



On Interpretability Between Some Weak Essentially Undecidable Theories

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Abstract. We introduce two essentially undecidable first-order theories WT and T. The intended model for the theories is a term model. We prove that WT is mutually interpretable with Robinson’s R. Moreover, we prove that Robinson’s Q is interpretable in T.

1 Introduction

A first-order theory T is *undecidable* if there is no algorithm for deciding if $T \vdash \phi$. If every consistent extension of an undecidable theory T also is undecidable, then T is *essentially undecidable*.

We introduce two first-order theories, WT and T, over the language $\mathcal{L}_T = \{\perp, \langle \cdot, \cdot \rangle, \sqsubseteq\}$ where \perp is a constant symbol, $\langle \cdot, \cdot \rangle$ is a binary function symbol and \sqsubseteq is a binary relation symbol. The intended model for these theories is a term model: The universe is the set of all variable-free \mathcal{L}_T -terms. Each term is interpreted as itself, and \sqsubseteq is interpreted as the subterm relation (s is a subterm of t iff $s = t$ or $t = \langle t_1, t_2 \rangle$ and s is a subterm of t_1 or t_2).

The non-logical axioms of WT are given by the two axiom schemes:

$$(WT_1) \quad s \neq t$$

where s and t are distinct variable-free terms.

$$(WT_2) \quad \forall x [x \sqsubseteq t \leftrightarrow \bigvee_{s \in \mathcal{S}(t)} x = s]$$

where t is a variable-free term and $\mathcal{S}(t)$ is the set of all subterms of t . There are no other non-logical axioms except those given by these two simple schemes, and at a first glance WT seems to be a very weak theory. Still it turns out that Robinson’s essentially undecidable theory R is interpretable in WT, and thus it follows that also WT is essentially undecidable. The theory T is given by the four axioms:

$$\begin{aligned} T_1 & \quad \forall xy [\langle x, y \rangle \neq \perp] \\ T_2 & \quad \forall x_1 x_2 y_1 y_2 [\langle x_1, x_2 \rangle = \langle y_1, y_2 \rangle \rightarrow (x_1 = y_1 \wedge x_2 = y_2)] \\ T_3 & \quad \forall x [x \sqsubseteq \perp \leftrightarrow x = \perp] \\ T_4 & \quad \forall xyz [x \sqsubseteq \langle y, z \rangle \leftrightarrow (x = \langle y, z \rangle \vee x \sqsubseteq y \vee x \sqsubseteq z)]. \end{aligned}$$

It is not difficult to see that T is a consistent extension of WT . Thus, since WT is essentially undecidable, we can conclude right away that also T is essentially undecidable. Furthermore, since every model of the finitely axiomatizable theory T is infinite, T cannot be interpretable in WT , and the obvious conjecture would be that T is mutually interpretable with Robinson's Q .

The Axioms of R

$$\begin{aligned} R_1 \quad \bar{n} + \bar{m} = \overline{n + m}; \quad R_2 \quad \bar{n} \times \bar{m} = \overline{nm}; \quad R_3 \quad \bar{n} \neq \bar{m} \text{ for } n \neq m; \\ R_4 \quad \forall x [x \leq \bar{n} \rightarrow x = 0 \vee \dots \vee x = \bar{n}]; \quad R_5 \quad \forall x [x \leq \bar{n} \vee \bar{n} \leq x] \end{aligned}$$

The Axioms of Q

$$\begin{aligned} Q_1 \quad \forall xy [Sx = Sy \rightarrow x = y]; \quad Q_2 \quad \forall x [Sx \neq 0]; \quad Q_3 \quad \forall x [x \neq 0 \rightarrow \exists y [x = Sy]]; \\ Q_4 \quad \forall x [x + 0 = x]; \quad Q_5 \quad \forall xy [x + Sy = S(x + y)]; \quad Q_6 \quad \forall x [x \times 0 = 0]; \\ Q_7 \quad \forall xy [x \times Sy = (x \times y) + x]; \quad Q_8 \quad \forall xy [x \leq y \leftrightarrow \exists z [x + z = y]] \end{aligned}$$

Fig. 1. The axioms of R are given by axiom schemes where $n, m \in \mathbb{N}$ and \bar{n} denotes the n^{th} numeral, that is, $\bar{0} \equiv 0$ and $\overline{n+1} \equiv S\bar{n}$.

The seminal theories R and Q are theories of arithmetic. The theory R is given by axiom schemes, and Q is a finitely axiomatizable extension of R , see Fig. 1 (Q is also known as Robinson arithmetic and is more or less Peano arithmetic without the induction scheme). It was proved in Tarski et al. [9] that R and Q are essentially undecidable. Another seminal essentially undecidable first-order theory is Grzegorzcyk's TC . This is a theory of concatenation. The language is $\{*, \alpha, \beta\}$ where α and β are constant symbols and $*$ is a binary function symbol. The standard TC model is the structure where the universe is $\{a, b\}^+$ (all finite nonempty strings over the alphabet $\{a, b\}$), $*$ is concatenation, α is the string a and β is the string b . It was proved in Grzegorzcyk and Zdanowski [3] that TC is essentially undecidable. It was later proved that TC is mutually interpretable with Q , see Visser [10] for further references. The theory $WTC^{-\epsilon}$ is a weaker variant of TC that has been shown to be mutually interpretable with R , see Higuchi and Horihata [4] for more details and further references. The axioms of TC and $WTC^{-\epsilon}$ can be found in Fig. 2.

The overall picture shows three finitely axiomatizable and essentially undecidable first-order theories of different character and nature: Q is a theory of arithmetic, TC is a theory of concatenation, and T is a theory of terms (it may also be viewed as a theory of binary trees). All three theories are mutually interpretable with each other, and each of them come with a weaker variant given by axiom schemes. These weaker variants are also essentially undecidable and mutually interpretable with each other.

The theory T has, in contrast to Q and TC , a purely universal axiomatization, that is, there are no occurrences of existential quantifiers in the axioms. Moreover, its weaker variant WT has a neat and very compact axiomatization compared to R and $WTC^{-\epsilon}$.

The Axioms of $WTC^{-\epsilon}$

$$\begin{aligned} WTC_1^{-\epsilon} & \forall xyz [(x * (y * z) \sqsubseteq \underline{t} \vee (x * y) * z \sqsubseteq \underline{t}) \rightarrow x * (y * z) = (x * y) * z] ; \\ WTC_2^{-\epsilon} & \forall xyz [x * y = z * u \wedge x * y \sqsubseteq \underline{t} \rightarrow ((x = z \wedge y = u) \vee \\ & \exists w [(x * w = z \wedge w * u = y) \vee (z * w = x \wedge w * y = u)])] ; \\ WTC_3^{-\epsilon} & \forall xy [\alpha \neq x * y] ; \quad WTC_4^{-\epsilon} \forall xy [\beta \neq x * y] ; \quad WTC_5^{-\epsilon} \alpha \neq \beta \end{aligned}$$

where $x \sqsubseteq y$ is defined by

$$x = y \vee \exists z_1 z_2 [z_1 * x = y \vee x * z_2 = y \vee (z_1 * x) * z_2 = y \vee z_1 * (x * z_2) = y] .$$

The Axioms of TC

$$\begin{aligned} TC_1 & \forall xyz [x * (y * z) = (x * y) * z] ; \\ TC_2 & \forall xyz [x * y = z * u \rightarrow ((x = z \wedge y = u) \vee \\ & \exists w [(x * w = z \wedge w * u = y) \vee (z * w = x \wedge w * y = u)])] ; \\ TC_3 & \forall xy [\alpha \neq x * y] ; \quad TC_4 \forall xy [\beta \neq x * y] ; \quad TC_5 \alpha \neq \beta \end{aligned}$$

Fig. 2. $WTC_1^{-\epsilon}$ and $WTC_2^{-\epsilon}$ are axiom schemes where $t \in \{a, b\}^+$ and \underline{t} is a term inductively defined by: $\underline{a} \equiv \alpha$, $\underline{b} \equiv \beta$, $\underline{au} \equiv \alpha * \underline{u}$ and $\underline{bu} \equiv \beta * \underline{u}$.

Another interesting theory which is known to be mutually interpretable with Q, and thus also with TC and T, is the adjunctive set theory AST. More on AST and adjunctive set theory can be found in Damjanovic [2]. For recent results related to the work in the present paper, we refer the reader to Jerabek [5], Cheng [1] and Kristiansen and Murwanashyaka [7].

The rest of this paper is fairly technical, and we will assume that the reader is familiar with first-order theories and the interpretation techniques introduced in Tarski et al. [9]. In Sect. 2 we prove that R and WT are mutually interpretable. In Sect. 3 we prove that Q is interpretable in T. We expect that T can be interpreted in Q by standard techniques available in the literature.

2 R and WT Are Mutually Interpretable

The theory R^- over the language of Robinson arithmetic is given by the axiom schemes

$$\begin{aligned} R_1^- & \bar{n} + \bar{m} = \overline{n + m} ; \quad R_2^- \bar{n} \times \bar{m} = \overline{nm} ; \quad R_3^- \bar{n} \neq \bar{m} \text{ for } n \neq m ; \\ R_4^- & \forall x [x \leq \bar{n} \leftrightarrow x = 0 \vee \dots \vee x = \bar{n}] \end{aligned}$$

where $n, m \in \mathbb{N}$. Recall that \bar{n} denotes the n^{th} numeral, that is, $\bar{0} \equiv 0$ and $\overline{n+1} \equiv S\bar{n}$.

We now proceed to interpret R^- in WT. We choose the domain $I(x) \equiv x = x$ (thus we can just ignore the domain). Furthermore, we translate the successor function $S(x)$ as the function given by $\lambda x. \langle x, \perp \rangle$, and we translate the constant 0 as $\langle \perp, \perp \rangle$. Let \bar{n}^* denote the translation of the numeral \bar{n} . Then we have $\overline{n+1}^* \equiv \langle \bar{n}^*, \perp \rangle$. It follows from WT_1 that the translation of each instance of R_3^- is a theorem of WT since \overline{m}^* and \bar{n}^* are different terms whenever $m \neq n$.

We translate $x \leq y$ as $x \sqsubseteq y \wedge x \neq \perp$. It is easy to see that

$$\text{WT} \vdash \forall x [x \sqsubseteq \bar{n}^* \wedge x \neq \perp \leftrightarrow \bigvee_{s \in \mathcal{T}(n)} x = s] \quad (1)$$

where $\mathcal{T}(n) = \mathcal{S}(\bar{n}^*) \setminus \{\perp\}$ and $\mathcal{S}(\bar{n}^*)$ denotes the set of all subterms of \bar{n}^* . We observe that $\mathcal{T}(n) = \{\bar{k}^* \mid k \leq n\}$ and that (1) indeed is the translation of the axiom scheme R_4^- . Hence we conclude that the translation of each instance of R_4^- is a theorem of WT.

Next we discuss the translation of $+$. The idea is to obtain $n + i$ through a formation sequence of length i . Such a sequence will be represented by a term of the form

$$\langle \dots \langle \langle \bar{n}^*, \bar{0}^* \rangle, \langle \overline{n+1}^*, \bar{1}^* \rangle \rangle, \langle \overline{n+2}^*, \bar{2}^* \rangle \rangle \dots, \langle \overline{n+i}^*, \bar{i}^* \rangle \rangle. \quad (2)$$

Accordingly we translate $x + y = z$ by the predicate $\text{add}(x, y, z)$ given by the formula

$$\begin{aligned} (y = \bar{0}^* \wedge z = x) \vee \left\{ y \neq \bar{0}^* \wedge \exists W [\langle x, \bar{0}^* \rangle \sqsubseteq W \wedge \right. \\ \left. \forall X \forall Y \sqsubseteq y [\langle X, Y \rangle \sqsubseteq W \wedge Y \neq y \wedge Y \neq \perp \rightarrow \right. \\ \left. (\langle \langle X, \perp \rangle, \langle Y, \perp \rangle \rangle \sqsubseteq W \wedge (\langle Y, \perp \rangle = y \rightarrow \langle X, \perp \rangle = z))] \right\}. \end{aligned}$$

Lemma 1. *For any $m, n \in \mathbb{N}$, we have*

$$\text{WT} \vdash \forall z [\text{add}(\bar{n}^*, \bar{m}^*, z) \leftrightarrow z = \overline{n+m}^*].$$

Proof. First we prove that $\text{WT} \vdash \text{add}(\bar{n}^*, \bar{m}^*, \overline{n+m}^*)$. This is obvious if $m = 0$. Assume $m > 0$. Let

$$S_0^n \equiv \langle \bar{n}^*, \bar{0}^* \rangle \quad \text{and} \quad S_{i+1}^n \equiv \langle S_i^n, \langle \overline{n+i+1}^*, \bar{i+1}^* \rangle \rangle$$

and observe that S_i^n is of the form (2). We will argue that we can choose the W in the definition of $\text{add}(x, y, z)$ to be the term S_m^n .

So let $W = S_m^n$. By the axioms of WT, we have $\langle \bar{n}^*, \bar{0}^* \rangle \sqsubseteq W$. Assume

$$\langle X, Y \rangle \sqsubseteq W \text{ and } Y \neq y = \bar{m}^* \text{ and } Y \sqsubseteq y = \bar{m}^* \text{ and } Y \neq \perp.$$

By the axioms of WT, we have that $Y \sqsubseteq \bar{m}^*$, $Y \neq \bar{m}^*$ and $Y \neq \perp$ imply $Y = \bar{k}^*$ for some $k < m$. Since $\langle X, Y \rangle \sqsubseteq W$, we know by WT_2 that $\langle X, Y \rangle$ is one of the subterms of W . By WT_1 and the form of S_m^n , we conclude that $X = \overline{n+k}^*$. Furthermore, the form of S_m^n and WT_2 then ensures that $\langle \langle X, \perp \rangle, \langle Y, \perp \rangle \rangle \sqsubseteq W = S_m^n$. Moreover, if $\langle Y, \perp \rangle = \bar{m}^*$, then by WT_1 , we must have $k = m - 1$, and thus, $\langle X, \perp \rangle = \langle \overline{n+(m-1)}^*, \perp \rangle = \overline{n+m}^*$. This proves that we can deduce $\text{add}(\bar{n}^*, \bar{m}^*, \overline{n+m}^*)$ from the axioms of WT, and thus we also have

$$\text{WT} \vdash \forall z [z = \overline{n+m}^* \rightarrow \text{add}(\bar{n}^*, \bar{m}^*, z)].$$

Next we prove that the converse implication $\text{add}(\overline{n^*}, \overline{m^*}, z) \rightarrow z = \overline{n+m^*}$ follows from the axioms of WT (and thus the lemma follows). This is obvious when $m = 0$. Assume $m \neq 0$ and $\text{add}(\overline{n^*}, \overline{m^*}, z)$. Then we have W such that $\langle \overline{n^*}, \overline{0^*} \rangle \sqsubseteq W$ and

$$\forall X \forall Y \sqsubseteq \overline{m^*} [\langle X, Y \rangle \sqsubseteq W \wedge Y \neq \overline{m^*} \wedge Y \neq \perp \rightarrow (\langle \langle X, \perp \rangle, \langle Y, \perp \rangle \rangle \sqsubseteq W \wedge (\langle Y, \perp \rangle = \overline{m^*} \rightarrow \langle X, \perp \rangle = z))]. \quad (3)$$

Since $\langle n, \overline{0^*} \rangle \sqsubseteq W$ and (3) hold, we have $\langle \overline{n+k+1^*}, \overline{k+1^*} \rangle \sqsubseteq W$ for any $k < m$. It also follows from (3) that $z = \overline{n+k+1^*}$ when $m = k+1$. \square

It follows from the preceding lemma that there for any $n, m \in \mathbb{N}$ exists a unique $k \in \mathbb{N}$ such that $\text{WT} \vdash \text{add}(\overline{n^*}, \overline{m^*}, \overline{k^*})$. We translate $x + y = z$ by the predicate ϕ_+ where $\phi_+(x, y, z)$ is the formula

$$(\exists ! u [\text{add}(x, y, u)] \wedge \text{add}(x, y, z)) \vee (\neg \exists ! u [\text{add}(x, y, u)] \wedge z = \perp). \quad (4)$$

The second disjunct of (4) ensures the functionality of our translation, that is, it ensures that $\text{WT} \vdash \forall xy \exists ! x \phi_+(x, y, z)$ (the same technique is used in [6]). By Lemma 1, we have $\text{WT} \vdash \phi_+(\overline{n^*}, \overline{m^*}, \overline{n+m^*})$. This shows that the translation of any instance of the axiom scheme R_1^- can be deduced from the axioms of WT.

We can also achieve a translation of $x \times y = z$ such that the translation of each instance of R_2^- can be deduced from the axioms of WT. Such a translation claims the existence of a term S_m^n where

$$S_1^n \equiv \langle \overline{n^*}, \overline{1^*} \rangle \quad \text{and} \quad S_{i+1}^n \equiv \langle S_i^n, \langle \overline{(i+1)n^*}, \overline{i+1^*} \rangle \rangle$$

and will more or less be based on the same ideas as our translation of $x + y = z$. We omit the details.

Theorem 2. *R and WT are mutually interpretable.*

Proof. We have seen how to interpret R^- in WT. It follows straightforwardly from results proved in Jones and Shepherdson [6] that R^- and R are mutually interpretable. Thus R is interpretable in WT. A result of Visser [11] states that a theory is interpretable in R if and only if it is locally finitely satisfiable, that is, each finite subset of the non-logical axioms has a finite model. Since WT clearly is locally finitely satisfiable, WT is interpretable in R. \square

3 Q is Interpretable in T

The language of the arithmetical theory Q^- is $\{0, S, M, A\}$ where 0 is a constant symbol, S is a unary function symbol, and A and M are ternary predicate symbols. The non-logical axioms of the first-order theory Q^- are the the following:

$$\begin{aligned} & A \forall xy z_1 z_2 [A(x, y, z_1) \wedge A(x, y, z_2) \rightarrow z_1 = z_2] ; \\ & M \forall xy z_1 z_2 [M(x, y, z_1) \wedge M(x, y, z_2) \rightarrow z_1 = z_2] ; \\ & Q_1 \forall xy [x \neq y \rightarrow Sx \neq Sy] ; \quad Q_2 \forall x [Sx \neq 0] ; \quad Q_3 \forall x [x = 0 \vee \exists y [x = Sy]] ; \\ & G_4 \forall x [A(x, 0, x)] ; \quad G_5 \forall xyu [\exists z [A(x, y, z) \wedge u = Sz] \rightarrow A(x, Sy, u)] ; \\ & G_6 \forall x [M(x, 0, 0)] ; \quad G_7 \forall xyu [\exists z [M(x, y, z) \wedge A(z, x, u)] \rightarrow M(x, Sy, u)] . \end{aligned}$$

Svejdar [8] proved that Q^- and Q are mutually interpretable. We will prove that Q^- is interpretable in T .

The first-order theory T^+ is T extended by the two non-logical axioms

$$T_5 \quad \forall x[x \sqsubseteq x] \quad \text{and} \quad T_6 \quad \forall xyz[x \sqsubseteq y \wedge y \sqsubseteq z \rightarrow x \sqsubseteq z].$$

Lemma 3. T^+ is interpretable in T .

Proof. We simply relativize quantification to the domain

$$I = \{ x \mid x \sqsubseteq x \wedge \forall uv[u \sqsubseteq v \wedge v \sqsubseteq x \rightarrow u \sqsubseteq x] \}.$$

Suppose $x_1, x_2 \in I$. We show that $\langle x_1, x_2 \rangle \in I$. Since $\langle x_1, x_2 \rangle = \langle x_1, x_2 \rangle$, we have $\langle x_1, x_2 \rangle \sqsubseteq \langle x_1, x_2 \rangle$ by T_4 . Suppose now that $u \sqsubseteq v \wedge v \sqsubseteq \langle x_1, x_2 \rangle$. We need to show that $u \sqsubseteq \langle x_1, x_2 \rangle$. By T_4 and $v \sqsubseteq \langle x_1, x_2 \rangle$, at least one of the following three cases holds: (a) $v = \langle x_1, x_2 \rangle$, (b) $v \sqsubseteq x_1$, (c) $v \sqsubseteq x_2$. *Case (a):* Since $u \sqsubseteq v$ and $v = \langle x_1, x_2 \rangle$, we have $u \sqsubseteq \langle x_1, x_2 \rangle$ by our logical axioms. *Case (b):* $u \sqsubseteq v \wedge v \sqsubseteq x_1$ implies $u \sqsubseteq x_1$ since $x_1 \in I$. By T_4 , we have $u \sqsubseteq \langle x_1, x_2 \rangle$. *Case (c):* We have $u \sqsubseteq \langle x_1, x_2 \rangle$ by an argument symmetric to the one used in Case (b). Hence, $\forall uv[u \sqsubseteq v \wedge v \sqsubseteq \langle x_1, x_2 \rangle \rightarrow u \sqsubseteq \langle x_1, x_2 \rangle]$.

This proves that I is closed under $\langle \cdot, \cdot \rangle$. It follows from T_3 that $\perp \in I$, and thus I satisfies the domain condition. Clearly, the translation of each non-logical axiom of T^+ is a theorem of T . \square

We now proceed to interpret Q^- in T^+ . We choose the domain N given by

$$N(x) \equiv x \neq \perp \wedge \forall y \sqsubseteq x[y = \perp \vee \exists z[y = \langle z, \perp \rangle]].$$

Lemma 4. We have (i) $T^+ \vdash N(\langle \perp, \perp \rangle)$, (ii) $T^+ \vdash \forall x[N(x) \rightarrow N(\langle x, \perp \rangle)]$ and (iii) $T^+ \vdash \forall yz[N(y) \wedge z \sqsubseteq y \rightarrow (z = \perp \vee N(z))]$.

Proof. It follows from T_1 , T_3 and T_4 that (i) holds. In order to see that (ii) holds, assume $N(x)$ (we will argue that $N(\langle x, \perp \rangle)$ holds). Suppose $y \sqsubseteq \langle x, \perp \rangle$. Now, $N(\langle x, \perp \rangle)$ follows from

$$y = \perp \vee \exists z[y = \langle z, \perp \rangle]. \tag{5}$$

Thus it is sufficient to argue that (5) holds. By T_4 , we know that $y \sqsubseteq \langle x, \perp \rangle$ implies $y = \langle x, \perp \rangle \vee y \sqsubseteq x \vee y \sqsubseteq \perp$. The case $y = \langle x, \perp \rangle$: We obviously have $\exists z[y = \langle z, \perp \rangle]$ and thus (5) holds. The case $y \sqsubseteq x$: (5) holds since $N(x)$ holds. The case $y \sqsubseteq \perp$: We have $y = \perp$ by T_3 , and thus (5) holds. This proves (ii).

We turn to the proof of (iii). Suppose $N(y) \wedge z \sqsubseteq y$ (we show $z = \perp \vee N(z)$). Assume $w \sqsubseteq z$. By T_6 , we have $w \sqsubseteq y$, moreover, since $N(y)$ holds, we have $w = \perp \vee \exists u[w = \langle u, \perp \rangle]$. Thus, we conclude that

$$\forall w \sqsubseteq z[w = \perp \vee \exists u[w = \langle u, \perp \rangle]]. \tag{6}$$

Now

$$z = \perp \vee \underbrace{(z \neq \perp \wedge \forall w \sqsubseteq z[w = \perp \vee \exists u[w = \langle u, \perp \rangle]])}_{N(z)}$$

follows tautologically from (6). \square

We interpret $\dot{0}$ as $\langle \perp, \perp \rangle$. We interpret the successor function Sx as $\lambda x.\langle x, \perp \rangle$. To improve the readability we will occasionally write $\dot{0}$ in place of $\langle \perp, \perp \rangle$, $\dot{S}t$ in place of $\langle t, \perp \rangle$ and $t \in N$ in place of $N(t)$. We will also write $\exists x \in N[\eta]$ and $\forall x \in N[\eta]$ in place of, respectively, $\exists x[N(x) \wedge \eta]$ and $\forall x[N(x) \rightarrow \eta]$. Furthermore, $\mathbf{Q}x_1, \dots, x_n \in N$ is shorthand for $\mathbf{Q}x_1 \in N \dots \mathbf{Q}x_n \in N$ where \mathbf{Q} is either \forall or \exists .

Lemma 5. *The translations of \mathbf{Q}_1 , \mathbf{Q}_2 and \mathbf{Q}_3 are theorems of T^+ .*

Proof. The translation of \mathbf{Q}_1 is $\forall x, y \in N[x \neq y \rightarrow \dot{S}x \neq \dot{S}y]$. By T_2 , we have $x \neq y \rightarrow \dot{S}x \neq \dot{S}y$ for any x, y , and thus, the translation of \mathbf{Q}_1 is a theorem of T^+ .

The translation of \mathbf{Q}_2 is $\forall x \in N[\dot{S}x \neq \dot{0}]$. Assume $x \in N$. Then we have $x \neq \perp$, and by T_2 , we have $\dot{S}x \equiv \langle x, \perp \rangle \neq \langle \perp, \perp \rangle \equiv \dot{0}$.

The translation of \mathbf{Q}_3 is $\forall x \in N[x = \dot{0} \vee \exists y \in N[x = \dot{S}y]]$. Assume $x \in N$, that is, assume

$$x \neq \perp \wedge \forall y \sqsubseteq x[y = \perp \vee \exists z[y = \langle z, \perp \rangle]] . \tag{7}$$

By T_5 , we have $x \sqsubseteq x$. By (7) and $x \sqsubseteq x$, we have

$$x \neq \perp \wedge (x = \perp \vee \exists z[x = \langle z, \perp \rangle])$$

and then, by a tautological inference, we also have $\exists z[x = \langle z, \perp \rangle]$. Thus, we have z such that $\langle z, \perp \rangle \equiv \dot{S}z = x \in N$. By Lemma 4 (iii), we have $z = \perp \vee z \in N$. If $z = \perp$, we have $x = \langle \perp, \perp \rangle \equiv \dot{0}$. If $z \in N$, we have $z \in N$ such that $x = \dot{S}z$. Thus, $\mathsf{T}^+ \vdash \forall x \in N[x = \dot{0} \vee \exists y \in N[x = \dot{S}y]]$. □

Before we give the translation of A , we will provide some intuition. The predicate $A(a, b, c)$ holds in the standard model for \mathbf{Q}^- iff $a + b = c$. Let $\widetilde{0} \equiv \dot{0}$ and $\widetilde{n+1} \equiv \dot{S}\widetilde{n}$, and observe that $a + b = c$ iff there exists an \mathcal{L}_{T} -term of the form

$$\langle \dots \langle \langle \perp, \langle \widetilde{a}, \widetilde{0} \rangle \rangle, \langle \widetilde{a+1}, \widetilde{1} \rangle \rangle, \langle \widetilde{a+2}, \widetilde{2} \rangle \rangle \dots, \langle \widetilde{a+b}, \widetilde{b} \rangle \rangle \tag{8}$$

where $c = a + b$. We will give a predicate ϕ_A such that $\phi_A(\widetilde{a}, \widetilde{b}, w)$ holds in T^+ iff w is of the form (8). Thereafter we will use ϕ_A to give the translation Ψ_A of A .

Let $\phi_A(x, y, w) \equiv$

$$(y = \dot{0} \rightarrow w = \langle \perp, \langle x, \dot{0} \rangle \rangle) \wedge \exists w' \exists z \in N[w = \langle w', \langle z, y \rangle \rangle] \wedge \forall u \forall Y, Z \in N[\theta_A(u, w, Y, Z)]$$

where $\theta_A(u, w, Y, Z) \equiv$

$$\langle u, \langle Z, Y \rangle \rangle \sqsubseteq w \wedge Y \neq \dot{0} \rightarrow \exists v \exists Y' Z' \in N[Z = \dot{S}Z' \wedge Y = \dot{S}Y' \wedge u = \langle v, \langle Z', Y' \rangle \rangle \wedge (Y' = \dot{0} \rightarrow (Z' = x \wedge v = \perp))] .$$

The translation Ψ_A of A is $\Psi_A(x, y, z) \equiv$

$$\exists w [\phi_A(x, y, w) \wedge \exists w' [w = \langle w', \langle z, y \rangle \rangle] \wedge \forall u [\phi_A(x, y, u) \rightarrow u = w]] .$$

Lemma 6.

$$\mathsf{T}^+ \vdash \forall x \in N \forall w [\phi_A(x, \dot{0}, w) \leftrightarrow w = \langle \perp, \langle x, \dot{0} \rangle \rangle] .$$

Proof. We assume $x \in N$ and prove the equivalence

$$\phi_A(x, \dot{0}, w) \leftrightarrow w = \langle \perp, \langle x, \dot{0} \rangle \rangle \quad (9)$$

The left-right direction of (9) follows straightforwardly from the definition of ϕ_A . To prove the right-left implication of (9), we need to prove $\phi_A(x, \dot{0}, \langle \perp, \langle x, \dot{0} \rangle \rangle)$. It is easy to see that $\phi_A(x, \dot{0}, \langle \perp, \langle x, \dot{0} \rangle \rangle)$ holds if

$$\forall u \forall Y, Z \in N [\theta_A(u, \langle \perp, \langle x, \dot{0} \rangle \rangle, Y, Z)] \quad (10)$$

holds, and to show (10), it suffices to show that

$$x, Y, Z \in N \text{ and } \langle u, \langle Z, Y \rangle \rangle \sqsubseteq \langle \perp, \langle x, \dot{0} \rangle \rangle \text{ and } Y \neq \dot{0} \quad (11)$$

is a contradiction. (If (11) is a contradiction, then (10) will hold as the antecedent of θ_A will be false for all $x, Y, Z \in N$ and all u .)

By T_4 and $\langle u, \langle Z, Y \rangle \rangle \sqsubseteq \langle \perp, \langle x, \dot{0} \rangle \rangle$ we have to deal with the following three cases: (a) $\langle u, \langle Z, Y \rangle \rangle = \langle \perp, \langle x, \dot{0} \rangle \rangle$, (b) $\langle u, \langle Z, Y \rangle \rangle \sqsubseteq \perp$ and (c) $\langle u, \langle Z, Y \rangle \rangle \sqsubseteq \langle x, \dot{0} \rangle$. *Case: (a):* We have $Y = \dot{0}$ by T_2 , but we have $Y \neq \dot{0}$ in (11). *Case (b):* We have $\langle u, \langle Z, Y \rangle \rangle = \perp$ by T_3 , and this contradicts T_1 . *Case (c):* By T_4 , this case splits into the three subcases: (a') $\langle u, \langle Z, Y \rangle \rangle = \langle x, \dot{0} \rangle$, (b') $\langle u, \langle Z, Y \rangle \rangle \sqsubseteq x$ and (c') $\langle u, \langle Z, Y \rangle \rangle \sqsubseteq \dot{0}$. *Case (a'):* We have $\langle u, \langle Z, Y \rangle \rangle = \langle x, \langle \perp, \perp \rangle \rangle$ since $\dot{0}$ is shorthand for $\langle \perp, \perp \rangle$. Thus, by T_2 , we have $Z = \perp$ and $Y = \perp$. This contradicts $Y, Z \in N$. *Case (b'):* We have $\langle u, \langle Z, Y \rangle \rangle \sqsubseteq x$ and $x \in N$. By Lemma 4 (iii), we have $\langle u, \langle Z, Y \rangle \rangle = \perp$ or $\langle u, \langle Z, Y \rangle \rangle \in N$. Now, $\langle u, \langle Z, Y \rangle \rangle = \perp$ contradicts T_1 . Furthermore, by our definitions, $\langle u, \langle Z, Y \rangle \rangle \in N$ implies that

$$\forall y_0 \sqsubseteq \langle u, \langle Z, Y \rangle \rangle [y_0 = \perp \vee \exists z_0 [y_0 = \langle z_0, \perp \rangle]] .$$

By T_5 , we have $\langle u, \langle Z, Y \rangle \rangle = \perp \vee \exists z_0 [\langle u, \langle Z, Y \rangle \rangle = \langle z_0, \perp \rangle]$, and this yields a contradiction together with T_1 and T_2 . Case (c') is similar to Case (a'), but a bit simpler. This completes the proof of the lemma. \square

Lemma 7.

$$\mathsf{T}^+ \vdash \forall x, y \in N \forall z w w' [w = \langle w', \langle z, y \rangle \rangle \wedge \phi_A(x, y, w) \rightarrow \phi_A(x, \dot{S}y, \langle w, \langle \dot{S}z, \dot{S}y \rangle \rangle)] .$$

Proof. We assume

$$x, y \in N \text{ and } w = \langle w', \langle z, y \rangle \rangle \text{ and } \phi_A(x, y, w) . \quad (12)$$

We need to prove $\phi_A(x, \dot{S}y, \langle w, \langle \dot{S}z, \dot{S}y \rangle \rangle) \equiv$

$$\begin{aligned} & (\dot{S}y = \dot{0} \rightarrow w = \langle \perp, \langle x, \dot{0} \rangle \rangle) \wedge \\ & \quad \exists w_0 \exists z_0 \in N [\langle w, \langle \dot{S}z, \dot{S}y \rangle \rangle = \langle w_0, \langle z_0, \dot{S}y \rangle \rangle] \wedge \\ & \quad \forall u \forall Y, Z \in N [\theta_A(u, \langle w, \langle \dot{S}z, \dot{S}y \rangle \rangle, Y, Z)] \end{aligned} \quad (13)$$

First we prove

$$z \in N \quad \text{and} \quad \dot{S}z \in N \quad (14)$$

Since $\phi_A(x, y, w)$ holds by our assumptions (12), we have $z_1 \in N$ and w_1 such that $w = \langle w_1, \langle z_1, y \rangle \rangle$. We have also assumed $w = \langle w', \langle z, y \rangle \rangle$. By T_2 , we have $z = z_1$, and thus $z \in N$. By Lemma 4 (ii), we have $\dot{S}z \in N$. This proves (14).

The second conjunct of (13) follows straightforwardly from (14). (simply let z_0 be $\dot{S}z$ and let w_0 be w). The first conjunct follows easily from T_2 and the assumption $y \in N$. Thus, we are left to prove the third conjunct of (13), namely

$$\begin{aligned} & \forall u \forall Y, Z \in N [\langle u, \langle Z, Y \rangle \rangle \sqsubseteq \langle w, \langle \dot{S}z, \dot{S}y \rangle \rangle \wedge Y \neq \dot{0} \rightarrow \\ & \quad \exists v \exists Y' Z' \in N [Z = \dot{S}Z' \wedge Y = \dot{S}Y' \wedge u = \langle v, \langle Z', Y' \rangle \rangle \wedge \\ & \quad \quad (Y' = \dot{0} \rightarrow (Z' = x \wedge v = \perp))]] \end{aligned} \quad (15)$$

In order to do so, we assume

$$Y, Z \in N \text{ and } \langle u, \langle Z, Y \rangle \rangle \sqsubseteq \langle w, \langle \dot{S}z, \dot{S}y \rangle \rangle \text{ and } Y \neq \dot{0} \quad (16)$$

and prove

$$\begin{aligned} & \exists v \exists Y' Z' \in N [Z = \dot{S}Z' \wedge Y = \dot{S}Y' \wedge u = \langle v, \langle Z', Y' \rangle \rangle \wedge \\ & \quad (Y' = \dot{0} \rightarrow (Z' = x \wedge v = \perp))]. \end{aligned} \quad (17)$$

By our assumptions (16), we have $\langle u, \langle Z, Y \rangle \rangle \sqsubseteq \langle w, \langle \dot{S}z, \dot{S}y \rangle \rangle$, and then T_4 yields three cases: (a) $\langle u, \langle Z, Y \rangle \rangle = \langle w, \langle \dot{S}z, \dot{S}y \rangle \rangle$, (b) $\langle u, \langle Z, Y \rangle \rangle \sqsubseteq w$ and (c) $\langle u, \langle Z, Y \rangle \rangle \sqsubseteq \langle \dot{S}z, \dot{S}y \rangle$. We prove that that (17) holds in each of these three cases.

Case (a): By T_2 , we have $u = w$, $Z = \dot{S}z$ and $Y = \dot{S}y$. By (14), we have $z \in N$. By (12), we have $y \in N$. Moreover, by (12), we also have $u = w = \langle w', \langle z, y \rangle \rangle$. Thus there exist v and $Y', Z' \in N$ such that

$$Z = \dot{S}Z' \wedge Y = \dot{S}Y' \wedge u = \langle v, \langle Z', Y' \rangle \rangle.$$

If $y = \dot{0}$, we must have $\langle v, \langle z, y \rangle \rangle = w = \langle \perp, \langle x, \dot{0} \rangle \rangle$ since $\phi_A(x, y, w)$ holds by our assumptions (12). By T_2 , this implies $z = x$ and $v = \perp$. This proves that (17) holds in Case (a).

Case (b): By our assumptions (12), we have $\phi_A(x, y, w)$, and thus we also have $\theta_A(u, w, Y, Z) \equiv$

$$\begin{aligned} & \langle u, \langle Z, Y \rangle \rangle \sqsubseteq w \wedge Y \neq \dot{0} \rightarrow \\ & \quad \exists v \exists Y' Z' \in N [Z = \dot{S}Z' \wedge Y = \dot{S}Y' \wedge u = \langle v, \langle Z', Y' \rangle \rangle \wedge \\ & \quad \quad (Y' = \dot{0} \rightarrow (Z' = x \wedge v = \perp))]. \end{aligned} \quad (18)$$

We are dealing with a case where the antecedent of (18) holds, and thus (17) holds.

Case (c): This case is not possible. By T_4 , this case splits into the subcases: (a') $\langle u, \langle Z, Y \rangle \rangle = \langle \dot{S}z, \dot{S}y \rangle$, (b') $\langle u, \langle Z, Y \rangle \rangle \sqsubseteq \dot{S}z$ and (c') $\langle u, \langle Z, Y \rangle \rangle \sqsubseteq \dot{S}y$. We prove that each of these subcases contradicts our axioms. Case (a'): Recall that $\dot{S}y$ is shorthand for $\langle y, \perp \rangle$. Thus, by T_2 , we have $Y = \perp$. This contradicts the assumption (12) that $Y \in N$. Case (b'): By Lemma 4 (iii), we have $\langle u, \langle Z, Y \rangle \rangle = \perp \vee N(\langle u, \langle Z, Y \rangle \rangle)$. Now, $\langle u, \langle Z, Y \rangle \rangle = \perp$ contradicts T_1 . Furthermore, $N(\langle u, \langle Z, Y \rangle \rangle)$ implies that there is z_0 such that $\langle u, \langle Z, Y \rangle \rangle = \langle z_0, \perp \rangle$. By T_2 , we have $\langle Z, Y \rangle = \perp$. This contradicts T_1 . Case (c') is similar to Case (b'). This proves that (17) holds, and thus we conclude that the lemma holds. \square

Lemma 8.

$$\begin{aligned} \mathsf{T}^+ \vdash \forall xy \in N \forall w [\phi_A(x, \dot{S}y, w) \rightarrow \\ \exists u \in N \exists w' [w = \langle w', \langle u, \dot{S}y \rangle \rangle \wedge \phi_A(x, y, w')]]. \end{aligned}$$

Proof. Let $x, y \in N$ and assume $\phi_A(x, \dot{S}y, w)$. Thus, we have w' and $z \in N$ such that

$$w = \langle w', \langle z, \dot{S}y \rangle \rangle \quad \text{and} \quad \forall u \forall Y, Z \in N [\theta_A(u, w, Y, Z)] \quad (19)$$

Use the assumptions (19) to prove that $\phi_A(x, y, w') \equiv$

$$\begin{aligned} (y = \dot{0} \rightarrow w' = \langle \perp, \langle x, \dot{0} \rangle \rangle) \wedge \exists w'' \exists z \in N [w' = \langle w'', \langle z, y \rangle \rangle] \wedge \\ \forall u \forall Y, Z \in N [\theta_A(u, w', Y, Z)] \quad (20) \end{aligned}$$

holds. We omit the details. \square

Lemma 9. *The translations of A , G_4 and G_5 are theorems of T^+ .*

Proof. The translation of the axiom A is

$$\forall x, y, z_1, z_2 \in N [\Psi_A(x, y, z_1) \wedge \Psi_A(x, y, z_2) \rightarrow z_1 = z_2] .$$

Assume $\Psi_A(x, y, z_1)$ and $\Psi_A(x, y, z_2)$. Then it follows straightforwardly from the definition of Ψ_A and T_2 that $z_1 = z_2$. Hence the translation is a theorem of T^+ .

The translation of G_4 is $\forall x \in N [\Psi_A(x, \dot{0}, x)]$, that is

$$\begin{aligned} \forall x \in N \exists w [\phi_A(x, \dot{0}, w) \wedge \exists w' [w = \langle w', \langle x, \dot{0} \rangle \rangle] \wedge \\ \forall u [\phi_A(x, \dot{0}, u) \rightarrow u = w]] . \end{aligned}$$

We have

$$\mathsf{T}^+ \vdash \phi_A(x, \dot{0}, \langle \perp, \langle x, \dot{0} \rangle \rangle) \quad \text{and} \quad \mathsf{T}^+ \vdash \forall u [\phi_A(x, \dot{0}, u) \rightarrow u = \langle \perp, \langle x, \dot{0} \rangle \rangle]$$

by Lemma 6, and it easy to see that the translation of G_4 is a theorem of T^+ .

The translation of G_5 is

$$\forall x, y, u \in N [\exists z \in N [\Psi_A(x, y, z) \wedge u = \dot{S}z] \rightarrow \Psi_A(x, \dot{S}y, u)]. \quad (21)$$

In order to prove that (21) can be deduced from the axioms of Γ^+ , we assume $\Psi_A(x, y, z) \wedge u = \dot{S}z$. Then we need to prove $\Psi_A(x, \dot{S}y, \dot{S}z) \equiv$

$$\begin{aligned} \exists w [\phi_A(x, \dot{S}y, w) \wedge \exists w' [w = \langle w', \langle \dot{S}z, \dot{S}y \rangle \rangle] \wedge \\ \forall u [\phi_A(x, \dot{S}y, u) \rightarrow u = w]]. \end{aligned} \quad (22)$$

By our assumption $\Psi_A(x, y, z)$ there is a unique w_1 such that $\phi_A(x, y, w_1)$ and $w_1 = \langle w_0, \langle z, y \rangle \rangle$ for some w_0 . By Lemma 7, we have $\phi_A(x, \dot{S}y, \langle w_1, \langle \dot{S}z, \dot{S}y \rangle \rangle)$. Thus, we have w_2 such that $\phi_A(x, \dot{S}y, w_2)$ and $w_2 = \langle w_1, \langle \dot{S}z, \dot{S}y \rangle \rangle$. It is easy to see that (22) holds if w_2 is unique. Thus we are left to prove the uniqueness of w_2 , more precisely, we need to prove that

$$\forall W_2 [\phi_A(x, \dot{S}y, W_2) \rightarrow W_2 = w_2]. \quad (23)$$

In order to prove (23), we assume $\phi_A(x, \dot{S}y, W_2)$ (we will prove $W_2 = w_2 = \langle w_1, \langle \dot{S}z, \dot{S}y \rangle \rangle$). By our assumption $\phi_A(x, \dot{S}y, W_2)$ and Lemma 8, we have $u_0 \in N$ and W_1 such that $W_2 = \langle W_1, \langle u_0, \dot{S}y \rangle \rangle$ and $\phi_A(x, y, W_1)$. We have argued that there is a unique $w_1 = \langle w_0, \langle z, y \rangle \rangle$ such that $\phi_A(x, y, w_1)$ holds. By this uniqueness, we have $W_1 = w_1 = \langle w_0, \langle z, y \rangle \rangle$. So far we have proved

$$w_2 = \langle \overbrace{\langle w_0, \langle z, y \rangle \rangle}^{w_1}, \langle \dot{S}z, \dot{S}y \rangle \rangle \text{ and } W_2 = \langle \overbrace{\langle w_0, \langle z, y \rangle \rangle}^{W_1}, \langle u_0, \dot{S}y \rangle \rangle$$

and then we are left to prove that $u_0 = \dot{S}z$. By our assumption $\phi_A(x, \dot{S}y, W_2)$, we have v and $Z', Y' \in N$ such that $u_0 = \dot{S}Z', \dot{S}y = \dot{S}Y'$ and $W_1 = \langle v, \langle Z', Y' \rangle \rangle$. Thus, $\langle v, \langle Z', Y' \rangle \rangle = \langle w_0, \langle z, y \rangle \rangle$. By T_2 , we have $z = Z'$, and thus, $u_0 = \dot{S}Z' = \dot{S}z$. This proves that (23) holds. \square

We will now give the translation Ψ_M of M . Let $\phi_M(x, y, w) \equiv$

$$\begin{aligned} (y = \dot{0} \rightarrow w = \langle \perp, \langle \dot{0}, \dot{0} \rangle \rangle) \wedge \exists w' \exists z \in N [w = \langle w', \langle z, y \rangle \rangle] \wedge \\ \forall u \forall Y, Z \in N \theta_M(u, w, Y, Z) \end{aligned}$$

where $\theta_M(u, w, Y, Z) \equiv$

$$\begin{aligned} \langle u, \langle Z, Y \rangle \rangle \sqsubseteq w \wedge Y \neq \dot{0} \rightarrow \exists v \exists Y', Z' \in N [\Psi_A(Z', x, Z) \wedge \\ Y = \dot{S}Y' \wedge u = \langle v, \langle Z', Y' \rangle \rangle \wedge (Y' = \dot{0} \rightarrow Z' = \dot{0} \wedge v = \perp)]. \end{aligned}$$

We let $\Psi_M(x, y, z) \equiv$

$$\exists w [\phi_M(x, y, w) \wedge \exists w' [w = \langle w', \langle z, y \rangle \rangle \wedge \forall u [\phi_M(x, y, u) \rightarrow u = w]].$$

The translations of M , G_6 and G_7 are

$$\begin{aligned} M & \forall x, y, z_1, z_2 \in N [\Psi_M(x, y, z_1) \wedge \Psi_M(x, y, z_2) \rightarrow z_1 = z_2] \\ G_6 & \forall x \in N [M(x, \dot{0}, \dot{0})] \\ G_7 & \forall x, y, u \in N [\exists z \in N [\Psi_M(x, y, z) \wedge \Psi_A(z, x, u)] \rightarrow \Psi_M(x, \dot{S}y, u)] . \end{aligned}$$

The proof of the next lemma follows the lines of the proof of Lemma 9. We omit the details.

Lemma 10. *The translations of M , G_6 and G_7 are theorems of T^+ .*

Theorem 11. *Q is interpretable in T .*

Proof. It is proved in Svejdar [8] that Q is interpretable in Q^- . It follows from the lemmas above that Q^- is interpretable in T^+ which again is interpretable in T . Hence the theorem holds. \square

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