the harmonic oscillator. The next section will present the 4 dimensionless incursive discrete equations.

## 9.3 The 4 Dimensionless Incursive Discrete Equations of the Harmonic Oscillator

For the discrete harmonic oscillator, let us use the dimensionless variables, $X$ and $V$, of Antippa and Dubois [16], for the variables, $x$ and $v$, as follows:

$$X(k) = [k/2]^{1/2}x(k) \text{ and } V(k) = [m/2]^{1/2}v(k) \quad (9.3.1a, b)$$

with the dimensionless time

$$\tau = \omega t \quad (9.3.2a)$$

where the pulsation (9.2.1c) is given by

$$\omega = [k/m]^{1/2} \quad (9.3.2b)$$

and with the dimensionless interval of time given by

$$\Delta\tau = \omega\Delta t = \omega h = H \quad (9.3.3)$$

So, the two incursive dimensionless harmonic oscillators are given by the following 4 first-order discrete equations: First Incursive Oscillator, from the dimensionless equations (9.2.4a, b):

$$\begin{aligned} X_1(k+1) &= X_1(k) + HV_1(k) \\ V_1(k+1) &= V_1(k) - HX_1(k+1) \end{aligned} \quad (9.3.4a, b)$$

Second Incursive Oscillator, from the dimensionless equations (9.2.5a, b):

$$\begin{aligned} V_2(k+1) &= V_2(k) - HX_2(k) \\ X_2(k+1) &= X_2(k) + HV_2(k+1) \end{aligned} \quad (9.3.5a, b)$$

These incursive discrete oscillators are non-recursive computing anticipatory systems.

Indeed, in Eq. (9.3.4b) of the first incursive oscillator, the velocity, $V_1(k+1)$, at the future next time step, $(k+1)$, is computed from the velocity, $V_1(k)$, at the current time step, $k$, and the position, $X_1(k+1)$, at the future next time step, $(k+1)$, which represents an anticipatory system represented by an anticipation of one time step, $k$. Similarly in Eq. (9.3.5b) of the second incursive oscillator, the position, $X_2(k+1)$, at the future next time step, $(k+1)$, is computed from the position, $X_2(k)$, at the current time step, $k$, and the velocity, $V_2(k+1)$, at the future next time step, $(k+1)$, which represents an anticipatory system represented by an anticipation of one time step, $k$.

These two incursive discrete harmonic oscillators define a discrete hyperincursive harmonic oscillator given by four incursive discrete equations.

A complete mathematical development of incursive and hyperincursive systems was presented in a series of papers by Adel F. Antippa and Daniel M. Dubois on the harmonic oscillator via the discrete path approach [9], on anticipation, orbital stability, and energy conservation in discrete harmonic oscillators [10], on the dual incursive system of the discrete harmonic oscillator [11], on the superposed hyperincursive system of the discrete harmonic oscillator [12], on the incursive discretization, system bifurcation, and energy conservation [13], on the hyperincursive discrete harmonic oscillator [14], on the synchronous discrete harmonic oscillator [15], on the discrete harmonic oscillator, a short compendium of formulas [16], on the time-symmetric discretization of the harmonic oscillator [17], and finally, on the discrete harmonic oscillator, evolution of notation and cumulative erratum [18].

The next section will present the constants of motion of the two incursive discrete harmonic oscillators.

## 9.4 The Constants of Motion of the Two Incursive Discrete Equations of the Harmonic Oscillator [33]

The constant of motion of the first incursive oscillator

$$\begin{aligned} X_1(k+1) &= X_1(k) + HV_1(k) \\ V_1(k+1) &= V_1(k) - HX_1(k+1) \end{aligned} \qquad (9.3.4a, b)$$

is given by

$$K_1(k) = X_1^2(k) + V_1^2(k) + HX_1(k)V_1(k) = K_1 = \text{constant} \qquad (9.4.1a)$$

**Theorem 9.1 [33]** *The expression* $K_1(k) = X_1^2(k) + V_1^2(k) + HX_1(k)V_1(k)$ *is a constant of motion of the first incursive equations* (9.3.4a, b).

***Proof*** Multiply the first Eq. (9.3.4a) by $X_1(k+1)$ at right and the second Eq. (9.3.4b) by $V_1(k+1)$ at left, then add the two equations, and one obtains successively

$$\begin{aligned} K_1(k+1) &= X_1(k+1)X_1(k+1) + V_1(k+1)V_1(k+1) + HX_1(k+1)V_1(k+1) \\ &= X_1(k+1)X_1(k) + HX_1(k+1)V_1(k) + V_1(k)V_1(k+1) \\ &= X_1(k)X_1(k) + HX_1(k)V_1(k) + HX_1(k+1)V_1(k) + V_1(k)V_1(k) - HV_1(k)X_1(k+1) \\ &= X_1(k)X_1(k) + HX_1(k)V_1(k) + V_1(k)V_1(k) = K_1(k) = K_1 = \text{constant} \end{aligned}$$

So the expression is constant because the expression is invariant in two successive temporal steps. ■

In replacing the expression of the velocity $V_1(k)$ from Eq. (9.3.4a) to the $H$ term in Eq. (9.4.1a), the term depending on $H$ disappears, as follows

$$X_1(k)X_1(k) + V_1(k)V_1(k) + X_1(k)[X_1(k+1) - X_1(k)] = K_1$$

or

$$X_1(k)X_1(k+1) + V_1(k)V_1(k) = K_1 \tag{9.4.1b}$$

which looks like the conservation of the energy.

The constant of motion of the second incursive oscillator

$$V_2(k+1) = V_2(k) - HX_2(k)$$
$$X_2(k+1) = X_2(k) + HV_2(k+1) \tag{9.3.5a, b}$$

is given by

$$K_2(k) = X_2^2(k) + V_2^2(k) - HX_2(k)V_2(k) = K_2 = \text{constant} \tag{9.4.2a}$$

**Theorem 9.2 [33]** *The expression* $K_2(k) = X_2^2(k) + V_2^2(k) - HX_2(k)V_2(k)$ *is a constant of motion of the second incursive equations* (9.3.5a, b).

***Proof*** Multiply the first Eq. (9.3.5a) by $V_2(k+1)$ at right and the second Eq. (9.3.5b) by $X_2(k+1)$ at left, then add the two equations, and one obtains successively

$$\begin{aligned}
K_2(k+1) &= X_2(k+1)X_2(k+1) + V_2(k+1)V_2(k+1) - HX_2(k+1)V_2(k+1)\\
&= X_2(k+1)X_2(k) - HX_2(k)V_2(k+1) + V_2(k)V_2(k+1)\\
&= X_2(k)X_2(k) + HV_2(k+1)X_2(k) - HX_2(k)V_2(k+1) + V_2(k)V_2(k) - HV_2(k)X_2(k)\\
&= X_2(k)X_2(k) - HX_2(k)V_2(k) + V_2(k)V_2(k) = K_2(k) = K_2 = \text{constant}
\end{aligned}$$

So the expression is constant because the expression is invariant in two successive temporal steps. ■

In replacing the expression of the position $X_2(k)$ from Eq. (9.3.5a) to the $H$ term in Eq. (9.4.2b), the term depending on $H$ disappears

$$X_2(k)X_2(k) + V_2(k)V_2(k) - [V_2(k) - V_2(k+1))]V_2(k) = K_2$$

or

$$X_2(k)X_2(k) + V_2(k+1)V_2(k) = K_2 \tag{9.4.2b}$$

that also looks like the conservation of the energy.

These constants of motion (9.4.1a) and (9.4.2a) differ with the inversion of the sign of $H$, as follows

$$+H = +\omega h = +\omega\Delta t, \text{ and } -H = -\omega h = -\omega\Delta t \tag{9.4.3a, b}$$

because the inversion of the discrete time interval of the first incursion gives the second incursion.

NB: It is very important to notice that there is a fundamental difference between an inversion of the sign of the discrete time, $\Delta t$, in the discrete equations and an inversion of the sign of the continuous time, $t$, in the differential equations.

Let us now consider a simple example of the solution of the discrete position and the discrete velocity of the dimensionless discrete harmonic oscillator, given by the following analytical solution (synchronous solution)

$$X_1(k) = \cos(2k\pi/N) \text{ and } V_1(k) = -\sin((2k+1)\pi/N) \tag{9.4.4a, b}$$

$$X_2(k) = \cos((2k+1)\pi/N) \text{ and } V_2(k) = -\sin(2k\pi/N) \tag{9.4.5a, b}$$

where $N$ is the number of iterations for a cycle of the oscillator,

with the index of iterations $k = 0, 1, 2, 3, \ldots$

The interval of discrete time $H$ depends of $N$ (for a synchronous solution):

$$H = 2\sin(\pi/N) \tag{9.4.6}$$

For $N = 6$, for example,

$$H = 2\sin(\pi/6) = 1 \tag{9.4.6a}$$

The two constants of motion, with the solutions (9.4.4a, b) and (9.4.5a, b) are given by

$$\cos^2(2k\pi/N) + \sin^2((2k+1)\pi/N) - H\cos(2k\pi/N)\sin((2k+1)\pi/N) = K_1$$
$$\cos^2((2k+1)\pi/N) + \sin^2(2k\pi/N) + H\cos((2k+1)\pi/N)\sin(2k\pi/N) = K_2$$

For $N = 6, H = 1, k = 0$, one obtains the same constant of motion for the two incursive oscillators:

$$\cos^2(0) + \sin^2(\pi/6) - \cos(0)\sin(\pi/6) = 1.0 + 0.25 - 0.5 = 0.75 = K_1 \tag{9.4.7a}$$

$$\cos^2(\pi/6) + \sin^2(0) + \cos(\pi/6)\sin(0) = 0.75 + 0.0 + 0.0 = 0.75 = K_2 \tag{9.4.7b}$$

And the averaged energy is a constant given by

$$\begin{aligned}[E_1(k) + E_2(k)]/2 =& [X_1^2(k) + V_1^2(k) + X_2^2(k) + V_2^2(k)]/2 \\ =& \left[\cos^2((2k+1)\pi/N) + \sin^2(2k\pi/N) + \cos^2(2k\pi/N) + \sin^2((2k+1)\pi/N)\right]/2 = 1\end{aligned}$$

A very interesting and important invariant, $INV_{12}$, is given by

$$INV_{12} = X_1(k)X_2(k) + V_2(k)V_1(k) = \text{constant} \tag{9.4.8}$$

With the values of the example, this gives a constant

$$\begin{aligned} INV_{12} &= X_1(k)X_2(k) + V_2(k)V_1(k) \\ &= \cos(2k/N)\cos((2k+1)/N) + \sin((2k+1)/N)\sin(2k/N) = \cos(\pi/N) \end{aligned}$$

For N = 6,

$$INV_{12} = \cos(\pi/6) = 3^{1/2}/2 = 0.8660 \tag{9.4.8a}$$

For large value of $N$,

$$INV_{12} \approx 1 \tag{9.4.8b}$$

In the next section, we will give a numerical simulation of the two incursive discrete harmonic oscillators in view of comparing with the analytical solutions that we have presented in this section.

## 9.5 Numerical Simulations of the Two Incursive Discrete Harmonic Oscillators

This section gives the numerical simulations of the two incursive harmonic oscillators [33].

Firstly, the parameters for the simulation are given as follows.

The number of iterations is given by

$$N = 6 \tag{9.5.1}$$

The interval of discrete time is then given by

$$H = 2\sin(\pi/N) = 2\sin(\pi/6) = 1 \tag{9.5.2}$$

And the boundary conditions are given by

$$X_1(0) = \cos(0) = 1 \text{ and } V_1(0) = -\sin(\pi/6) = -0.5 \tag{9.5.3a, b}$$

Table 9.1a gives the simulation of the first incursive discrete equations (9.3.4a, b) of the harmonic oscillator.

In Table 9.1a, we give the energy

**Table 9.1** (a) Simulation of the incursive discrete equations (9.3.4a, b). (b) Simulation of the discrete incursive equations (9.3.5a, b)

(a)

| First incursive Discrete harmonic oscillator | | | | | | | | Analytical solution | |
|---|---|---|---|---|---|---|---|---|---|
| $N$ | $H$ | $k$ | $X_1(k)$ | $V_1(k)$ | $E_1(k)$ | $E_{F1}(k)$ | $K_1(k)$ | $X_1(k) = \cos(2k\pi/N)$ | $V_1(k) = -\sin((2k+1)\pi/N)$ |
| 6 | 1 | 0 | 1.000 | −0.500 | 1.25 | −0.50 | 0.75 | $\cos(0\pi/6) = 1$ | $-\sin(1\pi/6)$ $-1/2$ |
| | | 1 | 0.500 | −1.000 | 1.25 | −0.50 | 0.75 | $\cos(2\pi/6) = 1/2$ | $-\sin(3\pi/6) = -1$ |
| | | 2 | −0.500 | −0.500 | 0.50 | 0.25 | 0.75 | $\cos(4\pi/6) = -1/2$ | $-\sin(5\pi/6) = -1/2$ |
| | | 3 | −1.000 | 0.500 | 1.25 | −0.50 | 0.75 | $\cos(6\pi/6) = -1$ | $-\sin(7\pi/6) = 1/2$ |
| | | 4 | −0.500 | 1.000 | 1.25 | −0.50 | 0.75 | $\cos(8\pi/6) = -1/2$ | $-\sin(9\pi/6) = 1$ |
| | | 5 | 0.500 | 0.500 | 0.50 | 0.25 | 0.75 | $\cos(10\pi/6) = 1/2$ | $-\sin(11\pi/6) = 1/2$ |
| | | 6 | 1.000 | −0.500 | 1.25 | −0.50 | 0.75 | $\cos(12\pi/6) = 1$ | $-\sin(13\pi/6) = -1/2$ |
| | | 7 | 0.500 | −1.000 | 1.25 | −0.50 | 0.75 | $\cos(14\pi/6) = 1/2$ | $-\sin(15\pi/6) = -1$ |

(b)

| Second incursive Discrete harmonic oscillator | | | | | | | | Analytical solution | |
|---|---|---|---|---|---|---|---|---|---|
| $N$ | $H$ | $k$ | $X_2(k)$ | $V_2(k)$ | $E_2(k)$ | $E_{B2}(k)$ | $K_2(k)$ | $X_2(k) = \cos((2k+1)\pi/N)$ | $V_2(k) = -\sin(2k\pi/N)$ |
| 6 | 1 | 0 | 0.866 | 0.000 | 0.75 | 0.00 | 0.75 | $\cos(1\pi/6) = \sqrt{3}/2$ | $-\sin(0\pi/6) = 0$ |
| | | 1 | 0.000 | −0.866 | 0.75 | 0.00 | 0.75 | $\cos(3\pi/6) = 0$ | $-\sin(2\pi/6) = -\sqrt{3}/2$ |
| | | 2 | −0.866 | −0.866 | 1.50 | −0.75 | 0.75 | $\cos(5\pi/6) = -\sqrt{3}/2$ | $-\sin(4\pi/6) = -\sqrt{3}/2$ |
| | | 3 | −0.866 | 0.000 | 0.75 | 0.00 | 0.75 | $\cos(7\pi/6) = -\sqrt{3}/2$ | $-\sin(6\pi/6) = 0$ |
| | | 4 | 0.000 | 0.866 | 0.75 | 0.00 | 0.75 | $\cos(9\pi/6) = 0$ | $-\sin(8\pi/6) = \sqrt{3}/2$ |
| | | 5 | 0.866 | 0.866 | 1.50 | −0.75 | 0.75 | $\cos(11\pi/6) = \sqrt{3}/2$ | $-\sin(10\pi/6) = \sqrt{3}/2$ |
| | | 6 | 0.866 | 0.000 | 0.75 | 0.00 | 0.75 | $\cos(13\pi/6) = \sqrt{3}/2$ | $-\sin(12\pi/6) = 0$ |

(continued)

**Table 9.1** (continued)

| (b) | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Second incursive<br>Discrete harmonic oscillator | | | | | | | | Analytical solution | |
| $N$ | $H$ | $k$ | $X_2(k)$ | $V_2(k)$ | $E_2(k)$ | $E_{B2}(k)$ | $K_2(k)$ | $X_2(k) = \cos((2k+1)\pi/N)$ | $V_2(k) = -\sin(2k\pi/N)$ |
| | | 7 | 0.000 | −0.866 | 0.75 | 0.00 | 0.75 | $\cos(15\pi/6) = 0$ | $-\sin(14\pi/6) = -\sqrt{3}/2$ |

$$E_1(k) = X_1^2(k) + V_1^2(k)$$

the forward energy

$$E_{F1}(k) = +HX_1(k)V_1(k)$$

and the constant of motion

$$K_1(k) = X_1^2(k) + V_1^2(k) + HX_1(k)V_1(k) = K_1 = \text{constant} \tag{9.4.1a}$$

The numerical values correspond exactly to the analytical solutions

$$X_1(k) = \cos(2k\pi/N)$$
$$V_1(k) = -\sin((2k+1)\pi/N) \tag{9.4.4a, b}$$

NB: see the correspondence of the variables with the hyperincursive harmonic oscillator at Table 9.3:

$$X_1(k) = X(2k),\ V_1(k) = V(2k+1) \tag{9.5.4}$$

Secondly, the parameters for the simulation are given as follows.
The number of iterations,

$$N = 6 \tag{9.5.5}$$

The interval of discrete time is then given by

$$H = 2\sin(\pi/N) = 2\sin(\pi/6) = 1 \tag{9.5.6}$$

The boundary conditions,

$$X_2(0) = \cos(\pi/6) = (3/4)^{1/2} = 0.8660$$
$$V_2(0) = -\sin(0) = 0 \tag{9.5.7a, b}$$

Table 9.1b gives the simulation of the second incursive discrete equations (9.3.5a, b) of the harmonic oscillator.

In Table 9.1b, we give the energy

$$E_2(k) = X_2^2(k) + V_2^2(k)$$

the backward energy

$$E_{B2}(k) = -HX_2(k)V_2(k)$$

and the constant of motion

$$K_2(k) = X_2^2(k) + V_2^2(k) - HX_2(k)V_2(k) = K_2 = \text{constant} \tag{9.4.2a}$$

The numerical values correspond exactly to the analytical solutions

$$X_2(k) = \cos((2k+1)\pi/N)$$
$$V_2(k) = -\sin(2k\pi/N) \tag{9.4.5a, b}$$

NB: see the correspondence of the variables with the hyperincursive harmonic oscillator at Table 9.3:

$$X_2(k) = X(2k+1),\ V_2(k) = V(2k) \tag{9.5.8}$$

In the next section, we demonstrate that the dimensionless hyperincursive discrete harmonic oscillator is separable into two incursive discrete harmonic oscillators.

## 9.6 The Dimensionless Hyperincursive Discrete Harmonic Oscillator Is Separable into Two Incursive Discrete Harmonic Oscillators

For the hyperincursive discrete harmonic oscillator, given by the Eqs. (9.2.11a, b), we use the dimensionless variables, $X$ and $V$, for the variables, $x$ and $v$, as follows:

$$X(k) = [k/2]^{1/2}x(k)$$
$$V(k) = [m/2]^{1/2}v(k) \tag{9.3.1a, b}$$

with the dimensionless time

$$\tau = \omega t \tag{9.3.2a}$$

where the pulsation (9.2.1c) is given by

$$\omega = [k/m]^{1/2} \tag{9.3.2b}$$

and with the dimensionless interval of time given by

$$\Delta\tau = \omega\Delta t = \omega h = H \tag{9.3.3}$$

So, the two Eqs. (9.2.11a, b) are then transformed to the following two dimensionless equations of the hyperincursive discrete harmonic oscillator

$$\begin{aligned} X(k+1) &= X(k-1) + 2HV(k) \\ V(k+1) &= V(k-1) - 2HX(k) \end{aligned} \tag{9.6.1a, b}$$

for $k = 1, 2, 3, \ldots$,

with the 4 even and odd boundary conditions, $X(0)$, $V(1)$, $V(0)$, $X(1)$.

This hyperincursive discrete harmonic oscillator is a recursive computing system that is separable into two independent incursive discrete harmonic oscillators [33], as shown in Tables 9.2a, b.

Table 9.2a gives the first iterations of the hyperincursive discrete equations (9.6.1a, b).

It is well seen that there are two independent series of iterations defining two incursive discrete harmonic oscillators, as given in Table 9.2b.

As well seen in Table 9.2b, the first incursive harmonic oscillator, with the boundary conditions,

$X(0)$, $V(1)$, is given by

$$\begin{aligned} X(2k) &= X(2k-2) + 2HV(2k-1) \\ V(2k+1) &= V(2k-1) - 2HX(2k) \end{aligned} \tag{9.6.2a, b}$$

and the second incursive harmonic oscillator, with the boundary conditions, $V(0)$, $X(1)$, is given by

$$\begin{aligned} V(2k) &= V(2k-2) - 2HX(2k-1) \\ X(2k+1) &= X(2k-1) + 2HV(2k) \end{aligned} \tag{9.6.3a, b}$$

for $k = 1, 2, 3, \ldots$

Let us remark that the difference between the two incursive oscillators represented by the Eqs. (9.3.4a, b, 9.3.5a, b) and these Eqs. (9.6.2a, b, 9.6.3a, b), holds in the labeling of the successive time steps. In the incursive oscillators, (9.3.4a, b, 9.3.5a, b), the position and velocity are computed at the same time step while in the incursive oscillators, (9.6.2a, b, 9.6.3a, b), the position and the velocity are computed at successive time steps, but the numerical simulations of both give the same values. Each incursive oscillator is the discrete time inverse, $+\Delta t \to -\Delta t$, and, $-\Delta t \to +\Delta t$ of the other incursive oscillator, defined by time forward and time backward derivatives. So the two incursive oscillators are not reversible. But the superposition of the two

**Table 9.2** (a) This table gives the first iterations of the hyperincursive discrete equations (9.6.1a, b). (b) This table shows the two independent incursive discrete harmonic oscillators

| (a) | | |
|---|---|---|
| Hyperincursive discrete harmonic oscillator | | |
| | $X(k+1) = X(k-1) + 2\,H\,V(k)$ | $V(k+1) = V(k-1) - 2\,H\,X(k)$ |
| | Boundary conditions: $X(0) = C_1$, $V(1) = C_2$, $V(0) = C_3$, $X(1) = C_4$ | |
| $k$ | Iterations | |
| 1 | $X(2) = X(0) + 2\,H\,V(1)$ | $V(2) = V(0) - 2\,H\,X(1)$ |
| 2 | $X(3) = X(1) + 2\,H\,V(2)$ | $V(3) = V(1) - 2\,H\,X(2)$ |
| 3 | $X(4) = X(2) + 2\,H\,V(3)$ | $V(4) = V(2) - 2\,H\,X(3)$ |
| 4 | $X(5) = X(3) + 2\,H\,V(4)$ | $V(5) = V(3) - 2\,H\,X(4)$ |
| 5 | $X(6) = X(4) + 2\,H\,V(5)$ | $V(6) = V(4) - 2\,H\,X(5)$ |
| 6 | $X(7) = X(5) + 2\,H\,V(6)$ | $V(7) = V(5) - 2\,H\,X(6)$ |
| … | – | – |

| (b) | | |
|---|---|---|
| | First incursive discrete harmonic oscillator | Second incursive discrete harmonic oscillator |
| | Boundary conditions: $X(0) = C_1$, $V(1) = C_2$ | Boundary conditions: $V(0) = C_3$, $X(1) = C_4$ |
| k | Iterations | Iterations |
| 1 | $X(2) = X(0) + 2\,H\,V(1)$ | $V(2) = V(0) - 2\,H\,X(1)$ |
| 2 | $V(3) = V(1) - 2\,H\,X(2)$ | $X(3) = X(1) + 2\,H\,V(2)$ |
| 3 | $X(4) = X(2) + 2\,H\,V(3)$ | $V(4) = V(2) - 2\,H\,X(3)$ |
| 4 | $V(5) = V(3) - 2\,H\,X(4)$ | $X(5) = X(3) + 2\,H\,V(4)$ |
| 5 | $X(6) = X(4) + 2\,H\,V(5)$ | $V(6) = V(4) - 2\,H\,X(5)$ |
| 6 | $V(7) = V(5) - 2\,H\,X(6)$ | $X(7) = X(5) + 2\,H\,V(6)$ |
| … | – | – |

incursive oscillators given by the hyperincursive discrete oscillator is reversible. In putting the expression of $V(k)$ from the Eq. (9.6.1b)

$$V(k) = [X(k+1) - X(k-1)]/2H \tag{9.6.4}$$

to the Eq. (9.6.1a), one obtains the second-order hyperincursive discrete harmonic oscillator

$$X(k+2) - 2X(k) + X(k-2) = -4H^2 X(k) \tag{9.6.5}$$

With the dimensionless variables, the dimensionless energy is given by

$$E(k) = X^2(k) + V^2(k) \tag{9.6.6}$$

The next section will give simulations of the hyperincursive discrete harmonic oscillator.

## 9.7 Numerical Simulations of the Hyperincursive Discrete Equations of the Harmonic Oscillator

This section gives the numerical simulations of the hyperincursive discrete harmonic oscillator.

Firstly, we will give explicitly the parameters for the simulation for the case corresponding to the simulations given at the preceding section for the two incursive discrete harmonic oscillators.

The number of iterations is given by,

$$N = 12 \tag{9.7.1}$$

The interval of discrete time is then given by

$$H = \sin(2\pi/N) = \sin(\pi/6) = 0.5 \tag{9.7.2}$$

NB: When $N$ is large,

$$H = \sin(2\pi/N) \approx 2\pi/N = \omega\Delta t = 2\pi\,\Delta t/T \tag{9.7.3}$$

so the period $T$ of the harmonic oscillator is

$$T = 2\pi/\omega = N\Delta t \tag{9.7.4}$$

The boundary conditions are given by

$$X(0) = C_1 = \cos(0) = 1 \text{ and } V(1) = C_2 = -\sin(\pi/6) = -0.5$$
$$V(0) = C_3 = -\sin(0) = 0 \text{ and } X(1) = C_4 = \cos(\pi/6) = (3)^{1/2}/2 = 0.8660$$
(9.7.5a, b, c, d)

Table 9.3 gives the simulation of the hyperincursive discrete equations (9.6.1a, b) of the harmonic oscillator.

NB: Let us remark that this hyperincursive discrete harmonic oscillator represents alternatively the values of the two incursive harmonic oscillators, given at Table 9.1a, b, with the following correspondence:

$$X_1(k) = X(2k),\ V_1(k) = V(2k+1), X_2(k) = X(2k+1),\ V_2(k) = V(2k) \tag{9.7.6}$$

**Table 9.3** Numerical simulation of the Eqs. (9.6.1a, b)

| Hyperincursive Discrete harmonic oscillator | | | | | | Analytical solution | |
|---|---|---|---|---|---|---|---|
| $N$ | $H$ | $k$ | $X(k)$ | $V(k)$ | $E(k)$ | $X(k) = \cos(2k\pi/N)$ | $V(k) = -\sin(2k\pi/N)$ |
| 12 | 0.5 | 0 | 1.0000 | 0.0000 | 1.0 | $\cos(0) = 1$ | $-\sin(0) = 0$ |
| | | 1 | 0.8660 | −0.5000 | 1.0 | $\cos(2\pi/12) = \sqrt{3}/2$ | $-\sin(2\pi/12) = -1/2$ |
| | | 2 | 0.5000 | −0.8660 | 1.0 | $\cos(4\pi/12) = 1/2$ | $-\sin(4\pi/12) = -\sqrt{3}/2$ |
| | | 3 | 0.0000 | −1.0000 | 1.0 | $\cos(6\pi/12) = 0$ | $-\sin(6\pi/12) = -1$ |
| | | 4 | −0.5000 | −0.8660 | 1.0 | $\cos(8\pi/12) = -1/2$ | $-\sin(8\pi/12) = -\sqrt{3}/2$ |
| | | 5 | −0.8660 | −0.5000 | 1.0 | $\cos(10\pi/12) = -\sqrt{3}/2$ | $-\sin(10\pi/12) = -1/2$ |
| | | 6 | −1.0000 | 0.0000 | 1.0 | $\cos(12\pi/12) = -1$ | $-\sin(12\pi/12) = 0$ |
| | | 7 | −0.8660 | 0.5000 | 1.0 | $\cos(14\pi/12) = -\sqrt{3}/2$ | $-\sin(14\pi/12) = 1/2$ |
| | | 8 | −0.5000 | 0.8660 | 1.0 | $\cos(16\pi/12) = -1/2$ | $-\sin(16\pi/12) = \sqrt{3}/2$ |
| | | 9 | 0.0000 | 1.0000 | 1.0 | $\cos(18\pi/12) = 0$ | $-\sin(18\pi/12) = 1$ |
| | | 10 | 0.5000 | 0.8660 | 1.0 | $\cos(20\pi/12) = 1/2$ | $-\sin(20\pi/12) = \sqrt{3}/2$ |
| | | 11 | 0.8660 | 0.5000 | 1.0 | $\cos(22\pi/12) = \sqrt{3}/2$ | $-\sin(22\pi/12) = 1/2$ |
| | | 12 | 1.0000 | 0.0000 | 1.0 | $\cos(24\pi/12) = 1$ | $-\sin(24\pi/12) = 0$ |
| | | 13 | 0.8660 | −0.5000 | 1.0 | $\cos(26\pi/12) = \sqrt{3}/2$ | $-\sin(26\pi/12) = -1/2$ |

Secondly, Figs. 9.1, 9.2, 9.3, 9.4, 9.5 and 9.6 give the simulations of the hyperincursive discrete harmonic oscillator from Eqs. 9.2.16 a, b, with $N = 3$, 4, 6, 12, 24 and 48 time steps.

The figures of the simulations of the hyperincursive discrete harmonic oscillator show the stability and the precision of the algorithm for values of time steps $N = 3$, 4, 6, 12, 24 and 48.

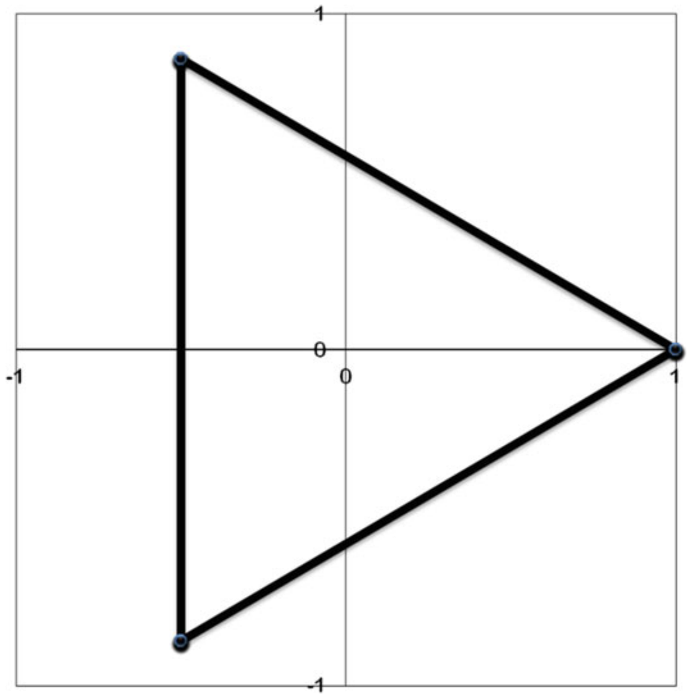


**Fig. 9.1** Simulation of the Eqs. 9.6.1a, b of the hyperincursive discrete harmonic oscillator with $N = 3$ time steps. The horizontal axis gives the position $X(k)$ and the vertical axis gives the velocity $V(k)$ of the oscillator

The representation of the harmonic oscillator tends to a circle when the number of time steps increases.

In a recent paper [8], I introduced the concept of deterministic anticipation. The general case of the discrete harmonic oscillator is taken as a typical example of a discrete deterministic anticipation given by the hyperincursive discrete oscillator that is separable into two incursive discrete oscillators. The hyperincursive oscillator shows a conservation of energy. The incursive oscillators do not show such a conservation of energy but show a deterministic anticipation. It is proposed to add, to the energy equation, a forward energy depending on the positive discrete time, $+H$, for the first incursive oscillator, and a backward energy depending on the negative discrete time, $-H$. The figures of the simulations of the hyperincursive discrete harmonic oscillation show the stability of the oscillator and the high precision of the numerical computed values, even for very small values of time steps.

In the next section, we will present a new derivation of recursive discrete harmonic oscillator based on a rotation of the incursive discrete harmonic oscillator.

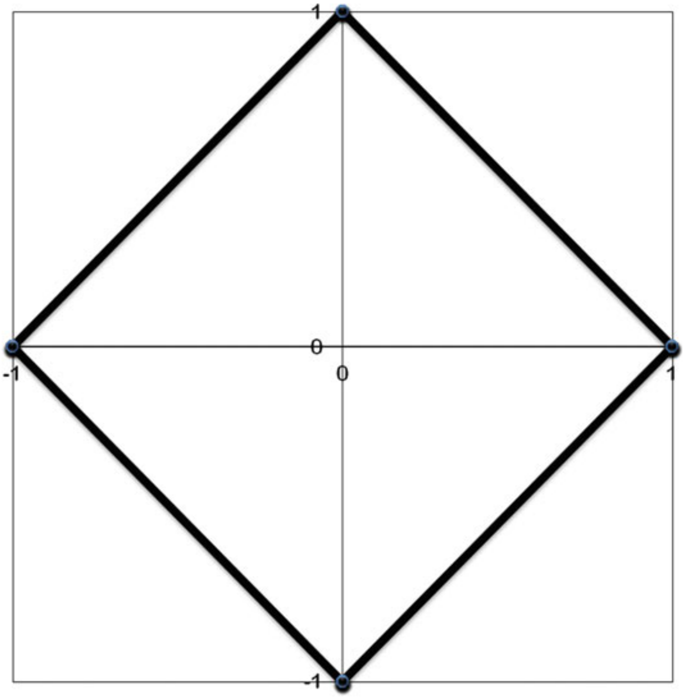


**Fig. 9.2** Continuation of Fig. 9.1 with $N = 4$ time steps

## 9.8 Rotation of the Incursive Harmonic Oscillators to Recursive Discrete Harmonic Oscillators

In the expression of the constant of motion of the first incursive harmonic oscillator, a rotation on the position and velocity variables gives rise to a pure quadratic expression of the constant of motion, similarly to the constant of energy of the classical continuous harmonic oscillator [33, 34].

The constant of motion (9.4.1a)

$$X_1(k)X_1(k) + HX_1(k)V_1(k) + V_1(k)V_1(k) = K_1 \tag{9.4.1a}$$

is an expression of a quadratic curve

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{9.8.1}$$

with

$$A = 1, B = H, C = 1, D = 0, E = 0, F = -K_1$$

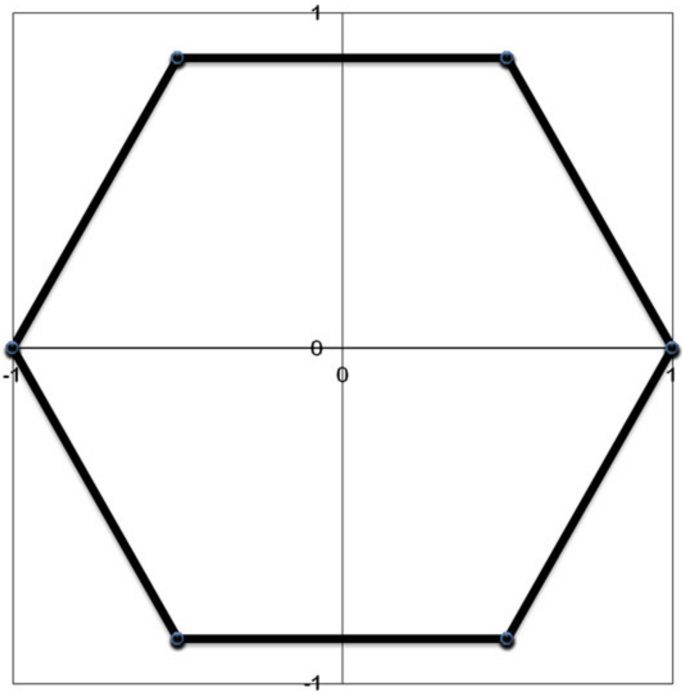


**Fig. 9.3** Continuation of Fig. 9.2 with $N = 6$ time steps

$$x = X_1(k), y = V_1(k) \tag{9.8.2}$$

The quantity

$$\Delta = B^2 - 4AC = INV \tag{9.8.3}$$

is an invariant under rotations and is known as the discriminant of Eq. (9.5.1).

The discriminant of the constant of motion is given by

$$\Delta = B^2 - 4AC = H^2 - 4 < 0 \tag{9.8.4}$$

which defines an ellipse.

NB: This inequality gives the maximum value of the discrete interval of time

$$H = \omega \Delta t < 2 \tag{9.8.5}$$

and this is exactly the maximum value for the discrete harmonic oscillator:

$$H = 2\sin(\pi/N) \tag{9.8.6}$$

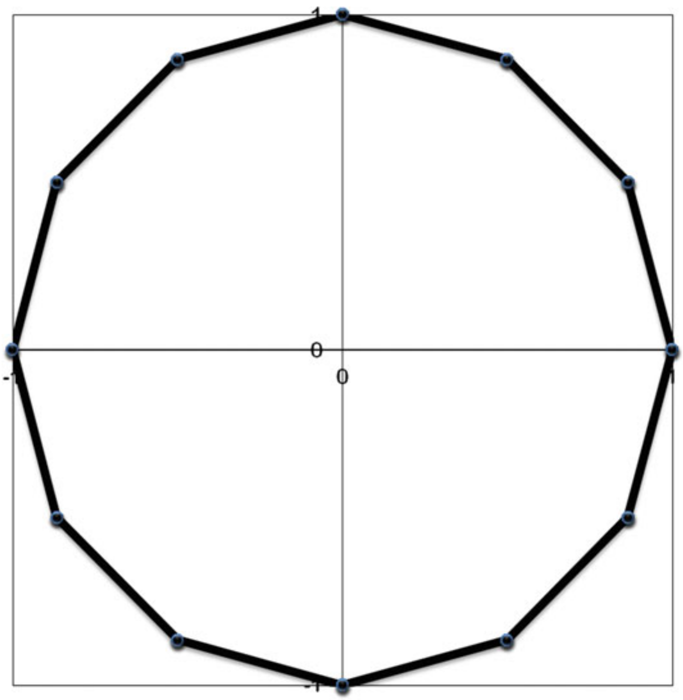


**Fig. 9.4** Continuation of Fig. 9.3 with $N = 12$ time steps. This case corresponds to the numerical values given in Table 9.3

The equations for the rotation are given by

$$X_1(k) = \cos(\theta)u_1(k) - \sin(\theta)v_1$$
$$V_1(k) = \sin(\theta)u_1(k) + \cos(\theta)v_1 \quad (9.8.7a, b)$$

With $A = C$, the angle $\theta$ is given by

$$\theta = \pi/4, \quad (9.8.8a)$$

so

$$\cos(\pi/4) = 2^{-1/2} = \rho \quad (9.8.8b)$$

and

$$\sin(\pi/4) = 2^{-1/2} = \rho \quad (9.8.8c)$$

With the Eqs. (9.8.8b, 9.8.8c) the Eqs. (9.8.7a, b) of the rotation transformed to

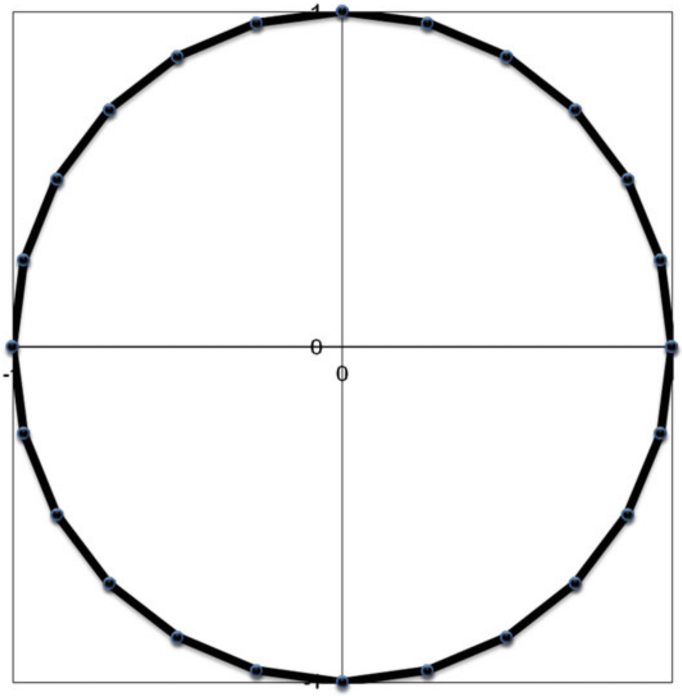

**Fig. 9.5** Continuation of Fig. 9.4 with $N = 24$ time steps

$$X_1(k) = \rho(u_1(k) - v_1(k))$$
$$V_1(k) = \rho(u_1(k) + v_1(k)) \quad (9.8.9a, b)$$

So the constant of motion becomes

$$(u_1(k) - v_1(k))^2 + H(u_1(k) - v_1(k))(u_1(k) + v_1(k)) + (u_1(k) + v_1(k))^2 = 2K_1$$
$$u_1^2(k) + v_1^2(k) - 2u_1(k)v_1(k) + Hu_1^2(k) - Hv_1^2(k) + u_1^2(k) + v_1^2(k) + 2u_1(k)v_1(k) = 2K_1$$
$$u_1^2(k) + v_1^2(k) + H[u_1^2(k) - v_1^2(k)]/2 = K_1(k) = K_1 \quad (9.8.10a)$$

For the second incursion, the constant of motion is obtained by inversion the sign of $H$:

$$u_2^2(k) + v_2^2(k) - H[u_2^2(k) - v_2^2(k)]/2 = K_2(k) = K_2 \quad (9.8.10b)$$

that is also a pure quadratic function.

Now let us give the discrete equations of the first oscillator

Let us make the rotation to the first incursive oscillator

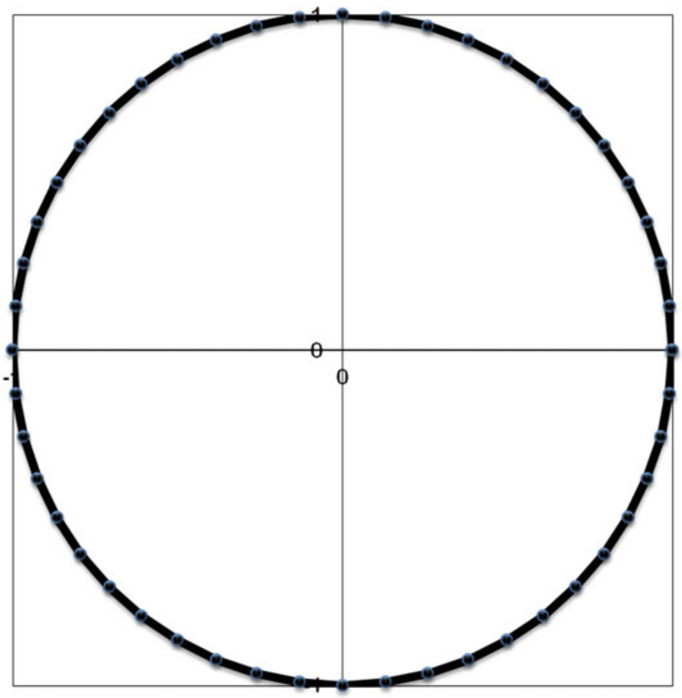


**Fig. 9.6** Continuation of Fig. 9.5 with $N = 48$ time steps

$$X_1(k+1) = X_1(k) + HV_1(k)$$
$$V_1(k+1) = V_1(k) - HX_1(k+1) = V_1(k) - HX_1(k) - H^2V_1(k)$$
$$\rho(u_1(k+1) - v_1(k+1)) = \rho(u_1(k) - v_1(k)) + H\rho(u_1(k) + v_1(k))$$
$$\rho(u_1(k+1) + v_1(k+1)) = \rho(u_1(k) + v_1(k)) - H\rho(u_1(k) - v_1(k)) - H^2\rho(u_1(k) + v_1(k)) \quad (9.3.4a, b)$$

Let us add the two equations

$$2\rho u_1(k+1) = 2\rho u_1(k) + 2H\rho v_1(k) - H^2\rho(u_1(k) + v_1(k))$$

and after division by $2\rho$,

we obtain the first rotated equation of the first incursive oscillator:

$$u_1(k+1) = u_1(k) + Hv_1(k) - H^2(u_1(k) + v_1(k))/2 \quad (9.8.11a)$$

Let us subtract the two equations

$$-2\rho v_1(k+1) = -2\rho v_1(k) + 2H\rho u_1(k) + H^2\rho(u_1(k) + v_1(k))$$

and after division by $-2\rho$,

we obtain the second rotated equation of the first incursive oscillator:

$$v_1(k+1) = v_1(k) - Hu_1(k) - H^2(u_1(k) + v_1(k))/2 \tag{9.8.11b}$$

With a similar rotation, the two equations of the second incursive oscillator

$$V_2(k+1) = V_2(k) - HX_2(k)$$
$$X_2(k+1) = X_2(k) + HV_2(k+1) \tag{9.3.5a, b}$$

are transformed to

$$v_2(k+1) = v_2(k) + Hu_2(k) - H^2(u_2(k) + v_2(k))/2 \tag{9.8.12a}$$

$$u_2(k+1) = u_2(k) - Hv_2(k) - H^2(u_2(k) + v_2(k))/2 \tag{9.8.12b}$$

These equations are the same as the equations of the first oscillator by inversion of the sign of $H$.

In conclusion, the 4 recursive equations of the discrete harmonic oscillator are given by [33]

$$u_1(k+1) = u_1(k) + Hv_1(k) - H^2(u_1(k) + v_1(k))/2 \tag{9.8.13a}$$

$$v_1(k+1) = v_1(k) - Hu_1(k) - H^2(u_1(k) + v_1(k))/2 \tag{9.8.13b}$$

$$u_2(k+1) = u_2(k) - Hv_2(k) - H^2(u_2(k) + v_2(k))/2 \tag{9.8.14a}$$

$$v_2(k+1) = v_2(k) + Hu_2(k) - H^2(u_2(k) + v_2(k))/2 \tag{9.8.14b}$$

with the corresponding constant of motion

$$u_1^2(k) + v_1^2(k) + H[u_1^2(k) - v_1^2(k)]/2 = K_1(k) = K_1 \tag{9.8.15}$$

$$u_2^2(k) + v_2^2(k) - H[u_2^2(k) - v_2^2(k)]/2 = K_2(k) = K_2 \tag{9.8.16}$$

This result is fundamental because it gives an explanation of the effect of the discretization of the time in discrete physics.

We have shown that the temporal discretization of the harmonic oscillator produces a rotation which gives rise to an anticipative effect with a reversible serial computation.

The information obtained from the discrete equations is richer than obtained by continuous physics.

## 9.9 The Space and Time-Symmetric Second-Order Hyperincursive Discrete Klein–Gordon Equation

In 1926, Klein [19] and Gordon [20] published independently their famous equation, called the Klein–Gordon equation.

The Klein–Gordon equation with the function $\varphi = \varphi(\mathbf{r}, t)$ in three spatial dimensions $\mathbf{r} = (x, y, z)$ and time $t$ is given by

$$\hbar^2 \partial^2 \varphi(r, t)/\partial t^2 = -\hbar^2 c^2 \nabla^2 \varphi(r, t) + m^2 c^4 \varphi(r, t) \tag{9.9.1}$$

or, in the explicit form of the nabla operator $\nabla$,

$$-\hbar^2 \partial^2 \varphi/\partial t^2 = -\hbar^2 c^2 \partial^2 \varphi/\partial x^2 - \hbar^2 c^2 \partial^2 \varphi/\partial y^2 - \hbar^2 c^2 \partial^2 \varphi/\partial z^2 + m^2 c^4 \varphi \tag{9.9.2}$$

where $\hbar$ is the constant of Plank, $c$ is the speed of light, and $m$ the mass.

As we will consider the discrete Klein–Gordon equation, we make the following usual change of variables

$$q(\mathbf{r}, t) = \varphi(\mathbf{r}, t) \tag{9.9.3}$$

$$a = \omega = mc^2/\hbar \tag{9.9.4}$$

where $\omega$ is a frequency, so the Klein–Gordon equation (9.9.2) becomes

$$\partial^2 q(\mathbf{r}, t)/\partial t^2 = +c^2 \partial^2 q(\mathbf{r}, t)/\partial x^2 + c^2 \partial^2 q(\mathbf{r}, t)/\partial y^2 + c^2 \partial^2 q(\mathbf{r}, t)/\partial z^2 - a^2 q(\mathbf{r}, t) \tag{9.9.5}$$

From the Klein–Gordon equation (9.5.5), the second-order hyperincursive discrete Klein–Gordon equation [31] is given by

$$\begin{aligned} &q(x, y, z, t + 2\Delta t) - 2q(x, y, z, t) + q(x, y, z, t - 2\Delta t) = \\ &\quad + B^2 \left[ q(x + 2\Delta x, y, z, t) - 2q(x, y, z, t) + q(x - 2\Delta x, y, z, t) \right] \\ &\quad + C^2 \left[ q(x, y + 2\Delta y, z, t) - 2q(x, y, z, t) + q(x, y - 2\Delta y, z, t) \right] \\ &\quad + D^2 \left[ q(x, y, z + 2\Delta z, t) - 2q(x, y, z, t) + q(x, y, z - 2\Delta z, t) \right] - A^2 q(x, y, z, t) \end{aligned} \tag{9.9.6}$$

where the following parameters $A$, $B$, $C$, and, $D$,

$$A = a(2\Delta t), B = c(2\Delta t)/(2\Delta x), C = c(2\Delta t)/(2\Delta y), D = c(2\Delta t)/(2\Delta z) \tag{9.9.7}$$

depend on the discrete interval of time $\Delta t$, and the discrete intervals of space, $\Delta x$, $\Delta y$, $\Delta z$, respectively. As usually made in computer science, let us now introduce the discrete time $t_k$, and the discrete spaces $x_l$, $y_{m,} z_n$, as follows

$$t_k = t_0 + k\Delta t,\ \ k = 0, 1, 2, \ldots, \tag{9.9.8}$$

where k is the integer time increment, and

$$x_l = x_0 + l\Delta x, l = 0, 1, 2, \ldots, y_m = y_0 + m\Delta y, m = 0, 1, 2, \ldots, z_n = z_0 + n\Delta z, n = 0, 1, 2, \ldots \tag{9.9.9}$$

where $l, m, n,$ are the integer space increments. So, with these time and space increments, the second-order hyperincursive discrete Klein–Gordon equation (9.2.6) becomes

$$\begin{aligned} &q(l, m, n, k + 2) - 2q(l, m, n, k) + q(l, m, n, k - 2) = \\ &\quad + B^2\big[q(l + 2, m, n, k) - 2q(l, m, n, k) + q(l - 2, m, n, k)\big] \\ &\quad + C^2\big[q(l, m + 2, n, k) - 2q(l, m, n, k) + q(l, m - 2, n, k)\big] \\ &\quad + D^2\big[q(l, m, n + 2, k) - 2q(l, m, n, k) + q(l, m, n - 2, k)\big] - A^2 q(l, m, n, k) \end{aligned} \tag{9.9.10}$$

This equation without spatial components, corresponding to a particle at rest, is similar to the harmonic oscillator. For a particle at rest, the Klein–Gordon equation (9.9.5), with the function $q(t)$ depending only on the time variable, is given by

$$\partial^2 q(t)/\partial t^2 = -a^2 q(t) \tag{9.9.11}$$

with the frequency $a = \omega = mc^2/\hbar$, given by the Eq. (9.9.4). This Eq. (9.9.11) is formally similar to the equation of the harmonic oscillator for which $q(t)$ would represent the position $a = \omega = mc^2/\hbar$ and $\partial q(t)/\partial t$ would represent the velocity $v(t) = \partial x(t)/\partial t$, as shown in Sect. 9.2. So, with only the temporal component, the second-order hyperincursive discrete Klein–Gordon equation (9.9.10) becomes

$$q(k + 2) - 2q(k) + q(k - 2) = -A^2 q(k) \tag{9.9.12}$$

that is similar to the second-order hyperincursive equation of the harmonic oscillator.

This hyperincursive equation (9.9.12) is separable into a first discrete incursive oscillator depending on two functions defined by $q_1(k)$, $q_2(k)$, and a second incursive oscillator depending on two other functions defined by $q_3(k)$, $q_4(k)$, given by first-order discrete equations.

So the first incursive equations are given by:

$$q_1(2k) = q_1(2k - 2) + Aq_2(2k - 1)$$

$$q_2(2k+1) = q_2(2k-1) - Aq_1(2k) \quad (9.9.13a, b)$$

where $q_1(2k)$ is defined on the even steps of the time, and $q_2(2k+1)$ is defined on the odd steps of the time. And the second incursive equations are given by:

$$\begin{aligned} q_3(2k) &= q_3(2k-2) - Aq_4(2k-1) \\ q_4(2k+1) &= q_4(2k-1) + Aq_3(2k) \end{aligned} \quad (9.9.14a, b)$$

where $q_3(2k)$ is defined of the even steps of the time, and $q_4(2k+1)$ is defined on the odd steps of the time. The second incursive system is the time reverse of the first incursive system in making the time inversion $\boldsymbol{T}$

$$\boldsymbol{T} : \Delta t \to -\Delta t \quad (9.9.15)$$

this gives an oscillator and its anti-oscillator.

In the next sections, we will present the bifurcation of this Eq. (9.9.10) to the 4 hyperincursive discrete Majorana real equations which bifurcate to the 4 hyperincursive discrete Dirac equations.

## 9.10 The Hyperincursive Discrete Majorana Equations and Continuous Majorana Real 4-Spinors

We deduced the following 4 hyperincursive discrete Majorana equations, depending on the discrete Majorana functions $\tilde{q}_j = \tilde{q}_j(x, y, z, t) = \tilde{q}_j(l, m, n, k), j = 1, 2, 3, 4,$ from the hyperincursive Klein–Gordon equation, [28–31],

$$\begin{aligned}
\tilde{q}_1(l,m,n,k+1) = \tilde{q}_1(l,m,n,k-1) &+ \widetilde{B}\big[\tilde{q}_4(l+1,m,n,k) - \tilde{q}_4(l-1,m,n,k)\big] \\
&- \widetilde{C}\big[\tilde{q}_1(l,m+1,n,k) - \tilde{q}_1(l,m-1,n,k)\big] + \widetilde{D}\big[\tilde{q}_3(l,m,n+1,k) - \tilde{q}_3(l,m,n-1,k)\big] \\
&- \widetilde{A}\tilde{q}_4(l,m,n,k) \\
\tilde{q}_2(l,m,n,k+1) = \tilde{q}_2(l,m,n,k-1) &+ \widetilde{B}\big[\tilde{q}_3(l+1,m,n,k) - \tilde{q}_3(l-1,m,n,k)\big] \\
&- \widetilde{C}\big[\tilde{q}_2(l,m+1,n,k) - \tilde{q}_2(l,m-1,n,k)\big] - \widetilde{D}\big[\tilde{q}_4(l,m,n+1,k) - \tilde{q}_4(l,m,n-1,k)\big] \\
&+ \widetilde{A}\tilde{q}_3(l,m,n,k) \\
\tilde{q}_3(l,m,n,k+1) = \tilde{q}_3(l,m,n,k-1) &+ \widetilde{B}\big[\tilde{q}_2(l+1,m,n,k) - \tilde{q}_2(l-1,m,n,k)\big] \\
&+ \widetilde{C}\big[\tilde{q}_3(l,m+1,n,k) - \tilde{q}_3(l,m-1,n,k)\big] + \widetilde{D}\big[\tilde{q}_1(l,m,n+1,k) - \tilde{q}_1(l,m,n-1,k)\big] \\
&- \widetilde{A}\tilde{q}_2(l,m,n,k) \\
\tilde{q}_4(l,m,n,k+1) = \tilde{q}_4(l,m,n,k-1) &+ \widetilde{B}\big[\tilde{q}_1(l+1,m,n,k) - \tilde{q}_1(l-1,m,n,k)\big] \\
&+ \widetilde{C}\big[\tilde{q}_4(l,m+1,n,k) - \tilde{q}_4(l,m-1,n,k)\big] - \widetilde{D}\big[\tilde{q}_2(l,m,n+1,k) - \tilde{q}_2(l,m,n-1,k)\big] \\
&+ \widetilde{A}\tilde{q}_1(l,m,n,k)
\end{aligned} \quad (9.10.1a, b, c, d)$$

with

$$\widetilde{A} = A = a(2\Delta t), \widetilde{B} = B = c\Delta t/\Delta x, \widetilde{C} = C = c\Delta t/\Delta y, \widetilde{D} = D = c\Delta t/\Delta z \quad (9.10.2a, b, c, d)$$

where $\Delta t$ and $\Delta x$, $\Delta y$, $\Delta z$ are the discrete intervals of time and space, respectively.

These 4 discrete equations (9.10.1a, b, c, d) can be transformed to partial differential equations.

Indeed, the discrete functions $\tilde{q}_j(x, y, z, t) = \tilde{q}_j(\mathbf{r}, t), j = 1, 2, 3, 4$ tend to the continuous functions

$\widetilde{\Psi}_j(x, y, z, t) = \widetilde{\Psi}_j(\boldsymbol{r}, t)$, when the discrete space and time intervals tend to zero. At the limit,

$$\widetilde{\Psi}_j = \widetilde{\Psi}_j(\boldsymbol{r}, t) = \lim_{\Delta \boldsymbol{r} \to 0, \Delta t \to 0} \tilde{q}_j(\mathbf{r}, t), j = 1, 2, 3, 4 \quad (9.10.3)$$

So, with the Majorana continuous functions

$$\widetilde{\Psi}_j = \widetilde{\Psi}_j(x, y, z, t), j = 1, 2, 3, 4, \quad (9.10.4)$$

Equations (9.10.1a, b, c, d) are transformed to the following 4 first-order partial differential equations

$$+\partial\tilde{\Psi}_1/\partial t = +c\partial\tilde{\Psi}_4/\partial x - c\partial\tilde{\Psi}_1/\partial y + c\partial\tilde{\Psi}_3/\partial z - (mc^2/\hbar)\tilde{\Psi}_4 \quad (9.10.5a)$$

$$+\partial\tilde{\Psi}_2/\partial t = +c\partial\tilde{\Psi}_3/\partial x - c\partial\tilde{\Psi}_2/\partial y - c\partial\tilde{\Psi}_4/\partial z + (mc^2/\hbar)\tilde{\Psi}_3 \quad (9.10.5b)$$

$$+\partial\tilde{\Psi}_3/\partial t = +c\partial\tilde{\Psi}_2/\partial x + c\partial\tilde{\Psi}_3/\partial y + c\partial\tilde{\Psi}_1/\partial z - (mc^2/\hbar)\tilde{\Psi}_2 \quad (9.10.5c)$$

$$+\partial\tilde{\Psi}_4/\partial t = +c\partial\tilde{\Psi}_1/\partial x + c\partial\tilde{\Psi}_4/\partial y - c\partial\tilde{\Psi}_2/\partial z + (mc^2/\hbar)\tilde{\Psi}_1 \quad (9.10.5d)$$

which are identical to the original Majorana equations [25], e.g., Eqs. (4a, b, c, d) in Pessa [26].

In 1937, Ettore Majorana published this last paper, before his mysterious disappearance.

## 9.11 The Bifurcation of the Majorana Real 4-Spinors to the Dirac Real 8-Spinors

Recently, we demonstrated that the Majorana 4-spinors equations bifurcate simply to the Dirac real 8-spinors equations [29, 30, 32]. First, let us consider the inverse parity space, in inversing the sign of the space variables in the Majorana equations (9.6.3a, b, c, d),

$$+\partial\tilde{\Psi}_1/\partial t = -c\partial\tilde{\Psi}_4/\partial x + c\partial\tilde{\Psi}_1/\partial y - c\partial\tilde{\Psi}_3/\partial z - (mc^2/\hbar)\tilde{\Psi}_4$$
$$+\partial\tilde{\Psi}_2/\partial t = -c\partial\tilde{\Psi}_3/\partial x + c\partial\tilde{\Psi}_2/\partial y + c\partial\tilde{\Psi}_4/\partial z + (mc^2/\hbar)\tilde{\Psi}_3$$
$$+\partial\tilde{\Psi}_3/\partial t = -c\partial\tilde{\Psi}_2/\partial x - c\partial\tilde{\Psi}_3/\partial y - c\partial\tilde{\Psi}_1/\partial z - (mc^2/\hbar)\tilde{\Psi}_2$$
$$+\partial\tilde{\Psi}_4/\partial t = -c\partial\tilde{\Psi}_1/\partial x - c\partial\tilde{\Psi}_4/\partial y + c\partial\tilde{\Psi}_2/\partial z + (mc^2/\hbar)\tilde{\Psi}_1 \quad (9.11.1a, b, c, d)$$

In defining the 2-spinors real functions,

$$\varphi_a = \begin{pmatrix} \widetilde{\Psi}_1 \\ \widetilde{\Psi}_2 \end{pmatrix}, \varphi_b = \begin{pmatrix} \widetilde{\Psi}_3 \\ \widetilde{\Psi}_4 \end{pmatrix}, \quad (9.11.2a, b)$$

the two Eqs. (9.11.1a, b) and (9.11.1c, d) are transformed to the two 2-spinors real equations

$$+\partial\varphi_a/\partial t = -c\sigma_1\partial\varphi_b/\partial x + c\sigma_0\partial\varphi_a/\partial y - c\sigma_3\partial\varphi_b/\partial z + (mc^2/\hbar)\sigma_2\varphi_b \quad (9.11.3a)$$

$$+\partial\varphi_b/\partial t = -c\sigma_1\partial\varphi_a/\partial x - c\sigma_0\partial\varphi_b/\partial y - c\sigma_3\partial\varphi_a/\partial z + (mc^2/\hbar)\sigma_2\varphi_a \quad (9.11.3b)$$

where the real 2-spinors matrices $\sigma_1, \sigma_2, \sigma_3$, are defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (9.11.4a, b, c)$$

and 2-Identity

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 \quad (9.11.4d)$$

With the inversion between $\sigma_0$ and $\sigma_2$, in introducing the tensor product by $-\sigma_2$, the functions $\widetilde{\Psi}_j$

$$\widetilde{\Psi}_j = \begin{pmatrix} \Psi_{j,1} \\ \Psi_{j,2} \end{pmatrix}, j = 1, 2, 3, 4, \quad (9.11.5)$$

bifurcate to two functions

$$-\sigma_2\Psi_j = -\sigma_2\begin{pmatrix} \Psi_{j,1} \\ \Psi_{j,2} \end{pmatrix} = -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \Psi_{j,1} \\ \Psi_{j,2} \end{pmatrix} = \begin{pmatrix} +\Psi_{j,2} \\ -\Psi_{j,1} \end{pmatrix}, j = 1, 2, 3, 4 \quad (9.11.6)$$

So the Majorana real 4-spinors equation bifurcates into the Dirac real 8-spinors equations

$$+\partial\Psi_{1,1}/\partial t = -c\partial\Psi_{4,1}/\partial x - c\partial\Psi_{4,2}/\partial y - c\partial\Psi_{3,1}/\partial z + (mc^2/\hbar)\Psi_{1,2} \quad (9.11.7a)$$

$$+\partial\Psi_{2,1}/\partial t = -c\partial\Psi_{3,1}/\partial x + c\partial\Psi_{3,2}/\partial y + c\partial\Psi_{4,1}/\partial z + (mc^2/\hbar)\Psi_{2,2} \quad (9.11.7b)$$

$$+\partial\Psi_{3,1}/\partial t = -c\partial\Psi_{2,1}/\partial x - c\partial\Psi_{2,2}/\partial y - c\partial\Psi_{1,1}/\partial z - (mc^2/\hbar)\Psi_{3,2} \quad (9.11.7c)$$

$$+\partial\Psi_{4,1}/\partial t = -c\partial\Psi_{1,1}/\partial x + c\partial\Psi_{1,2}/\partial y + c\partial\Psi_{2,1}/\partial z - (mc^2/\hbar)\Psi_{4,2} \quad (9.11.7d)$$

$$+\partial\Psi_{1,2}/\partial t = -c\partial\Psi_{4,2}/\partial x + c\partial\Psi_{4,1}/\partial y - c\partial\Psi_{3,2}/\partial z - (mc^2/\hbar)\Psi_{1,1} \quad (9.11.8a)$$

$$+\partial\Psi_{2,2}/\partial t = -c\partial\Psi_{3,2}/\partial x - c\partial\Psi_{3,1}/\partial y + c\partial\Psi_{4,2}/\partial z - (mc^2/\hbar)\Psi_{2,1} \quad (9.11.8b)$$

$$+\partial\Psi_{3,2}/\partial t = -c\partial\Psi_{2,2}/\partial x + c\partial\Psi_{2,1}/\partial y - c\partial\Psi_{1,2}/\partial z + (mc^2/\hbar)\Psi_{3,1} \quad (9.11.8c)$$

$$+\partial\Psi_{4,2}/\partial t = -c\partial\Psi_{1,2}/\partial x - c\partial\Psi_{1,1}/\partial y + c\partial\Psi_{2,2}/\partial z + (mc^2/\hbar)\Psi_{4,1} \quad (9.11.8d)$$

These 8 real first-order partial differential equations represent real 8-spinors equations that are similar to the original Dirac [21, 22] complex 4-spinors equations.

In defining the wave function

$$\Psi_j(x, y, z, t) = \Psi_j = \Psi_{j,1} + i\Psi_{j,2}, j = 1, 2, 3, 4, \quad (9.11.9)$$

with the imaginary number $i$, we obtain the original Dirac equation as a complex 4-spinors equation

$$+\partial\Psi_1/\partial t = -c\partial\Psi_4/\partial x + ic\partial\Psi_4/\partial y - c\partial\Psi_3/\partial z - i(mc^2/\hbar)\Psi_1 \quad (9.11.10a)$$

$$+\partial\Psi_2/\partial t = -c\partial\Psi_3/\partial x - ic\partial\Psi_3/\partial y + c\partial\Psi_4/\partial z - i(mc^2/\hbar)\Psi_2 \quad (9.11.10b)$$

$$+\partial\Psi_3/\partial t = -c\partial\Psi_2/\partial x + ic\partial\Psi_2/\partial y - c\partial\Psi_1/\partial z + i(mc^2/\hbar)\Psi_3 \quad (9.11.10c)$$

$$+\partial\Psi_4/\partial t = -c\partial\Psi_1/\partial x - ic\partial\Psi_1/\partial y + c\partial\Psi_2/\partial z + i(mc^2/\hbar)\Psi_4 \quad (9.11.10d)$$

Following our recent papers [35, 36], in the non-relativistic limit $p \ll mc$, the particles are at rest, with a momentum $p \cong 0$. Let us consider the following Dirac 2-spinors

$$\widehat{\Psi}(t) = \begin{pmatrix} \Psi_1(t) \\ \Psi_4(t) \end{pmatrix}, \quad (9.11.11)$$

for which the temporal non-relativistic Dirac equation is given by

$$\partial_t \tilde{\Psi}(t) = -i\left(mc^2/\hbar\right)\sigma_z \tilde{\Psi}(t) \tag{9.11.12}$$

where $\partial_t = \partial/\partial t$, and $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, is a Pauli matrix.

The analytical solution of the non-relativistic Dirac equation (9.11.12) is given by

$$\tilde{\Psi}(t) = \cos\left(mc^2 t/\hbar\right)\tilde{\Psi}(0) - i\sin\left(mc^2 t/\hbar\right)\sigma_z \tilde{\Psi}(0) \tag{9.11.13}$$

or in explicit form

$$\Psi_1(t) = \cos\left(mc^2 t/\hbar\right)\Psi_1(0) - i\sin\left(mc^2 t/\hbar\right)\Psi_1(0) \tag{9.11.14a}$$

$$\Psi_4(t) = \cos\left(mc^2 t/\hbar\right)\Psi_4(0) + i\sin\left(mc^2 t/\hbar\right)\Psi_4(0) \tag{9.11.14b}$$

We give, in the next section, the computing hyperincursive equations of the original Dirac complex 4-spinors equations [29].

## 9.12 The 4 Hyperincursive Discrete Dirac 4-Spinors Equations

Recently, we have presented the 4 hyperincursive discrete Dirac complex equations [29].

Let us define the discrete Dirac wave functions

$$Q_j(l, m, n, k) = Q_{j,1} + iQ_{j,2}, j = 1, 2, 3, 4, \tag{9.12.1a}$$

corresponding to the Dirac continuous wave functions (9.11.9), where $i$ is the imaginary number.

The 4 hyperincursive discrete Dirac equations of the discrete wave functions are then given by

$$\begin{aligned} Q_1(l, m, n, k+1) = Q_1(l, m, n, k-1) &- B\left[Q_4(l+1, m, n, k) - Q_4(l-1, m, n, k)\right] \\ &+ iC\left[Q_4(l, m+1, n, k) - Q_4(l, m-1, n, k)\right] - D\left[Q_3(l, m, n+1, k) - Q_3(l, m, n-1, k)\right] \\ &- iAQ_1(l, m, n, k) \end{aligned} \tag{9.12.2a}$$

$$\begin{aligned} Q_2(l, m, n, k+1) = Q_2(l, m, n, k-1) &- B\left[Q_3(l+1, m, n, k) - Q_3(l-1, m, n, k)\right] \\ &- iC\left[Q_3(l, m+1, n, k) - Q_3(l, m-1, n, k)\right] + D\left[Q_4(l, m, n+1, k) - Q_4(l, m, n-1, k)\right] \\ &- iAQ_2(l, m, n, k) \end{aligned} \tag{9.12.2b}$$

$$Q_3(l,m,n,k+1) = Q_3(l,m,n,k-1) - B\big[Q_2(l+1,m,n,k) - Q_2(l-1,m,n,k)\big] + iC\big[Q_2(l,m+1,n,k) - Q_2(l,m-1,n,k)\big] - D\big[Q_1(l,m,n+1,k) - Q_1(l,m,n-1,k)\big] + iAQ_3(l,m,n,k) \quad (9.12.2c)$$

$$Q_4(l,m,n,k+1) = Q_4(l,m,n,k-1) - B\big[Q_1(l+1,m,n,k) - Q_1(l-1,m,n,k)\big] - iC\big[Q_1(l,m+1,n,k) - Q_1(l,m-1,n,k)\big] + D\big[q_2(l,m,n+1,k) - Q_2(l,m,n-1,k)\big] + iAQ_4(l,m,n,k) \quad (9.12.2d)$$

with

$$A = 2\omega\Delta t, B = c\Delta t/\Delta x, C = c\Delta t/\Delta y, D = c\Delta t/\Delta z \quad (9.12.3)$$

where $\Delta t$ and $\Delta x$, $\Delta y$, $\Delta z$ are the discrete intervals of time and space, respectively.

## 9.13 The Hyperincursive Discrete Klein–Gordon Equation Bifurcates to the 16 Proca Equations

Let us show that there are 16 complex functions associated with this second-order hyperincursive discrete Klein–Gordon equation [29].

For a particle at rest, the Klein–Gordon equation (9.9.5), with the function $q(t)$ depending only on the time variable, is given by

$$\partial^2 q(t)/\partial t^2 = -a^2 q(t) \quad (9.13.1)$$

with the frequency, given by the Eq. (9.9.4).

This Eq. (9.13.1) is formally similar to the equation of the harmonic oscillator for which $q(t)$ would represent the position $x(t)$, and $\partial q(t)/\partial t$ would represent the velocity $v(t) = \partial x(t)/\partial t$.

So, with only the temporal component, the second-order hyperincursive discrete Klein–Gordon equation (9.9.10) becomes

$$q(k+2) - 2q(k) + q(k-2) = -A^2 q(k) \quad (9.13.2)$$

that is similar to the second-order hyperincursive discrete equation of the harmonic oscillator [36].

This hyperincursive equation (9.13.2) is separable into a first discrete incursive oscillator depending on two functions defined by $q_1(k)$, $q_2(k)$, and a second incursive oscillator depending on two other functions defined by $q_3(k)$, $q_4(k)$, given by first-order discrete equations.

So the first incursive equations are given by:

$$q_1(2k) = q_1(2k-2) + Aq_2(2k-1)$$

$$q_2(2k+1) = q_2(2k-1) - Aq_1(2k) \tag{9.13.3a, b}$$

where $q_1(2k)$ is defined of the even steps of the time, and $q_2(2k+1)$ is defined on the odd steps of the time. And the second incursive equations are given by:

$$\begin{aligned} q_3(2k) &= q_3(2k-2) - Aq_4(2k-1) \\ q_4(2k+1) &= q_4(2k-1) + Aq_3(2k) \end{aligned} \tag{9.13.4a, b}$$

where $q_3(2k)$ is defined of the even steps of the time, and $q_4(2k+1)$ is defined on the odd steps of the time. The second incursive system is the time reverse of the first incursive system in making the discrete time inversion $\boldsymbol{T}$

$$\boldsymbol{T} : \Delta t \rightarrow -\Delta t \tag{9.13.5}$$

which gives an oscillator and its anti-oscillator.

In defining the following 2 complex functions, where $i$ is the imaginary number,

$$\begin{aligned} q_{13}(2k) &= q_1(2k) + iq_3(2k) \\ q_{24}(2k+1) &= q_2(2k+1) - iq_4(2k+1) \end{aligned} \tag{9.13.6a, b}$$

the 4 real incursive equations (9.13.3a, b) and (9.13.4a, b) are transformed to 2 complex incursive equations

$$\begin{aligned} q_{13}(2k) &= q_{13}(2k-2) + Aq_{24}(2k-1) \\ q_{24}(2k+1) &= q_{24}(2k-1) - Aq_{13}(2k) \end{aligned} \tag{9.13.7a, b}$$

So the hyperincursive equation for a particle at rest shows a temporal bifurcation into an oscillatory equation and an anti-oscillatory equation.

For a moving particle, the 3 discrete space-symmetric terms in Eq. (9.9.10)

$$\begin{aligned} &q(l+2, m, n, k) - 2q(l, m, n, k) + q(l-2, m, n, k) \\ &q(l, m+2, n, k) - 2q(l, m, n, k) + q(l, m-2, n, k) \\ &q(l, m, n+2, k) - 2q(l, m, n, k) + q(l, m, n-2, k) \end{aligned}$$

are similar to the discrete time-symmetric term (9.13.2)

$$q(l, m, n, k+2) - 2q(l, m, n, k) + q(l, m, n, k-2).$$

The two complex functions (9.13.6a, b) bifurcate for even and odd steps of space $x$, giving 4 complex functions depending on 4 discrete incursive equations. These 4 complex functions bifurcate for even and odd steps of space $y$, giving 8 complex functions depending on 8 discrete incursive equations. Finally, these 8 complex functions bifurcate for even and odd steps of space $z$, giving 16 complex functions depending on 16 incursive discrete equations.

But if we consider the space variable as a set of the 3 space variables

$$\boldsymbol{r} = (x, y, z) \tag{9.13.8}$$

the two complex functions bifurcate for even and odd steps of the space variable $\boldsymbol{r} = (x, y, z)$, giving 4 complex functions depending on 4 discrete incursive equations, which correspond to a discrete parity inversion $\boldsymbol{P}$

$$\boldsymbol{P} : \Delta\boldsymbol{r} \rightarrow -\Delta\boldsymbol{r} \tag{9.13.9}$$

So, with the discrete time inversion and the parity, we define a group of 4 incursive discrete equations with 4 functions. This is in agreement with the thesis of Proca. Indeed, as demonstrated by Proca [23, 24] in 1930 and 1932, the Klein–Gordon equation admits in the general case a total of 16 functions. Classically, for the well-known Dirac equation, there are 4 complex wave functions. Proca demonstrated that there are 4 fundamental equations of 4 wave functions for the Dirac equation

$$\varphi_{r,s} \text{ for } r = 1, 2, 3, 4, \text{ and } s = 1 \tag{9.13.10}$$

and the other $3 \times 4$ other equations are similar to these 4 equations.

Proca classified the 16 equations in 4 groups of 4 functions:

I. 4 equations of the 4 functions $\varphi_{r,s}$ for $r = 1, 2, 3, 4,$ and $s = 1$
II. 4 equations of the 4 functions $\varphi_{r,s}$ for $r = 1, 2, 3, 4,$ and $s = 2$
III. 4 equations of the 4 functions $\varphi_{r,s}$ for $r = 1, 2, 3, 4,$ and $s = 3$
IV. 4 equations of the 4 functions $\varphi_{r,s}$ for $r = 1, 2, 3, 4,$ and $s = 4$

In each group, the 4 equations depend on 4 functions which are not separable except in particular cases.

In this chapter, we restricted our analysis to the first group of 4 functions in studying the case of the Majorana and Dirac equations.

## 9.14 Simulation of the Hyperincursive Discrete Quantum Majorana and Dirac Wave Equations

This last section deals with the numerical simulation of the hyperincursive discrete Majorana and Dirac wave equations depending on time and one spatial dimension (1D) and with a null mass.

The Majorana equations (9.10.5a, b, c, d) in one spatial dimension $z$ and with a null mass $m = 0$ are given by the 2 following Majorana wave equations

$$+\partial\widetilde{\Psi}_1/\partial t = +c\partial\widetilde{\Psi}_3/\partial z \tag{9.14.1a}$$

$$+\partial\widetilde{\Psi}_3/\partial t = +c\partial\widetilde{\Psi}_1/\partial z \tag{9.14.1b}$$

and

$$+\partial\widetilde{\Psi}_2/\partial t = -c\partial\widetilde{\Psi}_4/\partial z \tag{9.14.2a}$$

$$+\partial\widetilde{\Psi}_4/\partial t = -c\partial\widetilde{\Psi}_2/\partial z \tag{9.14.2b}$$

The corresponding hyperincursive discrete Majorana equations (9.10.1a, b, c, d) are given by the 2 following hyperincursive discrete wave equations

$$\tilde{q}_1(n, k+1) = \tilde{q}_1(n, k-1) + D\big[\tilde{q}_3(n+1, k) - \tilde{q}_3(n-1, k)\big] \tag{9.14.3a}$$

$$\tilde{q}_3(n, k+1) = \tilde{q}_3(n, k-1) + D\big[\tilde{q}_1(n+1, k) - \tilde{q}_1(n-1, k)\big] \tag{9.14.3b}$$

and

$$\tilde{q}_2(n, k+1) = \tilde{q}_2(n, k-1) - D\big[\tilde{q}_4(n+1, k) - \tilde{q}_4(n-1, k)\big] \tag{9.14.4a}$$

$$\tilde{q}_4(n, k+1) = \tilde{q}_4(n, k-1) - D\big[\tilde{q}_2(n+1, k) - \tilde{q}_2(n-1, k)\big] \tag{9.14.4b}$$

with

$$D = c\Delta t/\Delta z \tag{9.14.5}$$

where $\Delta t$ and $\Delta z$ are the discrete intervals of time and space, respectively.

The Dirac equations (9.11.10a, b, c, d) in one spatial dimension $z$ with a null mass $m = 0$ are given by the 2 following Dirac wave equations

$$+\partial\Psi_1/\partial t = -c\partial\Psi_3/\partial z \tag{9.14.6a}$$

$$+\partial\Psi_3/\partial t = -c\partial\Psi_1/\partial z \tag{9.14.6b}$$

and

$$+\partial\Psi_2/\partial t = +c\partial\Psi_4/\partial z \tag{9.14.7a}$$

$$+\partial\Psi_4/\partial t = +c\partial\Psi_2/\partial z \tag{9.14.7b}$$

which are similar to the Majorana wave equations (9.14.1a, b) and (9.14.2a, b), where the space variable $z$ is reversed to $-z$.

The corresponding hyperincursive discrete Dirac equations (9.12.2a, b, c, d) are given by the 2 following hyperincursive discrete wave equations

$$Q_1(n, k+1) = Q_1(n, k-1) - D\big[Q_3(n+1, k) - Q_3(n-1, k)\big] \tag{9.14.8a}$$

$$Q_3(n, k+1) = Q_3(n, k-1) - D\big[Q_1(n+1, k) - Q_1(n-1, k)\big] \tag{9.14.8b}$$

and

$$Q_2(n, k+1) = Q_2(n, k-1) + D\big[Q_4(n+1, k) - Q_4(n-1, k)\big] \tag{9.14.9a}$$

$$Q_4(n, k+1) = Q_4(n, k-1) + D\big[Q_2(n+1, k) - Q_2(n-1, k)\big] \tag{9.14.9b}$$

with

$$D = c\Delta t/\Delta z \tag{9.14.10}$$

where $\Delta t$ and $\Delta z$ are the discrete intervals of time and space, respectively.

For the numerical simulations, it is sufficient to simulate the 2 wave Eqs. (9.14.9a, b), in talking the value of

$$D = c\Delta t/\Delta z$$

and its reversed sign value

$$D = -c\Delta t/\Delta z.$$

The numerical values of $D$ is chosen as equal to

$$D = +1 \tag{9.14.11a}$$

and

$$D = -1 \tag{9.14.11b}$$

which correspond to the values of the interval of time given by

$$c\Delta t = \Delta z \tag{9.14.11c}$$

For the simulations, we will consider the following generic names of the variables

$$Q(n, k) = Q_2(n, k) \tag{9.14.12a}$$

$$P(n, k) = Q_4(n, k) \tag{9.14.12b}$$

So the generic computing algorithms of the hyperincursive discrete wave equations are given by

$$Q(n, k+1) = Q(n, k-1) + D[P(n+1, k) - P(n-1, k)] \tag{9.14.13a}$$

$$P(n, k+1) = P(n, k-1) + D\big[Q(n+1, k) - Q(n-1, k)\big] \tag{9.14.13b}$$

with the 2 values of the parameter $D$

$$D = +1 \tag{9.14.13c}$$

and

$$D = -1 \tag{9.14.13d}$$

which represent the hyperincursive discrete Dirac relativistic quantum equations (9.14.9a, b) and (9.14.8a, b) and also the hyperincursive discrete relativistic quantum Majorana equations (9.14.3a, b) and (9.14.4a, b).

With those two values $D = 1$, the simulations are numerically stable and give discrete space and time periodic solutions similar to the continuous analytical solutions of the continuous wave equation.

Now, we will give a few examples of simulation of these hyperincursive quantum algorithms.

Table 9.4a, b gives the simulation of the hyperincursive discrete algorithms of the discrete quantum wave equations (9.14.13a, b) with the parameter $D = +1$ and $D = -1$ of two particles in a periodic spatial domain.

Table 9.6a, b deals with the simulation of the hyperincursive discrete algorithms of the discrete quantum wave equations (9.14.13a, b) with the parameter $D = +1$ of two particles in a box. The two particles reflect to the two opposite walls of the box.

Table 9.6 shows the simulation of the hyperincursive discrete algorithms of the discrete quantum wave equations (9.14.13a, b) with the parameter $D = +1$ of a packet of particles in a periodic spatial domain that separates to two opposite packets.

Table 9.5a, b deals with the simulation of the hyperincursive discrete algorithms of the discrete quantum wave equations (9.14.13a, b) with the parameter $D = +1$ of a packet of particles in a box. The two opposite packets reflect to the two opposite walls of the box.

In Table 9.4a, the columns represent alternatively the values of the two wave functions, $Q(n, k)$ and $P(n, k)$ of the Eqs. (9.14.13a, b), depending on the space parameter n and the time parameter k,

Vertically, the parameter $k = 0$–$14$ represents the time steps.

Horizontally, the parameter $n = 0$–$11$ represents the spatial intervals.

**Table 9.4** (a) Simulation of the hyperincursive discrete algorithms of the discrete quantum wave equations (9.14.13a, b) with the parameter $D = +1$ of two particles. (b) Continuation of this table (a), with the parameter $D = -1$. Simulation of the hyperincursive discrete algorithms of the discrete quantum wave equations (9.14.13a, b) of two particles

(a)

Simulation of the hyperincursive algorithm
Of the discrete quantum wave equations with **D** = +1
Of two particles in a periodic space domain

| | n | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 8 | 9 | 9 | 10 | 10 | 11 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** |
| **k** | | | | | | | | | | | | | | | | | | | | | | | | | |
| 0 | | **0** | **0** | **0** | **0** | **2** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** |
| 1 | | **0** | **0** | **1** | **1** | **0** | **0** | **1** | **−1** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** |
| 2 | | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** |
| 4 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 |
| 5 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **−1** | 0 | 0 | **1** | **1** | 0 | 0 | 0 | 0 |
| 6 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **2** | **0** | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 | **1** | **−1** | 0 | 0 | 0 | 0 |
| 8 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **−1** | 0 | 0 |
| 9 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **−1** |
| 10 | | **1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

(continued)

**Table 9.4** (continued)

(a)

Simulation of the hyperincursive algorithm
Of the discrete quantum wave equations with **D** = +1
Of two particles in a periodic space domain

| | **n** | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 8 | 9 | 9 | 10 | 10 | 11 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** |
| **k** | | | | | | | | | | | | | | | | | | | | | | | | | |
| 11 | | 0 | 0 | **1** | **–1** | 0 | 0 | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 12 | | 0 | 0 | 0 | 0 | **2** | **0** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 13 | | 0 | 0 | **1** | **1** | 0 | 0 | **1** | **–1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 14 | | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **–1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

(b)

Simulation of the hyperincursive algorithm
Of the discrete quantum wave equations with **D** = −1
Of two particles in a periodic space domain

| | **n** | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 8 | 9 | 9 | 10 | 10 | 11 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** |
| **k** | | | | | | | | | | | | | | | | | | | | | | | | | |
| 0 | | **0** | **0** | **0** | **0** | **2** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** |
| 1 | | **0** | **0** | **1** | **–1** | **0** | **0** | **1** | **1** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** |
| 2 | | **1** | **–1** | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **–1** |

(continued)

**Table 9.4** (continued)

(b)

Simulation of the hyperincursive algorithm
Of the discrete quantum wave equations with $\mathbf{D} = -1$
Of two particles in a periodic space domain

| | **n** | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 8 | 9 | 9 | 10 | 10 | 11 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** |
| **k** | | | | | | | | | | | | | | | | | | | | | | | | | |
| 4 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **–1** | 0 | 0 |
| 5 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 | **1** | **–1** | 0 | 0 | 0 | 0 |
| 6 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **2** | **0** | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **–1** | 0 | 0 | **1** | **1** | 0 | 0 | 0 | 0 |
| 8 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **–1** | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 |
| 9 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **–1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** |
| 10 | | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **–1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 11 | | 0 | 0 | **1** | **1** | 0 | 0 | **1** | **–1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 12 | | 0 | 0 | 0 | 0 | **2** | **0** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 13 | | 0 | 0 | **1** | **–1** | 0 | 0 | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 14 | | **1** | **–1** | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

**Table 9.5** Simulation of the hyperincursive discrete algorithms of the discrete quantum wave equations (9.14.13a, b) with the parameter $D = +1$ of two particles in a box

(a)

Simulation of the hyperincursive algorithm
Of the discrete quantum wave equations with **D** = +1
Of two particles in a box

| | n | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 8 | 9 | 9 | 10 | 10 | 11 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** |
| **k** | | | | | | | | | | | | | | | | | | | | | | | | | |
| 0 | | **0** | **0** | **0** | **0** | **0** | **0** | **2** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** |
| 1 | | **0** | **0** | **0** | **0** | **1** | **1** | **0** | **0** | **1** | **–1** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** |
| 2 | | 0 | 0 | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **–1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **–1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **–1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **–1** | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | | 0 | 0 | **–1** | **–1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **–1** | 0 | 0 | 0 | 0 |
| 7 | | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **–1** | 0 | 0 |
| 8 | | 0 | 0 | 0 | 0 | 0 | 0 | **–1** | **–1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **–1** |
| 9 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **–1** | **–1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **–1** |
| 11 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | **–1** | **1** | 0 | 0 |
| 12 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **–1** | **–1** | 0 | 0 | **1** | **–1** | 0 | 0 | 0 | 0 |
| 13 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **2** | **0** | 0 | 0 | 0 | 0 | 0 |
| 14 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **–1** | 0 | 0 | **–1** | **–1** | 0 | 0 | 0 | 0 |

(continued)

**Table 9.5** (continued)

(a)

Simulation of the hyperincursive algorithm
Of the discrete quantum wave equations with **D** = +1
Of two particles in a box

| | **n** | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 8 | 9 | 9 | 10 | 10 | 11 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** |
| **k** | | | | | | | | | | | | | | | | | | | | | | | | | |
| 15 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **−1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 |
| 16 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **−1** | **−1** |
| 17 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **−1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 18 | | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **−1** | **−1** |
| 19 | | 0 | 0 | 0 | 0 | **−1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **−1** | **−1** | 0 | 0 |
| 20 | | 0 | 0 | **1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **−1** | **−1** | 0 | 0 | 0 | 0 |
| 21 | | **−1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **−1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 |
| 22 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **−1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 23 | | **−1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **−1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 24 | | 0 | 0 | **−1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | **−1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 25 | | 0 | 0 | 0 | 0 | **−1** | **1** | 0 | 0 | **−1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

(continued)

**Table 9.5** (continued)

(a)

Simulation of the hyperincursive algorithm
Of the discrete quantum wave equations with $\mathbf{D} = +1$
Of two particles in a box

| | **n** | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 8 | 9 | 9 | 10 | 10 | 11 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** |
| **k** | | | | | | | | | | | | | | | | | | | | | | | | | |
| 26 | | 0 | 0 | 0 | 0 | 0 | 0 | **−2** | **0** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 27 | | 0 | 0 | 0 | 0 | **−1** | **−1** | 0 | 0 | **−1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

(b)

Continuation of the
Simulation of the hyperincursive algorithm
Of the discrete quantum wave equations
Of two particles in a box

| | **n** | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 8 | 9 | 9 | 10 | 10 | 11 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** |
| **k** | | | | | | | | | | | | | | | | | | | | | | | | | |
| 26 | | 0 | 0 | 0 | 0 | 0 | 0 | **−2** | **0** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 27 | | 0 | 0 | 0 | 0 | **−1** | **−1** | 0 | 0 | **−1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

(continued)

**Table 9.5** (continued)

(b)

Continuation of the
Simulation of the hyperincursive algorithm
Of the discrete quantum wave equations
Of two particles in a box

| | **n** | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 8 | 9 | 9 | 10 | 10 | 11 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 28 | | 0 | 0 | **−1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 | **−1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 29 | | **−1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **−1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 30 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **−1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 31 | | **−1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **−1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 |
| 32 | | 0 | 0 | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **−1** | **1** | 0 | 0 | 0 | 0 |
| 33 | | 0 | 0 | 0 | 0 | **−1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **−1** | **1** | 0 | 0 |
| 34 | | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **−1** | **1** |
| 35 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **−1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 36 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **−1** | **1** |
| 37 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **−1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **−1** | 0 | 0 |
| 38 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 | **−1** | **1** | 0 | 0 | 0 | 0 |

(continued)

**Table 9.5** (continued)

(b)

Continuation of the
Simulation of the hyperincursive algorithm
Of the discrete quantum wave equations
Of two particles in a box

| | **n** | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 8 | 9 | 9 | 10 | 10 | 11 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 39 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **−2** | **0** | 0 | 0 | 0 | 0 | 0 |
| 40 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **−1** | **1** | 0 | 0 | **1** | **1** | 0 | 0 | 0 | 0 |
| 41 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 | **−1** | **−1** | 0 | 0 |
| 42 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **−1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** |
| 43 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 44 | | 0 | 0 | 0 | 0 | 0 | 0 | **−1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** |
| 45 | | 0 | 0 | 0 | 0 | **1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 |
| 46 | | 0 | 0 | **−1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 | 0 | 0 |
| 47 | | **1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 |
| 48 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 49 | | **1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

(continued)

**Table 9.5** (continued)

(b)

Continuation of the
Simulation of the hyperincursive algorithm
Of the discrete quantum wave equations
Of two particles in a box

| | **n** | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 8 | 9 | 9 | 10 | 10 | 11 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 50 | | 0 | 0 | **1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 51 | | 0 | 0 | 0 | 0 | **1** | **−1** | 0 | 0 | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 52 | | 0 | 0 | 0 | 0 | 0 | 0 | **2** | **0** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 53 | | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 | **1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

The initial conditions of $Q(n, k)$ and $P(n, k)$ are given by null values in all the space $n = 0$–11 for the time $k = 0$ and $k = 1$ except for the two particles

$Q(2, 0) = 2$ and $P(2, 0) = 0$

that represents two superposed particles, and

$Q(1, 1) = 1$ and $P(1, 1) = 1$

that represent the first particle moving to the left, and

$$Q(3, 1) = 1 \text{ and } P(3, 1) = -1$$

that represent the second particle moving to the right.

With the periodic boundary conditions of the space, the particles remain in the space domain.

The spatial domain is given by periodic boundary conditions: the first particle moving to the left moves from $n = 0, k = 2$ to $n = 11, k = 3$.

The second particle moving to the right moves from $n = 11, k = 9$ to $n = 0, k = 10$.

The two particles are superposed when they interfere, at $n = 2, k = 12$. The system is periodic in time, the values at times $k = 12$ and $k = 13$ are identical to the initial values at $k = 0$ and $k = 1$.

This Table 9.4b is the continuation of Table 9.4a, with the value of the parameter $D = -1$, for which the two particles move in the opposite directions.

The columns represent the two wave functions, $Q(n, k)$ and $P(n, k)$ of the Eqs. (9.14.13a, b).

The initial conditions of $Q(n, k)$ and $P(n, k)$ are given by null values in all the space $n = 0$–11 for the times $k = 0$ and $k = 1$ except for the two particles

$Q(2, 0) = 2$ and $P(2, 0) = 0$

that represent two superposed particles, and

$Q(3, 1) = 1$ and $P(3, 1) = 1$

that represent the first particle moving to the right, and

$$Q(1, 1) = 1 \text{ and } P(1, 1) = -1$$

that represent the second particle moving to the left.

With the periodic boundary conditions of the space, the particles remain in the space domain.

The spatial domain is given by periodic boundary conditions.

The first particle, of Table 9.4a, is now moving to the right and moves from $n = 11, k = 9$ to $n = 0, k = 10$. The second particle, of Table 9.4a, is now moving to the left and moves from $n = 0, k = 2$ to $n = 11, k = 3$.

The two particles are superposed when they interfere, at $n = 2, k = 12$.

The system is periodic in time, the values at times $k = 12$ and $k = 13$ are identical to the initial values at $k = 0$ and $k = 1$.

In this Table 9.5a, the boundary conditions of the two opposite walls of the 1D box are given by:

$$Q(-1, t) = 0, \quad P(-1, t) = 0, \quad Q(12, t) = 0, \quad P(12, t) = 0.$$

The two particles reflect on the two opposite walls of the box, and their values have reversed signs at $k = 26$ and $k = 27$. There is the continuation of this simulation at Table 9.5b.

In this Table 9.5b, the two particles reflect on the opposite walls of the box, and their values at $k = 52$ and $k = 53$ become identical to their initial values at $k = 0$ and $k = 1$ (see Table 9.5a).

In this Table 9.6, the spatial domain is given by periodic boundary conditions.

The initial packet of particles separates into two opposite packets of particles.

There is a stable propagation of the packets of particles.

Then the two packets of particles superpose and become the initial packet of particles.

The system is periodic in space and time, the values at times $k = 12$ and $k = 13$ are identical to the initial values at $k = 0$ and $k = 1$.

In this Table 9.7a, the boundary conditions of the two opposite walls of the 1D box are given by:

$$Q(-1, t) = 0, \quad P(-1, t) = 0, \quad Q(12, t) = 0, \quad P(12, t) = 0$$

The initial packet of particles separates into two opposite packets of particles.

The two packets of particles reflect on the two opposite walls of the box, and their values have reversed signs at $k = 26$ and $k = 27$. There is the continuation of the simulation at Table 9.7b.

In this Table 9.7b, the two packets of particles reflect on the opposite walls of the box. Then the two packets of particles become the initial packet of particles and their values at $k = 52$ and $k = 53$ become identical to the initial values at $k = 0$ and $k = 1$ (see Table 9.7a).

The simulations of the hyperincursive discrete algorithms of the quantum Majorana and Dirac wave equations presented in this last section demonstrate the power of these hyperincursive algorithms which are numerically stable.

Moreover these simulations are performed with discrete integer numbers.

## 9.15 Conclusion

This chapter presented algorithms for simulation of discrete space-time partial differential equations in classical physics and relativistic quantum mechanics.

We presented the second-order hyperincursive discrete harmonic oscillator that shows the conservation of energy. This recursive discrete harmonic oscillator is separable into two incursive discrete oscillators with the conservation of the constant of motion. The incursive discrete oscillators are related to forward and backward time derivatives and show anticipative properties. The incursive discrete oscillators are

**Table 9.6** Simulation of the hyperincursive discrete algorithms of the discrete quantum wave equations (9.14.13a, b) with the parameter $D = +1$ of a packet of particles

Simulation of the hyperincursive algorithm
Of the discrete quantum wave equations with **D** = +1
Of a packet of particles in a periodic space domain

| | n | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 8 | 9 | 9 | 10 | 10 | 11 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** |
| **k** | | | | | | | | | | | | | | | | | | | | | | | | | |
| 0 | | 0 | 0 | 0 | 0 | 0 | 0 | **2** | **0** | **4** | **0** | **8** | **0** | **4** | **0** | **2** | **0** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | | 0 | 0 | 0 | 0 | **1** | **1** | **2** | **2** | **5** | **3** | **4** | **0** | **5** | **−3** | **2** | **−2** | **1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | | 0 | 0 | **1** | **1** | **2** | **2** | **4** | **4** | **2** | **2** | **2** | **0** | **2** | **−2** | **4** | **−4** | **2** | **−2** | **1** | **−1** | 0 | 0 | 0 | 0 |
| 3 | | **1** | **1** | **2** | **2** | **4** | **4** | **2** | **2** | **1** | **1** | 0 | 0 | **1** | **−1** | **2** | **−2** | **4** | **−4** | **2** | **−2** | **1** | **−1** | 0 | 0 |
| 4 | | **2** | **2** | **4** | **4** | **2** | **2** | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **−1** | **2** | **−2** | **4** | **−4** | **2** | **−2** | **2** | **0** |
| 5 | | **5** | **3** | **2** | **2** | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **−1** | **2** | **−2** | **5** | **−3** | **4** | **0** |
| 6 | | **4** | **0** | **2** | **0** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **2** | **0** | **4** | **0** | **8** | **0** |
| 7 | | **5** | **−3** | **2** | **−2** | **1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | **2** | **2** | **5** | **3** | **4** | **0** |
| 8 | | **2** | **−2** | **4** | **−4** | **2** | **−2** | **1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | **2** | **2** | **4** | **4** | **2** | **2** | **2** | **0** |
| 9 | | **1** | **−1** | **2** | **−2** | **4** | **−4** | **2** | **−2** | **1** | **−1** | 0 | 0 | **1** | **1** | **2** | **2** | **4** | **4** | **2** | **2** | **1** | **1** | 0 | 0 |
| 10 | | 0 | 0 | **1** | **−1** | **2** | **−2** | **4** | **−4** | **2** | **−2** | **2** | **0** | **2** | **2** | **4** | **4** | **2** | **2** | **1** | **1** | 0 | 0 | 0 | 0 |
| 11 | | 0 | 0 | 0 | 0 | **1** | **−1** | **2** | **−2** | **5** | **−3** | **4** | **0** | **5** | **3** | **2** | **2** | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 |
| 12 | | 0 | 0 | 0 | 0 | 0 | 0 | **2** | **0** | **4** | **0** | **8** | **0** | **4** | **0** | **2** | **0** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 13 | | 0 | 0 | 0 | 0 | **1** | **1** | **2** | **2** | **5** | **3** | **4** | **0** | **5** | **−3** | **2** | **−2** | **1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 |
| 14 | | 0 | 0 | **1** | **1** | **2** | **2** | **4** | **4** | **2** | **2** | **2** | 0 | **2** | **−2** | **4** | **−4** | **2** | **−2** | **1** | **−1** | 0 | 0 | 0 | 0 |
| 15 | | **1** | **1** | **2** | **2** | **4** | **4** | **2** | **2** | **1** | **1** | 0 | 0 | **1** | **−1** | **2** | **−2** | **4** | **−4** | **2** | **−2** | **1** | **−1** | 0 | 0 |

**Table 9.7** Simulation of the hyperincursive discrete algorithms of the discrete quantum wave equations (9.14.13a, b) with the parameter $D = +1$ of a packet of particles in a box

(a)

Simulation of the hyperincursive algorithm
Of the discrete quantum wave equations with **D** = +1
Of a packet of particles in a box

| | n | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 8 | 9 | 9 | 10 | 10 | 11 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** |
| **k** | | | | | | | | | | | | | | | | | | | | | | | | | |
| 0 | | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 4 | 0 | 8 | 0 | 4 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | | 0 | 0 | 0 | 0 | 1 | 1 | 2 | 2 | 5 | 3 | 4 | 0 | 5 | −3 | 2 | −2 | 1 | −1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | | 0 | 0 | 1 | 1 | 2 | 2 | 4 | 4 | 2 | 2 | 2 | 0 | 2 | −2 | 4 | −4 | 2 | −2 | 1 | −1 | 0 | 0 | 0 | 0 |
| 3 | | 1 | 1 | 2 | 2 | 4 | 4 | 2 | 2 | 1 | 1 | 0 | 0 | 1 | −1 | 2 | −2 | 4 | −4 | 2 | −2 | 1 | −1 | 0 | 0 |
| 4 | | 2 | 2 | 4 | 4 | 2 | 2 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | −1 | 2 | −2 | 4 | −4 | 2 | −2 | 1 | −1 |
| 5 | | 5 | 5 | 2 | 2 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | −1 | 2 | −2 | 4 | −4 | 2 | −2 |
| 6 | | 4 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | −1 | 2 | −2 | 5 | −5 |
| 7 | | 5 | 5 | −2 | −2 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | −4 |
| 8 | | 2 | 2 | −4 | −4 | 2 | 2 | −1 | −1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | −1 | −2 | 2 | 5 | −5 |
| 9 | | 1 | 1 | −2 | −2 | 4 | 4 | −2 | −2 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | −1 | 1 | 2 | −2 | −4 | 4 | 2 | −2 |
| 10 | | 0 | 0 | −1 | −1 | 2 | 2 | −4 | −4 | 2 | 2 | −1 | −1 | 0 | 0 | 1 | −1 | −2 | 2 | 4 | −4 | −2 | 2 | 1 | −1 |
| 11 | | 0 | 0 | 0 | 0 | 1 | 1 | −2 | −2 | 4 | 4 | −2 | −2 | 0 | 2 | 2 | −2 | −4 | 4 | 2 | −2 | −1 | 1 | 0 | 0 |
| 12 | | 0 | 0 | 0 | 0 | 0 | 0 | −1 | −1 | 2 | 2 | −3 | −5 | 0 | 4 | 3 | −5 | −2 | 2 | 1 | −1 | 0 | 0 | 0 | 0 |
| 13 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | −4 | 0 | 8 | 0 | −4 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 |
| 14 | | 0 | 0 | 0 | 0 | 0 | 0 | 1 | −1 | −2 | 2 | 3 | −5 | 0 | 4 | −3 | −5 | 2 | 2 | −1 | −1 | 0 | 0 | 0 | 0 |

(continued)

**Table 9.7** (continued)

(a)

Simulation of the hyperincursive algorithm
Of the discrete quantum wave equations with $\mathbf{D} = +1$
Of a packet of particles in a box

| | n | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 8 | 9 | 9 | 10 | 10 | 11 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** |
| **k** | | | | | | | | | | | | | | | | | | | | | | | | | |
| 15 | | 0 | 0 | 0 | 0 | −1 | 1 | 2 | −2 | −4 | 4 | 2 | −2 | 0 | 2 | −2 | −2 | 4 | 4 | −2 | −2 | 1 | 1 | 0 | 0 |
| 16 | | 0 | 0 | 1 | −1 | −2 | 2 | 4 | −4 | −2 | 2 | 1 | −1 | 0 | 0 | −1 | −1 | 2 | 2 | −4 | −4 | 2 | 2 | −1 | −1 |
| 17 | | −1 | 1 | 2 | −2 | −4 | 4 | 2 | −2 | −1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | −2 | −2 | 4 | 4 | −2 | −2 |
| 18 | | −2 | 2 | 4 | −4 | −2 | 2 | 1 | −1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −1 | −1 | 2 | 2 | −5 | −5 |
| 19 | | −5 | 5 | 2 | −2 | −1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −4 | −4 |
| 20 | | −4 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −1 | −1 | −2 | −2 | −5 | −5 |
| 21 | | −5 | 5 | −2 | 2 | −1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −1 | −1 | −2 | −2 | −4 | −4 | −2 | −2 |
| 22 | | −2 | 2 | −4 | 4 | −2 | 2 | −1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | −1 | −1 | −2 | −2 | −4 | −4 | −2 | −2 | −1 | −1 |
| 23 | | −1 | 1 | −2 | 2 | −4 | 4 | −2 | 2 | −1 | 1 | 0 | 0 | −1 | −1 | −2 | −2 | −4 | −4 | −2 | −2 | −1 | −1 | 0 | 0 |
| 24 | | 0 | 0 | −1 | 1 | −2 | 2 | −4 | 4 | −2 | 2 | −2 | 0 | −2 | −2 | −4 | −4 | −2 | −2 | −1 | −1 | 0 | 0 | 0 | 0 |
| 25 | | 0 | 0 | 0 | 0 | −1 | 1 | −2 | 2 | −5 | 3 | −4 | 0 | −5 | −3 | −2 | −2 | −1 | −1 | 0 | 0 | 0 | 0 | 0 | 0 |

(continued)

**Table 9.7** (continued)

(a)

Simulation of the hyperincursive algorithm
Of the discrete quantum wave equations with **D** = +1
Of a packet of particles in a box

| | n | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 8 | 9 | 9 | 10 | 10 | 11 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** |
| **k** | | | | | | | | | | | | | | | | | | | | | | | | | |
| 26 | | 0 | 0 | 0 | 0 | 0 | 0 | **−2** | **0** | **−4** | **0** | **−8** | **0** | **−4** | **0** | **−2** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 27 | | 0 | 0 | 0 | 0 | **−1** | **−1** | **−2** | **−2** | **−5** | **−3** | **−4** | **0** | **−5** | **3** | **−2** | **2** | **−1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 |

(b)

Continuation of the simulation of the hyperincursive algorithm
Of the discrete quantum wave equations
Of a packet of particles in a box

| | n | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 8 | 9 | 9 | 10 | 10 | 11 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** |
| **k** | | | | | | | | | | | | | | | | | | | | | | | | | |
| 26 | | 0 | 0 | 0 | 0 | 0 | 0 | **−2** | **0** | **−4** | **0** | **−8** | **0** | **−4** | **0** | **−2** | **0** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 27 | | 0 | 0 | 0 | 0 | **−1** | **−1** | **−2** | **−2** | **−5** | **−3** | **−4** | **0** | **−5** | **3** | **−2** | **2** | **−1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 |

(continued)

**Table 9.7** (continued)

(b)

Continuation of the simulation of the hyperincursive algorithm
Of the discrete quantum wave equations
Of a packet of particles in a box

| | n | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 8 | 9 | 9 | 10 | 10 | 11 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** |
| **k** | | | | | | | | | | | | | | | | | | | | | | | | | |
| 28 | | 0 | 0 | −1 | −1 | −2 | −2 | −4 | −4 | −2 | −2 | −2 | 0 | −2 | 2 | −4 | 4 | −2 | 2 | −1 | 1 | 0 | 0 | 0 | 0 |
| 29 | | −1 | −1 | −2 | −2 | −4 | −4 | −2 | −2 | −1 | −1 | 0 | 0 | −1 | 1 | −2 | 2 | −4 | 4 | −2 | 2 | −1 | 1 | 0 | 0 |
| 30 | | −2 | −2 | −4 | −4 | −2 | −2 | −1 | −1 | 0 | 0 | 0 | 0 | 0 | 0 | −1 | 1 | −2 | 2 | −4 | 4 | −2 | 2 | −1 | 1 |
| 31 | | −5 | −5 | −2 | −2 | −1 | −1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −1 | 1 | −2 | 2 | −4 | 4 | −2 | 2 |
| 32 | | −4 | −4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −1 | 1 | −2 | 2 | −5 | 5 |
| 33 | | −5 | −5 | 2 | 2 | −1 | −1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −4 | 4 |
| 34 | | −2 | −2 | 4 | 4 | −2 | −2 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −1 | 1 | 2 | −2 | −5 | 5 |
| 35 | | −1 | −1 | 2 | 2 | −4 | −4 | 2 | 2 | −1 | −1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | −1 | −2 | 2 | 4 | −4 | −2 | 2 |
| 36 | | 0 | 0 | 1 | 1 | −2 | −2 | 4 | 4 | −2 | −2 | 1 | 1 | 0 | 0 | −1 | 1 | 2 | −2 | −4 | 4 | 2 | −2 | −1 | 1 |
| 37 | | 0 | 0 | 0 | 0 | −1 | −1 | 2 | 2 | −4 | −4 | 2 | 2 | 0 | −2 | −2 | 2 | 4 | −4 | −2 | 2 | 1 | −1 | 0 | 0 |
| 38 | | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | −2 | −2 | 3 | 5 | 0 | −4 | −3 | 5 | 2 | −2 | −1 | 1 | 0 | 0 | 0 | 0 |

(continued)

**Table 9.7** (continued)

(b)

Continuation of the simulation of the hyperincursive algorithm
Of the discrete quantum wave equations
Of a packet of particles in a box

| | n | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 8 | 9 | 9 | 10 | 10 | 11 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Q | P | Q | P | Q | P | Q | P | Q | P | Q | P | Q | P | Q | P | Q | P | Q | P | Q | P | Q | P |
| k | | | | | | | | | | | | | | | | | | | | | | | | | |
| 39 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −2 | 0 | 4 | 0 | −8 | 0 | 4 | 0 | −2 | 0 | 0 | 0 | 0 | 0 | 0 |
| 40 | | 0 | 0 | 0 | 0 | 0 | 0 | −1 | 1 | 2 | −2 | −3 | 5 | 0 | −4 | 3 | 5 | −2 | −2 | 1 | 1 | 0 | 0 | 0 | 0 |
| 41 | | 0 | 0 | 0 | 0 | 1 | −1 | −2 | 2 | 4 | −4 | −2 | 2 | 0 | −2 | 2 | 2 | −4 | −4 | 2 | 2 | −1 | −1 | 0 | 0 |
| 42 | | 0 | 0 | −1 | 1 | 2 | −2 | −4 | 4 | 2 | −2 | −1 | 1 | 0 | 0 | 1 | 1 | −2 | −2 | 4 | 4 | −2 | −2 | 1 | 1 |
| 43 | | 1 | −1 | −2 | 2 | 4 | −4 | −2 | 2 | 1 | −1 | 0 | 0 | 0 | 0 | 0 | 0 | −1 | −1 | 2 | 2 | −4 | −4 | 2 | 2 |
| 44 | | 2 | −2 | −4 | 4 | 2 | −2 | −1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | −2 | −2 | 5 | 5 |
| 45 | | 5 | −5 | −2 | 2 | 1 | −1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | 4 |
| 46 | | 4 | −4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 | 2 | 5 | 5 |
| 47 | | 5 | −5 | 2 | −2 | 1 | −1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 | 2 | 4 | 4 | 2 | 2 |
| 48 | | 2 | −2 | 4 | −4 | 2 | −2 | 1 | −1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 | 2 | 4 | 4 | 2 | 2 | 1 | 1 |
| 49 | | 1 | −1 | 2 | −2 | 4 | −4 | 2 | −2 | 1 | −1 | 0 | 0 | 1 | 1 | 2 | 2 | 4 | 4 | 2 | 2 | 1 | 1 | 0 | 0 |

(continued)

**Table 9.7** (continued)

(b)

Continuation of the simulation of the hyperincursive algorithm
Of the discrete quantum wave equations
Of a packet of particles in a box

| n | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 8 | 9 | 9 | 10 | 10 | 11 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** | **Q** | **P** |
| **k** | | | | | | | | | | | | | | | | | | | | | | | | |
| 50 | 0 | 0 | **1** | **−1** | **2** | **−2** | **4** | **−4** | **2** | **−2** | **2** | **0** | **2** | **2** | **4** | **4** | **2** | **2** | **1** | **1** | 0 | 0 | 0 | 0 |
| 51 | 0 | 0 | 0 | 0 | **1** | **−1** | **2** | **−2** | **5** | **−3** | **4** | **0** | **5** | **3** | **2** | **2** | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 |
| 52 | 0 | 0 | 0 | 0 | 0 | 0 | **2** | **0** | **4** | **0** | **8** | **0** | **4** | **0** | **2** | **0** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 53 | 0 | 0 | 0 | 0 | **1** | **1** | **2** | **2** | **5** | **3** | **4** | **0** | **5** | **−3** | **2** | **−2** | **1** | **−1** | 0 | 0 | 0 | 0 | 0 | 0 |

not recursive but time inverse of each other and are executed in series without the need of a work memory.

In simulation-based cyber-physical system studies, the main properties of the algorithms must meet the following constraints. The algorithms must be numerically stable and must be as compact as possible to be embedded in cyber-physical systems. Moreover the algorithms must be executed in real-time as quickly as possible without too much access to the memory.

The presented algorithms in this paper meet these conditions.

Then, we presented the second-order hyperincursive discrete Klein–Gordon equation given by space-time second-order partial differential equations for the simulation of the quantum Majorana real 4-spinors equations and of the relativistic quantum Dirac complex 4-spinors equations.

This chapter presented simulations of the hyperincursive discrete quantum Majorana and Dirac wave equations which are numerically stable.

One very important characteristic of these algorithms is the fact that they are space-time-symmetric, so the algorithms are fully invertible (reversible) in time and space.

The reversibility of the presented hyperincursive discrete algorithms is a fundamental condition to make quantum computing.

The development of simulation-based cyber-physical systems indeed evolves to quantum computing.

So the presented computing tools are well adapted to these future requirements.

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