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Galois Covers, Grothendieck-Teichmüller Theory and Dessins d'Enfants

Interactions between Geometry, Topology, Number Theory and Algebra, Leicester, UK, June 2018

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Preface

This proceedings volume presents original peer-reviewed contributions from the *London Mathematical Society Midlands Regional Meeting and Workshop on 'Galois Covers, Grothendieck-Teichmüller Theory and Dessins d'Enfants'*. The regional meeting and workshop were held at the University of Leicester in Leicester, United Kingdom from 4 to 7 June 2018 and were sponsored by the London Mathematical Society (LMS) with additional support from the University of Leicester, the Anglo-Franco-German Representation Theory Network (REPNET) and the Engineering and Physical Sciences Research Council (EPSRC). It was attended by well over 70 participants from the United Kingdom and other European countries.

The *LMS Midlands Regional Meeting* took place on 4 June 2018. The invited speakers were Minhyong Kim (University of Oxford, UK), Leila Schneps (Institut de Mathématiques de Jussieu Paris, France) and Fabrizio Catanese (University of Bayreuth, Germany). The meeting started with official society business chaired by the LMS president Caroline Series (University of Warwick, UK). In the opening talk entitled *Diophantine geometry and principal bundles* Minhyong Kim described fascinating new links between number theory, geometry and physics. In the second talk, Leila Schneps related Grothendieck-Teichmüller theory with number theory in her inspirational talk *Pro-unipotent Grothendieck-Teichmüller theory: surprising connections with number theory*. In the final talk of the LMS meeting Fabrizio Catanese gave an inspiring overview entitled *Mathematical Mysteries behind the Interplay of Algebra and Topology in Moduli Theory* on new connections between algebraic and topological aspects of the classification theory of algebraic varieties with symmetries.

The regional meeting was followed by an international workshop on *Galois covers, Grothendieck-Teichmüller theory and Dessins d'enfants* from 5 to 7 June 2018 and featured 16 invited international speakers, including the aforementioned three regional meeting speakers. This multidisciplinary workshop brought together world-leading experts as well as graduate students and early career researchers from geometry, topology, algebra and number theory. It also initiated new links and many future collaborations. Several of the talks featured introductions or overviews on the different interdisciplinary themes of the workshop. The invited speakers were: Fabrizio Catanese (University of Bayreuth, Germany), Anna Felikson (Durham University, UK), Paola Frediani (University of Pavia, Italy), Benoit Fresse (University of Lille, France), Christian Gleissner (University of Bayreuth, Germany), Pierre Guillot (IRMA, Strasbourg, France), Gareth Jones (University of Southampton, UK), Minhyong Kim (University of Oxford, UK), Bernhard Koeck (University of Southampton, UK), Michael Lönne (University of Bayreuth, Germany), Goran Malic (University of Manchester, UK), Marta Mazzocco (University of Birmingham, UK), Roberto Pignatelli (University of Trento, Italy), Mohamed Saïdi (University of Exeter, UK), Leila Schneps (Institut de Mathématiques de Jussieu Paris, France) and Richard Webb (University of Cambridge, UK).

We would like to thank the authors of these proceedings for their important contributions. We also thank the many referees for their work and suggestions to improve this volume. Furthermore, we thank all the speakers and participants who made this LMS regional meeting and workshop such a great success. Last but not least, we like to thank the London Mathematical Society, REPNET, EPSRC and the University of Leicester for their generous financial support. Finally, we thank Springer for the suggestion to include this proceedings volume in the Springer Proceedings in Mathematics & Statistics series. In particular, we would like to thank Banu Dhayalan and Remi Lodh from Springer who guided us smoothly and with expertise through the editing and publishing process.

Leicester, UK
April 2020

Frank Neumann
Sibylle Schroll

Contents

Galois Covers, Grothendieck-Teichmüller Theory and Dessins d'Enfants - An Introduction

Frank Neumann and Sibylle Schroll

Abstract In this introduction, we will give a brief overview of the themes and topics of the articles in this proceedings volume and summarise each individual contribution based on the abstracts and introduction.

Keywords Galois covers · Grothendieck-Teichmüller theory · Dessins d'enfants · Group actions · Representation theory

The LMS workshop *Galois covers, Grothendieck–Teichmüller theory and Dessins d'enfants* brought together many experts from the United Kingdom and abroad as well as graduate students and early career researchers centred around three main themes: Galois covers, Grothendieck–Teichmüller theory and dessins d'enfants. These themes bring about unexpected links between algebraic geometry, representation theory, number theory and algebraic topology. The proceedings of this workshop reflect the topics and scope of the lectures presented and explore through surveys and original research articles the many facets of these fascinating themes.

1 Overview of Themes

The absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is one of the most fundamental objects of study in number theory. In his groundbreaking work, Grothendieck [20] proposed a combinatorial approach to investigate $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ based on bipartite ribbon graphs which, because of their apparent simplicity, he called 'dessins d'enfants'. They arise as graphs embedded into Riemann surfaces and are related to algebraic curves defined over algebraic number fields. There exists a bijection between dessins d'enfants and

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conjugacy classes of finite index subgroups of $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})$, the fundamental group of the Riemann sphere minus the three points $\{0, 1, \infty\}$. At the origin of Grothendieck's development of the theory of dessins d'enfants is Belyi's famous theorem [8]. It essentially states that any non-singular algebraic curve, defined over an algebraic number field, represents a compact Riemann surface which is a ramified covering of the Riemann sphere, ramified at three points only, which after a Möbius transformation may be taken to be $\{0, 1, \infty\}$. Grothendieck's idea was to use the dessins d'enfants to construct combinatorial, topological and algebraic invariants that identify the orbits of this action (for overviews and surveys see for example [17, 21, 28–32]). Constructing these type of invariants is a central open problem in number theory and although much work in this direction has already been carried out, it is still unclear whether a complete list of such invariants exists and if there are finitely many of them. To study the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ more geometrically, Grothendieck in [20] also suggested to relate it to its action on the Teichmüller tower constructed from the fundamental groupoids of the moduli stacks of algebraic curves of genus g with n marked points, the so-called Teichmüller groupoids. Following this suggestion Drinfeld [14] introduced the closely related Grothendieck–Teichmüller group GT which is conjectured to be equal to the absolute Galois group. This initiated the study of geometric Galois actions through what is now known as Grothendieck–Teichmüller theory. There are also many fascinating links between the Grothendieck–Teichmüller tower and the theory of multiple zeta values as developed by Brown, Carr and Schneps [7] and in relation to the motivic fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ with its connections to diophantine geometry as studied by Kim [24, 25].

Galois covers appear naturally in relation with the famous Inverse Galois Problem, which asks if for an arbitrary finite group G, there exists a Galois field extension $L/\mathbb{Q}$ with Galois group isomorphic to G. A natural way to construct such extensions is by studying in algebro-geometric terms Galois covers of the projective line $\mathbb{P}^1$ for which the corresponding function field extension is a Galois extensions with Galois group being isomorphic to G. Furthermore, studying families of unramified coverings over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ leads to the rich geometry of Hurwitz moduli spaces [22, 32]. More recently, Bauer, Catanese and Grunewald [1, 2] have found a new geometric framework to study the absolute Galois group in geometric terms via its action on certain algebraic surfaces and related covering spaces, namely algebraic surfaces isogeneous to a product as well as Beauville surfaces. Beauville surfaces are a class of algebraic surfaces of general type, which can be described also in a purely algebraic and combinatorics way. They play now an important role in algebraic geometry, number theory and group theory [4]. There are many surprising relations between the classification theory of algebraic surfaces and their moduli spaces, the absolute Galois group, Galois coverings and Grothendieck's theory of dessins d'enfants [1, 16]. Several of these relations are explored in this proceedings volume.

The wide range of research articles published in these proceedings indicates the broad intradisciplinary nature of the workshop around the three themes and documents many new exciting connections and directions.

2 Summaries of Individual Contributions

As a quick orientation for the reader, we will now give brief overviews on the topics and main results of the individual articles in this proceedings volume based on the abstracts and introductions.

Scott Balchin, Frank Neumann: On the number of rational points of classifying stacks for Chevalley group schemes

Given an algebraic group G, one can associate to it the classifying stack $\mathscr{B}G$, which classifies principal G-bundles together with its automorphisms. In this proceedings article, Balchin and Neumann calculate the number of rational points of classifying stacks associated to some Chevalley group schemes using the Lefschetz–Grothendieck trace formula of Behrend [6] for ℓ-adic cohomology of algebraic stacks as well as appropriate analogues of classical theorems of Borel [9] for singular cohomology of classifying spaces of compact Lie groups. Finally, the authors also derive associated zeta functions for these classifying stacks.

Fabrizio Catanese: Cyclic Symmetry on Complex Tori and Bagnera–De Franchis Manifolds

Hyperelliptic manifolds are quotients of a complex torus T by a free action of a finite group G, which contains no translations. If G is a cyclic group, then such a quotient is called a Bagnera–De Franchis manifold [3, 12]. In this article, Catanese classifies all the possible linear actions of the cyclic group $G = \mathbb{Z}/n$ on a complex torus by using the cyclotomic exact sequence for the group algebra $\mathbb{Z}[G]$. As a main application he proves an important general structure theorem for Bagnera–De Franchis manifolds. Furthermore, he gives an interesting application to hypergeometric integrals, namely an explicit calculation of the homology of a cyclic covering of degree n of the projective line $\mathbb{P}^1$ as a $\mathbb{Z}[\mathbb{Z}/n]$-module, under the condition that there exists a point of full ramification.

Elisabetta Colombo, Paola Frediani: Second fundamental form of the Prym map in the ramified case

In this article, Colombo and Frediani study the second fundamental form of the Prym map $P_{g,r} : \mathcal{R}_{g,r} \to \mathcal{A}^{\delta}_{g-1+r}$ in the ramified case $r > 0$. Here $\mathcal{R}_{g,r}$ is the moduli space parametrising isomorphism classes of a complex projective curve C of genus g, together with a reduced effective divisor R of degree $2r$ and associated line bundle α on C such that $\alpha^2 = \mathcal{O}_C(R)$. $\mathcal{A}^{\delta}_{g-1+r}$ is the moduli space of abelian varieties of dimension $g - 1 + r$ with a polarization of type δ. The authors give an interpretation of this map in terms of the second fundamental form of the Torelli map of the covering curves. As an interesting application they use this description to obtain an upper bound for the dimension of a germ of a totally geodesic submanifold, and hence of a Shimura subvariety of $\mathcal{A}^{\delta}_{g-1+r}$ inside the Prym locus.

Ben Fairbairn: Strongly Real Beauville Groups III

Beauville surfaces were first defined by Catanese [11, 12] as a generalisation of an earlier example of Beauville [5]. They are a class of complex algebraic surfaces of general type defined by an action of a finite group G on a product of Riemann surfaces. These surfaces possess many attractive geometric properties several of which are dictated by group theoretical and combinatorical properties of the acting group G. A particularly interesting subclass are the strongly real Beauville surfaces that have an analogue of complex conjugation defined on them. Here Fairbairn gives a broad survey on this subclass and their defining groups called strongly real Beauville groups. Several interesting open problems, questions and conjectures related to strongly real Beauville groups are discussed along the way.

Filippo F. Favale, Christian Gleissner, Roberto Pignatelli: The pluricanonical systems of a product-quotient variety

In this article, Favale, Gleissner and Pignatelli study the geometry of product-quotient varieties, which are constructed by taking a minimal resolution of the singularities of a quotient of a finite product of algebraic curves by a diagonal action of a finite group. These product-quotient varieties provide a rich and important class of examples instrumental for the construction of K3-surfaces with particular types of symmetries [18] or of rigid not infinitesimally rigid compact complex manifolds [10]. The authors present a method for the computation of the plurigenera of product-quotient varieties and apply this to construct Calabi–Yau threefolds and to determine the minimal model of a product-quotient algebraic surface of general type.

Gareth Jones: Joining dessins together

Given a compact Riemann surface of genus $g > 1$, Hurwitz showed that its automorphism group has order less or equal to $84(g-1)$. A group is called a Hurwitz group if it can be realised as the automorphism group of a Riemann surface attaining this bound or equivalently, if it is given as a non-trivial finite quotient of the triangle group $\Delta(2, 3, 7)$. In this article, Jones reinterprets and generalises a joining operation for coset diagrams of a given group first introduced by Higman and later developed by Conder in connection with Hurwitz groups, as a connected sum operation on dessins d'enfants of a given type. The author illustrates this with numerous interesting examples.

Minhyong Kim: Arithmetic Chern–Simons Theory I

In several recent articles, Kim [26, 27] has suggested fascinating new links between arithmetic geometry and mathematical physics in what he calls arithmetic gauge theory. This proceedings article follows a similar theme, in which ideas of Dijkgraaf and Witten [15] on $2+1$ dimensional topological quantum field theory are applied to arithmetic geometry, namely to the theory of arithmetic curves, that is, the spectra of rings of integers in algebraic number fields. The main part is devoted to the construction and study of classical Chern–Simons functionals on moduli spaces of Galois representations and how to compute the arithmetic Chern–Simons invariants in several interesting examples. In a final section the author introduces new conjectural approaches for the use of arithmetic Chern–Simons theory to construct L-functions.

Michael Lönne: Branch stabilisation for the components of Hurwitz moduli spaces of Galois covers

In this article, Lönne, discusses group theoretical aspects crucial for the study of connected components of Hurwitz moduli spaces of Galois covers. The investigation of these Hurwitz spaces goes back to classical work by Clebsch [13] and Hurwitz [22]. He sets up a powerful algebraic framework in order to study the set of corresponding equivalence classes of monodromy maps and in particular to investigate geometric stabilisation by various Galois covers which are branched over the disc. The article analyses equivalence and stable equivalence in purely algebraic terms by means of homological invariants, which allows to distinguish equivalence classes of given boundary monodromy and Nielsen type.

Goran Malić, Sibylle Schroll: Dessins d'enfants and Brauer configuration algebras

In this article, Malić and Schroll analyse new connections between the theory of dessins d'enfants and the representation theory of associative algebras with the idea to develop new algebraic Galois invariants in this way. Namely, given a dessin d'enfant they associate to it an associative algebra, called a Brauer configuration algebra [19], which is a generalisation of a Brauer graph algebra. It is given via a quiver and relations induced by the monodromy of the dessin d'enfant. They show that the dimension of the Brauer configuration algebra associated to a dessin d'enfant and the dimension of the centre of this algebra are both invariant under the action of the absolute Galois group. This is illustrated through several examples of well-known algebras and associated dessins d'enfants. Finally, the authors also show that the Brauer configuration algebras of a dessin d'enfant and of its dual share the same path algebra.

Élise Raphael, Leila Schneps: On the elliptic Kashiwara–Vergne Lie algebra

In this article, Raphael and Schneps study certain higher genus analogues of the Grothendieck–Teichmüller Lie algebra introduced by Ihara [23]. Namely, they analyse and compare the constructions of two independently defined elliptic versions of the Kashiwara–Vergne Lie algebra $\mathfrak{krv}$. The first one is the Lie algebra $\mathfrak{krv}^{(1;1)}$ as constructed by A. Alekseev, N. Kawazumi, Y. Kuno and F. Naef, which arises from the study of graded formality isomorphisms associated to topological fundamental groups of surfaces. The second one is the Lie algebra $\mathfrak{krv}_{ell}$, which is defined by using mould theoretic techniques arising from multiple zeta theory as pioneered by the authors. The main result here establishes a canonical isomorphism between these two types of Lie algebras.

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On the Number of Rational Points of Classifying Stacks for Chevalley Group Schemes

Scott Balchin and Frank Neumann

Abstract We compute the number of rational points of classifying stacks of Chevalley group schemes using the Lefschetz–Grothendieck trace formula of Behrend for ℓ-adic cohomology of algebraic stacks. From this we also derive associated zeta functions for these classifying stacks.

Keywords Chevalley group schemes · algebraic groups · finite groups of Lie type · classifying stacks · l-adic cohomology

Introduction

Given an algebraic group or group scheme G, we can associate to it a classifying stack $\mathscr{B}G$, which is an algebraic stack classifying principal G-bundles or G-torsors together with their automorphisms. Classifying stacks of algebraic groups play a similar role in algebraic geometry as classifying spaces of topological or Lie groups in algebraic topology. For example, classifying spaces of compact Lie groups and their singular cohomology encode a vast amount of geometry and topology, whereas classifying stacks of algebraic groups and their ℓ-adic cohomology provide information about the geometry, arithmetic and representation theory of the algebraic group. Many cohomological results for compact Lie groups like Borel's fundamental theorems [4] on the singular cohomology of classifying spaces, have direct analogs in terms of ℓ-adic cohomology for classifying stacks of reductive algebraic groups.

In this article we will be studying the rational ℓ-adic cohomology of classifying stacks of a particular class of reductive algebraic groups over the field $\mathbb{F}_q$, namely

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the Chevalley groups. It turns out that the numbers of $\mathbb{F}_q$-rational points of their associated classifying stacks are precisely the inverses of the orders of the related finite groups of Lie type, which are obtained as the finite groups of $\mathbb{F}_q$-rational points of the original Chevalley group. The calculation of $\mathbb{F}_q$-rational points employs the trace formula for algebraic stacks due to Behrend [2, 3] involving the arithmetic Frobenius morphism acting on the ℓ-adic cohomology of the associated classifying stack. These numbers of rational points are in fact groupoid cardinalities as they take into account all the aforementioned automorphisms. We calculate these numbers in all cases given by the classification, namely for the untwisted and twisted Chevalley groups. Once we have computed the number of $\mathbb{F}_q$-rational points of these classifying stacks we can assemble them into a zeta function for the classifying stack, which as usual encodes a lot of arithmetic information. We will derive these zeta function in all the cases coming out of the classification. It can be expected that the ℓ-adic cohomology rings and the zeta functions of the classifying stacks will be of interest for the arithmetic and representation theory of Chevalley groups as well as for their associated finite groups of Lie type. Classifying spaces of compact Lie groups also have a rich homotopy theory and similarly algebraic stacks have étale homotopy types (see [1, 14]). The étale homotopy types of classifying stacks for reductive algebraic groups should have similar significance as the classical homotopy types for compact Lie groups. We will address this circle of homotopical questions in a follow-up article to this work.

This article features the following content. In the first section we recall the definition of quotient stacks and classifying stacks. We also collect the required facts about ℓ-adic cohomology of algebraic stacks. Furthermore, we describe the arithmetic Frobenius morphisms and their action on the cohomology and state Behrend's Lefschetz–Grothendieck trace formula, which counts the number of $\mathbb{F}_q$-rational points of algebraic stacks. Finally, we recall the zeta function for algebraic stacks and illustrate all the concepts for the fundamental example given by the multiplicative group scheme $\mathbb{G}_m$. In the second section we calculate the rational ℓ-adic cohomology of classifying stacks for connected reductive algebraic groups over $\mathrm{Spec}(\mathbb{F}_q)$ in analogy with the classical results of Borel on the singular cohomology of classifying spaces for connected compact Lie groups [4]. We state a general expression for the number of $\mathbb{F}_q$-rational points for these classifying stacks and derive as a corollary Steinberg's formula for the order of the finite group of $\mathbb{F}_q$-rational points of a given connective reductive algebraic group over $\mathbb{F}_q$ (see [33, 34]). In the final section, we apply this machinery to obtain explicit formulas for the number of $\mathbb{F}_q$-rational points and related zeta functions for the classifying stacks of Chevalley and Steinberg groups.

1 Classifying Stacks, ℓ-Adic Cohomology, Frobenius Morphisms and Zeta Functions

In this section we will recall briefly the notions of classifying stacks of algebraic groups, their ℓ-adic cohomology, and how to count $\mathbb{F}_q$-rational points using Behrend's

Grothendieck–Lefschetz trace formula for algebraic stacks. For the general theory of algebraic stacks we will refer to the book by Laumon and Moret-Bailly [23]. Other resources are [26, 27, 30] or the encyclopedic stacks project [31].

Definition 1.1 Let Z be a smooth scheme over a noetherian scheme S and G/S be a reductive group scheme of finite rank over S together with a (right) G-action $\mu : Z \times_S G \to Z$. The *quotient stack* $[Z/G]$ is defined via its groupoid of sections as follows: For a given scheme U/S over S, the groupoid $[Z/G](U)$ of U-valued points of $[Z/G]$ is the groupoid of principal G-bundles P over U together with a G-equivariant morphism $\alpha : P \to Z$ and isomorphisms of this data. In the special case that $Z = S$ being equipped with the trivial G-action, the resulting quotient stack $\mathscr{B}G := [S/G]$ is called the *classifying stack* of G.

The quotient stack $[Z/G]$, and in particular the classifying stack $\mathscr{B}G$, under the above general conditions are smooth algebraic stacks which are locally of finite type (see [3, 23, 26]).

The classifying stack $\mathscr{B}G$ should be viewed as an algebro-geometric analogue of the classifying space BG in algebraic topology. In the topological setting, principal G-bundles over a topological space X are classified by homotopy classes of maps into the classifying space BG, whereas in algebraic geometry, principal G-bundles–or G-torsors–over a scheme X can be classified by morphisms of stacks from X into the classifying stack $\mathscr{B}G$.

We will now recall briefly the machinery of ℓ-adic cohomology of algebraic stacks. As a general reference for the cohomology of algebraic stacks and its main properties we refer to the book of Laumon and Moret-Bailly [23]. Especially for the definition of ℓ-adic cohomology, its main properties and the general formalism of cohomology functors we refer to the work of Behrend [2, 3] and the subsequent articles by Laszlo and Olsson [21, 22]. An alternative systematic approach towards ℓ-adic cohomology for algebraic stacks was given recently as well by Gaitsgory and Lurie [15]. Here we will follow also closely the exposition and notations of [16].

Let $\mathscr{X}$ be an algebraic stack, which is smooth and locally of finite type over the base scheme $S = \mathrm{Spec}(\mathbb{F}_q)$, where $\mathbb{F}_q$ is the field with $q = p^s$ elements. Let ℓ be a prime different from p. The rational ℓ-adic cohomology $H^*(\mathscr{X}_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)$ over the lisse-étale site $\mathscr{X}_{\text{lis-ét}}$ of $\mathscr{X}$ is given as the limit of the ℓ-adic cohomology groups over all open substacks $\mathscr{U}$ of finite type of the algebraic stack $\mathscr{X}_{\overline{\mathbb{F}}_q} = \mathscr{X} \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\overline{\mathbb{F}}_q)$ over the algebraic closure $\overline{\mathbb{F}}_q$, i.e., we have

$$H^*(\mathscr{X}_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell) = \lim_{\substack{\mathscr{U} \subset \mathscr{X}_{\overline{\mathbb{F}}_q}, \\ \text{open, finite type}}} H^*(\mathscr{U}, \overline{\mathbb{Q}}_\ell).$$

In the following, we will normally write $H^*(\mathscr{X}, \overline{\mathbb{Q}}_\ell)$ instead of $H^*(\mathscr{X}_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)$ to simplify the notations.

The ℓ-adic cohomology has a Künneth decomposition [2, 3, 22]. That is, for $\mathscr{X}$, $\mathscr{Y}$ algebraic stacks as above, there is a natural isomorphism

$$H^*(\mathscr{X} \times \mathscr{Y}, \overline{\mathbb{Q}}_\ell) \simeq H^*(\mathscr{X}, \overline{\mathbb{Q}}_\ell) \otimes H^*(\mathscr{Y}, \overline{\mathbb{Q}}_\ell).$$

We also have an arithmetic Frobenius morphism, which acts on an algebraic stack $\mathscr{X}$ over $\mathrm{Spec}(\mathbb{F}_q)$ and its ℓ-adic cohomology. For this, let

$$\mathrm{Frob} : \overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q, \ a \mapsto a^q$$

be the classical Frobenius morphism given by a generator of the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ of the field extension $\overline{\mathbb{F}}_q/\mathbb{F}_q$. It induces an endomorphism of schemes

$$\mathrm{Frob}_{\mathrm{Spec}(\overline{\mathbb{F}}_q)} : \mathrm{Spec}(\overline{\mathbb{F}}_q) \to \mathrm{Spec}(\overline{\mathbb{F}}_q)$$

which naturally extends to an endomorphism of algebraic stacks defined as

$$\mathrm{Frob}_{\mathscr{X}} := id_{\mathscr{X}} \times \mathrm{Frob}_{\mathrm{Spec}(\overline{\mathbb{F}}_q)} : \mathscr{X}_{\overline{\mathbb{F}}_q} \to \mathscr{X}_{\overline{\mathbb{F}}_q}.$$

This endomorphism $\mathrm{Frob}_{\mathscr{X}}$ of algebraic stacks is called the *arithmetic Frobenius morphism*. By naturality, it induces an endomorphism $\Psi_q := \mathrm{Frob}^*_{\mathscr{X}}$ on the ℓ-adic cohomology of the algebraic stack $\mathscr{X}$:

$$\Psi_q = \mathrm{Frob}^*_{\mathscr{X}} : H^*(\mathscr{X}, \overline{\mathbb{Q}}_l) \to H^*(\mathscr{X}, \overline{\mathbb{Q}}_l).$$

Analogous to case of schemes X over a base $S = \mathrm{Spec}(\mathbb{F}_q)$ we can calculate the number $\mathscr{X}(\mathbb{F}_q) = \#\mathscr{X}(\mathrm{Spec}(\mathbb{F}_q)$ of $\mathbb{F}_q$-rational points for an algebraic stack $\mathscr{X}$ locally of finite type using a Lefschetz type trace formula. In the case of algebraic stacks though we need to modify this count to a 'stacky'count as we have to consider instead the groupoid cardinality of the groupoid $\mathscr{X}(\mathrm{Spec}(\mathbb{F}_q))$ of $\mathbb{F}_q$-rational points. The analogue of the Grothendieck–Lefschetz trace formula for algebraic stacks was conjectured and proved by Behrend, for particular cases in [2], and in full generality in [3].

Theorem 1.2 (Behrend–Grothendieck–Lefschetz trace formula) *Let $\mathscr{X}$ be a smooth algebraic stack of finite type over* $\mathrm{Spec}(\mathbb{F}_q)$ *and Ψ be the arithmetic Frobenius morphism. Then we have*

$$q^{dim(\mathscr{X})} \sum_{s \geq 0} (-1)^s \mathrm{tr}(\Psi_q | H^s(\mathscr{X}, \overline{\mathbb{Q}}_\ell)) = \#\mathscr{X}(\mathbb{F}_q).$$

Here $\#\mathscr{X}(\mathbb{F}_q) = \#\mathscr{X}(\mathrm{Spec}(\mathbb{F}_q))$ is the groupoid cardinality

$$\#\mathscr{X}(\mathrm{Spec}(\mathbb{F}_q)) = \sum_{x \in [\mathscr{X}(\mathrm{Spec}(\mathbb{F}_q))]} \frac{1}{\#Aut_{\mathscr{X}(\mathrm{Spec}(\mathbb{F}_q))}(x)}$$

of the groupoid $\mathscr{X}(\mathrm{Spec}(\mathbb{F}_q))$ of $\mathbb{F}_q$-rational points of the algebraic stack $\mathscr{X}$, where $\#Aut_{\mathscr{X}(\mathrm{Spec}(\mathbb{F}_q))}(x)$ is the order of the group of automorphisms of the isomorphism *class x in the groupoid* $\mathscr{X}(\mathrm{Spec}(\mathbb{F}_q))$ *and* $[\mathscr{X}(\mathrm{Spec}(\mathbb{F}_q))]$ *denotes the set of isomorphism classes of objects in the groupoid* $\mathscr{X}(\mathrm{Spec}(\mathbb{F}_q))$.

We are mostly interested in algebraic groups over the field $\mathbb{F}_q$ and their classifying stacks. By an algebraic group we understand here a smooth group scheme G of finite type over $\mathrm{Spec}(\mathbb{F}_q)$ (see [25]). The algebraic group G is called connected if it is geometrically connected in the sense of [13, Example VI.A, 2.1.1]. In the case of quotient stacks of group actions of algebraic groups on schemes or algebraic spaces over $\mathbb{F}_q$, the groupoid cardinality of the groupoid of of $\mathbb{F}_q$-rational points of the quotient stack can be identified as follows (see [2, Proposition 2.2.3])

Proposition 1.3 *Let X be a smooth algebraic space of finite type over* $\mathrm{Spec}(\mathbb{F}_q)$ *and G be a connected algebraic group over* $\mathrm{Spec}(\mathbb{F}_q)$ *acting on X. If* $\mathscr{X}$ *is the quotient stack* $\mathscr{X} = [X/G]$*, then we have*

$$\#\mathscr{X}(\mathbb{F}_q) = \frac{\#X(\mathbb{F}_q)}{\#G(\mathbb{F}_q)}.$$

In particular, for the classifying stack $\mathscr{B}G$ *of the algebraic group G we have*

$$\#\mathscr{B}G(\mathbb{F}_q) = \frac{1}{\#G(\mathbb{F}_q)}.$$

Proof This is essentially [2, Lemma 2.5.1] and the particular case follows by letting $X = \mathrm{Spec}(\mathbb{F}_q)$ be a point with trivial action of G. For the particular case, which is most relevant here, we can also argue directly in the following way. As G is connected, Lang's theorem [20] implies that each principal G-bundle on $\mathrm{Spec}(\mathbb{F}_q)$ is isomorphic to the trivial bundle and therefore the groupoid $\mathscr{B}G(\mathbb{F}_q)$ contains only a single object up to isomorphism, whose automorphism group is the finite group $G(\mathbb{F}_q)$. This gives the desired groupoid cardinality $\#\mathscr{B}G(\mathbb{F}_q)$. □

Remark 1.4 We could also employ a Lefschetz type trace formula for ℓ-adic cohomology with compact support using the action of the geometric Frobenius and Poincaré duality instead (see [35, Theorem 1.1]). This approach has certain advantages due to the existence of a full theory of six-operations for algebraic stacks [21, 22] and removes any smoothness condition. We will however follow here the original approach of Behrend using the arithmetic Frobenius morphism [2, 3].

Finally, for an algebraic stack $\mathscr{X}$ of finite type over $\mathbb{F}_q$ we can also define a zeta function incorporating all the counts of $\mathbb{F}_{q^i}$-rational points formally defined as (see [2])

Definition 1.5 Let $\mathscr{X}$ be a smooth algebraic stack of finite type over $\mathrm{Spec}(\mathbb{F}_q)$. The *zeta function* of $\mathscr{X}$ is the formal power series $\zeta_{\mathscr{X}}(t) \in \mathbb{Q}[[t]]$ given by

$$\zeta_{\mathscr{X}}(t) = \exp\left(\sum_{i=1}^{\infty} \#\mathscr{X}(\mathbb{F}_{q^i})\frac{t^i}{i}\right).$$

If $\mathscr{X}$ is a Deligne-Mumford stack, for example the moduli stack of elliptic curves Ell or more generally the moduli stack $\mathscr{M}_{g,n}$ of algebraic curves of genus g with n marked points, it can be shown that $\zeta_{\mathscr{X}}$ is a rational function. But in general, this zeta function for algebraic stacks might not be a rational function. Nevertheless, Behrend [2] showed that if $\mathscr{X}$ is given as a quotient stack of an algebraic space by a linear algebraic group, then the zeta function $\zeta_{\mathscr{X}}$ is a meromorphic function in the complex t-plane and can be calculated in many interesting cases (compare also [35]).

Let us finish this section with an instructive guiding example, for which we can study all the ingredients introduced above and their interplay. This can be viewed as a prelude for the general case and our explicit calculations in the following sections (compare also [3, 35]).

Example 1.6 (Multiplicative group scheme) Let $\mathbb{G}_m$ be the multiplicative group scheme over $\mathrm{Spec}(\mathbb{F}_q)$ i.e., given as

$$\mathbb{G}_m = \mathrm{Spec}(\mathbb{F}_q[x, x^{-1}]).$$

Let $\mathscr{B}\mathbb{G}_m$ be the associated classifying stack which classifies $\mathbb{G}_m$-torsors i.e., the classifying stack of line bundles. For the dimension of this algebraic stack we have

$$\dim(\mathscr{B}\mathbb{G}_m) = -\dim(\mathbb{G}_m) = -1$$

The number $\#\mathscr{B}\mathbb{G}_m(\mathrm{Spec}(\mathbb{F}_q))$ of $\mathbb{F}_q$-rational points of $\mathscr{B}\mathbb{G}_m$ is given as the number of line bundles (up to isomorphism) over the 'point' $\mathrm{Spec}(\mathbb{F}_q)$. As all line bundles over the 'point' $\mathrm{Spec}(\mathbb{F}_q)$ are trivial, there is therefore just one isomorphism class x in $\mathscr{B}\mathbb{G}_m(\mathrm{Spec}(\mathbb{F}_q))$.

Furthermore we have

$$\#\mathrm{Aut}_{\mathscr{B}\mathbb{G}_m(\mathrm{Spec}(\mathbb{F}_q))}(x) = \#\mathbb{G}_m(\mathbb{F}_q) = \#\mathbb{F}_q^* = q - 1$$

so in other words we have that

$$\#\mathscr{B}\mathbb{G}_m(\mathbb{F}_q) = \sum_{x \in [\mathscr{B}\mathbb{G}_m(\mathrm{Spec}(\mathbb{F}_q))]} \frac{1}{\#\mathrm{Aut}_{\mathscr{B}\mathbb{G}_m(\mathrm{Spec}(\mathbb{F}_q))}(x)} = \frac{1}{q-1}.$$

The ℓ-adic cohomology of $\mathscr{B}\mathbb{G}_m$ is basically the cohomology of an infinite projective space, namely we have

$$H^*(\mathscr{B}\mathbb{G}_m, \overline{\mathbb{Q}}_\ell) = \overline{\mathbb{Q}}_\ell[c],$$

with c a generator in degree 2 and so from the Behrend–Grothendieck–Lefschetz trace formula we therefore get

$$q^{\dim(\mathscr{B}\mathbb{G}_m)} \sum_{i \geq 0} \mathrm{tr}(\Psi_q | H^{2i}(\mathscr{B}\mathbb{G}_m, \mathbb{Q}_\ell)) = q^{-1}(1 + q^{-1} + q^{-2} + \cdots) = \frac{1}{q} \sum_{i=0}^{\infty} \frac{1}{q^i}.$$

This formal calculation therefore gives a 'stacky' proof for the well-known formula

$$\sum_{i=0}^{\infty} \frac{1}{q^{i+1}} = \frac{1}{q-1}.$$

We can then compute the zeta function as follows

$$\begin{aligned}\zeta_{\mathscr{B}\mathbb{G}_m}(t) &= \exp(\sum_{i=1}^{\infty} \#\mathscr{B}\mathbb{G}_m(\mathbb{F}_{q^i})\frac{t^i}{i}) = \exp(\sum_{i=1}^{\infty} \frac{1}{q^i-1}\frac{t^i}{i})\\ &= \exp(\sum_{i=1}^{\infty} \frac{t^i}{i}\sum_{k=1}^{\infty}\frac{1}{q^{ki}}) = \prod_{k=1}^{\infty}\exp(\sum_{i=1}^{\infty}\frac{(t/q^k)^i}{i})\\ &= \prod_{k=1}^{\infty}(1-q^{-k}t)^{-1}.\end{aligned}$$

From this one can also derive a functional equation for the zeta function, namely we have

$$\zeta_{\mathscr{B}\mathbb{G}_m}(qt) = \frac{1}{1-t}\zeta_{\mathscr{B}\mathbb{G}_m}(t)$$

and so the zeta function for $\mathscr{B}\mathbb{G}_m$ has a meromorphic continuation to the complex t-plane having simple poles at $t = q^n$ for all $n \geq 1$.

2 Cohomology of Classifying Stacks and the Theorems of Borel and Steinberg

Let us collect some facts about the ℓ-adic cohomology of classifying stacks, which are basically algebro-geometric analogues of classical theorems of Borel for classifying spaces of compact Lie groups [4] (compare also [2, 3, 18, 35]).

Let G be a connected algebraic group over the field $\mathbb{F}_q$. We will employ the Leray spectral sequence of the universal morphism of algebraic stacks $f : \mathrm{Spec}(\mathbb{F}_q) \to \mathscr{B}G$ with fibre G, which is the algebro-geometric analogue of the universal topological fibration over the classifying space of a compact Lie group. The spectral sequence is given as follows:

$$E_2^{s,t} \cong H^s(\mathscr{B}G, R^t f_*\overline{\mathbb{Q}}_\ell) \Rightarrow H^{s+t}(\mathrm{Spec}(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell).$$

Here $R^t f_*\overline{\mathbb{Q}}_\ell$ is a constructible sheaf on $\mathscr{B}G$ and we have

$$R^t f_*\overline{\mathbb{Q}}_\ell = a^* f^* R^t f_*\overline{\mathbb{Q}}_\ell = a^* H^t(G, \overline{\mathbb{Q}}_\ell),$$

where $a : \mathscr{B}G \to \mathrm{Spec}(\mathbb{F}_q)$ is the structure morphism of the classifying stack $\mathscr{B}G$ and $H^t(G, \overline{\mathbb{Q}}_\ell)$ the $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-module interpreted as a sheaf on $\mathrm{Spec}(\mathbb{F}_q)$ (see [3, Lemma 5.4]). The projection formula for the morphism f of algebraic stacks then implies for the E_2-term of the spectral sequence that (see [3, Corollary. 5.3, Proposition 5.5])

$$H^s(\mathscr{B}G, R^t f_* \overline{\mathbb{Q}}_\ell) \cong H^s(\mathscr{B}G, \overline{\mathbb{Q}}_\ell) \otimes_{\overline{\mathbb{Q}}_\ell} H^t(G, \overline{\mathbb{Q}}_\ell).$$

Furthermore, the spectral sequence converges to $H^*(\mathrm{Spec}(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$, which means that we have $E_\infty^{s,t} = 0$ if $(s, t) \neq (0, 0)$ and $H^0(\mathrm{Spec}(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell) \cong \overline{\mathbb{Q}}_\ell$. Therefore the spectral sequence simplifies as follows

$$E_2^{s,t} \cong H^s(\mathscr{B}G, \overline{\mathbb{Q}}_\ell) \otimes_{\overline{\mathbb{Q}}_\ell} H^t(G, \overline{\mathbb{Q}}_\ell) \Rightarrow \overline{\mathbb{Q}}_\ell.$$

For each $t \geq 1$, the differential $d_{t+1}^{0,t}$ is an isomorphism

$$d_{t+1}^{0,t} : E_{t+1}^{0,t} \xrightarrow{\cong} E_{t+1}^{t+1,0}.$$

on the $(t+1)$-page. It is a monomorphism resp., epimorphism because it is the last possible non-zero differential *from* $E_*^{0,t}$ resp., *into* $E_*^{t+1,0}$. So for each $t \geq 1$ we get an isomorphism from the transgressive subspace $N^t = E_{t+1}^{0,t}$ of $H^t(G, \overline{\mathbb{Q}}_\ell) \cong E_2^{0,t}$ to the quotient $E_{t+1}^{t+1,0}$ of $H^{t+1}(\mathscr{B}G, \overline{\mathbb{Q}}_\ell) \cong E_2^{t+1,0}$. In particular, the epimorphism $H^{t+1}(\mathscr{B}G, \overline{\mathbb{Q}}_\ell) \twoheadrightarrow N^t$ is induced by the differential $d_{t+1}^{0,t}$.

Let now $N = \bigoplus_{t \geq 1} N^t$ be the graded transgressive $\overline{\mathbb{Q}}_\ell$-vector space. Then we have the following from Borel's transgression theorem [4, Théorème 13.1] in its algebraic form (compare [3, Theorem 5.6], [35, Theorem 4.8]):

(i) If t is even, then $N^t = 0$.
(ii) The canonical map $\Lambda^* N \xrightarrow{\cong} H^*(G, \overline{\mathbb{Q}}_\ell)$ is an isomorphism of graded $\overline{\mathbb{Q}}_\ell$-algebras.
(iii) The spectral sequence induces an epimorphism of graded $\overline{\mathbb{Q}}_\ell$-vector spaces

$$H^*(\mathscr{B}G, \overline{\mathbb{Q}}_\ell) \twoheadrightarrow N[-1]$$

and every section gives an isomorphism

$$\mathrm{Sym}^*(N[-1]) \xrightarrow{\cong} H^*(\mathscr{B}G, \overline{\mathbb{Q}}_\ell).$$

Let us now assume that G is a connected reductive algebraic group over the field $\mathbb{F}_q$ of rank $\mathrm{rk}(G) = n$. Let $B = T \cdot U$ be a Borel subgroup of G with maximal torus T and unipotent radical U. Let $N_G(T)$ be the normaliser of T in G and $W = N_G(T)/T$ the Weyl group. Furthermore, let $X_*(T)$ be the lattice of cocharacters $\lambda : \mathbb{G}_m \to T$ of T and $\varepsilon_1', \dots \varepsilon_n'$ roots of unity given as the eigenvalues of the arithmetic Frobenius on the lattice $X_*(T)$ and therefore also on the character group $X^*(T)$. From a classical theorem of Chevalley [8], it follows that the subalgebra of W-invariant elements $\mathrm{Sym}^*(X_*(T) \otimes \mathbb{C})^W$ of the symmetric algebra $\mathrm{Sym}^*(X_*(T) \otimes \mathbb{C})$ is generated by homogeneous polynomials $I_1, \ldots, I_n$ with uniquely determined degrees $d_1, \ldots, d_n$ i.e., $\deg(I_j) = d_j$ for $j = 1, \ldots, n$. For example, for an n-dimensional torus $\mathbb{G}_m^n$ these degrees are simply $d_j = 1$ for all $j = 1, \ldots, n$. If G is semi-simple, then we have $d_j > 1$ for all $j = 1, \ldots, n$. The homogeneous generators I_j can be chosen as eigenvectors of the induced Frobenius action with their eigenvalues being roots of unity denoted by $\varepsilon_1, \ldots, \varepsilon_n$ (compare [12, 19, 33] and [9, Sommes trig. 8.2]).

We have the projection morphism $p : \mathscr{B}T \to \mathscr{B}G$ with fibre the flag variety G/T fitting into a 2-cartesian diagram of algebraic stacks

$$\begin{array}{ccc} G/T & \longrightarrow & \mathscr{B}T \\ \big\downarrow & & \big\downarrow {\scriptstyle p} \\ \mathrm{Spec}(\mathbb{F}_q) & \longrightarrow & \mathscr{B}G \end{array}$$

The above diagram plays a similar role as the analogous topological fibration for classifying spaces of compact Lie groups. The following theorem is an algebro-geometric analogue of another classical theorem of Borel for connected compact Lie groups (compare [4, 18]).

Theorem 2.1 *Let G be a connected reductive algebraic group over the field $\mathbb{F}_q$. Then we have:*

(i) The Leray spectral sequence for the projection morphism $p : \mathscr{B}T \to \mathscr{B}G$ is given as

$$E_2^{s,t} \cong H^s(\mathscr{B}G, \overline{\mathbb{Q}}_\ell) \otimes_{\overline{\mathbb{Q}}_\ell} H^t(G/T, \overline{\mathbb{Q}}_\ell) \Rightarrow H^{s+t}(\mathscr{B}T, \overline{\mathbb{Q}}_\ell)$$

and degenerates at the E_2-page. In particular, the induced homomorphism

$$H^*(\mathscr{B}T, \overline{\mathbb{Q}}_\ell) \to H^*(G/T, \overline{\mathbb{Q}}_\ell)$$

is an epimorphism. In fact, $H^*(G/T, \overline{\mathbb{Q}}_\ell)$ is the regular representation of the Weyl group W and generated by the Chern classes of those invertible sheaves $\mathcal{L}_\chi$ obtained as pushouts of the T-torsor $G \to G/T$ for the characters $\chi : T \to \mathbb{G}_m$.

(ii) The homomorphism $H^*(\mathscr{B}G, \overline{\mathbb{Q}}_\ell) \to H^*(\mathscr{B}T, \overline{\mathbb{Q}}_\ell)$ induced by the projection morphism $p : \mathscr{B}T \to \mathscr{B}G$ induces an isomorphism

$$H^*(\mathscr{B}G, \overline{\mathbb{Q}}_\ell) \cong H^*(\mathscr{B}T, \overline{\mathbb{Q}}_\ell)^W.$$

(iii) There is an isomorphism of graded $\overline{\mathbb{Q}}_\ell$-algebras

$$H^*(\mathscr{B}G, \overline{\mathbb{Q}}_\ell) \cong \overline{\mathbb{Q}}_\ell[c_1, \ldots c_n]$$

where the $c_i \in H^{2d_i}(\mathscr{B}G, \overline{\mathbb{Q}}_\ell)$ *are Chern class generators in even degrees* $2d_i$.

(iv) The arithmetic Frobenius morphism $\Psi_q = \mathrm{Frob}^*_{\mathscr{B}G}$ *acts on the* ℓ*-adic cohomology algebra as follows:*

$$\Psi_q(c_i) = \varepsilon_i q^{-d_i} c_i$$

Proof These results can be derived from the analogous classical results by Borel for connected compact Lie groups by lifting G to characteristic 0 following the scholium of [9, Sommes trig. 8.2] (see also [18, Example 4.11]). The degeneration of the spectral sequence

$$E_2^{s,t} \cong H^s(\mathscr{B}G, \overline{\mathbb{Q}}_\ell) \otimes_{\overline{\mathbb{Q}}_\ell} H^t(G/T, \overline{\mathbb{Q}}_\ell) \Rightarrow H^{s+t}(\mathscr{B}T, \overline{\mathbb{Q}}_\ell)$$

at E_2 results because $E_2^{s,t} = 0$ if s or t is odd. This follows from the Bruhat decomposition for the flag variety G/B for a Borel subgroup B containing T, which implies that $H^j(G/T, \overline{\mathbb{Q}}_\ell) = 0$ if j is odd. The spectral sequence inherits an action of the Weyl group W. It acts trivially on $H^*(\mathscr{B}G, \overline{\mathbb{Q}}_\ell)$ and via character theory as the regular representation on $H^*(G/T, \overline{\mathbb{Q}}_\ell)$. Taking the pushout of the principal T-bundle or T-torsor $T \to G \to G/T$ along a character $\chi\colon T \to \mathbb{G}_m$ as in the following diagram

$$\begin{array}{ccc} T & \xrightarrow{\chi} & \mathbb{G}_m \\ \downarrow & & \downarrow \\ G & \longrightarrow & E \\ \downarrow & & \downarrow \\ G/T & = & G/T \end{array}$$

gives a line bundle over G/T and the Chern classes of all these line bundles corresponding to such a character generate the cohomology $H^*(G/T, \overline{\mathbb{Q}}_\ell)$ (compare also [18, Example 4.11(c)]). Furthermore, the morphism $\mathscr{B}T \to \mathscr{B}B$ between classifying stacks induced by the inclusion of the maximal torus T into the Borel subgroup B has as fibres the classifying stack $\mathscr{B}U$ of the unipotent radical subgroup U of B. As U is $\mathbb{A}^n$ all higher cohomology vanishes and we get an induced isomorphism in cohomology. The fibres of the morphism $\mathscr{B}B \to \mathscr{B}G$ are given by the flag varieties G/B and therefore the morphism $\mathscr{B}T \to \mathscr{B}G$ induces a monomorphism

$$H^*(\mathscr{B}G, \overline{\mathbb{Q}}_\ell) \to H^*(\mathscr{B}T, \overline{\mathbb{Q}}_\ell)$$

which lands inside the polynomial invariants of the Weyl group W and for dimensional reasons we get an isomorphism

$$H^*(\mathscr{B}G, \overline{\mathbb{Q}}_\ell) \cong H^*(\mathscr{B}T, \overline{\mathbb{Q}}_\ell)^W.$$

The statement on the polynomial generators for the cohomology of $\mathscr{B}G$ and the action of the arithmetic Frobenius morphism holds first of all for tori. This follows using the Künneth theorem and because $H^*(\mathscr{B}\mathbb{G}_m, \overline{\mathbb{Q}}_\ell) \cong \overline{\mathbb{Q}}_\ell[c_1]$ with $\deg(c_1) = 2$, $d_1 = 1$ and action of the arithmetic Frobenius $\Psi_q(c_1) = \varepsilon_1 q^{-1}$. If T is a maximal torus, i.e., $T \cong \mathbb{G}_m^n$, then for the cohomology algebra of the classifying stack we have $H^*(\mathscr{B}T, \overline{\mathbb{Q}}_\ell) \cong \overline{\mathbb{Q}}_\ell[t_1, \ldots t_n]$, where the generators t_i have degrees $\deg(t_i) = 2$ for all $i = 1, \ldots n$. The Weyl group invariants $H^*(\mathscr{B}T, \overline{\mathbb{Q}}_\ell)^W$ form itself a polynomial algebra $\overline{\mathbb{Q}}_\ell[c_1, \ldots c_n]$ on homogeneous generators of degrees $2d_i$. Using the fact that the degrees d_i for $i = 1, \ldots, n$ of the homogeneous polynomial generators of the Weyl group invariants are precisely the degrees of the homogeneous generators of $H^*(\mathscr{B}T, \overline{\mathbb{Q}}_\ell)^W$ then implies the desired result on the cohomology of the classifying stack $\mathscr{B}G$ with the described action of the arithmetic Frobenius morphism (compare also [11, 16]). □

This allows us now to calculate the number of $\mathbb{F}_q$-rational points for the classifying stack of a given connected reductive algebraic group using the Behrend–Grothendieck–Lefschetz trace formula, which will be instrumental for our explicit calculations in the case of Chevalley groups.

Theorem 2.2 *Let G be a connected reductive algebraic group over the field $\mathbb{F}_q$ of rank* $\mathrm{rk}(G) = n$. *Then the number of $\mathbb{F}_q$-rational points of $\mathscr{B}G$ is given as*

$$\#\mathscr{B}G(\mathbb{F}_q) = q^{-\dim(G)} \prod_{i=1}^{n} (1 - \varepsilon_i q^{-d_i})^{-1}.$$

Proof We have $\dim(\mathscr{B}G) = -\dim(G)$ for the classifying stack $\mathscr{B}G$ of G. From the Behrend–Grothendieck–Lefschetz trace formula Theorem 1.2 we obtain from Theorem 2.1 using the explicit eigenvalues of the arithmetic Frobenius morphism for an adequate choice of a basis

$$\begin{aligned}
\#\mathscr{B}G(\mathbb{F}_q) &= q^{\dim(\mathscr{B}G)} \sum_{s\geq 0} (-1)^s \mathrm{tr}(\Psi_q | H^s(\mathscr{B}G, \overline{\mathbb{Q}}_\ell)) \\
&= q^{-\dim(G)} \sum_{s\geq 0} \mathrm{tr}(\Psi_q | H^{2s}(\mathscr{B}G, \overline{\mathbb{Q}}_\ell)) \\
&= q^{-\dim(G)} \prod_{i=1}^{n} (1 + \varepsilon_i q^{-d_i} + (\varepsilon_i q^{-d_i})^2 + (\varepsilon_i q^{-d_i})^3 + \cdots) \\
&= q^{-\dim(G)} \prod_{i=1}^{n} (1 - \varepsilon_i q^{-d_i})^{-1}.
\end{aligned}$$

This is the desired formula for the number of $\mathbb{F}_q$-rational points of the classifying stack. □

As a corollary we get immediately the following theorem of Steinberg (see [33] or [34, Theorem 25]), which calculates the order of the finite group $G(\mathbb{F}_q)$ of $\mathbb{F}_q$-rational points of the algebraic group G.

Theorem 2.3 (Steinberg) *Let G be a connected reductive algebraic group over the field $\mathbb{F}_q$ of rank* $\mathrm{rk}(G) = n$. *Then the number of $\mathbb{F}_q$-rational points of G is given as*

$$\#G(\mathbb{F}_q) = q^{\dim(G)} \prod_{i=1}^{n} (1 - \varepsilon_i q^{-d_i}).$$

Proof The formula follows at once from Theorem 2.2 using Proposition 1.3

$$\#G(\mathbb{F}_q) = \frac{1}{\#\mathscr{B}G(\mathbb{F}_q)} = q^{\dim(G)} \prod_{i=1}^{n} (1 - \varepsilon_i q^{-d_i}).$$

□

In the final section we will now use these formulas to calculate the number of $\mathbb{F}_q$-rational points and the related zeta functions for classifying stacks of certain algebraic groups intimately related to finite groups of Lie type.

3 Applications to Chevalley Group Schemes and Finite Groups of Lie Type

Let G be a connected compact Lie group. Associated to G is a reductive complex algebraic group $G(\mathbb{C})$, the complexification of G, which can be constructed as the algebraic group of $\mathbb{C}$-rational points of a group scheme $G_{\mathbb{C}}$ over $\mathbb{C}$ obtained via base change from the associated integral affine Chevalley group scheme $G_{\mathbb{Z}}$, i.e.,

$$G_{\mathbb{C}} = G_{\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{C}).$$

In fact, for any field k, taking the k-rational points of the Chevalley group scheme G_k over k given via base change as

$$G_k = G_{\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(k)$$

we obtain the Chevalley group

$$G(k) = \mathrm{Hom}_{\mathrm{Sch}/k}(\mathrm{Spec}(k), G_k),$$

where Sch/k is the category of schemes over the field k.

We are interested here in particular in the finite Chevalley group $G(\mathbb{F}_q)$ of $\mathbb{F}_q$-rational points, which can be constructed also as the fixed point set

Table 1 The Weyl group data for the Chevalley groups

Group	Dim/$\mathbb{C}$	Degrees of Weyl group invariants
SL_{n+1} (Type A_n)	$n(n+2)$	$2, 3, \ldots, n+1$
SO_{2n+1} (Type B_n)	$n(2n+1)$	$2, 4, \ldots, 2n$
Sp_{2n} (Type C_n)	$n(2n+1)$	$2, 4, \ldots, 2n$
SO_{2n} (Type D_n)	$n(2n-1)$	$n, 2, 4, \ldots, 2n-2$
G_2	14	2, 6
F_4	52	2, 6, 8, 12
E_6	78	2, 5, 6, 8, 9, 12
E_7	133	2, 6, 8, 10, 12, 14, 18
E_8	248	2, 8, 12, 14, 18, 20, 24, 30

$$G(\mathbb{F}_q) = G(\overline{\mathbb{F}}_q)^{\psi_q}$$

of the Frobenius morphism

$$\psi_q : G(\overline{\mathbb{F}}_q) \to G(\overline{\mathbb{F}}_q).$$

Here ψ_q is the arithmetic Frobenius homomorphism induced by the classical Frobenius homomorphism

$$\text{Frob} : \overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q, \ \ a \mapsto a^q.$$

See the references [12, 13, 25] for more information on the general theory of reductive group schemes.

The goal of this section is to compute the number of $\mathbb{F}_q$-rational points and the zeta functions for the classifying stacks of Chevalley group schemes over $\mathbb{F}_q$. Recall that such groups are determined by their complex dimension and the degrees of the fundamental Weyl group invariants (see for example [5]). We summarise the necessary data in Table 1. Our first aim is to rewrite Steinberg's theorem (Theorem 2.3) in terms of this data.

Proposition 3.1 *Let G be a Chevalley group scheme over $\mathbb{F}_q$. Then we have*

$$\#\mathscr{B}G(\mathbb{F}_q) = \sum_{i_1=0}^{\infty} \cdots \sum_{i_r=0}^{\infty} q^{-(i_1 d_1 + \cdots + i_r d_r + \dim(G))},$$

where the d_i are the degrees of the Weyl group invariants and r is the rank of G.

Proof We start by noting that these Chevalley group schemes are connected reductive algebraic groups over $\mathbb{F}_q$. Therefore, from Theorem 2.1 we have isomorphisms $H^*(\mathscr{B}G, \overline{\mathbb{Q}}_\ell) \cong H^*(\mathscr{B}T, \overline{\mathbb{Q}}_\ell)^W \cong \overline{\mathbb{Q}}_\ell[c_1, \ldots c_r]$. In particular the cohomology generators c_i are in degree $2d_i$, where the d_i are the degrees of the Weyl group invariants.

Comparing this with Theorem 2.2, we see that these degrees of the Weyl group invariants are exactly the d_i as in Steinberg's Theorem 2.3. The result then follows from Theorem 2.1 as we can identify the action of the arithmetic Frobenius on the ℓ-adic cohomology algebra of the classifying stack $\mathscr{B}G$ in terms of the degrees d_i of the Weyl group invariants. □

Example 3.2 Let G be the exceptional group G_2. We can use Proposition 3.1 to easily compute $\#\mathscr{B}G_2(\mathbb{F}_q)$ using the fact that $\dim(G) = 14$ and it has Weyl group invariants in degrees 2 and 6 respectively:

$$\#\mathscr{B}G_2(\mathbb{F}_q) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} q^{-(2i_1+6i_2+14)}.$$

One can computationally check that this infinite sum does indeed converge to

$$\left(q^{14}(1-q^{-2})(1-q^{-6})\right)^{-1} = \#G_2(\mathbb{F}_q)^{-1}$$

as expected.

From Proposition 3.1, along with some manipulation of complex functions, we can immediately give a general form for the zeta function of the classifying stack.

Proposition 3.3 *Let G be a Chevalley group scheme over $\mathbb{F}_q$. Then*

$$\zeta_{\mathscr{B}G(\mathbb{F}_q)}(t) = \prod_{k_1=1}^{\infty}\cdots\prod_{k_r=1}^{\infty}\left(1 - q^{-(k_1d_1+\cdots+k_rd_r+\left(\dim(G)-\Sigma_{i=1}^{r}d_i\right))}t\right)^{-1},$$

where the d_i are the degrees of the Weyl group invariants.

Proof This result is an exercise in manipulating complex functions. Recall that we have

$$\zeta_{\mathscr{X}}(t) = \exp\left(\sum_{i=1}^{\infty}\#\mathscr{X}(\mathbb{F}_{q^i})\frac{t^i}{i}\right).$$

We then use the result of Proposition 3.1 to substitute in the value of $\#\mathscr{X}(\mathbb{F}_{q^i})$

$$= \exp\left(\sum_{i=1}^{\infty}\sum_{k_1=0}^{\infty}\cdots\sum_{k_r=0}^{\infty} q^{-i(k_1d_1+\cdots+k_rd_r+\dim(G))}\frac{t^i}{i}\right),$$

Next we adjust the indices k_s so they start at 1 instead of 0

$$= \exp\left(\sum_{i=1}^{\infty}\sum_{k_1=1}^{\infty}\cdots\sum_{k_r=1}^{\infty} q^{-i(k_1d_1+\cdots+k_rd_r+\dim(G)-\sum_{s=1}^{r}d_s)}\frac{t^i}{i}\right).$$

Finally, we follow the manipulation for the case of $\mathbb{G}_m$ to get the desired result. In particular we have

$$= \prod_{k_1=1}^{\infty} \cdots \prod_{k_r=1}^{\infty} \exp(\sum_{i=1}^{\infty} q^{-i(k_1 d_1+\cdots+k_r d_r+\dim(G)-\sum_{s=1}^{r} d_s)} \frac{t^i}{i})$$
$$= \prod_{k_1=1}^{\infty} \cdots \prod_{k_r=1}^{\infty} \left(1 - q^{-(k_1 d_1+\cdots+k_r d_r+\left(\dim(G)-\Sigma_{s=1}^{r} d_s\right))} t\right)^{-1},$$

which is the desired expression. □

We now use Propositions 3.1 and 3.3 along with the datum of Table 1 to compute systematically the number of $\mathbb{F}_q$-rational points of the classifying stacks $\mathscr{B}G$ of all the classical, exceptional and twisted Chevalley group schemes G. We also give the corresponding zeta functions. The groups $G(\mathbb{F}_q)$ of $\mathbb{F}_q$-rational points of G are finite groups of Lie type and their orders are well known and can be found for example in [17, §1.5]. From these orders, we can then compute the invariants needed for the computation of the numbers of $\mathbb{F}_q$-rational points of the classifying stacks and the corresponding zeta functions.

3.1 Classical and Exceptional Chevalley Groups

We will first discuss the classical and exceptional Chevalley groups, as constructed by Chevalley and which give rise to the untwisted family of finite groups of Lie type (see [6, 7]).

3.1.1 Classical Chevalley Groups

$G = SL_{n+1}$

$$\#G(\mathbb{F}_q) = q^{n(n+2)}(1-q^{-2})(1-q^{-3})\cdots(1-q^{-(n+1)})$$
$$\#\mathscr{B}G(\mathbb{F}_q) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_n=0}^{\infty} q^{-(2i_1+3i_2+\cdots+(n+1)i_n+n(n+2))}$$
$$\zeta_{\mathscr{B}G}(t) = \prod_{k_1=1}^{\infty}\prod_{k_2=1}^{\infty}\cdots\prod_{k_n=1}^{\infty}\left(1 - q^{-\left(2k_1+3k_2+\cdots+(n+1)k_n+\frac{n(n+1)}{2}\right)} t\right)^{-1}$$

$\underline{G = SO_{2n+1}}$

$$\#G(\mathbb{F}_q) = q^{n(2n+1)}(1-q^{-2})(1-q^{-4})\cdots(1-q^{-2n})$$
$$\#\mathscr{B}G(\mathbb{F}_q) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_n=0}^{\infty} q^{-(2i_1+4i_2+\cdots+2ni_n+n(2n+1))}$$
$$\zeta_{\mathscr{B}G}(t) = \prod_{k_1=1}^{\infty}\prod_{k_2=1}^{\infty}\cdots\prod_{k_n=1}^{\infty}\left(1-q^{-\left(2k_1+4k_2+\cdots+2nk_n+n^2\right)}t\right)^{-1}$$

$\underline{G = Sp_{2n}}$

$$\#G(\mathbb{F}_q) = q^{n(2n+1)}(1-q^{-2})(1-q^{-4})\cdots(1-q^{-2n})$$
$$\#\mathscr{B}G(\mathbb{F}_q) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_n=0}^{\infty} q^{-(2i_1+4i_2+\cdots+2ni_n+n(2n+1))}$$
$$\zeta_{\mathscr{B}G}(t) = \prod_{k_1=1}^{\infty}\prod_{k_2=1}^{\infty}\cdots\prod_{k_n=1}^{\infty}\left(1-q^{-\left(2k_1+4k_2+\cdots+2nk_n+n^2\right)}t\right)^{-1}$$

$\underline{G = SO_{2n}}$

$$\#G(\mathbb{F}_q) = q^{n(2n-1)}(1-q^{-n})(1-q^{-2})(1-q^{-4})\cdots(1-q^{-(2n-2)})$$
$$\#\mathscr{B}G(\mathbb{F}_q) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_n=0}^{\infty} q^{-(ni_1+2i_2+4i_3+\cdots+(2n-2)i_n+n(2n-1))}$$
$$\zeta_{\mathscr{B}G}(t) = \prod_{k_1=1}^{\infty}\prod_{k_2=1}^{\infty}\cdots\prod_{k_n=1}^{\infty}\left(1-q^{-(nk_1+2k_2+4k_3+\cdots+(2n-2)k_n+n(n-1))}t\right)^{-1}$$

3.1.2 Exceptional Chevalley Groups

$\underline{G = G_2}$

$$\#G(\mathbb{F}_q) = q^{14}(1-q^{-2})(1-q^{-6})$$
$$\#\mathscr{B}G(\mathbb{F}_q) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} q^{-(2i_1+6i_2+14)}$$
$$\zeta_{\mathscr{B}G}(t) = \prod_{k_1=1}^{\infty}\prod_{k_2=1}^{\infty}\left(1-q^{-(2k_1+6k_2+6)}t\right)^{-1}$$

$\underline{G = F_4}$

$$\#G(\mathbb{F}_q) = q^{52}(1-q^{-2})(1-q^{-6})(1-q^{-8})(1-q^{-12})$$

$$\#\mathscr{B}G(\mathbb{F}_q) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty}\sum_{i_4=0}^{\infty} q^{-(2i_1+6i_2+8i_3+12i_4+52)}$$

$$\zeta_{\mathscr{B}G}(t) = \prod_{k_1=1}^{\infty}\prod_{k_2=1}^{\infty}\prod_{k_3=1}^{\infty}\prod_{k_4=1}^{\infty}\left(1-q^{-(2k_1+6k_2+8k_3+12k_4+24)}t\right)^{-1}$$

$\underline{G = E_6}$

$$\#G(\mathbb{F}_q) = q^{78}(1-q^{-2})(1-q^{-5})(1-q^{-6})(1-q^{-8})(1-q^{-9})(1-q^{-12})$$

$$\#\mathscr{B}G(\mathbb{F}_q) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_6=0}^{\infty} q^{-(2i_1+5i_2+6i_3+8i_4+9i_5+12i_6+78)}$$

$$\zeta_{\mathscr{B}G}(t) = \prod_{k_1=1}^{\infty}\prod_{k_2=1}^{\infty}\cdots\prod_{k_6=1}^{\infty}\left(1-q^{-(2k_1+5k_2+6k_3+8k_4+9k_5+12k_6+36)}t\right)^{-1}$$

$\underline{G = E_7}$

$$\#G(\mathbb{F}_q) = q^{133}(1-q^{-2})(1-q^{-6})(1-q^{-8})(1-q^{-10})(1-q^{-12})(1-q^{-14})(1-q^{-18})$$

$$\#\mathscr{B}G(\mathbb{F}_q) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_7=0}^{\infty} q^{-(2i_1+6i_2+8i_3+10i_4+12i_5+14i_6+18i_7+133)}$$

$$\zeta_{\mathscr{B}G}(t) = \prod_{k_1=1}^{\infty}\prod_{k_2=1}^{\infty}\cdots\prod_{k_7=1}^{\infty}\left(1-q^{-(2k_1+6k_2+8k_3+10k_4+12k_5+14k_6+18k_7+63)}t\right)^{-1}$$

$\underline{G = E_8}$

$$\#G(\mathbb{F}_q) = q^{248}(1-q^{-2})(1-q^{-8})(1-q^{-12})(1-q^{-14})(1-q^{-18})(1-q^{-20})(1-q^{-24})(1-q^{-30})$$

$$\#\mathscr{B}G(\mathbb{F}_q) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_8=0}^{\infty} q^{-(2i_1+8i_2+12i_3+14i_4+18i_5+20i_6+24i_7+30i_8+248)}$$

$$\zeta_{\mathscr{B}G}(t) = \prod_{k_1=1}^{\infty}\prod_{k_2=1}^{\infty}\cdots\prod_{k_8=1}^{\infty}\left(1-q^{-(2k_1+8k_2+12k_3+14k_4+18k_5+20k_6+24k_7+30k_8+120)}t\right)^{-1}$$

3.2 Twisted Chevalley or Steinberg Groups

Finally, we shall look at the twisted Chevalley groups, which are sometimes also referred to as the Steinberg groups and which were constructed by Steinberg [32] as a variation and generalisation of Chevalley's original construction and Lang's theorem. They give rise to the twisted family of finite groups of Lie type [6]. The twisted Chevalley groups are given as algebraic group schemes over $\mathbb{F}_q$ for particular prime powers q, and therefore fall into the context given by Steinberg's theorem (see [17]). These twisted groups can be classified as follows:

- Classical Steinberg groups [32]:
  (1) ${}^2A_n$ over $\mathbb{F}_q$ with $q = p^{2k}$.
  (2) ${}^2D_n$ over $\mathbb{F}_q$ with $q = p^{2k}$.
- Exceptional Steinberg groups [32]:
  (3) ${}^2E_6$ over $\mathbb{F}_q$ with $q = p^{2k}$ (constructed also independently by Tits [37]).
  (4) ${}^3D_4$ over $\mathbb{F}_q$ with $q = p^{3k}$.

Note that the *twisting* implies that the roots of unity ϵ_i appearing in the theorem of Steinberg are not always 1 here, which can be seen from the calculation of orders of the finite groups of $\mathbb{F}_q$-rational points.

3.2.1 Classical Steinberg Groups

$\underline{G = {}^2A_n \; over \; \mathbb{F}_q \; with \; q = p^{2k}}$

$$\#G(\mathbb{F}_q) = q^{n(n+2)}(1+q^{-2})(1-q^{-3})\cdots(1-(-1)^{n+1}q^{-(n+1)})$$

$\underline{For \; n = 4a}$

$$\#\mathscr{B}G(\mathbb{F}_q) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_n=0}^{\infty}(-1)^{(i_1+i_3+\cdots+i_{n-1})}q^{-(2i_1+3i_2+\cdots+(n+1)i_n+n(n+2))}$$

$$\zeta_{\mathscr{B}G}(t) = \prod_{k_1=1}^{\infty}\prod_{k_2=1}^{\infty}\cdots\prod_{k_n=1}^{\infty}\left(1-(-1)^{(k_1+k_3+\cdots+k_{n-1})}q^{-\left(2k_1+3k_2+\cdots+(n+1)k_n+\frac{n(n-1)}{2}-2\right)}t\right)^{-1}$$

$\underline{For \; n = 4a+1}$

$$\#\mathscr{B}G(\mathbb{F}_q) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_n=0}^{\infty}(-1)^{(i_1+i_3+\cdots+i_n)}q^{-(2i_1+3i_2+\cdots+(n+1)i_n+n(n+2))}$$

$$\zeta_{\mathscr{B}G}(t) = \prod_{k_1=1}^{\infty}\prod_{k_2=1}^{\infty}\cdots\prod_{k_n=1}^{\infty}\left(1+(-1)^{(k_1+k_3+\cdots+k_n)}q^{-\left(2k_1+3k_2+\cdots+(n+1)k_n+\frac{n(n-1)}{2}-2\right)}t\right)^{-1}$$

$\underline{For\ n = 4a + 2}$

$$\#\mathscr{B}G(\mathbb{F}_q) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_n=0}^{\infty}(-1)^{(i_1+i_3+\cdots+i_{n-1})}q^{-(2i_1+3i_2+\cdots+(n+1)i_n+n(n+2))}$$

$$\zeta_{\mathscr{B}G}(t) = \prod_{k_1=1}^{\infty}\prod_{k_2=1}^{\infty}\cdots\prod_{k_n=1}^{\infty}\left(1+(-1)^{(k_1+k_3+\cdots+k_{n-1})}q^{-\left(2k_1+3k_2+\cdots+(n+1)k_n+\frac{n(n-1)}{2}-2\right)}t\right)^{-1}$$

$\underline{For\ n = 4a + 3}$

$$\#\mathscr{B}G(\mathbb{F}_q) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_n=0}^{\infty}(-1)^{(i_1+i_3+\cdots+i_n)}q^{-(2i_1+3i_2+\cdots+(n+1)i_n+n(n+2))}$$

$$\zeta_{\mathscr{B}G}(t) = \prod_{k_1=1}^{\infty}\prod_{k_2=1}^{\infty}\cdots\prod_{k_n=1}^{\infty}\left(1-(-1)^{(k_1+k_3+\cdots+k_n)}q^{-\left(2k_1+3k_2+\cdots+(n+1)k_n+\frac{n(n-1)}{2}-2\right)}t\right)^{-1}$$

$\underline{G = {}^2D_n \text{ over } \mathbb{F}_q \text{ with } q = p^{2k}}$

$$\#G(\mathbb{F}_q) = q^{n(2n-1)}(1+q^{-n})(1-q^{-2})(1-q^{-4})\cdots(1-q^{-(2n-2)})$$

$$\#\mathscr{B}G(\mathbb{F}_q) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_n=0}^{\infty}(-1)^{i_1}q^{-(ni_1+2i_2+4i_3+\cdots+(2n-2)i_n+n(2n-1))}$$

$$\zeta_{\mathscr{B}G}(t) = \prod_{k_1=1}^{\infty}\prod_{k_2=1}^{\infty}\cdots\prod_{k_n=1}^{\infty}\left(1+(-1)^{k_1}q^{-(nk_1+2k_2+4k_3+\cdots+(2n-2)k_n+n(n-1))}t\right)^{-1}$$

3.2.2 Exceptional Steinberg Groups

$\underline{G = {}^2E_6 \text{ over } \mathbb{F}_q \text{ with } q = p^{2k}}$

$$\#G(\mathbb{F}_q) = q^{78}(1-q^{-2})(1+q^{-5})(1-q^{-6})(1-q^{-8})(1+q^{-9})(1-q^{-12})$$

$$\#\mathscr{B}G(\mathbb{F}_q) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_6=0}^{\infty}(-1)^{i_2+i_5}q^{-(2i_1+5i_2+6i_3+8i_4+9i_5+12i_6+78)}$$

$$\zeta_{\mathscr{B}G}(t) = \prod_{k_1=1}^{\infty}\prod_{k_2=1}^{\infty}\cdots\prod_{k_6=1}^{\infty}\left(1-(-1)^{(k_2+k_5)}q^{-(2k_1+5k_2+6k_3+8k_4+9k_5+12k_6+36)}t\right)^{-1}$$

$G = {}^3D_4$ *over* $\mathbb{F}_q$ *with* $q = p^{3k}$

$$\#G(\mathbb{F}_q) = q^{28}(1-q^{-2})(1-\xi q^{-4})(1-\xi^2 q^{-4})(1-q^{-6}) \text{ with } \xi^3 = 1, \xi \neq 1$$

$$\#\mathscr{B}G(\mathbb{F}_q) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty}\sum_{i_4=0}^{\infty} \xi^{(i_2+2i_3)} q^{-(2i_1+4i_2+4i_3+6i_4+28)}$$

$$\zeta_{\mathscr{B}G}(t) = \prod_{k_1=1}^{\infty}\prod_{k_2=1}^{\infty}\prod_{k_3=1}^{\infty}\prod_{k_4=1}^{\infty} \left(1 - \xi^{(k_2+2k_3)} q^{-(2k_1+4k_2+4k_3+6k_4+13)} t\right)^{-1}$$

Remark 3.4 It is well known, that there exists another exotic infinite class of finite simple groups of Lie type, namely the Suzuki and Ree groups constructed by Suzuki [36] and Ree [28, 29] respectively. They are described as follows:

(1) ${}^2B_2(\mathbb{F}_{q^2})$ with $q^2 = 2^{2n+1}$ [36].
(2) ${}^2G_2(\mathbb{F}_{q^2})$ with $q^2 = 3^{2n+1}$ [28].
(3) ${}^2F_4(\mathbb{F}_{q^2})$ with $q^2 = 2^{2n+1}$ [29].

These are again twisted versions of groups of Lie type, but they cannot be interpreted directly as groups of $\mathbb{F}_q$-rational points of some algebraic group in the way described above. However, one could heuristically consider, for example, the group ${}^2B_2$ as being an algebraic group over a "field" with $2^{(n+\frac{1}{2})}$ elements, which, of course, does not exist. On the other hand de Medts and Naert [24] have recently developed a general framework in which these groups can be obtained as groups of rational points of algebraic groups over a "twisted field" by appropriately extending the category of schemes to a category of "twisted schemes". Such a twisted field can then be interpreted like a "field with $\sqrt{p}$ elements". Following along this path, one could consider also a category of twisted algebraic stacks and related Lefschetz trace formula which would then allow to do similar considerations as above for classifying stacks of algebraic groups over "twisted fields". This will be part of a follow-up project.

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Cyclic Symmetry on Complex Tori and Bagnera–De Franchis Manifolds

Fabrizio Catanese

Abstract We describe the possible linear actions of a cyclic group $G = \mathbb{Z}/n$ on a complex torus, using the cyclotomic exact sequence for the group algebra $\mathbb{Z}[G]$. The main application is devoted to a structure theorem for Bagnera–De Franchis Manifolds (these are the quotients of a complex torus by a free action of a cyclic group), but we also give an application to hypergeometric integrals, namely, we describe the intersection product and Hodge structures for the homology of fully ramified cyclic coverings of the projective line.

Keywords Complex Tori · Hyperelliptic Manifolds · Bagnera-De Franchis Manifolds · Hodge Structures · Cyclic Coverings · Group Algebra · Factorial Rings · Cyclotomic Rings · Resultants of Cyclotomic Polynomials · Fundamental Groups

AMS Classification 14K99 · 14D99 · 32Q15 · 32M17 · 32Q57 · 11A07 · 11R18 · 13C05

1 Introduction

Classically, the word 'hyperelliptic' was used for two different ways of generalizing the class of elliptic curves, i.e. the complex tori of dimension 1.

Hyperelliptic curves are defined to be the curves who admit a map to $\mathbb{P}^1$ of degree 2, and are not Hyperelliptic Varieties according to the definition of the French school of Appell, Humbert, Picard, Poincaré.

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The French school defined the Hyperelliptic Varieties as those smooth projective varieties whose universal covering is biholomorphic to $\mathbb{C}^g$ (in particular the Abelian varieties are in this class). For $g = 1$ these are just the elliptic curves, whereas a prize, the Bordin prize, was offered for those mathematicians who would achieve the classification of the Hyperelliptic varieties of dimension 2. Enriques and Severi were awarded the Prize in 1907 [19], but they withdrew their first paper after discussion with De Franchis (replacing it with a second one); Bagnera and De Franchis were awarded the Bordin Prize in 1909, they gave a simpler proof [2] apart of a small gap; perhaps for this reason we prefer to call Bagnera–De Franchis surfaces the Hyperelliptic surfaces which are not Abelian surfaces.

Kodaira [21] showed that if we take the wider class of compact complex manifolds of dimension 2 whose universal covering is $\mathbb{C}^2$, then there are other non algebraic and non Kähler surfaces, called nowadays Kodaira surfaces.

Based on Kodaira's work, Iitaka conjectured that if a compact Kähler Manifold X has universal covering biholomorphic to $\mathbb{C}^g$, then necessarily X is a quotient $X = T/G$ of a complex torus T by the free action of a finite group G (which we may assume to contain no translations).

The conjecture by Iitaka had been proven in dimension 2 by Kodaira, and was much later proven in dimension 3 by Campana and Zhang [6]. Whereas it was shown in [14] that, if the abundance conjecture holds, then a projective smooth variety X with universal covering $\mathbb{C}^n$ is a Hyperelliptic variety according to the following definition.

Definition 1.1 A Hyperelliptic Manifold X is defined to be a quotient $X = T/G$ of a complex torus T by the free action of a finite group G which contains no translations.

We say that X is a Hyperelliptic Variety if moreover the torus T is projective, i.e., it is an Abelian variety A.

The study of Hyperelliptic manifolds in dimension three was begun in [28], continued in [23], who used earlier results of [20], and concluded then in [12, 13].

In all dimensions the Teichmüller spaces of Hyperelliptic Manifolds were described in [9].

If the group G is a cyclic group $\mathbb{Z}/n$, then such a quotient is called [3, 10] a Bagnera–De Franchis manifold (in dimension $g = 2$, G is necessarily cyclic, whereas in dimension $g \geq 3$ the only examples with G non Abelian were first discovered via a computer search in [15], have $G = D_4$ and were classified in [13] (for us D_4 is the dihedral group of order 8).

Indeed, (see for instance [11]) every Hyperelliptic Manifold is a deformation of a Hyperelliptic Variety, so that a posteriori the two notions are related to each other, in particular the set of underlying differentiable manifolds is the same.

By the so-called Bieberbach's third theorem [4, 5] concerning the finiteness of Euclidean crystallographic groups, Hyperelliptic manifolds of a fixed dimension g belong to a finite number of families.

We give in this paper an explicit boundedness result for the case of Bagnera–De Franchis Manifolds, for which G is a finite cyclic group (Theorem 6.1).

The theory of Bagnera–De Franchis Manifolds was introduced in [3], and also expounded in [10] (following ideas introduced first in [7]) and our main motivation here was to expand and improve the presentation given there, whereas we refer the reader to [16] for a classification of Bagnera–De Franchis Manifolds of low dimension.

In order to do so, it is necessary to treat linear actions of a finite group G on a complex torus, and we do this here for the case of a cyclic group. The subtle point is to describe these actions not only up to isogeny, but determining explicit the torsion subgroups involved in these isogenies.

The starting point is that a linear action of a group G on a complex torus consists in two steps:

(1) viewing $\Lambda := H_1(T, \mathbb{Z})$ as a $\mathbb{Z}[G]$-module
(2) choosing an appropriate Hodge decomposition on $\Lambda \otimes \mathbb{C} := \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$

$$\Lambda \otimes \mathbb{C} = H^{1,0} \oplus \overline{H^{1,0}}$$

which is invariant for the group action (i.e., $H^{1,0}$ is a G-invariant subspace).

While (2) uses, for G cyclic, just the eigenspace decomposition, (1) requires us to explain in detail some elementary and mostly well known facts about the group algebra of a cyclic group G, $\mathbb{Z}[G] = \mathbb{Z}[x]/(x^n - 1)$.

This is derived in Sect. 4 from an elementary generalization of the Chinese remainder theorem (Theorem 2.1), concerning quotients of factorial rings by principal ideals (these appear naturally in the intersection theory of divisors), and from some classical results about resultants of cyclotomic polynomials, explained in Sect. 3. The interesting result for our purposes is Proposition 4.3.

The first application that we give is related to hypergeometric integrals: we calculate explicitly the homology of a cyclic covering of degree n of the projective line $\mathbb{P}^1$ as a $\mathbb{Z}[\mathbb{Z}/n]$-module, under the assumption that there is a point of full ramification. The description is particularly nice for the case where there are two points of full ramification, Theorem 5.3 shows that we have a direct sum of certain cyclic modules naturally associated to the ramification indices.

Theorem 5.3 *Assume that we have a cyclic covering $f : C \to \mathbb{P}^1$ with group $\mathbb{Z}/n$ and with two points of full ramification.*

Then, if the order of the inertia groups are $n, r_1, \dots, r_k, n$, the $\mathbb{Z}[x]$-module $H_1(C, \mathbb{Z})$ is a direct sum of cyclic modules,

$$H_1(C, \mathbb{Z}) = \oplus_1^k \mathbb{Z}[x]/(1 + x^{n/r_j} + x^{2n/r_j} + \dots + x^{(r_j - 1)n/r_j}).$$

In the final Sect. 7 we determine also explicitly the intersection product for the first homology group $H_1(C, \mathbb{Z})$.

We pose the question of finding a simple description in the general case.

Section 6 is devoted to the second and main application, namely, the algebraic description of Bagnera–De Franchis Manifolds with group $\mathbb{Z}/n$.

The main result is Theorem 6.1 (see Sect. 3 for the notation used).

Theorem 6.1 *A Bagnera–De Franchis Manifold with group $G = \mathbb{Z}/n$ is completely determined by the following data:*

(1) the datum of torsion free R_d-modules Λ_d of finite rank, for all $d|n$, such that $\Lambda_1 \neq 0$ and with Λ_1, Λ_2 of even rank;
(2) the datum of a finite subgroup $\Lambda^0 \subset A' := \oplus_{d|n} A_d$, where $A_d := \Lambda_{d,\mathbb{R}}/\Lambda_d$,
(3) an element $\beta_1 \in A_1$ generating a subgroup $\langle \beta_1 \rangle$ of order exactly n, such that:
(4) (A) Λ^0 is stable for multiplication by the element x of the subring $R(n) \subset R'(n) := \oplus_{d|n} R_d$, and
(5) (B) $\Lambda^0 \cap A_d = 0 \; \forall d|n$,
(6) (C) the projection of Λ^0 into A_1 intersects the subgroup $\langle \beta_1 \rangle$ only in 0;
(7) the datum of a complex structure on each $\Lambda_d \otimes \mathbb{C}$, i.e., a Hodge decomposition

$$\Lambda_d \otimes \mathbb{C} = V(d) \oplus \overline{V(d)},$$

which allows to decompose $V(d) = \oplus_{j<d,(j.d)=1} V_j$ as a direct sum of eigenspaces for the action α of x.
(8) The properties (A) and (B) imply that $\Lambda^0 \subset \oplus_{d|n}(\frac{\Phi_d}{Q_n}\Lambda_d)/\Lambda_d$, hence, in particular, the number of such subgroups Λ^0 is finite.

Throughout the paper we have been trying to illustrate the concepts introduced, or discussed, via many concrete examples.

2 An Exact Sequence in Factorial Rings

Theorem 2.1 *Let $\mathcal{O}$ be a factorial ring, and assume that we have an integer $k \geq 2$ and elements $f_1, \dots, f_k \in \mathcal{O}$, such that*

(1) f_i is not a unit
(2) for $i \neq j$, f_i and f_j are relatively prime.
Then we have a natural exact sequence

$$0 \to R := \mathcal{O}/(f_1 f_2 \dots f_k) \to \oplus_1^k \mathcal{O}/(f_i) \to \oplus_{i<j} \mathcal{O}/(f_i, f_j) \to 0,$$

where, setting $R_i := \mathcal{O}/(f_i)$, $R_{i,j} := \mathcal{O}/(f_i, f_j)$ for $i \neq j$,
(3) $R \to \oplus_1^k R_i$ is induced by the natural surjections $R \to R_i = \mathcal{O}/(f_i)$, and where
(4) $(a_i) \in \oplus_1^k R_i \mapsto (a_i - a_j) \in \oplus_{i<j} \mathcal{O}/(f_i, f_j)$.

Remark 2.2 Observe that the hypothesis that f_i is not a unit is not really needed, since, if f_i is a unit, then $R_i = 0 = R_{i,j} \; \forall j$, and we have the same exact sequence as the one corresponding to the set obtained from the set $\{f_1, \dots, f_k\}$ by deleting f_i.

The proof follows essentially by induction from the standard special case $k = 2$:

Lemma 2.3 *Assume either that*

(I) *$\mathcal{O}$ is a factorial ring, and $f, g \in \mathcal{O}$ are relatively prime elements.*
Or assume that

(II) *$\mathcal{O}$ is any ring, the ideal (f) is prime, $g \notin (f)$.*
Then we have a natural exact sequence

$$0 \to R := \mathcal{O}/(fg) \to \mathcal{O}/(f) \oplus \mathcal{O}/(g) \to \mathcal{O}/(f, g) \to 0.$$

Proof We first prove the exactness of

$$\mathcal{O} \to \mathcal{O}/(f) \oplus \mathcal{O}/(g) \to \mathcal{O}/(f, g) \to 0.$$

Surjectivity is obvious, whereas if $(a \ (mod \ f), b \ (mod \ g))$ maps to 0, then

$$a - b \in (f, g) \leftrightarrow \exists \alpha, \beta \in \mathcal{O}, a - b = \alpha f + \beta g \leftrightarrow a - \alpha f = b + \beta g := c.$$

But then $c - a \in (f), c - b \in (g)$, proving exactness in the middle.

Finally, $c \in \mathcal{O} \mapsto (0, 0)$ if and only if $c \in (f) \cap (g)$.

In case (I), by unique factorization and since f, g are relatively prime, c is divisible by fg, hence the kernel of the first homomorphism is the principal ideal (fg).

In case (II), $c \in (g)$ implies the existence of d such that $c = dg$. Since $c \in f$, and (f) is prime, then necessarily either $g \in (f)$, or $d \in (f)$. The first possibility is excluded by our assumption, therefore there exists $e \in \mathcal{O}$ with $d = ef$, hence $c = efg$, and we are done. □

Corollary 2.1 *Let $\mathcal{O}$ be a factorial ring, and assume that we have an integer $k \geq 2$ and elements $f_1, \ldots, f_k \in \mathcal{O}$, such that*

(1) *f_i is not a unit*
(2) *for $i \neq j$, f_i and f_j are relatively prime.*

Then, setting $F := f_1 f_2 \ldots f_k$, the cokernel N of the following exact sequence:

$$0 \to R := \mathcal{O}/(F) \to \oplus_1^k \mathcal{O}/(f_i) \to N \to 0$$

has a filtration

$$0 := N_0 \subset N_1 \subset \cdots \subset N_{k-1} = N$$

such that $N_i/N_{i-1} \cong \mathcal{O}/(f_i, f_{i+1} \ldots f_k)$.

Proof Observe first of all that $R \to \oplus_1^k R_i$ is an inclusion, since our elements f_i are relatively prime.

We prove now the main assertion by induction on k, the case $k = 2$ being the content of Lemma 2.3.

Set $g := f_2 \ldots f_k$, and observe that $F = f_1 g$: by Lemma 2.3 we have an exact sequence

$$0 \to R = \mathcal{O}/(F) \to \mathcal{O}/(f_1) \oplus \mathcal{O}/(f_2 \dots f_k) \to \mathcal{O}/(f_1, f_2 \dots f_k) \to 0.$$

By induction we have an exact sequence

$$0 \to R' := \mathcal{O}/(f_2 \dots f_k) \to \oplus_2^k \mathcal{O}/(f_i) \to N' \to 0,$$

and a filtration $N'_0 \subset \cdots N'_{k-1} = N'$ with the desired properties.

Hence we have inclusions

$$0 \to R \to R_1 \oplus R' \to \oplus_1^k R_i$$

and, defining $N_1 := \mathcal{O}/(f_1, f_2 \dots f_k)$, we have $N' = N/N_1$, and it suffices to define N_i, for $i \geq 2$, to be the inverse image of N'_{i-1} inside N. □

Lemma 2.4 *Let $f, h, g \in \mathcal{O}$ and assume either that*

(i) the ring $\mathcal{O}$ is factorial, and f, g are relatively prime, or
(ii) the ideal (f) is a prime ideal, and $g \notin (f)$.
Then we have the exact sequence, where the first map is given by multiplication by g:

$$0 \to \mathcal{O}/(f, h) \to \mathcal{O}/(f, gh) \to \mathcal{O}/(f, g) \to 0.$$

Proof Set $A := \mathcal{O}/(f)$.

Then we have an exact sequence

$$A \to A/(gh) \to A/(g) \to 0,$$

and the kernel of the first map is the principal ideal generated by h, since

$$\{\phi \in A | \phi g \in (gh)\} = \{\phi | \exists \psi \ \phi g = gh\psi\} = \{\phi | \exists \psi \ \phi = h\psi\},$$

because g is not a zero divisor in A. □

Lemma 2.5 *Let $\mathcal{O}$ be a factorial ring, and assume that we have an integer $k \geq 2$ and elements $f_1, \dots, f_k \in \mathcal{O}$, such that*

(1) f_i is not a unit
(2) for $i \neq j$, f_i and f_j are relatively prime.
Then $\mathcal{O}/(f_1, f_2 \dots f_k)$ has a filtration whose graded quotient is

$$\oplus_{j=2}^k \mathcal{O}/(f_1, f_j).$$

Proof Apply Lemma 2.4 to $f := f_1$, $h := f_k$, and $g := f_2 \dots f_{k-1}$, and use induction. □

***Proof of Theorem* 2.1** We first of all observe that the map $\oplus_1^k R_i \to M := \oplus_{i<j} R_{i,j}$ factors through the quotient N.

We have shown that N has a filtration whose associated graded ring is exactly isomorphic to $\oplus_{i<j} R_{i,j}$, and from this we shall derive an isomorphism of N with $\oplus_{i<j} R_{i,j}$.

In fact, by induction, the homomorphism $N \to M$ induces an isomorphism $N' \cong \oplus_{i<j, i,j\geq 2} R_{i,j} := M'$.

Moreover, by the definition of the map, $R_1 \oplus R'$ maps to zero inside M'.

Hence it suffices to show that $(R_1 \oplus R')/R = N_1$ maps isomorphically to $M_1 := \oplus_{j\geq 2} R_{1,j}$, and this follows again by Corollary 2.5 observing that the homomorphism preserves the corresponding filtrations on both modules M_1, N_1 and that the homomorphism induces an isomorphism of associated graded modules: since, by induction (changing the order of the summands), for each j, N_1 surjects onto $R_{1,j}$.

Hence this homomorphism induces an isomorphism $N_1 \cong M_1$ and the proof is finished. □

We record here a result shown in the course of the proof of Theorem 2.1:

Corollary 2.2 *Let $\mathcal{O}$ be a factorial ring, and assume that we have an integer $k \geq 2$ and elements $f_1, \ldots, f_k \in \mathcal{O}$, such that*

(1) f_i is not a unit
(2) for $i \neq j$, f_i and f_j are relatively prime.
Then the $\mathcal{O}$-module $\mathcal{O}/(f_1, f_2 \ldots f_k)$ is isomorphic to

$$\oplus_{j=2}^k \mathcal{O}/(f_1, f_j).$$

3 An Arithmetic Application

In this section we consider the factorial ring $\mathcal{O} := \mathbb{Z}[x]$, set

$$Q_n := x^n - 1, \quad R(n) := \mathbb{Z}[x]/(Q_n).$$

We have, setting $\mu_n := \{\zeta \in \mathbb{C} | \zeta^n = 1\}$,

$$Q_n(x) = \Pi_{\zeta \in \mu_n}(x - \zeta),$$

and we have an irreducible decomposition in $\mathbb{Z}[x]$

$$Q_n(x) = \Pi_{d|n} \Phi_d(x),$$

where $\Phi_d(x)$ is the dth cyclotomic polynomial

$$\Phi_d(x) = \Pi_{\zeta \in \mu_n,\ ord(\zeta)=d}(x - \zeta).$$

We have (see [22], page 280),

$$\Phi_d(x) = \Pi_{d'|d}(x^{d/d'} - 1)^{\mu(d')},$$

where $\mu(d')$ is the Möbius function, such that

- $\mu(d') = 0$ if d' is not square-free
- $\mu(1) = 1$
- $\mu(p_1 \dots p_r) = (-1)^r$, if $p_1, \dots, p_r$ are distinct primes.

Definition 3.1 Define the **cyclotomic ring** as $R_d := \mathbb{Z}[x]/(\Phi_d)$, and, for integers $d \neq m$, $R_{d,m} := \mathbb{Z}[x]/(\Phi_d, \Phi_m)$.

In view of the above notation we obtain the important special case of (2) of Theorem 2.1, namely the following exact sequence:

$$0 \to R(n) \to R'(n) := \oplus_{d|n} R_d \to R^0(n) := \oplus_{d_1 < d_2} R_{d_1,d_2} \to 0 \tag{1}$$

We are going next to describe the rings R_{d_1,d_2}.

Example 3.2 Consider the polynomial $Q_6 = x^6 - 1 = (x^3 - 1)(x^3 + 1) = (x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1) = \Phi_1\Phi_3\Phi_2\Phi_6$.

We choose now $d = 3, m = 6$: then, since $\Phi_6 = \Phi_3 - 2x$, and $2(x^2 + x + 1) - 2x(x + 1) = 2$, we obtain that

$$(\Phi_3, \Phi_6) = (x^2 + x + 1, 2x) = (x^2 + x + 1, 2).$$

Hence $R_{3,6} = \mathbb{Z}/2[x]/(x^2 + x + 1) = \mathbb{F}_4$.

While $\mathbb{Z}/((\Phi_3, \Phi_6) \cap \mathbb{Z}) = \mathbb{Z}/2$.

Note that $r := Res_x(\Phi_3, \Phi_6)$ equals, by the interpolation formula, if ζ is a primitive third root of 1, $r = (\zeta^2 - \zeta + 1)(\zeta - \zeta^2 + 1) = (-2\zeta)(-2\zeta^2) = 4 = |R_{3,6}|$.

Instead, easily we get $R_{1,3} = \mathbb{Z}/3$, $R_{1,6} = 0$, $R_{1,2} = \mathbb{Z}/2$, $R_{2,3} = 0$, $R_{2,6} = \mathbb{Z}/3$.

Observe now in general that, since Φ_d, Φ_m are monic polynomials, and both irreducible, their resultant is a non zero integer $r = r_{d,m}$, such that, if

$$\mathbb{Z}/((\Phi_m, \Phi_d) \cap \mathbb{Z}) = \mathbb{Z}/(\beta_{d,m}),$$

then $\beta_{d,m} | r_{d,m}$ and the two numbers have the same radical.

It is easy to see that $R_{d,m} = 0$ if d, m are relatively prime: since then in the quotient we have $x^m \equiv 1, x^d \equiv 1 \Rightarrow (x - 1) \equiv 0$, hence $m \equiv 0 \Rightarrow R_{d,m} = 0$.

It is straightforward to calculate the discriminant of $Q_n = x^n - 1$ as the resultant of Q_n and its derivative: $Disc(Q_n) = n^n$.

However, up to ± 1, $Disc(Q_n) = \Pi_{0 \le i < j \le n-1}(\epsilon_n^i - \epsilon_n^j)$, where $\epsilon_n := exp(2\pi i/n)$.

Since $Q_n = \Pi_{d|n}\Phi_d$, follows that

$$n^n = Disc(Q_n) = \Pi_{d,m|n,\ d<m} Res(\Phi_d, \Phi_m) \Pi_{d|n} Disc(\Phi_d).$$

The clever calculation of all the factors of the above product was found by Emma Lehmer [24] in 1930 (in her terminology a simple integer is what is today called a square-free integer) and then reproven with different proofs by several authors [1, 17, 18]; in particular, the calculation of $\beta_{d,m}$ can be found in an article [18] by Gregory Dresden.

We summarize these results by Lehmer, Diederichsen, Apostol, Dresden with a minor addition (here $\phi(d)$ is the Euler function, i.e., $\phi(d) = \deg \Phi_d$):

Theorem 3.3 *Let $d, m \in \mathbb{Z}$, $d < m$. Then, if $\beta_{d,m}$ is defined by:*

$$\mathbb{Z}/((\Phi_m, \Phi_d) \cap \mathbb{Z}) = \mathbb{Z}/(\beta_{d,m}),$$

then $\beta_{d,m}$ is ± 1 unless $d|m$ and there exists a prime p such that

$$m = p^k d,$$

in which case $\beta_{d,p^k d} = p$.

Moreover, $Res_x(\Phi_d, \Phi_{p^k d}) = p^{\phi(d)}$ in the latter case, and 1 otherwise.

In particular, $R_{m,d} = 0$ unless $d|m$ and there exists a prime p such that $m = p^k d$, and in this case $R_{d,p^k d}$ is a direct sum of finite fields $\mathbb{F}_{p^\nu}$ if and only if d is not divisible by p.

Moreover, $R_{d,p^k d}$ is a field $\mathbb{F}_{p^{\phi(d)}}$ if and only if the class of p generates the group $(\mathbb{Z}/d)^*$.

Proof Only the last assertions need to be proven, since the rest is contained in the cited articles.

Clearly $R_{m,d} = 0$ if $\beta_{d,m}$ is ± 1, since then $1 \equiv 0$.

If instead $\beta_{d,m} = p$, $R_{d,p^k d}$ is an $\mathbb{F}_p = \mathbb{Z}/p$ module, and

$$R_{d,p^k d} = \mathbb{F}_p[x](\Phi_d, \Phi_{p^k d}) = \mathbb{F}_p[x]/(P),$$

where P is the G.C.D. of (the reductions $\Psi_d, \Psi_{p^k d}$ of) $\Phi_d, \Phi_{p^k d}$ inside $\mathbb{F}_p[x]$.

(i) If d is not divisible by p, then the polynomial $x^d - 1$ is square free, and $\mathbb{F}_p[x]/(x^d - 1)$ is a direct sum of fields. A fortiori $\mathbb{F}_p[x]/(P)$ is a direct sum of fields. Indeed we shall show next that $P = \Psi_d$.

(ii) The next question is to show that $\Psi_d | \Psi_{p^k d}$, so that $P = \Psi_d$ for d not divisible by p.

We use the previously cited formula for the cyclotomic polynomial Φ_D when $D = p^k d$ and p does not divide d (using the fact that only the terms with D' square-free occur):

$$\Phi_D(x) = \Pi_{D'|D}(x^{D/D'} - 1)^{\mu(D')} =$$

$$= \Pi_{d'|d}(x^{dp^k/d'} - 1)^{\mu(d')} \Pi_{d'|d}(x^{dp^{k-1}/d'} - 1)^{-\mu(d')}.$$

From this we derive, reducing modulo p,

$$\Psi_D(x) = \Psi_{p^k d}(x) = \Psi_d(x)^{(p^k - p^{k-1})} = \Psi_d(x)^{(p-1)p^{k-1}}.$$

(iii) In the case of $D = p^k d$, and where $p|d$, we write $d = d_1 p^h$, with d_1 not divisible by p.
The formulae we have just established in (ii) imply

$$\Psi_{p^k d}(x) = \Psi_{p^{k+h} d}(x) = \Psi_{d_1}(x)^{(p^{k+h-1}(p-1))},$$

$$\Psi_d(x) = \Psi_{p^h d_1}(x) = \Psi_{d_1}(x)^{(p^{h-1}(p-1))},$$

hence again the G.C.D. P equals $\Psi_d(x) = \Psi_{d_1}(x)^{(p^{h-1}(p-1))}$, and $\mathbb{F}_p[x]/(P)$ is an algebra with nilpotents.

(iv) Finally, remains to answer the question: when is the reduction Ψ_d irreducible? Certainly not in the case where $d|(p-1)$ and Ψ_d then splits as a product of linear factors (e.g. $\Psi_4 = (x^2+1) = (x+2)(x+3) \in \mathbb{F}_5[x]$).
In general, consider the splitting field of $(x^d - 1)$ as an extension of $\mathbb{F}_p$. It will be the smallest $\mathbb{F}_{p^k}$ which contains the dth roots of 1, hence k shall be the smallest integer such that $d|p^k - 1 \Leftrightarrow p^k \equiv 1 (mod\ d)$.
Hence Ψ_d irreducible iff the splitting field has degree $k = \phi(d)$, equivalently p is a generator of the group $(\mathbb{Z}/d)^*$. □

Example 3.4 (i) Consider $R_{4,8}$. It equals the algebra with nilpotents

$$\mathbb{Z}[x]/(x^2+1, x^4+1) = \mathbb{Z}[x]/(x^2+1, -x^2+1) = \mathbb{Z}[x]/(x^2+1, 2) = \mathbb{F}_2[x]/(1+x)^2.$$

4 $R(n) := \mathbb{Z}[C_n] = \mathbb{Z}[x]/(x^n - 1)$-Modules Which Are Torsion Free Abelian Groups

In this section C_n denotes the cyclic group with n elements, $C_n \cong \mathbb{Z}/n$.

Hence the group algebra $\mathbb{Z}[C_n]$ is isomorphic to

$$R(n) := \mathbb{Z}[x]/(Q_n) = \mathbb{Z}[x]/(x^n - 1)$$

and we can apply the results of the previous section.

Let Λ be an $R(n)$-module, and assume that Λ is a finitely generated torsion free Abelian group.

This hypothesis allows us to view Λ as a lattice in the $\mathbb{Q}$-vector space $\Lambda \otimes \mathbb{Q}$, which is therefore also an $R(n)$-module, and an $R(n) \otimes \mathbb{Q}$-module.

Since

$$R(n) \otimes \mathbb{Q} = \mathbb{Q}[x]/(Q_n) = \mathbb{Q}[x]/(\Pi_{d|n} \Phi_d) = \oplus_{d|n} \mathbb{Q}[x]/(\Phi_d) = \oplus_{d|n} R_d \otimes \mathbb{Q},$$

and accordingly (see for instance Lemma 24, page 313 of [10]) we have a splitting

$$\Lambda \otimes \mathbb{Q} = \oplus_{d|n} \Lambda_{d,\mathbb{Q}},$$

where $\Lambda_{d,\mathbb{Q}}$ is an $R_d \otimes \mathbb{Q}$-module, and an $R(n) \otimes \mathbb{Q}$-module via the projection $R(n) \otimes \mathbb{Q} \to R_d \otimes \mathbb{Q}$.

Definition 4.1 We define $\Lambda_d := \Lambda \cap \Lambda_{d,\mathbb{Q}}$. It is a lattice in $\Lambda_{d,\mathbb{Q}}$, so that we have an exact sequence

$$0 \to \oplus_{d|n} \Lambda_d \to \Lambda \to \Lambda^0 \to 0,$$

where Λ^0 is a finite Abelian group.

Λ_d is an R_d-module in view of the exact sequence (1) established in Sect. 3:

$$0 \to R(n) \to R'(n) = \oplus_{d|n} R_d \to R^0(n) = \oplus_{d_1 < d_2} R_{d_1,d_2} \to 0,$$

which shows that $R(n)$ acts on $\Lambda_{d,\mathbb{Q}}$ via the homomorphism $R(n) \to R_d$.

We can make the geometry of the above exact sequences more transparent if we introduce the associated real tori.

Definition 4.2 Given an $R(n)$-module Λ, which is a lattice, i.e., a free Abelian group of finite rank, we define the associated tori as:

(1) $A := (\Lambda \otimes \mathbb{R})/\Lambda$, and since
(2) $\Lambda \otimes \mathbb{R} = \oplus_{d|n} \Lambda_{d,\mathbb{R}}$, we define
(3) $A_d := \Lambda_{d,\mathbb{R}}/\Lambda_d$, hence
(4) we have an exact sequence

$$0 \to \Lambda^0 \to \oplus_{d|n} A_d \to A \to 0,$$

identifying the cokernel Λ^0 as a finite subgroup of the product torus $A' := \oplus_{d|n} A_d$, isogenous to A,
(5) $R'(n) := \oplus_{d|n} R_d$ acts on A', (see the notation of equation (1)).

Proposition 4.1 *The datum of an $R(n)$ module which is a lattice, i.e., a free Abelian group of finite rank, is equivalent to the datum of torsion free R_d-modules Λ_d of finite rank, and of a finite subgroup $\Lambda^0 \subset A' := \oplus_{d|n} A_d$, where $A_d := \Lambda_{d,\mathbb{R}}/\Lambda_d$, such that:*

(A) Λ^0 *is stable for the subring $R(n)$, which is equivalent to the requirement:* $x\Lambda^0 = \Lambda^0 (\Leftrightarrow x\Lambda^0 \subset \Lambda^0)$
(B) $\Lambda^0 \cap A_d = 0 \ \forall d|n$.
Properties (A) and (B) imply:
(C) $\frac{Q_n}{\Phi_d}(\lambda) \in \Lambda_d, \ \forall d|n, \forall \lambda \in \Lambda$;
hence, writing an element of Λ^0 as $(\lambda_d)_{d|n}$, we have

$$\lambda_d \in \left(\frac{\Phi_d}{Q_n}\Lambda_d\right)/\Lambda_d \cong \Lambda_d/\left(\frac{Q_n}{\Phi_d}\right),$$

and it follows that

(D) *The number of such finite subgroups Λ^0 is finite.*

Proof Clearly, Λ is determined by the subgroup $\Lambda^0 \subset A'$, and the property that Λ is stable for the subring $R(n)$ is equivalent to property (A) that $R(n)$ stabilizes Λ^0.

Property (B) ensures that $\Lambda \cap \Lambda_{d,\mathbb{R}} = \Lambda_d$.

Property (C) follows right away since $\frac{Q_n}{\Phi_d}(\lambda) \in \Lambda$, but its components in $\Lambda_{d',\mathbb{Q}}$ are $= 0$ for $d' \neq d$, hence this element lies in Λ_d.

Property (D) follows since $\Lambda^0 \subset \oplus_{d|n}(\frac{\Phi_d}{Q_n}\Lambda_d)/\Lambda_d$, which is finite group since $(\frac{\Phi_d}{Q_n}\Lambda_d)/\Lambda_d \cong \Lambda_d/(\frac{Q_n}{\Phi_d})$ is a finite module over the finite ring $\mathbb{Z}[x]/(\Phi_d, \frac{Q_n}{\Phi_d})$ that we have been describing in the previous section. □

The previous proposition is particularly useful in the case where the Dedekind ring R_d is a PID (Principal Ideal Domain), because then every torsion free R_d-module is free.

In fact, more generally (see [26]) every torsion free module over a Dedekind domain R is the direct sum of a free module with an ideal I, hence R is a PID iff every torsion free module is free.

However, how does the above description of $R(n)$-modules apply to the free module $R(n)$?

The answer is related to finding the inverse map in the Chinese remainder theorem, of which Theorem 2.1 is a generalization.

Proposition 4.2 *Consider the following module-homomorphism*

$$j : R'(n) = \oplus_{d|n} R_d \to R(n),$$

such that $j|_{R_d}$ is induced by multiplication with Q_n/Φ_d (recall that $R_d = \mathbb{Z}[x]/(\Phi_d)$).

(1) *Composing with the natural inclusion $i : R(n) \to R'(n)$ we obtain an injective map:*

$$\psi : R'(n) = \oplus_{d|n} R_d \to R'(n) = \oplus_{d|n} R_d,$$

which is of diagonal form.

(2) *We have $j(R_d) = R(n) \cap (R_d \otimes \mathbb{Q}) \subset R(n) \otimes \mathbb{Q}$.*

Proof (1) means that $\psi(R_d) \subset R_d$, which follows since $Q_n/\Phi_d \equiv 0 \in R_{d'}$ for $d' \neq d$.

(2) follows since $R(n) \cap (R_d \otimes \mathbb{Q})$ is the kernel of $R(n) \to \oplus_{d'|n, d' \neq d} R_{d'}$, hence it is an $R(n)$-module, hence an ideal in $R(n)$: and it must be the ideal generated by $\Pi_{d'|n,d'\neq d}\Phi_{d'} = Q_n/\Phi_d$. □

We also notice that $\psi \otimes \mathbb{Q}$ is an isomorphism, and that we have a surjection

$$Coker(\psi) \to Coker(i)$$

in view of the injective maps

$$j : R'(n) \to R(n), i : R(n) \to R'(n)$$

whose composition is ψ.

$Coker(\psi)$ is essentially the double of $Coker(i)$, since

$$Coker(\psi) = \oplus_{d|n} R_d/(Q_n/\Phi_d) = \oplus_{d|n}\mathbb{Z}[x]/(\Phi_d, Q_n/\Phi_d) = \oplus_{d|n}\mathbb{Z}[x]/(\Phi_d, \Pi_{d' \neq d}\Phi'_d).$$

Now,

$$R_d/(\Pi_{d' \neq d}\Phi'_d)$$

is by Corollary 2.2 isomorphic to the finite ring

$$\oplus_{d'|n, d' \neq d} R_{d,d'}.$$

Therefore, we have the surjection:

$$Coker(\psi) = \oplus_{d|n} R_d/(\Pi_{d' \neq d}\Phi'_d) \to Coker(i) = \oplus_{d,d'|n,\ d<d'} R_{d,d'}$$

and an exact sequence

$$0 \to Coker(j) \to Coker(\psi) \to Coker(i) \to 0,$$

and this shows that we have an isomorphism

$$Coker(j) \cong Coker(i) = \oplus_{d,d'|n,\ d<d'} R_{d,d'}.$$

We can summarize everything in the following

Proposition 4.3 *We have a sequence of inclusions:*

$$0 \to M' := \oplus_{d|n}(Q_n/\Phi_d)R_d \to R(n) \to R' := \oplus_{d|n} R_d,$$

such that

$$R^0(n) := R(n)/M' \subset R'/M' = \oplus_{d,d'|n,\ d \neq d'} R_{d,d'}$$

is identified as the submodule of R'/M',

$$R^0(n) = \{(a_{d,d'})|(a_{d,d'}) = (a_{d',d})\}.$$

Proof There remains only to prove the last assertion, which follows immediately from the observation that $R(n)$ is the kernel of the map to $\oplus_{d,d'|n,\ d<d'} R_{d,d'}$, given by taking differences $b_d - b_{d'}$ of the image of $b \in R(n)$ to R_d. □

Remark 4.3 Let us go back to Example 3.2, where the divisors of $n = 6$ are $1, 2, 3, 6$ and the only nonzero $R_{d,d'}$'s are:

- $R_{1,2} = \mathbb{Z}/2$ acted trivially by x, since $x \equiv 1$,
- $R_{1,3} = \mathbb{Z}/3$ acted trivially by x, since $x \equiv 1$,
- $R_{2,6} = \mathbb{Z}/3$, where x acts multiplying by -1, since $x + 1 \equiv 0$,
- $R_{3,6} = \mathbb{F}_4$, with $\mathbb{Z}/2$ basis $1, x$ and with $x^2 \equiv 1 + x$.

We can now apply the method of Proposition 4.1 to construct many $\Lambda^0 \subset R'/M' = \oplus_{d,d'|n,\ d\neq d'} R_{d,d'}$.

For instance, we may take

$$\Lambda^0 = \{(a_{d,d'}) | a_{1,2} = a_{2,1} = 0, a_{1,3} = a_{3,1}, a_{2,6} = a_{6,2}, a_{3,6} = a_{6,3} = 0\},$$

and we get a module different from $R(6)$.

5 Fully Ramified Cyclic Coverings of the Projective Line and Associated Hodge Structures

Let $f : C \to \mathbb{P}^1$ be a cyclic covering with Galois group

$$\mu_n = \{\zeta \in \mathbb{C} | \zeta^n = 1\} \cong \mathbb{Z}/n,$$

branched on $k + 2$ points $P_0 = 0$, $P_1 = 1$, $P_2, \dots, P_k, \infty$, and let us assume that C is the normalization of the affine curve of equation

$$y^n = x^{m_0}(x-1)^{m_1}(x-t_2)^{m_2} \dots (x-t_k)^{m_k},$$

where $P_i = \{x = t_i\}$ and where without loss of generality we may assume $1 \leq m_j < n$.

The local monodromy of the covering around the point P_i sends the standard local generator to the element $m_i \in \mathbb{Z}/n$, hence the inertia group of P_i is cyclic of order $r_i = \frac{n}{G.C.D.(n,m_i)}$.

Definition 5.1 One says that a Galois covering is **fully ramified** if there is a branch point whose inverse image consists of only one point.

In the case of curves, this implies that the Galois covering is cyclic with group $\mathbb{Z}/n$, and there is a branch point P_i with $r_i = n$.

In the following, we shall make the assumption that $f : C \to \mathbb{P}^1$ is fully ramified, and without loss of generality we may assume that $m_0 = 1$.

Our goal, in this section, is to describe $H_1(C, \mathbb{Z})$ as an $R(n) = \mathbb{Z}[\mu_n]$-module.

Recall that the fundamental group $\pi_1(C)$ is the kernel of the following exact sequence (see for instance [8], pages 101–104):

$$1 \to \pi_1(C) \to \mathcal{T} := T(0; r_0, r_1, \dots, r_k, r_\infty) \to \mathbb{Z}/n \to 0,$$

where the polygonal orbifold group $\mathcal{T} := T(0; r_0, r_1, \dots, r_k, r_\infty)$ has generators

$$\gamma_0, \gamma_1, \dots, \gamma_k, \gamma_\infty$$

and relations

$$\gamma_i^{r_i} = 1, \forall i, \;\; \gamma_0 \cdot \gamma_1 \cdot \dots \cdot \gamma_k \cdot \gamma_\infty = 1.$$

Breaking the symmetry, we shall see $\gamma_0, \gamma_1, \dots \gamma_k$ as generators for $\mathcal{T}$, and $\gamma_0^n = 1, \gamma_1^{r_1} = 1, \dots, \gamma_k^{r_k} = 1, (\gamma_0 \cdot \gamma_1 \cdot \dots \cdot \gamma_k)^{r_\infty} = 1$ as relations.

One sees then immediately that $\pi_1(C)$ is generated by $1 + n(k)$ elements, and, since $\gamma_0, \gamma_0^2, \dots, \gamma_0^{n-1}$ is a Schreier system, we can choose, by the Reidemeister–Schreier method ([25], theorem 2.7, page 89 and following) the following generators:

$$\gamma_0^n, \;\; \delta_{i,j} := \gamma_0^i \gamma_j \gamma_0^{-m_j - i}, i = 0, \dots, n-1, \; j = 1, \dots, k.$$

These generators are nice because the Galois group $\mathbb{Z}/n$ acts on $\pi_1(C)$ by conjugation of a lift of $i \in \mathbb{Z}/n$, hence by conjugation by γ_0^i. Hence these elements are permuted by the Galois group.

It is obvious that we can forget about the first generator γ_0^n in view of the relation $\gamma_0^n = 1$.

The Hurwitz formula calculates the genus g of the curve C as follows:

$$2g - 2 = n\left(-2 + \sum_{i=0,\dots,k,\infty} \frac{(r_i - 1)}{r_i}\right) = \sum_{i=0,\dots,k,\infty} \left(n - \frac{n}{r_i}\right) - 2n,$$

hence

$$2g = \sum_{i=1,\dots,k,\infty} \left(n - \frac{n}{r_i}\right) - n + 1 = \sum_{i=1,\dots,k} \left(n - \frac{n}{r_i}\right) + 1 - \frac{n}{r_\infty}$$

We shall further reduce the number of generators using the other relations, until we reach $2g$ generators: the classes of these in $H_1(C, \mathbb{Z}) = \pi_1(C)^{ab}$ will then give a basis for the first homology group.

Indeed, we can rewrite:

$$1 = \gamma_j^{r_j} = \gamma_0^i \gamma_j^{r_j} \gamma_0^{-i} = \delta_{i,j} \delta_{i+m_j,\, j} \delta_{i+2m_j,\, j} \dots \delta_{i+(r_j-1)m_j,\, j}.$$

Case $r_\infty = n$: We can eliminate in this way, since $(m_j) \subset \mathbb{Z}/n$ equals $(n/r_j) \subset \mathbb{Z}/n$, $\sum_{i=1,\dots,k} \frac{n}{r_i}$ generators, and we obtain the right number of generators, $\sum_{i=1,\dots,k} (n - \frac{n}{r_i}) = 2g$ generators. The rewriting of the relations coming from $\gamma_\infty^{r_\infty} = 1$ yields the standard relation for $\pi_1(C)$.

Definition 5.2 Define $D_{i,j}$ as the class of $\delta_{i,j}$ inside $H_1(C, \mathbb{Z})$, for $i = 0, \dots, n-1$, and $j = 1, \dots k$.

Then the previous relation rewrites as

$$\sum_{h=0,\dots,r_j-1} D_{i+hn/r_j, j} = 0.$$

We have therefore proven

Theorem 5.3 *Assume that we have a cyclic covering $f : C \to \mathbb{P}^1$ with group $\mathbb{Z}/n$ and with two points of full ramification.*

Then, if the order of the inertia groups are $n, r_1, \dots, r_k, n$, the $\mathbb{Z}[x]$-module $H_1(C, \mathbb{Z})$ is a direct sum of cyclic modules,

$$H_1(C, \mathbb{Z}) = \oplus_1^k (\mathbb{Z}[x]/(1 + x^{n/r_j} + x^{2n/r_j} + \dots + x^{(r_j-1)n/r_j}).$$

Example 5.4 This example is borrowed from the modular description of the Cartwright–Steger surface, which was first explained to me by Domingo Toledo.

Assume that we have $n = 12$, and $m_0 = 7, m_1 = m_2 = m_3 = 2, m_\infty = 11$, hence ramification indices (12, 6, 6, 6, 12).

Then $H_1(C, \mathbb{Z}) \cong \oplus_1^3 \mathbb{Z}[x]/(x^{10} + x^8 + \dots + 1)$.

Since $x^{10} + x^8 + \dots + 1 = (x^{12} - 1)/(x^2 - 1) = \Phi_3\Phi_4\Phi_6\Phi_{12}$, we see that, by the previous results, setting

$$M := \mathbb{Z}[x]/(x^{10} + x^8 + \dots + 1),$$

we have $H_1(C, \mathbb{Z}) \cong 3M$, and we have the usual sandwich

$$R_3 \oplus R_4 \oplus R_6 \oplus R_{12} \supset M \supset \Phi_4\Phi_6\Phi_{12}M \oplus \Phi_3\Phi_6\Phi_{12}M \oplus \Phi_3\Phi_4\Phi_{12}M \oplus \Phi_3\Phi_4\Phi_6M,$$

where of course we have an isomorphism of modules

$$\Phi_4\Phi_6\Phi_{12}M \oplus \Phi_3\Phi_6\Phi_{12}M \oplus \Phi_3\Phi_4\Phi_{12}M \oplus \Phi_3\Phi_4\Phi_6M \cong R_3 \oplus R_4 \oplus R_6 \oplus R_{12}.$$

An easy calculation shows that the ith dimensional eigenspace $V_j := H^0(\Omega_C^1)^j$ has dimension $v(j)$ with:

$$v(j) = 0, \text{for } j = 0, 5, 6, \;\; v(j) = 2, \text{for } j = 1, 2, 8, 9, \;\; v(j) = 1, \text{for } j = 3, 4, 7, 10, 11.$$

Now, we consider the cohomology $H^1(C, \mathbb{Z})$, again as a $\mathbb{Z}[x]$-module.

Then $H^1(C, \mathbb{Z}) = 3 \; Hom(M, \mathbb{Z})$, and by the exact sequence

$$0 \to M \to \sum_{d=3,4,6,12} (M/\Phi_d M = M \otimes_{\mathbb{Z}[x]} R_d = R_d) \to Coker \to 0,$$

we find that $Hom(M, \mathbb{Z})$ contains

$$\sum_{d=3,4,6,12} Hom(M/\Phi_d M, \mathbb{Z}) = \sum_{d=3,4,6,12} Hom(R_d, \mathbb{Z}) \cong \sum_{d=3,4,6,12} R_d,$$

where the duality $R_d \times R_d \to \mathbb{Z}$ is given by the product followed by the trace map.

It follows from our previous calculations that indeed we have a surjection $H^1(C, \mathbb{Z}) = 3\ Hom(M, \mathbb{Z}) \to 3R_{12}$.

Remark 5.5 Theorem 5.3 works if there are two points of full ramification; if there is exactly one, we are in the situation of
Case 2: $r_1, r_2, \dots r_k, r_\infty < n$.

Here we can use the relation

$$(\gamma_0 \cdot \gamma_1 \cdot \dots \cdot \gamma_k)^{r_\infty} = 1,$$

to rewrite

$$1 = \gamma_0^i (\gamma_0 \cdot \gamma_1 \cdot \dots \cdot \gamma_k)^{r_\infty} \gamma_0^{-i} =$$

$$= \delta_{i+1,1} \cdot \delta_{i+1+m_1,2} \cdot \dots \delta_{i+1+m_1+\dots+m_{k-1},k}\ \delta_{i+1-m_\infty,1} \dots \delta_{i-m_k,k} = 1.$$

These, since $\mathbb{Z}/(m_\infty) = \mathbb{Z}/(n/r_\infty)$, are n/r_∞ relations, as expected.

Passing to the Abelianization of $\pi_1(C)$, we get the extra relations:

$$\sum_{h=0,\dots r_\infty - 1} D_{i+1+hn/r_\infty,1} + D_{i+1+m_1+hn/r_\infty,2} + \dots + D_{i-m_k+hn/r_\infty,k} = 0.$$

This case should be easier to treat than the general one with no points of full ramification.

6 Structure Theorem for Bagnera–De Franchis Manifolds

The goal of this section is to give a complete structure theorem for Bagnera–De Franchis Manifolds, leaving aside the question of projectivity, which was treated in [11] (in the appendix it was shown that each BdF Manifold deforms to a projective one).

Set in this section $G = \mathbb{Z}/n$ and consider a Bagnera–De Franchis Manifold $X = A/G$, where A is a complex torus $A = V/\Lambda$ of dimension g. Here we use the letter A even if we have a torus, and not necessarily an Abelian variety, just in order to have a similar notation to [10] (where the letter T was used to denote some torsion subgroup).

Holomorphic maps $F : A \to A'$ of complex tori are affine maps, since their derivatives in the flat uniformizing parameters are constant: hence such holomorphic maps

$$F : A = V/\Lambda \to A' = V'/\Lambda'$$

can be represented as

$$F(v) = \alpha(v) + b \ (mod\ \Lambda'), \alpha : V \to V', \alpha(\Lambda) \subset \Lambda', b \in V'.$$

α is a linear map of vector spaces induced by $\alpha|_\Lambda$, which we still denote by α: indeed, any $\mathbb{Z}$-linear map $\alpha : (\Lambda) \to \Lambda'$ induces a complex linear map

$$\alpha \otimes \mathbb{C} : \Lambda \otimes \mathbb{C} = V \oplus \overline{V} \to \Lambda' \otimes \mathbb{C} = V' \oplus \overline{V'},$$

and α induces a homomorphism of complex tori if and only if

$$(\alpha \otimes \mathbb{C})(V) \subset V',$$

i.e. $\alpha \otimes \mathbb{C}$ is a homomorphism of Hodge structures.

We take now a generator $\gamma \in G$, and write $\gamma(v) = \alpha(v) + b$ (here $A' = A$).

Then we have a decomposition $V = \oplus_{\zeta \in \mu_n} V_\zeta$, where V_ζ is the eigenspace for the complex linear map α corresponding to the eigenvalue ζ.

The condition that γ has no fixed point means that there is no solution of the equation

$$\alpha(v) + b \equiv v (mod \Lambda) \Leftrightarrow (\alpha - Id)(v) + b \in \Lambda.$$

Writing $V = V_1 \oplus V_2$, $V_2 := \oplus_{\zeta \in \mu_n, \zeta \neq 1} V_\zeta$, we have that $(\alpha - Id)$ is invertible on V_2, hence after a change of the origin we may assume that $b \in V_1$, and that

$$\gamma(v_1, v_2) = (v_1 + b_1, \alpha_2(v_2)).$$

The condition that G operates freely on A amounts to:

$$(**) \ \exists \ (\lambda_1, \lambda_2) \in \Lambda \text{ such that } hb_1 = \lambda_1 \Leftrightarrow n|h.$$

α makes Λ an $R(n)$ module, hence we have (compare the notation introduced just before Proposition 4.1) the decomposition $\Lambda \otimes \mathbb{Q} = \oplus_{d|n} \Lambda_{d,\mathbb{Q}}$, and setting

$$\Lambda_d := \Lambda \cap \Lambda_{d,\mathbb{Q}}, \ \Lambda^0 := \Lambda / \oplus_{d|n} \Lambda_d, \ A_d := \Lambda_{d,\mathbb{R}}/\Lambda_d$$

we have an exact sequence

$$0 \to \Lambda^0 \to A' := \oplus_{d|n} A_d \to A \to 0.$$

Theorem 6.1 *A Bagnera–De Franchis Manifold with group $G = \mathbb{Z}/n$ is completely determined by the following data:*

(1) the datum of torsion free R_d-modules Λ_d of finite rank, for all $d|n$, such that $\Lambda_1 \neq 0$ and with Λ_1, Λ_2 of even rank;
(2) the datum of a finite subgroup $\Lambda^0 \subset A' := \oplus_{d|n} A_d$, where $A_d := \Lambda_{d,\mathbb{R}}/\Lambda_d$,
(3) an element $\beta_1 \in A_1$ generating a subgroup $\langle \beta_1 \rangle$ of order exactly n, such that:
(4) (A) Λ^0 is stable for multiplication by the element x of the subring $R(n) \subset R'(n) := \oplus_{d|n} R_d$, and
(5) (B) $\Lambda^0 \cap A_d = 0 \ \forall d|n$,
(6) (C) the projection of Λ^0 into A_1 intersects the subgroup $\langle \beta_1 \rangle$ only in 0;
(7) the datum of a complex structure on each $\Lambda_d \otimes \mathbb{C}$, i.e., a Hodge decomposition

$$\Lambda_d \otimes \mathbb{C} = V(d) \oplus \overline{V(d)},$$

which allows to decompose $V(d) = \oplus_{j<d,(j,d)=1} V_j$ as a direct sum of eigenspaces for the action α of x.
(8) The properties (A) and (B) imply that $\Lambda^0 \subset \oplus_{d|n}(\frac{\Phi_d}{Q_n}\Lambda_d)/\Lambda_d$, hence, in particular, the number of such subgroups Λ^0 is finite.

Proof According to Proposition 4.1 the data (1) and (2), provided that (4) and (5) hold, determine a lattice Λ which is an $R(n)$-module.

The conditions in (1) that Λ_1, Λ_2 have even rank are necessary for the existence of a complex structure on $\Lambda_d \otimes \mathbb{C}$ for $d = 1, 2$.

We can then choose $V_1 = V(1)$ to give a Hodge structure on $\Lambda_1 \otimes \mathbb{C}$ and, if n is even, $V(2) = V_{n/2}$ to give a complex structure on $\Lambda_2 \otimes \mathbb{C}$.

Whereas, for $d \geq 3$, $\Lambda_d \otimes \mathbb{C}$ splits as a direct sum of eigenspaces W_j, corresponding to the eigenvalues $\epsilon_n^{jn/d}$, for $j < d$, $(j, d) = 1$.

These eigenvalues come in conjugate pairs, hence it suffices to consider $W_j \oplus W_{n-j}$ and choose V_j, for $j < d/2$, to be any subspace of W_j, letting then $V_{n-j} \subset W_{n-j}$ be a subspace such that $W_j = V_j \oplus \overline{V_{n-j}}$.

We are done, since $W_{n-j} = \overline{W_j}$.

Finally, we take the transformation γ whose linear part is the linear map α corresponding to multiplication by x, and whose translation part is $\beta_1 \in A_1$. The condition that any power of γ with exponent $h < n$ has no fixed points means (see $(**)$) that the equation $(\gamma^h(v) - v) \in \Lambda$ has no solutions $v \in V$. Let β_1 be the class of $b \in V_1$: then this equation is equal to

$$hb + (\alpha^h - Id)(v) \in \Lambda.$$

Since the image of $(\alpha^h - Id)$ equals to V_2, this means that hb does not belong to the projection of Λ into V_1. This is equivalent to requiring that $h\beta_1$ does not belong to the projection of Λ^0 into A_1.

Property (8) was already shown in Proposition 4.1. □

Remark 6.2 (I) To relate the formulation given here with the content of Proposition 16 of [10], it suffices to define $\Lambda_2 := \Lambda \cap V_2$, and $T := \Lambda/(\Lambda_1 + \Lambda_2)$. Then T is isomorphic to the image of Λ^0 inside A_1, which was called T_1 in loc. cit. Hence one requires T_1 and $\langle \beta_1 \rangle$ to intersect only in 0, and clearly $X = A/G = (A_1 \times A_2)/G \times T$.

(II) On page 313, eight lines from the bottom of [10] there is a 'lapsus calami', asserting the splitting $R(m) = \oplus_{d|m} R_d$ without tensoring with $\mathbb{Q}$. However, fortunately, this wrong assertion is not used at all in [10].

7 The Intersection Product for the Homology of Fully Ramified Cyclic Coverings of the Line

In this section we use the presentation of the fundamental group of a cyclic covering $f : C \to \mathbb{P}^1$ as described in Sect. 5, and shall determine the intersection product map $H_1(C, \mathbb{Z}) \times H_1(C, \mathbb{Z}) \to \mathbb{Z}$ dual to the cup product for the first homology group.

We shall make the assumption that the ramification indices $r_0 = r_\infty = n$.

Then we have a set $\mathcal{G}_1$ of generators for $\pi_1(C)$, consisting of the $n \cdot k$ elements:

$$\delta_{i,j} := \gamma_0^i \gamma_j \gamma_0^{-m_j - i}, i = 0, \dots, n-1, j = 1, \dots, k.$$

As we saw, the $\sum_{i=1,\dots,k} \frac{n}{r_i}$ relations:

$$1 = \delta_{i,j} \delta_{i+m_j,\ j} \delta_{i+2m_j,\ j} \dots \delta_{i+(r_j-1)m_j,\ j},$$

allow to eliminate $\sum_{i=1,\dots,k} \frac{n}{r_i}$ of these generators, and we obtain a set $\mathcal{G}_2$ of generators, with the right number $\sum_{i=1,\dots,k}(n - \frac{n}{r_i}) = 2g$ of elements.

It is convenient to eliminate the generators $\delta_{i,j},\ i = 0, \dots, \frac{n}{r_i} - 1$, and each of them is then equal to the product

$$[\delta_{i+m_j,\ j} \delta_{i+2m_j,\ j} \dots \delta_{i+(r_j-1)m_j,\ j}]^{-1} = \delta^{-1}_{i+(r_j-1)m_j,\ j} \dots \delta^{-1}_{i+2m_j,\ j} \delta^{-1}_{i+m_j,\ j}.$$

Observe that this is the product of exactly $(r_j - 1)$ inverses of elements in the set $\mathcal{G}_2$ of $2g$ generators.

We consider now the relation which is the Reidemeister–Schreier rewriting of $\gamma_\infty^n = 1$.

We have:

$$1 = (\gamma_0 \gamma_1 \gamma_2 \dots \gamma_k)^n =$$

$$(\gamma_0 \gamma_1 \gamma_0^{-1})(\gamma_0 \gamma_2 \gamma_0^{-1}) \dots (\gamma_0 \gamma_k \gamma_0^{-1}) \cdot$$

$$(\gamma_0^2 \gamma_1 \gamma_0^{-2})(\gamma_0^2 \gamma_2 \gamma_0^{-2}) \dots (\gamma_0^2 \gamma_k \gamma_0^{-2}) \cdot$$

$$\dots$$

$$(\gamma_0^{n-1}\gamma_1\gamma_0^{-(n-1)})(\gamma_0^{n-1}\gamma_2\gamma_0^{-(n-1)})\dots(\gamma_0^{n-1}\gamma_k\gamma_0^{-(n-1)}).$$

$$\gamma_1\gamma_2\cdots\gamma_k.$$

We are then ready to rewrite in terms of the 'big' set $\mathcal{G}_1$ of generators:

$$1 = \delta_{1,\,1}\dots\delta_{1,\,k}\delta_{2,\,1}\dots\delta_{2,\,k}\dots\delta_{n-1,\,1}\dots\delta_{n-1,\,k}\delta_{0,\,1}\dots\delta_{0,\,k}.$$

This relation is then equal to the product of the $n\cdot k$ original generators.

Now, we replace, as indicated above, the generators $\delta_{i,j},\ i = 0,\dots\frac{n}{r_i}$ by the respective products of inverses of the final set $\mathcal{G}_2$ of $2g$ generators.

We are then left with a relation where do occur exactly the $2g$ generators in $\mathcal{G}_2$ ($2g = nk - \sum_{i=1,\dots,k}\frac{n}{r_i}$) with exponent equal to $+1$, and exactly

$$\sum_{i=1,\dots,k}\frac{n}{r_i}(r_i - 1) = \sum_{i=1,\dots,k} n - \frac{n}{r_i} = 2g$$

different generators with exponent equal to -1.

Hence the relation, even if not in the standard form, depicts a 2-dimensional manifold obtained attaching a 2-disk to a bouquet of $2g$ circles.

The recipe for the intersection product is then given in the following

Proposition 7.1 *Let π_g be a group with $2g$ generators, $a_1,\dots a_{2g}$, and with only one relation $W(a_1,\dots,a_{2g}) = 1$, where the word $W(a_1,\dots,a_{2g})$ consists of exactly $4g$ letters, of which $2g$ are exactly the letters $a_1,\dots a_{2g}$, and $2g$ are exactly the inverses of the letters $a_1,\dots a_{2g}$.*

Then π_g is the fundamental group of a closed Riemann surface C of genus g, and the intersection product on $H_1(C,\mathbb{Z}) = \pi_g^{ab}$ is determined as follows, considering the word as giving a cyclical order in the set (of cardinality $4g$) consisting of the generators a_i and of their inverses:

(1) $a_i\cdot a_j = 0$ if, removing a_i and a_i^{-1}, a_j and a_j^{-1} lie in the same of the two remaining intervals;
(2) $a_i\cdot a_j = 1$ if, removing a_i and a_i^{-1}, a_j lies in the interval going from a_i to a_i^{-1}, and a_j^{-1} lies in the other;
(3) $a_i\cdot a_j = -1$ if, removing a_i and a_i^{-1}, a_j^{-1} lies in the interval going from a_i to a_i^{-1}, and a_j lies in the other.

Proof Consider a bouquet of $2g$ circles meeting in one point P, and corresponding to the generators $a_1,\dots a_{2g}$.

We attach a 2-disk whose boundary is the word $W(a_1,\dots,a_{2g})$. Since in the word each generator a_i and its inverse a_i^{-1} appear exactly once, the space C that we obtain is smooth outside of P. At P however we have $4g$ segments coming in, and we fill in $4g$ angles, hence C is a topological manifold. It is also oriented since $H_2(C,\mathbb{Z}) = \mathbb{Z}$.

The two incoming segments corresponding to a_i and a_i^{-1} (who are oriented) locally separate C in a neighbourhood of P.

In case (1), two incoming segments corresponding to a_j and a_j^{-1} lie in the same half-plane, hence they can be deformed out of P until they do not intersect a_i at all. In cases (2) and (3), these segments lie in different halfplanes, hence the intersection product equals ± 1.

In case (2) the intersection is positively oriented, in case (3) it is negatively oriented, as one sees easily (compare the standard presentation where the word $W(a_1, \ldots, a_{2g}) = a_1 a_2 a_1^{-1} a_2^{-1} \ldots$). □

Example 7.1 Consider the Fermat elliptic curve E of affine equation

$$y^3 = x(x-1).$$

Here, $n = 3$ and $k = 1$, and we have generators

$$\delta_{0,\,1} = \gamma_1\gamma_0^{-1},\ \delta_{1,\,1} = \gamma_0\gamma_1\gamma_0^{-2},\ \delta_{2,\,1} = \gamma_0^2\gamma_1,$$

satisfying

$$\delta_{0,\,1}\delta_{1,\,1}\delta_{2,\,1} = 1,\ \delta_{1,\,1}\delta_{0,\,1}\delta_{2,\,1} = 1.$$

Eliminating $\delta_{2,\,1}$, we get generators $\delta_{0,\,1}\delta_{1,\,1}$ with relation

$$\delta_{0,\,1}\delta_{1,\,1} = \delta_{1,\,1}\delta_{0,\,1},$$

as fully expected.

Consider now the example of the curve C of affine equation

$$y^3 = x(x^4 - 1).$$

Here, $n = 3$ and $k = 4$, and we have generators

$$\delta_{0,\,j},\ \delta_{1,\,j},\ \delta_{2,\,j},\ j = 1, 2, 3, 4,$$

satisfying relations

$$\delta_{0,\,j}\delta_{1,\,j}\delta_{2,\,j} = 1,$$

$$\delta_{1,\,1}\delta_{1,\,2}\delta_{1,\,3}\delta_{1,\,4}\delta_{2,\,1}\ldots\delta_{2,\,4}\delta_{0,\,1}\ldots\delta_{0,\,4} = 1.$$

Set

$$\delta_{1,\,j} =: a_j,\ \delta_{2,\,j} =: b_j.$$

Then we have 8 generators satisfying the relation

$$a_1a_2a_3a_4b_1b_2b_3b_4b_1^{-1}a_1^{-1}b_2^{-1}a_2^{-1}b_3^{-1}a_3^{-1}b_4^{-1}a_4^{-1} = 1.$$

Using the rule we just described, we find:

$$a_i a_j = a_i b_j = b_i b_j = 1, \; 1 \le i < j \le 4,$$

$$b_1 a_j = 0, \; j = 1, 2, 3, 4, \; b_2 a_j = 0, \; j = 2, 3, 4, \; b_3 a_j = 0, \; j = 3, 4, \; a_4 b_4 = 0.$$

Remark 7.2 Together with Jong Hae Keum, Matthew Stover and Domingo Toledo, we proved that the Jacobian $J(C)$ of the above curve C is isomorphic to E^4, the product of four copies of the Fermat elliptic curve; and we also discovered other curves whose Jacobian is a product (or is isogenous to a product) of elliptic curves.

For the above curve C, however, we found later that the same example had been described by Ryo Nakajima in [27].

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Second Fundamental Form of the Prym Map in the Ramified Case

Elisabetta Colombo and Paola Frediani

Abstract In this paper we study the second fundamental form of the Prym map $P_{g,r} : \mathcal{R}_{g,r} \to \mathcal{A}^{\delta}_{g-1+r}$ in the ramified case $r > 0$. We give an expression of it in terms of the second fundamental form of the Torelli map of the covering curves. We use this expression to give an upper bound for the dimension of a germ of a totally geodesic submanifold, and hence of a Shimura subvariety of $\mathcal{A}^{\delta}_{g-1+r}$, contained in the Prym locus.

1 Introduction

Denote by $\mathcal{R}_{g,r}$ the moduli space parametrising isomorphism classes of pairs $[(C, \alpha, R)]$ where C is a smooth complex projective curve of genus g, R is a reduced effective divisor of degree $2r$ on C and α is a line bundle on C such that $\alpha^2 = \mathcal{O}_C(R)$. To such data it is associated a double cover of C, $\pi : \tilde{C} \to C$ branched on R, with $\tilde{C} = Spec(\mathcal{O}_C \oplus \alpha^{-1})$.

The Prym variety associated to $[(C, \alpha, R)]$ is the connected component containing 0 of the kernel of the norm map $\mathrm{Nm}_\pi : J\tilde{C} \to JC$. Notice that for $r > 0$, $\ker \mathrm{Nm}_\pi$ is connected. It is a polarized abelian variety of dimension $g - 1 + r$, denoted by $P(C, \alpha, R)$ or equivalently $P(\tilde{C}, C)$. The polarization Ξ is induced by restricting the principal polarization on $J\tilde{C}$ and it is of type $\delta = (1, \dots, 1, \underbrace{2, \dots, 2}_{g \text{ times}})$ for $r > 0$.

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For $r = 0, 1$ it is twice a principal polarization and we endow $P(\tilde{C}, C)$ with this principal polarization.

This defines the Prym map

$$P_{g,r} : \mathcal{R}_{g,r} \to \mathcal{A}^{\delta}_{g-1+r}, \ [(C, \alpha, R)] \mapsto [(P(C, \alpha, R), \Xi)],$$

where $\mathcal{A}^{\delta}_{g-1+r}$ is the moduli space of abelian varieties of dimension $g - 1 + r$ with a polarization of type δ. We define the Prym locus as the closure in $\mathcal{A}^{\delta}_{g-1+r}$ of the image of the map $P_{g,r}$.

The codifferential of $P_{g,r}$ at a generic point $[(C, \alpha, R)]$ is given by the multiplication map

$$(dP_{g,r})^* : S^2H^0(C, K_C \otimes \alpha) \to H^0(C, K_C^2(R)) \tag{1.1}$$

which is known to be surjective (see [11]), therefore $P_{g,r}$ is generically finite, if and only if

$$\dim \mathcal{R}_{g,r} \leq \dim \mathcal{A}^{\delta}_{g-1+r}.$$

This holds if: either $r \geq 3$ and $g \geq 1$, or $r = 2$ and $g \geq 3$, $r = 1$ and $g \geq 5$, $r = 0$ and $g \geq 6$.

If $r = 0$ the Prym map is generically injective for $g \geq 7$ ([9, 10]). If $r > 0$, Marcucci and Pirola [13], and later, for the missing cases, Marcucci and Naranjo [12] and Naranjo and Ortega [15] have proved the generic injectivity in all the cases except for $r = 2$, $g = 3$, which was previously studied by Nagaraj and Ramanan, and also by Bardelli, Ciliberto and Verra (see [1, 14]) and for which the degree of the Prym map is 3.

In this paper we study the second fundamental form of the restriction of the Prym map to the open set $\mathcal{R}^0_{g,r}$ where the Prym map is an immersion, with respect to the orbifold metric on $\mathcal{A}^{\delta}_{g-1+r}$ induced by the symmetric metric on the Siegel space $\mathcal{H}_{g-1+r}$.

In the unramified case $r = 0$, the second fundamental form of the Prym map was studied in [5] using the Hodge gaussian maps introduced in [8] and the flat structure on the degree 0 line bundle α on C. Here in Theorem 2.1 we give a description of the second fundamental form ρ_P for all $r \geq 0$ in terms of the second fundamental form $\tilde{\rho}$ of the Torelli map $j : \mathcal{M}_{2g-1+r} \to \mathcal{A}_{2g-1+r}$ described in [3, 6, 8]. At the point $[(C, \alpha, R)]$ we show that ρ_P is obtained from $\tilde{\rho}$ by first restricting to the kernel of $(dP_{g,r})^*$ and then projecting to $S^2H^0(K_C^2(R))$.

This also allows us to prove that the map ρ_P is a lifting of the second gaussian map of the line bundle $K_C \otimes \alpha$ (see Proposition 2.3). This was already proved in the unramified case $r = 0$ in [5] and previously in the case of the Torelli map in [8].

In the second part of the paper we use this description of ρ_P to study totally geodesic submanifolds in the Prym loci.

We recall that a conjecture by Coleman and Oort says that for big enough genus, there should not exist Shimura subvarieties of $\mathcal{A}_g$ generically contained in the Torelli locus, i.e. contained in the Torelli locus $\overline{j(\mathcal{M}_g)}$ and intersecting $j(\mathcal{M}_g)$. Shimura subvarieties are totally geodesic, hence it is possible to approach the conjecture studying the second fundamental of the Torelli map. This viewpoint was used in [6], where an upper bound on the dimension of a germ of a totally geodesic submanifold contained in the Torelli locus, depending on g, is given.

In [7] a question similar to the one of Coleman and Oort for Pryms was asked in the case $r = 0, 1$, i.e. when the Prym variety is principally polarised and the Prym loci $\overline{P_{g,r}(\mathcal{R}_{g,r})}$ contain the Torelli locus. More precisely, the question is about the existence, for big enough genus, of Shimura subvarieties generically contained in the Prym loci. We say that a subvariety $Z \subset \mathcal{A}_{g-1+r}$ is generically contained in the Prym locus if $Z \subset \overline{P_{g,r}(\mathcal{R}_{g,r})}$, $Z \cap P_{g,r}(\mathcal{R}_{g,r}) \neq \emptyset$ and Z intersects the locus of indecomposable polarised abelian varieties. In [7] examples in low dimension of Shimura curves generically contained in the Prym loci for $r = 0, 1$ are given, using Galois covers of $\mathbb{P}^1$.

In [4] we gave an upper bound on the dimension of a germ of a totally geodesic submanifold contained in the Prym locus when $r = 0$, depending only on g, which is similar to the estimate in [6] for the Torelli locus, that we achieved via the second fundamental form.

Here we generalise the above question for any $r \geq 0$ and we find an upper bound for the dimension of a germ of a totally geodesic submanifold contained in the Prym loci which depends on g and r (Theorem 3.4). This is obtained as the generic case of a bound depending also on the gonality k, given for a germ of a totally geodesic submanifold contained in the Prym loci passing through a point $[(C, \alpha, R)]$, with C a k-gonal curve (Theorem 3.2).

2 The 2nd Fundamental Form of the Prym Map

Let C be a smooth complex projective curve of genus g, R a reduced divisor of degree $2r$ on C and α a line bundle on C such that $\alpha^2 = \mathcal{O}_C(R)$. To such data corresponds a double cover $\pi : \tilde{C} \to C$ branched on R. In the ramified case $r > 0$, the Prym variety $P(C, \alpha, R)$ associated to this data is the polarised abelian variety given by the kernel of the norm map $Nm_\pi : J\tilde{C} \to JC$. For $r = 0$ it is its connected component containing the origin. For $r > 1$ the polarisation is given by $\Xi := \Theta_{\tilde{C}|P(C,\alpha,R)}$, where $\Theta_{\tilde{C}}$ is a theta divisor of $J\tilde{C}$. For $r = 0, 1$ the polarisation $\Theta_{\tilde{C}|P(C,\alpha,R)}$ is twice a principal polarisation Ξ. Consider the Prym map $P_{g,r} : \mathcal{R}_{g,r} \to \mathcal{A}^{\delta}_{g-1+r}$, which associates to a point $[(C, \alpha, R)] \in \mathcal{R}_{g,r}$ the isomorphism class of its Prym variety $P(C, \alpha, R)$ with the polarisation Ξ.

As recalled in the introduction, the codifferential of the Prym map (1.1) is given by the multiplication map and it is surjective at the generic point if either $r \geq 3$ and $g \geq 1$, or $r = 2$ and $g \geq 3$, $r = 1$ and $g \geq 5$, $r = 0$ and $g \geq 6$ (see [11]).

In these cases we denote by $\mathcal{R}^0_{g,r}$ the non empty open subset of $\mathcal{R}_{g,r}$ where the Prym map $P_{g,r}$ is an immersion.

Consider the orbifold tangent bundle exact sequence of the Prym map

$$0 \to T_{\mathcal{R}^0_{g,r}} \to P^*_{g,r} T_{\mathcal{A}^\delta_{g-1+r}} \to \mathcal{N}_{\mathcal{R}^0_{g,r}/\mathcal{A}^\delta_{g-1+r}} \to 0 \tag{2.1}$$

On $\mathcal{A}^\delta_{g-1+r}$ we consider the orbifold metric induced by the symmetric metric on the Siegel space $\mathcal{H}_{g-1+r}$ and the associated second fundamental form with respect to the metric connection of the above exact sequence. Denote its dual by

$$\rho_P : \mathcal{N}^*_{\mathcal{R}^0_{g,r}/\mathcal{A}^\delta_{g-1+r}} \to S^2\Omega^1_{\mathcal{R}^0_{g,r}}. \tag{2.2}$$

To describe this second fundamental form we study the second fundamental form of the Torelli map of the covering curves $\tilde{C}$. Since our computations will be local, we will restrict to an open set U of $\mathcal{R}^0_{g,r}$ where there is a universal family $\tilde{f} : \tilde{\mathcal{C}} \to U$ and the differential of the modular map $U \to \mathcal{M}_{\tilde{g}}$, $[\tilde{C} \to C] \mapsto [\tilde{C}]$ is injective.

Denote by $\tilde{H} := R^1\tilde{f}_*\mathbb{C}$, $\tilde{H}_b = H^1(\tilde{C}_b, \mathbb{C})$, by $\tilde{\mathcal{F}} = \tilde{f}_*\omega_{\tilde{\mathcal{C}}|U}$ the Hodge bundle, $\tilde{\mathcal{F}}_b = H^0(\tilde{C}_b, K_{\tilde{C}_b})$. The $\mathbb{Z}/2\mathbb{Z}$ action corresponding to the $2:1$ covering gives a decomposition $\tilde{H} = \tilde{H}^+ \oplus \tilde{H}^-$ in ± 1 eigenspaces and an analogous decomposition $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}^+ \oplus \tilde{\mathcal{F}}^-$. We have $\tilde{\mathcal{F}}^+_b \cong H^0(C_b, K_{C_b})$, $\tilde{\mathcal{F}}^-_b \cong H^0(C_b, K_{C_b} \otimes \alpha_b)$, $\forall b \in U$.

The Gauss–Manin connection ∇_{GM} on $\tilde{H}$ induces a connection $\nabla^{1,0}$ on the Hodge bundle $\tilde{\mathcal{F}}$. Both connections are $\mathbb{Z}/2\mathbb{Z}$ invariant, hence we have a connection ∇^- on $\tilde{\mathcal{F}}^-$, and an induced connection ∇ on $S^2\tilde{\mathcal{F}}^-$. We have the identifications on U: $\Omega^1_{\mathcal{M}_{\tilde{g}|U}} \cong (\tilde{f}_*\omega^2_{\tilde{\mathcal{C}}|U})$ and Ω^1_U can be identified with the subbundle preserved by the $\mathbb{Z}/2\mathbb{Z}$ action: $\Omega^1_U = (\tilde{f}_*\omega^2_{\tilde{\mathcal{C}}|U})^+$. Note in fact that at a point $b := [\tilde{C} \to C] \in U$, $\Omega^1_{U\,b} = H^0(C, K^2_C(R)) \cong H^0(K^2_{\tilde{C}})^+$, as one can easily check via the projection formula. Moreover $P^*\Omega^1_{\mathcal{A}^\delta_{g-1+r}} \cong S^2\tilde{\mathcal{F}}^-$ and the connection ∇ corresponds to the connection associated to the Siegel metric on $\mathcal{A}^\delta_{g-1+r}$.

Consider the exact sequence

$$0 \to \mathcal{I}_2 \to S^2\tilde{\mathcal{F}} \stackrel{m}{\to} \tilde{f}_*\omega^2_{\tilde{\mathcal{C}}|U} \to 0 \tag{2.3}$$

where the map m is the multiplication map and it is the dual of the differential of the Torelli map of the curves $\tilde{C}$ on U.

The second fundamental form of the exact sequence (2.3) is a map

$$\Psi : \mathcal{I}_2 \to \tilde{f}_*\omega^2_{\tilde{\mathcal{C}}|U} \otimes \Omega^1_U \cong \tilde{f}_*\omega^2_{\tilde{\mathcal{C}}|U} \otimes (\tilde{f}_*\omega^2_{\tilde{\mathcal{C}}|U})^+ \tag{2.4}$$

Since the multiplication map m is $\mathbb{Z}/2\mathbb{Z}$ equivariant, we also have the exact sequence:

$$0 \to \mathcal{I}^+_2 \to (S^2\tilde{\mathcal{F}})^+ \stackrel{m}{\to} (\tilde{f}_*\omega^2_{\tilde{\mathcal{C}}|U})^+ \to 0 \tag{2.5}$$

Clearly we have $(S^2\tilde{\mathcal{F}})^+ \cong S^2\tilde{\mathcal{F}}^+ \oplus S^2\tilde{\mathcal{F}}^-$ and the restriction of the multiplication map m to $S^2\tilde{\mathcal{F}}^-$ is the dual of the differential of the Prym map P (see [11]). More precisely, with the identifications on U: $P^*\Omega^1_{\mathcal{A}^\delta_{g-1+r}} \cong S^2\tilde{\mathcal{F}}^-$, $\Omega^1_U = (\tilde{f}_*\omega^2_{\tilde{\mathcal{C}}|U})^+$, the dual of the exact sequence (2.1) on U can be written as

$$0 \to \mathcal{G} \to S^2\tilde{\mathcal{F}}^- \xrightarrow{m} (\tilde{f}_*\omega^2_{\tilde{\mathcal{C}}|U})^+ \to 0 \tag{2.6}$$

where $\mathcal{G} = S^2\tilde{\mathcal{F}}^- \cap \mathcal{I}_2^+\mathcal{A}^\delta_{g-1+r} \cong \mathcal{N}^*_{U/\mathcal{A}^\delta_{g-1+r}}$. So the dual of the second fundamental form of the Prym map is a map

$$\rho_P : \mathcal{G} \to (\tilde{f}_*\omega^2_{\tilde{\mathcal{C}}|U})^+ \otimes (\tilde{f}_*\omega^2_{\tilde{\mathcal{C}}|U})^+, \tag{2.7}$$

which is symmetric, since it is the second fundamental form of an immersion. Clearly we have

$$\rho_P = p \circ \Psi_{|\mathcal{G}} \tag{2.8}$$

where $p : (\tilde{f}_*\omega^2_{\tilde{\mathcal{C}}|U}) \otimes (\tilde{f}_*\omega^2_{\tilde{\mathcal{C}}|U})^+ \to S^2(\tilde{f}_*\omega^2_{\tilde{\mathcal{C}}|U})^+$ is the natural projection.

Denote by $\tilde{g} := 2g - 1 + r$ and consider now the Torelli map $\tilde{j} : \mathcal{M}^0_{\tilde{g}} \to \mathcal{A}_{\tilde{g}}$, where $\mathcal{M}^0_{\tilde{g}}$ is the complement of the hyperelliptic locus. Then $\tilde{j}$ is an immersion and we denote by $\tilde{\rho}$ the dual of the second fundamental form of $\tilde{j}$. Since we are working in a local setting, we can assume that we have a modular map $\mu : U \to V$ where V is an open subset of $\mathcal{M}^0_{\tilde{g}}$ on which there exists a universal family $\varphi : \tilde{\mathcal{C}}_V \to V$. The family $\tilde{\mathcal{C}}$ is the pullback of $\tilde{\mathcal{C}}_V$ via the map μ.

On V we have

$$0 \to \mathcal{I}_2(\omega_{\tilde{\mathcal{C}}_V|V}) \to S^2(\varphi_*\omega_{\tilde{\mathcal{C}}_V|V}) \xrightarrow{m} \varphi_*\omega^2_{\tilde{\mathcal{C}}_V|V} \to 0 \tag{2.9}$$

and the multiplication map m is the dual of the differential of the Torelli map.

On V the dual of the second fundamental form of the Torelli map is a map

$$\tilde{\rho} : \mathcal{I}_2(\omega_{\tilde{\mathcal{C}}_V|V}) \to \varphi_*\omega^2_{\tilde{\mathcal{C}}_V|V} \otimes \varphi_*\omega^2_{\tilde{\mathcal{C}}_V|V} \tag{2.10}$$

The pullback of the above exact sequence on U via μ is the exact sequence (2.3). Hence we have

$$\Psi = q \circ \mu^*\tilde{\rho} \tag{2.11}$$

where $q : (\tilde{f}_*\omega^2_{\tilde{\mathcal{C}}|U}) \otimes (\tilde{f}_*\omega^2_{\tilde{\mathcal{C}}|U}) \to (\tilde{f}_*\omega^2_{\tilde{\mathcal{C}}|U}) \otimes (\tilde{f}_*\omega^2_{\tilde{\mathcal{C}}|U})^+$ is the natural projection.

We have the following

Theorem 2.1 *The dual of the second fundamental form of the Prym map on U is obtained as $\rho_P = p' \circ (\mu^*\tilde{\rho})_{|\mathcal{G}}$, where $p' : S^2(\tilde{f}_*\omega^2_{\tilde{\mathcal{C}}|U}) \to S^2((\tilde{f}_*\omega^2_{\tilde{\mathcal{C}}|U})^+)$ is the natural projection.*

Proof By (2.8) and (2.11) we have $\rho_P = p \circ \Psi_{|\mathcal{G}} = p \circ q \circ (\mu^* \tilde{\rho})_{|\mathcal{G}} = p' \circ (\mu^* \tilde{\rho})_{|\mathcal{G}}$. □

At the point $b_0 := [(C, \alpha, R)] \in U$ corresponding to the $2:1$ cover $\pi : \tilde{C} \to C$, the space $P^*_{g,r} \Omega^1_{\mathcal{A}^\delta_{g-1+r}, b_0}$ is isomorphic to $S^2 H^0(K_C \otimes \alpha)$, $\Omega^1_{\mathcal{R}^0_{g,r}, b_0}$ is isomorphic to $H^0(K_C^2(R))$, $\mathcal{G}_{b_0} \cong I_2(K_C \otimes \alpha)$, and the dual of the exact sequence (2.1) at the point b_0, that is the exact sequence (2.6) at b_0 becomes

$$0 \to I_2(K_C \otimes \alpha) \to S^2 H^0(K_C \otimes \alpha) \stackrel{m}{\to} H^0(K_C^2(R)) \to 0.$$

The dual of the second fundamental form of the Prym map at the point b_0 is a map

$$\rho_P : I_2(K_C \otimes \alpha) \to S^2 H^0(K_C^2(R)) \tag{2.12}$$

Observe that $\forall Q \in I_2(K_C \otimes \alpha) \stackrel{\pi^*}{\hookrightarrow} I_2(K_{\tilde{C}})^+$, $\forall v_1, v_2 \in H^1(T_C(-R)) \cong H^1(T_{\tilde{C}})^+$, by Theorem (2.1) we have:

$$\rho_P(Q)(v_1 \odot v_2) = \tilde{\rho}(\pi^* Q)(v_1 \odot v_2). \tag{2.13}$$

Denote by

$$\tilde{\mu}_2 : I_2(K_{\tilde{C}}) \to H^0(K_{\tilde{C}}^4) \tag{2.14}$$

the second Gaussian map of the canonical bundle $K_{\tilde{C}}$ and by

$$\mu_2 := \mu_{2, K_C \otimes \alpha} : I_2(K_C \otimes \alpha) \to H^0(K_C^4(R)) \tag{2.15}$$

the second Gaussian map of the line bundle $K_C \otimes \alpha$ (for the definition and a local expression of the second Gaussian maps see e.g. [2], Sect. 2). Notice that $\tilde{\mu}_2$ is equivariant, hence it induces a map $\tilde{\mu}_2 : I_2(K_{\tilde{C}})^+ \to H^0(K_{\tilde{C}}^4)^+ \cong H^0(K_C^4(2R))$.

We have the following

Lemma 2.2 *For every $Q \in I_2(K_C \otimes \alpha)$, $\mu_2(Q) = \tilde{\mu}_2(\pi^* Q)$ via the inclusion $H^0(K_C^4(R)) \subset H^0(K_C^4(2R))$.*

Proof We show the equality by a local computation. For a point $P \notin R$, take local coordinates z in a neighbourhood V of P and w in a neighbourhood U of a point $T \in \pi^{-1}(P)$ such that the local expression of $\pi : U \to V$ is $w \mapsto w = z$. Since $\alpha^2 = \mathcal{O}_C(R)$, $\alpha^2_{|V} = \mathcal{O}_V$ and we choose a local frame a of α on V such that $a^2 = 1$ and $(\pi^* a)_{|U} = 1$. Fix a basis $\{\omega_i\}$ of $H^0(K_C \otimes \alpha)$, so locally $\omega_i = f_i(z) dz \otimes a$. Then on U we have $\pi^*(\omega_i) = f_i(w) dw$. Take a quadric $Q = \sum_{i,j} a_{i,j} \omega_i \odot \omega_j \in I_2(K_C \otimes \alpha)$, hence locally $\pi^* Q = \sum_{i,j} a_{i,j} f_i(w) dw \odot f_j(w) dw$ and we have $\tilde{\mu}_2(\pi^* Q) = -\sum_{i,j} a_{i,j} f_i'(w) f_j'(w)(dw)^4$. On the other hand on V, $\mu_2(Q) = -\sum_{i,j} a_{i,j} f_i'(z) f_j'(z)(dz)^4 a^2$, hence the statement follows, since $a^2 = 1$.

Observe that the equality can also be checked locally around a critical point T over a point $P \in R$. So we can assume that the map $\pi : U \to V$ is of the form $w \mapsto w^2 = z$. Now $\alpha^2_{|V} = \mathcal{O}_V(P)$ and $(\pi^* \alpha)_{|U} = \mathcal{O}_U(T)$ so we choose a local frame

a of α on V such that $a^2 = \frac{1}{z}$ and $(\pi^* a)_{|U} = \frac{1}{w}$. Now locally $\pi^*(\omega_i) = 2 f_i(w^2) dw$ and $\pi^*(Q) = 4 \sum_{i,j} a_{i,j} f_i(w^2) dw \odot f_j(w^2) dw$. So we have

$$\tilde{\mu}_2(\pi^* Q) = -4 \sum_{i,j} a_{i,j} f_i'(w^2) f_j'(w^2) 4w^2 (dw)^4 = -\sum_{i,j} a_{i,j} f_i'(z) f_j'(z) \frac{(dz)^4}{z}.$$

On the other hand locally $\mu_2(Q) = -\sum_{i,j} a_{i,j} f_i'(z) f_j'(z) (dz)^4 a^2 = \tilde{\mu}_2(\pi^* Q)$. □

We recall the definition of a Schiffer variation of a line bundle L on a curve C at a point $p \in C$. Consider the evaluation map $v : H^0(K_C \otimes L) \otimes \mathcal{O}_C \to K_C \otimes L$ and its dual

$$\xi_L : T_C \otimes L^{-1} \to H^1(L^{-1}) \otimes \mathcal{O}_C.$$

A Schiffer variation in P is a generator of the image of the map

$$(T_C \otimes L^{-1})_P \to H^1(L^{-1}).$$

Note that this map can be seen as the coboudary map of the exact sequence

$$0 \to L^{-1} \to L^{-1}(P) \to L^{-1}(P)_{|P} \to 0, \tag{2.16}$$

with the identification $T_{C|P} = \mathcal{O}_C(P)_{|P}$.

If we fix a local coordinate z centred in P, we choose the Schiffer variation $\xi_{P,L}$ in P to be the Doulbeault class of $\theta_{P,L} = \frac{\bar{\partial} b_P}{z} \otimes l^{-1} \in A^{0,1}(L^{-1})$, where b_P is a bump function at P which is equal to one in a neighborhood of P and l is a local frame of L.

Proposition 2.3 *We have the following commutative diagram*

$$\begin{array}{ccc} I_2(K_C \otimes \alpha) & \xrightarrow{\rho_P} & S^2(H^0(K_C^2(R))) \\ \downarrow{\scriptstyle -4\pi i \mu_2} & & \downarrow{\scriptstyle m} \\ H^0(K_C^4(R)) & \longrightarrow & H^0(K_C^4(2R)) \end{array}$$

Proof We recall that we have a similar statement for $\tilde{\rho}$ and $\tilde{\mu_2}$, namely $m \circ \tilde{\rho} = -2\pi i \tilde{\mu}_2$ (see [8, Thm. 3.1], [6, Thm. 2.2]). Denote as usual by $\pi : \tilde{C} \to C$ the double cover and by σ the covering involution on $\tilde{C}$, take a point $T \in \tilde{C}$ with $T \neq \sigma(T)$ and fix a local coordinate on $\tilde{C}$ around T (and correspondingly around $\sigma(T)$) and on C around $P := \pi(T)$. For a point $S \in \tilde{C}$, denote by $\xi_S := \xi_{S,K_{\tilde{C}}}$.

Then by [8, Thm. 3.1], [6, Thm. 2.2] we have:

$$\tilde{\rho}(\pi^* Q)(\xi_T \odot \xi_{\sigma(T)}) = -4\pi i (\pi^* Q)(T, \sigma(T)) \cdot \tilde{\eta}_T(\sigma(T)) = -4\pi i Q(P, P) \cdot \tilde{\eta}_T(\sigma(T)) = 0, \tag{2.17}$$

since $Q(P, P) = 0$ and $\tilde{\eta}_T \in H^0(K_{\tilde{C}}(2T))$ has only one double pole in T, so it is holomorphic around $\sigma(T)$. Set $v := \xi_T + \xi_{\sigma(T)} \in H^1(T_{\tilde{C}})^+$. By (2.17) and by the $\mathbb{Z}/2\mathbb{Z}$ equivariance of $\tilde{\rho}$ we have:

$$2\tilde{\rho}(\pi^* Q)(\xi_T \odot \xi_T) = \tilde{\rho}(\pi^* Q)(v \odot v) = \rho_P(Q)(v \odot v) \tag{2.18}$$

where the last equality is (2.13).

By [8, Thm. 3.1], [6, Thm. 2.2] we have

$$\tilde{\rho}(\pi^* Q)(\xi_T \odot \xi_T) = m(\tilde{\rho}(\pi^*(Q))(T) = -2\pi i \tilde{\mu}_2(\pi^* Q)(T) = -2\pi i \mu_2(Q)(P), \tag{2.19}$$

by Lemma (2.2).

Consider the line bundle $L = K_C(R)$, fix a local coordinate z in P and the corresponding Schiffer variation $\xi_{P,L} \in H^1(T_C(-R))$. Choose in T and $\sigma(T)$ the local coordinates determined by z and take the associated Schiffer variations $\xi_T, \xi_{\sigma(T)}$ of $K_{\tilde{C}}$ at T and $\sigma(T)$. Then with the identification of $H^1(T_C(-R))$ with $H^1(T_{\tilde{C}})^+$, the Schiffer variation $\xi_{P,L}$ corresponds to $v = \xi_T + \xi_{\sigma(T)}$. This can be checked as follows: take the exact sequence

$$0 \to T_{\tilde{C}} \to T_{\tilde{C}}(T + \sigma(T)) \to T_{\tilde{C}}(T + \sigma(T))_{|T+\sigma(T)} \to 0. \tag{2.20}$$

The image of its coboundary map $H^0(T_{\tilde{C}}(T + \sigma(T))_{|T+\sigma(T)}) \to H^1(T_{\tilde{C}})$ is the subspace generated by $\{\xi_T, \xi_{\sigma(T)}\}$ and the $\mathbb{Z}/2\mathbb{Z}$-invariant part of this image is generated by $v = \xi_T + \xi_{\sigma(T)}$. Since π is finite, $R^1\pi_* T_{\tilde{C}} = 0$, so, if we apply π_* to (2.20), using the projection formula and the isomorphism $\pi_*\mathcal{O}_{\tilde{C}} \cong \mathcal{O}_C \oplus \alpha^{-1}$, we get the exact sequence

$$0 \to (T_C \otimes \alpha^{-1}) \oplus T_C(-R) \to (T_C \otimes \alpha^{-1}(P)) \oplus T_C(-R)(P) \to \pi_*(T_{\tilde{C}}(T + \sigma(T))_{|T+\sigma(T)}) \to 0. \tag{2.21}$$

Taking the invariant part, we get the exact sequence (2.16) for $L = K_C(R)$:

$$0 \to T_C(-R) \to T_C(-R)(P) \to T_C(-R)(P)_{|P} \to 0. \tag{2.22}$$

In particular we have the isomorphism $H^0((\pi_*(T_{\tilde{C}}(T + \sigma(T))_{|T+\sigma(T)}))^+) \cong H^0(T_C(-R)(P)_{|P})$. Recall that the Schiffer variation $\xi_{P,L}$ is a generator of the image of the coboundary map $H^0(T_C(-R)(P)_{|P}) \to H^1(T_C(-R)) \cong H^1(T_{\tilde{C}})^+$ of the exact sequence (2.22).

Form this it follows that the subspace $\langle v \rangle \subset H^1(T_{\tilde{C}})^+ \cong H^1(T_C(-R))$ is identified with the subspace generated by $\xi_{P,L}$. By the choice we made of the coordinates around P, T and $\sigma(T)$ one can check the identification of v with $\xi_{P,L}$.

So we have

$$\rho_P(Q)(v \odot v) = \rho_P(Q)(\xi_{P,L} \odot \xi_{P,L}) = (m(\rho_P(Q)))(P), \tag{2.23}$$

by the definition of $\xi_{P,L}$, and putting together (2.18), (2.19), (2.23) we get

$$(m(\rho_P(Q)))(P) = -4\pi i \mu_2(Q)(P) \tag{2.24}$$

for all P which is not a critical value of π. Hence $m \circ \rho_P(Q)$ and $-4\pi i \mu_2(Q)$ are two sections of $H^0(K_C^4(R))$ that coincide in the complement of a finite set in C, so they coincide. □

3 Totally Geodesic Submanifolds

In this section, following the ideas of [4, 6], we give an upper bound for the dimension of a germ of a totally geodesic submanifold of $\mathcal{A}^{\delta}_{g-1+r}$ contained in $P_{g,r}(\mathcal{R}^0_{g,r})$.

Proposition 3.1 *Assume that* $[(C, \alpha, R)] \in \mathcal{R}^0_{g,r}$ *is such that* C *is a* k*-gonal curve of genus* g*, with* $g + r \geq k + 3$ *and* $\alpha^2 = \mathcal{O}_C(R)$.

(1) If $r > k + 1$*, then there exists a quadric* $Q \in I_2(K_C \otimes \alpha)$ *such that* rank $\rho(Q) \geq 2g - 2 - k + r$.
(2) If $r \leq k + 1$*, then there exists a quadric* $Q \in I_2(K_C \otimes \alpha)$ *such that* rank $\rho(Q) \geq 2g - 2k - 4 + 2r$.

Proof Let F be a line bundle on C such that $|F|$ is a g^1_k and choose a basis $\{x, y\}$ of $H^0(F)$. Set $M = K_C \otimes \alpha \otimes F^{-1}$ and denote by B the base locus of $|M|$. By Riemann Roch

$$h^0(M) = h^0(M(-B)) = h^0(F \otimes \alpha^{-1}) + g - 1 - k + r \geq g - k + r - 1 \geq 2 \tag{3.1}$$

by assumption.

Note that in case (1) $B = \emptyset$, since $deg(M) = 2g - 2 + r - k > 2g - 1$.

Take a pencil $\langle t_1, t_2 \rangle$ in $H^0(M)$. If $B \neq \emptyset$, write $t_i = t'_i s$ for a section $s \in H^0(C, \mathcal{O}_C(B))$ with $\operatorname{div}(s) = B$. Then $\langle t'_1, t'_2 \rangle$ is a base point free pencil in $|M(-B)|$. Let $\psi : C \to \mathbb{P}^1$ be the morphism induced by this pencil and $\tilde{\psi} = \psi \circ \pi : \tilde{C} \to \mathbb{P}^1$ and set $d := deg(\psi) = deg(M(-B))$. Denote by φ the morphism induced by the pencil $|F|$ and and $\tilde{\varphi} = \varphi \circ \pi : \tilde{C} \to \mathbb{P}^1$.

Consider the rank 4 quadric $Q := xt_1 \odot yt_2 - xt_2 \odot yt_1$. Clearly $Q \in I_2(K_C \otimes \alpha)$. We want to show that $rk\tilde{\rho}(\pi^* Q) \geq d$.

Consider the set $E := \psi(R \cup Crit(\varphi) \cup Crit(\psi) \cup B)$ where $Crit(\varphi)$ (resp. $Crit(\psi)$) denote the set of critical points of φ (resp. ψ). Let $z \in \mathbb{P}^1 \setminus E$ and let $\{P_1, \ldots, P_d\}$ be the fibre of ψ over z. By changing coordinates on $\mathbb{P}^1$ we can assume $z = [0, 1]$, i.e. $t'_1(P_i) = 0$ for $i = 1, \ldots d$. Then clearly $t_1(P_i) = 0$, so $Q(P_i, P_j) = 0$ for all i, j. Set $\{T_i, \sigma(T_i)\} = \pi^{-1}(P_i)$, so $\pi^* Q(T_i, T_j) = \pi^* Q(T_i, \sigma(T_j)) = Q(P_i, P_j) = 0$.

Let us fix a local coordinate at the relevant points and write $\xi_T := \xi_{T,K_{\tilde{C}}}$ for a Schiffer variation of $\tilde{C}$ at T. Set $v_i := \xi_{T_i} + \xi_{\sigma(T_i)}$. Clearly $v_i \in H^1(T_{\tilde{C}})^+ \simeq H^1(T_C(-R))$, so by (2.13) we have $\rho_P(Q)(v_i \odot v_j) = \tilde{\rho}(\pi^* Q)(v_i \odot v_j)$.

Hence, by [6, Thm. 2.2], for $i \neq j$

$$\begin{aligned}\rho_P(Q)(v_i \odot v_j) = \tilde{\rho}(\pi^*Q)(v_i \odot v_j) = 2\tilde{\rho}(\pi^*Q)(\xi_{T_i} \odot \xi_{T_j}) + 2\tilde{\rho}(\pi^*Q)(\xi_{\sigma(T_i)} \odot \xi_{T_j}) = \\ -8\pi i(\pi^*Q)(T_i, T_j)\tilde{\eta}_{T_j}(T_i) - 8\pi i(\pi^*Q)(\sigma(T_i), T_j)\tilde{\eta}_{T_j}(\sigma(T_i)) = 0\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\tilde{\rho}(\pi^*Q)(v_i \odot v_i) = 2\tilde{\rho}(\pi^*Q)(\xi_{T_i} \odot \xi_{T_i}) + 2\tilde{\rho}(\pi^*Q)(\xi_{\sigma(T_i)} \odot \xi_{T_i}) = \\ -4\pi i\tilde{\mu_2}(\pi^*Q)(T_i) - 8\pi i(\pi^*Q)(\sigma(T_i), T_i)\tilde{\eta}_{T_i}(\sigma(T_i)) = -4\pi i\tilde{\mu_2}(\pi^*Q)(T_i) = -4\pi i\mu_2(Q)(P_i).\end{aligned} \tag{3.3}$$

For a rank 4 quadric the second Gaussian map can be computed as follows: $\mu_2(Q) = \mu_{1,F}(x \wedge y)\mu_{1,M}(t_1 \wedge t_2)$, where $\mu_{1,F}$ and $\mu_{1,M}$ are the first Gaussian maps of the line bundles F and M (see [2, Lemma 2.2]). Now $\mu_{1,F}(x \wedge y)(P_i) \neq 0$, because $P_i \notin Crit(\varphi)$ by the choice of z. Moreover P_i is not in the base locus B. On $C \setminus B$ the morphism ψ coincides with the map associated to $\langle t_1, t_2\rangle$. Since $P_i \notin Crit(\psi)$, it is not a critical point for the latter map. Therefore also $\mu_{1,M}(t_1 \wedge t_2)(P_i) \neq 0$. Thus $\mu_2(Q)(P_i) = \mu_{1,F}(x \wedge y)(P_i)\mu_{1,M}(t_1 \wedge t_2)(P_i) \neq 0$ for every $i = 1, \dots, d$.

We claim that the vectors $\{v_1, \dots, v_d\}$ are linearly independent in $H^1(T_{\tilde{C}})^+ \simeq H^1(T_C(-R))$.

In fact we show that the subspace $W := \langle \xi_{T_1}, \xi_{\sigma(T_1)}, \dots, \xi_{T_d}, \xi_{\sigma(T_d)}\rangle$ of $H^1(\tilde{C}, T_{\tilde{C}})$ has dimension $2d$. This is equivalent to say that the annihilator $Ann(W)$ of W in $H^0(\tilde{C}, K^2_{\tilde{C}})$ has codimension $2d$. Observe that $Ann(W) = H^0(\tilde{C}, K^2_{\tilde{C}}(-D))$, where $D = T_1 + \sigma(T_1) + \dots + T_d + \sigma(T_d)$. Then by Riemann Roch we have $h^0(K^2_{\tilde{C}}(-D)) = h^0(T_{\tilde{C}}(D)) + 4(\tilde{g} - 1) - 2d - (\tilde{g} - 1) = 3\tilde{g} - 3 - 2d$, since $deg(T_{\tilde{C}}(D)) = -2(\tilde{g} - 1) + 2d = -4g - 2r + 4 + 2d \leq -4g - 2r + 4 + 2(2g - 2 + r - k - deg(B)) = -2k - 2deg(B) < 0$.

This shows that $\{\xi_{T_1}, \xi_{\sigma(T_1)}, \dots, \xi_{T_d}, \xi_{\sigma(T_d)}\}$ are linearly independent in $H^1(\tilde{C}, T_{\tilde{C}})$ and hence also $\{v_1, \dots, v_d\}$ are linearly independent in $H^1(T_{\tilde{C}})^+$.

By (3.2), (3.3) one immediately obtains that the restriction of $\tilde{\rho}(Q)$ to the subspace $W' := \langle v_1, \dots, v_d\rangle$ is represented in the basis $\{v_1, \dots, v_d\}$ by a diagonal matrix with entries

$-4\pi i\mu_2(\pi^*Q)(T_i) = -4\pi i\mu_2(Q)(P_i) \neq 0$ on the diagonal. So $\rho_P(Q)$ has rank at least d.

In case (1) B is empty, hence $d = 2g - 2 + r - k$. In case (2), by Clifford Theorem we have:

$$2(h^0(M(-B)) - 1) \leq deg(M(-B)) = 2g - 2 + r - k - deg(B),$$

hence

$$deg(B) \leq 2g - 2 + r - k - 2h^0(M(-B)) + 2 \leq 2g + r - k - 2(g - k + r - 1) = k - r + 2$$

and $d = 2g - 2 + r - k - \deg(B) \geq 2g + 2r - 2k - 4$. $\square$

Theorem 3.2 *Assume that* $[(C, \alpha, R)] \in \mathcal{R}^0_{g,r}$ *where* C *is a* k*-gonal curve of genus* g *with* $g + r \geq k + 3$*. Let* Y *be a germ of a totally geodesic submanifold of* $\mathcal{A}^\delta_{g-1+r}$ *which is contained in* $P_{g,r}(\mathcal{R}^0_{g,r})$ *and passes through* $P(C, \alpha, R)$*. Then*

(1) If $r > k + 1$*, then* $\dim Y \leq 2g - 2 + \frac{3r+k}{2}$.
(2) If $r \leq k + 1$*, then* $\dim Y \leq 2g + r + k - 1$.

Proof Since Y is totally geodesic, for any $v \in T_{[(C,\alpha,R)]}Y$ we must have that $\rho(Q)(v \odot v) = 0$ for any Q in $I_2(K_C \otimes \alpha)$. Hence if a quadric Q is such that the rank of $\rho(Q)$ is at least m

$$\dim T_{[C]}Y \leq (3g - 3 + 2r) - \frac{m}{2}.$$

The result then follows by the existence of a quadric $Q \in I_2(K_C \otimes \alpha)$ shown in Theorem 3.1, with $rank(\rho(Q)) \geq m$, where $m = 2g - 2 - k + r$ in case (1) and $m = 2g - 2k - 4 + 2r$ in case (2). $\square$

Remark 3.3 In case (2) of Proposition 3.1 if $|M|$ is base point free we have the same estimate as in case (1), namely rank $\rho(Q) \geq 2g - 2 - k + r$. So in this case the bound on the dimension of a germ of a totally geodesic submanifold Y contained in the Prym locus and passing through $P(C, \alpha, R)$ with C a k-gonal curve such that $|M|$ is base point free becomes: $\dim Y \leq 2g - 2 + \frac{3r+k}{2}$.

We will now give a bound on the dimension of a germ a totally geodesic submanifold of $\mathcal{A}^\delta_{g-1+r}$ contained in $P_{g,r}(\mathcal{R}^0_{g,r})$ which does not depend on the gonality.

Theorem 3.4 *Let* Y *be a germ of a totally geodesic submanifold of* $\mathcal{A}^\delta_{g-1+r}$ *which is contained in* $P_{g,r}(\mathcal{R}^0_{g,r})$*.*

(1) If $g < 2r - 5$*, then* $\dim Y \leq \frac{9}{4}g + \frac{3}{2}r - \frac{5}{4}$.
(2) If $g \geq 2r - 5$*, then* $\dim Y \leq \frac{5}{2}g + r + \frac{1}{2}$.

Proof Recall that the gonality k of a genus g curve is at most $[(g + 3)/2]$.

If $g < 2r - 5$, then $k \leq [(g + 3)/2] < r - 1$, hence statement (1) follows immediately by Theorem 3.2 (1).

If $g \geq 2r - 5, r \leq [(g + 3)/2] + 1$. If $r \leq k + 1$, we apply inequality (2) of Theorem 3.2 and we get statement (2). If $k < [(g + 3)/2]$ and $r > k + 1$, then estimate (1) of Theorem 3.2 applies, so $\dim Y \leq 2g - 2 + \frac{3r+k}{2} < \frac{9}{4}g + \frac{3}{2}r - \frac{5}{4} \leq \frac{5}{2}g + r + \frac{1}{2}$. $\square$

Remark 3.5 In [7] examples of Shimura curves (hence totally geodesic) of $\mathcal{A}_{g-1+r}$ contained in $P_{g,r}(\mathcal{R}_{g,r})$ when $r = 0, 1$ have been constructed using families of Galois covers of $\mathbb{P}^1$. The examples when $r = 0$ are all contained in $\mathcal{A}_{g-1}$ with $g \leq 13$, while the ones with $r = 1$ are all contained in $\mathcal{A}_g$ with $g \leq 8$.

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Strongly Real Beauville Groups III

Ben Fairbairn

Abstract Beauville surfaces are a class of complex surfaces defined by letting a finite group G act on a product of Riemann surfaces. These surfaces possess many attractive geometric properties several of which are dictated by properties of the group G. A particularly interesting subclass are the 'strongly real' Beauville surfaces that have an analogue of complex conjugation defined on them. In this survey we discuss these objects and in particular the groups that may be used to define them. *En route* we discuss several open problems, questions and conjectures and in places make some progress made on addressing these.

Keywords Beauville group · Beauville surface · Symmetric Riemann surface · Strongly real Beauville surface · Group actions

1 Introduction

The reader asking 'where are the first two instalments of this series?' should note the following. Morally the first instalment of this series (i.e. 'Strongly Real Beauville Groups I') is [29] whilst its sequel (i.e. 'Strongly Real Beauville Groups II') is [30]. Each instalment is fairly self-contained (to the point of having a fair amount of overlap in their introductory sections) and so hopefully the reader will lose little if they have neither read nor have to hand copies of these.

Roughly speaking (precise definitions will be given in the next section), a Beauville surface is a complex surface $\mathcal{S}$ defined by taking a pair of complex curves, i.e. Riemann surfaces, $\mathcal{C}_1$ and $\mathcal{C}_2$ and letting a finite group G act freely on their product to define $\mathcal{S}$ as a quotient $(\mathcal{C}_1 \times \mathcal{C}_2)/G$. These surfaces have a wide variety of attractive geometric properties: they are surfaces of general type; their automorphism

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groups [53] and fundamental groups [7, 23] are relatively easy to compute (being closely related to G); they are rigid surfaces in the sense of admitting no nontrivial deformations [9] and thus correspond to isolated points in the moduli space of surfaces of general type [42].

Much of this good behaviour stems from the fact that the surface $(\mathscr{C}_1 \times \mathscr{C}_2)/G$ is uniquely determined by a particular pair of generating sets of G known as a 'Beauville structure'. This converts the study of Beauville surfaces to the study of groups with Beauville structures, i.e. Beauville groups.

Beauville surfaces were first defined by Catanese in [19] as a generalisation of an earlier example of Beauville [13, Exercise X.13(4)] (native English speakers may find the English translation [14] somewhat easier to read and get hold of) in which $\mathscr{C}_1 = \mathscr{C}_2$ and the curves are both the Fermat curve defined by the equation $X^5 + Y^5 + Z^5 = 0$ being acted on by the group $(\mathbb{Z}/5\mathbb{Z}) \times (\mathbb{Z}/5\mathbb{Z})$ (this choice of group may seem somewhat odd at first, but the reason will become clear later). Bauer, Catanese and Grunewald went on to use these surfaces to construct examples of smooth regular surfaces with vanishing geometric genus [10]. Early motivation came from the consideration of the 'Friedman–Morgan speculation'—a technical conjecture concerning when two algebraic surfaces are diffeomorphic which Beauville surfaces provide counterexamples to. More recently, they have been used to construct interesting orbits of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (connections with Gothendeick's theory of *dessins d'enfants* make it possible for this group to act on the set of all Beauville surfaces). Indeed one of the more impressive applications of these surfaces is the proof by González-Diez and Jaikin-Zapirain in [44] that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the set of regular dessins by showing that it acts regularly on the set of Beauville surfaces.

Furthermore, Beauville's original example has also been used by Galkin and Shinder in [40] to construct examples of exceptional collections of line bundles.

Like any survey article, the topics discussed here reflect the research interests of the author. Slightly older surveys discussing related geometric and topological matters are given by Bauer et al. in [11, 12]. Other notable works in the area include [6, 28, 54, 67, 73].

In Sect. 2 we provide preliminary information and in particular give specific definitions for the concepts we have only talked about very vaguely until now. In Sect. 3 we will discuss the finite simple groups before considering the more general case of characteristically simple groups in Sect. 4. In Sect. 5 we move to the opposite extreme by considering abelian and nilpotent groups. In Sect. 6 we will discuss recent work on Doubly Hurwitz Beauville groups and related constructions before finally in Sect. 7 discussing a number of smaller and more minor matters.

2 Preliminaries

We give the main definition.

Definition 1 A surface $\mathcal{S}$ is a *Beauville surface of unmixed type* if

- the surface $\mathcal{S}$ is isogenous to a higher product, that is, $\mathcal{S} \cong (\mathcal{C}_1 \times \mathcal{C}_2)/G$ where $\mathcal{C}_1$ and $\mathcal{C}_2$ are complex algebraic curves of genus at least 2 and G is a finite group acting faithfully on $\mathcal{C}_1$ and $\mathcal{C}_2$ by holomorphic transformations in such a way that it acts freely on the product $\mathcal{C}_1 \times \mathcal{C}_2$, and
- each $\mathcal{C}_i/G$ is isomorphic to the projective line $\mathbb{P}_1(\mathbb{C})$ and the corresponding covering map $\mathcal{C}_i \to \mathcal{C}_i/G$ is ramified over three points.

There also exists a concept of Beauville surfaces of mixed type in which the action of G interchanges the two curves $\mathcal{C}_1$ and $\mathcal{C}_2$ but these are much harder to construct and we shall not discuss these here. (For further details of the mixed case, the most up-to-date information at the time of writing may be found in the work of the author and Pierro in [36].)

In the first of the above conditions the genus of the curves in question needs to be at least 2. It was later proved by Fuertes, González-Diez and Jaikin-Zapirain in [38] that in fact we can take the genus as being at least 6. The second of the above conditions implies that each $\mathcal{C}_i$ carries a regular dessin in the sense of Grothendieck's theory of *dessins d'enfants* (children's drawings) [47]. Furthermore, by Belyĭ's Theorem [15] this ensures that $\mathcal{S}$ is defined over an algebraic number field in the sense that when we view each Riemann surface as being the zeros of some polynomial we find that the coeffcients of that polynomial belong to some number field. Equivalently they admit an orientably regular hypermap [58], with G acting as the orientation-preserving automorphism group. A modern account of *dessins d'enfants* and proofs of Belyĭ's theorem may be found in the recent book of Girondo and González-Diez [43].

These constructions can also be described instead in terms of uniformization and using the language of Fuchsian groups [46, 71].

What makes this class of surfaces so good to work with is the fact that all of the above definition can be 'internalised' into the group. It turns out that a group G can be used to define a Beauville surface if and only if it has a certain pair of generating sets known as a Beauville structure.

Definition 2 Let G be a finite group. For $x, y \in G$ let

$$\Sigma(x, y) := \bigcup_{i=1}^{|G|} \bigcup_{g \in G} \{(x^i)^g, (y^i)^g, ((xy)^i)^g\}.$$

An *unmixed Beauville structure* for the group G is a set of pairs of elements $\{\{x_1, y_1\}, \{x_2, y_2\}\} \subset G \times G$ with the property that $\langle x_1, y_1\rangle = \langle x_2, y_2\rangle = G$ such that

$$\Sigma(x_1, y_1) \cap \Sigma(x_2, y_2) = \{e\}.$$

If G has a Beauville structure we say that G is a *Beauville group*. Furthermore we say that the structure has *type*

$$((o(x_1), o(y_1), o(x_1y_1)), (o(x_2), o(y_2), o(x_2y_2))).$$

In some parts of the literature authors have defined the above structure in terms of so-called 'spherical systems of generators of length 3', meaning $\{x, y, z\} \subset G$ with $xyz = e$, but we omit $z = (xy)^{-1}$ from our notation in this survey. (The reader is warned that this terminology is a little misleading since the underlying geometry of Beauville surfaces is hyperbolic thanks to the below constraint on the orders of the elements.) Furthermore, many earlier papers on Beauville structures add the condition that for $i = 1, 2$ we have that

$$\frac{1}{o(x_i)} + \frac{1}{o(y_i)} + \frac{1}{o(x_iy_i)} < 1,$$

but this condition was subsequently found to be unnecessary following Bauer, Catanese and Grunewald's investigation of the wall-paper groups in [8]. A triple of elements and their orders satisfying this condition are said to be hyperbolic. Geometrically, the type gives us considerable amounts of geometric information about the surface: the Riemann–Hurwitz formula

$$g(\mathscr{C}_i) = 1 + \frac{|G|}{2}\left(1 - \frac{1}{o(x_i)} - \frac{1}{o(y_i)} - \frac{1}{o(x_iy_i)}\right)$$

tells us the genus of each of the curves used to define the surface $\mathscr{S}$ and by a theorem of Zeuthen-Segre this in turn gives us the Euler number of the surface $\mathscr{S}$ since

$$e(\mathscr{S}) = 4\frac{(g(\mathscr{C}_1) - 1)(g(\mathscr{C}_2) - 1)}{|G|}$$

which in turn gives us the holomorphic Euler-Poincaré characteristic of $\mathscr{S}$ since $4\chi(\mathscr{S}) = e(\mathscr{S})$ (see [19, Theorem 3.4]). On a more practical and group theoretic note, the type is often useful for verifying that the critical condition that $\Sigma(x_1, y_1) \cap \Sigma(x_2, y_2) = \{e\}$ is satisfied since this will clearly hold whenever the number $o(x_1)o(y_1)o(x_1y_1)$ is coprime to the number $o(x_2)o(y_2)o(x_2y_2)$.

The abelian Beauville groups were essentially classified by Catanese in [19, pp. 24] and the full argument is given explicitly in [8, Theorem 3.4] where the following is proved.

Theorem 1 *Let G be an abelian group. Then G is a Beauville group if, and only if, $G = (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})$ where $n > 1$ is coprime to* 6.

This explains why Beauville's original example used the group $(\mathbb{Z}/5\mathbb{Z}) \times (\mathbb{Z}/5\mathbb{Z})$ — it is the smallest abelian Beauville group.

Given any complex surface $\mathscr{S}$ it is natural to consider the complex conjugate surface $\overline{\mathscr{S}}$. In particular, it is natural to ask whether or not these two surfaces are biholomorphic.

Definition 3 Let $\mathscr{S}$ be a complex surface. We say that $\mathscr{S}$ is *real* if there exists a biholomorphism $\sigma : \mathscr{S} \to \overline{\mathscr{S}}$ such that σ^2 is the identity map.

(We remark that strictly speaking the above definition is not quite right, it being impossible to compose σ with itself. It is more accurate to talk of the composition $\sigma \circ \overline{\sigma}$ where $\overline{\sigma} : \overline{\mathscr{S}} \to \mathscr{S}$.)

As is often the case with Beauville surfaces, the above geometric condition can be translated into purely group theoretic terms.

Definition 4 Let G be a Beauville group and let $X = \{\{x_1, y_1\}, \{x_2, y_2\}\}$ be a Beauville structure for G. We say that G and X are *strongly real* if there exists an automorphism $\phi \in \text{Aut}(G)$ and elements $g_i \in G$ for $i = 1, 2$ such that

$$g_i \phi(x_i) g_i^{-1} = x_i^{-1} \text{ and } g_i \phi(y_i) g_i^{-1} = y_i^{-1}$$

for $i = 1, 2$.

In practice we can always replace one generating pair by some generating pair that is conjugate to it and so we can take $g_1 = g_2 = e$ and this is often what is done in practice.

In [8] Bauer, Catanese and Grunewald show that a Beauville surface is real if, and only if, the corresponding Beauville group and structure are strongly real. This all comes from the study of the following related concept in the theory of Riemann surfaces. In Singerman's nomenclature of [66], a Riemann surface with a function behaving like the function σ in Definition 3 is said to be symmetric. The relationship with automorphisms of the corresponding group critically depends on the main result of [66]. The reader is warned, however, that some notable errors in [66] were subsequently found and are corrected by Jones et al. in [59]. More specifically, the condition that an automorphism like the above exists is sufficient but it is not necessary. This is corrected by Jones, Singerman and Watson by giving a complete list of conditions that are both necessary and sufficient in [59, Theorem 1.1]. We thus repeat the question first posed by the author as [30, Question 1].

Question 1 Are there interesting strongly real Beauville surfaces arising from the conditions given in [59, Theorem 1.1] but not [66, Theorem 2]?

We remark that symmetric Riemann surfaces are also connected to the theory of Klein surfaces. Real algebraic curves and compact Klein surfaces are equivalent in the same way that the categories of complex algebraic curves and compact Riemann surfaces are equivalent. Indeed, just as a compact, connected, orientable surface admits the structure of a complex analytic manifold of dimension 1 (this is, a Riemann surface structure) then a compact connected surface that is not necessarily orientable admits the structure of a complex *dianalytic* manifold of dimension 1, that is, a Klein surface structure. See [65] for an introductory discussion and [17] for a recent survey of these surfaces.

By way of immediate easy examples, note that the function $x \mapsto -x$ is an automorphism of any abelian group and so every Beauville group given by Theorem 1 is an example of a strongly real Beauville group. More generally the following question is the main subject of this article.

Question 2 Which groups are strongly real Beauville groups?

3 The Finite Simple Groups

Naturally, a necessary condition for being a strongly real Beauville group is being a Beauville group. Furthermore, a necessary condition for being a Beauville group is being 2-generated: we say that a group G is 2-generated if there exist two elements $x, y \in G$ such that $\langle x, y\rangle = G$. It is an easy exercise for the reader to show that the alternating groups A_n for $n \geq 3$ are 2-generated (see the work of Miller in [63]). In [68] Steinberg proved that all of the simple groups of Lie type are 2-generated and in [1] Aschbacher and Guralnick used cohomological methods to show that the larger of the sporadic simple groups are 2-generated, the smaller ones having been dealt with by numerous previous authors. These results rely heavily on the classification of finite simple groups. We thus have that all of the non-abelian finite simple groups are 2-generated making them natural candidates for Beauville groups. This led Bauer, Catanese and Grunewald to conjecture that aside from A_5, which is easily seen to not be a Beauville group, every non-abelian finite simple group is a Beauville group — see [8, Conjecture 1] and [9, Conjecture 7.17]. This suspicion was later proved correct [26, 27, 41, 48], indeed the full theorem proved by the author, Magaard and Parker in [27] is actually a more general statement about quasisimple groups (recall that a group G is quasisimple if it is generated by its commutators and the quotient by its center $G/Z(G)$ is a simple group.). A sketch of the proof of this Theorem is given by the author in [28, Sect. 3].

Having found that all but one of the non-abelian finite simple groups are Beauville groups, it is natural to ask which of the finite simple groups are strongly real Beauville groups. In [8, Sect. 5.4] Bauer, Catanese and Grunewald wrote

> There are 18 finite simple nonabelian groups of order $\leq$ 15000. By computer calculations we have found strongly [real] Beauville structures on all of them with the exceptions of A_5, $PSL_2(7)$, A_6, A_7, $PSL_3(3)$, $U_3(3)$ and the Mathieu group M_{11}.

On the basis of this they made the following conjecture.

Conjecture 1 (*The Weak Strongly Real Conjecture*) All but finitely many of the finite simple groups are Strongly Real Beauville Groups.

In addition to the above, further 'circumstantial evidence' for this conjecture come from the following recent theorem of Malcolm [62] which suggests that if x and y

can be simultaneously inverted by an inner automorphism, then we have plenty of control over $\Sigma(x, y)$.

Theorem 2 *Every element of every non-abelian finite simple group is a product of two strongly real elements.*

Subsequently, various infinite families of simple groups were shown to satisfy Conjecture 1 (including some alluded to above) and computations performed by the author lead to the following [29, Conjecture 1].

Conjecture 2 (*The Strong Strongly Real Conjecture*) All non-abelian finite simple groups apart from A_5, M_{11} and M_{23} are strongly real Beauville groups.

As far as the author is aware no advances in proving the conjecture has been made since [29] appeared so we refer the reader there for the specific information on the most recent progress on this conjecture.

4 Characteristically Simple Groups

Another class of finite groups that has recently been studied from the viewpoint of Beauville constructions, and appears to be fertile ground for providing further examples of strongly real Beauville groups, are the characteristically simple groups that we define as follows (the definition commonly given is somewhat different from the below but in the case finite groups it can easily be shown to be equivalent to the below).

Definition 5 A finite group G is said to be *characteristically simple* if G is isomorphic to some direct product S^k where S is a finite simple group.

For example, as we saw in Theorem 1, if $p > 3$ is prime then the abelian Beauville groups isomorphic to $(\mathbb{Z}/p\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z})$ are characteristically simple as is every finite simple group.

The study of characteristically simple Beauville groups was initiated by Jones in [55, 56] where the following conjecture is discussed.

Conjecture 3 Let G be a finite non-abelian characteristically simple group. Then G is a Beauville group if and only if it is a 2-generated group not isomorphic to A_5.

At the time that the previous installment of this series [30] was written the author was skeptical about 2-generated characteristically simple groups being strongly real Beauville groups. Consequently some calculations showing that S^k is strongly real for small values of k when $|S| < 100\,000$ was all that was given in [30, Sect. 4]. The author also gave a rather pithy conjecture regarding the groups of the form $S \times S$. Since then in [35] the author and Jones have generalised these results substantially with the following Theorem.

Theorem 3 *The 2-generated characteristically simple group S^k is a strongly real Beauville group if S is any of the following groups.*

(a) The alternating group A_n apart from $(n, k) = (5, 1)$;
(b) The groups $L_2(q)$ apart from $(q, k) = (4, 1)$ or $(5, 1)$;
(c) The sporadic groups apart from $(S, k) = (M_{11}, 1)$ or $(M_{23}, 1)$ and
(d) Simple groups of order at most 10,000,000.

Simple groups of order at most 10,000,000 includes the groups $L_n(q)$ of order less than or equal to $|L_4(4)|$ and the ten smallest sporadic groups as well as the Tits group, ${}^2F_4(2)'$, among many others. A more comprehensive list may be found [22, pp. 239–240].

The above was proved as a step towards verifying the following, a substantial extension of Conjecture 2.

Conjecture 4 (*The Strongly Strong Strongly Real Conjecture*) Every 2-generated characteristically simple group S^k is a strongly real Beauville group apart from $(S, k) = (\mathrm{A}_5, 1)$, $(\mathrm{M}_{11}, 1)$ or $(\mathrm{M}_{23}, 1)$.

5 Abelian and Nilpotent Groups

Recall that the abelian Beauville groups were classified in Theorem 1 and that an immediate corollary of this result is that every abelain Beauville group is strongly real.

Theorem 1 has been put to great use by González-Diez, Jones and Torres-Teigell in [45] where several structural results concerning the surfaces defined by abelian Beauville groups are proved. For each abelian Beauville group they describe all the surfaces arising from that group, enumerate them up to isomorphism and impose constraints on their automorphism groups. As a consequence they show that all such surfaces are defined over $\mathbb{Q}$.

After the abelian groups, the next most natural class of finite groups to consider are the nilpotent groups. In [2, Lemma 1.3] Barker, Boston and the author note the following easy Lemma.

Lemma 1 *Let G and G' be Beauville groups and let $\{\{x_1, y_1\}, \{x_2, y_2\}\}$ and $\{\{x_1', y_1', \}, \{x_2', y_2'\}\}$ be their respective Beauville structures. Suppose that*

$$gcd(o(x_i), o(x_i')) = gcd(o(y_i), o(y_i')) = 1$$

for $i = 1, 2$. Then $\{\{(x_1, x_1',), (y_1, y_1')\}, \{(x_2, x_2'), (y_2, y_2')\}\}$ is a Beauville structure for the group $G \times G'$.

Recall that a finite group is nilpotent if, and only if, it is isomorphic to the direct product of its Sylow subgroups. It thus follows that Lemma 1, and its easy to prove converse, reduces the study of nilpotent Beauville groups to that of Beauville p-groups. Note that Theorem 1 gives us infinitely many examples of Beauville p-groups for every prime $p > 3$: simply let n be any power of p. Early examples of Beauville 2-groups and 3-groups were constructed by Fuertes et al. in [38] where a Beauville group of order 2^{12} and another of order 3^{12} were constructed. Even earlier than this, two (mixed) Beauville 2-groups of order 2^8 arose as part of a classification due to Bauer, Catanese and Grunewald in [10] of certain classes of surfaces of general type, which give rise to an example of an (unmixed) Beauville 2-group of order 2^7.

Subsequently, in [2] Barker, Boston and the author classified the Beauville p-groups of order at most p^4 and made substantial progress on the cases of groups of order p^5 and p^6. Later, in [69] Stix and Vdovina have constructed another infinite series of Beauville p-groups. In particular this gives the first examples of non-abelian Beauville p-groups of arbitrarily large order and any prime $p \geq 5$. To do this they make use of the theory of pro-p groups and in doing so provide generalisations of examples from [2]. The first explicit construction of an infinite family of Beauville 3-groups was recently given by Fernández-Alcober and Gül in [37] where they consider homomorphic images of the famous Nottingham group as well as providing other general constructions for Beauville p-groups. In doing so they settled several conjectures made in [2].

The earliest explicit infinite family of Beauville 2-groups were constructed by Barker et al. in [3–5] where, again, more general constructions are also considered. The most comprehensive surveys on Beauville p-groups in general are given by Boston in [16] and more recently by the author in [24].

Few of the known examples of Beauville p-groups are known to either be strongly real/non-strongly real. As far as the author is aware the earliest examples of non-abelian strongly real Beauville p-groups to be discovered were an isolated pair of examples of 2-groups constructed by the author in [30, Sect. 7] namely the groups

$$\langle u, v \mid (u^i v^j)^4,\ i, j = 0, 1, 2, 3 \rangle$$

which has order 2^{14} and

$$\langle u, v \mid u^8, v^8, [u^2, v^2], (u^i v^j)^4,\ i, j = 1, 2, 3 \rangle$$

which has order 2^{13}.

Recently in [50] Gül constructed the first known infinite family of non-abelian strongly real Beauville p-groups and in particular discovered the first examples in which p is odd. More specifically, the main result of [50] is the following.

Theorem 4 *Let $F = \langle x, y \mid x^p, y^p \rangle$ be the free product of two cyclic groups of order p for an odd prime p and let $i = k(p-1)+1$ for $k \geq 1$. Then the quotient $F/\gamma_{i+1}(F)$ is a strongly real Beauville group.*

Subsequently in [51] Gül constructed further examples by considering quotients of certain triangle groups. More specifically Gül prove that there are non-abelian strongly real Beauville p-groups of order p^n for every $n \geq 3$, 5 or 7 for the primes $p \geq 5$, $p = 3$ and $p = 7$ respectively.

At around the same time the author constructed another infinite family of non-abelian strongly real Beauville p-groups for p odd in [32, 33] by proving the following.

Theorem 5 *Let p be an odd prime and let q and r be powers of p. If q and r are sufficiently large, then groups $C_q \wr C_r / Z(C_q \wr C_r)$ are strongly real Beauville groups.*

Unlike the groups given by Theorem 4 this theorem gives multiple non-isomorphic examples for infinitely many orders. For example when $(q, r) = (3^{28}, 3^3)$ or $(q, r) = (3^3, 3^5)$ we obtain groups of order 3^{731} which cannot be isomorphic since they have centers of different orders.

By way of a new result we have the following.

Proposition 1 *For every prime p, the smallest non-abelian Beauville p-group is a strongly real Beauville group.*

Proof The smallest non-abelian Beauville p-groups were determined by Barker, Boston and the author in [2]. For $p = 2$ the smallest example is the group defined by the following presentation.

$$G := \langle u, v \mid (u^i v^j)^4 \text{ for } i, j = 0, \ldots, 3,\ (u^2 v^2)^2,\ [u, v]^2,\ (uvuv^3)^2 \rangle$$

For an automorphism, we consider the group defined by the above presentation with an additional generator that we call t along with the new relations $t^2,\ u^t u,\ v^t v$. It is easy to see that if we take

$$x_1 := u, \qquad y_1 := v, \qquad x_2 := uvu \quad \text{and} \quad y_2 := uvuvu,$$

then $\langle x_1, y_1 \rangle = \langle x_2, y_2 \rangle = G$; conjugation by t inverts all of these elements and the conjugacy condition is easily checked computationally.

The case $p = 3$ is similar.

The cases $p \geq 5$ are a special case of [34, Proposition 11]. □

One obvious place providing fertile ground for new examples of strongly real Beauville p-groups are subgroups of larger known Beauville p-groups because the exponent of a subgroup is at most that of the original group. The aforementioned group constructed in [30, Sect. 7] has an automorphism group of order 2^{25} suggesting its subgroup structure morally should work. Alas this good idea quickly falls down. None of the proper subgroups of order greater than 2^9 are even 2-generated, let alone are Beauville groups. As mentioned earlier in this section, no subgroup of order less than 2^7 is even a Beauville group, let alone a strongly real one suggesting that the subgroup structure of this group provides little in the way of new examples.

6 Doubly Hurwitz/Minimal Beauville Groups

We recall the following.

Definition 6 A finite group G is a *Hurwitz group* if it can be generated by an element of order 2 and an element of order 3 such that their product has order 7.

The study of these objects is motivated by Hurwitz's automorphisms theorem which states that the automorphism group of a compact Riemann surface of genus $g \geq 2$ has order at most $84(g-1)$ with equality if and only if the automorphism group is a Hurwitz group. It is easy to show that a Hurwitz group is necessarily perfect making simple groups the natural starting point for investigating these objects.

Recently in [57] Jones and Pierro addressed a question of Zvonkin asking if there exist groups that act as Hurwitz groups in two essentially different ways, that is, which have two generating triples that together provide a Beauville structure.

Definition 7 A *doubly Hurwitz Beauville group* or *dHB group* is a Beauville group of type ((2,3,7),(2,3,7)).

The main results of [57] are summed up in the following.

Theorem 6 *(a) The following are doubly Hurwitz Beauville groups.*

(i) The alternating group A_n for all $n \geq 589$.
(ii) The groups $SL_n(q)$ and $L_n(q)$ for all $n \geq 631$ and prime powers q.

(b) None of the following are doubly Hurwitz Beauville groups.

(i) The sporadic simple groups.
(ii) The groups $L_n(q)$ for $n \leq 7$, ${}^2G_2(3^r)$, ${}^2F_4(2)'$, $G_2(q)$ and ${}^3D_4(q)$.

The basic question of which groups are doubly Hurwitz Beauville groups remains far from resolved, the content of [57] being just a first step, but despite this the following harder question still seems worth asking.

Question 3 Which groups have strongly real Beauville structures of type ((2,3,7), (2,3,7))?

For general discussions of the current knowledge of Hurwitz groups and their corresponding surfaces, see the two excellent surveys of Conder [20, 21] and the more historically-oriented survey of MacBeath [61].

Of course not all groups are Hurwitz groups however every group is the automorphism group of various Riemann surfaces and every group will attain the minimum genus on some surface. Given a group G its *strong symmetric genus* is the minimum genus of a compact Riemann surface on which G acts as a group of automorphisms preserving orientation. For groups that are not Hurwitz groups we can ask the more general analogous question replacing (2,3,7) with whatever type achieves the strong symmetric genus of the group.

Table 1 The sporadic simple groups that are not Hurwitz groups and the types of their generators that attain their symmetric genus

G	Type	G	Type	G	Type
M_{11}	(2,4,11)	M_{12}	(2,3,10)	M_{22}	(2,5,7)
M_{23}	(2,4,23)	HS	(2,3,11)	J_3	(2,4,5)
M_{24}	(3,3,4)	McL	(2,5,8)	Suz	(2,4,5)
O'N	(2,3,8)	Co_2	(2,3,11)	Fi_{23}	(2,3,8)
Co_1	(2,3,8)	$\mathbb{B}$	(2,3,8)		

Definition 8 A *doubly minimal Beauville group* or *dmB group* is a Beauville group G of type $((a, b, c), (a, b, c))$ where (a, b, c) attains the strong symmetric genus of G.

Question 4 Which Beauville groups are dmB groups?

For the sporadic groups we have the following.

Lemma 2 *None of the sporadic groups, except possibly the baby monster group* $\mathbb{B}$*, define Beauville surfaces corresponding to Riemann surfaces that attain their strong symmetric genus.*

Proof The sporadic groups that are Hurwitz are dealt with in [57]. The generators that attain the strong symmetric genus of the remaining groups are given in Table 1. Each of M_{11}, M_{23}, J_3, McL and O'N have only one class of involutions. Similar arguments rule out the groups M_{12}, M_{22}, HS and Co_2. In the groups Co_1 and Fi_{23} all elements of order 8 power up to the same class of involutions. The group M_{24} is easily ruled out computationally (naively scrolling through the elements of the group, it is small enough to do this, shows that a (3,3,4) generating pair necessarily uses elements of class 3B). □

The Baby Monster group $\mathbb{B}$ is famously computationally difficult to deal with: its lowest degree representation is in 4371 dimensions and its lowest degree permutation representation is on around 10^{10} points owing to having order around 10^{34}. Worse, the Baby Monster has far more conjugacy classes for us to worry about than the smaller cases. In [72] Wilson showed that the Baby Monster is not a Hurwitz group but is (2,3,8) generated. There are four classes of involutions; two classes of elements of order 3 and fourteen classes of elements of order 8! Structure constant calculations naively rule out very few cases without a detailed investigation of its 30 classes of maximal subgroups most of which contain elements of all of these orders. More worryingly we do not know the character tables of most of the maximal subgroups and calculating them in the larger cases a computationally taxing problem. Generating triples of classes $(2C, 3B, 8X)$ where X is any of N, M and K and $(2D, 3B, 8X)$ where X is any of N, M, K and I do exist but none using class $3A$ are known. It is unlikely that there are any suggesting that this case is the same as the other sporadic groups.

Problem 1 Settle the case of the Baby Monster.

One class of low-rank groups of Lie type not yet dealt with are ruled out by the following.

Lemma 3 *The Suzuki groups* ${}^2B_2(2^{2n+1})$ *are never dmB groups.*

Proof These groups are well known to not be Hurwitz groups since they contain no elements of order 3. In [70] Suzuki showed that his now eponymous groups were (2,4,5) generated and it is this type that gives the strong symmetric genus of these groups. These groups however have only have one class of involutions making it impossible to have a Beauville structure of type ((2,4,5),(2,4,5)). □

We pose the analogue of Question 3 for dmB groups.

Question 5 Which groups have strongly real Beauville structures that make them dmB groups?

7 Miscelenia

In this final short section we briefly discuss a number of more minor matters.

7.1 Purity

In [34] the author initiated the study of the following.

Definition 9 A finite group G is a *Purely Strongly Real Beauville Group* if G is a Beauville group such that every Beauville structure of G is strongly real. A finite group G is a *Purely Non-Strongly Real Beauville Group* if G is a Beauville group such that none of its Beauville structures are strongly real.

The main results of [34] focus on constructing examples of these concepts and posing questions about them that are summarised as follows. We first highlight the fact that most Beauville groups appear to fit into neither category and various examples among the finite simple groups are constructed. For infinitely many examples we have the following.

Proposition 2 *The following are purely strongly real Beauville groups.*

(a) The groups $L_2(q)$ *where* $q > 4$ *is even;*
(b) abelian Beauville groups;
(c) the groups

$$\langle x, y, z \mid x^{p^n},\ y^{p^n}, z^{p^r}\ [x, y] = z,\ [x, z],\ [y, z]\rangle$$

of order p^{2n+r} *and where* $p \geq 5$ *is prime and* $n \geq r \geq 1$ *are integers.*

Observe that the above gives no examples that are 2-groups or 3-groups.

Problem 2 Find other examples of purely strongly real Beauville groups.

In particular, we have the following question.

Question 6 Do there exist purely strongly real Beauville 2-groups and 3-groups?

The following provides us with infinitely many examples of purely non-strongly real Beauville groups.

Proposition 3 *If G and H are Beauville groups of coprime order, such that G is a purely non-strongly real Beauville group, then $G \times H$ is a purely non-strongly real Beauville group.*

As observed earlier (though not in the terminology defined in this section) the Matheiu groups M_{11} and M_{23} are purely non-strongly real Beauville groups which combined with the numerous examples of Beauville p-groups discussed earlier provides infinitely many examples of non-strongly real Beauville groups.

7.2 Higher Dimensional Analogues

In various parts of the literature many mathematicians have considered varieties isogenous to a higher product $(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n)/G$, the case $n = 2$ (i.e. surfaces) simply being the most frequently studied. The property of Rigidity that makes Beauville surfaces stand out can easily be generalised to higher dimensions. The following definition was given by Jones in [52].

Definition 10 Let G be a finite group. The *Beauville dimension* of G is the least positive integer d such that there exist generating pairs $(x_1, y_1), \ldots, (x_d, y_d) \in G^2$ such that

$$\Sigma(x_1, y_1) \cap \cdots \cap \Sigma(x_d, y_d) = \{e\}.$$

We write $d_B(G)$ for the Beauville dimension of G. If no such integer exists then we say that the group has infinite Beauville dimension.

Beauville groups are simply groups of Beauville dimension 2. Groups with higher Beauville dimension correspond to higher dimensional complex varieties that also enjoy many of the nice properties of Beauville surfaces such as being rigid, being defined over algebraic number fields etc. By way of an easy example, we noted in Sect. 3 the only finite simple group with Beauville dimension not equal to 2 is the alternating group A_5 which has infinite Beauville dimension since every sigma set must contain elements from the only class of cyclic subgroups of order 5. The following example is the earliest known given in [52].

Example 1 Consider the group

$$(\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z}) = \langle x,\ y \mid x^3, y^3, [x, y]\rangle.$$

The non-trivial elements of this group naturally partition into the four cyclic subgroups $\langle x\rangle$, $\langle y\rangle$, $\langle xy\rangle$ and $\langle xy^2\rangle$. The sigma set of any generating set contains members from three of these subgroups. However we also have that

$$\Sigma(x, y) \cap \Sigma(x, y^2) \cap \Sigma(x, xy) \cap \Sigma(y, xy) = \{e\}$$

and so $d_B(G) = 4$.

For many years this and close relatives of it were the only known examples of groups with finite Beauville dimension greater than 2. Recent work of the author's PhD student, Ludovico Carta, to appear in [18], extends this example to infinite families of groups with Beauville dimensions 3 and 4.

Question 7 Are there any groups G such that $d_B(G)$ is finite and $d_B(G) > 4$?

Observe that the variety constructed in Example 1 combined with the earlier observations about abelian groups suggests the following.

Problem 3 Construct examples of groups G such that $d_B(G) > 2$ is finite and the corresponding variety is strongly real.

7.3 Reflection Groups

Another class of 2-generated finite groups that have only been partially investigated from the viewpoint of Beauville constructions are reflection groups. In [31] the author proves the following.

Theorem 7 *Every finite irreducible Coxeter group is a strongly real Beauville group aside from the groups of type:*

(a) A_n *for* $n \leq 3$;
(b) B_n *for* $n \leq 4$;
(c) D_n *for* $n \leq 4$;
(d) F_4, H_3 *and*
(e) $I_2(k)$ *for any* k.

Corollary 1 *No product of three or more irreducible Coxeter groups is a Beauville group. Furthermore,* $K_1 \times K_2$ *is a strongly real Beauville group if* K_1 *and* K_2 *are strongly real irreducible Coxeter Beauville groups not of type* B_n.

Corollary 2 *An irreducible Coxeter group is a Beauville group if and only if it is a strongly real Beauville group.*

Altogether the above goes most of the way to classifying which of the real reflection groups are strongly real Beauville groups, however completing the task is more difficult. Several examples are given in [31, Sect. 5] showing that $K_1 \times K_2$ can be a strongly real Beauville group, even when K_1 and/or K_2 are not.

Question 8 Which real reflection groups are Strongly Real Beauville Groups?

As far as the author is aware, nowhere in the literature considers the more general question of which finite reflection groups of any kind (the above completely ignores objects such as complex reflections groups and quarternionic reflection groups) are strongly real Beauville groups. For example, this wider class of groups includes all cyclic groups (which are clearly not even Beauville groups) but it also gives us examples like the following.

Example 2 In the Sheppard–Todd classification of complex reflection groups, the group denoted G_{24}, also denoted $W(\mathrm{J}_3(4))$, is isomorphic to the group $\mathrm{L}_2(7) \times C_2$. It is easily verified that if we take

$$x_1 := (1,4,8,3,6,5,7)(9,10),\ y_1 := (2,5,7,6,3,4,8)(9,10),$$

$$x_2 := (1,7,2,4)(3,6,8,5)(9,10),\ y_2 := (1,8,2,5)(3,6,4,7)$$

$$\text{and } t := (3,6)(4,7)(5,8),$$

then these permutations give a Beauville structure of type ((14,14,7),(4,4,4)) such that $x_i^t = x_i^{-1}$ and $y_i^t = y_i^{-1}$ for $i = 1, 2$ showing that this is a Strongly Real Beauville Group.

Question 9 Which finite reflection groups are (strongly real) Beauville groups?

7.4 Beauville Spectra

The following definition was first made by Fuertes, González-Diez and Jaikin-Zapirain in [38, Definition 11].

Definition 11 Let G be a finite group. The *Beauville genus spectrum* of G, denoted $Spec(G)$, is the set of pairs of integers (g_1, g_2) such that $g_1 \leq g_2$ and there are curves $\mathscr{C}_1$ and $\mathscr{C}_2$ of genera g_1 and g_2 with the action of G on $\mathscr{C}_1 \times \mathscr{C}_2$ such that $(\mathscr{C}_1 \times \mathscr{C}_2)/G$ is a Beauville surface.

They went on in [38] to determine the Beauville genus spectra for the symmetric group S_5, the linear group $L_2(7)$ and abelian Beauville groups as well as showing that $Spec(S_6) \neq \varnothing$ (though clearly this last result has been generalised by any theorem proving that other groups are Beauville group). These calculations were later pushed further to other small almost-simple groups by Pierro in his PhD thesis [64], the largest group he considered being the Suzuki group Sz(8) (whose order is just 29120) there being 73 such pairs for this group. As the number of conjugacy classes of the groups grows, the size of the corresponding Beauville genus spectrum also grows making it difficult to push these calculations for almost-simple groups much further. Computer programmes written in GAP [39] can also be found in [64].

Imposing a restriction on the Beauville structures clearly makes this set smaller and thus the problem of determining such a spectrum is more tractable. The following natural definition was first made by the author in [24].

Definition 12 The *strongly real Beauville genus spectrum* of G, that we shall denote $SRSpec(G)$ is the set of pairs of integers (g_1, g_2) such that $g_1 \leq g_2$ and there are curves $\mathscr{C}_1$ and $\mathscr{C}_2$ of genera g_1 and g_2 with the action of G on $\mathscr{C}_1 \times \mathscr{C}_2$ such that $(\mathscr{C}_1 \times \mathscr{C}_2)/G$ is a real Beauville surface.

Since elements of larger order tend to have the property that no automorphism will map them to their inverses it seems likely to the author that $SRSpec(G)$ will in general be much smaller than $Spec(G)$ for most groups. In particular, if determining $Spec(G)$ for a given group G is difficult owing to its size, then the problem of performing the same task for $SRSpec(G)$ may be much more tractable.

Problem 4 Determine the strongly real Beauville genus spectrum of Beauville groups.

Many of the other problems raised here can be described in terms of this quantity. For example, determining if a group G is a strongly real Beauville group is the same as determining when $SRSpec(G) \neq \varnothing$; if G is a purely strongly real Beauville group, then it has the property that $Spec(G) = SRSpec(G)$ and if G is a purely non-strongly real Beauville group, then $SRSpec(G) = \varnothing$.

For most Beauville groups it is likely that $|SRSpec(G)| < |Spec(G)|$. This motivates the following interesting question first posed in the specific case of p-groups by the author as [24, Question 6.14].

Question 10 For a Beauville group G how does the size of $SRSpec(G)$ compare to $Spec(G)$? A little more specifically, how does $|SRSpec(G)|/|Spec(G)|$ behave as $|G| \to \infty$?

7.5 Beauville Graphs

In recent years there has been a growing trend towards encoding generational problems for finite groups in graphs in the hope of using graph-theoretic techniques to address group-theoretic matters. The most common being the following.

Definition 13 Given a finite group G its *generating graph* is the graph $\Gamma(G)$ defined as follows. The vertices of $\Gamma(G)$ are the non-trivial elements of G with two elements x and y being adjoined by an edge if and only if $\langle x, y\rangle = G$.

It seems natural to translate the study of Beauville structures into graph-theoretic language. We thus make the following definition.

Definition 14 Given a finite group G its *Beauville generating graph* is the graph $\Gamma_B(G)$ defined as follows. The vertices of $\Gamma_B(G)$ are the sets $\Sigma(x, y)$ where $x, y \in G$ generate the group with two vertices $\Sigma(x, y)$ and $\Sigma(x', y')$ being adjoined by an edge if and only if $\Sigma(x, y) \cap \Sigma(x', y') = \{e\}$ or equivalently if $\{\{x, y\}, \{x', y'\}\}$ is a Beauville structure for G.

Compared to the generating graph, the Beauville generating graph has far fewer vertices meaning that in principle it should be somewhat easier to study.

Example 3 Recall the classification of abelian Beauville groups in Theorem 1. In particular if $p \geq 5$ is prime, then $(\mathbb{Z}/p\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z})$ is a Beauville group. Similar to Example 1 the non-trivial elements of this group are partitioned into the $p + 1$ sets of non-trivial elements of the cyclic subgroups. Any $\Sigma(x, y)$ consists of three of these and all possible combinations can be achieved. It follows that this graph is isomorphic to what graph theorists sometimes call the Kneser graph $K^{(3)}_{p+1}$ and general theorems regarding these objects reveal a flurry of properties.

- The graph is connected if and only $p > 5$ (the $p = 5$ case being a set of three disjoint edges.)
- These graphs are regular (that is, every vertex has the same degree) of degree $\binom{p-2}{3}$. Kneser graphs in general are in fact both vertex transitive and edge transitive (this is, any pair of vertices can be sent to each other by some automorphism of the graph and similarly for the edges).
- The chromatic number (the smallest number of colours that can be used to colour the vertices in such a way that adjacent vertices have different colours) is exactly $p - 3$.
- If $p > 7$, then the girth (the length of the smallest cycle in the graph) is 3, in the case $p = 7$ it is 4.
- An application of the Erdős-Ko-Rado theorem tells us that the independence number (the largest size of a set of vertices such that no two are adjacent) is $\binom{p}{2}$.

For Beauville graphs more generally it is unlikely that properties anything like as nice as the above list will hold. Nonetheless the Beauville graphs of other Beauville groups may be of interest.

Lemma 4 *The Beauville graphs of the groups $L_2(p^r)$ are never connected.*

Proof If q is odd then there is always a pair of type $((q + 1)/2, (q - 1)/2, p)$ by straightforward structure constant calculations. For such a pair we have that $\Sigma(x, y) = G$ and so this pair is guaranteed to generate the group and this vertex of the graph will be isolated. The case q even is similar. □

Example 4 If G is the Mathieu group M_{11}, then the graph $\Gamma_B(G)$ is not connected since M_{11} is (5,6,11) generated and since the only primes dividing the order of the group are 2, 3, 5 and 11 and there is only one class of elements of order 2 so such a generating pair corresponds to an isolated vertex.

Question 11 Which Beauville groups have connected Beauville graphs? What other properties do these have?

Of course we can easily consider a more sparse graph that keep the strongly real condition in mind.

Definition 15 Let G be a finite group. The *strongly real Beauville graph* denoted $\Gamma_{SRB}(G)$ is defined the same way as the Beauville graph except adjacency is now defined by $\{\{x, y\}, \{x', y'\}\}$ being a strongly real Beauville structure for G

Example 5 If G is a purely non-strongly real Beauville group, then $\Gamma_{SRB}(G) = \Gamma_B(G)$ so in particular some of these graphs are discussed in some detail in Example 3. If G is a purely non-strongly real Beauville group, then $\Gamma_{SRB}(G)$ is empty regardless of any properties of $\Gamma_B(G)$.

Problem 5 Investigate the properties of these graphs.

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The Pluricanonical Systems of a Product-Quotient Variety

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Abstract We give a method for the computation of the plurigenera of a product-quotient manifold, and two different types of applications of it: to the construction of Calabi-Yau threefolds and to the determination of the minimal model of a product-quotient surface of general type.

Keywords Product-quotient manifolds · Finite group actions · Invariants

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1 Introduction

Product-quotient varieties are varieties obtained by taking a minimal resolution of the singularities of a quotient $X := \left(\prod_1^n C_i\right)/G$, the *quotient model*, where G is a finite group acting diagonally, i.e. as $g(x_1, \ldots, x_n) = (gx_1, \ldots, gx_n)$. Usually the genera of the curves C_i are assumed to be at least 2: for the sake of simplicity, we will assume this implicitly from now on.

The notion of product-quotient variety has been introduced in [1] in the first nontrivial case $n = 2$, as a generalization of the *varieties isogenous to a product of unmixed type*, where the action of the group is assumed to be free.

Product-quotient varieties have proved in the last decade to form a very interesting class, because even if they are relatively easy to construct, there are several objects with interesting properties among them. Indeed they have been a fruitful source of examples with applications in different areas of algebraic geometry.

For example [2] constructs in this way several $K3$ surfaces with automorphisms of prime order that are not symplectic. A completely different application is the construction of rigid not infinitesimally rigid compact complex manifolds obtained in [3], answering a question about 50 years old.

A classical problem is the analysis of the possible behaviors of the canonical map of a surface of general type. Reference [4] provides upper bounds for both the degree of the map and the degree of its image, but very few examples realizing values near those bounds are in literature. The current best values have been recently attained respectively in [5, 6] with this technique.

Last but not least, product-quotient surfaces have been used to construct several new examples of surfaces of general type S with $\chi(\mathcal{O}_S) = 1$, the minimal possible value, see [7] and the references therein. Restricting for the sake of simplicity to the regular case, the minimal surfaces of general type with geometric genus $p_g = 0$, whose classification is a long standing problem known as *Mumford's dream*, we now have dozens of families of them constructed as product-quotient surfaces, see [1, 8–11], a huge number when compared with the examples constructed by other techniques, see [12]. In higher dimensions, a complete classification of threefolds isogenous to a product with $\chi(\mathcal{O}_X) = -1$, the maximal possible value, has been achieved recently see [13, 14].

It is very likely that the list of product-quotient surfaces of general type with $p_g = 0$ in [11] is complete, but we are not able to prove it. The main obstruction to get a full classification is that it is very difficult to determine the minimal model of a regular product-quotient variety. Indeed the list was produced by a computer program able to classify all regular product-quotient surfaces S with $p_g = 0$ and a given value of K^2. The surfaces of general type S with $p_g = 0$ have, by standard inequalities, $1 \leq K_S^2 \leq 9$ when minimal, but a minimal resolution of the singularities of a product-quotient surface may be not minimal and then have $K_S^2 \leq 0$. Detecting the rational curves with self-intersection -1 in one of these surfaces may be very difficult, see for example the *fake Godeaux surface* in [1, Sect. 5].

More generally, in birational geometry one would like to know, given an algebraic variety, one of the"simplest" variety in its birational class, a "minimal" one. This is the famous Minimal Model Program, producing a variety with nef canonical system and at worse terminal singularities, or a Mori fiber space. At the moment we are not able to run a minimal model program explicitly for a general product-quotient variety even in dimension 2. Anyhow, knowing all plurigenera $h^0(dK)$ of an algebraic variety gives a lot of information on its minimal models.

Actually the main result of this paper is a method for computing all plurigenera of a product-quotient variety. We first prove the following

Theorem *Let Y be a smooth quasi-projective variety, let G be a finite subgroup of* $\mathrm{Aut}(Y)$ *and let $\psi\colon \widehat{X} \to Y/G =: X$ be a resolution of the singularities. Then there exists a normal variety $\widetilde{Y}$, a proper birational morphism $\phi\colon \widetilde{Y} \to Y$ and a finite surjective morphism $\epsilon\colon \widetilde{Y} \to \widehat{X}$ such that the following diagram commutes:*

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{\epsilon} & \widehat{X} \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi} \\ Y & \xrightarrow[\pi]{} & X \end{array}$$

Setting $R := K_Y - \pi^* K_X$ and $E := K_{\hat{X}} - \psi^* K_X$ there is a natural isomorphism

$$H^0\big(\widehat{X}, \mathcal{O}_{\widehat{X}}(dK_{\widehat{X}})\big) \simeq H^0\left(Y, \mathcal{O}_Y(dK_Y) \otimes \mathcal{I}_d\right)^G$$

for all $d \geq 1$, where $\mathcal{I}_d$ is the sheaf of ideals $\mathcal{O}_Y(-dR) \otimes \phi_* \mathcal{O}_{\widetilde{Y}}(\epsilon^* dE)$.

and then we show how to compute $\phi_* \mathcal{O}_{\widetilde{Y}}(\epsilon^* dE)$ when X has only isolated cyclic quotient singularities. It should be mentioned that we need an explicit basis of $H^0\left(Y, \mathcal{O}_Y(dK_Y)\right)$ to use the theorem. So in our applications we will work with equations defining Y.

The second motivation for this paper was to investigate methods to construct Calabi-Yau threefolds systematically. Indeed, most of the known Calabi-Yau threefolds are constructed by taking the resolution of a generic anticanonical section of a toric Fano fourfold. This idea stems from Batyrev's seminal paper [15] but the complete list of this topologically distinct Calabi-Yau threefolds which one can obtain with this method was obtained with the help of the computer (see [16]) with the classification of the 473.800.776 reflexive polytopes in dimension 4. Apart from these Calabi-Yau threefolds, very few examples are known and their construction involves ad hoc methods such as quotients by group actions (see, for example, [17–20]).

Hence, the idea of using the well-known machinery of the product-quotient varieties could prove to be effective in finding new examples of Calabi-Yau threefolds. We first prove that no product-quotient variety can be Calabi-Yau. Still, as in [2] for dimension 2, they may be birational to a Calabi-Yau. We then introduce the concept of a numerical Calabi-Yau variety, that is a variety whose Hodge numbers are compatible with a possible Calabi-Yau minimal model. Then we show that a numerical

Calabi-Yau product-quotient threefold is birational to a Calabi-Yau threefold if and only if all its plurigenera are equal to 1. We construct 12 families of numerical Calabi-Yau threefolds as product-quotient variety and use our above mentioned Theorem to compute, for two of them, their plurigenera and then determine if they are birational to a Calabi-Yau threefold or not.

Finally we apply our method to show the minimality of several product-quotient surfaces whose quotient model has several noncanonical singularities, thus disproving a conjecture of I. Bauer and the third author, namely [11, Conjecture 1.5].

The paper is organized as follows.

The first two sections are devoted to some possible applications of a formula for the plurigenera of product-quotient manifolds.

In Sect. 2 we discuss conditions for a product-quotient variety to be minimal. Then we concentrate in the case of dimension 2, giving an explicit formula for the number of curves contracted by the morphism onto the minimal model in terms of the plurigenera.

In Sect. 3 we move to dimension 3, discussing the product-quotient threefolds birational to Calabi-Yau threefolds.

In Sect. 4 we produce, with the help of the computer program MAGMA, 12 families of numerical Calabi-Yau threefolds.

In Sect. 5 we prove our main Theorem above, in Proposition 5.1 and Theorem 5.5.

In Sect. 6, we show how to compute $\mathcal{I}_d$ when all stabilizers are cyclic, as in the case of product-quotient varieties.

In Sects. 7 and 8 we apply our theorem to two of the numerical Calabi-Yau threefolds produced in Sect. 4, showing that one is birational to a Calabi-Yau threefold and the other is not.

Finally, in Sect. 9, we discuss the mentioned application of our theorem to certain product-quotient surfaces and explain why this application would be difficult to achieve with existing techniques.

Notation All algebraic varieties in this article are complex, quasi-projective and integral, so irreducible and reduced.

A curve is an algebraic variety of dimension 1, a surface is an algebraic variety of dimension 2.

For every projective algebraic variety X we consider the dimensions $q_i(X) := h^i(X, \mathcal{O}_X)$ of the cohomology groups of its structure sheaf for all $1 \leq i \leq \dim X$. For $i = n$ this is the *geometric genus* $p_g(X) := q_n(X)$, for $i < n$ they are called *irregularities*. If X is smooth, by Hodge Theory $q_i(X) = h^{i,0}(X) := h^0(X, \Omega^i_X)$. If X is a curve, there are no irregularities and the geometric genus is the usual genus $g(X)$. If S is a surface the unique irregularity $q_1(S)$ is usually denoted by $q(S)$.

A normal variety X is Gorenstein if its dualizing sheaf ω_X ([21, III.7]) is a line bundle. If X is Gorenstein in codimension 1 then ω_X is a Weil divisorial sheaf and we denote by K_X a canonical divisor, so $\omega_X \cong \mathcal{O}_X(K_X)$. We then define its d-th plurigenus $P_d(X) = h^0(X, \mathcal{O}_X(dK_X))$. By Serre duality $P_1(X) = p_g(X)$. X is $\mathbb{Q}$-Gorenstein if K_X is $\mathbb{Q}$-Cartier i.e. if there exists $d \in \mathbb{N}$ such that dK_X is Cartier. A normal variety is factorial, resp. $\mathbb{Q}$-factorial if every integral Weil divisor is Cartier, resp. $\mathbb{Q}$-Cartier.

We use the symbols $\sim_{lin}$ for linear equivalence of Cartier divisors, $\sim_{num}$ for numerical equivalence of $\mathbb{Q}$-Cartier divisors.

We write $\mathbb{Z}_m$ for the cyclic group of order m, $\mathcal{D}_m$ for the dihedral group of order $2m$, $\mathfrak{S}_m$ for the symmetric group in m letters.

For $a, b, c \in \mathbb{Z}$, $a \equiv_b c$ if and only if b divides $a - c$.

2 Minimal Models of Quotients of Product of Two Curves

Consider a product $\prod_{i=1}^n C_i$ of smooth curves. If the genus of each curve is at least 2, then $K_{\prod C_i}$ is ample. Moreover, if G is a finite group acting freely in codimension 1 on $\prod_{i=1}^n C_i$, as in the case of product quotient varieties (of dimension at least 2), K_X is ample too, where we have set $X := \left(\prod_{i=1}^n C_i\right)/G$.

In particular, if G acts freely then X is smooth and K_X is ample, so X is a smooth minimal variety of general type.

If the action of G is free in codimension 1 and X has at worst canonical singularities, then we can take a *terminalization* of X, i.e. a crepant resolution $\widehat{X} \to X$ of the canonical singularities of X such that $\widehat{X}$ has terminal singularities. Then $K_{\widehat{X}}$ is automatically nef and therefore $\widehat{X}$ is a minimal model of X.

If $q(\widehat{X}) \neq 0$, the Albanese morphism of $\widehat{X}$ gives some obstructions to the existence of $K_{\widehat{X}}$-negative curves, since it contracts every rational curve. Indeed the first example of a quotient $X = \left(\prod_{i=1}^n C_i\right)/G$ of general type such that a minimal resolution of the singularities $\widehat{X}$ of X is not a minimal variety is the product-quotient surface studied in [22, 6.1].

We find worth mentioning here that in the similar case of *mixed quotients*, i.e. for minimal resolutions S of singularities of a quotient $C \times C/G$ where G exchange the factors, there are results guaranteeing the minimality of S if S is irregular (and some more assumptions, see [23, Theorem 3] and [24, Theorem 4.5] for the exact statements).

If $q(\widehat{X}) = 0$ we have no Albanese morphism and then determining the minimal model is much more difficult. The first example in the literature of a product-quotient variety $\widehat{X}$ that is not minimal with $q(\widehat{X}) = 0$ is the *fake Godeaux surface* in [1, Sect. 5], whose minimal model is determined by a complicated *ad hoc* argument.

See also [11, Sect. 6] for some conjectures and partial results about sufficient conditions for the minimality of $\widehat{X}$ when $q(\widehat{X}) = 0$.

On the other hand, a lot of information on the birational class of X can be obtained without running an explicit minimal model program for it, by computing some of the birational invariants of X. The geometric genus and the irregularities of $\widehat{X}$ are its simplest birational invariants. They are not difficult to compute for product-quotient varieties. The next natural birational invariants to consider are the plurigenera $P_d(\widehat{X})$, $\forall d \in \mathbb{N}$. They determine a very important birational invariant, the Kodaira dimension

$\mathrm{kod}(\widehat{X})$. If X is of general type, i.e. $\mathrm{kod}(\widehat{X}) = \dim(\widehat{X})$, an important role in the classification theory is played by the volume

$$\mathrm{vol}(K_{\widehat{X}}) := (\dim X)! \limsup_{m\to\infty} \frac{P_m(X)}{m^{\dim X}}$$

of its canonical divisor, that is a birational invariant determined by the plurigenera.

Indeed, let us now restrict for the sake of simplicity to the case $n = 2$. If $\widehat{X}$ is a surface of general type, then it is well known that it has a unique minimal model X_{min}. The natural map of $\widehat{X}$ on its minimal model is the composition of r elementary contractions, where $r = \mathrm{vol}(K_{\widehat{X}}) - K^2_{\widehat{X}}$, and $\mathrm{vol}(K_{\widehat{X}})$ equals the self intersection of a canonical divisor of the minimal model.

By [25, Proposition 5.3] a surface of general type S is minimal if and only if $h^1(\mathcal{O}_S(dK_S)) = 0$ for all $d \geq 2$. Then, by Riemann-Roch, $P_d(S) = \chi(\mathcal{O}_S) + \binom{d}{2}K^2_S$, and therefore

$$\binom{d}{2}K^2_S = P_d(S) + q(S) - p_g(S) - 1. \tag{2.1}$$

Since the right-hand side of (2.1) is a birational invariant it follows that if $\widehat{X}$ is of general type, then

$$\mathrm{vol}(K_{\widehat{X}}) = \frac{P_3(\widehat{X}) - P_2(\widehat{X})}{2} = P_2(\widehat{X}) + q(\widehat{X}) - p_g(\widehat{X}) - 1.$$

By the Enriques-Kodaira classification and Castelnuovo rationality criterion, every surface $\widehat{X}$ with $K^2_{\widehat{X}} > 0$ and $P_2(\widehat{X}) \neq 0$ is of general type, so we have the following well known proposition:

Proposition 2.1 *Assume $\widehat{X}$ is a surface with $K^2_{\widehat{X}} > 0$ and $P_2(\widehat{X}) \neq 0$.*
Then $\widehat{X}$ is a surface of general type and

$$\mathrm{vol}(K_{\widehat{X}}) = P_2(\widehat{X}) + q(\widehat{X}) - p_g(\widehat{X}) - 1 = \frac{P_3(\widehat{X}) - P_2(\widehat{X})}{2}.$$

Similarly, we can compute the volume of the canonical divisor of $\hat{X}$ if we know any pair of plurigenera P_d, $d \geq 2$, or one of its plurigenera, geometric genus and all irregularities. Once we compute $K^2_{\hat{X}}$, an easy computation, we immediately deduce whether $\hat{X}$ is minimal and more generally the number r of irreducible curves of $\hat{X}$ contracted on the minimal model.

3 Product Quotient Varieties Birational to Calabi-Yau Threefolds

The Beauville-Bogomolov theorem has been recently extended to the singular case [26], requiring an extension of the notion of Calabi-Yau to minimal models. The following is the natural definition, a bit more general than the one necessary for the Beauville-Bogomolov decomposition in [26].

Definition 3.1 A complex projective variety Z with at most terminal singularities is called Calabi-Yau if it is Gorenstein,

$$K_Z \sim_{lin} 0 \qquad \text{and} \qquad q_i(Z) = 0 \qquad \forall 1 \leq i \leq \dim Z - 1.$$

Calabi-Yau varieties of dimension 2 are usually called $K3$ surfaces.

We first show that there is no Calabi-Yau product-quotient variety.

Proposition 3.2 *Let $X = (C_1 \times \ldots \times C_n)/G$ be the quotient model of a product-quotient variety and let $\rho \colon \widehat{X} \to X$ be a partial resolution of the singularities of X such that $\widehat{X}$ has at most terminal singularities.*

Then $K_{\widehat{X}} \not\sim_{num} 0$.

Proof Let $\pi \colon \prod C_i \to X$ be the quotient map. Then π is unramified in codimension 1. Since $K_{\prod C_i}$ is ample, then K_X is ample too, so it has strictly positive intersection with every curve of X. Since codim Sing $X \geq 2$ one can easily find a curve C in X not containing any singular point of X: for example a general fibre of the projection $X = (C_1 \times \ldots \times C_n)/G \to (C_2 \times \ldots \times C_n)/G$. Set $\widehat{C} = \rho^* C$. Then $K_{\widehat{X}}\widehat{C} = K_X C \neq 0$ and therefore $K_{\widehat{X}} \not\sim_{num} 0$. □

So there is no hope to construct a Calabi-Yau variety directly as partial resolution of the singularities of a product-quotient variety, but one can still hope to get something birational to a Calabi-Yau variety. Reference [2] constructed several $K3$ surfaces that are birational to product-quotient varieties. Their method starts by constructing product-quotient surfaces with $p_g = 1$ and $q = 0$.

We follow a similar approach for constructing Calabi-Yau threefolds. This leads to the following definition:

Definition 3.3 A normal threefold $\widehat{X}$ is a numerical Calabi-Yau if

$$p_g(\widehat{X}) = 1, \qquad q_i(\widehat{X}) = 0 \qquad \text{for} \qquad i = 1, 2.$$

Proposition 3.4 *Let $\widehat{X}$ be a product-quotient threefold. Assume that $\widehat{X}$ is birational to a Calabi-Yau threefold. Then $\widehat{X}$ is a numerical Calabi-Yau threefold.*

Proof Let Z be a Calabi-Yau threefold birational to $\widehat{X}$. To prove that $\widehat{X}$ is a numerical Calabi-Yau, we take a common resolution $\widehat{Z}$ of the singularities of Z and of $\widehat{X}$. Since $\widehat{X}$ and Z have terminal singularities, and terminal singularities are rational (see [27]), it follows that

$$p_g(Z) = p_g(\widehat{Z}) = p_g(\widehat{X}) \quad \text{and} \quad q_i(Z) = q_i(\widehat{Z}) = q_i(\widehat{X})$$

by the Leray spectral sequence. □

Remark 3.5 It follows that the quotient model of a numerical Calabi-Yau product-quotient threefold has at least one singular point that is not canonical. Indeed, since the quotient map $\pi\colon C_1 \times C_2 \times C_3 \to X$ is quasi-étale and the curves C_i have genus at least two, then K_X is ample. If X had only canonical singularities, then $\widehat{X}$ would be of general type, and so would be Z, a contradiction.

Remark 3.6 Let X be the quotient model of a numerical Calabi-Yau product-quotient threefold and $\rho\colon \widehat{X} \to X$ be a resolution, then $p_g(\widehat{X}) = 1 \Rightarrow \kappa(\widehat{X}) \neq -\infty$. Now we run a Minimal Model Program on $\widehat{X}$. Assume that it ends with a Mori fibre space, then $\kappa(\widehat{X}) = -\infty$ according to [28, Theorem 3-2-3]) which is impossible. Therefore, the Minimal Model Program ends with a threefold Z with terminal singularities and K_Z nef.

We close this section with its main result, a criterion to decide whether a numerical Calabi-Yau product-quotient threefold is birational to a Calabi-Yau threefold.

Proposition 3.7 *Let $\widehat{X}$ be a numerical Calabi-Yau product-quotient threefold. If $P_d(\widehat{X}) = 1$ for all $d \geq 1$, then $\widehat{X}$ is birational to a Calabi-Yau threefold.*

Proof Let Z be a minimal model of $\widehat{X}$. It suffices to show that K_Z is trivial. According to Kawamata's abundance for minimal threefolds [29], some multiple m_0K_Z is base point free. By assumption $h^0(m_0K_Z) = h^0(m_0K_{\widehat{X}}) = 1$, which implies that m_0K_Z is trivial. In particular $m_0K_{Z^0}$ is trivial, where $Z^0 = Z \setminus Sing(Z)$ is the smooth locus. Since Z has terminal singularities $h^0(K_{Z^0}) = h^0(K_{\widehat{X}}) = 1$ and it follows that K_{Z^0} is trivial. By normality, K_Z must be also trivial. □

4 Examples of Numerical Calabi-Yau Product-Quotient Threefolds

In this section we present an algorithm that allows us to systematically search for numerical Calabi-Yau threefolds. We use a MAGMA implementation of this algorithm to produce a list of examples of such threefolds. For a detailed account about classification algorithms and the language of product quotients, we refer to [30].

To describe the idea of the algorithm, suppose that the quotient model of a numerical Calabi-Yau threefold

$$X = \big(C_1 \times C_2 \times C_3\big)/G$$

is given. Then $C_i/G \cong \mathbb{P}^1$ and we have three G-covers $f_i\colon C_i \to \mathbb{P}^1$. Let $b_{i,1}, \dots, b_{i,r_i}$ be the branch points of f_i and denote by $T_i := [m_{i,1}, \dots, m_{i,r_i}]$ the three unordered lists of branching indices, these will be called the types in the sequel.

Proposition 4.1 *The type* $T_i := [m_{i,1}, \ldots, m_{i,r_i}]$ *satisfies the following properties:*

(i) $m_{i,j} \leq 4g(C_i) + 2$,
(ii) $m_{i,j}$ *divides the order of* G,
(iii) $r_i \leq \dfrac{4\big(g(C_i) - 1\big)}{n} + 4$,
(iv) $2g(C_i) - 2 = |G|\Bigg(-2 + \sum_{j=1}^{r_i} \dfrac{m_{i,j} - 1}{m_{i,j}}\Bigg)$

Proof (ii) follows from the fact that the $m_{i,j}$ are the orders of the stabilizers of the points above the branch points $b_{i,j}$.

(i) is an immediate consequence of the classical bound of Wiman [31] for the order of an automorphism of a curve of genus at least 2, since the stabilizers are cyclic.

(iv) is the Riemann-Hurwitz formula. (iii) follows from (iv) and $m_{i,j} \geq 2$. □

1st Step: The first step of the algorithm is based on the proposition above. As an input value we fix an integer g_{max}. The output is a full list of numerical Calabi-Yau product-quotient threefolds, such that the genera of the curves C_i are bounded from above by g_{max}.

According to the Hurwitz bound on the automorphism group, we have

$$|Aut(C_i)| \leq 84(g_{max} - 1).$$

Consequently there are only finitely many possibilities for the order n of the group G. On the other hand, for fixed $g_i \leq g_{max}$ and fixed group order n, there are only finitely many possibilities for integers $m_{i,j} \geq 2$ fulfilling the constraints from the proposition above. We wrote a MAGMA code, that returns all admissible combinations

$$[g_1, g_2, g_3, n, T_1, T_2, T_3].$$

2nd Step: For each tuple $[g_1, g_2, g_3, n, T_1, T_2, T_3]$ determined in the first step, we search through the groups G of order n and check if we can realize three G covers $f_i : C_i \to \mathbb{P}^1$ with branching indices $T_i := [m_{i,1}, \ldots, m_{i,r_i}]$. By *Riemann's existence theorem* such covers exist if and only if there are elements $h_{i,j} \in G$ of order $m_{i,j}$, which generate G and fulfill the relations

$$\prod_{j=1}^{r_i} h_{i,j} = 1_G \qquad \text{for each} \quad 1 \leq i \leq 3.$$

Let X be the quotient of $C_1 \times C_2 \times C_3$ by the diagonal action of G. The singularities

$$\frac{1}{n}(1, a, b)$$

of X can be determined using the elements $h_{i,j}$ cf. [1, Proposition 1.17]. The same is true for the invariants p_g and q_i of a resolution cf. [13, Sect. 3], since they are given as the dimensions of the G-invariant parts of $H^0(\Omega^p_{C_1\times C_2\times C_3})$, which can be determined using the formula formula of *Chevalley-Weil* see [13, Theorem 2.8]. The threefolds with only canonical singularities are discarded as well as those with invariants different from $p_g = 1, q_1 = q_2 = 0$. As an output we return the following data of X: the group G, the types T_i, the set of canonical singularities $\mathcal{S}_c$ and the set of non-canonical singularities $\mathcal{S}_{nc}$.

We run our MAGMA implementation of the algorithm for $g_{max} = 6$ and the additional restriction that the $f_i\colon C_i \to \mathbb{P}^1$ are branched in only three points i.e. $r_i = 3$. The output is in Table 1.

They may be birational to a Calabi-Yau threefold or not. Both cases occur, as we will see in Sects. 7 and 8.

5 The Sheaves of Ideals $\mathcal{I}_d$ on a Smooth Projective Variety with a Finite Group Action

Proposition 5.1 *Let Y be a smooth quasi-projective variety, let G be a finite subgroup of* $\mathrm{Aut}(Y)$ *and let $\psi\colon \widehat{X} \to Y/G$ be a resolution of the singularities. Then there exists a normal variety $\widetilde{Y}$, a proper birational morphism $\phi\colon \widetilde{Y} \to Y$ and a finite surjective morphism $\epsilon\colon \widetilde{Y} \to \widehat{X}$ such that the following diagram commutes:*

$$\begin{CD} \widetilde{Y} @>{\epsilon}>> \widehat{X} \\ @V{\phi}VV @VV{\psi}V \\ Y @>>{\pi}> Y/G \end{CD}$$

Up to isomorphism $\widetilde{Y}$ is the normalisation of the fibre product $Y \times_{Y/G} \widehat{X}$ and ϕ and ϵ are the natural maps.

The proof of this proposition is just a combination of the universal property of the fibre product and the universal property of the normalisation.

Remark 5.2 Note that $\widetilde{Y}$ in general fails to be smooth cf. [32, Example 2.30].

Proposition 5.3 *The G action on Y lifts to an action on $\widetilde{Y}$ such that $\widehat{X}$ is the quotient.*

Proof Consider the natural G action on $Y \times_{Y/G} \widehat{X}$. By the universal property of the normalisation it lifts to an action on $\widetilde{Y}$. The birational map $\widetilde{Y}/G \to \widehat{X}$ induced by ϵ is finite and therefore an isomorphism by Zariski's Main theorem. □

Table 1 Some numerical Calabi-Yau product-quotient threefolds. Each row corresponds to a threefold, each column to one of the data of the construction: from left to right the group G, its Id in the MAGMA database of finite groups, the three types, the canonical singularities and the singularities that are not canonical. The symbol $\frac{(a,b)}{n}^{\lambda}$ used in the last two columns of the table denotes λ cyclic quotient singularities of type $\frac{1}{n}(1, a, b)$. We recall the definition in Sect. 6

No.	G	Id	T_1	T_2	T_3	$\mathcal{S}_c$	$\mathcal{S}_{nc}$
1	$\mathbb{Z}_6$	$\langle 6, 2\rangle$	[3, 6, 6]	[3, 6, 6]	[3, 6, 6]	$\frac{(1,1)}{3}^{4}, \frac{(2,2)}{3}^{24}$	$\frac{(1,1)}{6}^{8}$
2	$\mathbb{Z}_8$	$\langle 8, 1\rangle$	[2, 8, 8]	[2, 8, 8]	[4, 8, 8]	$\frac{(1,1)}{2}^{32}, \frac{(1,3)}{4}^{3}$	$\frac{(1,1)}{4}, \frac{(1,1)}{8}^{2}, \frac{(1,3)}{8}^{6}$
3	$\mathbb{Z}_{10}$	$\langle 10, 2\rangle$	[2, 5, 10]	[2, 5, 10]	[5, 10, 10]	$\frac{(1,1)}{2}^{14}, \frac{(1,4)}{5}, \frac{(2,2)}{5}^{8}, \frac{(2,3)}{5}^{4}$	$\frac{(1,2)}{5}^{4}, \frac{(1,1)}{10}^{2}$
4	$\mathbb{Z}_{12}$	$\langle 12, 2\rangle$	[2, 12, 12]	[2, 12, 12]	[3, 4, 12]	$\frac{(1,1)}{2}^{40}, \frac{(1,1)}{3}, \frac{(2,2)}{3}^{3}$	$\frac{(1,1)}{4}^{4}, \frac{(1,1)}{12}, \frac{(5,5)}{12}^{3}$
5	$(\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2$	$\langle 32, 11\rangle$	[2, 4, 8]	[2, 4, 8]	[2, 4, 8]	$\frac{(1,1)}{2}^{24}$	$\frac{(1,1)}{4}^{6}, \frac{(1,1)}{8}, \frac{(1,5)}{8}^{3}$
6	$(\mathcal{D}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_4$	$\langle 64, 8\rangle$	[2, 4, 8]	[2, 4, 8]	[2, 4, 8]	$\frac{(1,1)}{2}^{60}$	$\frac{(1,1)}{4}^{6}, \frac{(1,1)}{8}^{4}$
7	$\mathfrak{S}_4 \times \mathbb{Z}_3$	$\langle 72, 42\rangle$	[2, 3, 12]	[2, 3, 12]	[2, 3, 12]	$\frac{(1,1)}{2}^{36}, \frac{(1,1)}{3}^{17}$	$\frac{(1,1)}{12}, \frac{(1,7)}{12}^{3}$
8	$\mathbb{Z}_2^4 \rtimes \mathbb{Z}_5$	$\langle 80, 49\rangle$	[2, 5, 5]	[2, 5, 5]	[2, 5, 5]	$\frac{(1,4)}{5}^{6}$	$\frac{(1,1)}{5}^{2}$
9	$\mathbb{Z}_2^4 \rtimes \mathbb{Z}_5$	$\langle 80, 49\rangle$	[2, 5, 5]	[2, 5, 5]	[2, 5, 5]	$\frac{(1,3)}{5}^{2}, \frac{(3,4)}{5}^{4}$	$\frac{(1,2)}{5}^{2}$
10	$\mathbb{Z}_4^2 \rtimes \mathfrak{S}_3$	$\langle 96, 64\rangle$	[2, 3, 8]	[2, 3, 8]	[2, 3, 8]	$\frac{(1,1)}{2}^{16}, \frac{(1,1)}{3}, \frac{(2,2)}{3}^{3}$	$\frac{(1,1)}{4}^{6}, \frac{(1,1)}{8}, \frac{(1,5)}{8}^{3}$
11	$GL(3, \mathbb{F}_2)$	$\langle 168, 42\rangle$	[2, 3, 7]	[2, 3, 7]	[2, 3, 7]	$\frac{(1,1)}{2}^{16}, \frac{(1,1)}{3}, \frac{(2,2)}{3}^{3}, \frac{(2,4)}{7}^{2}$	$\frac{(1,1)}{7}, \frac{(1,4)}{7}^{3}, \frac{(4,4)}{7}^{3}$
12	G_{192}	$\langle 192, 181\rangle$	[2, 3, 8]	[2, 3, 8]	[2, 3, 8]	$\frac{(1,1)}{2}^{28}, \frac{(1,1)}{3}^{4}, \frac{(2,2)}{3}^{12}$	$\frac{(1,1)}{4}^{6}, \frac{(1,1)}{8}^{4}$

Remark 5.4 Let Y be a normal quasi-projective variety and $G < \mathrm{Aut}(Y)$ be a finite group, then the quotient map $\pi\colon Y \to X := Y/G$ induces an isomorphism

$$\pi^*\colon H^0(X, \mathcal{L}) \simeq H^0(Y, \pi^*\mathcal{L})^G$$

for any line bundle $\mathcal{L}$ on X. The quotient $X := Y/G$ is a normal $\mathbb{Q}$-factorial quasi-projective variety. In particular K_X is $\mathbb{Q}$-Cartier. Let $\psi\colon \widehat{X} \to X$ be a resolution of singularities and K_X be a canonical divisor, then

$$K_{\widehat{X}} = \psi^* K_X + E,$$

where E is a $\mathbb{Q}$-divisor supported on the exceptional locus $\mathrm{Exc}(\psi)$. Since π is finite and $Sing(X) \subset X$ has codimension ≥ 2, Hurwitz formula holds:

$$K_Y = \pi^* K_X + R.$$

We point out that Y is smooth, and thus the ramification divisor R is a Cartier divisor.

Theorem 5.5 *Under the assumptions from Proposition 5.1, there is a natural isomorphism*

$$H^0\big(\widehat{X}, \mathcal{O}_{\widehat{X}}(dK_{\widehat{X}})\big) \simeq H^0\left(Y, \mathcal{O}_Y(dK_Y) \otimes \mathcal{I}_d\right)^G$$

for all $d \geq 1$, where $\mathcal{I}_d$ is the sheaf of ideals $\mathcal{O}_Y(-dR) \otimes \phi_*\mathcal{O}_{\widetilde{Y}}(\epsilon^* dE)$.

Remark 5.6 If we write $E = P - N$, where P, N are effective without common components, then $\mathcal{I}_d \cong \mathcal{O}_Y(-dR) \otimes \phi_*\mathcal{O}_{\widehat{X}}(-\epsilon^* dN)$.

Proof Using Remark 5.4, we compute

$$\begin{aligned}
\epsilon^* dK_{\widehat{X}} &= \epsilon^*(\psi^* dK_X + dE) \\
&= \epsilon^*\psi^* dK_X + \epsilon^* dE \\
&= \epsilon^*\psi^* dK_X + \epsilon^* dE \\
&= \phi^*\pi^* dK_X + \epsilon^* dE \\
&= \phi^*(dK_Y - dR) + \epsilon^* dE.
\end{aligned}$$

Since the divisors $\epsilon^* dK_{\widehat{X}}$ and $\phi^*(dK_Y - dR)$ are Cartier, the divisor $\epsilon^* dE$ is also Cartier and we obtain the isomorphism of line bundles

$$\mathcal{O}_{\widetilde{Y}}(\epsilon^* dK_{\widehat{X}}) \cong \mathcal{O}_{\widetilde{Y}}(\phi^*(dK_Y - dR)) \otimes \mathcal{O}_{\widetilde{Y}}(\epsilon^* dE)$$

According to Proposition 5.3 $\widehat{X}$ is the quotient of $\widetilde{Y}$ by G. By Remark 5.4

$$H^0\big(\widehat{X}, \mathcal{O}_{\widehat{X}}(dK_{\widehat{X}})\big) \simeq H^0\big(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}}(\phi^*(dK_Y - dR)) \otimes \mathcal{O}_{\widetilde{Y}}(\epsilon^* dE)\big)^G$$

Using the projection formula:

$$H^0\big(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}}(\phi^*(dK_Y - dR)) \otimes \mathcal{O}_{\widetilde{Y}}(\epsilon^* dE)\big)^G = H^0\big(Y, \mathcal{O}_Y(dK_Y - dR) \otimes \phi_*\mathcal{O}_{\widetilde{Y}}(\epsilon^* dE)\big)^G.$$

□

Theorem 5.5 gives a method to compute the plurigenus $P_d(\widehat{X})$, if we can determine the sheaf of ideals $\phi_*\mathcal{O}_{\widetilde{Y}}(\epsilon^* dE)$ and know a basis of $H^0\,(Y, \mathcal{O}_Y(dK_Y))$ explicitly. In the next section we explain how to compute these ideals, under the assumption that X has only isolated cyclic quotient singularities.

6 The Sheaves of Ideals $\mathcal{I}_d$ for Cyclic Quotient Singularities

In this section we specialize to the case of a G-action, where the fixed locus of every automorphism $g \in G$ is isolated and the stabilizer of each point $y \in Y$ is cyclic. Under this assumption, each singularity of $X = Y/G$ is an isolated cyclic quotient singularity

$$\frac{1}{m}(a_1, \ldots, a_n),$$

i.e. locally in the analytic topology, around the singular point the variety Y/G is isomorphic to a quotient $\mathbb{C}^n/H$, where $H \simeq \mathbb{Z}_m$ is a cyclic group generated by a diagonal matrix

$$\operatorname{diag}\left(\xi^{a_1}, \ldots, \xi^{a_n}\right), \quad \text{where} \quad \xi := \exp\left(\frac{2\pi\sqrt{-1}}{m}\right) \quad \text{and} \quad \gcd(a_i, m) = 1.$$

In the sequel, we use toric geometry to construct a resolution $\widehat{X}$ of the quotient Y/G and give a local description of the variety $\widetilde{Y}$ in Proposition 5.1. We start by collecting some basics about cyclic quotient singularities from the toric point of view. For details we refer to [33, Chap. 11].

Remark 6.1
- As an affine toric variety, the singularity $\frac{1}{m}(a_1, \ldots, a_n)$ is given by the lattice

$$N := \mathbb{Z}^n + \frac{\mathbb{Z}}{m}(a_1, \ldots, a_n) \quad \text{and the cone} \quad \sigma := \operatorname{cone}(e_1, \ldots, e_n),$$

  where the vectors e_i are the euclidean unit vectors. We denote this affine toric variety by U_σ.
- The inclusion $i : (\mathbb{Z}^n, \sigma) \to (N, \sigma)$ induces the quotient map

$$\pi : \mathbb{C}^n \to \mathbb{C}^n/\mathbb{Z}_m.$$

- There exists a subdivision of the cone σ, yielding a fan Σ such that the toric variety $\widehat{X_\Sigma}$ is smooth and the morphism $\psi : \widehat{X_\Sigma} \to U_\sigma$ induced by the identity map of the lattice N is a resolution of U_σ i.e. birational and proper.

Now, the local construction of $\widetilde{Y}$ as a toric variety is straightforward. Observe that the fan Σ is also a fan in the lattice $\mathbb{Z}^n$. We define $\widetilde{Y_\Sigma}$ to be the toric variety associated to $(\mathbb{Z}^n, \Sigma)$. The commutative diagram

$$\begin{array}{ccc} (\mathbb{Z}^n, \Sigma) & \longrightarrow & (N, \Sigma) \\ \downarrow & & \downarrow \\ (\mathbb{Z}^n, \sigma) & \longrightarrow & (N, \sigma) \end{array}$$

of inclusions induces a commutative diagram of toric morphisms, which is the local version of the diagram from Proposition 5.1:

$$\begin{array}{ccc} \widetilde{Y_\Sigma} & \xrightarrow{\ \epsilon\ } & \widehat{X_\Sigma} \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \psi} \\ \mathbb{C}^n & \xrightarrow[\pi]{} & U_\sigma = \mathbb{C}^n/\mathbb{Z}_m \end{array}$$

Indeed, the following proposition holds:

Proposition 6.2 *The map $\epsilon \colon \widetilde{Y_\Sigma} \to \widehat{X_\Sigma}$ is finite and surjective and $\phi \colon \widetilde{Y_\Sigma} \to \mathbb{C}^n$ is birational and proper.*

Proof We need to show that $\mathbb{C}[N^\vee \cap \tau^\vee] \subset \mathbb{C}[\mathbb{Z}^n \cap \tau^\vee]$ is a finite ring extension for all cones τ in Σ. Clearly, any element of the form $c\chi^q \in \mathbb{C}[\mathbb{Z}^n \cap \tau^\vee]$ is integral over $\mathbb{C}[N^\vee \cap \tau^\vee]$, because $mq \in N^\vee \cap \tau^\vee$ and $c\chi^q$ solves the monic equation $x^m - c^m\chi^{mq} = 0$. The general case follows from the fact that any element in $\mathbb{C}[\mathbb{Z}^n \cap \tau^\vee]$ is a finite sum of elements of the form $c\chi^q$ and finite sums of integral elements are also integral. Since Σ is a refinement of σ the morphism ϕ is birational and proper according to [33, Theorem 3.4.11]. □

For the next step, we describe how to determine the discrepancy divisor in $\widehat{X}$ over each singular point of the quotient Y/G and its pullback under the morphism ϵ.

Proposition 6.3 (*[33, Proposition 6.2.7 and Lemma 11.4.10]*)

- *The exceptional prime divisors of the birational morphisms*

$$\psi \colon \widehat{X_\Sigma} \to U_\sigma \quad \text{and} \quad \phi \colon \widetilde{Y_\Sigma} \to \mathbb{C}^n$$

  are in one to one correspondence with the rays $\rho \in \Sigma \setminus \sigma$.
- *Write $v_\rho \in N$ for the primitive generator of the ray ρ and $E_\rho \subset \widehat{X_\Sigma}$ for the corresponding prime divisor, then $K_{X_\Sigma} = \psi^* K_{U_\sigma} + E$, where*

$$E := \sum_{\rho \in \Sigma \setminus \sigma} \big(\langle v_\rho, e_1 + \ldots + e_n\rangle - 1\big) E_\rho.$$

- *Write $w_\rho \in \mathbb{Z}^n$ for the primitive generator of the ray ρ and $F_\rho \subset \widetilde{Y_\Sigma}$ for the corresponding prime divisor, then*

$$\epsilon^* E_\rho = \lambda_\rho F_\rho \quad \textit{where} \quad \lambda_\rho > 0 \quad \textit{such that} \quad w_\rho = \lambda_\rho v_\rho.$$

In particular

$$\epsilon^* E = \sum_{\rho \in \Sigma \setminus \sigma} \big(\langle w_\rho, e_1 + \ldots + e_n \rangle - 1\big) F_\rho.$$

It remains to determine the pushforward $\phi_* \mathcal{O}_{\widetilde{Y_\Sigma}}(\epsilon^* dE)$ for $d \geq 1$. We provide a recipe to compute $\phi_* \mathcal{O}_{\widetilde{Y_\Sigma}}(\epsilon^* D)$ for a general Weil divisor D supported on the exceptional locus of ϕ.

Proposition 6.4 *Let $\phi\colon \widetilde{Y_\Sigma} \to \mathbb{C}^n$ be the birational morphism from above and*

$$D = \sum_{\rho \in \Sigma \setminus \sigma} u_\rho F_\rho, \qquad u_\rho \in \mathbb{Z}$$

be a Weil divisor, supported on the exceptional locus of ϕ. For each integer $k \geq 1$, we define the sheaf of ideals $\mathcal{I}_{kD} := \phi_* \mathcal{O}(kD)$, then:

i) The ideal of global sections $I_{kD} \subset \mathbb{C}[x_1, \ldots, x_n]$ is given by

$$I_{kD} = \bigoplus_{\alpha \in k P_D \cap \mathbb{Z}^n} \mathbb{C} \cdot \chi^\alpha,$$

where $P_D := \{u \in \mathbb{R}^n \mid u_i \geq 0, \quad \langle u, w_\rho \rangle \geq -u_\rho\}$ is the polyhedron associated to D.

(ii) Let $l = (l_1, \ldots, l_n)$ be a tuple of positive integers such that $l_i \cdot e_i \in P_D$ and define

$$\square_l := \{y \in \mathbb{R}^n \mid 0 \leq y_i \leq l_i\}.$$

Then, the set of monomials χ^α, where α is a lattice point in the polytope $k(\square_l \cap P_D)$ generate I_{kD}.

Proof (i) By definition of the pushforward and the surjectivity of ϕ, we have

$$I_{kD} = \phi_* \mathcal{O}_{\widetilde{Y_\Sigma}}(kD)(\mathbb{C}^n) = H^0(\widetilde{Y_\Sigma}, \mathcal{O}_{\widetilde{Y_\Sigma}}(kD)).$$

According to [33, Proposition 4.3.3], it holds

$$H^0(\widetilde{Y_\Sigma}, \mathcal{O}(kD)) = \bigoplus_{\alpha \in P_{kD} \cap \mathbb{Z}^n} \mathbb{C} \cdot \chi^\alpha$$

and the claim follows since $k P_D = P_{kD}$. Note that the inequalities $u_i \geq 0$ imply

$$\chi^\alpha \in \mathbb{C}[x_1, \ldots, x_n] \quad \text{for all} \quad \alpha \in P_{kD} \cap \mathbb{Z}^n.$$

(ii) Let χ^α be a monomial, such that the exponent $\alpha = (\alpha_1, \ldots, \alpha_n) \in P_{kD} \cap \mathbb{Z}^n$ is not contained in the polytope

$$k(\Box_l \cap P_D) = \Box_{kl} \cap P_{kD},$$

say $kl_1 < \alpha_1$. Then we define $\beta_1 := \alpha_1 - kl_1$ and write χ^α as a product

$$\chi^\alpha = \chi^{(\beta_1, \alpha_2, \ldots, \alpha_n)} \chi^{kl_1 e_1}.$$

□

Remark 6.5 • Note that the inequalities $\langle u, w_\rho \rangle \geq -ku_\rho$ in the definition of the polyhedron

$$P_{kD} = \{u \in \mathbb{R}^n \mid u_i \geq 0, \quad \langle u, w_\rho \rangle \geq -ku_\rho\}$$

are redundant if $u_\rho \geq 0$.
- For $D = \epsilon^* E$ we have $u_\rho = \lambda_\rho\big(\langle v_\rho, e_1 + \ldots + e_n \rangle - 1\big)$. This integer is, according to Proposition 6.3, equal to the discrepancy of E_ρ multiplied by $\lambda_\rho > 0$. In particular, in the case of a canonical singularity, the ideal $\mathcal{I}_{\epsilon^* kE}$ is trivial, since all $u_\rho \geq 0$.
- The ideal I_{kD} has a unique minimal basis, because it is a monomial ideal.

Remark 6.6 If we perform the star subdivision of the cone σ along all rays generated by a primitive lattice point v_ρ with

$$\langle v_\rho, e_1 + \ldots + e_n \rangle - 1 < 0$$

we obtain a fan Σ' that is not necessarily smooth. However, there is a subdivision of Σ' yielding a smooth fan Σ. Since the new rays $\rho \in \Sigma \setminus \Sigma'$ do not contribute to the polyhedra of $\epsilon^* kE$, there is no need to compute Σ explicitly.

From the description of the ideal I_{kD}, it follows that $(I_D)^k \subset I_{kD}$ for all positive integers k. However, this inclusion is in general not an equality. The reason is that the polytope $\Box_l \cap P_D$ may not contain enough lattice points. We can solve this problem by replacing D with a high enough multiple:

Proposition 6.7 *Let D be a divisor as in Proposition 6.4. Then, there exists a positive integer s such that*

$$(I_{sD})^k = I_{skD} \quad \textit{for all} \quad k \geq 1.$$

Proof Let $l = (l_1, \ldots, l_n)$ be a tuple of positive integers such that $l_i \cdot e_i \in P_D$. According to Proposition 6.4 (ii) the monomials χ^α with

$$\alpha \in k\big(\Box_l \cap P_D\big) \cap \mathbb{Z}^n$$

generate I_{kD} for all $k \geq 1$. Since the vertices of the polytope $\square_l \cap P_D$ have rational coordinates, there is a positive integer s' such that $s'\big(\square_l \cap P_D\big)$ is a lattice polytope i.e. the convex hull of finitely many lattice points. We define $s := s'(n-1)$, then $s\big(\square_l \cap P_D\big)$ is a normal lattice polytope (see [33, Theorem 2.2.12]), which means that

$$\big(ks'\big(\square_l \cap P_D\big)\big) \cap \mathbb{Z}^n = k\big(s'\big(\square_l \cap P_D\big) \cap \mathbb{Z}^n\big) \qquad \text{for all} \qquad k \geq 1.$$

Clearly, this implies $(I_{sD})^k = I_{skD}$ for all $k \geq 1$. □

Remark 6.8 According to the proof of Proposition 6.7 we can take $s = (n-1)s'$ where s' is the smallest positive integer such that all the vertices of $s'P_D$ have integral coordinates.

Listing 1 Computation of the ideal $I_{k\epsilon^* E}$ for the singularity $1/n(1, a, b)$

```

// The first function determines the lattice points that we need to blow up according
// to Computational Rem 5.6. It returns the primitive generators "w_rho" of these points
// according to Prop 5.3 and the discrepancy of the pullback divisor eps^{\ast} E.

Vectors:=function(n,a,b)
Ve:={};
for i in [1..n-1] do
    x:=i/n;
    y:=(i*a mod n)/n;
    z:=(i*b mod n)/n;
    d:=x+y+z-1;
        if d lt 0 then
            lambda:=Lcm([Denominator(x),Denominator(y),Denominator(z)]);
            Include(~Ve,[lambda*x,lambda*y,lambda*z,lambda*d]);
        end if;
    end for;
    return Ve;
end function;

// The function "IntPointsPoly" determines a basis for the monomial ideal,
// according to Proposition 5.4. However, this basis is not necessarily minimal.
// The subfunction "MinMultPoint" is used to determine the cube in ii) of Proposition 5.4.

MinMultPoint:=function(P,v)
    n:=1;
    while n*v notin P do
        n:= n+1;
    end while;
    return n;
end function;

IntPointsPoly:=function(n,a,b,k)
    L:=ToricLattice(3);
    La:=Dual(L);
    e1:=L![1,0,0]; e2:=L![0,1,0]; e3:=L![0,0,1];
    P:=HalfspaceToPolyhedron(e1,0) meet
       HalfspaceToPolyhedron(e2,0) meet
       HalfspaceToPolyhedron(e3,0);
    Vec:=Vectors(n,a,b);
    for T in Vec do
        w:=L![T[1],T[2],T[3]];
        u:=T[4];
        P:= P meet HalfspaceToPolyhedron(w,-k*u);
    end for;
    multx:=MinMultPoint(P,La![1,0,0]);
    multy:=MinMultPoint(P,La![0,1,0]);
```

```
    multz:=MinMultPoint(P,La![0,0,1]);
    P:=P meet HalfspaceToPolyhedron(L![−1,0,0],−k∗multx) meet
            HalfspaceToPolyhedron(L![0,−1,0],−k∗multy) meet
            HalfspaceToPolyhedron(L![0,0,−1],−k∗multz);
    return Points(P);
end function;

// The next functions are used to find the (unique) minimal monomial basis of the ideal.

IsMinimal:=function(Gens)
    test:=true; a:=0;
    for a_i in Gens do
        for a_j in Gens do
            if a_i ne a_j then
                d:=a_j−a_i;
                if d.1 ge 0 and d.2 ge 0 and d.3 ge 0 then
                    a:=a_i; test:=false;
                    break a_i;
                end if;
            end if;
        end for;
    end for;
    return test, a;
end function;

SmallerGen:=function(Gens,a)
    Set:=Gens;
    for b in Gens do
        d:=b−a;
        if d.1 ge 0 and d.2 ge 0 and d.3 ge 0 and b ne a then
            Exclude(~Set,b);
        end if;
    end for;
    return Set;
end function;

MinBase:=function(n,a,b,k)
    F:=RationalField();
    PL<x1,x2,x3>:=PolynomialRing(F,3);
    test:=false;
    Gens:=IntPointsPoly(n,a,b,k);
    while test eq false do
        test, a:=IsMinimal(Gens);
        if test eq false then
            Gens:=SmallerGen(Gens,a);
        end if;
    end while;
    MB:={};
    for g in Gens do
        Include(~MB,PL.1^g.1∗PL.2^g.2∗PL.3^g.3);
    end for;
    return MB;
end function;
```

7 A Calabi-Yau 3-Fold

In this section we apply Theorem 5.5 to the first numerical Calabi-Yau threefold listed in Sect. 4, Table 1.

We start by giving an explicit description of the threefold by writing the canonical ring of the curve $C := C_1 \cong C_2 \cong C_3$ and the group action on it.

We consider the hyperelliptic curve

$$C := \{y^2 = x_0^6 + x_1^6\} \subset \mathbb{P}(1, 1, 3)$$

of genus 2, together with the $\mathbb{Z}_6$-action generated by the automorphism g defined by

$$g((x_0 : x_1 : y)) = (x_0 : \omega x_1 : y), \qquad \text{where} \qquad \omega := e^{\frac{2\pi i}{6}}.$$

By adjunction there is an isomorphism of graded rings between $R(C, K_C) := \bigoplus_d H^0(C, \mathcal{O}_C(dK_C))$ and $\mathbb{C}[x_0, x_1, y]/(y^2 - x_0^6 - x_1^6))$, where $\deg x_i = 1$ and deg $y = 3$.

Lemma 7.1 *The action of g on $R(C, K_C)$ induced by the pull-back of holomorphic differential forms is*

$$x_0 \mapsto \omega x_0 \qquad\qquad x_1 \mapsto \omega^2 x_1 \qquad\qquad y \mapsto \omega^3 y = -y$$

Proof Consider the smooth affine chart $x_0 \neq 0$ with local coordinates $u := \frac{x_1}{x_0}$ and $v := \frac{y}{x_0^3}$. In this chart C is the vanishing locus of $f := v^2 - u^6 - 1$. By adjunction the monomials $x_0, x_1, y \in R(C, K_C)$ correspond respectively to the forms that, in this chart, are

$$x_0 \mapsto \frac{du}{\frac{\partial f}{\partial v}} = \frac{du}{2v} \qquad\qquad x_1 \mapsto u\frac{du}{2v} \qquad\qquad y \mapsto v\left(\frac{du}{2v}\right)^{\otimes 3}$$

The statement follows since g acts on the local coordinates as $(u, v) \mapsto (\omega u, v)$. $\square$

Proposition 7.2 *The threefold $X := C^3/\mathbb{Z}_6$, where the group $\mathbb{Z}_6$ acts as above on each copy of C, is a numerical Calabi-Yau threefold.*

There are 8 non canonical singularities on X, all of type $\frac{1}{6}(1, 1, 1)$.

Proof The points on C with non-trivial stabilizer subgroup of $\mathbb{Z}_6$ are the four points p_0, p_1, p_2, p_3 with the following weighted homogeneous coordinates $(x_0 : x_1 : y)$:

$$p_0 = (1 : 0 : 1) \qquad p_1 = (1 : 0 : -1) \qquad p_2 = (0 : 1 : 1) \qquad p_3 = (0 : 1 : -1)$$

In the table below, for each point p_j, we give a generator of its stabilizer, and the action of the generator on a local parameter of the curve C near p_j.

point	$p_{0/1} = (1 : 0 : \pm 1)$	$p_{2/3} = (0 : 1 : \pm 1)$
generator of the stabilizer	g	g^2
local action	$x \mapsto \omega x$	$x \mapsto \omega^4 x$

p_0 and p_1 are then stabilized by the whole group $\mathbb{Z}_6$, forming then two orbits of cardinality 1, whereas p_2 and p_3 are stabilized by the index two subgroup of $\mathbb{Z}_6$, and form a single orbit.

Consequently the points with nontrivial stabilizer are the 64 points $p_{i_1} \times p_{i_2} \times p_{i_3}$ forming 8 orbits of cardinality 1, the points $p_{i_1} \times p_{i_2} \times p_{i_3}$ with $i_j \in \{0, 1\}$, and and 28 of cardinality 2. So $C^3/\mathbb{Z}_6$ has 36 singular points:

- 8 singular points of type $\frac{1}{6}(1, 1, 1)$, the classes of the points $p_{i_1} \times p_{i_2} \times p_{i_3}$ with $i_j \in \{0, 1\}$: these are not canonical;
- 4 singular points of type $\frac{1}{3}(1, 1, 1)$, the classes of the points $p_{i_1} \times p_{i_2} \times p_{i_3}$ with $i_j \in \{2, 3\}$: these have a crepant resolution;
- 24 singular points of type $\frac{1}{3}(1, 1, 2)$, the classes of the remaining points $p_{i_1} \times p_{i_2} \times p_{i_3}$: these are terminal singularities.

We prove now that a resolution $\rho\colon \widehat{X} \to X = C^3/\mathbb{Z}_6$ has invariants $p_g(\widehat{X}) = 1$, $q_1(\widehat{X}) = q_2(\widehat{X}) = 0$ using representation theory and the fact that

$$H^0(\widehat{X}, \Omega^i_{\widehat{X}}) \simeq H^0(C^3, \Omega^i_{C^3})^G.$$

By Lemma 7.1 the character of the natural representation $\varphi\colon \mathbb{Z}_6 \to GL\big(H^0(K_C)\big)$ is $\chi_\varphi = \chi_\omega + \chi_{\omega^2}$. By Künneth's formula the characters χ_i of the $\mathbb{Z}_6$ representations on $H^0(C^3, \Omega^i_{C^3})$ are respectively

$$\chi_3 = \chi_\varphi^3 \qquad \chi_2 = 3\chi_\varphi^2 \qquad \chi_1 = 3\chi_\varphi.$$

The claim follows, since χ_3 contains exactly one copy of the trivial character whereas χ_2 and χ_1 do not contain the trivial character at all. □

We write coordinates

$$((x_{01} : x_{11} : y_1), (x_{02} : x_{12} : y_2), (x_{03} : x_{13} : y_3))$$

on $\mathbb{P}(1, 1, 3)^3$, so that C^3 is the locus defined by the ideal $\left(y_j^2 - x_{0j}^6 - x_{1j}^6, j{=}1, 2, 3\right)$.

Künneth's formula yields a basis for $H^0(dK_{C^3})$:

$$\left\{ \prod_{i=1}^{3} x_{0i}^{a_i} x_{1i}^{b_i} y_i^{c_i} \mid a_i + b_i + 3c_i = d, \quad c_i = 0, 1 \right\}.$$

on which g acts as

$$\prod_{i=1}^{3} x_{0i}^{a_i} x_{1i}^{b_i} y_i^{c_i} \mapsto \omega^{\sum_i (a_i+2bi+3c_i)} \prod_{i=1}^{3} x_{0i}^{a_i} x_{1i}^{b_i} y_i^{c_i}$$

By the proof of Proposition 7.2, writing $p_i = (1:0:(-1)^i) \in \mathbb{P}(1,1,3)$ for $i = 0, 1$, the eight points

$$p_{i_1} \times p_{i_2} \times p_{i_3}, \qquad i_j = 0, 1$$

are precisely those that descend to the eight singularities of type $\frac{1}{6}(1,1,1)$.

To determine the plurigenenera of X we need the following lemma.

Lemma 7.3 *For all $d \geq 1$, the sheaf of ideals $\mathcal{I}_d$ equals $\mathcal{P}^{3d}$, where $\mathcal{P}$ is the ideal of the reduced scheme $\{p_{i_1} \times p_{i_2} \times p_{i_3} | i_j = 0\}$.*

Proof As already mentioned, all non-canonical singularities are of type $\frac{1}{6}(1,1,1)$. These singularities are resolved by a single toric blowup along the ray ρ generated by $v := \frac{1}{6}(1,1,1)$. The polyhedron associated to the divisor $\epsilon^* dE = -3dF_\rho$ is

$$P_{-3dF_\rho} = \{u \in \mathbb{R}^3 \mid u_i \geq 0, \quad u_1 + u_2 + u_3 \geq 3d\},$$

so the corresponding ideal is just the $3d$-th power of the maximal ideal. □

Then we can prove

Proposition 7.4 *$X = C^3/\mathbb{Z}_6$ is birational to a Calabi-Yau threefold.*

Proof By Proposition 3.7 we only need to prove that all plurigenera are equal to 1, so, by Theorem 5.5, that, $\forall d \geq 1$,

$$H^0\left(C^3, \mathcal{O}_{C^3}(dK_{C^3}) \otimes \mathcal{I}_d\right)^G \cong \mathbb{C}$$

The vector space $H^0\left(C^3, \mathcal{O}_{C^3}(dK_{C^3})\right)$ is contained in the $\mathbb{Z}^3$-graded ring

$$R := \mathbb{C}[x_{01}, x_{11}, y_1, x_{02}, x_{12}, y_2, x_{03}, x_{13}, y_3] / \left(y_i^2 - x_{i0}^6 - x_{i1}^6, i = 1, 2, 3\right)$$

with gradings

$$\begin{aligned} \deg x_{01} &= (1,0,0) & \deg x_{11} &= (1,0,0) & \deg y_1 &= (2,0,0) \\ \deg x_{02} &= (0,1,0) & \deg x_{12} &= (0,1,0) & \deg y_2 &= (0,2,0) \\ \deg x_{03} &= (0,0,1) & \deg x_{13} &= (0,0,1) & \deg y_3 &= (0,0,2) \end{aligned}$$

as the subspace $R_{d,d,d}$ of the homogeneous elements of multidegree (d,d,d). By Lemma 7.1 the natural action of G on $H^0\left(C^3, \mathcal{O}_{C^3}(dK_{C^3})\right)$ is induced by the restriction of the following action of its generator g on R:

$$x_{0i} \mapsto \omega x_{0i} \qquad x_{1i} \mapsto \omega^2 x_{1i} \qquad y_i \mapsto \omega^3 y_i$$

By Lemma 7.3, since the elements of R vanishing on the reduced scheme $\{p_{i_1} \times p_{i_2} \times p_{i_3} | i_j = 0\}$ form the ideal (x_{11}, x_{21}, x_{31})

$$H^0\left(C^3, \mathcal{O}_{C^3}(dK_{C^3}) \otimes \mathcal{I}_d\right) = R_{d,d,d} \cap (x_{11}, x_{12}, x_{13})^{3d} = \langle (x_{11}x_{12}x_{13})^d \rangle$$

is one dimensional.

Since its generator $x_{11}x_{12}x_{13}$ is $G-$invariant, the proof is complete. □

8 A Fake Calabi-Yau 3-Fold

We consider the hyperelliptic curves

$$C_2 := \{y^2 = x_0x_1(x_0^4 + x_1^4)\} \subset \mathbb{P}(1, 1, 3) \quad \text{and} \quad C_3 := \{y^2 = x_0^8 + x_1^8\} \subset \mathbb{P}(1, 1, 4)$$

of respective genus two and three, together with the $\mathbb{Z}_8$-actions $g(x_0 : x_1 : y) = (x_0 : \omega^2 x_1 : \omega y)$ on C_2 and $g(x_0 : x_1 : y) = (x_0 : \omega x_1 : y)$ on C_3, where $\omega = e^{\frac{2\pi i}{8}}$.

Proposition 8.1 *The threefold $X = (C_2^2 \times C_3)/G$, where $G = \mathbb{Z}_8$ acts diagonally, is a numerical Calabi-Yau threefold. X has exactly* 44 *singular points and more precisely*

$$6 \times \frac{1}{8}(1, 1, 3), \quad 2 \times \frac{1}{8}(1, 1, 1), \quad 3 \times \frac{1}{4}(1, 1, 3), \quad 1 \times \frac{1}{4}(1, 1, 1). \quad 32 \times \frac{1}{2}(1, 1, 1).$$

Proof The points with non-trivial stabilizer on C_2 are $q_0 := (0 : 1 : 0)$ and $q_1 := (1 : 0 : 0)$ with the full group as stabilizer and the points

$$p_i := (1 : x_i : 0), \quad \text{where} \quad x_i^4 = -1$$

with stabilizer $\langle g^4 \rangle \cong \mathbb{Z}_2$.

Next, we compute the local action around the points p_i and q_i.

The points q_1 and p_i are contained in the smooth affine chart $x_0 \neq 0$ of $\mathbb{P}(1, 1, 3)$, with affine coordinates $u = \frac{x_1}{x_0}$ and $v = \frac{y}{x_0^2}$. Here, the curve is the vanishing locus of the polynomial $f := v^2 - u^5 - u$ and the group acts via $(u, v) \mapsto (\omega^2 u, \omega v)$.

Since $\frac{\partial f}{\partial u}(q_1) = -1$ and $\frac{\partial f}{\partial v}(p_i) = 4$, by the implicit function theorem, v is a local parameter for C_2 near these points. In particular g acts around q_1 as the multiplication by ω and g^4 acts around p_i as the multiplication by $\omega^4 = -1$.

A similar computation on the affine chart $x_1 \neq 0$ shows that g acts around q_0 as the multiplication by ω^3. The table below summarizes our computation.

point	$q_0 = (0:1:0)$	$q_1 = (1:0:0)$	$p_i = (1:x_i:0)$ $x_i^4 = -1$
Stab	$\langle g \rangle$	$\langle g \rangle$	$\langle g^4 \rangle$
local action	$x \mapsto \omega^3 x$	$x \mapsto \omega x$	$x \mapsto -x$

Similarly, for C_3, we obtain

points	$s_1 = (1:0:1)$, $s_2 = (1:0:-1)$	$s_3 = (0:1:1)$, $s_4 = (0:1:-1)$
Stab	$\langle g \rangle$	$\langle g^2 \rangle$
local action	$x \mapsto \omega x$	$x \mapsto \omega^6 x$

Then the diagonal action on $C_2^2 \times C_3$ admits $6 \cdot 4 \cdot 4 = 144$ points with non-trivial stabilizer. The 8 points of the form

$$q_i \times q_j \times s_k, \quad \text{where} \quad i, j \in \{0, 1\} \quad \text{and} \quad k \in \{1, 2\}.$$

are stabilized by the full group. Therefore, they are mapped to 8 singular points on the quotient. These singularities are

$$\begin{cases} 2 \times \frac{1}{8}(1,1,1) & \text{for } i = j = 0 \\ 4 \times \frac{1}{8}(1,1,3), & \text{for } i \neq j \\ 2 \times \frac{1}{8}(1,3,3) & \text{for } i = j = 1 \end{cases}$$

The 8 points

$$q_i \times q_j \times s_k, \quad \text{where} \quad i, j \in \{0, 1\} \quad \text{and} \quad k \in \{3, 4\}$$

have $\langle g^2 \rangle \cong \mathbb{Z}_4$ as stabilizer. These map to 4 singular points on the quotient:

$$\begin{cases} 1 \times \frac{1}{4}(1,1,1) & \text{for } i = j = 0 \\ 3 \times \frac{1}{4}(1,1,3), & \text{else} \end{cases}$$

The remaining 128 points have stabilizer $\mathbb{Z}_2$. These points yield 32 terminal singularities of type $\frac{1}{2}(1,1,1)$ on the quotient.

To show that X is numerical Calabi-Yau, we verify that

$$p_g(\widehat{X}) = 1, \qquad \text{and} \qquad q_2(\widehat{X}) = q_1(\widehat{X}) = 0$$

for a resolution $\widehat{X}$ of X like in the proof of Proposition 7.2. □

This example is not birational to a Calabi-Yau threefold.

Proposition 8.2 *Let $\rho\colon \widehat{X} \to X$ be a resolution of the singularities of X and Z be a minimal model of $\widehat{X}$.*

Then Z is not Calabi-Yau.

Proof We show that $P_2(\widehat{X}) \geq 3$. A monomial basis of $H^0(2K_{C_2^2\times C_3})$ is

$$\left\{ \prod_{i=1}^{3} x_{0i}^{a_i} x_{1i}^{b_i} y_i^{c_i} \mid a_1 + b_1 + 3c_1 = a_2 + b_2 + 3c_2 = 2,\ a_3 + b_3 + 4c_3 = 4 \right\}.$$

The table below displays all points on $C_2^2 \times C_3$ with non-trivial stabilizer, that descend to a non-canonical singularity and the germ of the plurisection

$$\prod_{i=1}^{3} x_{0i}^{a_i} \cdot x_{1i}^{b_i} \cdot y_i^{c_i}$$

in local coordinates up to a unit as well as the stalks of the ideal $\mathcal{I}_2$ in these points.

point	singularity	germ	stalk
$(q_0, q_0, s_{1/2})$	$\frac{1}{8}(1,1,3)$	$y_1^{2a_1+c_1} y_2^{2a_2+c_2} x_3^{b_3}$	$\left((y_1, y_2)^3 + (x_3)\right)^2$
$(q_1, q_1, s_{1/2})$	$\frac{1}{8}(1,1,1)$	$y_1^{2b_1+c_1} y_2^{2b_2+c_2} x_3^{b_3}$	$(y_1, y_2, x_3)^{10}$
$(q_0, q_1, s_{1/2})$	$\frac{1}{8}(1,3,3)$	$y_1^{2a_1+c_1} y_2^{2b_2+c_2} x_3^{b_3}$	$\left((y_2, x_3)^3 + (y_1)\right)^2$
$(q_1, q_0, s_{1/2})$	$\frac{1}{8}(1,3,1)$	$y_1^{2b_1+c_1} y_2^{2a_2+c_2} x_3^{b_3}$	$\left((y_1, x_3)^3 + (y_2)\right)^2$
$(q_0, q_0, s_{3/4})$	$\frac{1}{4}(1,1,1)$	$y_1^{2a_1+c_1} y_2^{2a_2+c_2} x_3^{a_3}$	$(y_1, y_2, x_3)^2$

With the help of MAGMA we found the following three monomial sections of $H^0(2K_{C_2^2\times C_3} \otimes \mathcal{I}_2)$:

$$x_{11}^2 x_{12}^2 x_{03}^2 x_{13}^2, \quad x_{01}x_{11}x_{12}^2 x_{13}^4 \quad \text{and} \quad x_{11}^2 x_{02} x_{12} x_{13}^4.$$

Using the same argument as in Lemma 7.1, we obtain the action on the canonical ring $R(C_2, K_{C_2})$ as

$$x_0 \mapsto \eta^2 x_0, \quad x_1 \mapsto \eta^6 x_1, \quad y \mapsto \eta^8 y, \quad \text{where} \quad \eta^2 = \omega$$

and on $R(C_3, K_{C_3})$ as

$$x_0 \mapsto \eta x_0, \quad x_1 \mapsto \eta^3 x_1, \quad y \mapsto \eta^4 y.$$

We conclude that the three sections above are also $\mathbb{Z}_8$ invariant, in particular $P_2(\widehat{X}) \geq 3$. □

Note that each of the three monomials in the proof of Proposition 8.2 contains a variable that does not appear in the other two. This implies that the subring of the canonical ring of $\widehat{X}$ generated by the three monomials is isomorphic to the ring of polynomials in three variables. In particular $\mathrm{kod}(\widehat{X}) \geq 2$.

9 Some Minimal Surfaces of General Type

In this section we construct some product-quotient surfaces with several singular points and investigate their minimality.

The construction is as follows.

Definition 9.1 Let $a, b \in \mathbb{N}$ such that $\gcd(ab, 1-b^2) = 1$, $ab \geq 4$ and $b \geq 3$. Define $n = ab$ and let $1 \leq e \leq n-1$ be the unique integer such that $e \cdot (1-b^2) \equiv_n 1$ (i.e. e represents the inverse modulo n of $1-b^2$). For example, one can take $a = b \geq 3$. Define

$$\omega = \mathrm{e}^{\frac{2\pi i}{n}} \qquad \text{and} \qquad \lambda = \mathrm{e}^{\frac{2\pi i}{n(n-3)}} .$$

Consider the Fermat curve C of degree n in $\mathbb{P}^2$, i.e. the plane curve

$$x_0^n + x_1^n + x_2^n = 0.$$

where x_i are projective coordinates on $\mathbb{P}^2$. Consider the natural action ρ_1 of $G := \mathbb{Z}_n \oplus \mathbb{Z}_n$ on C generated by

$$g_1 \cdot (x_0 : x_1 : x_2) = (\lambda x_0 : \lambda\omega x_1 : \lambda x_2) \quad h_1 \cdot (x_0 : x_1 : x_2) = (\lambda x_0 : \lambda x_1 : \lambda\omega x_2). \tag{9.1}$$

Define

$$g_2 := g_1 h_1^b \qquad h_2 := g_1^{-b} h_1^{-1} \qquad (\text{ and } k_2 = g_2^{-1} h_2^{-2} = g_1^{b-1} h_1^{1-b}). \tag{9.2}$$

Under the above assumptions, g_2 and h_2 are generators of G, inducing a second G-action ρ_2 on C by

$$g_2 \cdot (x_0 : x_1 : x_2) := g_1 \cdot (x_0 : x_1 : x_2) \quad h_2 \cdot (x_0 : x_1 : x_2) := h_1 \cdot (x_0 : x_1 : x_2)$$

The diagonal action $\rho_1 \times \rho_2$ on $C \times C$ gives a product quotient surface $\widehat{X}_{a,b}$ with quotient model $X_{a,b}$.

The action ρ_1 has 3 orbits where the action is not free:

$$\begin{aligned} \mathrm{Fix}(g_1) &= \{(1 : 0 : -\eta) \mid \eta^n = 1\} \\ \mathrm{Fix}(h_1) &= \{(1 : -\eta : 0) \mid \eta^n = 1\} \\ \mathrm{Fix}(k_1) &= \{(0 : 1 : -\eta) \mid \eta^n = 1\} \end{aligned}$$

respectively stabilized by $\langle g_1\rangle$, $\langle h_1\rangle$ and $\langle k_1 := g_1^{-1}h_1^{-1}\rangle$. Notice $g_1 = g_2^e h_2^{2b}$ and $h_1 = g_2^{-eb}h_2^{-e}$. The following relations hold:

$$\langle g_1\rangle \cap \langle g_2\rangle = \langle g_1^a\rangle \quad \langle g_1\rangle \cap \langle h_2\rangle = \langle 1\rangle \quad \langle g_1\rangle \cap \langle k_2\rangle = \langle 1\rangle$$
$$\langle h_1\rangle \cap \langle h_2\rangle = \langle h_1^a\rangle \quad \langle h_1\rangle \cap \langle k_2\rangle = \langle 1\rangle \quad \langle k_1\rangle \cap \langle k_2\rangle = \begin{cases} \langle 1\rangle & \text{if } n \text{ is odd} \\ \langle (g_1h_1)^{n/2}\rangle & \text{if } n \text{ is even} \end{cases}.$$

The only points of $C \times C$ with non trivial stabilizer are

Fixed points	#Points	Stabilizer	Type of singularity on X	
$\mathrm{Fix}(g_1)^2$	n^2	$\langle g_1^a \times g_1^a\rangle \simeq \mathbb{Z}_b$	$\frac{1}{b}(1,1)$	Any n
$\mathrm{Fix}(h_1)^2$	n^2	$\langle h_1^a \times h_1^{-a}\rangle \simeq \mathbb{Z}_b$	$\frac{1}{b}(1,b-1)$	Any n
$\mathrm{Fix}(k_1)^2$	n^2	$\langle k_1^{n/2} \times k_1^{n/2}\rangle \simeq \mathbb{Z}_2$	$\frac{1}{2}(1,1)$	n even

In particular, the only non canonical singularities of $X_{a,b}$ are b points of type $\frac{1}{b}(1,1)$.

Since $C/G \simeq \mathbb{P}^1$ then $q(\widehat{X}_{a,b}) = 0$. Moreover, we have, by the formulas in [1],

$$K^2_{X_{a,b}} = \frac{8(g(C)-1)^2}{\#G} = 2(n-3)^2$$

and, as we have exactly b singular points of type $\frac{1}{b}(1,1)$,

$$r^*K_{X_{a,b}} = K_{\widehat{X}_{a,b}} + \frac{b-2}{b}(E_1 + \cdots + E_b)$$

where E_i are the exceptional divisors introduced by the resolution over the non-canonical points. These are disjoint rational curves with selfintersection $-b$ so

$$K^2_{\widehat{X}_{a,b}} = K^2_{X_{a,b}} - b^2\frac{(b-2)^2}{b^2} = 2(n-3)^2 - (b-2)^2. \tag{9.3}$$

Remark 9.2 Notice that $K^2_{\widehat{X}_{a,b}} \geq 2$ for all (a,b) satisfying our assumptions, unless $(a,b) \in \{(1,4),(1,5)\}$.

There is an isomorphism between $H^0(K_C)$ and

$$H^0(\mathcal{O}_C(n-3)) = H^0(\mathcal{O}_{\mathbb{P}^2}(n-3)) = \mathbb{C}[x_0,x_1,x_2]_{n-3}.$$

Then ρ_1 induces a G-action on $H^0(\omega_C)$ via pull-back of holomorphic forms on C. We wrote ρ_1 so that this action coincides with the natural action induced by (9.1) on monomials of degree $n-3$. Explicitly, if $m_0 + m_1 + m_2 = n-3$ we have

$$g_1 \cdot x_0^{m_0} x_1^{m_1} x_2^{m_2} = (g_1^{-1})^*(x_0^{m_0} x_1^{m_1} x_2^{m_2}) = \\ = \lambda^{-m_0-m_1-m_2} \omega^{-m_1} x_0^{m_0} x_1^{m_1} x_2^{m_2} = \omega^{-m_1-1} x_0^{m_0} x_1^{m_1} x_2^{m_2}$$

and

$$h_1 \cdot x_0^{m_0} x_1^{m_1} x_2^{m_2} = (h_1^{-1})^*(x_0^{m_0} x_1^{m_1} x_2^{m_2}) = \\ = \lambda^{-m_0-m_1-m_2} \omega^{-m_2} x_0^{m_0} x_1^{m_1} x_2^{m_2} = \omega^{-m_2-1} x_0^{m_0} x_1^{m_1} x_2^{m_2}.$$

The canonical action induced by ρ_2 and the bicanonical action follow accordingly. Working as in the previous sections we computed $p_g(\widehat{X}_{a,b}) = h^0(K_{C\times C})^G$ and $h^0(2K_{C\times C})^G$ for the case $a = b$. The values are all in Table 9. We stress that for $a = b \geq 3$ we always get $K^2_{\widehat{X}_{a,b}} > 0$ and $p_g(\widehat{X}_{a,b}) > 0$ so $\widehat{X}_{a,b}$ is of general type. As the only non-canonical singular points are of type $\frac{1}{b}(1,1)$ we have

$$P_2(\widehat{X}_{b,b}) = H^0(2K_{\widehat{X}_{b,b}}) \simeq H^0(K_{C\times C} \otimes \mathcal{I}_{R_{nc}}^{2b-4})^G$$

where $\mathcal{I}_{R_{nc}}$ is the ideal sheaf of functions vanishing at order at least $2b-4$ in all the points of

$$R_{nc} = \mathrm{Fix}(g_1)^2 = \{(1:0:-\eta_1) \times (1:0:-\eta_2) \quad | \quad \eta_1^n = \eta_2^n = 1\}.$$

Using the embedding of $C \times C$ in $\mathbb{P}^2 \times \mathbb{P}^2$ we have $H^0(2K_{C\times C}) = H^0(\mathcal{O}_{C\times C}(2n-6, 2n-6))$. To simplify the computation, we just look for the invariant monomials with the right vanishing order on R_{nc}: in principle their number is only a lower bound for $P_2(\widehat{X}_{b,b})$; the vanishing order of $x_0^{m_0} x_1^{m_1} x_2^{m_2} y_0^{n_0} y_1^{n_1} y_2^{n_2}$ with $0 \leq m_1, n_1, m_1 + m_2, n_1 + n_2 \leq 2n-6$ and $0 \leq m_2, n_2 \leq n-1$ equals $m_1 + n_1$.

We prove

Proposition 9.3 *Assume $a = b \geq 3$. Then $H^0(2K_Y \otimes \mathcal{I}_{R_{nc}})^G$ is generated by invariant monomials. Moreover, the codimension of $H^0(2K_Y \otimes \mathcal{I}_{R_{nc}})^G$ in $H^0(2K_Y)^G$ is $b(b-3)$.*

Proof We give only a sketch of the proof.

The invariant bicanonical monomials are those of the form $x_0^{m_0} x_1^{m_1} x_2^{m_2} y_0^{n_0} y_1^{n_1} y_2^{n_2}$ with

$$\begin{cases} (I): & m_1 + 4 + 2b + n_1 + bn_2 \equiv_n 0 \\ (II): & m_2 - 2b - bn_1 - n_2 \equiv_n 0 \\ 0 \leq m_1, n_1, m_1+m_2, n_1+n_2 \leq 2n-6 \\ 0 \leq m_2, n_2 \leq n-1 \\ m_0 + m_1 + m_2 = n_0 + n_1 + n_2 = 2b-6 \end{cases} \tag{9.4}$$

We prove that if $\nu := m_1 + n_1 \leq 2b-4$ then $b \geq 4$ and $\nu = b-4$. Hence the Proposition is true for $b = 3$ and we can assume $b \geq 4$. In this case we solve (9.4) under the assumption $\nu = b - 4$, finding $b - 3$ possibilities for the pair (m_1, n_1). We denote by W_b the vector space generated by the invariant bicanonical monomials and with $W_b^{(m_1,n_1)}$ its subspace generated by monomials which have assigned exponents for the variables x_1 and y_1. Monomials in $W_b^{(m_1,n_1)}$ satisfy $m_2, n_2 \in \{-3 + kb \mid 1 \leq k \leq b\}$ and if $m_2 = -3 + k_m b$ and $n_2 = -3 + k_n b$ then $k_m \equiv_b k_n + n_1 + 2$. Using these observations, we prove that the dimension of $W_b^{(m_1,n_1)}$ is b which implies then that W_b has codimension $b(b-3)$ in $H^0(2K_Y)^G$.

It remains to prove that a polynomial whose monomials have $\nu = b - 4$ cannot vanish in all the points of R_{nc} with order at least $2b - 4$. A polynomial p in W_b is a linear combination of polynomials living in $W_b^{(m_1,n_1)}$. In affine coordinates $z_i = x_i/x_0$, $w_i = y_i/y_0$ we have

$$p = \sum_{j=0}^{b-4} z_1^j w_1^{b-4-j} p_j(z_2, w_2)$$

with $z_1^j w_1^{b-4-j} p_j(z_2, w_2) \in W_b^{(m_1,n_1)}$. In the second part of the proof we prove that if a polynomial $p \in W_b$ vanishes at order at least $2b - 4$ in the points of R_{nc} then, necessarily $p_j(-\eta, -\mu) = 0$ for all pairs of n-th roots of 1. This is obtained by implicit differentiation and by keeping track of the order of vanishing of the various terms of the sum.

In the third and final part of the proof we prove that if $z_1^j w_1^{b-4-j} q(z_2, w_2) \in W_b^{(m_1,n_1)}$ and $q(-\eta, -\mu) = 0$ for enough pairs of n-th roots of 1 then q is actually 0. More precisely, we prove that if $q(-1, \mu_i) = 0$ for $\mu_1, \ldots, \mu_b$ with $\mu^n = 1$ and $\mu_i^b \neq \mu_j^b$ for $i \neq j$, then $q = 0$. This can be seen as follows. If $[x]_b$ is the only representative of x modulo b in the range $[1, b]$ one can choose the monomials $f_k = z_1^{m_1} w_1^{b-4-m_1} q_k$ with

$$q_k = (-1)^{b(k+[k+2+n_1]_b)} z_2^{-3+b[k+2+n_1]_b} w_2^{-3+kb} \qquad 1 \leq k \leq b$$

as basis for $W_b^{(m_1,n_1)}$. The coefficient is simply to get easier computations. If $q = \sum_{k=1}^{b} \lambda_k q_k$ then

$$q(-1 - \mu) = \mu^{-3} \sum_k \lambda_k (\mu^b)^k.$$

We know that $q(-1, -\mu_i) = 0$ for $\mu_1, \ldots, \mu_b$. Then either $\lambda_k = 0$ for all k or the matrix $A = ((\mu_i^b)^k)_{1 \leq i,k \leq b}$ has determinant 0. But A is a Vandermonde-type matrix associated to $\{\mu_1^b, \ldots, \mu_b^b\}$ and its determinant is zero if and only if there is a pair (i, j) with $i \neq j$ such that $\mu_i^b = \mu_j^b$. But this contradicts the hypothesis so we have, finally, $p = 0$. □

Having a way to compute P_2 also means that we have a way to determine whether our surfaces are minimal or not. Indeed, by Proposition 2.1, we have

$$\mathrm{vol}(K_{\widehat{X}_{a,b}}) = P_2(\widehat{X}_{a,b}) - \chi(\mathcal{O}_S) \geq K^2_{\widehat{X}_{a,b}}$$

with equality if and only if S is already minimal. Here we summarize the invariants for the product-quotient surfaces obtained for $3 \le a = b \le 12$.

b	$g(C)$	$K^2_{X_{b,b}}$	$K^2_{\widehat{X}_{b,b}}$	$p_g(\widehat{X}_{b,b})$	$\chi(\mathcal{O}_{\widehat{X}_{b,b}})$	$h^0(2K_{C\times C})^G$	$P_2(\widehat{X}_{b,b})$	vol $K_{\widehat{X}_{b,b}}$	vol $K_{\widehat{X}_{b,b}} - K^2_{\widehat{X}_{b,b}}$
3	28	72	71	9	10	81	81	71	0
4	105	338	334	43	44	382	378	334	0
5	276	968	959	122	123	1092	1082	959	0
6	595	2178	2162	274	275	2455	2437	2162	0
7	1128	4232	4207	531	532	4767	4739	4207	0
8	1953	7442	7406	933	934	8380	8340	7406	0
9	3160	12168	12119	1524	1525	13698	13644	12119	0
10	4851	18818	18754	2356	2357	21181	21111	18754	0
11	7140	27848	27767	3485	3486	31341	31253	27767	0
12	10153	39762	39662	4975	4976	44746	44638	39662	0

Hence we can conclude

Proposition 9.4 *For all* $3 \le b \le 12$, $\widehat{X}_{b,b}$ *is a regular minimal surface of general type.*

We notice that this result would be difficult to achieve with the techniques of [1, 11] since both the minimality criteria there e.g. [1, Proposition 4.7] and [11, Lemma 6.9] require that at most two of the exceptional divisors of the resolution of the singularities of the quotient model have self-intersection different to -2 and -3, whereas in the last example we have 12 curves of self-intersection -12.

This disproves the conjecture [11, Conjecture 1.5], proved in [11] for surfaces with $p_g = 0$. Indeed, all these surfaces have invariant γ ([11, Definition 2.3]) equal to zero, so $p_g + \gamma = p_g \neq 0$, whereas [11, Conjecture 1.5] suggests that all minimal product-quotient surfaces should have $p_g + \gamma = 0$.

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Joining Dessins Together

Gareth A. Jones

Abstract An operation of joining coset diagrams for a given group, introduced by Higman and developed by Conder in connection with Hurwitz groups, is reinterpreted and generalised as a connected sum operation on dessins of a given type. A number of examples are given.

Keywords Dessin d'enfant · Coset diagram · Triangle group · Monodromy group

Subject Class Primary 14H57 · Secondary 05C10 · 20B25

1 Introduction

Around 1970 in Oxford, in a series of unpublished lectures on Generators and Relations, Graham Higman introduced a technique which he called 'sewing coset diagrams together'. The basic idea was to combine two transitive permutation representations of a group Γ, of finite degrees n_1 and n_2, to give a transitive permutation representation of Γ of degree $n_1 + n_2$. In certain cases this could be done by taking permutation diagrams for these two representations, that is, coset diagrams for point-stabilisers with respect to certain generators for Γ, and joining them to obtain a larger connected diagram, also representing Γ, by replacing some pairs of fixed points of a generator of order 2 with 2-cycles. Under suitable conditions, this could be iterated to give permutation representations of Γ of even larger degrees. One aim for using this technique was to obtain relatively concise presentations for various finite groups, such as $\mathrm{PSL}_2(q)$ for certain values of q, and the sporadic simple group J_1 of order 175,560 which had been discovered by Janko [8, 9] a few years earlier. Another aim was to obtain finite quotients of certain finitely presented infinite groups, such as various triangle groups (including the modular and extended modular group), and Coxeter's groups $G^{p,q,r}$ (see [4, Sect. 7.5]) with presentations

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$$\langle A, B, C \mid A^p = B^q = C^r = (AB)^2 = (BC)^2 = (CA)^2 = (ABC)^2 = 1\rangle.$$

Higman gave an outline argument, using this technique, that the alternating group A_n is a Hurwitz group, that is, a quotient of the triangle group[1]

$$\Delta = \Delta(3, 2, 7) = \langle X, Y, Z \mid X^3 = Y^2 = Z^7 = XYZ = 1\rangle,$$

for all sufficiently large values of n. This was fully proved, extended and made more precise by Higman's student Marston Conder in his 1980 D. Phil. thesis and his first published paper [1]: he showed that A_n is a Hurwitz group for all $n \geq 168$, and also for a specific set of smaller values of n, ranging from 15 to 166. (See [3] for a useful recent survey of Hurwitz groups, also by Conder.) He also applied this technique to realise all sufficiently large alternating and symmetric groups as quotients of the extended triangle group $\Delta[3, 2, 7]$.

This method of joining coset diagrams together was also applied by Wilson Stothers [23, 24] in 1974 to the modular group $\Gamma = PSL_2(\mathbb{Z})$, and in 1977 to the triangle group Δ, in order to study the possible specifications (essentially signature, plus cusp-split in the case of Γ) for their subgroups of finite index. This work arose out of his Cambridge PhD thesis, written in 1972 under the supervision of Peter Swinnerton-Dyer.

Conder's technique for finding alternating quotients of Δ was later extended by Pellegrini and Tamburini [21], who showed that the double cover $2.A_n$ of A_n is a Hurwitz group for all $n \geq 231$. It was transferred by Lucchini, Tamburini and Wilson [19] from permutations to matrices in order to show that the groups $SL_n(q)$ are all Hurwitz groups for $n \geq 287$, while Lucchini and Tamburini [18] proved a similar result for various other families of classical groups. Conder's coset diagrams have also been used in [13] to construct Beauville surfaces from pairs of Hurwitz dessins.

In [2], Conder extended the technique to generalise his results in [1] to triangle groups of type $(3, 2, k)$ for all $k \geq 7$; his diagrams have been used in [11] to show that such groups have uncountably many maximal subgroups, and in [12] to realise all countable groups as automorphism groups of maps and hypermaps of various types. By extending the joining operation to Dyck groups, Everitt [5] showed that each non-elementary finitely generated Fuchsian group has all sufficiently large alternating groups as quotients.

The theory developed by Higman and Conder was entirely 1-dimensional: their basic tools were graphs, with edges implicitly labelled and directed. However, when the theory is applied to triangle groups (including extended triangle groups), as in many of Higman's examples, and in almost all of the others discussed above, the coset diagrams can be interpreted as maps on compact oriented surfaces (often the sphere), and therefore, following Grothendieck [7], as dessins d'enfants, that is, as 'pictures' of algebraic curves defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers. Moreover, their joining operations correspond to certain type-preserving connected sum operations

[1]For consistency with dessin notation, we prefer $\Delta(3, 2, 7)$ to the isomorphic $\Delta(2, 3, 7)$.

on dessins, allowing Hurwitz dessins of arbitrary size and genus to be created from simple planar ingredients. Thus Conder's main result in [1] can be interpreted as using connected sums to construct, for each integer $n \geq 168$ (and many smaller n), a planar dessin with monodromy group A_n, so that this group is realised as a Hurwitz group on its Galois cover, of genus $1 + n!/168$.

The aim of this paper is to explain these reinterpretations, with illustrative examples, and to generalise the joining operations as connected sum operations on surfaces. In particular, whereas Higman and Conder based their technique on using the fixed points of a generator of order 2 of a triangle group, we shall extend this to fixed points of a generator of any order.

2 Preliminaries

We refer the reader to [6, 15, 17] for background on dessins. Here we summarise a few special aspects of dessins and other topics to be used later.

2.1 Hurwitz Dessins

Hurwitz showed that if X is a compact Riemann surface of genus $g > 1$ then $|\mathrm{Aut}\, X| \leq 84(g-1)$. A group G is defined to be a *Hurwitz group* if $G \cong \mathrm{Aut}\, X$ for some surface X attaining this bound, or equivalently, G is a non-trivial finite quotient of the triangle group $\Delta := \Delta(3, 2, 7)$. In this case there is a bijection, given by $X \cong \mathbb{H}/N$, between the isomorphism classes of such surfaces X and the normal subgroups N of Δ with $\Delta/N \cong G$.

Any transitive finite permutation representation $\theta : \Delta \to G \leq \mathrm{Sym}\,\Omega$ of Δ gives a Hurwitz group G. Hence so does any subgroup M of finite index in Δ, where $\theta : \Delta \to G$ is the permutation representation of Δ on the cosets of M, and $N = \ker\theta$ is the core of M in Δ, the intersection of its conjugates.

In such cases θ or M induces a *Hurwitz dessin* $\mathcal{M}$ on the Riemann surface $\mathbb{H}/M$; this is a bipartite map with the edges corresponding to the elements of Ω, and the black and white vertices and faces corresponding to the cycles of the images x, y and z in G of the standard generators of Δ, so that G acts as the monodromy group of $\mathcal{M}$. Similarly the subgroup N, which corresponds to the regular representation of G, induces a regular Hurwitz dessin $\mathcal{N}$ on the Hurwitz surface $\mathbb{H}/N$; this dessin $\mathcal{N}$ is the Galois cover, or minimal regular cover $\tilde{\mathcal{M}}$ of $\mathcal{M}$, with automorphism group $\mathrm{Aut}\,\mathcal{N} \cong G$.

If $|\Delta : M| = n$, and x, y and z (of orders 3, 2 and 7) have α, β and γ fixed points on the cosets of M, then it follows from a result of Singerman [22] that M has signature $\sigma = (g; 3^{[\alpha]}, 2^{[\beta]}, 7^{[\gamma]})$, where the Riemann–Hurwitz formula implies that $\mathcal{M}$ has genus

$$g = 1 + \frac{1}{84}(n - 28\alpha - 21\beta - 36\gamma). \tag{1}$$

This means that M has a standard presentation as a Fuchsian group, with generators

$$A_i, B_i \ (i = 1, \ldots, g), \quad X_j \ (j = 1, \ldots, \alpha), \quad Y_k \ (k = 1, \ldots, \beta), \quad Z_l \ (l = 1, \ldots, \gamma)$$

and defining relations

$$\prod_i [A_i, B_i] \cdot \prod_j X_j \cdot \prod_k Y_k \cdot \prod_l Z_l = X_j^3 = Y_k^2 = Z_l^7 = 1.$$

We will also refer to σ as the signature of the dessin $\mathcal{M}$ corresponding to M.

The values of α, β and γ can be computed as follows. Any transitive permutation group G can be regarded as permuting the cosets of a point-stabiliser H. A simple counting argument shows that the number of fixed points of an element $h \in G$ is $|h^G \cap H|.|C_G(h)|/|H|$, where h^G and $C_G(h)$ denote the conjugacy class and centraliser of h in G, so that $|h^G|.|C_G(h)| = |G|$ by the Orbit-Stabiliser Theorem. Of course, since the periods 3, 2 and 7 of Δ are all prime, the parameters n, α, β and γ uniquely determine the cycle-structures of x, y and z, and hence the signatures of M and $\mathcal{M}$.

2.2 Jordan's Theorem

Many Hurwitz dessins of degree n have monodromy group G isomorphic to the alternating group A_n. To prove this in specific cases, one can often use the following theorem (see [26, Theorem 13.9]):

Theorem 2.1 (Jordan) *If G is a primitive permutation group of degree n containing a cycle of prime length $l \leq n - 3$ then $G \geq A_n$.*

We will also use the following extension of Jordan's Theorem [10], where the primality condition is omitted. (The proof uses the classification of finite simple groups; this is the only part of this paper which depends upon it.)

Theorem 2.2 *If G is a primitive permutation group of degree n containing a cycle of length $l \leq n - 3$ then $G \geq A_n$.*

In either case, if G is the monodromy group of a Hurwitz dessin, then G is a quotient of Δ and so must be perfect; thus $G \neq S_n$ and hence $G = A_n$.

2.3 From Coset Diagrams to Dessins

In order to construct permutation representations of Δ, Higman and Conder used the vertices of small triangles to represent 3-cycles of X, with their cyclic order

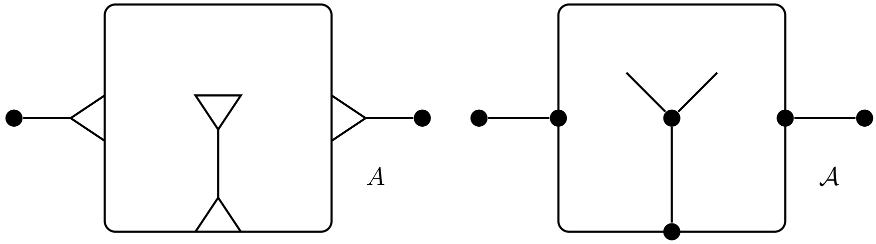


Fig. 1 The diagram A and dessin $\mathcal{A}$

given by the positive (anticlockwise) orientation of the page, and heavy dots for its fixed points; they used long edges to indicate 2-cycles of Y. Any connected diagram formed in this way is a coset diagram for a point-stabiliser $M \leq \Delta$. For example the left-hand part of Fig. 1 shows the coset diagram A from [1]; here M has index 14, and X and Y have cycle structures $1^{[2]}, 3^{[4]}$ and $1^{[2]}, 2^{[6]}$.

Any coset diagram D for Δ can be converted into a Hurwitz dessin $\mathcal{D}$ by shrinking each triangle representing a 3-cycle of X to a single vertex, with the cyclic order giving the local orientation of the surface, and then adding a free edge for each fixed point of Y. Thus the directed edges of $\mathcal{D}$ correspond to the vertices of D, with the same actions of X and Y. The faces of the dessin correspond to the cycles of Z. This transformation is illustrated in Fig. 1, which shows the spherical dessin $\mathcal{A}$ corresponding to the coset diagram A. The process can be reversed by truncating a Hurwitz dessin $\mathcal{D}$, so that each trivalent vertex is replaced with a small triangle, then removing the free edges and ignoring the faces, to give the coset diagram D.

Conder's diagrams in [1] all have bilateral symmetry, so they can be regarded as coset diagrams for subgroups of the extended triangle group

$$\Delta[3, 2, 7] = \langle X, Y, T \mid X^3 = Y^2 = (XY)^7 = (XT)^2 = (YT)^2 = 1 \rangle,$$

with T permuting vertices by reflection in the vertical axis. The motivation for this was that Conder wanted to realise S_n and A_n as quotients of $\Delta[3, 2, 7]$ for all sufficiently large n. Here we do not have this ambition, so we will not restrict our diagrams and dessins to those with bilateral symmetry.

The dessins $\mathcal{A}, \ldots, \mathcal{N}$ corresponding to Conder's basic diagrams $A, \ldots, N$ are all drawn in [13], whereas only $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{G}$ are used in this paper; their basic properties are listed in the Appendix.

2.4 Macbeath–Hurwitz Curves

Being perfect, every Hurwitz group is a covering of a non-abelian finite simple group, which is also a Hurwitz group. Up to any given finite order, most non-abelian finite

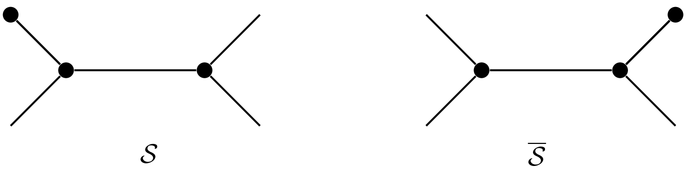


Fig. 2 Dessins $\mathcal{S}$ and $\overline{\mathcal{S}}$

simple groups have the form $\mathrm{PSL}_2(q)$ for prime-powers q, so it is natural that many of our examples will also have this form.

Theorem 2.3 (Macbeath [20]) $\mathrm{PSL}_2(q)$ *is a Hurwitz group if and only if one of the following conditions is satisfied:*

1. $q = 7$ *or* $q = p^3$ *for a prime* $p \equiv \pm 2$ *or* $\pm 3 \bmod (7)$*, with one associated Hurwitz curve;*
2. q *is a prime* $p \equiv \pm 1 \bmod (7)$*, with three associated Hurwitz curves distinguished by the choice of* $\pm\mathrm{tr}(z)$.

In case (1) the uniqueness of the corresponding regular dessin shows that it is defined over $\mathbb{Q}$. Streit [25] showed that in case (2) the three dessins are defined over the real cyclotomic field $\mathbb{Q}(\cos(2\pi/7)) = \mathbb{Q}(e^{2\pi i/7}) \cap \mathbb{R}$, and that for each q they form an orbit under the Galois group C_3 of that field. This orbit consists of one reflexible dessin and a chiral pair.

For odd q, $\mathrm{PSL}_2(q)$ has order $q(q^2-1)/2$, so the genus of the corresponding curve or curves is

$$g = 1 + \frac{q(q^2-1)}{168}.$$

On the other hand $|\mathrm{PSL}_2(8)| = 504$, so in this case the curve has genus 7.

3 Some Small Dessins

We now introduce some small Hurwitz dessins to be used later as examples.

Example The dessin $\mathcal{S}$ and its mirror image $\overline{\mathcal{S}}$, shown in Fig. 2, are the smallest Hurwitz dessins, having degree $n = 7$. They both have genus 0. In each case the monodromy group is $\mathrm{PGL}_3(2)$, in its natural action on the points or the lines of the Fano plane $\mathbb{P}^2(\mathbb{F}_2)$; these are equivalent to the actions of the isomorphic group $\mathrm{PSL}_2(7)$ on its two conjugacy classes of subgroups $H \cong \mathrm{S}_4$. These dessins correspond to two conjugacy classes of subgroups M of Δ, of index 7 and signature $(0; 2, 2, 2, 3)$.

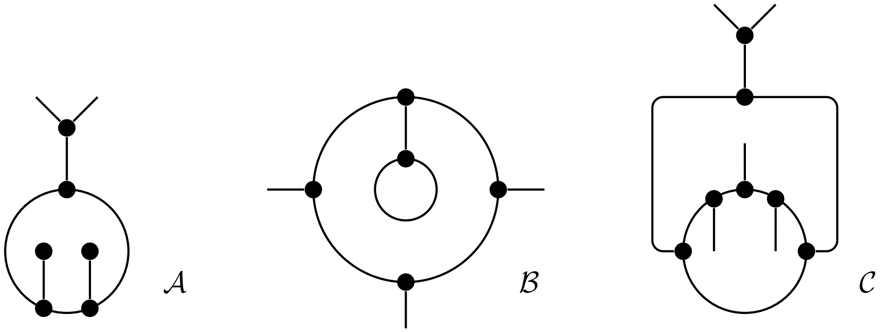


Fig. 3 Dessins $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$

Example Figure 3 shows three Hurwitz dessins $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, of genus 0 and of degrees 14, 15 and 21, based on Conder's coset diagrams A, B and C in [1]. Note that $\mathcal{A}$ has been redrawn from Fig. 1 so that the free edges are now in the outer face; the reason for this will become clear in the next section, when we consider joining operations (see Fig. 5).

The monodromy group of $\mathcal{A}$ is $\mathrm{PSL}_2(13)$ in its natural representation of degree 14 on the projective line $\mathbb{P}^1(\mathbb{F}_{13})$; the point-stabilisers are the Sylow 13-normalisers $H \cong \mathrm{C}_{13} \rtimes \mathrm{C}_6$. One can take $x : t \mapsto 1/(1-t)$, $y : t \mapsto -t$ and $z : t \mapsto (t+1)/t$ as standard generators. This dessin corresponds to a conjugacy class of subgroups M of index 14 in Δ; since $\mathcal{A}$ has genus 0, and x and y each have two fixed points while z has none, these subgroups M are quadrilateral groups, with signature $(0; 2, 2, 3, 3)$. We have $\mathcal{A} \cong \widetilde{\mathcal{A}}/H$ where $\widetilde{\mathcal{A}}$ is the Galois cover of $\mathcal{A}$, corresponding to the core K of M in Δ; this is the unique reflexible dessin in the Galois orbit of three Macbeath–Hurwitz dessins of genus 14 with automorphism group $\mathrm{PSL}_2(13)$. (See [16] for further details of this Galois orbit and the corresponding quotient dessins.)

In $\mathcal{B}$, the commutator $[x, y]$ has cycle structure 3, 5, 7, so the monodromy group G contains a 3-cycle, a 5-cycle and a 7-cycle. A transitive group of degree 15 with an element of order 7 must be primitive (otherwise it would be contained in a wreath product $\mathrm{S}_5 \wr \mathrm{S}_3$ or $\mathrm{S}_3 \wr \mathrm{S}_5$, of order coprime to 7), and a primitive group containing a 3-cycle contains the alternating group (see [26, Theorem 13.3]). Since x, y and z are even, $G = \mathrm{A}_{15}$. (As Conder showed in [1], this is the smallest alternating group which is a Hurwitz group.) In this case the corresponding subgroups M of Δ have signature $(0; 2, 2, 2, 7)$.

The monodromy group of $\mathcal{C}$ is $\mathrm{PGL}_3(2)$, isomorphic to $\mathrm{PSL}_2(7)$, in its imprimitive action of degree 21 on the flags (incident point-line pairs) of the Fano plane, or equivalently its action by conjugation on its involutions. The point-stabilisers are the Sylow 2-subgroups $H \cong \mathrm{D}_4$. The corresponding subgroups M of Δ have signature $(0; 2, 2, 2, 2, 2)$.

4 Joins for Hurwitz Dessins

From now on, unless stated otherwise, all dessins will be Hurwitz dessins, that is, finite oriented hypermaps of type (3, 2, 7). In this section we will consider the handles and joins introduced by Higman and used by Conder, reinterpreting them as operations on dessins.

4.1 y-Handles and y-Joins

The joins used by Higman and Conder are based on fixed points of the generating involution Y for Δ (or X in their notation; the symbols are transposed here for more convenient explanation of links with dessins). Let $\mathcal{D}$ be a Hurwitz dessin with monodromy group

$$G = \langle x, y, z \mid x^3 = y^2 = z^7 = xyz = \cdots = 1\rangle.$$

This gives a faithful transitive permutation representation of G. As usual, cycles of x, y and z are represented as vertices, edges and faces. If y has two fixed points (free edges) in the same face, we can uniquely label them a and b so that $b = ax$, $axyx$ or $axyxyx$, that is, $b = az^{1-k}x$ for some $k = 1, 2$ or 3; the ordered pair (a, b) is then called a (k)-*handle*. These are illustrated in Fig. 4. If we do not wish to specify the value of k, we will sometimes call these y-*handles*, to distinguish them from the x-handles to be defined later.

Example In each of the dessins $\mathcal{S}$ and $\overline{\mathcal{S}}$ in Fig. 2, Y has three fixed points. One ordered pair of them forms a (1)-handle, and another forms a (2)-handle; there are no (3)-handles.

Example In the dessin $\mathcal{A}$ in Fig. 3, Y has two fixed points, forming a (1)-handle. In $\mathcal{B}$, Y has three fixed points; two ordered pairs form (2)-handles, and a third forms a (3)-handle. In $\mathcal{C}$, Y has five fixed points, providing a (1)-handle and a (3)-handle.

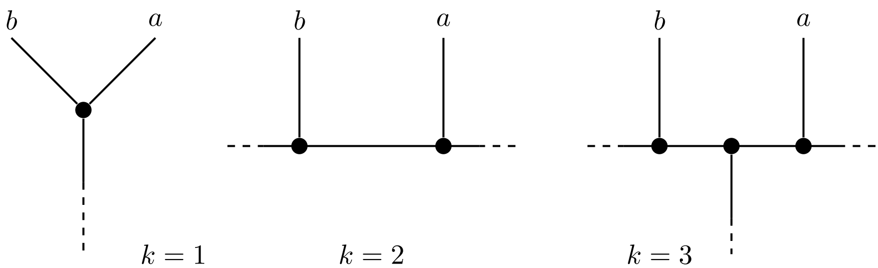


Fig. 4 (k)-handles for $k = 1, 2, 3$

Lemma 4.1 *Let $\mathcal{D} \to \overline{\mathcal{D}}$ be a covering of Hurwitz dessins. Then any (k)-handle in $\mathcal{D}$ projects onto a (k)-handle in $\overline{\mathcal{D}}$.*

Proof If $ay = a$, $by = b$ and $b = az^{1-k}x$ in $\mathcal{D}$, then the images $\overline{a}$ and $\overline{b}$ of a and b in $\overline{\mathcal{D}}$ satisfy the corresponding equations. If $\overline{a} = \overline{b}$ then the covering reduces the number of sides of the face containing the handle (a, b), so its image has just one side. Thus Z fixes $\overline{a}$, and hence so does $\langle Y, Z\rangle = \Delta$, contradicting the transitivity of Δ. Hence $\overline{a} \neq \overline{b}$, so $(\overline{a}, \overline{b})$ is a (k)-handle. □

The following partial converse is obvious:

Lemma 4.2 *Let $\tilde{\mathcal{D}} \to \mathcal{D}$ be a d-sheeted covering of Hurwitz dessins. Then any (k)-handle (a, b) in $\mathcal{D}$ lifts to d disjoint (k)-handles in $\tilde{\mathcal{D}}$, provided the covering is unbranched over a and b.* □

Dessins with arbitrarily many (k)-handles can be constructed by applying Lemma 4.2 to a dessin $\mathcal{D}$ with at least one (k)-handle (a, b), where the covering $\tilde{\mathcal{D}} \to \mathcal{D}$ corresponds to a suitable index d inclusion $\tilde{M} \leq M$ of map subgroups. To avoid branching over a and b we require the elliptic generators of order 2 of M corresponding to the fixed points a and b of Y to be elements of $\tilde{M}$. In all except a few small cases, such subgroups $\tilde{M}$ exist.

Example Let $\mathcal{D} = \mathcal{C}$, so that M has signature $(0; 2^{[5]})$. Let Y_1 and Y_2 be the elliptic generators corresponding to the (1)-handle, Y_3 and Y_4 those corresponding to the (3)-handle, and let Y_5 be the fifth, chosen so that $Y_1 \ldots Y_5 = 1$. For any integer $d \geq 2$ define an epimorphism

$$\theta : M \to \mathrm{D}_d = \langle u, v \mid u^2 = v^2 = (uv)^d = 1\rangle$$

by $Y_1, Y_2 \mapsto u$, $Y_3, Y_4 \mapsto v$, and $Y_5 \mapsto 1$. Then $\tilde{M} = \theta^{-1}(\langle u\rangle)$ has index d in M, and contains Y_1 and Y_2, so it corresponds to a covering of $\mathcal{C}$ with d disjoint (1)-handles, while $\theta^{-1}(\langle v\rangle)$ corresponds to a covering with d disjoint (3)-handles. Similarly, examples with arbitrarily many disjoint (2)-handles can be constructed as coverings of $\mathcal{B}$.

We now define joins of dessins. Suppose that Hurwitz dessins $\mathcal{D}_i$ $(i = 1, 2)$ of degree d_i and genus g_i have (k)-handles (a_i, b_i) for the same value of k. We can form a (k)-*join* $\mathcal{D}_1(k)\mathcal{D}_2$ by defining the cycles of x and y to be those they have on $\mathcal{D}_1$ or on $\mathcal{D}_2$, except that the four fixed points a_i, b_i of y are replaced with two 2-cycles (a_1, a_2) and (b_1, b_2). (This is called a k-*composition* in [1], but we prefer to reserve the word 'composition' for a different method of combining dessins, by composing their Belyĭ functions.) As with handles, if we do not wish to specify the value of k we sometimes will call these y-*joins*, to distinguish them from x-joins defined later. This operation is illustrated in Fig. 5 in the case $k = 1$; the cases $k = 2$ and 3 are similar. Topologically this is a connected sum operation, where the two dessins are joined across cuts between the free ends of the half-edges representing the fixed points a_i and b_i, with the cuts staying in the faces containing the two k-handles. If

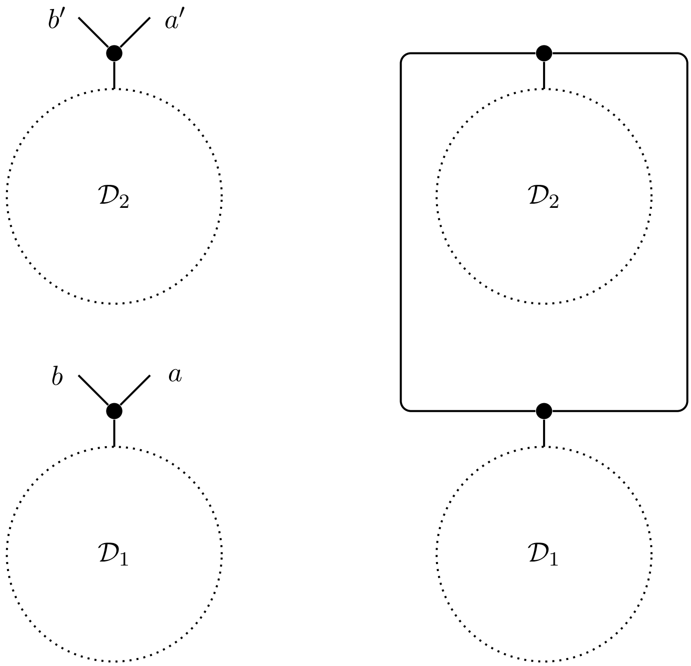


Fig. 5 Construction of $\mathcal{D}_1(1)\mathcal{D}_2$

either dessin $\mathcal{D}_i$ has more than one (k)-handle, one may need to specify the choice of handles in order to define their (k)-join uniquely.

The following simple result records the additivity properties of joins.

Theorem 4.3 *If Hurwitz dessins $\mathcal{D}_i$ $(i = 1, 2)$ of degree n_i and signature $(g_i; 3^{[\alpha_i]}, 2^{[\beta_i]}, 7^{[\gamma_i]})$ have a (k)-join $\mathcal{D}$, then $\mathcal{D}$ has degree n and signature $(g; 3^{[\alpha]}, 2^{[\beta]}, 7^{[\gamma]})$ where*

$$n = n_1 + n_2,\ g = g_1 + g_2,\ \alpha = \alpha_1 + \alpha_2,\ \beta = \beta_1 + \beta_2 - 4 \textit{ and } \gamma = \gamma_1 + \gamma_2.$$

Proof The equations for n, α, β and γ follow immediately from the definition of a join, and that for g follows from applying the Riemann–Hurwitz formula (1) to these three dessins. □

Example The dessins $\mathcal{A}$ and $\mathcal{C}$ in Fig. 3 each have a unique (1)-handle, so we can form the join $\mathcal{A}(1)\mathcal{C}$ of degree $14 + 21 = 35$, as shown in Fig. 6. This dessin has monodromy group $G = \mathrm{A}_{35}$. To prove this, we first show that G is primitive. If not, it has a blocks of size b, so G is embedded in the wreath product $\mathrm{S}_b \wr \mathrm{S}_a = (\mathrm{S}_b)^a \rtimes \mathrm{S}_a$, where $ab = 35$ and hence $\{a, b\} = \{5, 7\}$. We cannot have $a = 5$ since G would then act on the blocks as a Hurwitz subgroup of S_5, so $a = 7$ and $b = 5$. This implies

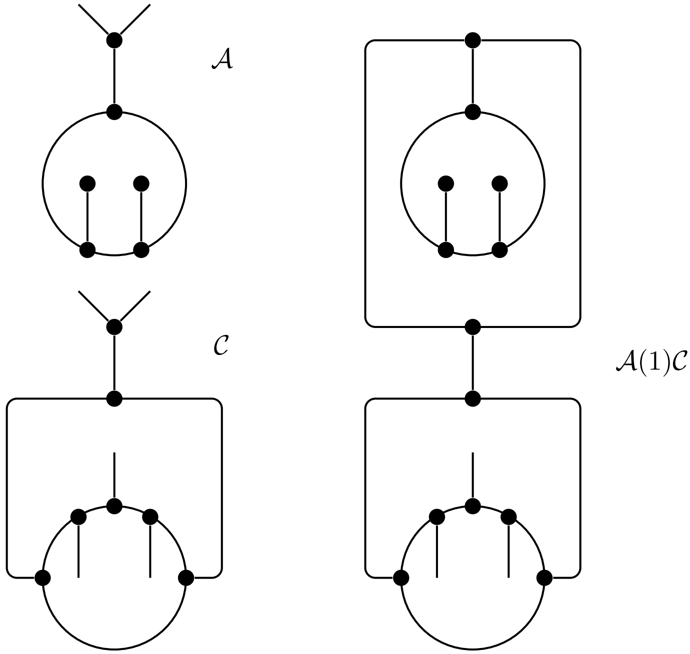


Fig. 6 Construction of $\mathcal{A}(1)\mathcal{C}$

that the Sylow 7-subgroups of G, which are mutually conjugate, are isomorphic to C_7, so all elements of order 7 have the same cycle structure. However, inspection of Fig. 6 shows that $w := [x, y]$ has cycle structure $1^{[2]}, 2^{[2]}, 4^{[2]}, 21$, so w^{12} has cycle structure $1^{[14]}, 7^{[3]}$, whereas for z it is $7^{[5]}$. Thus G is primitive. Now w^4 is a 21-cycle, so $G \geq A_{35}$ by Theorem 2.2, with equality since G is perfect. (This has been confirmed using GAP.) This example illustrates how the monodromy group of a join (in this case of order approximately $5 \cdot 17 \times 10^{39}$) can be much larger than those of the two factors (here of orders 1092 and 168). (In principle, it could be smaller, but it is hard to think of an example.)

Warning When making a y-join, it is important that a_1 should be paired with a_2, and b_1 with b_2, rather than a_1 with b_2 and a_2 with b_1. For example, the dessin $\mathcal{S}$ in Fig. 2 has a (1)-handle, so we can form a Hurwitz dessin $\mathcal{S}(1)\mathcal{S}$ (as we will see in Fig. 8), but if we choose the wrong pairing for the fixed points of y in the two copies of $\mathcal{S}$, as in Fig. 7, we obtain a dessin of type (3, 2, 12) rather than (3, 2, 7), with faces of valency 2 and 12; its monodromy group, of order $2688 = 2^7.3.7$, is solvable, so it cannot be a Hurwitz group. More generally, applying such 'incorrect' pairings to Hurwitz dessins gives dessins which have quotients of the modular group

$$\Gamma = PSL_2(\mathbb{Z}) = \Delta(3, 2, \infty) = \langle X, Y \mid X^3 = Y^2 = 1 \rangle,$$

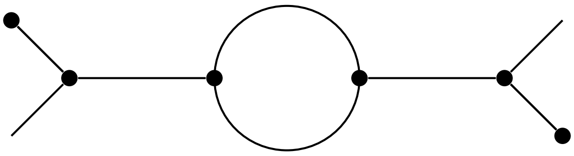


Fig. 7 Joining copies of $\mathcal{S}$ incorrectly

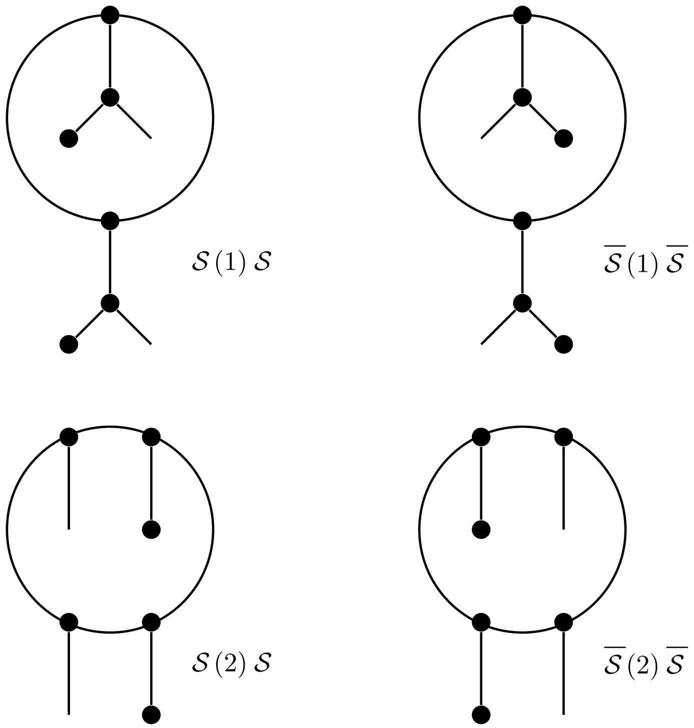


Fig. 8 The dessins $\mathcal{S}(k)\mathcal{S}$ and $\overline{\mathcal{S}}(k)\overline{\mathcal{S}}$

rather than $\Delta(3, 2, 7)$, as monodromy groups. We will consider this in more detail in Sect. 6, devoted to Γ.

It should be clear from the definition that joining is commutative and associative, in the sense that there are obvious isomorphisms $\mathcal{D}_1(k)\mathcal{D}_2 \cong \mathcal{D}_2(k)\mathcal{D}_1$ and $\mathcal{D}_1(k)(\mathcal{D}_2(k')\mathcal{D}_3) \cong (\mathcal{D}_1(k)\mathcal{D}_2)(k')\mathcal{D}_3$ whenever these dessins are well-defined. In the case of associativity this requires the chosen (k)- and (k')-handles in $\mathcal{D}_2$ to be mutually disjoint. The following lemma ensures that, as long as we do not use copies of $\mathcal{B}$, $\mathcal{S}$ or $\overline{\mathcal{S}}$, distinct y-handles in the same dessin will be in distinct faces, so they will be mutually disjoint and can therefore be used independently of each other:

Lemma 4.4 *The following conditions on a Hurwitz dessin $\mathcal{D}$ are equivalent:*

1. *$\mathcal{D}$ has two y-handles with a common fixed point;*
2. *$\mathcal{D}$ has two y-handles in the same face;*
3. *$\mathcal{D}$ is isomorphic to $\mathcal{B}$, $\mathcal{S}$ or $\overline{\mathcal{S}}$.*

Proof The implication (1) $\Rightarrow$ (2) is obvious, and (3) $\Rightarrow$ (1) is evident from Figs. 2 and 3. The implication (2) $\Rightarrow$ (3) follows from straightforward case-by-case analysis, using the fact that a (k)-handle in a face uses four, three or four of its seven sides as $k = 1, 2$ or 3 (see Fig. 4). □

4.2 The Double Cover $\mathcal{D}(k)\mathcal{D}$

If a dessin $\mathcal{D}$ has a (k)-handle, one can form the join $\mathcal{D}^* = \mathcal{D}(k)\mathcal{D}$, a double cover of $\mathcal{D}$ branched over two points, namely those corresponding to the fixed points a and b of Y in the chosen handle.

There is an obvious automorphism of order 2 of $\mathcal{D}^*$, interchanging the two copies of $\mathcal{D}$; the quotient is isomorphic to $\mathcal{D}$. The double covering $\mathcal{D}^* \to \mathcal{D}$ shows that the monodromy group G^* of $\mathcal{D}^*$ must be imprimitive, with blocks of size 2 permuted as the monodromy group G of $\mathcal{D}$, so it is isomorphic to a subgroup of the wreath product $S_2 \wr G$.

If $\mathcal{D}$ has degree n and signature $(g; 3^{[\alpha]}, 2^{[\beta]}, 7^{[\gamma]})$, then it follows from Theorem 4.3 that $\mathcal{D}^*$ has degree $n^* = 2n$ and signature

$$(g^*; 3^{[\alpha^*]}, 2^{[\beta^*]}, 7^{[\gamma^*]}) = (2g; 3^{[2\alpha]}, 2^{[2\beta-4]}, 7^{[2\gamma]}).$$

If M and M^* are the subgroups of Δ corresponding to $\mathcal{D}$ and $\mathcal{D}^*$ as in Sect. 2.1, then since M^* has index 2 in M, it contains the subgroup $M'M^2$ generated by the commutators and squares of the elements of M. Now the standard presentation of M (see Sect. 2.1) shows that $M/M'M^2$ is an elementary abelian 2-group of rank $r = 2g + \beta - 1$, so M has $2^r - 1$ subgroups of index 2, each corresponding to a double covering of $\mathcal{D}$. (For instance, $\mathcal{D}^*$ corresponds to the normal closure in M of all its standard generators except the two involutions Y_k corresponding to the fixed points of Y in the chosen handle.) This immediately leads to the following:

Lemma 4.5 *Suppose that a planar Hurwitz dessin $\mathcal{D}$ has exactly two fixed points for Y, forming a (k)-handle; then $\mathcal{D}^*$ is the unique Hurwitz dessin which is a double covering of $\mathcal{D}$. If, in addition, the point stabiliser H in the monodromy group G of $\mathcal{D}$ has a subgroup H^* of index 2, then $\mathcal{D}^*$ also has monodromy group G, with point stabiliser H^*.* □

However, if $g > 0$ or $\beta > 2$ then $\mathcal{D}^*$ is one of several double coverings of $\mathcal{D}$, and its monodromy group could be G or a proper covering group of G contained in $S_2 \wr G$ (as must happen if H has no subgroup of index 2).

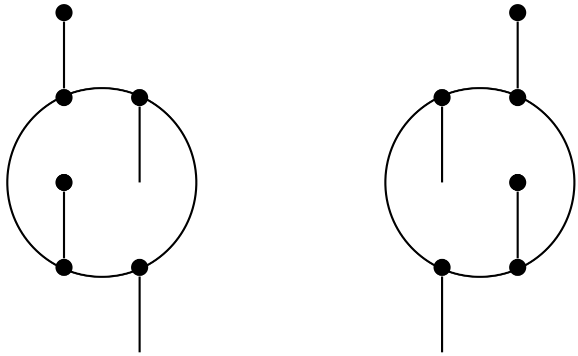


Fig. 9 The third double coverings of $\mathcal{S}$ and $\overline{\mathcal{S}}$

Example Let $\mathcal{D} = \mathcal{S}$ or $\overline{\mathcal{S}}$ (see Fig. 2), with monodromy group $G = \mathrm{PGL}_3(2)$ acting on the points or lines of the Fano plane, so that H belongs to one of the two conjugacy classes of subgroups of G isomorphic to S_4. Then $g = 0$ but $\beta = 3$, so Lemma 4.5 does not apply. Since $r = 2$ there are three double coverings of $\mathcal{D}$. Taking $k = 1$ we obtain the chiral pair of planar dessins $\mathcal{S}^* = \mathcal{S}(1)\mathcal{S}$ and $\overline{\mathcal{S}}^* = \overline{\mathcal{S}}(1)\overline{\mathcal{S}}$ in the top row of Fig. 8. The monodromy group of each is the unique Hurwitz group G^* of genus 17, a non-split extension of an elementary abelian group of order 8 by its automorphism group $\mathrm{GL}_3(2) \cong G$. There is a chiral pair of regular Hurwitz dessins associated with this group, and these two dessins are their quotients.

Taking $k = 2$ we obtain the dessins $\mathcal{S}(2)\mathcal{S}$ and $\overline{\mathcal{S}}(2)\overline{\mathcal{S}}$ in the bottom row of Fig. 8. They have the same monodromy group G as $\mathcal{S}$ and $\overline{\mathcal{S}}$, but now in its imprimitive representations on the cosets of the unique subgroup H^* of index 2 in H, isomorphic to A_4; the two conjugacy classes of such subgroups give the two representations.

The third double covering of $\mathcal{D}$ in this example does not arise from a (k)-join as defined earlier. However, like the coverings $\mathcal{D}(k)\mathcal{D}$ for $k = 1, 2$ it is branched over two of the three fixed points of y in $\mathcal{D}$, in this case satisfying $b = axyx^{-1}$ rather than $b = a(xy)^{k-1}x$. Figure 9 shows the chiral pair of Hurwitz dessins resulting from taking $\mathcal{D} = \mathcal{S}$ and $\overline{\mathcal{S}}$; their monodromy group is the Hurwitz group G^* of genus 17 associated earlier with $\mathcal{D}(1)\mathcal{D}$.

Example Figure 10 shows the dessins $\mathcal{A}^* = \mathcal{A}(1)\mathcal{A}$, $\mathcal{B}^* = \mathcal{B}(3)\mathcal{B}$ and $\mathcal{C}^* = \mathcal{C}(1)\mathcal{C}$, of degrees 28, 30 and 42. Since $\mathcal{A}$ is planar and has $\beta = 2$, the above argument shows that $\mathcal{A}^*$ has the same monodromy group $\mathrm{PSL}_2(13)$ as $\mathcal{A}$, but now in its imprimitive representation of degree 28 on the cosets of the unique subgroup $H^* \cong \mathrm{C}_{13} \rtimes \mathrm{C}_3$ of index 2 in $H \cong \mathrm{C}_{13} \rtimes \mathrm{C}_6$. On the other hand, since the monodromy group of $\mathcal{B}$ is A_{15}, and a point stabiliser $H \cong \mathrm{A}_{14}$ in A_{15} has no subgroup of index 2, the monodromy group of $\mathcal{B}^*$ must be a proper covering group of A_{15}; according to GAP it is an extension of an elementary abelian group of order 2^{14} by A_{15}, which must be the imprimitive group $(\mathrm{S}_2 \wr \mathrm{A}_{15}) \cap \mathrm{A}_{30}$. Similarly, although the monodromy group of $\mathcal{C}$ is $\mathrm{PGL}_3(2) \cong \mathrm{PSL}_2(7)$, that of $\mathcal{C}^*$ is a covering of this group by an elementary abelian group of order 2^6; as an abstract group, and as the automorphism group of

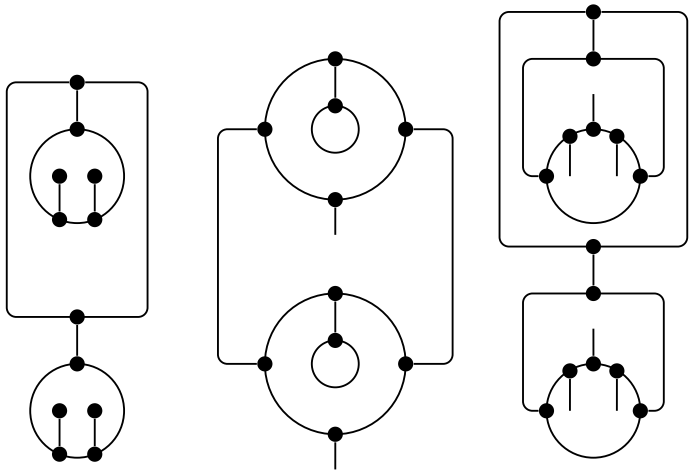


Fig. 10 The dessins $\mathcal{A}(1)\mathcal{A}$, $\mathcal{B}(3)\mathcal{B}$ and $\mathcal{C}(1)\mathcal{C}$

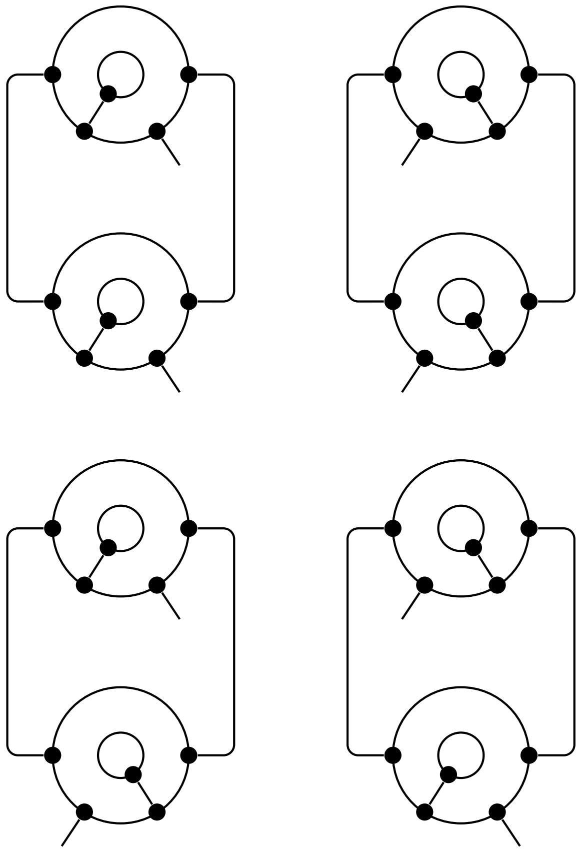
the regular cover of $\mathcal{C}^*$, this is the unique Hurwitz group of genus 257, corresponding to the normal subgroup $K'K^2$ of Δ, where K corresponds to the regular Hurwitz dessin of genus 3. These dessins $\mathcal{A}^*$, $\mathcal{B}^*$ and $\mathcal{C}^*$ are all reflexible, because $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are reflexible, invariant under a reflection which preserves the chosen handle (as an unordered pair of fixed points).

Example The dessin $\mathcal{B}$ also has two (2)-handles, transposed by its reflection. We can therefore form four dessins $\mathcal{B}(2)\mathcal{B}$ of degree 30, shown in Fig. 11, by choosing one of these handles in each copy of $\mathcal{B}$. The two dessins in the top row, where the same handle has been chosen from each copy, clearly form a chiral pair; GAP shows that their monodromy group is $(S_2 \wr A_{15}) \cap A_{30}$. The two dessins in the bottom row, where different handles have been chosen from each copy, are also mirror images of each other, but they are in fact isomorphic, by a half-turn of the sphere transposing the two copies of $\mathcal{B}$, so that this dessin is invariant under the antipodal symmetry of the sphere. In this case GAP shows that the monodromy group is A_{30}.

As a variation on the concept of a double covering $\mathcal{D}(k)\mathcal{D}$, if $\mathcal{D}$ has a (k)-handle then so has $\overline{\mathcal{D}}$, and even if $\mathcal{D} \ncong \overline{\mathcal{D}}$ we can form the dessin $\mathcal{D}(k)\overline{\mathcal{D}} \cong \overline{\mathcal{D}}(k)\mathcal{D}$. If $\mathcal{D}$ has degree n and signature $(g; 3^{[\alpha]}, 2^{[\beta]}, 7^{[\gamma]})$, then so has $\overline{\mathcal{D}}$, so as in the case of $\mathcal{D}^*$ it follows from Theorem 4.3 that $\mathcal{D}(k)\overline{\mathcal{D}}$ has degree $n^* = 2n$ and signature $(2g; 3^{[2\alpha]}, 2^{[2\beta-4]}, 7^{[2\gamma]})$.

Example If we join the dessins $\mathcal{S}$ and $\overline{\mathcal{S}}$ in Fig. 2, we see a phenomenon similar to that in the preceding example, as shown in Fig. 12. The dessins $\mathcal{S}(1)\overline{\mathcal{S}}$ and $\overline{\mathcal{S}}(1)\mathcal{S}$ in the first row, clearly mirror images of each other, are also isomorphic under a rotation

Fig. 11 The dessins $\mathcal{B}(2)\mathcal{B}$



of order 2, and are each invariant under the antipodal isometry of the sphere. Although the monodromy group of $\mathcal{S}$ and $\overline{\mathcal{S}}$ is $\mathrm{PGL}_3(2) \cong \mathrm{PSL}_2(7)$, and for $\mathcal{S}(1)\mathcal{S}$ and $\overline{\mathcal{S}}(1)\overline{\mathcal{S}}$ it is a covering of this group, this dessin has monodromy group $\mathrm{PSL}_2(13)$, in its natural representation. It is, in fact, one of a Galois orbit of three dessins corresponding to this action; the others are $\mathcal{A}$, shown in Fig. 3, and the dessin $\mathcal{S}(2)\overline{\mathcal{S}} \cong \overline{\mathcal{S}}(2)\mathcal{S}$, also invariant under the antipodal isometry, shown in the second row of Fig. 12.

4.3 Multiple Joins

If dessins $\mathcal{D}_i$ $(i = 1, 2)$ each have several mutually disjoint handles, then one can use these to make a multiple join $\mathcal{D}$ by joining m_k (k)-handles in $\mathcal{D}_1$ to the same number in $\mathcal{D}_2$ (provided that many exist) for $k = 1, 2, 3$. If each $\mathcal{D}_i$ has degree n_i and signature $(g_i; 2^{[\alpha_i]}, 3^{[\beta_i]}, 7^{[\gamma_i]})$, then $\mathcal{D}$ has degree $n_1 + n_2$ and signature $(g; 2^{[\alpha]}, 3^{[\beta]}, 7^{[\gamma]})$

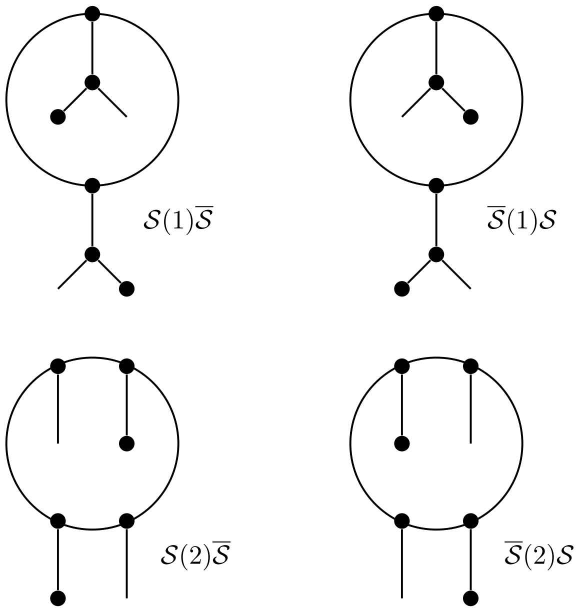


Fig. 12 The dessins $\mathcal{S}(k)\overline{\mathcal{S}}$ and $\overline{\mathcal{S}}(k)\mathcal{S}$ for $k = 1, 2$

where $g = g_1 + g_2 + m - 1$ ($m := \sum_k m_k$), $\alpha = \alpha_1 + \alpha_2$, $\beta = \beta_1 + \beta_2 - 4m$ and $\gamma = \gamma_1 + \gamma_2$.

In this way, Hurwitz dessins of arbitrary genus g can be formed from a pair of Hurwitz dessins of genus 0 with at least $g + 1$ compatible handles. For instance, since the dessin $\mathcal{G}$ in Fig. 13 (based on Conder's diagram G in [1]) has genus 0 and three disjoint (1)-handles, we could take each $\mathcal{D}_i$ to be the composition $\mathcal{G}(1)\cdots(1)\mathcal{G}$ of $g - 1$ copies of $\mathcal{G}$. (Using such a composition as a 'stem', on which to attach further Hurwitz dessins, is a basic idea in [1] and several subsequent papers.)

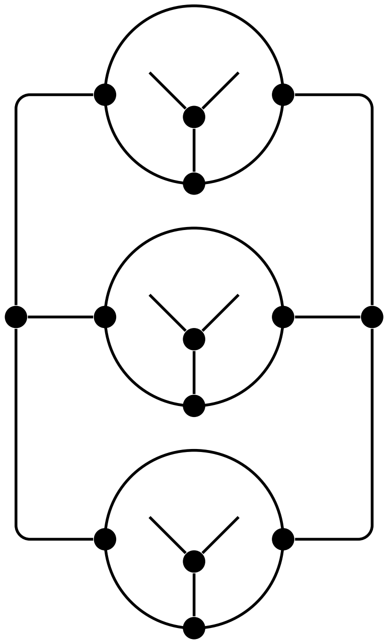
4.4 y-Handles from $\mathbf{PSL_2(q)}$

Motivated by the example of the dessin $\mathcal{A}$, it is natural to look for other instances of y-handles arising from the Hurwitz groups $G = \mathrm{PSL}_2(q)$ in their natural representation. Based on Macbeath's classification [20] of the Hurwitz groups of this form, as explained in Sect. 2.4, we have the following:

Theorem 4.6 *A Hurwitz dessin corresponding to the natural representation of the group* $G = \mathrm{PSL}_2(q)$, $q = p^e$, *has a* (k)*-handle if and only if either*

- $q = p = 13$, *with* $\mathrm{tr}(z) = \pm 5$ *and* $k = 1$, *or*
- $q = p = 29$, *with* $\mathrm{tr}(z) = \pm 3$ *and* $k = 2$, *or*
- $q = p = 41$, *with* $\mathrm{tr}(z) = \pm 11$ *and* $k = 3$.

Fig. 13 The dessin $\mathcal{G}$



Proof If a (k)-handle exists then y has two fixed points, so q must be odd and y must have order dividing $(q-1)/2$ where $q \equiv 1 \bmod (4)$. By applying an automorphism of G we may assume that y has the form $t \mapsto -t$, fixing $a = 0$ and $b = \infty$, or equivalently, that y corresponds to the pair of elements

$$\pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

of $\mathrm{SL}_2(q)$, where $i^2 = -1$. An element of G has order 3 if and only if it has trace ± 1, so the most general form for x is

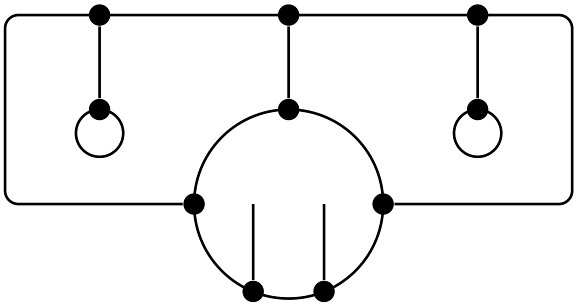
$$\pm \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}$$

where $a(1-a) - bc = 1$. It then follows that xy has trace

$$t = \pm(2a-1)i.$$

Now any element of order 7, such as xy, has trace $\pm\tau_j$ where $\tau_j = \lambda^j + \lambda^{-j}$ for $j = 1, 2, 3$ and λ is a primitive 7th root of 1 (possibly in an extension field). One easily checks that $\tau_1 + \tau_2 + \tau_3 = -1$, $\tau_1\tau_2 + \tau_2\tau_3 + \tau_3\tau_1 = -2$ and $\tau_1\tau_2\tau_3 = 1$, so the elements $\pm\tau_j$ are respectively the roots of the polynomials

Fig. 14 The dessin $\mathcal{F}$



$$p_{\pm}(t) := t(t^2 - 2) \pm (t^2 - 1).$$

First we look for (1)-handles. These require $x : 0 \mapsto \infty$, so $a = 0$ and xy has trace $t = \pm i$. Thus i must be a root of one of the polynomials $p_{\pm}(t)$, so $3i = \pm 2$ and hence $p = 13$ with $t = \pm 5$. By Macbeath's classification, it follows that $q = 13$. The dessin is that shown earlier as $\mathcal{A}$ in Fig. 3.

In the case of a (2)-handle we require $xyx : 0 \mapsto \infty$. Writing x and y as above we find that

$$xyx : 0 \mapsto \frac{(1-2a)b}{2a^2 - 3a + 2},$$

so this is equivalent to

$$2a^2 - 3a + 2 = 0.$$

Putting $t = \delta(2a-1)i$ with $\delta^2 = 1$ gives $t^2 = -4a^2 + 4a - 1 = -2a + 3$, so

$$\begin{aligned} 0 = p_{\pm}(t) &= \delta(2a-1)i(-2a+1) \pm (-2a+2) \\ &= \delta i(-4a^2 + 4a - 1) \pm 2(1-a) \\ &= \delta i(-2a+3) \pm 2(1-a), \end{aligned}$$

and hence $0 = (2a-3)^2 + 4(a-1)^2 = 8a^2 - 20a + 13 = -8a + 5$, that is, $a = 5/8$. This gives $0 = 50 - 120 + 128 = 58$ and hence $q = p = 29$. The roots of $2a^2 - 3a + 2$ in $\mathbb{F}_{29}$ are $a = -3$ and -10, and we have $i = \pm 12$, so xy has trace $t = \delta(2a-1)i = \pm 3$ or ± 9. Now the roots of $p_+(t)$ are 3, 12 and 13, so $a = -3$, giving $t = \pm 3$.

It is now straightforward to check that the resulting dessin is the planar dessin $\mathcal{F}$ in Fig. 14, corresponding to Conder's diagram F in [1].

In the case of a (3)-handle we require $(xy)^2 x : 0 \mapsto \infty$. Writing x and y as above we find that this is equivalent to

$$4a^3 - 8a^2 + 8a - 3 = 0.$$

Putting $t = \delta(2a-1)i$ with $\delta^2 = 1$ gives $t^2 = -4a^2 + 4a - 1$, so

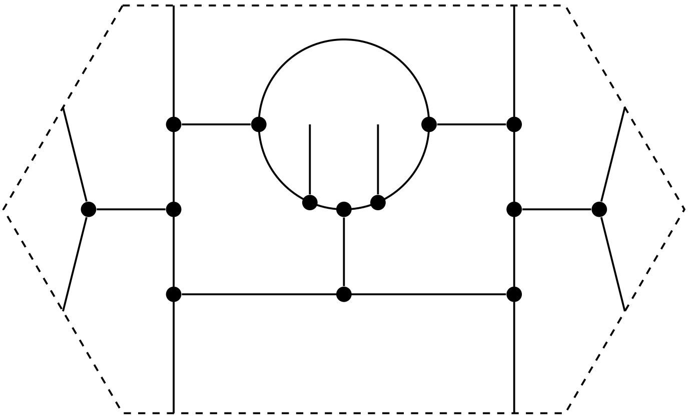


Fig. 15 A torus dessin $\mathcal{T}$ with monodromy group $\mathrm{PSL}_2(41)$

$$\begin{aligned} 0 = p_{\pm}(t) &= t(t^2-2) \pm (t^2-1) \\ &= \delta(2a-1)i(-4a^2+4a-3) \pm (-4a^2+4a-2) \\ &= \delta i(-8a^3+12a^2-10a+3) \pm (-4a^2+4a-2) \\ &= \delta i(-4a^2+6a-3) \pm (-4a^2+4a-2) \end{aligned}$$

so that

$$\begin{aligned} 0 &= (-4a^2+6a-3)^2 + (-4a^2+4a-2)^2 \\ &= 32a^4 - 80a^3 + 92a^2 - 52a + 13. \end{aligned}$$

Using the cubic to eliminate the leading term gives

$$-16a^3 + 28a^2 - 28a + 13 = 0.$$

Repeating this gives

$$-4a^2 + 4a + 1 = 0.$$

Thus $4a^2 = 4a + 1$, so $4a^3 = 4a^2 + a = 5a + 1$, and hence the cubic becomes $5a - 4 = 0$, that is, $a = 4/5$. Substituting this in the above quadratic gives $64 = 105$, so $q = p = 41$, with $a = 9$. Here $i = \pm 9$ so $t = \pm 11$. As a check, $p_+(-11) = 0$, so $-11 = \lambda + \lambda^{-1}$ for some primitive 7th root of 1 in $\mathbb{F}_{41^3}$.

The corresponding dessin $\mathcal{T}$ has genus $g = 1$. It is shown in Fig. 15 with opposite sides of the outer hexagon identified to form a torus. □

The dessin $\mathcal{F}$ with monodromy group $\mathrm{PSL}_2(29)$ which appears in this theorem corresponds to a conjugacy class of subgroups M of index 30 in Δ with signature $(0; 2, 2, 7, 7)$. As in the case $q = 13$, it follows from Lemma 4.5 that the dessin $\mathcal{F}^* = \mathcal{F}(2)\mathcal{F}$, shown on the right in Fig. 16, also has monodromy group $\mathrm{PSL}_2(29)$, acting on the cosets of the unique subgroup $H^* = H^2 \cong \mathrm{C}_{29} \rtimes \mathrm{C}_7$ of index 2 in a natural point stabiliser $H \cong \mathrm{C}_{29} \rtimes \mathrm{C}_{14}$.

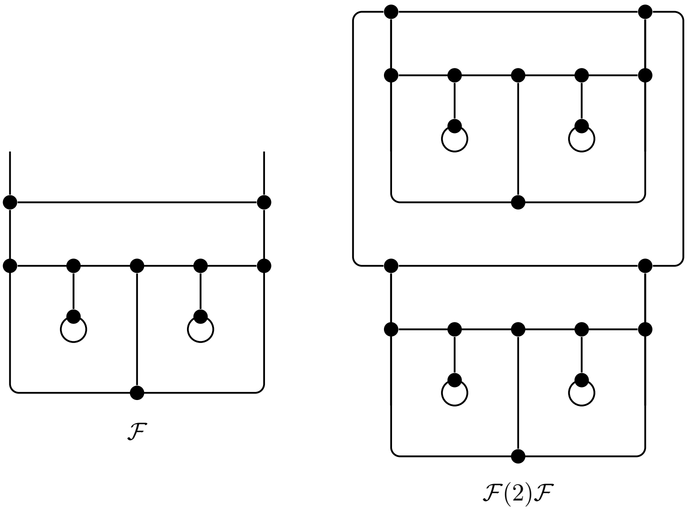


Fig. 16 $\mathcal{F}$ and $\mathcal{F}(2)\mathcal{F}$

However, Lemma 4.5 does not apply to the dessin $\mathcal{T}$ with monodromy group $\mathrm{PSL}_2(41)$, since this has genus 1. Indeed, the dessin $\mathcal{T}^* = \mathcal{T}(3)\mathcal{T}$, which has genus 2 and has no fixed points for y, cannot be that corresponding to the unique subgroup $H^2 \cong \mathrm{C}_{29} \rtimes \mathrm{C}_{10}$ of index 2 in $H \cong \mathrm{C}_{29} \rtimes \mathrm{C}_{20}$, since the latter dessin is an unbranched covering of $\mathcal{T}$, of genus 1, with four fixed points for y forming two (3)-handles.

Theorem 4.6 shows that although the number β of fixed points of y is invariant under the absolute Galois group $\mathbb{G} = \mathrm{Gal}\,\overline{\mathbb{Q}}/\mathbb{Q}$ (as indeed are the cycle structures of x, y and z for any dessin, see [14]), the property of having a (k)-handle is not invariant. As Macbeath showed in Theorem 2.3, for each prime $p \equiv \pm 1 \bmod (7)$ there are three Hurwitz dessins with the natural representation of $\mathrm{PSL}_2(p)$ as their monodromy group, distinguished by the values of $\pm\mathrm{tr}(z)$. In each case, as shown by Streit [25] they form an orbit of $\mathbb{G}$, but for $p = 13$, 29 and 41 only one of them has a handle, by Theorem 4.6. For the other two dessins in each orbit the two fixed points of y lie in different faces, as shown in Fig. 17 for $p = 13$.

Corollary 4.7 *Suppose that a Hurwitz dessin $\mathcal{D}$ has a monodromy group $G = \mathrm{PSL}_2(q)$ in a representation which covers the natural representation. Then $\mathcal{D}$ has a (k)-handle if and only if*

- *one of the three conclusions of Theorem 4.6 holds, and*
- *the point stabilisers for $\mathcal{D}$ have even order.*

Proof If $\mathcal{D}$ has a (k)-handle, then by Lemma 4.1 so does its quotient dessin $\overline{\mathcal{D}}$ corresponding to the natural representation of G, so one of the three conclusions of Theorem 4.6 applies, with $q = p = 13$, 29 or 41. The point stabilisers for $\mathcal{D}$ are

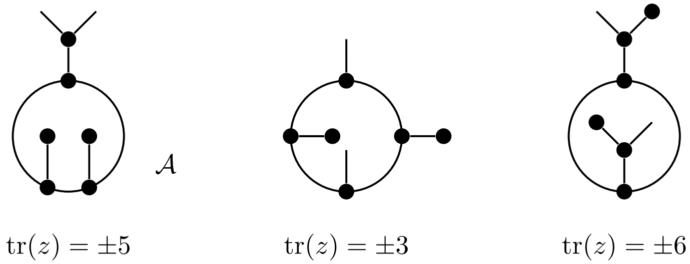


Fig. 17 The Galois orbit containing $\mathcal{A}$

subgroups H_0 of natural point stabilisers $H \cong C_p \rtimes C_{(p-1)/2}$ in G. If H_0 has odd order then $\mathcal{D}$ has no fixed points for y and hence no handles, whereas if H_0 has odd index in H, then the handle in $\overline{\mathcal{D}}$ must lift to at least one handle in $\mathcal{D}$. This deals with the cases $q = 13$ or 29, since every subgroup of H has odd order or odd index. In the case $q = 41$ there are also subgroups H_0 of even order and even index; however, these are all subgroups of odd index in H^2, which correspoinds to a dessin with two handles, so their corresponding dessins also have handles. □

In each of the three cases in Theorem 4.6 the natural point stabilisers H are Frobenius groups $H \cong C_p \rtimes C_{(p-1)/2}$. The subgroups H_0 of H therefore consist of one subgroup $C_p \rtimes C_f$ and a conjugacy class of p subgroups C_f for each factor f of $(p-1)/2$. By Corollary 4.7, the subgroups corresponding to dessins with y-handles are those for which f is even. When $p = 13$, 29 or 41 these values of f are 2 and 6, or 2 and 14, or 2, 4, 10 and 20. Of course, we need to exclude $H_0 = H$ in order to obtain non-identity coverings.

Example When $q = p = 13$ in Corollary 4.7 the point stabilisers $H_0 < H$ giving y-handles consist of a normal subgroup $C_{13} \rtimes C_2 \cong D_{13}$, and two conjugacy classes of 13 subgroups C_2 and C_6. The first case corresponds to the 3-sheeted regular covering $\mathcal{G}$ of $\mathcal{A}$ of degree 42 shown in Fig. 13, branched over the two fixed points of x and considered later in Sect. 5.2 as an example of an x-join. In this case, and also when $H_0 = C_2$, the numbers α, β and γ of fixed points of x, y and z are 0, 6 and 0, with three (1)-handles; the dessins have genus $g = 0$ and 6 respectively. When $H_0 = C_6$ we have $\alpha = \beta = 2$ and $\gamma = 0$; the dessin has genus 2, and there is one (1)-handle. The last two dessins, of degrees 546 and 182, are too large for us to draw. When $q = p = 29$ the subgroups $H_0 < H$ providing handles consist of a normal subgroup $C_{29} \rtimes C_2 \cong D_{29}$, and two conjugacy classes of 29 subgroups C_2 and C_{14}. The first corresponds to a 7-sheeted regular covering of $\mathcal{F}$ of degree 210, branched over the two fixed points of z. Here, and also when $H_0 = C_2$, we have $\alpha = \gamma = 0$ and $\beta = 14$, with seven (2)-handles, but with $g = 0$ and 70 respectively. When $H_0 = C_{14}$ we have $\alpha = 0$ and $\beta = \gamma = 2$, so there is one (2)-handle and the genus is 10. When $q = p = 41$ the relevant subgroups $H_0 < H$ are three normal subgroups $C_{41} \rtimes C_f$ for $f = 2$, 4 or 10, and four conjugacy classes of 41 subgroups C_f for $f = 2, 4, 10$ or 20. The

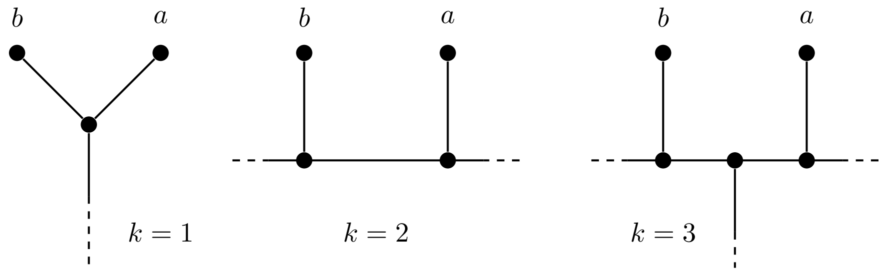


Fig. 18 (k)-handles for x, with $k = 1, 2, 3$

corresponding dessins have $\alpha = \gamma = 0$ and $\beta = 40/f$, with $20/f$ (3)-handles. The normal subgroups correspond to $20/f$-sheeted unbranched regular coverings of the dessin in Fig. 15, of genus 1. The dessins corresponding to subgroups C_f have genus $1 + 400/f$.

Of course, this leaves open the question of what handles can appear in other representations of Hurwitz groups $PSL_2(q)$. As above, Lemma 4.1 shows that it is sufficient to classify those handles arising from primitive representations, where the point-stabilisers are maximal subgroups.

5 *x*-Joins

Instead of using fixed points of y, one can also define handles, and a similar joining operation, based on fixed points a and b of the standard generator x of order 3. In this case, we will say that they form a (k)*-handle for* x if $b = a(yx)^k y$. We may assume that $k = 1, 2$ or 3, as illustrated in Fig. 18.

If $k = 1$, it is easy to see that the dessin is unique: it must be as in Fig. 19, shown with the fixed points in the outer or inner face, since no further vertices, edges or faces can be added. This dessin, of degree 8 and genus 0, corresponds to the natural action of $PSL_2(7)$ on the projective line over $\mathbb{F}_7$, with point stabiliser $H \cong C_7 \rtimes C_3$.

Figure 20 shows that the cases $k = 2$ and $k = 3$ are equivalent: a (2)-handle for x must be as shown on the left, so by transposing the roles of a and b it can also be regarded as a (3)-handle as on the right, and vice versa. We will therefore always assume that $k = 1$, as in Fig. 19, or $k = 2$; we will call these trivial and non-trivial x-handles respectively.

If Hurwitz dessins $\mathcal{D}_i$ $(i = 1, 2, 3)$ have (k)-handles (a_i, b_i) for x, with the same value of k, we can form an *x-join* $\mathcal{D}_1(x)\mathcal{D}_2(x)\mathcal{D}_3$, by defining the cycles of x and y to be those they have on $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$, except that the six fixed points a_i, b_i of x become two 3-cycles $a = (a_1, a_2, a_3)$ and $b = (b_3, b_2, b_1)$ (note the reverse cyclic ordering of subscripts). The result is a connected Hurwitz dessin, and again this can be regarded as a connected sum operation, joining surfaces across cuts.

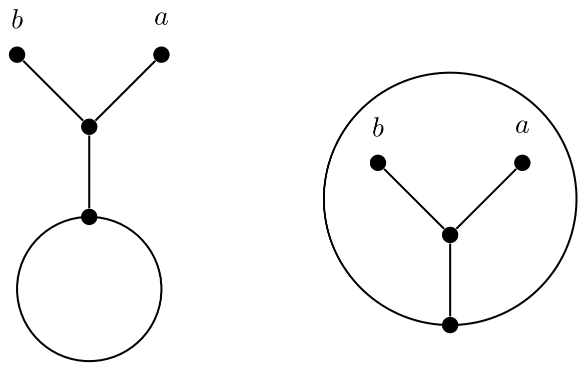


Fig. 19 The unique dessin with a (1)-handle for x

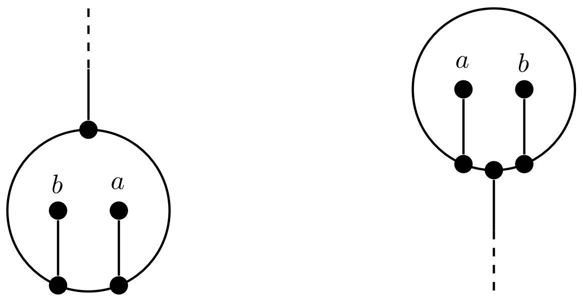


Fig. 20 Equivalence of (2)- and (3)-handles for x

The analogue of Theorem 4.3 for x-joins is as follows:

Theorem 5.1 *If Hurwitz dessins $\mathcal{D}_i$ $(i = 1, 2, 3)$ of degree n_i and signature $(g_i; 3^{[\alpha_i]}, 2^{[\beta_i]}, 7^{[\gamma_i]})$ have an x-join $\mathcal{D}$, then $\mathcal{D}$ has degree n and signature $(g; 3^{[\alpha]}, 2^{[\beta]}, 7^{[\gamma]})$ where*

$$n = \sum n_i,\ g = \sum g_i,\ \alpha = \sum \alpha_i - 6,\ \beta = \sum \beta_i \text{ and } \gamma = \sum \gamma_i.$$

Proof The proof is similar to that for Theorem 4.3. □

In the trivial case $k = 1$, the only possible x-join involves three copies $\mathcal{D}_i$ of the dessin $\mathcal{D}$ of degree 8 in Fig. 19. The resulting dessin $\mathcal{D}^* = \mathcal{D}(x)\mathcal{D}(x)\mathcal{D}$ is shown in Fig. 21. It has degree 24 and genus 0, and is a 3-sheeted regular covering of $\mathcal{D}$. The monodromy group of $\mathcal{D}^*$ is described in the Example in Sect. 5.1. Since neither x nor y has fixed points, this dessin cannot be used to form further joins.

In the more general case $k = 2$, a typical x-join is shown in Fig. 22, with the handles of each dessin drawn in the outer face. If we restrict attention to the faces containing the points a_i and b_i, we have a 3-sheeted covering, branched over the points a and b. Whereas in the case of y-joins we had the same monodromy permutation, a 2-cycle, at each of the two the branch-points, here we use two mutually inverse 3-cycles; the important point is that in both cases the product of the two monodromy

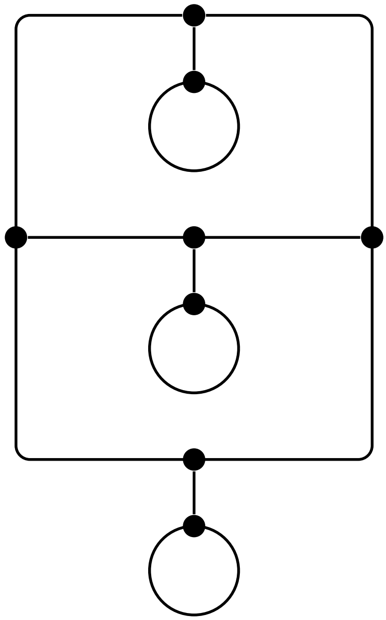


Fig. 21 The trivial x-join, with $k = 1$

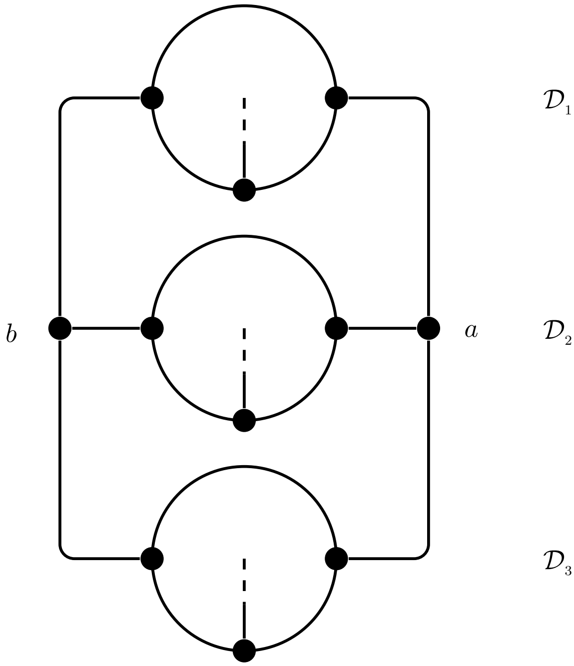


Fig. 22 The general dessin $\mathcal{D}_1(x)\mathcal{D}_2(x)\mathcal{D}_3$, with $k = 2$

permutations is the identity. The resulting dessin depends only on the chosen cyclic order of the three dessins $\mathcal{D}_i$; reversing this gives the mirror image of $\overline{\mathcal{D}}_1(x)\overline{\mathcal{D}}_2(x)\overline{\mathcal{D}}_3$, where $\overline{\mathcal{D}}_i$ denotes the mirror image of $\mathcal{D}_i$. It follows that if each $\mathcal{D}_i$ is reflexible the result is completely independent of the order of the three dessins. If the three dessins $\mathcal{D}_i$ are mutually isomorphic, by isomorphisms which match up the selected handles, then $\mathcal{D}_1(x)\mathcal{D}_2(x)\mathcal{D}_3$ has an automorphism of order 3, permuting the dessins $\mathcal{D}_i$ cyclically and fixing a and b.

5.1 The Triple Cover $\mathcal{D}(x)\mathcal{D}(x)\mathcal{D}$

If a dessin $\mathcal{D}$ has an x-handle, one can form the join $\mathcal{D}^* = \mathcal{D}(x)\mathcal{D}(x)\mathcal{D}$, a triple cover of $\mathcal{D}$ branched over two points, corresponding to the fixed points a and b of x in the chosen handle. If $\mathcal{D}$ has degree n and signature $(g; 3^{[\alpha]}, 2^{[\beta]}, 7^{[\gamma]})$, then $\mathcal{D}^*$ has degree $3n$ and signature $(3g; \alpha^{[3\alpha-6]}, 3^{[3\beta]}, 7^{[3\gamma]})$.

The covering $\mathcal{D}^* \to \mathcal{D}$ is regular, induced by the obvious automorphism of order 3 of $\mathcal{D}^*$, so M^* is a normal subgroup of M. (It is, in fact, the normal closure in M of all its standard generators apart from the two generators X_j corresponding to the fixed points of X in the chosen handle.) Thus M^* contains the subgroup $M'M^3$ generated by the commutators and cubes of the elements of M. Now the signature of M shows that $M/M'M^3$ is an elementary abelian 3-group of rank $r = 2g + \alpha - 1$, so M has $(3^r - 1)/2$ subgroups of index 3, each corresponding to a regular triple covering of $\mathcal{D}$. This leads to the following analogue of Lemma 4.5:

Lemma 5.2 *Suppose that a planar Hurwitz dessin $\mathcal{D}$ has exactly two fixed points for X, forming a (k)-handle; then $\mathcal{D}^* = \mathcal{D}(x)\mathcal{D}(x)\mathcal{D}$ is the unique Hurwitz dessin which is a triple covering of $\mathcal{D}$. If, in addition, the point stabiliser H in the monodromy group G of $\mathcal{D}$ has a subgroup H^* of index 3, then $\mathcal{D}^*$ also has monodromy group G, with point stabiliser H^*.* □

Example This applies to the dessin $\mathcal{D}$ in Fig. 19, showing that the monodromy group of $\mathcal{D}^*$ (shown in Fig. 21) is $\mathrm{PSL}_2(7)$, like that of $\mathcal{D}$, but now acting on the cosets of a Sylow 7-subgroup $H^* \cong \mathrm{C}_7$.

However, if $g > 0$ or $\alpha > 2$ then $\mathcal{D}^*$ is one of several triple coverings of $\mathcal{D}$, and its imprimitive monodromy group could be isomorphic to G or a proper covering group of G contained in $\mathrm{S}_3 \wr G$. The latter must happen if H has no subgroup of index 3, for example if G is A_n acting naturally.

5.2 A Non-trivial Example of an x-Join

The triple cover $\mathcal{A}^* = \mathcal{A}(x)\mathcal{A}(x)\mathcal{A}$ of the dessin $\mathcal{A}$ of degree 14 in Figs. 1 and 3 is a Hurwitz dessin of degree 42 and genus 0. This is the dessin $\mathcal{G}$ based on Conder's diagram G in [1] and shown in Fig. 13. It is a 3-sheeted regular covering of $\mathcal{A}$, branched over the two points corresponding to the fixed points a and b of x, so that $\mathcal{A} \cong \mathcal{G}/\mathrm{C}_3$ where the group C_3 of automorphisms of $\mathcal{G}$ permutes the three copies of $\mathcal{A}$ cyclically. It follows from Lemma 5.2 that, like $\mathcal{A}$, this dessin $\mathcal{G}$ also has monodromy group $\mathrm{PSL}_2(13)$, but now in its imprimitive representation of degree 42 on the cosets of a dihedral subgroup $H^* \cong \mathrm{D}_{13} = \mathrm{C}_{13} \rtimes \mathrm{C}_2$.

This example illustrates an important general point about x-joins, analogous to a point made earlier about y-joins (see Fig. 7 and the accompanying text). In Fig. 13 there appear to be several different ways of joining three copies of the dessin $\mathcal{A}$ together: some of them could first be rotated through a half-turn, or equivalently reflected in a horizontal axis, so that the 'Y' containing the y-handle is inverted. However, unless we do this to all three copies, this would break the rule that vertices a_i in the x-handles must be identified with each other to form a new vertex a, with the same applying to the vertices b_i: there cannot be any 'mixing' of vertices a_i and b_i. Equivalently, there must be combinatorial isomorphisms between the oriented faces containing the x-handles in the three dessins, so that xy still has order 7 after the joining operation. Incorrect identifications here could result in faces of valency other than 1 or 7, giving a transitive representation of $\Delta(3, 2, \infty)$, isomorphic to the modular group $\mathrm{PSL}_2(\mathbb{Z})$, rather than Δ (see Sect. 6).

5.3 x-Handles from $\mathbf{PSL_2}(q)$

The following is the analogue of Theorem 4.6 for x-handles:

Theorem 5.3 *A dessin corresponding to the natural representation of the group* $G = \mathrm{PSL}_2(q)$ *has an* x*-handle if and only if* $q = 13$*, with* $\mathrm{tr}(z) = \pm 5$.

Proof If an x-handle exists then x has two fixed points, so it must be an elliptic element, of order dividing $(q-1)/2$ where $q \equiv 1 \bmod (3)$. By applying an automorphism of G we may assume that x has the form $t \mapsto \omega t$, fixing $a = 0$ and $b = \infty$, where $\omega^2 + \omega + 1 = 0$, or equivalently, that x corresponds to the pair of elements

$$\pm \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}$$

of $\mathrm{SL}_2(q)$. An element of G has order 2 if and only if it has trace 0, so the most general form for y is

$$\pm \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

where $a^2 + bc + 1 = 0$. It then follows that xy has trace

$$\pm t = \pm a(\omega^2 - \omega) = \pm a\sqrt{-3}.$$

As before, $t = \pm\tau_j$ where $\tau_j = \lambda^j + \lambda^{-j}$ where λ is a primitive 7th root of 1, and the elements $\pm\tau_j$ are respectively the roots of the polynomials

$$p_{\pm}(t) := t(t^2 - 2) \pm (t^2 - 1).$$

Now 0 and ∞ form an x-handle if and only if $yxyxy : 0 \mapsto \infty$. Writing x and y as above we find that this is equivalent to

$$\frac{abc(\omega - 1)}{bc + a^2\omega^2} = a.$$

Now $a \neq 0$ (since otherwise $\langle x, y\rangle \cong \mathrm{S}_3$), so this simplifies to

$$3a^2 = \omega - 2.$$

Thus $t^2 = -3a^2 = 2 - \omega$, so

$$0 = -p_{\pm}(t) = t(t^2 - 2) \pm (t^2 - 1) = -t\omega \pm (1 - \omega)$$

and hence

$$t^2\omega^2 = (1 - \omega)^2.$$

This simplifies to $\omega = 3$, so that $3^2 + 3 + 1 = 0$. Thus $p = 13$, so $q = 13$ with $t = \pm 5$ and $a = \pm 3$. □

The dessin characterised by this theorem is $\mathcal{A}$, shown in Figs. 1 and 3.

Corollary 5.4 *Suppose that a Hurwitz dessin* $\mathcal{D}$ *has a monodromy group* $G = \mathrm{PSL}_2(q)$ *in a representation which covers the natural representation. Then* $\mathcal{D}$ *has an* x*-handle if and only if*

- $q = 13$, *with* $\mathrm{tr}(z) = \pm 5$, *and*
- *the point stabilisers have order divisible by* 3.

Proof The proof is similar to that of Corollary 4.7 for y-handles, with the prime 3 replacing 2. □

The possible point-stabilisers in $\mathrm{PSL}_2(13)$ satisfying these conditions are therefore those isomorphic to $\mathrm{C}_{13} \rtimes \mathrm{C}_6$, $\mathrm{C}_{13} \rtimes \mathrm{C}_3$, C_6, or C_3. The first two correspond to the planar dessins $\mathcal{A}$ and $\mathcal{A}(1)\mathcal{A}$ considered earlier (see Figs. 1, 3 and 10); these have one x-handle and two, respectively, as have the dessins corresponding to the last two, which are 13-sheeted branched coverings of these, of genus 2 and 4.

As with y-handles, this leaves open the question of which other representations of Hurwitz groups $\mathrm{PSL}_2(q)$ provide x-handles. As before, it is sufficient to consider those handles arising from primitive representations.

6 Handles and Joins for the Modular Group

In [2], Conder extended the technique of joining coset diagrams which he and Higman had developed for $\Delta(3, 2, 7)$ to the triangle groups $\Delta(3, 2, r)$ for all integers $r \geq 7$, obtaining similar results to those in the case $r = 7$.

One can obtain a further extension to the modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z}) = \Delta(3, 2, \infty) \cong \mathrm{C}_3 * \mathrm{C}_2$, by omitting the relation $Z^7 = 1$, so that faces of any valency are allowed, while vertices all have valency 3 or 1. In this case, any ordered pair of fixed points of x or of y in the same face form an x- or y-handle, so that three or two such dessins with handles can be joined as described earlier to produce another, but this time with no requirement that the handles should be in isomorphic faces.

In any such trivalent dessin, the order of z is the level of the corresponding subgroup M of Γ; in the case of a congruence subgroup, this coincides with the level as defined number-theoretically, by a result of Wohlfahrt [27].

As an example of dessins rising from congruence subgroups, there is a natural action of Γ by Möbius transformations on $\mathbb{P}^1(\mathbb{Q})$. For any prime p, reducing this action mod (p) gives the natural action of $\mathrm{PSL}_2(p)$ on $\mathbb{P}^1(\mathbb{F}_p)$. The generating set

$$X : t \mapsto \frac{1}{1-t}, \quad Y : t \mapsto \frac{-1}{t}, \quad Z : t \mapsto t+1$$

for Γ, with defining relations

$$X^3 = Y^2 = XYZ = 1,$$

reduces mod (p) to give a generating triple (x, y, z) of type $(3, 2, p)$ for $G(p) := \mathrm{PSL}_2(p)$, so the natural action of $G(p)$ on the projective line $\mathbb{P}^1(p)$, with stabilisers $H \cong \mathrm{C}_p \rtimes \mathrm{C}_{(p-1)/2}$ for odd p, gives a dessin $\mathcal{P}(p)$ of that type and of degree $p + 1$ with monodromy group $G(p)$. There are two faces, of valencies p and 1; there are two free edges (giving a y-handle) or none as $p \equiv \pm 1$ mod (4), and for $p > 3$ there are two 1-valent vertices (giving an x-handle) or none as $p \equiv \pm 1$ mod (3).

Example By Dirichlet's theorem there are infinitely many primes $p \equiv 7$ mod (12). For such primes, $\mathcal{P}(p)$ has $(p + 5)/3$ vertices (two of valency 1, corresponding to the fixed points of x, the rest of valency 3), $(p + 1)/2$ edges (none of them free), and two faces, so it has genus $g = (p - 7)/12$. (See Fig. 19 for $\mathcal{P}(7)$, and Fig. 23 for $\mathcal{P}(19)$, with opposite sides of the outer parallelogram identified to form a torus.) The two vertices of valency 1 both lie in the face of valency p, so they form a handle. Given three primes $p_i \equiv 7$ mod (12), the join $\mathcal{D} = \mathcal{P}(p_1)(x)\mathcal{P}(p_2)(x)\mathcal{P}(p_3)$ has degree $s + 3$, where $s = p_1 + p_2 + p_3$, and has genus $(s - 21)/12$.

If we take each $p_i = p$ we obtain a regular triple cover $\mathcal{P}(p)^*$ of $\mathcal{P}(p)$, with $G(p)$ as its monodromy group, now acting on the cosets of a subgroup $H^* \cong \mathrm{C}_p \rtimes \mathrm{C}_{(p-1)/6}$. However, if the primes p_i are not all equal, then the monodromy group can be very different from the groups $G(p_i)$. For instance, the torus dessin

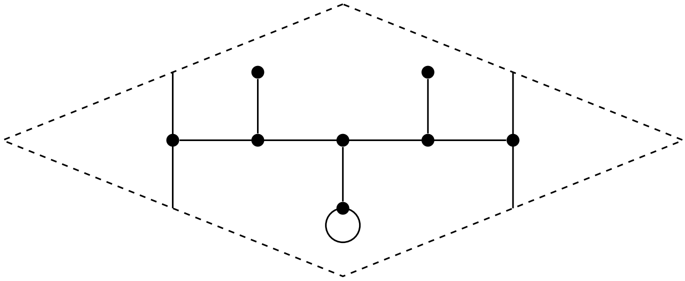


Fig. 23 The dessin $\mathcal{P}(19)$

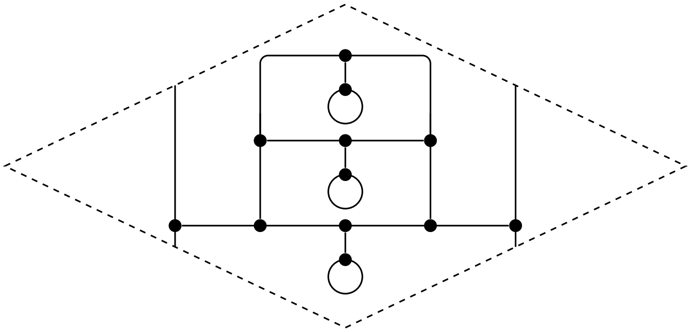


Fig. 24 The dessin $\mathcal{P}(7)(x)\mathcal{P}(7)(x)\mathcal{P}(19)$

$\mathcal{P}(7)(x)\mathcal{P}(7)(x)\mathcal{P}(19)$ shown in Fig. 24 has monodromy group $G \cong \mathrm{A}_{36}$; this fact, confirmed by GAP, can be proved as follows, using the cycle structure $1^{[3]}, 7, 9, 17$ of z.

If G is imprimitive it has a blocks of size b, where $ab = 36$ and $a, b > 1$, so it can be embedded in the wreath product $\mathrm{S}_b \wr \mathrm{S}_a$. This has order $(b!)^a a!$ with $a, b \leq 18$, so we require $\{a, b\} = \{18, 2\}$ for it to have order divisible by 17. Since G is generated by elements x and z of odd order it cannot have two blocks, so $a = 18$ and $b = 2$. However, if the group $\mathrm{S}_2 \wr \mathrm{S}_{18} = (\mathrm{S}_2)^{18} \rtimes \mathrm{S}_{18}$ has an element of order 7.9.17 then so has its quotient S_{18}; this is clearly false, so G is primitive. Applying Jordan's Theorem (see Sect. 2.2) to the 7-cycle z^{153} gives $G \cong \mathrm{A}_{36}$ or S_{36}, and as before the latter is impossible since $G = \langle x, z \rangle$.

7 Handles and Joins for Dessins of All Types

Here we briefly sketch how to define handles and joins for dessins of any type (p, q, r). It is sufficient to define handles and joins only for x, generalising the construction given earlier for them in the case $p = 3, q = 2$.

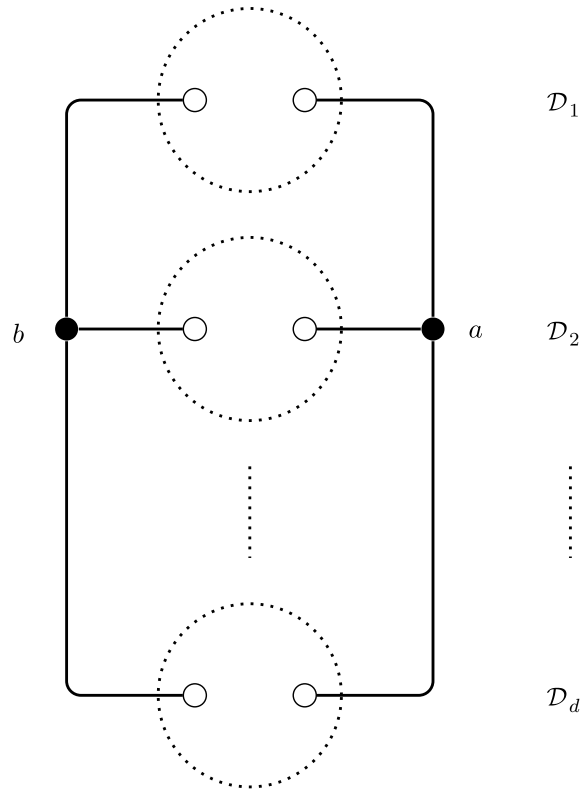


Fig. 25 The x-join $\mathcal{D}_1(x)\mathcal{D}_2(x)\cdots(x)\mathcal{D}_d$

We will now represent a dessin $\mathcal{D}$ in the usual way as a bipartite map, with x and y rotating edges around their incident black and white vertices, and with faces corresponding to the cycles of z. (Up to now, with $q = 2$, we have omitted the white vertices, but now, with arbitrary q, we need them.) As before, a (k)-handle for x is a pair a, b of its fixed points with $b = a(yx)^k y$, so that the corresponding 1-valent black vertices lie in the same face.

Let dessins $\mathcal{D}_i$ $(i = 1, 2, \ldots, d)$ of type (p, q, r) have (k)-handles (a_i, b_i) for x, with the same k, where d divides p. An *x-join* $\mathcal{D}_1(x)\mathcal{D}_2(x)\cdots(x)\mathcal{D}_d$ is formed by defining the cycles of x and y to be those they have in the dessins $\mathcal{D}_i$, except that the fixed points a_i, b_i of x become two cycles $a = (a_1, a_2, \ldots, a_d)$ and $b = (b_d, b_{d-1} \ldots, b_1)$. The result is a dessin $\mathcal{D}$ of type (p, q, r), formed by joining the dessins $\mathcal{D}_i$ across cuts (see Fig. 25). As before, this operation is additive in the degrees and genera of the dessins $\mathcal{D}_i$. The elliptic periods in the signature of $\mathcal{D}$ are those for the dessins $\mathcal{D}_i$, except that $2d$ periods p corresponding to the fixed points a_i, b_i are replaced with two periods p/d corresponding to a and b (these can be omitted if $d = p$).

A similar joining operation with $k = 1$ was used by Everitt in [5] for coset diagrams for the Dyck groups (cocompact groups of genus 0, that is, of signature

(0; $m_1, \ldots, m_r$) for integers $m_i \geq 2$). This was an important ingredient in his proof that each non-elementary Fuchsian group Γ has almost all alternating groups among its quotients. It is hoped to use a combination of these joining operations to extend the results in [11, 12] on maximal subgroups from certain triangle groups to all such groups Γ.

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Appendix: The Dessins Based on Conder's Diagrams

In [1], Conder used 14 coset diagrams $A, \ldots, N$ for subgroups of finite index n in Δ. As we have explained, his diagrams can be interpreted as planar Hurwitz dessins $\mathcal{A}, \ldots, \mathcal{N}$ of degree n. We have already used the dessins $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{G}$ as examples of Hurwitz dessins: see Fig. 3 for $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, and Figs. 14 and 13 for $\mathcal{F}$ and $\mathcal{G}$; for the other dessins, see [13].

In Table 1 we give the degree n and the monodromy group G (in most cases obtained by using GAP) for each dessin. When G is A_n or $PSL_2(n-1)$, it always acts naturally; such an action is primitive, so the automorphism group of the dessin is trivial. For $\mathcal{C}$, $\mathcal{E}$ and $\mathcal{G}$, where G does not act naturally, we give a point-stabiliser H in the form $G > H$. The automorphism group is trivial for $\mathcal{C}$ and $\mathcal{E}$, and isomorphic to C_3

Table 1 Dessins $\mathcal{A}, \ldots, \mathcal{N}$ based on Conder's diagrams $A, \ldots, N$

Dessin	Degree n	Monodromy group G	Number of (k)-handles	α, β, γ
$\mathcal{A}$	14	$PSL_2(13)$	1, 0, 0	2, 2, 0
$\mathcal{B}$	15	A_{15}	0, 1, 1	0, 3, 1
$\mathcal{C}$	21	$PGL_3(2) > D_4$	1, 0, 1	0, 5, 0
$\mathcal{D}$	22	A_{22}	0, 1, 0	1, 2, 1
$\mathcal{E}$	28	$PSL_2(8) > D_9$	1, 1, 0	1, 4, 0
$\mathcal{F}$	30	$PSL_2(29)$	0, 1, 0	0, 2, 2
$\mathcal{G}$	42	$PSL_2(13) > D_{13}$	3, 0, 0	0, 6, 0
$\mathcal{H}$	42	A_{42}	1, 0, 1	0, 6, 0
$\mathcal{I}$	57	A_{57}	0, 2, 0	0, 5, 1
$\mathcal{J}$	72	$(S_2 \wr A_{36}) \cap A_{72}$	2, 0, 0	0, 4, 2
$\mathcal{K}$	72	A_{72}	1, 0, 0	0, 4, 2
$\mathcal{L}$	102	A_{102}	0, 1, 0	0, 2, 4
$\mathcal{M}$	108	A_{108}	1, 1, 0	0, 4, 3
$\mathcal{N}$	108	A_{108}	1, 0, 1	0, 4, 3

and C_2 for $\mathcal{G}$ and $\mathcal{J}$. We also give the number of disjoint (k)-handles for $k = 1, 2, 3$ and the numbers α, β and γ of fixed points of x, y and z, which determine the signature of the dessin (they all have genus 0).

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Arithmetic Chern–Simons Theory I

Minhyong Kim

N.B. This paper has some overlap with [4], but was written before. The decision to submit it for publication now comes from the wish to set down some earlier motivation for the constructions of that paper as well as of [3].

Abstract In this paper, we apply ideas of Dijkgraaf and Witten [7, 27] on 2+1 dimensional topological quantum field theory to arithmetic curves, that is, the spectra of rings of integers in algebraic number fields. In the first three sections, we define classical Chern–Simons functionals on spaces of Galois representations. In the highly speculative Sect. 6, we consider the far-fetched possibility of using Chern–Simons theory to construct L-functions.

Keywords Topological quantum field theory · Number theory

1991 Mathematics Subject Classification 14G10 · 11G40 · 81T45

1 The Arithmetic Chern–Simons Action: Basic Case

We wish to move rather quickly to a concrete definition in this first section. The reader is directed to Sect. 5 for a motivational discussion of L-functions.

Let $X = \mathrm{Spec}(\mathcal{O}_F)$, the spectrum of the ring of integers in a number field F. We assume that F is totally imaginary, for simplicity of exposition. Denote by $\mathbb{G}_m$ the

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étale sheaf that associates to a scheme the units in the global sections of its coordinate ring. We have the following canonical isomorphism ([22], p. 538):

$$\mathrm{inv}: H^3(X, \mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z}. \qquad (*)$$

This map is deduced from the 'invariant' map of local class field theory. We will use the same name for a range of isomorphisms having the same essential nature, for example,

$$\mathrm{inv}: H^3(X, \mathbb{Z}_p(1)) \simeq \mathbb{Z}_p, \qquad (**)$$

where $\mathbb{Z}_p(1) = \varprojlim_i \mu_{p^i}$, and $\mu_n \subset \mathbb{G}_m$ is the sheaf of nth roots of 1. This follows from the exact sequence

$$0 \to \mu_n \to \mathbb{G}_m \stackrel{(\cdot)^n}{\to} \mathbb{G}_m \to \mathbb{G}_m/(\mathbb{G}_m)^n \to 0.$$

That is, according to loc. cit.,

$$H^2(X, \mathbb{G}_m) = 0,$$

while by op. cit., p. 551, we have

$$H^i(X, \mathbb{G}_m/(\mathbb{G}_m)^n) = 0$$

for $i \geq 1$. If we break up the above into two short exact sequences,

$$0 \to \mu_n \to \mathbb{G}_m \stackrel{(\cdot)^n}{\to} \mathcal{K}_n \to 0,$$

and

$$0 \to \mathcal{K}_n \to \mathbb{G}_m \to \mathbb{G}_m/(\mathbb{G}_m)^n \to 0,$$

we deduce

$$H^2(X, \mathcal{K}_n) = 0,$$

from which it follows that

$$H^3(X, \mu_n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z},$$

the n-torsion inside $\mathbb{Q}/\mathbb{Z}$. Taking the inverse limit over $n = p^i$ gives the second isomorphism above. The pro-sheaf $\mathbb{Z}_p(1)$ is a very familiar coefficient system for étale cohomology and (**) is reminiscent of the fundamental class of a compact oriented three manifold for singular cohomology. Such an analogy was noted by Mazur around 50 years ago [23] and has been developed rather systematically by a number of mathematicians, notably, Masanori Morishita [24]. Within this circle of ideas is included the analogy between knots and primes, whereby the map

$$\mathrm{Spec}(\mathcal{O}_F/\mathfrak{P}_v) \rightarrowtail X$$

from the residue field of a prime $\mathfrak{P}_v$ should be similar to the inclusion of a knot. Let F_v be the completion of F at the place v and $\mathcal{O}_{F_v}$ its valuation ring. If one takes this analogy seriously (as did Morishita), the map

$$\mathrm{Spec}(\mathcal{O}_{F_v}) \to X,$$

should be similar to the inclusion of a handle-body around the knot, whereas

$$\mathrm{Spec}(F_v) \to X$$

resembles the inclusion of its boundary torus.[1] Given a finite set S of primes, we can look at the scheme

$$X_S := \mathrm{Spec}(\mathcal{O}_F[1/S]) = X \setminus \{\mathfrak{P}_v\}_{v \in S}.$$

Since a link complement is homotopic to the complement of a tubular neighbourhood, the analogy is then forced on us between X_S and a three manifold with boundary given by a union of tori, one for each 'knot' in S. These of course are basic morphisms in 2+1 dimensional topological quantum field theory [1]. From this perspective, perhaps the coefficient system $\mathbb{G}_m$ of the first isomorphism should have reminded us of the S^1-coefficient important in Chern–Simons theory [7, 27]. A more direct analogue of $\mathbb{G}_m$ is the sheaf $\mathcal{O}_M^\times$ of invertible analytic functions on a complex variety M. However, for compact Kaehler manifolds, the comparison isomorphism

$$H^1(M, S^1) \simeq H^1(M, \mathcal{O}_M^\times)_0,$$

where the subscript refers to the line bundles with trivial Chern class, is a consequence of Hodge theory. This indicates that in the étale setting with no natural constant sheaf of S^1's, the familiar $\mathbb{G}_m$ has a topological nature, and can be regarded as a substitute.[2] One problem, however, is that the $\mathbb{G}_m$-coefficient computed directly gives divisible torsion cohomology, whence the need for considering coefficients like $\mathbb{Z}_p(1)$ in order to get functions of geometric objects having an analytic nature as arise, for example, in the theory of torsors for motivic fundamental groups [5, 15–18].

Let

$$\pi = \pi_1(X, b),$$

[1] It is not clear to me that the topology of the boundary should really be a torus. (In fact, M. Kapranov has remarked that it is closer to a Klein bottle.) A torus boundary is reasonable if one thinks of the ambient space as a three-manifold. On the other hand, perhaps it's possible to have a notion of a knot in a *homology three-manifold* that has an exotic tubular neighbourhood?

[2] Recall, however, that it is of significance in Chern–Simons theory that one side of this isomorphism is purely topological while the other has an analytic structure.

the profinite étale fundamental group of X, where we take

$$b : \mathrm{Spec}(\bar{F}) \to X$$

to be the geometric point coming from an algebraic closure of F. Assume now that the group $\mu_n(\bar{F})$ of nth roots of 1 is in F. Fix an isomorphism $\zeta_n : \frac{1}{n}\mathbb{Z}/\mathbb{Z} \simeq \mu_n$. Then

$$\mathrm{inv} : H^3(X, \mathbb{Z}/n\mathbb{Z}) \simeq H^3(X, \mu_n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

Now let A be a finite group and fix a class $c \in H^3(A, \mathbb{Z}/n\mathbb{Z})$. Let

$$\mathcal{M}(A) := \mathrm{Hom}_{cont}(\pi, A)/A$$

be the set of isomorphism classes of principal A-bundles over X. Here, the subscript refers to continuous homomorphisms, on which A is acting by conjugation. For $[\rho] \in \mathcal{M}(A)$, we get a class

$$\rho^*(c) \in H^3(\pi, \mathbb{Z}/n\mathbb{Z})$$

that depends only on the isomorphism class $[\rho]$. Denoting by inv also the composed map

$$H^3(\pi, \mathbb{Z}/n\mathbb{Z}) \to H^3(X, \mathbb{Z}/n\mathbb{Z}) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

We get thereby a function

$$CS_c : \mathcal{M}(A) \to \frac{1}{n}\mathbb{Z}/\mathbb{Z};$$

$$[\rho] \mapsto \mathrm{inv}(\rho^*(c)).$$

This is the basic and easy case of the classical Chern–Simons functional in the arithmetic setting.

Examples might be constructed along the following lines. Let $A = \mathbb{Z}/n\mathbb{Z}$, $\alpha \in H^1(A, \mathbb{Z}/n\mathbb{Z})$ the class of the identity, and $\beta \in H^2(A, \mathbb{Z}/n\mathbb{Z})$ the class of the extension

$$0 \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{n} \mathbb{Z}/n^2\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

Then $\beta = \delta\alpha$, where $\delta : H^1(A, \mathbb{Z}/n\mathbb{Z}) \to H^2(A, \mathbb{Z}/n\mathbb{Z})$ is the boundary map arising from the extension. From the cohomology theory of finite cyclic groups ([26], I.7), we know that

$$(\cdot) \cup \beta : H^1(A, \mathbb{Z}/n\mathbb{Z}) \to H^3(A, \mathbb{Z}/n\mathbb{Z})$$

is an isomorphism. Put

$$c := \alpha \cup \beta = \alpha \cup \delta\alpha \in H^3(A, \mathbb{Z}/n\mathbb{Z}).$$

Then

$$CS_c([\rho]) = \text{inv}[\rho^*(\alpha) \cup \delta\rho^*(\alpha)],$$

in close analogy to the formulas of abelian Chern–Simons theory.

In [3], this formula is applied to the study of arithmetic linking numbers.

2 The Arithmetic Chern–Simons Action: Boundaries

Let n be a natural number and S a finite set of primes in $\mathcal{O}_F$. We assume in this section that all primes of F dividing n are in S. Let

$$\pi_S := \pi_1(X_S, b)$$

and

$$\pi_v = \text{Gal}(\bar{F}_v/F_v),$$

equipped with maps

$$i_v : \pi_v \to \pi_S$$

given by choices of embeddings $\bar{F} \rightarrowtail \bar{F}_v$. The collection

$$\{i_v\}_{v \in S}$$

will be denoted by i_S. Let

$$Y_S(A) := \text{Hom}_{cont}(\pi_S, A)$$

and denote by $\mathcal{M}_S(A)$ the action groupoid whose objects are the elements of $Y_S(A)$ with morphisms given by the conjugation action of A. We also have the local version

$$Y_S^{loc}(A) = \prod_{v \in S} \text{Hom}_{cont}(\pi_v, A)$$

as well as the action groupoid $\mathcal{M}_S^{loc}(A)$ with objects $Y_S^{loc}(A)$ and morphisms given by the action of $A^S := \prod_{v \in S} A$ conjugating the separate components in the obvious sense. Thus, we have the restriction functor

$$r_S : \mathcal{M}_S(A) \to \mathcal{M}_S^{loc}(A),$$

where a homomorphism $\rho : \pi_S \to A$ is restricted to the collection

$$i_S^*\rho := (\rho \circ i_v)_{v\in S}$$

and A is embedded diagonally in A^S.

We will now employ a cocycle $c \in Z^3(A, \mathbb{Z}/n\mathbb{Z})$ to associate a $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$-torsor to each point of $Y_{loc}^S(A)$ in an A^S-equivariant manner. This will be a finite arithmetic version of the Chern–Simons line bundle [9] over $\mathcal{M}_{loc}^S$. We use the notation

$$C_S^i := \prod_{v\in S} C^i(\pi_v, \mathbb{Z}/n\mathbb{Z})$$

for the continuous cochains,

$$Z_S^i := \prod_{v\in S} Z^i(\pi_v, \mathbb{Z}/n\mathbb{Z}) \subset C_S^i$$

for the cocycles, and

$$B_S^i := \prod_{v\in S} B^i(\pi_v, \mathbb{Z}/n\mathbb{Z}) \subset Z_S^i \subset C_S^i$$

for the coboundaries. In particular, we have the coboundary map (see Appendix A for the sign convention)

$$d : C_S^2 \to Z_S^3.$$

Let $\rho_S := (\rho_v)_{v\in S} \in Y_S^{loc}(A)$ and put

$$c \circ \rho_S := (c \circ \rho_v)_{v\in S},$$

$$c \circ Ad_a := (c \circ Ad_{a_v})_{v\in S}$$

for $a = (a_v)_{v\in S} \in A^S$, where Ad_{a_v} refers to the conjugation action. To define the arithmetic Chern–Simons line associated to ρ_S, we need the intermediate object

$$H(\rho_S) := d^{-1}(c \circ \rho_S)/B_S^2 \subset C_S^2/B_S^2.$$

This is non-empty because H^3 of a local field is zero, and is a torsor for

$$H_S^2 := \prod_{v\in S} H^2(G_v, \mathbb{Z}/n\mathbb{Z}) \simeq \prod_{v\in S} \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

([26], Theorem (7.1.8).) We then use the sum map

$$\Sigma : \prod_{v\in S} \frac{1}{n}\mathbb{Z}/\mathbb{Z} \to \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

to push this out to a $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$-torsor. That is, define

$$L(\rho_S) := \Sigma_*[H(\rho_S)].$$

The natural map $H(\rho_S) \to L(\rho_S)$ will also be denoted by the sum symbol Σ.

In fact, L extends to a functor from $\mathcal{M}_S^{loc}(A)$ to the category of $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$-torsors. To carry out this extension, we just need to extend H to a functor to H_S^2-torsors. According to Appendices A and B, for $a = (a_v)_{v \in S} \in A^S$ and each v, there is an element $h_{a_v} \in C^2(A, \mathbb{Z}/n)/B^2(A, \mathbb{Z}/n)$ such that

$$c \circ Ad_{a_v} = c + dh_{a_v}.$$

Also,

$$h_{a_v b_v} = h_{a_v} \circ \mathrm{Ad}_{b_v} + h_{b_v}.$$

Hence, given $a : \rho_S \to \rho'_S$, so that $\rho'_S = \mathrm{Ad}_a \circ \rho_S$, we define

$$H(a) : H(\rho_S) \to H(\rho'_S)$$

to be the map induced by

$$x \mapsto x' = x + (h_{a_v} \circ \rho_v)_{v \in S}.$$

Then

$$dx' = dx + (d(h_{a_v} \circ \rho_v))_{v \in S} = (c \circ \rho_v)_{v \in S} + ((dh_{a_v}) \circ \rho_v)_{v \in S} = (c \circ \mathrm{Ad}_{a_v} \circ \rho_v)_{v \in S}.$$

So

$$x' \in d^{-1}(c \circ \rho'_S)/B_S^2,$$

and by the formula above, it is clear that H is a functor. That is, ab will send x to

$$x + h_{ab} \circ \rho_S,$$

while if we apply b first, we get

$$x + h_b \circ \rho_S \in H(\mathrm{Ad}_b \circ \rho_S),$$

which then goes via a to

$$x + h_b \circ \rho_S + h_a \circ \mathrm{Ad}_b \circ \rho_S.$$

Thus,

$$H(ab) = H(a)H(b).$$

Defining

$$L(a) = \Sigma_* \circ H(a)$$

turns L into a functor from $\mathcal{M}_S^{loc}$ to $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$-torsors. Even though we are not explicitly laying down geometric foundations, it is clear that L defines thereby an A^S-equivariant $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$-torsor on $Y_S^{loc}(A)$, or a $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$-torsor on the stack $\mathcal{M}_S^{loc}(A)$.

We can compose the functor L with the restriction $r_S : \mathcal{M}_S(A) \to \mathcal{M}_S^{loc}(A)$ to get an A-equivariant functor L^{glob} from $\mathcal{M}_S(A)$ to $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$-torsors.

Lemma 2.1 *Let $\rho \in Y_S(A)$ and $a \in Aut(\rho)$. Then $L^{glob}(a) = 0$.*

Proof By assumption, $\mathrm{Ad}_a\rho = \rho$, and hence, $dh_a \circ \rho = 0$. That is, $h_a \circ \rho \in H^2(\pi_S, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$. Hence, by the reciprocity law for $H^2(\pi_S, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$ ([26], Theorem (8.1.17)), we get

$$\Sigma_*(h_a \circ \rho) = 0.$$

□

By the argument of [9], p. 439, we see that there is a $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$-torsor

$$L^{\text{inv}}([\rho])$$

of invariant sections for the functor L^{glob} depending only on the orbit $[\rho]$. This is the set of families of elements

$$x_{\rho'} \in L^{glob}(\rho')$$

as ρ' runs over $[\rho]$ with the property that every morphism $a : \rho_1 \to \rho_2$ takes x_{ρ_1} to x_{ρ_2}. Alternatively, $L^{\text{inv}}([\rho])$ is the inverse limit of the $L^{glob}(\rho')$ with respect to the indexing category $[\rho]$. Alternatively, what Lemma 2.1 shows is that there are canonical isomorphisms between the fibres of the torsor over points in the same orbit.

Since S contains all primes dividing n, we have

$$H^3(\pi_S, \frac{1}{n}\mathbb{Z}/\mathbb{Z}) = H^3(X_S, \frac{1}{n}\mathbb{Z}/\mathbb{Z}) = 0,$$

([26], Proposition (8.3.18)) and the cocycle $c \circ \rho$ is a coboundary

$$c \circ \rho = d\beta$$

for $\beta \in C^2(\pi_S, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$. This element defines a class

$$CS_c([\rho]) := \Sigma([i_S^*(\beta)]) \in L^{\text{inv}}([\rho]).$$

A different choice β' will be related by

$$\beta' = \beta + z$$

for a 2-cocycle $z \in Z^2(\pi_S, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$, which vanishes when mapped to $L((\rho \circ i_v)_{v \in S})$. Thus, the class $CS_c([\rho])$ is independent of the choice of β and defines a global section

$$CS_c \in \Gamma(\mathcal{M}_S(A), L^{glob}).$$

Within the context of this paper, a 'global section' should just be interpreted as an assignment of $CS_c([\rho])$ as above for each orbit $[\rho]$.

3 The Arithmetic Chern–Simons Action: The *p*-adic Case

Now fix a prime p and assume all primes of F dividing p are contained in S. Fix a compatible system $(\zeta_{p^n})_n$ of p-power roots of unity, giving us an isomorphism

$$\zeta : \mathbb{Z}_p \simeq \mathbb{Z}_p(1) := \varprojlim_n \mu_{p^n}.$$

In this section, we will be somewhat more careful with this isomorphism. Also, it will be necessary to make some assumptions on the representations that are allowed.

Let A be a p-adic Lie group, e.g., $GL_n(\mathbb{Z}_p)$. Assume A is equipped with an open homomorphism $t : A \to \Gamma := \mathbb{Z}_p^\times$ and define A^n to be the kernel of the composite map

$$A \to \mathbb{Z}_p^\times \to (\mathbb{Z}/p^n\mathbb{Z})^\times =: \Gamma_n.$$

Let

$$A^\infty = \cap_n A^n = \mathrm{Ker}(t).$$

In this section, we denote by $Y_S(A)$ the continuous homomorphisms

$$\rho : \pi_S \to A$$

such that $t \circ \rho$ is a power χ^s of the p-adic cyclotomic character of π_S by a p-adic unit s. (We note that s itself is allowed to vary.) Of course this condition will be satisfied by any geometric Galois representation or natural p-adic families containing one.

As before, A acts on $Y_S(A)$ by conjugation. But in this section, we will restrict the action to A^∞ and use the notation $\mathcal{M}_S(A)$ for the corresponding action groupoid.

Similarly, we denote by Y_S^{loc} the collections of continuous homomorphisms $\rho_S = (\rho_v : \pi_v \to A)_{v \in S}$ for which there exists a p-adic unit s such that $t \circ \rho_v = (\chi|\pi_v)^s$ for all v. $\mathcal{M}_S^{loc}(A)$ then denotes the action groupoid defined by the product $(A^\infty)^S$ of the conjugation action on the ρ_S.

We now fix a continuous cohomology class

$$c \in H^3(A, \mathbb{Z}_p[[\Gamma]]),$$

where

$$\mathbb{Z}_p[[\Gamma]] = \varprojlim_n \mathbb{Z}_p[\Gamma_n].$$

We represent c by a cocycle in $Z^3(A, \mathbb{Z}_p[[\Gamma]])$, which we will also denote by c. Given $\rho \in Y_S(A)$, we can view $\mathbb{Z}_p[[\Gamma]]$ as a continuous representation of π_S, where the action is left multiplication via $t \circ \rho$. We denote this representation by $\mathbb{Z}_p[[\Gamma]]_\rho$. The isomorphism $\zeta : \mathbb{Z}_p \simeq \mathbb{Z}_p(1)$, even though it's not π_S-equivariant, does induce a π_S-equivariant isomorphism

$$\zeta_\rho : \mathbb{Z}_p[[\Gamma]]_\rho \simeq \Lambda := \mathbb{Z}_p[[\Gamma]] \otimes \mathbb{Z}_p(1).$$

Here, $\mathbb{Z}_p[[\Gamma]]$ written without the subscript refers to the action via the cyclotomic character of π_S (with $s = 1$ in the earlier notation). The isomorphism is defined as follows. If $t \circ \rho = \chi^s$, then we have the isomorphism

$$\mathbb{Z}_p[[\Gamma]] \simeq \mathbb{Z}_p[[\Gamma]]_\rho$$

that sends γ to γ^s. On the other hand, we also have

$$\mathbb{Z}_p[[\Gamma]] \simeq \Lambda$$

that sends γ to $\gamma \otimes \gamma\zeta(1)$. Thus, ζ_ρ can be taken as the inverse of the first followed by the second.

Combining these considerations, we get an element

$$\zeta_\rho \circ \rho^* c = \zeta_\rho \circ c \circ \rho \in Z^3(\pi_S, \Lambda).$$

Similarly, if $\rho_S := (\rho_v)_{v \in S} \in Y_S^{loc}$, we can regard $\mathbb{Z}_p[[\Gamma]]_{\rho_v}$ as a representation of π_v for each v, and we get π_v equivariant isomorphisms

$$\zeta_{\rho_v} : \mathbb{Z}_p[[\Gamma]]_{\rho_v} \simeq \Lambda.$$

We also use the notation

$$\zeta_{\rho_S} : \prod_{v \in S} \mathbb{Z}_p[[\Gamma]]_{\rho_v} \simeq \prod_{v \in S} \Lambda$$

for the isomorphism given by the product of the ζ_{ρ_v}.

It will be convenient to again denote by $C^i_S(\Lambda)$ the product $\prod_{v\in S} C^i(\pi_v, \Lambda)$ and use the similar notations $Z^i_S(\Lambda)$, $B^i_S(\Lambda)$ and $H^i_S(\Lambda)$. The element $\zeta_{\rho_S} \circ \rho_S^* c$ is an element in $Z^3_S(\Lambda)$. We then put

$$H(\rho_S, \Lambda) := d^{-1}((\zeta_{\rho_S} \circ \rho_S^* c))/B^2_S(\Lambda) \subset C^2_S(\Lambda)/B^2_S(\Lambda).$$

This is a torsor for

$$H^2_S(\Lambda) \simeq \prod_{v\in S} H^2(\pi_v, \Lambda).$$

The augmentation map

$$a : \Lambda \to \mathbb{Z}_p(1)$$

for each v can be used to push this out to a torsor

$$a_*(H(\rho_S, \Lambda))$$

for the group

$$\prod_{v\in S} H^2(\pi_v, \mathbb{Z}_p(1)) \simeq \prod_{v\in S} \mathbb{Z}_p,$$

which then can be pushed out with the sum map

$$\Sigma : \prod_{v\in S} \mathbb{Z}_p \to \mathbb{Z}_p$$

to give us a $\mathbb{Z}_p$-torsor

$$L(\rho_S, \mathbb{Z}_p) := \Sigma_*(a_*(H(\rho_S, \Lambda))).$$

As before, we can turn this into a functor $L(\cdot, \mathbb{Z}_p)$ on $\mathcal{M}^{loc}_S(A)$, taking into account the action of $(A^\infty)^S$. By composing with the restriction functor

$$r_S : \mathcal{M}_S(A) \to \mathcal{M}^{loc}_S(A),$$

we also get a $\mathbb{Z}_p$-torsor $L^{glob}(\cdot, \mathbb{Z}_p)$ on $\mathcal{M}_S(A)$.

We now choose an element $\beta \in C^2(\pi_S, \Lambda)$ such that

$$d\beta = \zeta_\rho \circ c \circ \rho \in Z^3(\pi_S, \Lambda) = B^3(\pi_S, \Lambda)$$

to define the p-adic Chern–Simons action

$$CS_c([\rho]) := \Sigma_* a_* i_S^*(\beta) \in L^{glob}([\rho], \mathbb{Z}_p).$$

The argument that this action is independent of β and equivariant is also the same as before, giving us an element

$$CS_c \in \Gamma(\mathcal{M}_S(A), L^{glob}(\cdot, \mathbb{Z}_p)).$$

4 Remarks

1. The restrictions (1) and (2) on the representations ρ that make up $Y_S(A)$ in Sect. 3 might seem rather stringent. However, if we take A to be the image of some fixed p-adic geometric Galois representation ρ_0, this includes all twists $\rho_0(s)$ of ρ_0 by unit powers χ^s of the p-adic cyclotomic character. Thus, we are in effect constructing with the cocycle c a section of a line bundle on the entire p-adic weight space $\mathbb{Z}_p^\times$. In the next section, we will discuss the motivation coming from the theory of L-functions. The ability to construct such a section is already promising from this point of view.
2. We have dealt with the p-adic theory assuming S is non-empty. It is straightforward to get a p-adic function on the moduli space for X, the case 'without boundary'. But according to the Fontaine-Mazur conjecture, an infinite p-adic Lie group should not be possible as the image of a representation of $\pi_1(X, b)$. Indeed, since $CS_c(\rho)$ is a p-adic invariant of such a represention, plausible applications to questions of existence and distribution could be considered.
3. In the p-adic theory, no changes are necessary for F with a real embedding provided we take $p \neq 2$. Indeed, even though the duality theorems involving the sheaf $\mathbb{G}_m$ become somewhat more complicated because of the contribution from real places, such contributions all vanish for p-adic coefficient sheaves if p is odd. However, if one were to imagine a Chern–Simons theory for complex L-functions, the Archimedean places should be expected to play an essential role.
4. In the first two sections, we assumed the field F contained the nth roots of 1 so as to trivialize the sheaf μ_n. This allowed us to construct functions out of constant cohomology classes for A. Similarly, in Sect. 3, we obtained $\mathbb{Z}_p(1)$ cohomology classes from $\mathbb{Z}_p$-classes by a twisting trick familiar in Iwasawa theory. To avoid this, one could have regarded the group A as a constant sheaf and used cohomology classes in $H^3(BA, \mu_n)$ from the beginning. But it is hard to imagine constructing such classes other than by twisting classes with constant coefficients. This is essentially equivalent to our approach.
5. We are not giving at present any examples. For finite groups A, it is not hard to get classes in H^3, for example, starting from cyclic subgroups. On the other hand, a norm compatible sequence of classes for infinite p-adic Lie groups seems to be harder to construct. In subsequent work, we will study this question systematically from the viewpoint of Lazard's theory of analytic groups and duality for groups like $GL_n(\mathbb{Z}_p)$ [12].

6. It is unfortunate that the p-adic case does not include $A = \mathbb{Z}_p$ for reasons of cohomological dimension. Even in topological Chern–Simons theory, the abelian case seems to have a nature different from groups like $SU(2)$. One way of getting around this difficult for $A \simeq \mathbb{Z}_p^r$ might be to use classes in $H^1(A, \mathbb{Z}_p)$ pulled back to π_S, from which one could take Massey products to end up with 3-cocycles. Another possibility, following a pattern familiar in Iwasawa theory, would be to find a sequence of $\mathbb{Z}/p^n\mathbb{Z}$ classes that are congruent in a somewhat subtle sense, to which one applies the construction at the end of Sect. 1.
7. One notable difference from the usual Chern–Simons theory is that the Chern–Simons line of this paper is presented as an additive torsor, rather than a multiplicative one. However, note that we are using an isomorphism $\frac{1}{n}\mathbb{Z}/\mathbb{Z} \simeq \mu_n$, and the latter is multiplicative. Thus, our finite torsors can also be thought of as multiplicative μ_n-torsors, in closer parallel to the topological setting.
   However, the p-adic Chern–Simons line does seem to be genuinely additive. As will be explained in the next section, the values of p-adic L-functions should also lie in the fibers of a line bundle. Thus, if there is a connection between the two, the arithmetic Chern–Simons invariant should be related to the *logarithm* of the p-adic L-function.
8. In this paper, we are defining only the classical Chern–Simons functional. Speculating wildly, one might hope that twists of the value of a classical functional by a family of cyclotomic characters represent a kind of semi-classical approximation. In any case, it would be interesting to construct a quantum wavefunction in the arithmetic setting. For the finite-coefficient case of Sects. 1 and 2, this is in principle easy to define. The (more important) p-adic coefficients present a greater challenge.
9. Since the Spec(F_v) are playing the role of boundary tori, moduli spaces of local Galois representations should make up the classical phase spaces of arithmetic Chern–Simons theory. In the topological case, the corresponding moduli space has an interpretation using either holomorphic vector bundles or Higgs bundles, depending on the group. In this regard, it is interesting to take note of recent developments in p-adic Hodge theory defining a functor from Galois representations to vector bundles on a p-adic curve [8]. The moduli space of vector bundles that arises admits a uniformization by an infnite-dimensional Grassmannian in essentially the same manner as for complex Riemann surfaces. The possibility of using this construction to study determinant line bundles following the pattern of conformal field theory appears to be an interesting avenue of investigation in the study of local moduli spaces.
10. It is somewhat unforunate in this regard that work of Kapustin and Witten [13] on the geometric Langlands programme doesn't make use of Chern–Simons theory, but rather, S-duality for 4D gauge theory. Since the Langlands programme is another source of L-functions in arithmetic, a pleasant coincidence might have been for topological Chern–Simons theory to play a critical role also in the geometric Langlands programme. In any case, the analogy between Chern–Simons functions and L-functions suggests a possibility for defining L-functions in geometric Langlands, usually thought not to admit such a formalism. That is, the

L-function on the geometric Galois side should have the structure of a wave-function over a character variety. The role of automorphic forms in geometric Langlands is played by D-modules on moduli spaces of principal bundles that are Hecke eigensheaves in a suitable sense. The theory of automorphic L-functions should then assign an amplitude to such a D-module, possibly using a path integral over objects on a three manifold that have the given D-module as a boundary value.

5 Towards Computation

In this section, we indicate how one might go about computing the Chern–Simons invariant in the unramified case with finite coefficients. That is, we assume we are in the setting of Sect. 1. The ideas of this section have been developed in the paper [4].

Let $X = \mathrm{Spec}(\mathcal{O}_F)$ and M a continuous representation of $\pi = \pi_1(X)$ regarded as a locally constant sheaf on X. Assume $M = \varprojlim M_n$ with M_n finite representations such that there is a finite set T of primes in $\mathcal{O}_F$ containing all primes dividing the order of any $|M_n|$. Let $U = \mathrm{Spec}(\mathcal{O}_{F,T})$, $G_T = \pi_1(U)$, and $G_v = \mathrm{Gal}(\bar{F}_v/F_v)$ for a place v of F. Write m_v for the maximal ideal of $\mathcal{O}_F$ corresponding to the place v and r_v for the restriction map of cochains or cohomology classes from G_T to G_v.

Denote by $C^*_c(G_T, M)$ the complex defined as a mapping fiber

$$C^*_c(G_T, M) := \mathrm{Fiber}[C^*(G_T, M) \to \prod_{v \in T} C^*(G_v, M)].$$

So

$$C^n_c(G_T, M) = C^n(G_T, M) \times \prod_{v \in T} C^{n-1}(G_v, M),$$

and

$$d(a, (b_v)) = (da, (r_v(a) - db_v))$$

for $(a, (b_v)) \in C^n_c(G_T, M)$. As in [10], p. 20, since there are no real places in F, there is a quasi-isomorphism

$$C^*_c(G_T, M) \simeq R\Gamma(U, j_!(M)),$$

where $j : U \to X$ is the inclusion. But there is also an exact sequence

$$0 \to j_! j^*(M) \to M \to i_* i^*(M) \to 0,$$

where $i : T \to X$ is the closed immersion complementary to j. Thus, we get an exact sequence

$$\prod_{v\in T} H^2(\mathrm{Spec}(\mathcal{O}_F/m_v), i^*(M)) \to H^3(C_c(G_T, M)) \to H^3(X, M) \to \prod_{v\in T} H^3(\mathrm{Spec}(\mathcal{O}_F/m_v),$$

from which we get an isomorphism

$$H^3(C_c(G_T, M)) \simeq H^3(X, M),$$

since $\mathrm{Spec}(\mathcal{O}_F/m_v)$ has cohomological dimension 1.

We interpret this as a statement that the cohomology of X

$$H^3(X, M)$$

can be identified with cohomology of a 'compactification' of U with respect to the 'boundary', that is, the union of the $\mathrm{Spec}(F_v)$ for $v \in T$. This means that a class $z \in H^3(X, M)$ is represented by $(c, (b_v)_{v\in T})$, where $c \in Z^3(G_T, M)$ and $b_v \in C^2(G_v, M)$ in such a way that

$$db_v = c|G_v.$$

There is also the exact sequence

$$\to H^2(G_T, M) \to \prod_{v\in T} H^2(G_v, M) \to H^3_c(U, M) \to 0,$$

the last zero being $H^3(U, M) = 0$. We can use this to compute the invariant of z when $M = \mu_n$. We have to lift z to a collection of classes $x_v \in H^2(G_v, \mu_n)$ and then take the sum

$$\mathrm{inv}(z) = \sum_v \mathrm{inv}(x_v).$$

This is independent of the choice of the x_v by the reciprocity law. The lifting process may be described as follows. The map

$$\prod_{v\in T} H^2(G_v, \mu_n) \to H^3_c(U, \mu_n)$$

just takes a tuple of 2-cocycles $(x_v)_{v\in T}$ to $(0, (x_v)_{v\in T})$. But by the vanishing of $H^3(U, \mu_n)$, given $z = (c, (b_v))$, we can find a global cochain $a \in C^2(G_T, \mu_n)$ such that $da = c$. We then put $x_v := b_v - r_v(a)$.

When we start with a class $z \in H^3(\pi, \mu_n)$ let $c \in Z^3(\pi, \mu_n)$ represent z. Let $I_v \in G_v$ be the inertia subgroup. We now can trivialise $c|G_v$ by first trivialising it over G_v/I_v to which it factors (since c is a globally unramified class). That is, the b_v as above can be chosen as cochains factoring through G_v/I_v. This is possible because $H^3(G_v/I_v, \mu_n) = 0$. The class $(c, (b_v))$ chosen this way is independent of the choice of the b_v. This is because $H^2(G_v/I_v, \mu_n)$ is also zero. The point is that the representation of z as $(c, (b_v))$ with unramified b_v is essentially canonical. More precisely, given $c|(G_v/I_v) \in Z^3(G_v/I_v, \mu_n)$, there is a canonical

$$b_v \in C^2(G_v/I_v, \mu_n)/B^2(G_v/I_v, \mu_n)$$

such that $db_v = c|(G_v/I_v)$. This can then be lifted to a canonical class in $C^2(G_v, \mu_n)$ $/B^2(G_v, \mu_n)$. Now we trivialise $c|G_T$ globally as above, that is, by the choice of $a \in C^2(G_T, \mu_n)$ such that $da = c|G_T$. Then $((b_v - r_v(a))_{v\in T}$ will be cocycles, and we compute

$$\mathrm{inv}(z) = \sum_v \mathrm{inv}(b_v - r_v(a)).$$

A few remarks about this method:

1. Underlying this is the fact that the the compact support cohomology $H^3(U, \mu_n)$ can be computed relative to the somewhat fictitious boundary of U or as relative cohomology $H^3(X, T; \mu_n)$. Choosing the unramified local trivialisations corresponds to this latter representation.
2. To summarise the main idea again, starting from a cocycle $c \in Z^3(\pi, \mu_n)$ we have canonical unramified trivalisations at each v and a non-canonical global ramified trivialisation.

   The invariant of z measures the discrepancy between the unramified local trivialisations and a ramified global trivialisation.

   The fact that the non-canonicality of the global trivialisation is unimportant follows from the reciprocity law.
3. The description above that computes the invariant by comparing the local unramified trivialisation with the global ramified one is a precise analogue of the so-called 'glueing formula' for Chern–Simons invariants when applied to $\rho^*(c)$ for a representation $\rho : \pi \to \frac{1}{n}\mathbb{Z}/\mathbb{Z}$ and a 3-cocycle c on $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$. A systematic treatment with explicit examples is presented in the work [4].

For the moment, we content ourselves with some ideas for the case of $\mathrm{Hom}(\pi, \mathbb{Z}/p)$.

Recall from Sect. 1 that a 3-cocycle on $\mathbb{Z}/p$ can be obtained as $\delta\alpha \cup \alpha$, where $\alpha \in H^1(\mathbb{Z}/p, \mathbb{Z}/p)$ is the identity map and δ is the boundary map coming from the extension

$$E : 0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0.$$

If we have a homomorphism

$$f : N \to \mathbb{Z}/p,$$

a trivialisation of $f^*(\delta\alpha \cup \alpha)$ may be obtained by trivialising $\delta\alpha$. That is, if $db = f^*(\delta\alpha)$, for a cochain b on N, then

$$d(-\alpha \cup b) = \alpha \cup \delta\alpha.$$

Another way of putting this is that a splitting of the sequence $f^*(E)$ will give a trivialisation. That is, if there is a lifting $\tilde{f} : N \to \mathbb{Z}/p^2$ of f, then we can construct a trivialisation. An explicit description goes like this. Choose a set-theoretic splitting $s : \mathbb{Z}/p \to \mathbb{Z}/p^2$, for example, in the standard way that sends the class of $i \mod p$ to that of $i \mod p^2$. Then $\delta\alpha = ds$. Suppose $\tilde{f}$ exists as above. Then the trivialisation of $f^*\delta\alpha$ is given by

$$b := s \circ f - \tilde{f},$$

so that $-\alpha \cup (s \circ f - \tilde{f})$ is a trivialisation of $\alpha \cup \delta\alpha$. Now, if $N = G_v/I_v \simeq \hat{\mathbb{Z}}$, it suffices to choose $\tilde{f}$ in any manner. So the key point is the lifting $\tilde{f}$ in the case where $N = G_T$ and $f : G_T \to \mathbb{Z}/p$ is the composition of a representation $\rho : \pi \to \mathbb{Z}/p$ with the quotient map $k : G_T \to \pi$. To construct examples, here is a simple starting point. Take F an totally imaginary field such that the class group $C_F \simeq \mathbb{Z}/p$. I believe there are many examples where the Hilbert class field of F has been constructed as a Kummer extension, even though we need to look through the literature on explicit class field theory (say with $F = \mathbb{Q}(\mu_{p^2})$). Let $H = F(h^{1/p})$ and let $\rho : \pi \to \mathbb{Z}/p$ be the corresponding Kummer character. With these assumptions, of course there can't be a lift $\tilde{\rho} : \pi \to \mathbb{Z}/p^2$. However, by taking T to be the ramified places of the character corresponding to h^{1/p^2}, $f := \rho \circ k$ does lift to $\tilde{f} : G_T \to \mathbb{Z}/p^2$. This then gives the trivialisation of $f^*(\delta\alpha)$ as above.

6 Motivation: L-Functions

In the following, the ring R can be provisionally thought of as either $\mathbb{C}$, $\mathbb{Z}_p$, or $\mathbb{Q}_p$ for some primes p. However, one can, and needs to, allow more general coefficients, such as an extension field of $\mathbb{Q}_p$, or the profinite group rings of Iwasawa theory ([10], 1.4.1). It is conceivable that more general rings are appropriate for the complex theory as well. However, for concreteness, it is all right to keep in mind these simple cases.

The theory of L-functions, still largely conjectural, assigns a canonical *L-amplitude*

$$L(X, \mathcal{F})$$

to a pair consisting of a scheme X of finite type over $\mathbb{Z}$ and a constructible sheaf $\mathcal{F}$ of finitely-generated R-modules in the étale topology of X. It is convenient to allow also elements of bounded derived categories of such $\mathcal{F}$ as coefficients. This amplitude is sometimes a number in R, but is expected in general to be an element of a determinant line. The proposal that an amplitude of the right sort can always be defined is known as the Hasse-Weil conjecture for complex L-functions and Iwasawa's main conjecture for p-adic L-functions. The main difficulty can be thought of as a problem of regularizing an infinite product. Since this point of view may not be entirely familiar to physicists, we give a brief overview of the theory described in [10, 14].

Associated to $(X, \mathcal{F})$, there are the cohomology groups with compact support

$$H_c^i(X, \mathcal{F}),$$

which are finitely generated R-modules. We denote by $D(X, \mathcal{F})$ the dual of the determinant of cohomology

$$D(X, \mathcal{F}) := \otimes_i \det H^i(X, \mathcal{F})^{(-1)^{i+1}},$$

a projective R-module of rank 1 [19]. Hence, if $\mathcal{M}$ is a moduli space of sheaves on X, the $D(X, \mathcal{F})$ will vary over points $[\mathcal{F}] \in \mathcal{M}$ and come together to form a line bundle[3]

$$\mathcal{D} \to \mathcal{M}.$$

Note here that $\mathcal{M}$ will be like the representation varieties in complex geometry, and hence, have the structure of a scheme, formal scheme, or an analytic space over $\mathrm{Spec}(R)$.

The L-amplitude is conjectured to be a generator

$$L(X, \mathcal{F}) \in D(X, \mathcal{F}),$$

which should patch together to a trivialisation of $\mathcal{D}$ over $\mathcal{M}$. Thus, the theory of L-functions proposes the existence of a canonical section

$$L(X, \cdot) \in \Gamma(\mathcal{M}, \mathcal{D})$$

for suitable moduli spaces $\mathcal{M}$ of sheaves. The techniques of arithmetic geometry have so far provided essentially ad hoc methods for constructing such sections in limited settings. Thus, the availability of solutions to entirely analogous problems in quantum field theory is the main motivation for an attempt to develop a parallel arithmetic theory.

A sheaf $\mathcal{F}$ is *acyclic* if $H_c^i(X, \mathcal{F}) = 0$ for all i. For an acyclic sheaf $\mathcal{F}$, there is a canonical trivialisation

$$D(X, \mathcal{F}) \simeq R$$

corresponding to the fact that the determinant of the zero module is R. For acyclic sheaves, the L-amplitude can be regarded as an element of R. Furthermore, over the locus $\mathcal{M}_{acyc} \subset \mathcal{M}$ of acylic sheaves, we expect the determinant line bundle to have a canonical trivialization

$$\mathcal{D}|\mathcal{M}_{acyc} \simeq \mathcal{O}_{\mathcal{M}_{acyc}}.$$

Thus, over $\mathcal{M}_{acyc}$, the L-amplitude can be regarded as a function.

[3]For this motivational discussion, the precise conditions necessary for the geometric statement to hold will be left unstated.

For coefficient rings like $R = \mathbb{Z}_p$, even when $\mathcal{F}$ is not acyclic, $\mathcal{F} \otimes \mathbb{Q}_p$ may be acyclic. So even when an element in $D(X, \mathcal{F})$ may not canonically be an element of R, it may sometimes be regarded as an element of $R \otimes \mathbb{Q}_p$. A related phenomenon is the following. Suppose

$$\mathcal{M} = \mathrm{Spec}(T)$$

and the locus of non-acyclic sheaves form a divisor with equation $f = 0$. Then $\mathcal{D}$ can be regarded as a T-module. And

$$\mathcal{D}[1/f] = \mathcal{D} \otimes T[1/f]$$

is canonically trivial. Let s be the section of $\mathcal{D}[1/f]$ corresponding to 1 under this trivialization. Then, in favorable circumstances, for example, if $\mathcal{M}$ is regular, the section

$$(1/f)s$$

extends over all of $\mathcal{M}$ and can be regarded as a trivializing section of $\mathcal{D}$. This is the way in which characteristic elements that occur in classical formulations of the Iwasawa main conjecture become interpreted as trivializing sections of determinant lines (cf. [10], Example 2.5).

The L-amplitude is conjectured to satisfy some natural conditions ([14], conjecture 3.2.2, modified by [10], conjecture 2.3.2):

(1) Multiplicativity: If

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is a exact sequence, then the canonical isomorphism

$$D(X, \mathcal{F}_2) \simeq D(X, \mathcal{F}_2) \otimes D(X, \mathcal{F}_2)$$

takes $L(X.\mathcal{F}_2)$ to $L(X.\mathcal{F}_1) \otimes L(X.\mathcal{F}_3)$.

(2) Compatibility change of coefficient rings: If R' is an R-algebra and $\mathcal{F}' = \mathcal{F} \otimes^L R'$, then the natural isomorphism

$$D(X, \mathcal{F}) \otimes_R R' \simeq D(X, \mathcal{F}')$$

takes $L(X, \mathcal{F}) \otimes 1$ to $L(X, \mathcal{F}')$. (The base-change considered in [10] is more general to accommodate the possibility of non-commutative coefficient rings.)

(3) Two normalisation conditions: an easy one for sheaves over a finite field, and a very hard one having to do with conjectures on L-amplitude of motives.

We comment on (1) and (3). The most important case of (1) is

$$0 \to j_!(j^{-1}\mathcal{F}) \to \mathcal{F} \to i_*(i^{-1}(\mathcal{F})) \to 0,$$

where $i : Z \rightarrowtail X$ is a closed embedding and $j : U \rightarrowtail X$ is the complement. Then the required multiplicativity is

$$L(X, \mathcal{F}) = L(U, \mathcal{F}) \otimes L(Z, \mathcal{F}),$$

where we omit the inverse images for notational convenience. Note that when all three are acyclic, the tensor product becomes a product of numbers and this is a literal equality.

The easy normalisation condition in (3) is when $X = \mathrm{Spec}(\mathbb{F}_q)$, the spectrum of a finite field with $q = p^d$ elements. In that case, the stalk $\mathcal{F}_x$ at a geometric point

$$x : \mathrm{Spec}(\bar{\mathbb{F}}_q) \to \mathrm{Spec}(\mathbb{F}_q)$$

carries an action of the geometric Frobenius

$$Fr_x : \mathrm{Spec}(\bar{\mathbb{F}}_q) \to \mathrm{Spec}(\bar{\mathbb{F}}_q)$$

(the dual to the map $a \mapsto a^{q^{-1}}$). Thus, we get an exact sequence

$$0 \to H^0(\mathcal{F}) \to \mathcal{F}_x \xrightarrow{I - Fr_x} \mathcal{F}_x \to H^1(\mathcal{F}) \to 0,$$

inducing an isomorphism

$$D(\mathrm{Spec}(\mathbb{F}_q), \mathcal{F}) \simeq \det(\mathcal{F}_x)^* \otimes \det(\mathcal{F}_x) \simeq R.$$

Then $L(\mathrm{Spec}(\mathbb{F}_q), \mathcal{F})$ is defined to be the inverse image of 1. When $\mathcal{F}_x$ is R-free and $\mathcal{F}$ is acyclic, this gives the normalization

$$L(\mathrm{Spec}(\mathbb{F}_q), \mathcal{F}) = \frac{1}{\det([I - Fr_x]|\mathcal{F}_x)}.$$

When $X = \mathrm{Spec}(\mathbb{F}_q)$, the category of sheaves of R-modules is equivalent to the category of continuous representations of $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ on R-modules. This Galois group is topologically generated by Fr_x. The formalism of the Weil-étale topology [20, 21] allows us to view arbitrary representations of the Weil group $W_{\mathbb{F}_q} \subset \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$, that is, the group of integer powers of Fr_x, as sheaves on schemes over $\mathbb{F}_q$. Since $W_{\mathbb{F}_q} \simeq \mathbb{Z}$, the one-dimensional complex characters of the Weil group of $\mathrm{Spec}(\mathbb{F}_q)$ are parametized by $\mathbb{C}^\times$. So they can all be written as

$$Fr_x \mapsto q^{-s},$$

for some $s \in \mathbb{C}$. (The reason we parametrize the characters this way is because it is the description that's compatible with the norm character on the global idele class group.) We denote the 1-dim representation corresponding to this character $\mathbb{C}(s)$.

When $\mathcal{F}$ is a sheaf of $\mathbb{C}$-vector spaces, we denote by $\mathcal{F}(s)$ the sheaf corresponding to the representation $\mathcal{F}_x \otimes \mathbb{C}(s)$. If $\mathcal{F}(s)$ is acyclic, we get

$$L(\mathrm{Spec}(\mathbb{F}_q), \mathcal{F}(s)) = \frac{1}{\det([I - p^{-s} Fr_x]|\mathcal{F}_x)}.$$

This is the way in which the analytic L-factors that arise in the complex theory of L-functions come up naturally as we vary a representation in a canonical one-parameter family.

For general X, let S be a finite subset of X_0, the set of closed points of X, and $U_S = X \setminus S$. Then the multiplicative property of the L-amplitude gives

$$L(X, \mathcal{F}) = L(U_S, \mathcal{F}) \prod_{y \in S} L(\mathrm{Spec}(k(y)), \mathcal{F}_y),$$

where $k(y)$ is the (finite) residue field at y. If the limit as S grows large exists, we should have

$$L(X, \mathcal{F}) = L(\text{generic}, \mathcal{F}) \prod_{y \in X_0} L(\mathrm{Spec}(k(y)), \mathcal{F}_y),$$

where the factor $L(\text{generic}, \mathcal{F})$ can sometimes be determined. In substantial generality, it can be shown that the limit exists when we replace $\mathcal{F}$ by $\mathcal{F}(s)$ for $Re(s)$ sufficiently large, forcing on us essentially the familiar definition of an L-amplitude as an infinite product. There is also a formalism for making sense of this for coefficient rings more general than $\mathbb{C}$ (subject to hard conjectures and theorems about Weil sheaves associated to l-adic sheaves). The usual Hasse-Weil conjecture asserts that when $\mathcal{F}$ is motivic, one can define $L(X, \mathcal{F}(s))$ in a way that's meromorphic in s, with poles contributed only by trivial sheaves.

The hard (and important) normalisation condition would require lengthy prerequisites, and will not be discussed here at all. The reader is referred to [10, 14].

Now we specialise to the situation where $X = \mathrm{Spec}(\mathcal{O}_F)$ as in the earlier sections, and $X_S = \mathrm{Spec}(\mathcal{O}_F[1/S])$ for a finite set of primes S. As indicated above, a p-adic L-function is supposed to be a section of $\mathcal{D}$ on $\mathcal{M}_S$:

$$L(X, \cdot) \in \Gamma(\mathcal{M}_S, \mathcal{D}).$$

In this paper, we have constructed in Sect. 3

$$CS_c(\cdot)$$

an additive version of such a section, at least for a restricted family. The optimistic wish referred to in the abstract is a comparison

$$CS_c(\cdot) \sim \log L(X, \cdot).$$

To effect such a comparison, one would obviously have to relate the $\mathbb{Z}_p$-torsors constructed in an elementary fashion to the determinant line bundles. I am told by Dan Freed that such a comparison is not available even in topological Chern–Simons theory, and may be rather difficult. Nevertheless, the strong analogy between the multiplicativity of L-functions and the glueing formula seems worth investigating in detail.

Bruce Bartlett has emphasised to me the importance of Reidemeister torsion within this circle of ideas. Indeed, Witten [27] had already noted that the square root of Reidemeister torsion appears as the main contribution to the semi-classical Chern–Simons wavefunction by a classical minimum. Since there has been for some time a folklore analogy in number theory between L-functions and Reidemeister torsion (cf. [6]), a reasonable avenue of investigation might be a definition of an arithmetic Reidemeister torsion using the arithmetic Chern–Simons functional, which could then be compared to the L-amplitude.

The main point is important enough to be worth repeating: it is a major unsolved problem of arithmetic geometry to define global sections of determinant line bundles satisfying the natural properties outlined above. The speculations of this section were motivated by the wishful thought that ideas from physics could be employed to effect such a definition. The constructions of the first three sections can be regarded as small beginning steps in this direction.

7 Appendix: Conjugation on Group Cochains

We compute cohomology of a topological group G with coefficients in a topological abelian group M with continuous G-action using the complex whose component of degree i is $C^i(G, M)$, the continuous maps from G^i to M. The differential

$$d : C^i(G, M) \to C^{i+1}(G, M)$$

is given by

$$df(g_1, g_2, \dots, g_{i+1})$$

$$= g_1 f(g_2, \dots, g_{i+1}) + \sum_{k=1}^{i} f(g_1, \dots, g_{k-1}, g_k g_{k+1}, g_{k+2}, \dots, g_{i+1}) + (-1)^{i+1} f(g_1, g_2, \dots, g_i).$$

We denote by

$$B^i(G, M) \subset Z^i(G, M) \subset C^i(G, M)$$

the images and the kernels of the differentials, the coboundaries and the cocycles, respectively. The cohomology is then defined as

$$H^i(G, M) := Z^i(G, M)/B^i(G, M).$$

There is a natural right action of G on the cochains given by

$$a : c \mapsto c^a := a^{-1} c \circ Ad_a,$$

where Ad_a refers to the conjugation action of a on G^i.

Lemma 7.1 *The G action on cochain commutes with d:*

$$d(c^a) = (dc^a)$$

for all $a \in G$.

Proof If $c \in C^i(G, M)$, then

$$d(c^a)(g_1, g_2, \dots, g_{i+1}) = g_1 a^{-1} c(Ad_a(g_2), \dots, Ad_a(g_{i+1}))$$

$$+\sum_{k=1}^{i} a^{-1} c(Ad_a(g_1), \dots, Ad_a(g_{k-1}), Ad_a(g_k)Ad_a(g_{k+1}), Ad_a(g_{k+2}), \dots, Ad_a(g_{i+1}))$$

$$+(-1)^{i+1} a^{-1} c(Ad_a(g_1), Ad_a(g_2), \dots, Ad_a(g_i))$$

$$= a^{-1} Ad_a(g_1) c(Ad_a(g_2), \dots, Ad_a(g_{i+1}))$$

$$+\sum_{k=1}^{i} a^{-1} c(Ad_a(g_1), \dots, Ad_a(g_{k-1}), Ad_a(g_k)Ad_a(g_{k+1}), Ad_a(g_{k+2}), \dots, Ad_a(g_{i+1}))$$

$$+(-1)^{i+1} a^{-1} c(Ad_a(g_1), Ad_a(g_2), \dots, Ad_a(g_i))$$

$$= a^{-1}(dc)(Ad_a(g_1), Ad_a(g_2), \dots, Ad_a(g_{i+1})$$

$$= (dc)^a(g_1, g_2, \dots, g_{i+1}).$$

□

We use also the notation $(g_1, g_2, \dots, g_i)^a := Ad_a(g_1, g_2, \dots, g_i)$. It is well known that this action is trivial on cohomology. We wish to show the construction of explicit h_a with the property that

$$c^a = c + dh_a$$

for cocycles of degree 1, 2, and 3. The first two are relatively straightforward, but degree 3 is somewhat delicate. In degree 1, first note that $c(e) = c(ee) = c(e) + ec(e) = c(e) + c(e)$, so that $c(e) = 0$. Next, $0 = c(e) = c(gg^{-1}) = c(g) + gc(g^{-1})$, and hence, $c(g^{-1}) = -g^{-1}c(g)$. Therefore,

$$c(aga^{-1}) = c(a) + ac(ga^{-1}) = c(a) + ac(g) + agc(a^{-1}) = c(a) + ac(g) - aga^{-1}c(a).$$

From this, we get

$$c^a(g) = c(g) + a^{-1}c(a) - ga^{-1}c(a).$$

That is,

$$c^a = c + dh_a$$

for the zero cochain $h_a(g) = a^{-1}c(a)$.

Lemma 7.2 *For each $c \in Z^i(G, M)$ and $a \in G$, we can associate an*

$$h_a^{i-1}[c] \in C^{i-1}(G, M)/B^{i-1}(G, M)$$

in such a way that

$$(1) \quad c^a - c = dh_a^{i-1}[c];$$

$$(2) \quad h_{ab}^{i-1}[c] = (h_a^{i-1}[c])^b + h_b^{i-1}[c].$$

Proof This is clear for $i = 0$ and we have shown above the construction of $h_a^0[c]$ for $c \in Z^1(G, M)$ satisfying (1). Let us check the condition (2):

$$h_{ab}^0[c](g) = (ab)^{-1}c(ab)$$

$$= b^{-1}a^{-1}(c(a) + ac(b)) = b^{-1}h_a^0[c](Ad_b(g)) + h_b^0[c](g) = (h_a^0[c])^b(g) + h_b^0[c](g).$$

We prove the statement using induction on i, which we now assume to be ≥ 2. For a module M, we have the exact sequence

$$0 \to M \to C^1(G, M) \to N \to 0,$$

where $C^1(G, M)$ has the right regular action of G and $N = C^1(G, M)/M$. Here, we give $C^1(G, M)$ the topology of pointwise convergence. There is a canonical linear splitting $s : N \to C^1(G, M)$ with image the group of functions f such that $f(e) = 0$, using which we topologise N. According to [25], proof of 2.5, the G-module $C^1(G, M)$ is acyclic,[4] that is,

$$H^i(G, C^1(G, M)) = 0$$

for $i > 0$. Therefore, given a cocycle $c \in Z^i(G, M)$, there is an $F \in C^{i-1}(G, C^1(G, M))$ such that its image $f \in C^{i-1}(G, N)$ is a cocycle and $dF = c$. Hence, $d(F^a - F) = c^a - c$. Also, by induction, there is a $k_a \in C^{i-2}(G, N)$ such that

[4]The notation there for $C^1(G, M)$ is $F_0^0(G, M)$. One difference is that Mostow uses the complex $E^*(G, M)$ of equivariant homogeneous cochains in the definition of cohomology. However, the isomorphism $E^n \to C^n$ that sends $f(g_0, g_1, \ldots, g_n)$ to $f(1, g_1, g_1g_2, \ldots, g_1g_2 \cdots g_n)$ identifies the two definitions. This is the usual comparison map one uses for discrete groups, which clearly preserves continuity.

$f^a - f = dk_a$ and $k_{ab} = (k_a)^b + k_b + dl$ for some $l \in C^{i-3}(G, N)$ (zero if $i = 2$). Let $K_a = s \circ k_a$ and put

$$h_a = F^a - F - dK_a.$$

Then the image of h_a in N is zero, so h_a takes values in M, and $dh_a = c^a - c$. Now we check property (2). Note that

$$K_{ab} = s \circ k_{ab} = s \circ (k_a)^b + s \circ k_b + s \circ dl.$$

But $s \circ (k_a)^b - (s \circ k_a)^b$ and $s \circ dl - d(s \circ l)$ both have image in M. Hence, $K_{ab} = K_a^b + K_b + d(s \circ l) + m$ for some cochain $m \in C^{i-2}(G, M)$. From this, we deduce

$$dK_{ab} = (dK_a)^b + dK_b + dm,$$

from which we get

$$h_{ab} = F^{ab} - F - dK_{ab} = (F^a)^b - F^b + F^b - F - (dK_a)^b - dK_b - dm = (h_a)^b + h_b + dm.$$

□

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Branch Stabilisation for the Components of Hurwitz Moduli Spaces of Galois Covers

Michael Lönne

Abstract We consider components of Hurwitz moduli space of G-Galois covers and set up a powerful algebraic framework to study the set of corresponding equivalence classes of monodromy maps. Within that we study geometric stabilisation by various G-covers branched over the disc. Our results addresses the problem to decide equivalence and stable equivalence algebraically. We recover a homological invariant, which we show to distinguish the equivalence classes of given boundary monodromy and Nielsen type, if the latter is sufficiently large in the appropriate sense.

Keywords Monodromy · Galois cover · Hurwitz space · Branch stabilisation Braid group

1 Introduction

In this article we want to discuss at length some group theoretical aspects crucial to the study of connected components of Hurwitz spaces of coverings, which can be traced back to [4, 11].

In the narrow sense, the objective is the study of morphisms $p\colon C \to C/G =: C'$ induced by an effective action of a finite group G. The existence of such an effective action provides information about the complex structure of the curve and also about the group. In the broader sense, we include branched covers over Riemann surfaces, usually with at least one boundary component.

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Let us recall first that the geometry of the covering p encodes several numerical invariants that are constant under deformation: the genus g' of the base C' and its number of boundary components, the number d of branch points $y_1, \dots, y_d \in C'$ and the orders $m_1, \dots, m_d$ of the local monodromies—strictly speaking as an unordered multi-set. These invariants form the *primary numerical type* and have been the object of intensive studies since long.

A refined invariant is obtained from the monodromy $\mu : \pi_1(C' \setminus \{y_1, \dots, y_d\}) \to G$ of the regular unramified cover given by restriction of p to the complement of the branch points, $p^{-1}(C' \setminus \{y_1, \dots, y_d\})$. Instead of keeping track only of the orders m_i of the elements in G associated to the local monodromies at the punctures y_i, we consider the multi-set of their conjugacy classes. This is given by the *Nielsen type*, cf. [17], the class function ν which on each conjugacy class $\mathcal{C}$ in G takes the cardinality of local monodromies in $\mathcal{C}$ as its value. The function ν can be characterised also without taking recourse to the quotient C'. For each conjugacy class $\mathcal{C}$ just count the number of G-orbits on C such that an element is in $\mathcal{C}$ which rotates a disc around some point of the orbit by the smallest possible angle.

While many groups have been shown to allow classification by primary numerical type and Nielsen type, we are interested into arbitrary groups and get the motivation by the progress in the case of genus stabilisation.

Let us briefly recall the main results in the case of free actions. There the second homology group $H_2(G)$ was shown

- to classify equivalence classes of unbranched G-coverings for abelian and metabelian groups [6, 7].
- to classify stable equivalence classes for every group [14].
- to classify equivalence classes for every group if the genus g' is sufficiently large [5].

An analogous result to the last in the case of non-free action was proved with Catanese and Perroni [3], where the second homology group had to be replaced by a quotient $H_{2,\Gamma}$.

The case of branch stabilisation is more involved. While genus stabilisation corresponds to connected sum with a trivial G-cover over the torus, summing a branched cover does involve making non-trivial choices for the monodromies at the branch points. In fact we will stabilise by boundary connected sum with a punctured disc, but consider very general choices for the branch monodromies.

We will succeed following the above program in the following sense

- we classify various sets of equivalence classes of G-covers by elements of a monoid or a set with a monoid action.
- we classify certain sets of stable equivalence classes of G-covers by elements of a set, which can be distinguished by the primary numerical type, the Nielsen type and using $H_{2,\Gamma}$.
- we classify equivalence classes of G-covers by their stable equivalence classes, if the Nielsen type is sufficiently large.

For example we prove the algebraic version, Theorem 4.9 of the following geometric result.

Theorem 1.1 *Suppose Γ is a union of conjugacy classes generating G, then there exists an integer N such that*

> *the number of equivalence classes of G-covers of the disc with local monodromies in Γ and fixed Nielsen type ν is independent of ν, if ν takes a value at least N on each conjugacy class.*

Let us give a short overview on the content of this article: In the next section we recall Hopf formula and explain its validity in the setting of crossed modules. We then take some care to motivate the definition of a G-crossed module taking a union Γ of conjugacy classes into account, which gives rise to a finite abelian group which is later shown to be equal to $H_{2,\Gamma}$.

In Sect. 3 we equip the set of isomorphism classes of G-covers with an algebraic structure of monoid. This provides the necessary tool to investigate the geometric notion of stable equivalence by algebraic means. In the following section we address the stabilisation of G-covers branched over the disc, and determine conditions on the Nielsen types, such that stabilisation is surjective, resp. bijective.

Section 5 considers various geometric generalisations of G-covers and their equivalence classes. The corresponding algebraic setting is then presented and explored.

In the final Sect. 6 we revise the definition of tautological central extension and recall the definition of quotient $H_{2,\Gamma}$ of $H_2(G)$ from [2]. We will then see how it fits very well with the set-up of the previous sections and deduce a classification result for G-covers with sufficiently large Nielsen type.

2 Algebraic Setting

The formula of Hopf describes the second homology group $H_2(G, \mathbb{Z})$ as the kernel of a natural map associated to a given group G. To avoid unnecessary generality we restrict to the case of a finite group of order $n := \mathrm{ord}(G)$. In that case there is a natural finite presentation of G for any map $S \to G$, such that the image generates G as a group. It is expressed in a short exact sequence

$$1 \quad \to \quad R \quad \to \quad F \quad \to \quad G \quad \to \quad 1\,,$$

where $F = F_S$ is the free group freely generated by elements of S and $R = R_S \subset F$ is the free subgroup of relations.

The Hopf formula [10] then states

$$H_2(G, \mathbb{Z}) \quad = \quad \frac{R \cap [F, F]}{[F, R]}.$$

The same information is conveyed in the following exact sequences

$$1 \to H_2(G, \mathbb{Z}) \to \frac{F}{[F, R]} \to \frac{F}{[F, F]} \times G \to 1$$

$$1 \to H_2(G, \mathbb{Z}) \to \frac{R}{[F, R]} \to \frac{F}{[F, F]} \to \frac{F}{R[F, F]} \to \quad 1$$

However, there is another setting which also provides an approach to the second homology of G using a free object, the category of *G-crossed modules*. Let us quickly recall the basic definition.

Definition 2.1 A group homomorphism $\partial : C \to G$ with an action

$$\begin{aligned} C \times G &\to C \\ (c, g) &\mapsto c^g \end{aligned}$$

is called a *G-crossed module*, if

XM1: ∂ is G-equivariant for the conjugation action of G on itself,

$$\partial(c^g) \quad = \quad (\partial c)^g \quad = \quad g^{-1}(\partial c)g, \qquad \forall\, c \in C,\ g \in G.$$

XM2: the *Peiffer identities* hold

$$ca \quad = \quad ac^{\partial a}, \qquad \forall\, a, c \in C.$$

Example 2.2 If $S \subset G$ generates G then

$$\partial_S \ : \ \frac{F_S}{[F_S, R_S]} \quad \longrightarrow \quad G$$

is a G-crossed module which is a free G-crossed module on the inclusion $S \to G$, cf. [18]. If S is invariant under conjugation the G-action is simply induced by

$$\begin{aligned} G \times F_S &\longrightarrow F_S \\ (g, \hat{a}) \ &\mapsto \ \widehat{g^{-1}ag} \end{aligned}$$

which readily gives the G-equivariance of ∂_S:

$$\partial_S(\hat{a}^g) \quad = \quad \partial_S(\widehat{g^{-1}ag}) \quad = \quad g^{-1}ag \quad = \quad g^{-1}(\partial_S a)g.$$

We note that the denominator of the Hopf formula has been incorporated into the crossed module. So it remains to perform the intersection of its numerator, between a derived subgroup and a kernel. This can be done with lots of G-crossed modules with still the second homology of G dropping out.

Theorem 2.3 ([8]) *If $\partial : C \to G$ is a projective G-crossed module (in particular if it is a free G-crossed module) with ∂ surjective, then*

$$H_2(G, \mathbb{Z}) \quad = \quad \ker \partial \cap [C, C].$$

We will now define a G-crossed module depending on a union $\Gamma \subset G$ of conjugacy classes. To this end, we first relax the notion of G-crossed module, dispensing with the group structure of the domain.

Definition 2.4 A map $\varepsilon : Q \to G$ from a G-set Q to the group G is called an *augmentation* and Q an *augmented quandle* if

AQ1: ε is G-equivariant for the conjugation of G on itself, (Q is a G-crossed set)

$$\varepsilon(p^g) \quad = \quad g^{-1}\varepsilon(p)g, \quad \forall\, p \in Q, g \in G$$

AQ2: idempotency holds in the sense

$$p^{\varepsilon(p)} \quad = \quad p, \quad \forall\, p \in Q.$$

G-crossed modules are augmented quandles, but also, more importantly, the set Γ.

Example 2.5 The union of conjugacy classes $\Gamma \subset G$ is a G set for the action by conjugation. Therefore the injection $\varepsilon : \Gamma \to G$ is G-equivariant in the sense of **AQ1**. Property **AQ2** also holds since any element in G is unchanged under conjugation with itself.

We do not want to recall the story of the notion of quandle here, but refer for this and more on augmented quandles to the source [12]. Instead we take the quickest path back to G-crossed modules. It leads via the following definition.

Definition 2.6 Let Q be an augmented quandle then the *adjoint group* Adj Q is the group presented by

$$\text{Adj}\, Q \quad = \quad \langle e_q, q \in Q \mid e_p e_q = e_q e_{p^{\varepsilon(q)}} \rangle.$$

It has a unique group homomorphism to G compatible with the augmentation ε

$$\partial_Q : \text{Adj}\, Q \to G, \quad e_q \mapsto \varepsilon(q)$$

since Adj Q has the universal property for quandle maps to groups. And it serves our purpose, thanks to

Proposition 2.7 *Suppose $\varepsilon : Q \to G$ is an augmented quandle, then*

$$\partial_Q : \text{Adj}\, Q \quad \to \quad G$$

is a crossed module over G with respect to the tautological action of G on Adj Q *induced by the action of G on Q*

$$\begin{aligned} \text{Adj } Q \times G &\to \text{Adj } Q \\ (e_q, g) &\mapsto {e_q}^g \quad = \quad e_{q^g} \end{aligned}$$

Proof First we check that ∂_Q is G-equivariant:

$$\partial_Q({e_q}^g) = \partial_Q(e_{q^g}) = \partial_Q(\varepsilon_Q(q^g)) = \varepsilon(q^g) = g^{-1}\varepsilon(q)g = \partial_Q(\varepsilon_Q(q))^g = \partial_Q(e_q)^g.$$

Second we have to check the Peiffer identities:

$$e_p^{\partial_Q(e_q)} = e_p^{\partial_Q(\varepsilon_Q(q))} = e_p^{\varepsilon(q)} = e_{p^{\varepsilon(q)}} = e_{p*q} = e_q^{-1} e_p e_q$$

□

In analogy to the Hopf formula we get an abelian group

$$H(\Gamma, G) \quad := \quad \ker \partial_\Gamma \cap [\text{Adj } \Gamma, \text{Adj } \Gamma]. \tag{1}$$

As we will see, that intersection is a proper quotient of the second homology in general.

3 The Hurwitz Monoid

To associate to a G-cover of the punctured disc an algebraic object, we recall the following geometric set-up:

Let D be the closed unit disc, $(p_n) \subset D$ a sequence of distinct points in the interior, $p_0 \in \partial D$ a base point and let (γ_n) be a geometric basis for $\pi_1(D \setminus (p_n), p_0)$.

Any G-cover of D—connected or not—which is unbranched outside $p_1, \dots, p_d$ has a well-defined monodromy map relative p_0. To the homotopy class of a closed path it assigns the unique bijection acting from the right on the fibre at p_0 which gives the same map as path-lifting.

Only after choosing a point $\tilde{p}$ in the fibre, i.e. for a *pointed G-cover*, a monodromy map μ with values in G is well-defined in general[1]: Via the (left) G-action the fibre at p_0 is identified with $\{g\tilde{p} | g \in G\}$ and μ takes the value g if path-lifting maps $\tilde{p}$ to $g\tilde{p}$.

From now we tacitly assume all G-covers and their isomorphisms to be pointed and only put an occasional (pointed) to remind the reader of this fact.

[1]Points in the same orbit under the center of G determine G-equivariantly isomorphic covers, in particular the choice is superfluous in case of abelian G.

In these circumstances, the chosen datum associates to a G-cover a monodromy tuple of elements in G.

$$\mu(\gamma_1), \dots, \mu(\gamma_d).$$

As in previous papers, see [2, 3], we want to call such a tuple a Hurwitz vector of genus 0 according to

Definition 3.1 For any $d > 1$ the set G^d is a Br_d—set by the well-known Hurwitz action of the braid group. An element of this Br_d—set is called a $(0, d)$-*Hurwitz vector*, while an element of $\coprod G^d$ is simply called a *Hurwitz vector of genus* 0.

Two Hurwitz vectors v, w are *braid equivalent* if they belong to the same braid group orbit,

$$v \quad \approx \quad w.$$

Since on the geometric side we only consider isomorphism classes of G-covers, i.e. up to G-equivariant (pointed) maps covering a map preserving (p_n) as a set, we get a one to one correspondence

$$\{G\text{-covers of } D, \text{ branched outside } p_1, \dots, p_d\} /_{\mathrm{iso}} \quad \overset{1:1}{=\!=} \quad G^d / \mathrm{Br}_d \ .$$

Now on the algebraic side we have a natural composition $G^d \times G^e \to G^{d+e}$ given by juxtaposition which obviously is associative. It is equivariant under the inclusions $\mathrm{Br}_d \to \mathrm{Br}_{d+e}$ and $\mathrm{Br}_e \to \mathrm{Br}_{d+e}$, where in the second case a braid on strands one to e is moved to the corresponding braid on strands $d + 1$ to $d + e$. The resulting monoid is called the *Hurwitz class monoid* of G:

$$\mathcal{H}_G \quad := \quad \coprod_{d \geqslant 0} G^d / \mathrm{Br}_d$$

where the unit is understood to be represented by the empty tuple.

This composition can also be constructed on the geometric side. Given a pair of G-covers unbranched outside $p_1, \dots, p_d$, respectively $p_1, \dots, p_e$, there is a unique class of G-covers unbranched outside $p_1, \dots, p_{d+e}$, which is isomorphic to the first over a regular neighbourhood of $\gamma_1 \cup \dots \cup \gamma_d$ and to the second over a regular neighbourhood of $\gamma_{d+1} \cup \dots \cup \gamma_{d+e}$, pointed by the same point $\tilde{p}$ over p_0.

On a moments thought, we may restrict on the geometric side to G-covers with local monodromies in a union of conjugacy classes $\Gamma \subset G$ and get on the algebraic side a submonoid

$$\mathcal{H}_{G,\Gamma} \quad := \quad \coprod_{d \geqslant 0} \Gamma^d / \mathrm{Br}_d \, .$$

We next define equivalence of G-covers with respect to stabilisation by a G-cover C_u corresponding to some $u \in G^e$.

Definition 3.2 Two G-covers are called C_u—*stably equivalent* if composition of either with the same number ℓ of copies of C_u yield isomorphic G-covers.

The corresponding elements $v, w \in \coprod G^d$ are called *u—stably equivalent* if $vu^\ell \approx wu^\ell$, for some ℓ.

So u-stable equivalence of v, w is equality in the Hurwitz class monoid $\mathcal{H}_G$ of elements represented by vu^ℓ, wu^ℓ for some ℓ. In fact we can push this further to an equality in some monoid of fractions, but we need to recall the evaluation map on $\mathcal{H}_G$ first.

Lemma 3.3 *The* evaluation map *defined on* $\coprod G^d$ *with values in* G

$$v = (v_1, \dots, v_d) \quad \mapsto \quad ev(v) := v_1 \cdots v_d \in G$$

has the following properties:

(i) *ev is a monoid homomorphism.*
(ii) *ev is constant on braid group orbits.*
(iii) *ev induces an monoid homomorphism on the monoid $\mathcal{H}_G$, also called* evaluation.

The proof is easy and left to the reader.

Now $ev(u) \in G$ has finite order—say n—so we deduce that the elements $u^{\ell n}$ are central in $\mathcal{H}_G$ thanks to the following lemma proved in [2].

Lemma 3.4 *If v, w are Hurwitz vectors of genus* 0 *and $ev(v) = 1 \in G$, then vw and wv are braid equivalent.*

In particular these elements form a central submonoid of $\mathcal{H}_G$, which is the same as a central multiplicative set. The following technical result will provide the existence of corresponding monoids of fractions and expose their relation with the *enveloping group* of $\mathcal{H}_G$, defined by the universal property, that every monoid homomorphism to a group factors uniquely through the monoid homomorphism to the enveloping group.

Proposition 3.5 *Suppose $\mathcal{S} \subset \mathcal{H}_G$ is a submonoid, equal to $\mathcal{H}_G$ or central and generated by an element u^ℓ with $ev(u)$ of order dividing ℓ, then*

(i) *there is an equivalence relation on $\mathcal{H}_G \times \mathcal{S}$ given by*

$$(v, s) \quad \sim \quad (v', s') \qquad \Longleftrightarrow \qquad \exists \tilde{v} \in \mathcal{H}_G, \tilde{s} \in \mathcal{S} : v\tilde{r} = v'\tilde{s}, \; s\tilde{r} = s'\tilde{s}$$

Equivalence classes in case $\mathcal{S}$ generated by u^ℓ are written $[v/s]_u$.

(ii) *there is a well-defined monoid structure on equivalence classes induced by multiplication in $\mathcal{H}_G$, which for $\mathcal{S}$ central is*

$$\left[\tfrac{v}{u^{n\ell}}\right]_u \left[\tfrac{w}{u^{m\ell}}\right]_u = \left[\tfrac{vw}{u^{(n+m)\ell}}\right]_u .$$

This monoid is denoted by $\mathcal{H}_G\mathcal{S}^{-1}$.

(iii) *the* total monoid of fractions $\mathcal{H}_G\mathcal{H}_G^{-1}$ *together with the map is uniquely isomorphic to the enveloping group of $\mathcal{H}_G$*

(iv) *the monoid of fractions associated to u is isomorphic to the enveloping group if*
$$g \in G \quad \Longrightarrow \quad \exists v \in \mathcal{H}_G, n > 0 : gv = u^{n\ell} \in \mathcal{H}_G$$

Proof The first two claims follow from the following two properties, well-know from localisation of rings, see [19].

- $s \in \mathcal{S}, v \in \mathcal{H}_G \implies s\mathcal{H}_G \cap v\mathcal{S} \neq \emptyset$
- $s \in \mathcal{S}, v, w \in \mathcal{H}_G, sv = sw \implies \exists s' \in \mathcal{S} : vs' = ws'$

Both are immediate for $\mathcal{S}$ central. The argument in the case $\mathcal{S} = \mathcal{H}_G$ is only slightly more difficult. In fact we note that every element has a power that is central, since its evaluation in the finite group G has finite order and by Lemma 3.4.

$$\begin{aligned} s \in \mathcal{S}, v \in \mathcal{H}_G &\implies sv^{\operatorname{ord} ev(v)} = v^{\operatorname{ord} ev(v)}s \in s\mathcal{H}_G \cap v\mathcal{S} \\ s \in \mathcal{S}, v, w \in \mathcal{H}_G, sv = sw &\implies s^{\operatorname{ord} ev(s)}v = s^{\operatorname{ord} ev(s)}w \\ &\implies vs^{\operatorname{ord} ev(s)} = ws^{\operatorname{ord} ev(s)} \end{aligned}$$

To get the last two claims we first notice that for the monoids of fractions $\mathcal{H}_G\mathcal{S}^{-1}$ by construction any monoid homomorphism to a group factors. Thus we only need to show that every element has an inverse. This is trivially true for the the total monoid of fraction. In the case of claim iv) the elements of $\mathcal{H}_G$ given by a single letter $g \in G$ have an inverse $[v/u^{n\ell}]$ by hypothesis. Since these elements generate $\mathcal{H}_G$ every element in the monoid has an inverse. □

We followed this route of progressive abstraction to have a criterion for stably equivalence in terms of an equality in a monoid of fraction which even is a group in the cases we are mostly interested in:

Corollary 3.6 *Suppose $u \in \mathcal{H}_G$ has $ev(u) \in G$ of order ℓ, then the following are equivalent*

(i) *$v, w \in \mathcal{H}_G$ are u—stably equivalent*
(ii) $[v/1]_u = [w/1]_u \in \mathcal{H}_G\mathcal{S}^{-1}$

Obviously, the proof of the proposition works also to proof analogous claims for submonoids of the monoids $\mathcal{H}_{G,\Gamma}$. However we skip the details and only spell out the corresponding corollary.

Corollary 3.7 *Suppose $u \in \mathcal{H}_{G,\Gamma}$ has $ev(u) \in G$ of order ℓ and $\mathcal{S} = \{u^{n\ell}\}$, then the following are equivalent*

(i) *$v, w \in \mathcal{H}_{G,\Gamma}$ are u—stably equivalent*
(ii) $[v/1]_u = [w/1]_u \in \mathcal{H}_{G,\Gamma}\mathcal{S}^{-1}$

In particular, if we want to decide u-stable equivalence in $\mathcal{H}_{G,\Gamma}$ under the hypothesis that the monoid of fractions is a group, then we can do so in the finitely presented group Adj Γ.

Proposition 3.8 *Suppose $u \in \mathcal{H}_{G,\Gamma}$ has $ev(u) \in \Gamma$ of order ℓ and $\mathcal{S} = \{u^{n\ell}\}$ is a denominator set such that $\mathcal{H}_{G,\Gamma}S^{-1}$ is a group, then*

$$\begin{aligned} & v, w \in \Gamma^d \textit{ are } u - \textit{stably equivalent} \\ \iff \quad & e_{v_1} \cdots e_{v_d} \ = \ e_{w_1} \cdots e_{w_d} \ \in \ \mathrm{Adj}\,\Gamma \end{aligned}$$

Proof By the corollary above it suffices to show that the second claim of the proposition is equivalent to the second claim of the corollary. This follows since both groups involved are the enveloping group of the monoid $\mathcal{H}_G$, and the element involved are mapped to each other by the unique map provided by the universal property. For the monoid of fractions this was shown above, for the adjoint group it is proved by Kamada, Matsumoto [13]. It uses the fact that the universal properties of the enveloping group and the adjoint group provide mutually inverse group homomorphisms between the two. □

4 The Genus Zero Case

We investigate the relation between equivalence and stable equivalence of (pointed) G-covers branched over the disc, i.e. with $g = 0$. They will be shown to be equal for the G-covers with sufficiently 'rich' branching and detectable in monoids of fractions from the last section. This involves studying conditions on the Hurwitz vector u such that stability holds for geometric stabilisation by the corresponding G-cover C_u according to

Definition 4.1 We say that *stability holds* for geometric stabilisation by C_u, if there exists a positive integer m, such that

$$C_u - \text{stable equivalence} \quad = \quad G - \text{cover equivalence}$$

on the G-covers equivalent to covers obtained by m iterations of the C_u-stabilisation.

The condition of the definition easily translates into a condition on the algebraic side. Stability holds if

$$u - \text{stable equivalence} \quad = \quad \text{braid equivalence}$$

on the set of tuples braid equivalent to some vu^m. We are thus bound to study properties of Hurwitz vectors of the last kind in more detail.

Before entering into the discussion of stability, let us review some important tools, giving the definitions and providing their basis properties.

Definition 4.2 On the set $\coprod G^d$ of Hurwitz vectors for the finite group G

(i) the relation

$$v \leqslant w \quad :\Longleftrightarrow \quad \exists u \, : \, vu = w$$

is called the *prefix order* on $\coprod G^d$.

(ii) the relation

$$v \lessapprox w \quad :\Longleftrightarrow \quad \exists u \, : \, vu \approx w$$

is called the *weak prefix order* on $\coprod G^d$, and so is the induced order on $\mathcal{H}_G$.

(iii) an element $v \in G^d$ is said to *generate* the subgroup $H \subset G$ if

$$\langle v \rangle \quad := \quad \langle v_1, \dots, v_d \rangle \; = \; H.$$

Now let $\{\mathcal{C}_i\}$ denote the set of equivalence classes for conjugacy on G, then

$$\mathbb{Z}\big(G/_{\sim}\big) \quad = \quad \oplus_i \, \mathbb{Z}\,\mathcal{C}_i \quad \text{with} \quad \nu \leqslant \nu' \quad :\Longleftrightarrow \quad \nu_i \leqslant \nu_i', \; \forall i$$

is a free abelian group with a partial order. Note that more generally the free abelian group on a union Γ of conjugacy classes of G is naturally isomorphic to the abelianisation of the adjoint group for the conjugation quandle Γ

$$\oplus_{\mathcal{C}_i \subset \Gamma} \mathbb{Z}\,\mathcal{C}_i \quad = \quad H_1(\mathrm{Adj}(\Gamma), \mathbb{Z}).$$

Next we recall the basic properties of the *Nielsen type* and the subgroup generated by a Hurwitz vector, but leave the proofs to the reader, see also [2, 3].

Lemma 4.3 *The* Nielsen map *defined on* $\coprod G^d$ *with values in* $\oplus_{\mathcal{C}_i} \mathbb{Z}\,\mathcal{C}_i = \mathbb{Z}(G/_{\sim})$,

$$v = (v_1, \dots, v_d) \mapsto \nu(v) := \sum \nu_i \mathcal{C}_i$$
$$\nu_i \; = \; \#\{ j \mid v_j \in \mathcal{C}_i \}$$

has the following properties:

(i) ν *is a monoid homomorphism.*
(ii) ν *is constant on braid group orbits.*
(iii) ν *induces a monoid homomorphism on* $\mathcal{H}_G$ *also called* Nielsen map.
(iv) the Nielsen map is order preserving for both prefix orders

$$v \leqslant w \quad \Longrightarrow \quad v \lessapprox w \quad \Longrightarrow \quad \nu(v) \leqslant \nu(w),$$

Lemma 4.4 *The map* $\langle \; \rangle : \coprod G^d \to \{H \mid H \subset G\}$ *to the subgroups of G partially ordered by inclusion has the following properties:*

(i) $\langle \; \rangle$ *is order preserving for both prefix orders on* $\coprod G^d$.
(ii) $\langle \; \rangle$ *is constant on braid group orbits.*

(iii) $\langle\ \rangle$ *induces an order preserving map on $\mathcal{H}_G$ with respect to the weak prefix order.*

A special role will be given to the Hurwitz vector defined in terms of $\Gamma \subset G$

$$u_\Gamma = (\underbrace{g_1, \ldots, g_1}_{\text{ord } g_1}, \ldots, \underbrace{g_r, \ldots, g_r}_{\text{ord } g_r}),$$

where $g_1, \ldots, g_r$ is an enumeration of the elements of Γ. Its braid equivalence class is independent of choices thanks to Lemma 3.4, since $ev(u_\Gamma) = 1_G$. The corresponding denominator set generated by u_Γ will be denoted by $\mathcal{S}_\Gamma$.

In the sequel, our argument is motivated by the approach of Conway and Parker, see [9]. The first step is the first part of [15, Lemma 6.9].

Lemma 4.5 *Suppose g_1, g_2 are conjugate elements of order n in a finite group G and v is a Hurwitz vector which generates G, then*

$$vg_1^n \quad \approx \quad vg_2^n.$$

The second part of that lemma inspired our next result:

Lemma 4.6 *Let w be a Hurwitz vectors generating G and u a Hurwitz vector with entries in Γ, then*

$$\nu(w) \geqslant \nu(u_\Gamma u) \quad \Longrightarrow \quad \exists v \ : \ w \approx vu, \ v \textit{ generates } G$$

Proof Thanks to induction, it suffices to prove the claim in case u consists of a single entry g. By hypothesis the conjugacy class of g occurs so often that by the pigeonhole principle there is a g_1 conjugate to g which occurs at least $n = \text{ord } g + 1$ times. By Hurwitz moves we can obtain a Hurwitz vector $w'g_1^{n+1}$ braid equivalent to w, hence $w'g_1$ generates G. We apply Lemma 4.5 and obtain $w \approx w'g_1g^n$. Thus our claim follows for $u = g$ and $v = w'g_1g^{n-1}$. □

Now we are ready to prove a bunch of bijectivity results for certain map induced by stabilisation maps.

Lemma 4.7 *Let u be a Hurwitz vector and suppose $\nu_0 \geqslant \nu(u_{\Gamma(u)})$, then for all $n \geqslant 1$*

$$\begin{array}{ccc} \cdot u : \left\{ [v]_\approx \;\middle|\; \begin{array}{l} \nu(v) = n\nu(u) + \nu_0 \\ \langle u \rangle \subset \langle v \rangle \end{array} \right\} & \longrightarrow & \left\{ [w]_\approx \;\middle|\; \begin{array}{l} \nu(w) = (n+1)\nu(u) + \nu_0 \\ \langle u \rangle \subset \langle w \rangle \end{array} \right\} \\ [v]_\approx & \mapsto & [vu]_\approx \end{array}$$

is surjective and there exists $m = m(u, \nu_0)$, such that it is bijective for $n \geqslant m$.

Proof To prove surjectivity we want to apply Lemma 4.6. Indeed, for w with $[w]$ in the range, we let $G = \langle w \rangle$ and $\Gamma = \Gamma(u) \cap G$. Then u has entries in the set Γ and $\nu(w) \geqslant \nu(u) + \nu_0 \geqslant \nu(u_\Gamma u)$, since $u_\Gamma \lessapprox u_{\Gamma(n)}$. Hence we get the conclusion, that $[w] = [w'u]$ with $\langle w' \rangle = G \supset \langle u \rangle$.

Bijectivity follows from the fact, that an infinite chain of surjective maps between finite sets must eventually stabilise. □

Proposition 4.8 *Suppose u_Γ generates G invariably (i.e. $\nu(u'_\Gamma) = \nu(u_\Gamma)$ implies $\langle u'_\Gamma \rangle = G$), then there exists an integer $m = m(G)$ such that for*

- *any $\nu_1 \geqslant \nu_0 \geqslant m\nu(u_\Gamma)$ with $\nu_0, \nu_1 \in \nu(\coprod \Gamma^d)$*
- *any Hurwitz vector u with $\nu(u) + \nu_0 = \nu_1$*

juxtaposition of u induces a bijective map

$$\cdot u : \left\{ [v]_\approx \,\middle|\, \nu(v) = \nu_0 \right\} \longrightarrow \left\{ [v]_\approx \,\middle|\, \nu(v) = \nu_1 \right\}$$
$$[w]_\approx \mapsto [wu]_\approx$$

Proof Apply Lemma 4.7 and get $m = m(u_\Gamma, \nu(u_\Gamma))$ such that $u_\Gamma \cdot$ is bijective for $n \geqslant m$. There are now Hurwitz vectors u_0, u' such that

$$\nu(u_0) + m\nu(u_\Gamma) = \nu_0, \quad u_0 u u' \approx u_\Gamma^k \text{ for some } k > 0.$$

Then we get a factorisation of the bijective map u_Γ^k:

$$\left\{\nu(v) = m\nu_\Gamma\right\} \xrightarrow{\cdot u_0} \left\{\nu(v) = \nu_0\right\} \xrightarrow{\cdot u} \left\{\nu(v) = \nu_1\right\} \xrightarrow{\cdot u'} \left\{\nu(v) = k\nu_\Gamma\right\}$$

Since u_Γ generates G invariantly, every element in the given sets generates G, hence Lemma 4.7 gives surjectivity of all three maps. Thus we conclude that each of the surjective maps, in particular $\cdot u$ is bijective. □

Theorem 4.9 *Suppose u_Γ generates G, then there exists an integer $m = m(G)$ such that*

(i) for

- *any $\nu_1 \geqslant \nu_0 \geqslant m\nu(u_\Gamma)$ with $\nu_0, \nu_1 \in \nu(\coprod \Gamma^d) = \mathbb{N}(\Gamma/_\sim)$*
- *any subgroup H of G,*
- *any Hurwitz vector $u \in \coprod H^d$ with $\nu(u) + \nu_0 = \nu_1$*

juxtaposition of u induces a bijective map

$$\cdot u : \left\{ [v]_\approx \,\middle|\, \begin{array}{r} \nu(v) = \nu_0 \\ \langle v \rangle = H \end{array} \right\} \longrightarrow \left\{ [v]_\approx \,\middle|\, \begin{array}{r} \nu(v) = \nu_1 \\ \langle v \rangle = H \end{array} \right\}$$
$$[w] \mapsto [wu]$$

(ii) for any $\nu_1, \nu_0 \geqslant m\nu(u_\Gamma)$ *with* $\nu_0, \nu_1 \in \nu(\coprod \Gamma^d)$

$$\#\left\{[v]_\approx \,\middle|\, \nu(v) = \nu_0\right\} \quad = \quad \#\left\{[v]_\approx \,\middle|\, \nu(v) = \nu_1\right\}$$

Proof Let m be a common stability bound for the finitely many Hurwitz vectors u'_Γ with $\nu(u'_\Gamma) = \nu(u_\Gamma)$. Then we can handle the claim of i).

In case $\nu_0 \notin \nu(\coprod H^d)$, also ν_1 is not contained, so both sides are empty sets and the claim is trivially true. Otherwise pick $u_0 \in \coprod H^d$ with $\nu(u_0) = \nu_0$ and $u'_\Gamma \in \coprod H^d$ with $\nu(u'_\Gamma) = \nu(u_\Gamma)$ a multiple of $u_{\Gamma \cap H}$. Claim i) then follows as in the previous proofs.

The second claim is an immediate corollary, since the sets involved decompose over all possible subgroups H of G into the sets of the first claim and shown there to be bijective. □

Definition 4.10 Call m as in the theorem a Γ-*stability bound.*

Proposition 4.11 *Suppose* v, w *are Hurwitz vectors generating* G *with entries in* Γ*, and* $\nu(v) \geqslant \nu(u_\Gamma^m)$*, where* m *is a* Γ*-stability bound, then*

$$v \approx w \quad \Longleftrightarrow \quad [v/1] = [w/1] \in \mathcal{H}_{G,\Gamma}\mathcal{S}_\Gamma^{-1},$$

Proof We have only the prove the reverse implication. The first hypotheses on the right implies

$$\begin{aligned} v &\approx_u w && u_\Gamma\text{-stably} \\ \Longrightarrow \quad vu_\Gamma^\ell &\approx wu_\Gamma^\ell && \text{for some } \ell \\ \Longrightarrow \quad v &\approx w && \text{by theorem 4.9} \end{aligned}$$

□

In Corollary 3.6 we showed that elements with the same invariant are stably equivalent. In this proposition we show that elements with sufficiently high Nielsen class *belong to the stable range*, ie. where stable equivalence implies equivalence. The existence of a stability range expresses the fact that stability holds.

5 Generalisations

In this section we will aim for some generalisations going beyond the case of G-covers of the disc branched at a finite set of points up to G-equivariant covering isomorphisms.

Though we still want to understand stabilisation under 'geometric' composition with a G-cover C_u over the disc corresponding to some $u \in G^e$, there are several different direction open to generalisation:

- include G-covers over surfaces of higher genus or higher number of boundary components, one boundary component is needed at least to perform geometric composition.
- modify the notion of isomorphism on the base: restrict the induced maps to preserve isotopy classes of appropriate geometric objects.
- modify the notion of isomorphism on the fibres: restrict to maps preserving G-markings of a set of fibres.

To get the flavour of these generalisations, let us look at some examples and the corresponding algebraic structure.

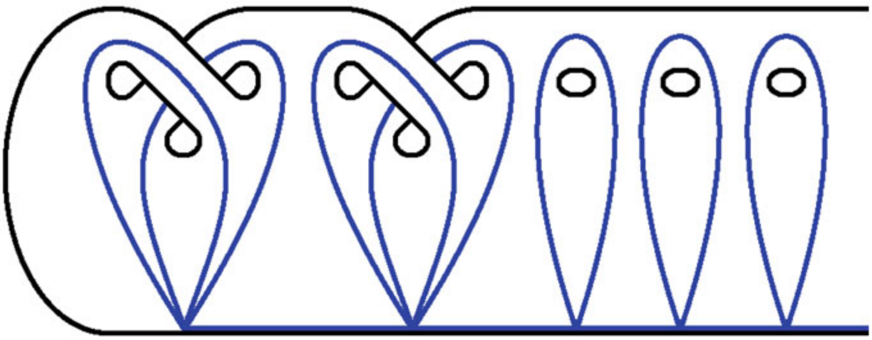
Example 5.1 On the geometric side consider a Riemann surface $\Sigma = \Sigma_g^1$ of genus g with one boundary component. Let p_0 be a point on the boundary, $(p_n) \subset \Sigma$ be a sequence of distinct points in the interior. Finally let α_i, β_i corresponding to handles, and a sequence (γ_n) corresponding to the interior points be elements of a geometric basis for $\pi_1(\Sigma \setminus (p_n), p_0)$

As before any (pointed) G-cover of Σ, which is unbranched outside $p_1, \ldots, p_d$ has a monodromy map that gives rise to the monodromy tuple of elements in G

$$\mu(\alpha_1), \mu(\beta_1), \ldots, \mu(\alpha_g), \mu(\beta_g), \mu(\gamma_1), \ldots, \mu(\gamma_d),$$

In the present case, such tuples are naturally acted on by the mapping class group $\mathcal{M}ap_{g,d}^1 := \mathcal{M}ap(\Sigma_g^1, \{p_1, \ldots, p_d\})$, and we recall that elements of G^{2g+d} are called (g, d)-Hurwitz vectors, since the Cartesian product is considered as a $\mathcal{M}ap_{g,d}^1$-set. We get the one to one correspondence

$$\{G\text{-covers of } \Sigma, \text{ branched outside } p_1, \ldots, p_d\} /_{\text{iso}} \quad \stackrel{1:1}{=\!=} \quad G^{2g+d}/\mathcal{M}ap_{g,d}^1$$

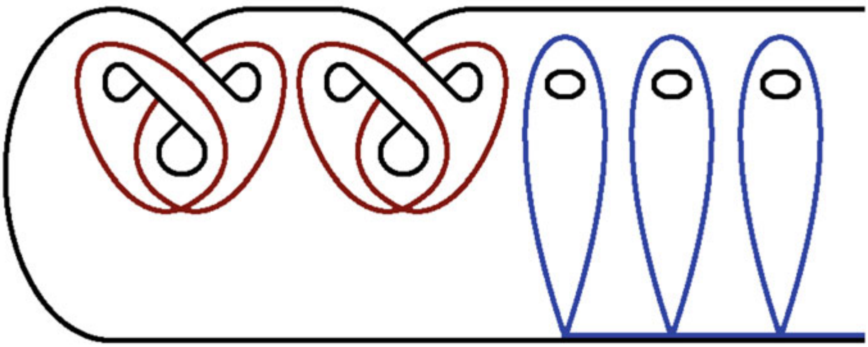


base with geometric basis

Example 5.2 Let us consider the G-covers of Σ as in the previous example. However we define two G-covers to be *isomorphic preserving chains* if they are isomorphic via some G-equivariant covering map such that the induced map preserves the free isotopy classes of the α_i, β_i.

Accordingly on the algebraic side we no longer have the action of the full mapping class group but rather that of the subgroup preserving the said isotopy classes. This subgroup can be identified with the mapping class group $\mathcal{M}ap_{0,d}^{g+1}$ of a subsurface Σ' of genus 0 with d punctures and $g+1$ boundary components obtained from Σ by cutting a regular neighbourhood of simple curves representing the given isotopy classes, thus

$$\{G\text{-covers of } \Sigma, \text{ branched outside } p_1, \ldots, p_d\} /_{\text{iso}_{\alpha,\beta}} \quad \stackrel{1:1}{=\!=} \quad G^{2g+d} / \mathcal{M}ap_{0,d}^{g+1}$$



base with curves representing two chains

Example 5.3 For the third kind of generalisation we look again at G-covers of the disc D equipped with distinct interior points (p_n), a geometric basis (γ_n) with respect to p_0 on the boundary. Let $q_0, \ldots, q_k$ be distinct points on the boundary with $q_0 = p_0$. We introduce a *q-marking* to be given by G-equivariant maps ℓ_i from G to the fibres over the q_i.

Then we get an extended monodromy map defined on the fundamental groupoid of the punctured base relative to the finite set of points q_i.

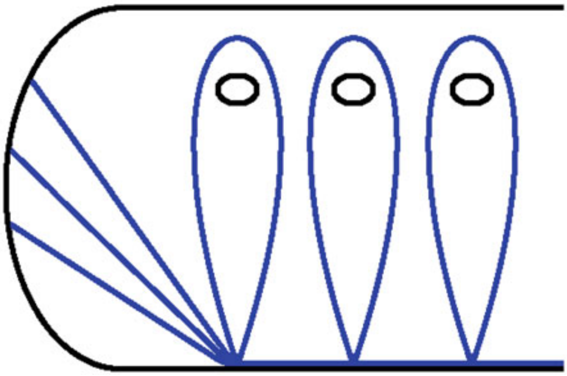
$$\mu : \pi_1^{gr-oid}(D \setminus \{p_1, \ldots, p_d\}, \{q_0, \ldots, q_k\}) \longrightarrow G$$

To the homotopy class of a path from q_i to $q_{i'}$ it associates the unique element g in G such that $g\ell_{i'}(1)$ gives the same point in the fibre at $q_{i'}$ as path lifting to $\ell_i(1)$.

The domain is a free groupoid on $d+k$ generators, and we end up with the one to one correspondence

$$\{\text{marked } G\text{-covers of } D, \text{ branched outside } p_1, \ldots, p_d\} /_{\text{iso}_{\text{marked}}} \quad \stackrel{1:1}{=\!=} \quad G^{k+d} / \text{Br}_d$$

The set G^{k+d} in this case is a Br_d-set as the Cartesian product of the trivial Br_d-set G^k and the Br_d-set G^d of Hurwitz vectors of genus 0.



base with geometric base of groupoid

The common features of these generalisations—and others—on the algebraic side should be well noted:

- we classify orbits of G-tuples,
- the action is by some geometrically distinguished mapping class group,
- Br_d acts as the mapping class group of a suitable neighbourhood of $\gamma_1, \ldots, \gamma_d$.

They will allow to get an algebraic model for composition with G-covers of genus 0 on the geometric side, which relies on the notion in the following definition:

Definition 5.4 Let M be a monoid and S be a set together with a map

$$\rho : S \times M \to S, \quad (s, m) \mapsto sm$$

such that

$$s(m_1 m_2) = (sm_1)m_2, \quad s1_M = s$$

Then we will call ρ a M*-action* and S a M*-set* (instead of the more common M*-act*).

Let us formulate the hypothesis in more abstract terms

Proposition 5.5 *Suppose there is an array of nested groups*

$$\begin{array}{ccccccc} \mathcal{M}_0 \subset & \mathcal{M}_1 & \subset & \mathcal{M}_2 & \cdots & \cdots & \mathcal{M}_d \cdots \\ & \cup & & \cup & & & \cup \\ & \mathrm{Br}_1 & \subset & \mathrm{Br}_2 & \cdots & \cdots & \mathrm{Br}_d \cdots \end{array}$$

with $\mathcal{M}_d$ *acting on* G^{r+d} *such that as a* Br_d*-set* G^{r+d} *is the Cartesian product of a trivial factor* G^r *and the* Br_d*-set* G^d *of Hurwitz vectors of genus* 0*, then*

(i) $\coprod_d G^{r+d}$ *is a* $\coprod_d G^d$*-set for concatenation,*

(ii) there is an induced action of the Hurwitz class monoid $\mathcal{H}_G$ on the $\mathcal{M}$-orbits

$$\mathcal{Q}_{G,\mathcal{M}} \quad := \quad \coprod_d G^{r+d}/\mathcal{M}_d$$

Proof Of course, the monoid $\coprod G^d$ is a $\coprod G^d$-set and the action is by concatenation. Thus the first claim follows from the observation that $\coprod_d G^{r+d}$ is an invariant subset.

For the second claim we have to show that the action on equivalence classes does not depend on the representatives. In fact

$$x\mathcal{M}_e = y\mathcal{M}_e, v\,\mathrm{Br}_d = w\,\mathrm{Br}_d \implies xv\mathcal{M}_{e+d} = yw\mathcal{M}_{e+d}$$

since the action of Br_d on G^d is the same as that on the second factor of the product $G^{r+e+d} = G^{r+e} \times G^d$ via the action of $\mathrm{Br}_{e+d} \subset \mathcal{M}_{e+d}$ provided Br_d is considered as the subgroup of Br_{e+d} braiding only the last d strands. □

There is still one more essential feature:

- the number of local monodromies in each conjugacy class is invariant under the equivalence.

In fact a choice of a geometric free basis can be made in such a way, that the local monodromies correspond to the entries of a tuple in G^{r+d} except for the first s entries, with $s \leqslant r$. Hence in extension of the Nielsen type we get

Definition 5.6 The *Nielsen map* ν is defined on G^{r+d} with values in the abelian group $\oplus_{\mathcal{C}_i}\mathbb{Z}\,\mathcal{C}_i = \mathbb{Z}(G/_\sim)$,

$$v = (v_1, \dots, v_{r+d}) \mapsto \nu(v) := \sum \nu_i \mathcal{C}_i$$
$$\nu_i \;=\; \#\{j > s \mid v_j \in \mathcal{C}_i\}$$

At this point we have collected enough information to turn back to the previous sections and see that their results generalise all along the way we have gone.

Remark 5.7 The action of $\mathcal{H}_G$ on $\mathcal{Q}_{G,\mathcal{M}}$ can be 'localised' at a denominator set $\mathcal{S}$ to yield a group action of the group $\mathcal{H}_G\mathcal{S}^{-1}$ on the set $\mathcal{Q}_{G,\mathcal{M}}\mathcal{S}^{-1}$.

Moreover if $u \in \mathcal{H}_G$ has $ev(u) \in G$ of order ℓ and $\mathcal{S} = \{u^{n\ell}\}$, then the following are equivalent

(i) $v, w \in \mathcal{Q}_{G,\mathcal{M}}$ are u—stably equivalent
(ii) $[v/1]_u = [w/1]_u \;\in\; \mathcal{Q}_{G,\mathcal{M}}\mathcal{S}^{-1}$

We will say more about stabilisation. However, instead of pushing the generalisation to the limits, we fix $\Gamma = \Gamma_G = G \setminus \{1_G\}$ which contains all non-trivial elements of G and generates G invariably.

Lemma 5.8 *Let u be a Hurwitz vector and suppose* $\nu_0 \geqslant (r-s+1)\nu_\Gamma$*, then*

$$\begin{array}{ccc} \left\{ v\mathcal{M} \,\middle|\, \nu(v) = n\nu(u) + \nu_0 \right\} & \xrightarrow{\cdot u} & \left\{ w\mathcal{M} \,\middle|\, \nu(w) = (n+1)\nu(u) + \nu_0 \right\} \\ v\mathcal{M} & \mapsto & vu\mathcal{M} \end{array}$$

is surjective for $n > 0$ *and there exists* $m = m(u, \nu_0)$*, such that it is bijective for* $n \geqslant m$.

Proof The larger bound for ν_0 is needed to make sure that for $w \in G^{r+d}$ its *tail* ${}^\tau w \in G^d$ has $\nu({}^\tau w) \geqslant \nu(u) + \nu_\Gamma$. Then we proceed as in the proof of Lemma 4.7 using Lemma 4.6 and that every Hurwitz vector with $\nu \geqslant \nu_\Gamma$ generates G by our general assumption $\Gamma = \Gamma_G$. □

Theorem 5.9 *Let* $\Gamma = \Gamma_G$*, then there exists an integer* $m = m(G)$ *such that for*

- *any* $\nu_1 \geqslant \nu_0 \geqslant m\nu(u_\Gamma)$
- *any Hurwitz vector u with* $\nu(u) + \nu_0 = \nu_1$

juxtaposition of u induces a bijective map

$$\begin{array}{ccc} \left\{ [v]_\mathcal{M} \,\middle|\, \nu(v) = \nu_0 \right\} & \xrightarrow{\cdot u} & \left\{ [v]_\mathcal{M} \,\middle|\, \nu(v) = \nu_1 \right\} \\ v\mathcal{M} & \mapsto & vu\mathcal{M} \end{array}$$

Proof Apply Lemma 5.8 to $u = u_\Gamma$ to get $m(u_\Gamma, (r+1)\nu_\Gamma)$ and let $m(G) = m(u_\Gamma, \nu(u_\Gamma)) + r + 1$ such that $u_\Gamma\cdot$ is bijective for $n \geqslant m$ on

$$\left\{ v\mathcal{M} \,\middle|\, \nu(v) = n\nu_\Gamma \right\}$$

There are now Hurwitz vectors u_0, u' such that

$$\nu(u_0) + m\nu(u_\Gamma) = \nu_0, \quad u_0uu' \approx u_\Gamma^k \text{ for some } k > 0.$$

Then we can conclude as in the proof of Proposition 4.8. □

This is the important task, since the analogous result for genus stabilisation will be applied in the argument for homological stability.

6 The Tautological Lift

In the final section we revise the definition of tautological central extension and recall some information, see [2], in particular the definition and properties of a quotient $H_{2,\Gamma}$ of $H_2(G)$ which proved to be crucial in our classification of curves with dihedral group of automorphisms.

We will then see how to recover that group in the set-up of the previous sections and deduce a classification result for (pointed) G-covers in the stable range.

Definition 6.1 Let G be a finite group and let $F = F_G$, $R = R_G$ be as before. For any union of conjugacy classes $\Gamma \subset G$, define

(i) the *tautological lift* $G \to F_G$, which maps $g \mapsto \hat{g}$,
(ii) the *tautological map* on $\coprod G^d : v = (v_1, \dots, v_d) \mapsto \hat{v}_1 \cdots \hat{v}_d$
(iii) the normal subgroup R_Γ normally generated by commutators in $[F, R]$ and tautological lifts of conjugacy relations for elements in Γ:

$$R_\Gamma \quad = \quad \langle\!\langle\, [F, R], \hat{a}\hat{b}\hat{c}^{-1}\hat{b}^{-1} \,\big|\, \forall a \in \Gamma, b \in G, c = b^{-1} ab \,\rangle\!\rangle$$

(Note that the given elements generate this normal subgroup as subgroup.)
(iv) the quotient group of F by R_Γ

$$G_\Gamma \quad = \quad F\big/R_\Gamma \, .$$

(v) the *boundary homomorphism* $\alpha : G_\Gamma \to G$, induced by $\hat{a} \mapsto a$ with kernel K_Γ.

Lemma 6.2 *With the notation just introduced, $R_\Gamma \subset R$ and $K_\Gamma = R\big/R_\Gamma$. In particular K_Γ is contained in the centre of G_Γ and the short exact sequence*

$$1 \to \frac{R}{R_\Gamma} \to G_\Gamma \to G \to 1$$

is a central extension.

Proof $[F, R] \subset R$ because R is normal in F. Moreover $\hat{a}\hat{b}\hat{c}^{-1}\hat{b}^{-1} \in R$ for any $a, b, c \in G$ with $ab = bc$, therefore $R_\Gamma \subset R$. By the definition of α we have that $K_\Gamma = \frac{R}{R_\Gamma}$. Finally, K_Γ is in the centre of G_Γ because $[F, R] \subset R_\Gamma$. □

The *tautological lift* $G \to G_\Gamma$, $a \mapsto \hat{a}$ is not a group homomorphism in general, but every element in G_Γ with image $g \in G$ can be written as $\hat{g}z = z\hat{g}$, with $z \in K_\Gamma$. Here, by abuse of notation, $\hat{g}$ denotes also the class of $\hat{g} \in F$ in $G_\Gamma = F\big/R_\Gamma$.

Remark 6.3 Let $\Gamma \subset G$ be the union of distinct conjugacy classes $C_1, \dots, C_t$ and let $g_1, \dots, g_r$ be the elements of $G \setminus \Gamma$, then the abelianisation G_Γ^{ab} of G_Γ is the free abelian group on $t + r$ generators

$$G_\Gamma^{ab} \quad \cong \quad \mathbb{Z}C_1 \oplus \cdots \oplus \mathbb{Z}C_t \oplus \mathbb{Z}g_1 \oplus \cdots \oplus \mathbb{Z}g_r .$$

The Nielsen map ν on $\coprod \Gamma^d$ factors through the tautological map and the abelianisation as

$$\textstyle\coprod_d \Gamma^d \quad \longrightarrow \quad G_\Gamma \quad \longrightarrow \quad G_\Gamma^{ab} \quad \longrightarrow \quad \oplus_i \mathbb{Z}C_i$$

Definition 6.4 Let $\Gamma \subset G$ be a union of non-trivial conjugacy classes of G. We define

$$H_{2,\Gamma}(G) = \ker\left(G_\Gamma \to G \times G_\Gamma^{ab}\right) ,$$

where $G_\Gamma \to G \times G_\Gamma^{ab}$ is the morphism with first component the boundary map α and second component the abelianisation.

Let us recall from [2] the precise relation between $H_2(G, \mathbb{Z})$ and $H_{2,\Gamma}(G)$.

Lemma 6.5 *Let G be a finite group and let $\Gamma \subset G$ be a union of nontrivial conjugacy classes. Write $G = \frac{F}{R}$ and $G_\Gamma = \frac{F}{R_\Gamma}$. Then, there is a short exact sequence*

$$1 \to \frac{R_\Gamma \cap [F, F]}{[F, R]} \to H_2(G, \mathbb{Z}) \to H_{2,\Gamma}(G) \to 1 .$$

In particular $H_{2,\Gamma}(G)$ is abelian.

Remark 6.6 The Schur multiplier is often interpreted as a cohomology group

$$H^2(G, \mathbb{Q}/\mathbb{Z})$$

which is algebraically dual to $H_2(G, \mathbb{Z})$. In case $\Gamma = G$ Moravec [16] identified the group $H_{2,G}(G)$ with a subgroup of $H^2(G, \mathbb{Q}/\mathbb{Z})$ introduced by Bogomolov and justly called Bogomolov multiplier by Moravec.

The algebraic object from the previous to enter the stage is the adjoint group associated a quandle from Definition 2.6. To emphasise its importance for the pair G, Γ we give a more specific name:

Definition 6.7 Suppose G is a finite group and $\Gamma \subset G$ a union of conjugacy classes, then the adjoint group for the quandle Γ

$$\mathrm{Adj}\,\Gamma \quad := \quad \langle e_g, g \in \Gamma \mid e_a e_b = e_b e_{a^b}, a, b \in \Gamma \rangle$$

is called the *tautological crossed module* of the pair G, Γ. The structural maps are

$$\partial_\Gamma : e_g \mapsto g, \quad \mathrm{Adj}\,\Gamma \times G \to \mathrm{Adj}\,\Gamma : (e_a, g) \mapsto e_{a^g} .$$

(It is a crossed module thanks to Proposition 2.7.)

The two following results establish the close relation between the 'old' tautological central extension and the 'new' tautological crossed module.

Proposition 6.8 *The map $e_a \mapsto \hat{a}$ for all $a \in \Gamma$ extends to an injective group homomorphism*

$$\mathrm{Adj}\,\Gamma \quad \longrightarrow \quad F/R_\Gamma$$

with left inverse.

Proof For the extension we only need to check that the relations of the domain map to R_Γ:

$$e_a e_b e_{a^b}^{-1} e_{b^{-1}} \quad \mapsto \quad \hat{a}\hat{b}\hat{c}^{-1}\hat{b}^{-1}, \text{ with } c = a^b = b^{-1}ab.$$

For the left inverse we first define a map $F \to \operatorname{Adj}\Gamma$ by

$$\hat{h} \quad \mapsto \quad e_{g_1} \cdots e_{g_r}$$

where for each $h \in G$ a unique factorisation $h = g_1 \cdots g_r$, $r = r(h)$ has been chosen, with $r = 1$, $g_1 = h$ if $h \in \Gamma$. Since the composition of the two maps is the identity on Γ, it induces the identity map on $\operatorname{Adj}\Gamma$ if R_Γ is in the kernel of this second map.

Let us note first, that by construction the map $F \to G$ factors through ∂_Γ. Accordingly R maps to $\ker \partial_\Gamma$. Now recall that for any crossed module $\partial : C \to G$ the kernel is central in C. In fact, conjugation by any element in the kernel is trivial by the Peiffer identity. We infer, that R maps to the centre of $\operatorname{Adj}\Gamma$ and hence $[F, R]$ maps to the identity.

We complete the proof by showing that this is true also for the remaining elements generating R_Γ:

$$\begin{aligned} \hat{g}\hat{h} \quad &\mapsto \quad e_g e_{g_1} \cdots e_{g_r} = e_{g_1} e_{g^{g_1}} e_{g_2} \cdots e_{g_r} = e_{g_1} \cdots e_{g_r} e_{g^h} \\ \hat{h}\widehat{h^{-1}gh} \quad &\mapsto \quad e_{g_1} \cdots e_{g_r} e_{g^h} \end{aligned}$$

□

Proposition 6.9 *If* Γ *generates* G*, then*

$$H_{2,\Gamma} \quad = \quad H(\Gamma, G) \;:=\; \ker \partial_\Gamma \cap [\operatorname{Adj}\Gamma, \operatorname{Adj}\Gamma]$$

Proof Since the maps from the previous proposition induce the identity on the quotient G and maps on the respective abelianisations, they also induce maps on the given groups. Therefore the group of the right is a subgroup of the other and it remains to show, that the kernel N of $G_\Gamma \to \operatorname{Adj}\Gamma$ intersects $H_{2,\Gamma}$ trivially.

By the construction of the map, N is normally generated by elements

$$\hat{g}_1 \ldots \hat{g}_r \hat{h}^{-1}, \quad \text{for all } h \notin \Gamma$$

They all map to $1 \in G$, so they are central and generate a free abelian subgroup in G_Γ of rank $|G \setminus \Gamma|$. The image in $\mathbb{Z}(G \setminus \Gamma)$ via G_Γ^{ab} is of the same rank, since each generator is mapped to a standard generator. Thus N maps injectively to G_Γ^{ab} and thus intersects trivially with $H_{2,\Gamma}$. □

To pursue the proof of the following classification result, we first need to introduce another Nielsen map.

Definition 6.10 The canonical extension of the Nielsen map $\mathcal{H}_G \to \oplus_i \mathbb{Z}\mathcal{C}_i$ to the localisation

$$\begin{aligned} \nu : \mathcal{H}_G \mathcal{S}_\Gamma^{-1} &\longrightarrow \oplus_i \mathbb{Z}\mathcal{C}_i \\ \left[v/u_\Gamma^\ell\right] &\mapsto \nu(v) - \ell\nu(u_\Gamma) \end{aligned}$$

is also called *Nielsen map*.

Theorem 6.11 *Suppose u_Γ generates G, then there exists an integer $m = m(G)$ such that for any $\nu_0 \geqslant m\nu(u_\Gamma)$ with $\nu_0 \in \nu(\coprod \Gamma^d)$*

$$\#\left\{ [v]_\approx \;\middle|\; \begin{array}{c} \nu(v) = \nu_0 \\ \langle v \rangle = G \end{array} \right\} = \#H_{2,\Gamma} \times [G, G]$$

Proof First we note that there is an exact sequence induced from the central extension in Lemma 6.2

$$H_{2,\Gamma} \longrightarrow \ker \nu \longrightarrow [G, G]$$

Moreover, any $[v/u_\Gamma^\ell] \in \ker \nu$ gives two maps

$$\cdot u_\Gamma, \cdot v : \left\{ [v]_\approx \;\middle|\; \begin{array}{c} \nu(v) = \nu_0 \\ \langle v \rangle = G \end{array} \right\} \longrightarrow \left\{ [v]_\approx \;\middle|\; \begin{array}{c} \nu(v) = \nu_0 + \ell\nu_\Gamma \\ \langle v \rangle = G \end{array} \right\}$$

Both are bijective since we are above the stability bound, hence we can define a well-defined action

$$[w]_\approx [v/u_\Gamma^\ell] \quad = \quad [w'], \quad \text{such that } wv \approx w' u_\Gamma^\ell.$$

Again using the fact, that we are in the stable range, the set is mapped injectively to the enveloping group $\operatorname{Adj}\Gamma$. This can be exploited to show that the action is free and transitive, because the action is now identified with multiplication inside $\operatorname{Adj}\Gamma$ by the subgroup $\ker \nu$.

While freeness is immediate, we are left to check, that any two elements w, w' are in one orbit: There is a unique element in the group $\operatorname{Adj}\Gamma$ which maps one to the other, which again we can write as $[v/u_\Gamma^\ell]$. Under the Nielsen map, it must map to 0, since ν is a homomorphism on $\operatorname{Adj}\Gamma$. Thus this element is in $\ker \nu$ and we have also proved transitivity. The claim on the cardinality of the set is then obvious. □

Rephrased in more geometrical terms the statement of the theorem tells us:

> In the stable range connected (pointed) G-covers are classified up to equivalence by the Nielsen type, the evaluation, and an element in $H_{2,\Gamma}$.

Do not miss the caveat: the homological information is not canonical, but depends on the choice of an element in each fibre of $\operatorname{Adj}\Gamma \to G \times \oplus_i \mathbb{Z}\mathcal{C}_i$.

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Dessins d'Enfants and Brauer Configuration Algebras

Goran Malić and Sibylle Schroll

Abstract In this paper we associate to a dessin d'enfant an associative algebra, called a Brauer configuration algebra. This is an algebra given by quiver and relations induced by the monodromy of the dessin d'enfant. We show that the dimension of the Brauer configuration algebra associated to a dessin d'enfant and the dimension of the centre of this algebra are invariant under the action of the absolute Galois group. We give some examples of well-known algebras and their dessins d'enfants. Finally we show that the Brauer configuration algebra of a dessin d'enfant and its dual share the same path algebra.

Keywords Dessins d'enfants · Galois invariant · Absolute Galois group · Finite dimensional algebra · Brauer configuration algebra

2000 Mathematics Subject Classification Primary 16P10 · 11G32 · 14H57

1 Introduction

In this paper we associate a newly defined class of finite dimensional algebras called Brauer configuration algebras [9] to dessins d'enfants. This new algebraic structure generalises Brauer graph algebras, a well-understood class of algebras of tame representation type. Brauer graph algebras arise from clean dessins d'enfants, that is, those dessins d'enfants for which the ramification indices above 1 are all equal to 2;

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by relaxing this restriction we obtain Brauer configuration algebras. The connection between Brauer graph algebras and clean dessins d'enfants is established in [17].

There is an action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on Brauer configuration algebras induced by the natural action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on dessins d'enfants. We prove that this action keeps invariant certain algebraic data, such as the dimension of the Brauer configuration algebra A associated to a dessin d'enfant, as well as the dimension of the centre of A. More invariants in the special case of clean dessins d'enfants and Brauer graph algebras are shown in [17]. Our hope is that the results in this paper will open the theory of dessins d'enfants to the techniques of representation theory and homological algebra with the idea of eventually leading to new Galois invariants.

A dessin d'enfant (dessin for short) is a classical combinatorial object given by a cellular embedding of a bipartite graph on a connected, closed and orientable surface. However, a dessin induces on its underlying surface the structure of an algebraic curve defined over the algebraic numbers $\overline{\mathbb{Q}}$ and conversely, any algebraic curve over $\overline{\mathbb{Q}}$ corresponds to a dessin. Consequently, there is a natural faithful action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the set of dessins; both results are due to Belyĭ [2]. Dessins were introduced by Grothendieck in his influential Esquisse d'un Programme [11] where they are considered as a starting point to the study of the structure of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ as the automorphism group of a tower of so-called Teichmüller groupoids of moduli spaces of curves in any genus. One of the main goals of the theory of dessins d'enfants is to understand the invariants of this action and consequently to be able to distinguish between any two orbits. Several invariants that distinguish between certain orbits are known, but a complete answer does not seem to be within reach.

Brauer configuration algebras are a new class of associative symmetric algebras (equipped with a non-degenerate symmetric form) generalising Brauer graph algebras [9]. Brauer graph algebras originate in the modular representation theory of finite groups [12] and their representation theory is well understood. While Brauer graph algebras are of tame representation type, that is, in every dimension the isomorphism classes of their indecomposable representations can be parametrised by finitely many one parameter families, Brauer configuration algebras are, in general, of wild representation type. An algebra A is of wild representation type if for any finite dimensional algebra there exists a representation embedding into the module category of A, that is in some sense their representation theory contains the representation theory of any finite dimensional algebra. It is generally believed that the finite dimensional representations of a wild algebra cannot be classified in a meaningful way. However, for the class of Brauer configuration algebras, a precise structural description of all representations has been given in [8]. Furthermore, the dimensions of both the Brauer configuration algebras [9] and of the zeroth Hochschild cohomology space of a Brauer configuration algebra [22] are known.

We start with a rapid introduction to both Grothendieck's theory of dessins d'enfants in Sect. 2 and to associative algebras given by quivers and relations in Sect. 3. In Sect. 4 we introduce Brauer configuration algebras defined in [9] and show how they naturally arise from dessins d'enfants. In Sect. 5 we give two examples of well-known algebras that arise as Brauer configuration algebras of specific dessins d'enfants, the symmetric Nakayama algebras and quotients of preprojective algebras of type $\tilde{A}$. In Sect. 6 we discuss the Brauer configuration algebra associated to the dessin D^* dual to D and show that D^* and D have the same quiver when D has no faces and black vertices of degree 1. In Sect. 7 we discuss the Galois action on Brauer configuration algebras, and show that the centre of a Brauer configuration algebra as well as the dimension of the Brauer configuration algebra are invariant under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

2 Dessins d'Enfants

We start by briefly recalling Grothendieck's theory of dessins d'enfants. For a more detailed introduction to the topic see for example [6, 13, 15, 20]. We closely follow the exposition in [16].

A *dessin d'enfant* (dessin for short) is a finite bipartite connected graph G (with multiple edges allowed) embedded cellularly on a connected, closed and orientable surface X. In this embedding the vertices of G are points on X coloured in black or white, the edges are curved segments on X intersecting only at the vertices so that each edge ends in exactly one black and one white vertex, and the complement of the embedding is a finite union of connected components homeomorphic to an open 2-cell, which we call *faces* of the dessin. Two dessins (G_1, X_1) and (G_2, X_2) are isomorphic if there is an orientation preserving homeomorphism $X_1 \to X_2$ that restricts to a bipartite graph isomorphism $G_1 \to G_2$.

Equivalently, a dessin is a pair (X, f) where X is a compact Riemann surface and $f : X \to \mathbb{CP}^1$ is a holomorphic ramified covering of the Riemann sphere, ramified over a subset of $\{0, 1, \infty\}$. The pair (X, f) is also called a *Belyĭ pair* and the map f is called a *Belyĭ map* or a *Belyĭ function*. Two dessins (X_1, f_1) and (X_2, f_2) are isomorphic if they are isomorphic as ramified covers, i.e. if there is an orientation preserving homeomorphism $h: X_1 \to X_2$ such that $f_1 = f_2 \circ h$.

The equivalence between the two definitions is obtained as follows: given a dessin (X, f), the preimage $f^{-1}([0, 1])$ of the closed unit interval corresponds to a cellular embedding of a bipartite graph on the underlying topological surface of X such that the black and white vertices correspond to $f^{-1}(0)$ and $f^{-1}(1)$ respectively, and the edges correspond to the preimages of the open unit interval.

Conversely, given a dessin on a topological surface X, add a single new vertex to the interior of each face. To distinguish it from the black and white vertices, we will represent these vertices by diamonds $\diamond$. Now triangulate X by connecting the diamonds with the black and white vertices that are on the boundaries of the corresponding faces. Following the orientation of X, call the triangles with vertices oriented as $\bullet$-$\circ$-$\diamond$-$\bullet$ positive, and call other triangles negative (see Fig. 1).

Now send the positive and negative triangles to the upper and lower half-plane of $\mathbb{C}$, respectively, and send the sides of the triangles to the real line so that black, white and diamond vertices are sent to 0, 1 and ∞, respectively. As a result, we obtain

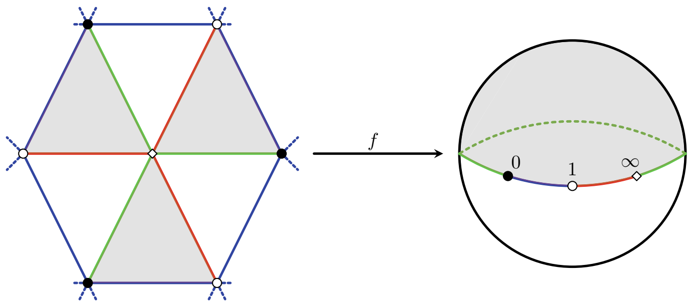


Fig. 1 The positive (shaded) and negative triangles are mapped to the upper and lower-half plane, respectively. The sides of the triangles are mapped to $\mathbb{R} \cup \{\infty\}$ so that the black and white vertices map to 0 and 1, respectively, and the face centres map to ∞

a ramified cover $f : X \to \mathbb{CP}^1$, ramified only over a subset of $\{0, 1, \infty\}$. We now impose on X the unique Riemann surface structure which makes f holomorphic.

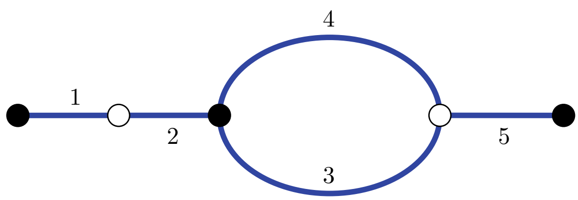
2.1 A Permutation Representation of Dessins

The monodromy action induced by lifts under f of simple closed loops on $\mathbb{CP}^1$ based at $1/2$ and circling around 0 and 1 gives one further definition of dessins via group theory: a dessin is the conjugacy class of a 2-generated transitive subgroup $\langle\sigma, \alpha\rangle$ of S_n, where n is the degree of f. Two dessins $\langle\sigma_1, \alpha_1\rangle$ and $\langle\sigma_2, \alpha_2\rangle$ are isomorphic if they are isomorphic as permutation groups.

Throughout this section let (X, f) be a dessin with n half-edges (or, equivalently, such that f is a degree n covering). In this section we describe how each such dessin can be represented by a triple $(\sigma, \alpha, \varphi)$ of permutations in the symmetric group S_n. However, we first fix the following notation which we will keep throughout the paper.

Convention 1 We label the half-edges of a dessin with the elements of the set $\{1, \ldots, n\}$ so that, when standing at a black vertex, and looking towards an adjacent white vertex, the label is placed on the 'left side' of the half-edge. See Fig. 2 for an example.

Fig. 2 Labelling of half-edges. The labels are always on the left when looking from a black vertex to its adjacent white vertices



Following Convention 1, label the half-edges of a dessin arbitrarily. Now let σ and α denote the permutations which record the cyclic (counter-clockwise) orderings of the labels around black and white vertices, respectively, and let φ denote the permutation which records the counter-clockwise ordering of the labels within each face.

Example 2.1 For the dessin in Fig. 2 we have $\sigma = (1)(2\,3\,4)(5)$, $\alpha = (1\,2)(3\,5\,4)$ and $\varphi = (1\,4\,5\,2)(3)$. The cycles of length 1 are usually dropped. Note that the cycle corresponding to the 'outer face' is, from the reader's perspective, recorded clockwise. This does not violate our convention since the labels of that face should be viewed from the opposite side of the sphere and switch orientation once the face is unfolded into a disc.

A change of labels corresponds to simultaneous conjugation of σ, α and φ by some element in S_n. Therefore, any dessin can be represented, up to conjugation, by a triple of permutations.

Definition 2.2 The length of a cycle in σ or α corresponding to a black or a white vertex, respectively, is called the *degree* of the vertex. The length of a cycle in φ corresponding to a face is called the *degree* of the face. Thus, the degree of a vertex is the number of half-edges incident to it, while the degree of a face is half the number of half-edges on its boundary.

A triple $(\sigma, \alpha, \varphi)$ representing a dessin $D = (X, f)$ satisfies the following properties:

- the group $\langle \sigma, \alpha, \varphi \rangle$ acts transitively on the set $\{1, \ldots, n\}$ and
- $\sigma\alpha\varphi = 1$.

The first property above is due to the fact that dessins are connected while the second is due to the following: consider three non-trivial simple loops γ_0, γ_1 and γ_∞ in $\pi_1(\mathbb{CP}^1 \backslash \{0, 1, \infty\}, 1/2)$ based at $1/2$ and going around 0, 1 and ∞ once, respectively. The lifts of these loops under f correspond to paths on X that start and end at a (possibly the same) point in $f^{-1}(1/2)$. We observe the following.

- Every half-edge of D contains precisely one element of $f^{-1}(1/2)$ since f is unramified at $1/2$.
- The cardinality of $f^{-1}(1/2)$ is precisely n. Hence there is a bijection $f^{-1}(1/2) \to \{1, \ldots, n\}$.
- With respect to this bijection, σ, α and φ can be thought of as permutations of $f^{-1}(1/2)$.

Therefore the loops γ_0, γ_1 and γ_∞ induce σ, α and φ. Since the product $\gamma_0\gamma_1\gamma_\infty$ is trivial in $\pi_1(\mathbb{CP}^1 \backslash \{0, 1, \infty\}, 1/2)$, the corresponding permutation $\sigma\alpha\varphi$ must be trivial as well.

We have now seen that to every dessin with n half-edges we can assign a triple of permutations in S_n such that their product is trivial and the group that they generate acts transitively on the set $\{1, \ldots, n\}$.

In a similar way we can show that this assignment works in the opposite direction as well: given three permutations σ, α and φ in S_n such that $\sigma\alpha\varphi = 1$ and such that the group that they generate acts transitively on $\{1, \dots, n\}$, we can construct a dessin with n half-edges respecting the order imposed by the permutations σ, α and φ. Therefore, up to simultaneous conjugation, a dessin is uniquely represented by a transitive triple $(\sigma, \alpha, \varphi)$ with $\sigma\alpha\varphi = 1$, and such a triple recovers a unique dessin up to isomorphism.

Remark 2.3 Dessins correspond to 2-generated transitive permutation groups since we can set $\varphi = (\sigma\alpha)^{-1}$. However, we prefer to emphasise all three permutations.

We will use the notation $D = (\sigma, \alpha, \varphi)$ to denote that a dessin D is represented by the triple $(\sigma, \alpha, \varphi)$.

2.2 Belyĭ's Theorem

Belyĭ's theorem is the starting point of Grothendieck's remarkable Esquisse d'un Programme [11] in which he sketches an approach towards understanding the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over the rationals as an automorphism group of a certain topological object. We restate the theorem here.

Theorem 2.4 (Belyĭ) *Let* X *be a smooth projective algebraic curve defined over* $\mathbb{C}$. *Then* X *is defined over* $\overline{\mathbb{Q}}$ *if and only if there is a holomorphic ramified covering* $f : X \to \mathbb{CP}^1$, *ramified at most over a subset of* $\{0, 1, \infty\}$.

Aside from Belyĭ's own papers [2, 3], various other proofs can be found in, for example, [23, Theorem 4.7.6] or [6, Chap. 3] or the recent proof in [7]. Belyĭ himself concluded that the above theorem implies that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ embeds into the outer automorphism group of the profinite completion of the fundamental group of $\mathbb{CP}^1 \backslash \{0, 1, \infty\}$. However it was Grothendieck who observed that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ must therefore act faithfully on the set of dessins as well. This interplay between algebraic, combinatorial and topological objects is what prompted Grothendieck to develop Esquisse d'un Programme. For more detail, see [21] or [23].

2.3 Galois Action on Dessins

Let $D = (X, f)$ be a dessin. If X is of genus 0, then necessarily $X = \mathbb{CP}^1$ and $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ is a rational map with critical values in the set $\{0, 1, \infty\}$. If $f = p/q$, where $p, q \in \mathbb{C}[z]$, then Belyĭ's theorem implies that $p, q \in \overline{\mathbb{Q}}[z]$. Moreover, the coefficients of both p and q generate a finite Galois extension K of $\mathbb{Q}$. Therefore $p, q \in K[z]$. Then $\mathrm{Gal}(K/\mathbb{Q})$ acts on f by acting on the coefficients of p and q, that is if $\theta \in \mathrm{Gal}(K/\mathbb{Q})$ and

$$f(z) = \frac{a_0 + a_1 z + \cdots + a_m z^m}{b_0 + b_1 z + \cdots b_n z^n},$$

$$\text{then } f^{\theta}(z) = \frac{\theta(a_0) + \theta(a_1)z + \cdots + \theta(a_m)z^m}{\theta(b_0) + \theta(b_1)z + \cdots \theta(b_n)z^n}.$$

If X is of positive genus, then as an algebraic curve it is defined by the zero-set of an irreducible polynomial F in $\mathbb{C}[x, y]$. This time we must take into consideration the coefficients of both F and f which, due to Belyĭ's theorem again, generate a finite Galois extension K of $\mathbb{Q}$. Similarly as in the genus 0 case, $\mathrm{Gal}(K/\mathbb{Q})$ acts on D by acting on the coefficients of both F and f simultaneously.

It is not immediately clear that the action of some automorphism in $\mathrm{Gal}(K/\mathbb{Q})$ on a Belyĭ map f will produce a Belyĭ map. However, this is indeed the case and we refer the reader to the discussion in [15, Sect. 2.4.2].

Since any $\mathbb{Q}$-automorphism of K extends to an $\mathbb{Q}$-automorphism of $\overline{\mathbb{Q}}$ [4, Chap. 3], we truly have an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the set of dessins.

Definition 2.5 We will denote by $D^{\theta} = (X^{\theta}, f^{\theta})$ the dessin that is the result of the action of $\theta \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $D = (X, f)$. We will also say that D^{θ} is *conjugate* to D.

The following example is borrowed from [15, Ex. 2.3.3].

Example 2.6 Let $D = (X, f)$ be a dessin where X is the elliptic curve

$$y^2 = x(x-1)(x-(3+2\sqrt{3})),$$

and $f : X \to \mathbb{CP}^1$ is the composition $g \circ \pi_x$, where $\pi_x : X \to \mathbb{CP}^1$ is the projection to the first coordinate and $g : \mathbb{CP}^1 \to \mathbb{CP}^1$ is given by

$$g(z) = -\frac{(z-1)^3(z-9)}{64z}.$$

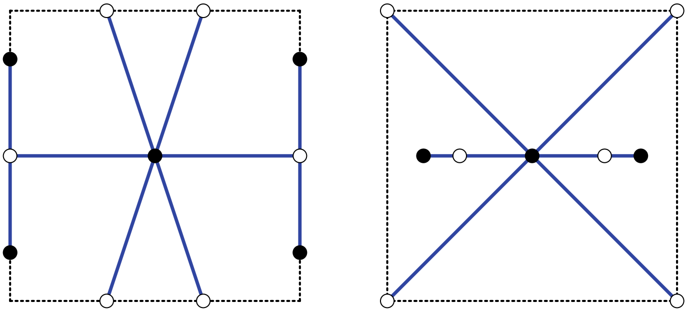
The corresponding bipartite map is depicted on the left in Fig. 3.

Note that we must consider $g \circ \pi_x$ and not just π_x since π_x is not a Belyĭ map; it is ramified over four points, namely 0, 1, $3 + 2\sqrt{3}$ and ∞. However, g maps these four points onto the set $\{0, 1, \infty\}$ and therefore $g \circ \pi_x$ is a true Belyĭ map.

The Galois extension that the coefficients of X and f generate is $K = \mathbb{Q}(\sqrt{3})$ and the corresponding Galois group has only one non-trivial automorphism, namely $\theta : \sqrt{3} \mapsto -\sqrt{3}$. Therefore X^{θ} is the elliptic curve $y^2 = x(x-1)(x-(3-2\sqrt{3}))$. The curve X^{θ} is non-isomorphic to X, which can easily be seen by computing the j-invariants of both.

What about f^{θ}? In this case, $\pi_x : X^{\theta} \to \mathbb{CP}^1$ is unramified over $3 + 2\sqrt{3}$ and ramified over $3 - 2\sqrt{3}$. However, g maps $3 - 2\sqrt{3}$ to 0 as well, and since g is defined over $\mathbb{Q}$, the Belyĭ functions f and f^{θ} coincide.

The bipartite map corresponding to (X^{θ}, f^{θ}) is depicted on the right in Fig. 3.



$X\colon y^2 = x(x-1)(x-(3+2\sqrt{3}))$ $\quad$ $X^\theta\colon y^2 = x(x-1)(x-(3-2\sqrt{3}))$

$f\colon (x,y) \mapsto x \mapsto -\frac{(x-1)^3(x-9)}{64x}$ $\quad$ $f^\theta\colon (x,y) \mapsto x \mapsto -\frac{(x-1)^3(x-9)}{64x}$

Fig. 3 The two dessins (X, f) and (X^θ, f^θ) from Example 2.6. The dotted lines indicate the boundary of the polygon representation of a genus 1 surface with the usual identification of the left-and-right and top-and-bottom sides

This action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on dessins is faithful already on the set of *trees*, that is the genus 0 dessins with precisely one face and Shabat polynomials as Belyĭ functions. However, this is not straight-forward (proofs can be found in [6, 20]) and, surprisingly, it is much easier to show faithfulness in genus 1 [6, Sect. 4.5.2]. Moreover, the action is faithful in every genus [6, Sect. 4.5.2].

3 From a Dessin d'Enfant to a Path Algebra

Recall that a quiver $Q = (Q_0, Q_1, s, t)$ is given by a finite set of vertices Q_0 and a finite set of oriented edges Q_1, called arrows, and functions $s, t : Q_1 \to Q_0$ where for $a \in Q_1$, $s(a)$ denotes the start of a and $t(a)$ denotes its target.

Consider a dessin D in its permutation representation $(\sigma, \alpha, \varphi)$. The permutation σ induces a quiver $\mathring{Q}$ in the following way. The vertices $\mathring{Q}_0$ of the quiver correspond to the white vertices of D and the set of arrows $\mathring{Q}_1$ is induced by the permutation σ in the following way: if $(i_1 \cdots i_k)$ is a non-trivial cycle in σ and if $\alpha_{i_1}, \ldots, \alpha_{i_k}$ denote the vertices of the quiver corresponding to the white vertices in D incident with $i_1, \ldots, i_k$ respectively, then we define a cycle of arrows in the quiver by setting $\alpha_{i_1} \to \alpha_{i_2} \to \cdots \to \alpha_{i_k} \to \alpha_{i_1}$. Note that the trivial cycles of σ induce loop arrows in $\mathring{Q}$.

Example 3.1 The dessin in Fig. 4 is given by the permutations

$$\sigma = (1\ 2\ 3\ 4\ 5)(6\ 7)(8\ 9)(10)(11),$$
$$\alpha = (1)(2\ 10\ 11)(3\ 6\ 9)(4\ 5)(7\ 8)$$

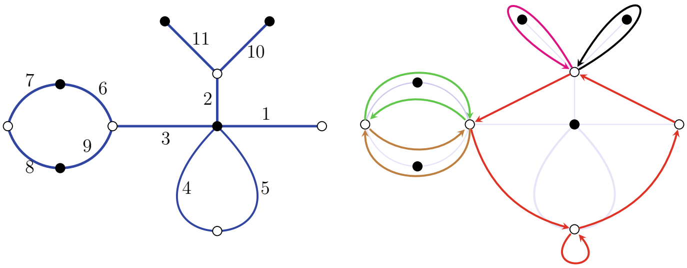


Fig. 4 A dessin D (left) and its quiver on the right. The differently coloured arrows in the quiver correspond to the different non-trivial cycles of the permutation σ of D

in S_{11}. The corresponding quiver is shown on the right of Fig. 4. The differently coloured arrows correspond to the different cycles of the permutation σ.

Let K be a field. Recall that given a quiver Q, the *path algebra* KQ, has vector space basis given by all possible finite paths in Q including a trivial path e_j for every vertex $j \in Q_0$. The multiplication of two paths is given by concatenation if possible and zero otherwise. The multiplicative identity of KQ is given by $\sum_{j \in Q_0} e_j$, the sum of the trivial paths in Q. If Q has an oriented cycle then KQ is an infinite dimensional algebra. For further details on paths algebras of quivers, see for example [1, 19].

Example 3.2 The quiver $\mathring{Q}$ associated to the trivial dessin $D = (\mathbb{CP}^1, z)$ is given by a single vertex and a single loop arrow a. A basis for $K\mathring{Q}$ is given by the paths $\{e, a, a^2, \ldots, a^n, \ldots\}$ and $K\mathring{Q}$ is therefore isomorphic to the polynomial algebra $K[x]$.

Quivers associated to dessins always have at least one oriented cycle and therefore the associated path algebras are always infinite dimensional.

3.1 *Admissible Ideals of KQ and Bound Quiver Algebras*

Let Q be a quiver and $\mathcal{R}$ be the two-sided ideal of KQ generated by all arrows in Q. This ideal decomposes as

$$KQ_1 \oplus \cdots \oplus KQ_l \oplus \cdots ,$$

where KQ_l is the subspace of KQ generated by all paths of length l. The *l-th power* $\mathcal{R}^l$ *of* $\mathcal{R}$ is the subspace with a basis of all paths of length at least l with decomposition

$$\mathcal{R}^l = \bigoplus_{r \geq l} KQ_r.$$

Definition 3.3 We say that a two-sided ideal I of KQ is *admissible* if there exists $n \geq 2$ such that

$$\mathcal{R}^n \subseteq I \subseteq \mathcal{R}^2.$$

If I is an admissible ideal of KQ we say that the quotient algebra KQ/I is a *bound quiver algebra*.

Bound quiver algebras are finite dimensional. They are indecomposable if and only if the quiver is connected.

The introduction of admissible ideals and bound quiver algebras is far from arbitrary. It is a well known result in representation theory due to Gabriel [5] stating that if K is algebraically closed, every connected finite dimensional K-algebra is Morita equivalent to a bound algebra KQ/I for a unique quiver Q and an admissible ideal I of KQ.

4 From a Dessin d'Enfant to a Brauer Configuration Algebra

In this section, given a dessin D we will define a quiver Q_D and an admissible ideal I_D of KQ_D so that the quotient algebra KQ_D/I_D is finite dimensional. Let D be a dessin with m black vertices σ_j, for $j = 1, \dots, m$ and n white vertices α_k, for $k = 1, \dots, n$. Let $\mathcal{L}$ be the set of loop arrows in $\mathring{Q}_1$ induced by a black vertex of degree 1. We will refer to the elements in $\mathcal{L}$ as *formal loop arrows*. We recall that the quiver $\mathring{Q}$ has vertices corresponding to the white vertices of D and if $D = (\sigma, \alpha, \varphi)$ then the arrows of $\mathring{Q}$ are induced by the permutation σ.

Definition 4.1 The quiver $Q_D = (Q_0, Q_1)$ associated to D is the quiver $Q_D = (Q_0, Q_1)$ with $Q_0 = \mathring{Q}_0$ as its vertex set and $Q_1 = \mathring{Q}_1 \setminus \mathcal{L}$ as its arrow set.

Definition 4.2 Let σ_j be a black vertex of degree $\deg \sigma_j \geq 2$. We call a cycle $a_{i_1} a_{i_2} \cdots a_{i_{\deg \sigma_j}}$ in Q_D induced by σ_j a *special* σ_j*-cycle*. When a black vertex is unspecified, the corresponding cycle in Q_D will be called a special σ-cycle. Black vertices of degree 1 by construction of Q_D do not contribute any cycles.

Furthermore, a special σ-cycle starting at the white vertex α_k is called *a special* σ*-cycle at* α_k.

Example 4.3 Suppose that σ_j is a black vertex of degree 5 such that the white vertices incident to it are labelled by $\alpha_1, \dots, \alpha_4$ in the counter-clockwise order, as in Fig. 5. Furthermore, let $a_1, \dots, a_5$ be the arrows as in Fig. 5. Then the special σ_j-cycles in Fig. 5 are given by $a_1a_2a_3a_4a_5$, $a_2a_3a_4a_5a_1$, $a_3a_4a_5a_1a_2$, $a_4a_5a_1a_2a_3$ and $a_5a_1a_2a_3a_4$. For the vertices α_1, α_3 and α_4 there is exactly one special σ_j-cycle at α_1, α_2 and α_3, respectively given by

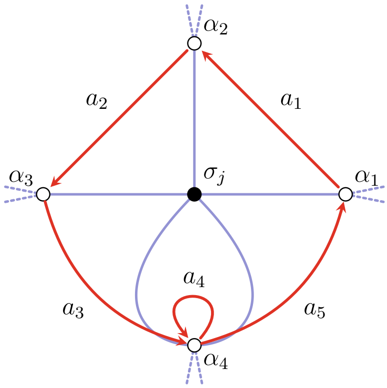


Fig. 5 A vertex σ_j of degree 5 in a dessin with the corresponding cycle in Q_D. Note that a_4 is a loop since α_4 shares two (consecutive) edges with σ_j

$$\text{at } \alpha_1 : a_1a_2a_3a_4a_5,$$
$$\text{at } \alpha_2 : a_2a_3a_4a_5a_1,$$
$$\text{at } \alpha_3 : a_3a_4a_5a_1a_2.$$

However, there are two special σ_j-cycles at α_4 given by $a_4a_5a_1a_2a_3$ and $a_5a_1a_2a_3a_4$.

The special σ-cycles at α_s will contribute relations to *the generating set of relations* ρ_D of *the admissible ideal* I_D of the path algebra KQ_D. The set ρ_D consists of three types of relations:

Relations of type one. For each white vertex α_s and each pair σ_j and σ_k of black vertices of degree at least 2 incident to α_s, all relations of the form

$$C_j - C_k,$$

where C_j and C_k are the special σ_j and σ_k-cycles at α_s are in ρ_D.

Relations of type two. For all σ_r all relations of the type Ca are in ρ_D, where C ranges across all special σ_r-cycles and a is the first arrow of C.

Relations of type three. All paths ab of length 2 which are not subpaths of any special cycle are relations in ρ_D.

Example 4.4 Let D be as the dessin from Fig. 4 with an additional half-edge attached to white vertex α_1. Label the arrows as in Fig. 6.

Relations of type one are given by:

$$\alpha_1 : \text{none.}$$
$$\alpha_2 : \text{none.}$$
$$\alpha_3 : r_3r_4r_5r_1r_2 - g_1g_2,\ r_3r_4r_5r_1r_2 - b_1b_2,\ g_1g_2 - b_1b_2.$$
$$\alpha_4 : r_4r_5r_1r_2r_3 - r_5r_1r_2r_3r_4.$$
$$\alpha_5 : b_2b_1 - g_2g_1.$$

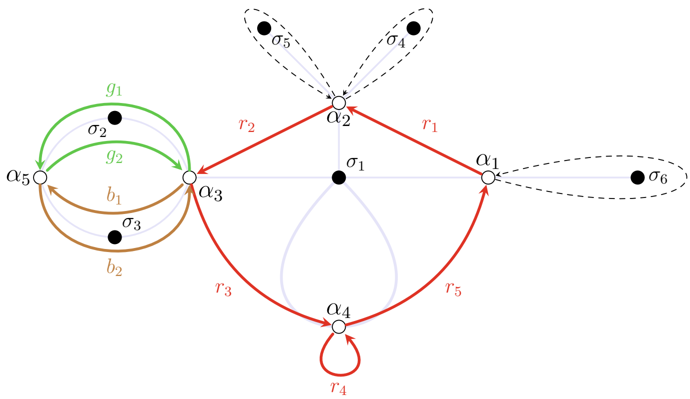


Fig. 6 The quiver Q_D from Example 4.4. Arrows are labelled according to colour (r for red, g for green, b for brown). The dashed loop arrows are formal

Relations of type two are given by:

$$\sigma_1\colon\ \overbrace{r_1r_2r_3r_4r_5r_1}^{\sigma_1\text{-cycle at }\alpha_1},\ \overbrace{r_2r_3r_4r_5r_1r_2}^{\sigma_1\text{-cycle at }\alpha_2},\ \overbrace{r_4r_5r_1r_2r_3r_4}^{\sigma_1\text{-cycle at }\alpha_3},\ \overbrace{r_4r_5r_1r_2r_3r_4,\ r_5r_1r_2r_3r_4r_5}^{\sigma_1\text{-cycles at }\alpha_4}$$

$$\sigma_2\colon\ \underbrace{g_1g_2g_1}_{\sigma_2\text{-cycle at }\alpha_3},\ \underbrace{g_2g_1g_2.}_{\sigma_2\text{-cycle at }\alpha_5}$$

$$\sigma_3\colon\ \underbrace{b_1b_2b_1}_{\sigma_3\text{-cycle at }\alpha_3},\ \underbrace{b_2b_1b_2.}_{\sigma_3\text{-cycle at }\alpha_5}$$

$\sigma_4, \sigma_5, \sigma_6\colon$ none.

Relations of type three are given by:

$$r_2g_1,\ r_2b_1,\ r_3r_5,\ r_4^2,\ g_1b_2,\ g_2b_1,\ g_2r_3,\ b_1g_2,\ b_2g_1,\ b_2r_3.$$

The bound quiver algebra KQ_D/I_D is called a *Brauer configuration algebra*. More generally, a *Brauer configuration* as defined in [9] is a tuple $(\Gamma_0, \Gamma_1, \mu, \mathfrak{o})$ where Γ_0 is the set of vertices, Γ_1 the set *polygons*, i.e. multisets of vertices, $\mu\colon \Gamma_0 \to \mathbb{N}$ a function into the positive integers called the *multiplicity*, and $\mathfrak{o}$ is the function specifying at every vertex a cyclic ordering of the polygons incident with that vertex. A Brauer configuration algebra is constructed from a Brauer configuration via a quiver and an ideal generated by relations of type one, two and three as defined above, with additional constraints on the relations given by the multiplicity function μ.

In the language of dessins d'enfants, Γ_0 is the set of black vertices, Γ_1 is the set of white vertices, $\mu\colon \Gamma_0 \to \mathbb{N}$ is the function $\mu(\sigma_j) = 1$, and $\mathfrak{o}$ is the counter-clockwise orientation induced by the orientation of the underlying Riemann surface of D. When a dessin is *clean*, i.e. when all white vertices have degree 2, then the corresponding

Brauer configuration algebra is a Brauer graph algebra; the relation between clean dessins and Brauer graph algebras has been studied in [17].

The relations ρ_D are not necessarily a minimal set of relations for I_D. The relations of type one and three are always minimal, however relations of type two are often redundant; this is a generalisation of a similar result for Brauer graph algebras [10].

Remark 4.5 The reason why the loop arrows induced by black vertices of degree 1 are not included in Q_D is because the ideal I_D would no longer be admissible if the corresponding relations were to be added to ρ_D. However, the quotient algebra $K\mathring{Q}_D/I$, where I is the ideal generated by the relations of type 1, 2 and 3, including the relations induced by formal loop arrows where each loop is its own special cycle, is isomorphic to the Brauer configuration algebra KQ_D/I_D.

An example of this are the two different Brauer configuration algebras associated to the dessin in Fig. 4 to which we can associate a quiver Q_1 with formal loop arrows as in Fig. 4 and a quiver Q_2 without the formal loop arrows as in Fig. 5. The algebras $K\mathring{Q}_{D_1}/I_1$ and $K\mathring{Q}_{D_2}/I_2$, where I_1 and I_2 are the ideals generated by the relations of type 1, 2 and 3 where in the case of I_1 every formal loop arrow is its own special cycle, are isomorphic.

Note that black vertices of degree 1 correspond to *truncated vertices* in [9].

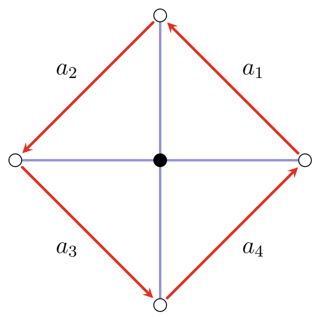
5 Examples of (Well-Known) Algebras and Their Dessins

5.1 Symmetric Nakayama Algebras

Let $D_n = (\mathbb{CP}^1, f_n)$ be the family of dessins given by the Belyĭ maps $f_n : z \mapsto z^n$ for $n \geq 2$. The quiver Q_{D_n} is represented by an oriented regular n-gon as in Fig. 7, with relations $a_1 \cdots a_n a_1, \ldots, a_n a_1 \cdots a_n$. In this case the only relations in I_{D_n} are relations of type two and they are necessarily minimal.

The corresponding Brauer configuration algebras KQ_{D_n}/I_{D_n} are symmetric Nakayama algebras.

Fig. 7 The quiver Q_{D_4} corresponding to the dessin $D_4 = (\mathbb{CP}^1, z^4)$



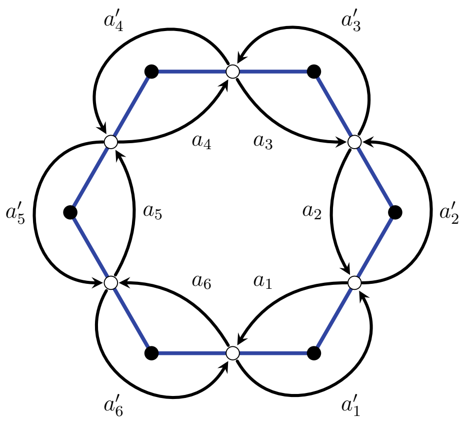


Fig. 8 The dessin D_6 and its associated quiver. The special cycles are given by the paths $a_i a_i'$ and $a_i' a_i$ for $i = 1, \ldots, 6$

5.2 *Koszul Brauer Configuration Algebras*

Let $D_n = (\mathbb{CP}^1, f_n)$ be the family of dessins given by the Belyĭ maps $f_n : z \mapsto (z^n + 1)^2/4z^n$ for $n \geq 3$. The zeros of f_n and $f_n - 1$ are at the n roots of -1 and 1, respectively, each appearing with multiplicity 2, and the two poles are at 0 and ∞, both of degree n. These dessins are regular polygons of degree n on the sphere.

Note in Fig. 8 that the elements generating I_{D_6} are all of length 2. Evidently this observation extends to I_{D_n}. Ideals generated by elements of length 2 are called *quadratic* and the corresponding Brauer configuration algebra is a Brauer graph algebra and by [10] it is Koszul.

6 Brauer Configuration Algebra of the Dual Dessin

6.1 *Paths Formed by the Faces of a Dessin*

In this subsection we analyse the paths formed by the faces of a dessin.

Proposition 6.1 *The faces of a dessin $D = (\sigma, \alpha, \varphi)$ decompose Q_D into disjoint cycles oriented clockwise. Moreover, if we count the formal loop arrows, then the lengths of these cycles correspond to the degrees of the faces.*

Proof Consider a face of D corresponding to the cycle $\varphi_j = (i_1 \ \cdots \ i_k)$ of φ. Let α_h denote the white vertex of D with h as its half-edge. Each half-edge i_m for $m = 1, \ldots, k$ induces a single arrow (possibly a formal loop arrow) whose source and target are the white vertices α_{i_m} and $\alpha_{i_m^\sigma}$, respectively. Note that if $\alpha_{i_m} \neq \alpha_{i_m^\sigma}$, then $\alpha_{i_m^\sigma}$ follows α_{i_m} in the counter-clockwise order around φ_j because $i_m^\sigma = i_m^{\varphi^{-1}\alpha^{-1}}$ and $\alpha_{i_m^{\varphi^{-1}\alpha^{-1}}} = \alpha_{i_m^{\varphi^{-1}}}$. Therefore, φ_j induces a cycle of arrows oriented clockwise. As i_m

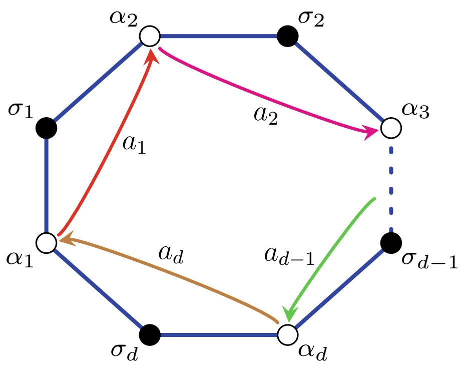


Fig. 9 In a polygonal face of degree $d \geq 2$ pairs of consecutive arrows belong to distinct σ-cycles

induces exactly one arrow, and each i_m belongs to exactly one cycle of φ, the cycles of arrows obtained from the faces of D are disjoint with lengths equal to the degrees of the faces. □

When a dessin D has a polygonal face, that is a face of degree $d \geq 2$ such that each black and each white vertex contributes exactly 2 half-edges, then the corresponding paths are formed by concatenations of relations of type 3.

Proposition 6.2 *The polygonal faces of D of degree $d \geq 2$ give rise to cyclic permutations of paths of the type $a_1a_2a_3 \cdots a_{d-1}a_d$ such that there are no repeating arrows, and a_ia_{i+1} and a_da_1 are relations of type 3, for all $i = 1, \ldots, d-1$.*

Proof A polygonal face of degree $d \geq 2$ is incident to d black vertices $\sigma_1, \ldots, \sigma_d$, and d white vertices $\alpha_1, \ldots, \alpha_d$ ordered counter-clockwise, so that σ_1 follows α_1 in this order. Let a_i be the arrow with α_i as source and α_{i+1} as target belonging to the special σ_i-cycle at α_i, for $i = 1, \ldots, d-1$, and let a_d be the arrow with α_d as source and α_1 as targed belonging to the special σ_d-cycle at α_d, see Fig. 9. Then a_ia_{i+1} and a_da_1 are not subpaths of any special cycle and hence a relation of type 3, for all $i = 1, \ldots, d-1$. □

Note that not every cyclic path in which consecutive arrows are type 3 relations corresponds to a face, see Fig. 10 for an example.

In general, a face of D with vertices $\sigma_1, \ldots, \sigma_k$ of degree at least 2 gives rise to paths of type $A_1A_2 \cdots A_k$, where A_j $(j = 1, \ldots, k)$ is a path of arrows in the corresponding σ_j-cycle, and A_j and A_{j+1} $(j = 1, \ldots, k-1)$ are connected by a relation of type 3, as well as A_k and A_1.

6.2 *Dual Dessins*

Consider a dessin $D = (X, f)$. Its *dual* dessin D^* is defined as the dessin corresponding to the Belyĭ pair $(X, 1/f)$.

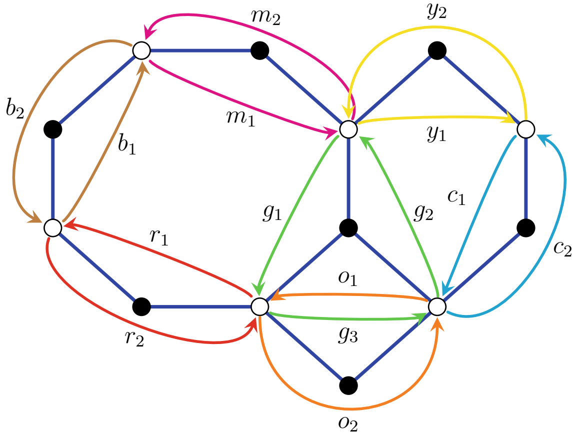


Fig. 10 The path $r_1b_1m_1y_1c_1o_1$ has no repeated arrows, and all of its subpaths of length 2 as well as o_1r_1 are relations of type 3, yet it corresponds to no face of the underlying dessin

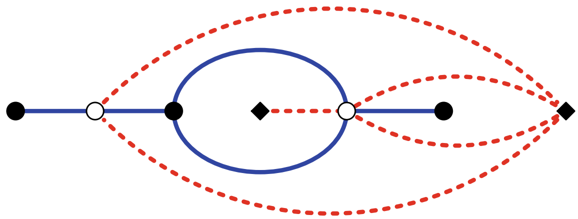


Fig. 11 A dessin (full) and its dual (dashed). The black vertices of the dual are indicated by ♦

In terms of permutation representations, if $D = (\sigma, \alpha, \varphi)$, then D^* will have the triple $(\varphi^{-1}, \alpha^{-1}, \sigma^{-1})$ as its permutation representation. Geometrically this means that the black vertices and the face centres of the dual are the face centres and the black vertices of D, respectively, while the white vertices remain unchanged, except for the orientation of the labels. The half-edges of D^* are the curved segments that connect the face centres and the white vertices of D, see Fig. 11 for an example.

Theorem 6.3 *Let D be a dessin and let D^* be its dual dessin. Then the quivers $\mathring{Q}_D$ and $\mathring{Q}_{D^*}^{op}$ are equal. Furthermore, if D has no vertices and no faces of degree 1 then the quivers Q_D and $Q_{D^*}^{op}$ are equal.*

Proof Consider a face of the dessin D. By Proposition 6.1 the cycle of arrows in $\mathring{Q}_D$ corresponding to this face is oriented counter-clockwise around the black vertex of D^* dual to this face, see Fig. 12. Therefore, in order to obtain the quiver of $\mathring{Q}_{D^*}$ we have to reverse the direction of every arrow in Q_D.

If D has no vertices and no faces of degree 1, then formal loops do not appear in Q_D and Q_{D^*} and therefore $Q_D = \mathring{Q}_D$ and $Q_{D^*} = \mathring{Q}_{D^*}$. □

Example 6.4 Let D be the dessin given by the bipartite 6-gon, as in Fig. 13. The set of arrows of its quiver is decomposed into two disjoint cycles $a_1b_1c_1$ and $a_2c_2b_2$. Its dual dessin D^* is shown in the same Figure. The quiver of D^* is not shown in the

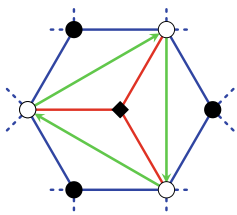


Fig. 12 The arrows in a face of a dessin (in green) form a cycle oriented clockwise around the corresponding dual vertex. The dual half-edges are shown in red

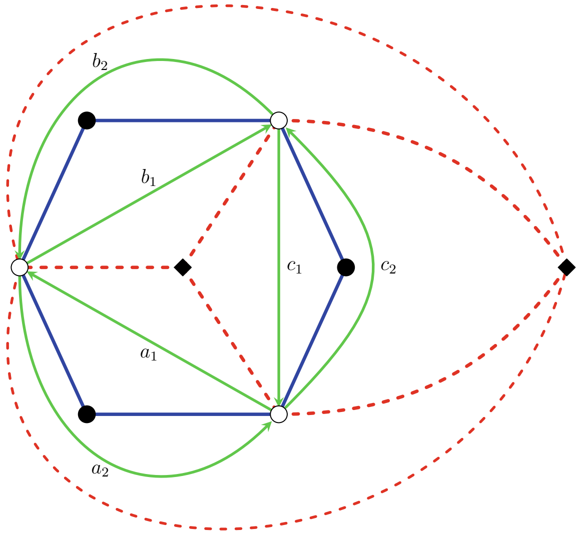


Fig. 13 The 6-gon dessin (blue), its quiver (green), and its dual dessin (dashed red)

Figure, but the reader can verify that its quiver is obtained by reversing the orientation of the arrows in Q_D, and the set of arrows of Q_{D^*} is decomposed into three disjoint cycles $a_1^{\text{op}}a_2^{\text{op}}$, $b_1^{\text{op}}b_2^{\text{op}}$ and $c_1^{\text{op}}c_2^{\text{op}}$.

Example 6.5 The dessins $D_n = (\mathbb{CP}^1, z^n)$ are self-dual, i.e. $D_n^* \cong D_n$. The Brauer configuration algebra $KQ_{D_n^*}/I_{D_n^*}$ associated to the dual D^* of D is again a symmetric Nakayama algebra. In fact, we have that $KQ_{D_n}/I_{D_n} \cong KQ_{D_n^*}/I_{D_n^*}$.

7 Galois Action on Brauer Configuration Algebras

Definition 7.1 Let $\mathfrak{S}$ denote the set of all σ-cycles. We say that a σ_i-cycle and a σ_j-cycle are equivalent if one is a cyclic permutation of the other. An equivalence class represented by a σ_j-cycle will be denoted $\overline{\sigma_j}$ and let $\overline{\mathfrak{S}}$ be the set of equivalence classes of σ_j-cycles.

Lemma 7.2 *Let D be a dessin and $Q_D = (Q_0, Q_1)$ its quiver. The following are Galois invariants:*

(i) The number of vertices $|Q_0|$ of Q.
(ii) The number of arrows $|Q_1|$ of Q.
(iii) The cardinal $|\overline{\mathfrak{S}}|$ of the set $\overline{\mathfrak{S}} = \{\overline{\sigma_1}, \ldots, \overline{\sigma_r}\}$ of equivalence classes of σ-cycles.
(iv) The set $\{|\overline{\sigma_1}|, \ldots, |\overline{\sigma_r}|\}$ recording the number of arrows in representatives of equivalence classes of σ-cycles.

Proof That (i–iii) are Galois invariants follows from $|Q_0|$, $|Q_1|$ and $|\mathfrak{S}|$ being equal to the number of white vertices, the total number of half-edges, and the number of black vertices of D, respectively, all of which are Galois invariants. Likewise, (iv) is a Galois invariant because $|\sigma_j|$ is equal to the degree of the black vertex σ_j, and the degree sequence of D is a Galois invariant. □

Let $D = (\sigma, \alpha, \varphi)$ be a dessin, $\Lambda_D = KQ_D/I_D$ its associated Brauer configuration algebra. A basis of Λ_D is given by the set

$$\{p + I_D \mid p \text{ is a subpath of some } \sigma_j\text{-cycle in } Q_D\}.$$

The dimension of Λ_D is given by the sum

$$\dim_K \Lambda_D = 2|Q_0| + \sum_{\overline{C_i} \in \overline{\mathfrak{S}}} |\overline{C_i}|(|\overline{C_i}| - 1).$$

For a detailed discussion about a basis of a Brauer configuration algebra in general see [9].

Example 7.3 Let D be the dessin from Example 4.4, let $\Lambda_D = KQ_D/I_D$ be the associated Brauer configuration algebra, and for a path p in Q_D let $\overline{p} = p + I_D$. A basis of Λ_D is given by the trivial paths $\overline{e_1}$, $\overline{e_2}$, $\overline{e_3}$, $\overline{e_4}$ and $\overline{e_5}$, one for each vertex of Q_D, and the non-trivial paths

(i) $\overline{r_1}, \overline{r_1r_2}, \overline{r_1r_2r_3}, \overline{r_1r_2r_3r_4}$;
$\overline{r_2}, \overline{r_2r_3}, \overline{r_2r_3r_4}, \overline{r_2r_3r_4r_5}$;
$\overline{r_3}, \overline{r_3r_4}, \overline{r_3r_4r_5}, \overline{r_3r_4r_5r_1}$;
$\overline{r_4}, \overline{r_4r_5}, \overline{r_4r_5r_1}, \overline{r_4r_5r_1r_2}$;
$\overline{r_5}, \overline{r_5r_1}, \overline{r_5r_1r_2}, \overline{r_5r_1r_2r_3}$;
$\overline{g_1}, \overline{g_2}$;
$\overline{b_1}, \overline{b_2}$;
(ii) $\overline{r_1r_2r_3r_4r_5}, \overline{r_2r_3r_4r_5r_1}, \overline{r_3r_4r_5r_1r_2}, \overline{r_4r_5r_1r_2r_3}, \overline{g_2g_1}$.

The paths (i) are images of proper subpaths of σ_j-cycles, however note that in (ii) we have $\overline{r_3r_4r_5r_1r_2} = \overline{g_1g_2} = \overline{b_1b_2}$, $\overline{r_4r_5r_1r_2r_3} = \overline{r_5r_1r_2r_3r_4}$ and $\overline{g_2g_1} = \overline{b_2b_1}$. The dimension of Λ_D is $2 \cdot 5 + 5 \cdot 4 + 2 + 2 = 34$.

Theorem 7.4 *Let $D = (\sigma, \alpha, \varphi)$ be a dessin and $\Lambda_D = KQ_D/I_D$ the associated Brauer configuration algebra. The dimension* $\dim_K \Lambda_D$ *of* Λ_D *is a Galois invariant.*

Proof The dimension $\dim_K \Lambda_D$ is given by the sum

$$\dim_K \Lambda_D = 2|Q_0| + \sum_{\overline{C_i} \in \overline{\mathfrak{S}}} |\overline{C_i}|(|\overline{C_i}| - 1).$$

By the lemma above, the numbers $|Q_0|$ and $|\overline{C_i}|$ are Galois invariant. Therefore, the dimension of $\Lambda_D = KQ_D/I_D$ is a Galois invariant. □

A basis of the centre $Z(\Lambda_D)$ of $\Lambda_D = KQ_D/I_D$ is given by the set

$$\{1_\Lambda + I_D\} \cup \{p + I_D \mid p \text{ is a special cycle in } Q_D\} \cup \mathscr{L}_D,$$

where $1_\Lambda = e_1 + \cdots + e_{|Q_0|}$ is the identity of Λ_D and $\mathscr{L}_D$ is the set of loops in Q_D. The dimension $\dim_K Z(\Lambda_D)$ is given by the sum

$$\dim_K Z(\Lambda_D) = 1 + |Q_0| + |\mathscr{L}_D|.$$

For a detailed discussion about the centre of a Brauer configuration algebra see [22].

Example 7.5 Let D once again be the dessin from Example 4.4, let $\Lambda_D = KQ_D/I_D$ be the associated Brauer configuration algebra, and, for a path p in Q_D, let $\overline{p} = p + I_D$. A basis of $Z(\Lambda_D)$ is given by the identity $\overline{1}_\Lambda$, the loop $\overline{r_4}$ and the special cycles

$$\overline{r_1r_2r_3r_4r_5}, \overline{r_2r_3r_4r_5r_1}, \overline{r_3r_4r_5r_1r_2}, \overline{r_4r_5r_1r_2r_3}, \overline{g_2g_1}.$$

Note that $\overline{r_3r_4r_5r_1r_2} = \overline{b_1b_2} = \overline{g_1g_2}$, $\overline{r_4r_5r_1r_2r_3} = \overline{r_5r_1r_2r_3r_4}$ and $\overline{g_2g_1} = \overline{b_2b_1}$. The dimension is therefore $1 + 5 + 1 = 7$.

Proposition 7.6 *Let D be a dessin and $\Lambda_D = KQ_D/I_D$ the associated Brauer configuration algebra. The dimension of the centre* $\dim_K Z(\Lambda_D)$ *of* Λ_D *is a Galois invariant.*

Proof The dimension $\dim_K Z(\Lambda_D)$ is given by the sum

$$\dim_K Z(\Lambda_D) = 1 + |Q_0| + |\mathscr{L}_D|.$$

By Lemma 7.2, all the summands are invariants and therefore the dimension $\dim_K Z(\Lambda_D)$ of the centre $Z(\Lambda_D)$ of Λ_D is a Galois invariant. □

In fact, a stronger result holds, namely Galois-conjugate dessins have isomorphic centres. In order to prove this, we need the following technical result.

Lemma 7.7 *Let $\Lambda = kQ/I$ be a Brauer configuration algebra with multiplicity function $\mu = 1$. Let B be the basis of $Z(\Lambda)$ given by the identity, the loops and the special cycles of Q. If $\bar{p}, \bar{q} \in B$ such that neither is the identity, then $\bar{p}\bar{q} = \bar{0}$.*

Proof If p and q are not loops and $t(p) = s(q)$ then $p - q$ is necessarily a relation in I. Therefore $\bar{p} = \bar{q}$ and $\bar{p}\bar{q} = \bar{p}\bar{p} = 0$ as pp necessarily contains a type 2 relation.

If p is a loop and $t(p) = s(q)$ or $t(q) = s(p)$, then q is a basis element with p as either the first or the final arrow. The product $\bar{p}\bar{q}$ is $\bar{0}$ in the former because p^2 is a relation, and in the latter because pq is a type 2 relation. □

Theorem 7.8 *Let D_1 and D_2 be Galois-conjugate dessins with Brauer configuration algebras $\Lambda_1 = kQ_1/I_1$ and $\Lambda_2 = kQ_2/I_2$, respectively. Then $Z(\Lambda_1) \cong Z(\Lambda_2)$.*

Proof Let B_1 and B_2 be the bases of $Z(\Lambda_1)$ and $Z(\Lambda_2)$, respectively, given by the identity, the loops and the special cycles. By Proposition 7.6 we have $\dim Z(\Lambda_1) = \dim Z(\Lambda_2) = n$ so there is a vector-space isomorphism $f : Z(\Lambda_1) \to Z(\Lambda_2)$ mapping B_1 to B_2 so that loops are mapped to loops and special cycles to special cycles.

Let $\bar{p}_i \in B_1$ for $i = 2, \ldots, n$ be the basis elements of $Z(\Lambda_1)$ which are not $\bar{1}_{\Lambda_1}$. Then for $v, w \in Z(\Lambda_1)$ we have

$$v = \alpha\bar{1}_{\Lambda_1} + \sum_{i=2}^{n} \alpha_i \bar{p}_i,$$
$$w = \beta\bar{1}_{\Lambda_1} + \sum_{i=2}^{n} \beta_i \bar{p}_i,$$

where α, β, α_i and β_i for $i = 2, \ldots, n$ are scalars. Then

$$vw = \alpha\beta\bar{1}_{\Lambda_1} + \sum_{i=2}^{n} (\alpha_i\beta + \alpha\beta_i)\bar{p}_i$$

due to Lemma 7.7. Then

$$\begin{aligned} f(vw) &= \alpha\beta\bar{1}_{\Lambda_2} + \sum_{i=2}^{n} (\alpha_i\beta + \alpha\beta_i) f(\bar{p}_i) \\ &= \left(\alpha f(\bar{1}_{\Lambda_1}) + \sum_{i=2}^{n} \alpha_i f(\bar{p}_i)\right)\left(\beta f(\bar{1}_{\Lambda_1}) + \sum_{i=2}^{n} \beta_i f(\bar{p}_i)\right) = f(v)f(w) \end{aligned}$$

again due to Lemma 7.7. □

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On the Elliptic Kashiwara–Vergne Lie Algebra

Élise Raphael and Leila Schneps

Abstract We recall the definitions of two independently defined elliptic versions of the Kashiwara–Vergne Lie algebra $\mathfrak{krv}$, namely the Lie algebra $\mathfrak{krv}^{(1,1)}$ constructed by Alekseev, Kawazumi, Kuno and Naef arising from the study of graded formality isomorphisms associated to topological fundamental groups of surfaces, and the Lie algebra $\mathfrak{krv}_{ell}$ defined using mould theoretic techniques arising from multiple zeta theory by Raphael and Schneps, and show that they coincide.

Keywords Lie algebras · Derivations · Elliptic Kashiwara-Vergne problem

1 Introduction

From its inception in Grothendieck's *Esquisse d'un Programme* [9], Grothendieck–Teichmüller theory was intended to study the automorphism groups of the profinite mapping class groups—the fundamental groups of moduli spaces of Riemann surfaces of all genera and any number of marked points—with the goal of discovering new properties of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. However, due to the ease of study of the genus zero mapping class groups, which are essentially braid groups, the genus zero case garnered most of the attention, starting from the definition of the Grothendieck–Teichmüller group $\widehat{GT}$ by Drinfel'd [5] and the simultaneous construction by Ihara of the Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}$ [12, 13] in 1991. The extension of the definition to a Grothendieck–Teichmüller group acting on the profinite mapping class groups in all genera was subsequently given in 2000

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by Hatcher, Lochak, Schneps and Nakamura (cf. [11, 14]). The higher genus profinite Grothendieck–Teichmüller group satisfies the two-level principle articulated by Grothendieck, which states that the subgroup of $\widehat{GT}$ consisting of automorphisms that extend to the genus one mapping class groups with one or two marked points will automatically extend to automorphisms of the higher mapping class groups.

It has proven much more difficult to extend the Lie algebra Grothendieck–Teichmüller construction to higher genus. Indeed, while the genus zero mapping class groups have a natural Lie algebra analog in the form of the braid Lie algebras, there is no good Lie algebra analog of the higher genus mapping class groups. The only possible approach for the moment seems to be to replace the higher genus mapping class groups by their higher genus braid subgroups, which do have good Lie algebra analogs.[1] An early piece of work due to Tsunogai [19] in 2003 computed the relations that must be satisfied by a derivation acting on the genus one 1-strand braid Lie algebra $\mathfrak{lie}^{(1,1)}$ (which is free on two generators) to ensure that it extends to a derivation on the genus one 2-strand braid Lie algebra, in analogy with the derivations in $\mathfrak{grt}$, shown by Ihara to be exactly those that act on the genus zero 4-strand braid Lie algebra (also a free Lie algebra on two generators) and extend to derivations of the 5-strand braid Lie algebra.

After this, the next real breakthrough in the higher genus Lie algebra situation came with the work of Enriquez ([7], 2014) based on his previous joint work with Calaque and Etingof ([4], 2009). In particular, using the same approach as Tsunogai of replacing the higher genus mapping class groups with their higher genus braid subgroups, Enriquez in [7] was able to extend the definition of $\mathfrak{grt}$ to an elliptic version $\mathfrak{grt}_{ell}$, which he identified with an explicit Lie subalgebra of the algebra of derivations of the algebra of the genus one 1-strand braid Lie algebra $\mathfrak{lie}^{(1,1)}$ that extend to derivations of the 2-strand genus one braid Lie algebra. He showed in particular that there is a canonical surjection $\mathfrak{grt}_{ell} \twoheadrightarrow \mathfrak{grt}$, and a canonical section of this surjection, $\gamma : \mathfrak{grt} \hookrightarrow \mathfrak{grt}_{ell}$.

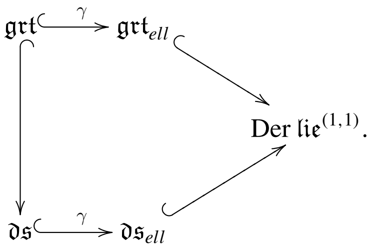
The Grothendieck–Teichmüller Lie algebra is closely related to two other Lie algebras, the double shuffle Lie algebra that arises from the theory of multiple zeta values and the Kashiwara–Vergne Lie algebra that arises from solutions to the (linearized) Kashiwara–Vergne problem. Indeed, there exist injective Lie algebra morphisms

$$\mathfrak{grt} \hookrightarrow \mathfrak{ds} \hookrightarrow \mathfrak{krv},$$

by work of Furusho [8] for the first injection, Écalle and Schneps [6, 16] for the second and Alekseev and Torossian [3] for a direct proof that $\mathfrak{grt}$ maps into $\mathfrak{krv}$. In fact, these three algebras are conjectured to be isomorphic, a conjecture that is supported by computation of the graded parts up to weight about 20. Thus it was a natural consequence of the work of Enriquez to consider the possibility of extending

[1] Another approach would be to replace the higher genus mapping class groups by their Torelli subgroups, which also have good Lie algebraic analogs determined by Hain [10]. In particular, this would include the key case of higher genus with 0 marked points, which have no associated braid groups. However, there has been no development of Lie Grothendieck–Teichmüller theory in this context as yet.

also these other Lie algebras from genus zero to genus one. An answer was proposed for the double shuffle Lie algebra in [18], which proposes a definition of an elliptic double shuffle Lie algebra $\mathfrak{ds}_{ell}$ based on mould theory and an elliptic interpretation of a major theorem of Écalle (cf. [6, 17, 18]). This elliptic double shuffle Lie algebra admits a section $\gamma : \mathfrak{ds} \hookrightarrow \mathfrak{ds}_{ell}$ which extends Enriquez's section in the sense that the following diagram commutes;



One interesting aspect of the mould theoretic approach is that it reveals a close relationship between the elliptic double shuffle Lie algebra and the associated graded of the usual double shuffle Lie algebra for the depth filtration. In the article [15], the authors of this paper showed that an analogous approach works to construct an elliptic version of $\mathfrak{krv}$, denoted $\mathfrak{krv}_{ell}$, which is given by two defining mould theoretic properties, and again has the key features of

- being naturally identified with a Lie subalgebra of the derivation algebra of the free Lie algebra on two generators;
- being equipped with an injective Lie algebra morphism $\gamma : \mathfrak{krv} \hookrightarrow \mathfrak{krv}_{ell}$ which extends the Grothendieck–Teichmüller and double shuffle sections;
- having a structure closely related to that of the associated graded of $\mathfrak{krv}$ for the depth filtration.

In independent work, Alekseev et al. [1, 2] took a different approach to the construction of higher genus Kashiwara–Vergne Lie algebras $\mathfrak{krv}^{(g,n)}$ for all $g, n \geq 1$, following the classical approach to the Kashiwara–Vergne problem which focuses on determining graded formality isomorphisms between prounipotent fundamental groups of surfaces and their graded counterparts (i.e. the exponentials of the associated gradeds of their associated Lie algebras).

More precisely, if Σ denotes a compact oriented surface of genus g with $n+1$ boundary components, the space $g(\Sigma)$ spanned by free homotopy classes of loops in Σ carries the structure of a Lie bialgebra equipped with the Goldman bracket and the Turaev cobracket. The Goldman–Turaev formality problem is the construction of a Lie bialgebra homomorphism θ from $g(\Sigma)$ to its associated graded $\mathrm{gr}\, g(\Sigma)$ such that $\mathrm{gr}\, \theta = \mathrm{id}$. In order to solve this problem, Alekseev et al. defined a family $KV(g, n+1)$ of Kashiwara–Vergne problems. In the particular situation where $(g, n) = (1, 0)$, the surface Σ is of genus 1 with one boundary component, and its fundamental group is free on two generators A, B, with the boundary loop being given by $C = (A, B)$. The associated prounipotent fundamental group is then free on two generators e^a, e^b

with a boundary element e^c satisfying $e^c = (e^a, e^b) = e^a e^b e^{-a} e^{-b}$. The associated Lie algebra is free on generators a, b. Since we have $c = \log(e^a e^b e^{-a} e^{-b})$, the explicit formula for c in the Lie algebra is

$$c = CH\Big(CH\big(CH(a,b), -a\big), -b\Big) = [a,b] + \text{higher order terms},$$

where CH denotes the Campbell-Hausdorff law on $\mathfrak{lie}^{(1,1)} \simeq \mathrm{Lie}[a,b]$. To define the genus one Kashiwara–Vergne Lie algebra $\mathfrak{krv}^{(1,1)}$, Alekseev et al. first defined the space of derivations d of $\mathrm{Lie}[a,b]$ that annihilate the element c and further satisfy a certain non-commutative divergence condition (see Sect. 2 for more detail), and then took $\mathfrak{krv}^{(1,1)}$ to be the associated graded of the above space. In fact this essentially comes down to using the same defining conditions but replacing c by its lowest graded component $[a,b]$. They showed that the resulting space is a Lie algebra under the bracket of derivations, and also that, like $\mathfrak{krv}_{ell}$, it is equipped with an injective Lie algebra morphism $\mathfrak{krv} \hookrightarrow \mathfrak{krv}^{(1,1)}$ that extends the Enriquez section $\gamma : \mathfrak{grt} \hookrightarrow \mathfrak{grt}_{ell}$.

The main result of this article is the equivalence of these two definitions of the elliptic Kashiwara–Vergne Lie algebra.

Main Theorem. *There is a canonical isomorphism* $\mathfrak{krv}^{(1,1)} \simeq \mathfrak{krv}_{ell}$.

It is an easy consequence of known results that the first defining property of $\mathfrak{krv}_{ell}$ corresponds to the annihilation of $[a,b]$. The proof of the theorem thus consists essentially in comparing the second defining properties of the two algebras. The article is organised as follows. In Sect. 2, we recall the definition of $\mathfrak{krv}^{(1,1)}$, in particular the divergence property, and in Sect. 3, we give a new reformulation of the divergence property. In Sect. 4 we recall the definition of $\mathfrak{krv}_{ell}$ and show that translating its second mould theoretic defining property back to a property of derivations on $\mathfrak{lie}^{(1,1)}$, it coincides with the reformulated version of the divergence property of $\mathfrak{krv}^{(1,1)}$ given in Sect. 3, which completes the proof.

2 The Elliptic Kashiwara–Vergne Lie Algebra from [AKKN]

Let $\mathfrak{lie}^{(1,1)}$ be the free Lie algebra on two generators $\mathrm{Lie}[a,b]$, to be thought of as the Lie algebra associated to the fundamental group of the once-punctured torus. Let $\mathfrak{lie}_n^{(1,1)}$ denote the weight n part of $\mathfrak{lie}^{(1,1)}$, where the weight is the total degree in a and b, and let $\mathfrak{lie}_{n,r}^{(1,1)}$ denote the weight n, depth r part of $\mathfrak{lie}^{(1,1)}$, where the depth is the b-degree. From this point on, we use the notation $c := [a,b]$ in $\mathfrak{lie}^{(1,1)}$ (this comes down to replacing the previous c by its minimal weight part in the associated graded).

Let $\mathfrak{der}^{(1,1)}$ denote the Lie subalgebra of $\mathrm{Der}\,\mathfrak{lie}^{(1,1)}$ of derivations d such that $d(c) = 0$. Let $\mathfrak{der}_n^{(1,1)}$ denote the subspace of $\mathfrak{der}^{(1,1)}$ of derivations d such that $d(a), d(b) \in \mathfrak{lie}_n^{(1,1)}$.

We define the push-operator on $\mathfrak{lie}^{(1,1)}$ as follows. We can write any monomial in the form $a^{i_0}b\cdots ba^{i_r}$, where the $i_j \geq 0$. The push-operator acts on monomials by

$$push(a^{i_0}ba^{i_1}b\cdots ba^{i_r}) = a^{i_r}ba^{i_0}b\cdots ba^{i_{r-1}}, \tag{1}$$

i.e. by cyclically permuting the powers of a between the b's. The operator extends to polynomials by linearity. We say that an element $f \in \mathfrak{lie}^{(1,1)}$ is *push-invariant* if $push(f) = f$.

Let $f \in \mathfrak{lie}_n^{(1,1)}$ for $n > 1$. It is shown in Theorem 21 of [16] that there exists an element $g \in \mathfrak{lie}_n^{(1,1)}$ satisfying $[a, g] + [b, f] = 0$ if and only if f is push-invariant, and if this is the case then g is unique. This condition is equivalent to the existence of a g such that the derivation determined by $a \mapsto f$, $b \mapsto -g$ annihilates the bracket $[a, b]$. Thus, f is the value on a of a derivation $d \in \mathfrak{der}_n^{(1,1)}$ if and only if f is push-invariant, in which case $d(b)$ is uniquely defined.

Any element $f \in \mathfrak{lie}^{(1,1)}$ can be decomposed uniquely as

$$f = f_a a + f_b b = af^a + bf^b = af_a^a a + af_b^a b + bf_a^b a + bf_b^b b \tag{2}$$

with $f_a, f_b, f^a, f^b, f_a^a, f_b^a, f_a^b, f_b^b \in \mathbb{Q}\langle a, b\rangle$. Let Tr_2 be the quotient of the free associative algebra $\mathbb{Q}\langle a, b\rangle$ (identified with the universal enveloping algebra of $\mathfrak{lie}^{(1,1)} \simeq \text{Lie}[a, b]$) by the equivalence relation: two words w and w' are equivalent if one can be obtained from the other by cyclic permutation of the letters. We write $tr : \mathbb{Q}\langle a, b\rangle \to Tr_2$ for this quotient map, called the *trace map*.

The *elliptic divergence* map $div : \mathfrak{der}^{(1,1)} \to Tr_2$ is defined in [AKKN] by

$$div(u) = tr(f_a + g_b)$$

where $d \in \mathfrak{der}^{(1,1)}$ satisfies $d(a) = f$ $d(b) = g$. Since $d([a, b]) = [a, g] + [f, b] = 0$, we have

$$ag_a a + ag_b b - ag^a a - bg^b a = bf_a a - af^a b - bf^b b + bf_b b.$$

Comparing the terms on both sides that start with a and end with b shows that $g_b = -f^a$. Thus we can write the divergence condition as a function of just f:

$$div(d) = tr(f_a - f^a).$$

In fact, using the decomposition (2), we have

$$tr(f_a - f^a) = tr(af_a^a + bf_a^b - f_a^a a - f_b^a b) = tr\big((f_a^b - f_b^a)b\big),$$

so

$$div(d) = tr\big((f_a^b - f_b^a)b\big).$$

Definition The elliptic Kashiwara–Vergne Lie algebra $\mathfrak{krv}^{(1,1)}$ defined in [AKKN] is the $\mathbb{Q}$-vector space spanned by the derivations $d \in \mathfrak{der}_n^{(1,1)}$, $n \geq 3$ having the property that there exists $K \in \mathbb{Q}$ such that

$$div(d) = \begin{cases} K\, tr\big([a,b]^{\frac{n-1}{2}}\big) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases} \tag{3}$$

It is closed under the bracket of derivations.

3 A Reformulation of the *div* Condition

The *div* condition is related in a natural way to the push-operator defined in (1). For any word w, let $C(w)$ denote its trace class in Tr_2, i.e. the set of words obtained from w by cyclically permuting its letters. We also write $C^b(w)$ (resp. $C_b(w)$) for the subset of $C(w)$ of words starting (resp. ending) with b. For any word $u = a^{i_0}b\cdots ba^{i_{r-1}}$ of depth $r-1$, set

$$P(u) = \{push^i(u) \,\big|\, 0 \leq i \leq r-1\}.$$

Note that we have $C_b(ub) = \{push^i(u)b \,\big|\, 0 \leq i \leq r-1\}$, i.e. $C_b(ub) = P(u)\cdot b$, and $|P(u)| = |C_b(ub)|$. The fact that $|P(u)|$ can be less than r is due to the possible symmetries in the word u with respect to the push-operator. For example, if $u = abbab$, we have $r = 4$ but $P(u) = \{abbab, babba\}$, $C_b(ub) = \{abbabb, babbab\}$.

Set

$$pushsym(u) = \sum_{v \in P(u)} w.$$

We extend the operator $pushsym$ to all of $\mathbb{Q}\langle a,b\rangle$ by linearity.

Let $(f|w)$ denote the coefficient of a word w in the polynomial f. We also write $\big(tr(f)|C(w)\big)$ for the coefficient of the trace class $C(w)$ in $tr(f) \in Tr_2$.

Let $f \in \mathbb{Q}\langle a,b\rangle$. Then for any word u in a,b, setting $w = ub$, we have the equality

$$\begin{aligned} \Big(tr\big((f_a^b - f_b^a)b\big) \,\big|\, C(w)\Big) &= \sum_{vb \in C(w)} \Big((f_a^b - f_b^a)b \,\big|\, vb\Big) \\ &= \sum_{vb \in C_b(ub)} \Big((f_a^b - f_b^a)b \,\big|\, vb\Big) \\ &= \sum_{v \in P(u)} \Big(f_a^b - f_b^a \,\big|\, v\Big) \\ &= \Big(pushsym(f_a^b - f_b^a) \,\big|\, u\Big). \end{aligned} \tag{4}$$

Since $w = ub$, we have $C(w) = C(ub) = C(bu)$. Indeed, the first equality holds because since the polynomial $(f_a^b - f_b^a)b$ ends in b, we only need to consider the coefficients of words in $C(w)$ ending in b; the second holds because the subset of words in $C(w) = C(ub)$ ending in b is equal to $C_b(ub)$, and the third holds because $C_b(ub) = P(u) \cdot b$ as noted above.

Equation (4) allows us to rewrite the divergence condition (3) on an element $f \in \mathfrak{lie}_n^{(1,1)}$ as the following condition: there exists $K \in \mathbb{Q}$ such that for every word u of weight $n-2$ and depth $r-1$, we have

$$\big(pushsym(f_a^b - f_b^a) \,\big|\, u\big) = \begin{cases} K \displaystyle\sum_{v \in C(ub)} \big([a,b]^r \,\big|\, v\big) & \text{if } n = 2r+1 \\ 0 & \text{if } n \neq 2r+1. \end{cases} \tag{5}$$

This is the version of the divergence condition that we will use for comparison with the Lie algebra $\mathfrak{krv}_{ell}$.

4 The Mould Theoretic $\mathfrak{krv}_{ell}$ from [15]

Recall that a mould is a family $A = (A_r)_{r \geq 0}$ where $A_r(u_1, \ldots, u_r)$ is a function of r commutative variables. We restrict our attention here to rational-function moulds with coefficients in $\mathbb{Q}$. These form a $\mathbb{Q}$-vector space under componentwise addition and multiplication by scalars. When the number of variables is specified, we drop the subscript r, for instance we write $A(u_1, \ldots, u_r) = A_r(u_1, \ldots, u_r)$.

A mould is said to be *alternal* if $A(\emptyset) = 0$ and

$$\sum_{w \in sh\big((u_1,\ldots,u_k),(u_{k+1},\ldots,u_r)\big)} A_r(w) = 0$$

for $r \geq 2$ and $1 \leq k \leq r-1$.

Let us define a few mould operators. The $swap$, $push$, $circ$ and Δ-operators on moulds are given by

$$\begin{aligned} swap(A)(v_1, \ldots, v_r) &= A(v_r, v_{r-1} - v_r, \ldots, v_1 - v_2) \\ push(A)(u_1, \ldots, u_r) &= A(u_2, \ldots, u_r, -u_1 - \cdots - u_r) \\ circ(A)(v_1, \ldots, v_r) &= A(v_r, v_1, \ldots, v_{r-1}) \\ \Delta(A)(u_1, \ldots, u_r) &= (u_1 + \cdots + u_r)u_1 \cdots u_r\, A(u_1, \ldots, u_r). \end{aligned}$$

There is no difference between the use of the commutative variables u_i or v_i, however the v_i's are traditionally used for operators and relations concerning the swap of a mould.

There is a direct connection between power series in a, b (having no constant term in a) and polynomial-valued moulds. Let $c_i = ad(a)^{i-1}(b)$ for $i \geq 1$, and consider Lie algebra $Lie[c_1, c_2, \ldots]$ inside the polynomial algebra $\mathbb{Q}\langle c_1, c_2, \ldots\rangle$. By Lazard elimination, these algebras are free and all polynomials in $\mathfrak{lie}^{(1,1)}$ having no linear term in a can be written as Lie polynomials in the c_i.

There is a bijection between the space of polynomials in the c_i and the space of polynomial-valued moulds, coming from linearly extending the map

$$c_{a_1} \cdots c_{a_r} \to (-1)^{n+r} u_1^{a_1-1} \cdots u_r^{a_r-1}, \tag{6}$$

where $n = a_1 + \cdots + a_r$. It is well-known that under this map, the subspace Lie $[c_1, c_2, \ldots]$ of $\mathfrak{lie}^{(1,1)}$, which consists of all Lie polynomials having no linear term in a, maps bijectively onto the space of alternal polynomial-valued moulds. In other words, when speaking of polynomial moulds, alternality corresponds precisely to the condition that the associated polynomial in a, b should be primitive for the standard coproduct $\Delta(a) = a \otimes 1 + 1 \otimes a$ $\Delta(b) = b \otimes 1 + 1 \otimes b$, i.e. should be a Lie polynomial.

Writing $f \in \mathfrak{lie}_m^{(1,1)}$ as

$$f = \sum_{\underline{k}} c_{\underline{k}}\, a^{k_0} b \cdots b a^{k_r}$$

and F for the corresponding mould, then $swap(F)$ is explicitly given by

$$swap(F)(v_1, \ldots, v_r) = (-1)^{m-1} \sum_{\underline{k} \text{ s.t. } k_r=0} c_{\underline{k}}\, v_1^{k_0} \cdots v_r^{k_{r-1}} \tag{7}$$

(cf. [[17], §3]).

A mould A is said to be *push-invariant* if $push(A) = A$, and *circ-neutral* if for all $r \geq 2$, we have

$$\sum_{i=0}^{r-1} circ^i(A)(v_1, \ldots, v_r) = 0.$$

We say that A is circ*-neutral if it is becomes circ-neutral after adding on a constant-valued mould.

Definition The mould version of $\mathfrak{krv}_{ell}$ consists of all polynomial-valued moulds F that are alternal and push-invariant and such that $swap\big(\Delta^{-1}(F)\big)$ is circ*-neutral.

The space $\mathfrak{krv}_{ell}$ is bigraded for the depth and the degree. Let $F \in \mathfrak{krv}_{ell}$ be a mould of depth r and degree d, so that it corresponds under the bijection (6) to a polynomial $f \in \mathfrak{lie}_{n,r}^{(1,1)}$ with $n = d + r$. The mould push-invariance property of a polynomial-valued mould F is equivalent to the polynomial push-invariance $push(f) = f$ (cf. [[15], Prop. 12]). In turn, the polynomial push-invariance of f implies that there exists a unique polynomial $g \in \mathfrak{lie}_{n,r+1}^{(1,1)}$ such that setting $d(a) = f$, $d(b) = g$, we obtain a derivation $d \in \mathfrak{der}_n^{(1,1)}$. The Lie bracket on $\mathfrak{krv}_{ell}$ corresponds to the Lie bracket on $\mathfrak{krv}^{(1,1)}$, namely bracketing of the derivations d. Thus, in order to prove that $\mathfrak{krv}_{ell}$ is in bijection with $\mathfrak{krv}^{(1,1)}$, it remains only to prove that the circ*-neutrality condition on $swap\big(\Delta^{-1}(F)\big)$ is equivalent to the divergence condition (5) on f.

Since

$$\Delta^{-1}(F)(u_1,\ldots,u_r)=\frac{1}{(u_1+\cdots+u_r)u_1\cdots u_r}F(u_1,\ldots,u_r),$$

we have

$$swap\big(\Delta^{-1}(F)\big)(v_1,\ldots,v_r)=\frac{1}{v_1(v_1-v_2)\cdots(v_{r-1}-v_r)v_r}swap(F)(v_1,\ldots,v_r),$$

so the circ*-neutrality condition is given explicitly as the existence of a constant $K\in\mathbb{Q}$ such that

$$\frac{swap(F)(v_1,\ldots,v_r)}{v_1(v_1-v_2)\cdots(v_{r-1}-v_r)v_r}+\frac{swap(F)(v_2,\ldots,v_1)}{v_2(v_2-v_3)\cdots(v_r-v_1)v_1}+\cdots+$$
$$+\frac{swap(F)(v_r,\ldots,v_{r-1})}{v_r(v_r-v_1)\cdots(v_{r-2}-v_{r-1})v_{r-1}}=\begin{cases}0 & n\neq 2r+1\\ (-1)^rKr & n=2r+1.\end{cases} \quad (8)$$

Indeed, we note that the only possibility for the sum to be equal to a constant is when the degrees of the numerator and denominator are equal, which can only happen when the degree $n-r$ of the mould $F=ma(f)$ in depth r is equal to the degree $r+1$ of the Δ-denominator in depth r, i.e. when $n=2r+1$. We write $(-1)^rKr$ for the constant rather than simply K, in order for the value of K in this formula to correspond to the value in the definition of the divergence condition given in (5) when we prove that the two conditions are equal.

Putting the left-hand side of (8) over a common denominator and multiplying both sides by that denominator gives the equivalent equality

$$swap(F)(v_1,v_2,\ldots,v_r)v_2\ldots v_{r-1}(v_r-v_1)+swap(F)(v_2,\ldots,v_r,v_1)v_3\cdots v_r(v_1-v_2)+\cdots$$

$$+swap(F)(v_r,\ldots,v_{r-1})v_1\ldots v_{r-2}(v_{r-1}-v_r)=(-1)^rKr\,v_1\cdots v_r(v_1-v_2)\cdots(v_r-v_1),$$

where $K=0$ unless $n=2r+1$. The left-hand side of this expands to

$$v_2\ldots v_{r-1}v_r\,swap(F)(v_1,\ldots,v_r)-v_1v_2\ldots v_{r-1}\,swap(F)(v_1,\ldots,v_r)$$
$$+v_1v_3\ldots v_r\ swap(F)(v_2,\ldots,v_r,v_1)-v_2v_3..v_r\,swap(F)(v_2,..,v_r,v_1)+\cdots$$
$$+v_1\ldots v_{r-1}\ swap(F)(v_r,\ldots,v_{r-1})-v_1\ldots v_{r-2}v_r\ swap(F)(v_r,\ldots v_{r-1}). \quad (9)$$

Fix a monomial $v_1^{i_1+1}v_2^{i_2+1}\ldots v_r^{i_r+1}$. Calculating its coefficient in (9) yields

$$\begin{aligned}
&\big(swap(F)(v_1,\dots,v_r)|v_1^{i_1+1}v_2^{i_2}\dots v_r^{i_r}\big)-\big(swap(F)(v_1,\dots,v_r)|v_1^{i_1}v_2^{i_2}\dots v_r^{i_r+1}\big)\\
&\quad+\big(swap(F)(v_2,\dots,v_r,v_1)|v_1^{i_1}v_2^{i_2+1}\dots v_r^{i_r}\big)-\big(swap(F)(v_2,\dots,v_r,v_1)|v_1^{i_1+1}v_2^{i_2}\dots v_r^{i_r}\big)+\cdots\\
&\quad\quad+\big(swap(F)(v_r,v_1,\dots,v_{r-1})|v_1^{i_1}\dots v_r^{i_r+1}\big)-\big(swap(F)(v_r,v_1,\dots,v_{r-1})|v_1^{i_1}\dots v_{r-1}^{i_{r-1}+1}v_r^{i_r}\big)\\
&=\big(swap(F)(v_1,\dots,v_r)|v_1^{i_1+1}v_2^{i_2}\dots v_r^{i_r}\big)-\big(swap(F)(v_1,\dots,v_r)|v_1^{i_1}v_2^{i_2}\dots v_r^{i_r+1}\big)\\
&\quad+\big(swap(F)(v_1,\dots,v_r)|v_1^{i_2+1}v_2^{i_3}\dots v_r^{i_1}\big)-\big(swap(F)(v_1,\dots,v_r)|v_1^{i_2}v_2^{i_3}\dots v_r^{i_1+1}\big)+\cdots\\
&\quad\quad+\big(swap(F)(v_1,\dots,v_r)|v_1^{i_r+1}\dots v_r^{i_{r-1}}\big)-\big(swap(F)(v_1,\dots,v_r)|v_1^{i_r}\dots v_r^{i_{r-1}+1}\big),
\end{aligned}$$

where the equality is obtained by bringing every term back to a coefficient of a word in $swap(F)(v_1,\dots,v_r)$.

The circ*-neutrality condition on $swap\big(\Delta^{-1}(F)\big)$ can thus be expressed by the family of relations for every tuple $(i_1,\dots,i_r)$:

$$\begin{aligned}
&\big(swap(F)(v_1,\dots,v_r)|v_1^{i_1+1}v_2^{i_2}\dots v_r^{i_r}\big)-\big(swap(F)(v_1,\dots,v_r)|v_1^{i_1}v_2^{i_2}\dots v_r^{i_r+1}\big)\\
&\quad+\big(swap(F)(v_1,\dots,v_r)|v_1^{i_2+1}v_2^{i_3}\dots v_r^{i_1}\big)-\big(swap(F)(v_1,\dots,v_r)|v_1^{i_2}v_2^{i_3}\dots v_r^{i_1+1}\big)+\cdots\\
&\quad\quad+\big(swap(F)(v_1,\dots,v_r)|v_1^{i_r+1}\dots v_r^{i_{r-1}}\big)-\big(swap(F)(v_1,\dots,v_r)|v_1^{i_r}\dots v_r^{i_{r-1}+1}\big)\\
&=(-1)^r Kr\big((v_1-v_2)\dots(v_r-v_1)\,|\,v_1^{i_1}\dots v_r^{i_r}\big)\\
&=(-1)^r Kr\big((v_1-v_2)\dots(v_{r-1}-v_r)v_r\,|\,v_1^{i_1}\dots v_r^{i_r}\big)-(-1)^r Kr\big(v_1(v_1-v_2)\dots(v_{r-1}-v_r)\,|\,v_1^{i_1}\dots v_r^{i_r}\big)\\
&=(-1)^r Kr\big((v_1-v_2)\dots(v_{r-1}-v_r)v_r\,|\,v_1^{i_1}\dots v_r^{i_r}\big)+Kr\big((v_1-v_2)\dots(v_{r-1}-v_r)v_r\,|\,v_1^{i_r}v_2^{i_{r-1}}\dots v_r^{i_1}\big).
\end{aligned}\tag{10}$$

We now translate this equality back into polynomial terms. We start with the right-hand side. The right-hand side is zero unless $n=2r+1$, so let us compute it in the case $n=2r+1$. We have $[a,b]=ad(a)(b)=c_2$, so $[a,b]^r=c_2^r$, and by formula (6), the polynomial-valued mould corresponding to $[a,b]^r$ is thus given by

$$A(u_1,\dots,u_r)=(-1)^r u_1\cdots u_r. \tag{11}$$

The swap of this mould is given by

$$swap(A)(v_1,\dots,v_r)=(-1)^r(v_1-v_2)\cdots(v_{r-1}-v_r)v_r.$$

The moulds A and $swap(A)$ are concentrated in degree r in depth r. Thus the right-hand side of (10) is zero unless $i_1+\cdots+i_r=r$, in which case it can be written as

$$Kr\Big(swap(A)(v_1,\dots,v_r)\,\big|\,v_1^{i_1}v_2^{i_2}\cdots v_r^{i_r}\Big)+(-1)^r Kr\Big(swap(A)(v_1,\dots,v_r)\,\big|\,v_1^{i_r}v_2^{i_{r-1}}\cdots v_r^{i_1}\Big). \tag{12}$$

So by (7), this expression translates back to polynomials as

$$-Kr\Big([a,b]^r\,|\,a^{i_1}b\cdots a^{i_r}b\Big)+(-1)^{r-1}Kr\Big([a,b]^r\,|\,a^{i_r}b\cdots a^{i_1}b\Big), \tag{13}$$

since here $m=2r$ is the degree of $[a,b]^r$.

Using (7) to directly translate the left-hand side of (10) in terms of the polynomial f, we thus obtain the following expression equivalent to the circ-neutrality property (10):

$$\begin{aligned}
&\big(f \mid a^{i_1+1}ba^{i_2}b\cdots ba^{i_r}b\big) - \big(f \mid a^{i_1}ba^{i_2}b\cdots ba^{i_r+1}b\big)\\
&\quad + \big(f \mid a^{i_2+1}ba^{i_3}b\cdots ba^{i_1}b\big) - \big(f \mid a^{i_2}ba^{i_3}b\cdots ba^{i_1+1}b\big) + \cdots\\
&\qquad + \big(f \mid a^{i_r+1}ba^{i_1}b\cdots ba^{i_{r-1}}b\big) - \big(f \mid a^{i_r}ba^{i_1}b\cdots ba^{i_{r-1}+1}b\big)\\
&= -Kr\Big([a,b]^r \mid a^{i_1}b\cdots a^{i_r}b\Big) - (-1)^r Kr\Big([a,b]^r \mid a^{i_r}b\cdots a^{i_1}b\Big).
\end{aligned} \tag{14}$$

Since f is push-invariant, we have $(f|ub) = (f|bu)$ for every word u, so we can modify the negative terms in (14):

$$\begin{aligned}
&\big(f \mid a^{i_1+1}ba^{i_2}b\cdots ba^{i_r}b\big) - \big(f \mid ba^{i_1}ba^{i_2}b\cdots ba^{i_r+1}\big)\\
&\quad + \big(f \mid a^{i_2+1}ba^{i_3}b\cdots ba^{i_1}b\big) - \big(f \mid ba^{i_2}ba^{i_3}b\cdots ba^{i_1+1}\big) + \cdots\\
&\qquad + \big(f \mid a^{i_r+1}ba^{i_1}b\cdots ba^{i_{r-1}}b\big) - \big(f \mid ba^{i_r}ba^{i_1}b\cdots ba^{i_{r-1}+1}\big)\\
&= -Kr\Big([a,b]^r \mid a^{i_1}b\cdots a^{i_r}b\Big) - (-1)^r Kr\Big([a,b]^r \mid a^{i_r}b\cdots a^{i_1}b\Big).
\end{aligned} \tag{15}$$

Now all words in the positive terms start in a and end in b, and all words in the negative terms start in b and end in a, so we can remove these letters and write

$$\begin{aligned}
&\big(f_b^a \mid a^{i_1}ba^{i_2}b\cdots ba^{i_r}\big) - \big(f_a^b \mid a^{i_1}ba^{i_2}b\cdots ba^{i_r}\big)\\
&\quad + \big(f_b^a \mid a^{i_2}ba^{i_3}b\cdots ba^{i_1}\big) - \big(f_a^b \mid a^{i_2}ba^{i_3}b\cdots ba^{i_1}\big) + \cdots\\
&\qquad + \big(f_b^a \mid a^{i_r}ba^{i_1}b\cdots ba^{i_{r-1}}\big) - \big(f_a^b \mid a^{i_r}ba^{i_1}b\cdots ba^{i_{r-1}}\big)\\
&= -Kr\Big([a,b]^r \mid a^{i_1}b\cdots a^{i_r}b\Big) - (-1)^r Kr\Big([a,b]^r \mid a^{i_r}b\cdots a^{i_1}b\Big).
\end{aligned} \tag{16}$$

Consider now a word $u = a^{i_1}ba^{i_2}b\cdots ba^{i_r}$ of degree (weight) $n-2$ and depth $r-1$, and let u' denote u written backwards. Using the previous notation m for the degree of $ub = a^{i_1}b\cdots a^{i_r}b$, we have $n-2 = m-1$, i.e. $m = n-1$. The left-hand side of (16) is equal to

$$\sum_{i=0}^{r-1}\Big((f_b^a - f_a^b)\,\big|\,push^i(u)\Big) = \frac{r}{|P(u)|}\Big(pushsym(f_b^a - f_a^b)\,\big|\,u\Big).$$

Changing the sign of both sides of (16) in order to compare with (5), it becomes

$$\frac{1}{|P(u)|}\Big(pushsym(f_a^b - f_b^a)\,\big|\,u\Big) = \begin{cases} 0 & n \neq 2r+1\\ K\Big([a,b]^r \mid ub\Big) + (-1)^r K\Big([a,b]^r \mid u'b\Big) & n = 2r+1. \end{cases} \tag{17}$$

Since the left-hand sides of (5) and (17) are identical, in order to prove that they give the same condition, we only need to check that the two right-hand sides are equal. Cancelling the factor Kr from each, this reduces to the following lemma.

Lemma *Let u be a word of depth $r-1$ and weight $n=2r-1$, let u' be u written backwards, and let $C(ub)$ denote the set of cyclic permutations of ub. Then*

$$\sum_{v\in C(ub)} \big([a,b]^r \mid v\big) = |P(u)|\big([a,b]^r \mid ub\big) + (-1)^r|P(u)|\big([a,b]^r \mid u'b\big). \quad (18)$$

Proof Observe that if $([a,b]^r|ub)\neq 0$, then ub must satisfy the *parity property* that, writing $ub=u_1\cdots u_{2r}$ where each u_i is letter a or b, the pair $u_{2i-1}u_{2i}$ must be either ab or ba for $0\le i\le r$. The coefficient of the word ub in $[a,b]^r$ is equal to $(-1)^j$ where j is the number of pairs $u_{2i-1}u_{2i}$ in ub that are equal to ba. In other words, if a word w appears with non-zero coefficient in $[a,b]^r$, then letting $U=ba$ and $V=ab$, we must be able to write w as a word in U,V, and the coefficient of w in $[a,b]^r$ is $(-1)^m$ where m denotes the number of times the letter U occurs.

If $w=ub=V^r=(ab)^r$, then $u'b=ub$. The coefficient of V^r in $[a,b]^r$ is equal to 1, so the right-hand side of (18) is equal to 2 if r is even and 0 if r is odd. For the left-hand side, $C(ub)=\{V^r,U^r\}$ and $C_b(ub)=\{V^r\}$, so $|C_b(ub)|=1$. The coefficient of U^r in $[a,b]^r$ is equal to $(-1)^r$, so the left-hand side is again equal to 2 if r is even and 0 if r is odd. Since $|C_b(ub)|=|P(u)|$, this proves (18) in the case $ub=V^r$.

Suppose now that $ub\neq V^r$ but that it satisfies the parity property. Write $ub=U^{a_1}V^{b_1}\cdots U^{a_s}V^{b_s}$ in which all the $a_i,b_i\ge 1$ except for a_1, which may be 0. Then $u'b$ is equal to $aU^{b_s-1}V^{a_s}\cdots U^{b_1}V^{a_1}b$. If $b_s>1$, then the pair $u_{2(b_s-1)+1}u_{2(b_s-1)+2}$ is aa, so $([a,b]^r|u'b)=0$. If $b_s=1$, then the word $u'b$ begins with aa and thus does not have the parity property, so again $([a,b]^r|u'b)=0$. This shows that if $([a,b]^r|ub)\neq 0$ then $([a,b]^r|u'b)=0$ and vice versa.

This leaves us with three possibilities for $ub\neq V^r$.

Case 1: $([a,b]^r|ub)\neq 0$. Then ub has the parity property, so we write $ub=U^{a_1}V^{b_1}\cdots U^{a_s}V^{a_s}$ as above. The right-hand side of (18) is then equal to $(-1)^j$ where $j=a_1+\cdots+a_s$. For the left-hand side, we note that the only words in the cyclic permutation class $C(ub)$ that have the parity property are the cyclic shifts of ub by an even number of letters, otherwise a pair aa or bb necessarily occurs as above. These are the same as the cyclic permutations of the word ub written in the letters U,V. All these cyclic permutations obviously have the same number of occurrences j of the letter U. Thus, the words in $C(ub)$ for which $[a,b]^r$ has a non-zero coefficient are the cyclic permutations of the word ub in the letters U,V, and the coefficient is always equal to $(-1)^j$. These words are exactly half of the all the words in $C(ub)$, so the sum in the left-hand side is equal to $(-1)^j|C(ub)|/2$. But $|C_b(ub)|=|P(u)|=|C(ub)|/2$, so the left-hand side is equal to $(-1)^j$, which proves (18) for words ub having the parity property.

Case 2: $([a,b]^r|u'b)\neq 0$. In this case it is $u'b$ that has the parity property, and the right-hand side of (18) is equal to $(-1)^{r+j'}$ where j' is the number of occurrences of U in the word $u'b=U^{a_1}V^{b_1}\cdots U^{a_s}V^{b_s}$, i.e. $j'=a_1+\cdots+a_s$. We have

$ub = aU^{b_s-1}V^{a_s}\cdots U^{b_1}V^{b_1}b$. The word $w = U^{b_s-1}V^{a_s}\cdots U^{b_1}V^{b_1}U$ then occurs in $C(ub)$, and the number of occurrences of the letter U in ub is equal to $j = b_1 + \cdots + b_{s-1} + b_s$. Since $a_1 + b_1 + \cdots + a_s + b_s = r$, we have $j + j' = r$ so $j' = r - j$ and the right-hand side of (18) is equal to $(-1)^j$. The number of words in $C(ub)$ that have non-zero coefficient in $[a,b]^r$ is $|C(ub)|/2 = |C_b(ub)| = |P(u)|$ as above, these words being exactly the cyclic permutations of w written in U, V, and the coefficient is always equal to $(-1)^j$. So the left-hand side of (18) is equal to $(-1)^j$, which proves (18) in the case where $u'b$ has the parity property.

Case 3: $([a,b]^r|ub) = ([a,b]^r|u'b) = 0$. The right-hand side of (18) is zero. For the left-hand side, consider the words in $C(ub)$. If there are no words in $C(ub)$ whose coefficient in $[a,b]^r$ is non-zero, then the left-hand side of (18) is also zero and (18) holds. Suppose instead that there is a word $w \in C(ub)$ whose coefficient in $[a,b]^r$ is non-zero. Then as we saw above, w is a cyclic shift of ub by an odd number of letters, and since all cyclic shifts of w by an even number of letters then have the same coefficient in $[a,b]^r$ as w, we may assume that w is the cyclic shift of ub by one letter, i.e. taking the final b and putting it at the beginning. Since w has non-zero coefficient in $[a,b]^r$, we can write $w = U^{a_1}V^{b_1}\cdots U^{a_s}V^{b_s}$, where $a_1 > 0$ since w now starts with b, but b_s may be equal to 0 since w may end with a. Then $ub = aU^{a_1-1}V^{b_1}\cdots U^{a_s}V^{b_s}b$, so we can write $u'b = U^{b_s}V^{a_s}\cdots U^{b_1}V^{a_1-1}ab = U^{b_s}V^{a_s}\cdots U^{v_1}V^{a_1}$. But then $u'b$ satisfies the parity property, so its coefficient in $[a,b]^r$ is non-zero, contradicting our assumption. Thus under the assumption, all words in $C(ub)$ have coefficient zero in $[a,b]^r$, which completes the proof of the Lemma. $\diamond$

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