Models of Domination in Graphs

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1 Introduction

As we have said before, a set $S \subseteq V$ is a *dominating set* of a graph $G = (V, E)$ if every vertex $v \in V$ is either an element of S or is adjacent to an element of S. In this chapter, we will take a look at dominating sets from a variety of different perspectives. Each perspective suggests a variation in the domination theme and different types or aspects of dominating sets.

We will not attempt to be comprehensive here, only to provide a sufficient number of different models to reveal domination in a much broader view. Chapter 11 in the book *Fundamentals of Domination in Graphs* [5] presents ten logical structures or frameworks where the concept of domination naturally arises. The suggested frameworks range from integer programming to hypergraphs. We repeat a few of these frameworks in this chapter.

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In the most general sense, we are interested in sets of vertices in a graph having some property $\mathcal{P}$, called $\mathcal{P}$-sets. We are interested in finding $\mathcal{P}$-sets of minimum and maximum cardinalities, using a notational system of the form $a(G)$ for the minimum cardinality of a $\mathcal{P}$-set, and upper case $A(G)$ for the maximum cardinality of a $\mathcal{P}$-set. These parameters are sometimes referred to as lower and upper parameters.

But we will make a further distinction. As an example, recall that a set $S \subseteq V$ is *independent* if no two vertices in S are adjacent. Since independence is an *hereditary property* (henceforth denoted H), meaning that every subset of an independent set is also independent, it would not make any sense to seek a minimum cardinality independent set, and so instead we seek a minimum cardinality maximal independent set (denoted *minimax*). Similarly, the property of being a dominating set is *superhereditary* (henceforth denoted SH), meaning that any superset of a dominating set is also a dominating set. Thus, it would not make any sense to seek a maximum cardinality dominating set, since the entire vertex set of any graph is a dominating set. So, instead, we seek the maximum cardinality of a minimal dominating set (denoted *maximin*). For concepts that are neither hereditary nor superhereditary, we generally seek a minimum cardinality $\mathcal{P}$-set, denoted *min*, and sometimes a maximum cardinality $\mathcal{P}$-set, denoted *max*.

To illustrate these examples, consider the double star $S(r, s)$ for $1 \le r \le s$ with two adjacent vertices u and v, where u is adjacent to r leaves and v is adjacent to s leaves. Let $L(u)$ and $L(v)$ denote the set of leaves adjacent to u and v, respectively. We note that $S(r, s)$ has exactly four minimal dominating sets, namely $S_1 = \{u, v\}$, $S_2 = L(u) \cup \{v\}$, $S_3 = L(v) \cup \{u\}$, and $S_4 = L(u) \cup L(v)$. It follows that S_1 is a minimum dominating set and S_4 is a maximin dominating set. Thus, the domination number $\gamma(S(r, s)) = |S_1| = 2$ and the upper domination number $\Gamma(S(r, s)) = |S_4| = r + s$. Further, the maximal independent sets of $S(r, s)$ are precisely the sets S_2, S_3, and S_4. Hence, S_4 is a maximum independent set and S_2 is a minimax independent set, and so the independent domination number $i(S(r, s)) = |S_2| = r + 1$ and the independence number $\alpha(S(r, s)) = |S_4| = r + s$.

In the remaining sections of this chapter, we will use the following notation, where recall for a vertex v in G, the set $N_G(v)$ denotes the set of neighbors of v in G, and the degree of v in G is denoted by $d_G(v) = |N_G(v)|$. Further recall that for a subset of vertices $S \subseteq V$, the degree of v in S, denoted $d_S(v)$, is the number of vertices in S adjacent to the vertex v; that is, $d_S(v) = |N_G(v) \cap S|$. In particular, if $S = V$, then $d_S(v) = d_G(v)$. If the graph G is clear from context, we simply write $N(v)$ and $d(v)$ rather than $N_G(v)$ and $d_G(v)$, respectively. For $k \ge 1$ an integer, we use the standard notation $[k] = \{1, \ldots, k\}$ and $[k]_0 = [k] \cup \{0\} = \{0, 1, \ldots, k\}$. At a glance,

$\overline{S} = V \setminus S$, that is, $\overline{S}$ denotes the vertices in V but not in S, called the *complement* of S in G.
$d_S(v) = |N(v) \cap S|$.
$d_S[v] = |N[v] \cap S|$.
$d_{\overline{S}}(v) = |N(v) \cap \overline{S}|$.
$d_{\overline{S}}[v] = |N[v] \cap \overline{S}|$.

$G[S]$, the subgraph of G *induced by* S.
$\delta(G) = \min\{d(v) \mid v \in V\}$.
$\Delta(G) = \max\{d(v) \mid v \in V\}$.

Also, to avoid excessive repetition, we will frequently list domination parameters in the following abbreviated format:

parameter name: concept definition; designation of being hereditary H, or superhereditary SH; if neither hereditary nor superhereditary no designation is given; notation for lower parameter and type of $\mathcal{P}$-set (min or minimax), notation for upper parameter and type $\mathcal{P}$-set (max or maximin).

For example, domination, where $\gamma(G)$ is the domination number and $\Gamma(G)$ is the upper domination number, is listed as:

domination: $N[S] = V$, that is, for every $v \in \overline{S}$, $d_S(v) \geq 1$; SH; $\gamma(G)$ (min), $\Gamma(G)$ (maximin).

2 Fundamental Domination Parameters

In this section, we present what are arguably the most basic of all parameters related to domination in graphs. From these basic parameters all others are derived in one way or another. We begin with a list of five fundamental domination parameters. Thereafter, we list seven related parameters. The designations H hereditary and SH superhereditary are given whenever they apply. Unless otherwise stated, S always denotes a subset of V and F always denotes a subset of E.

2.1 Domination Parameters

(a) *domination*: $N[S] = V$, that is, for every $v \in \overline{S}$, $d_S(v) \geq 1$; SH; $\gamma(G)$ (min), $\Gamma(G)$ (maximin).
(b) *independent domination*: $N[S] = V$ and S is independent; $i(G)$ (min), $\alpha(G)$ (max).
(c) *total domination*: $N(S) = V$, that is, for every $v \in V$, $d_S(v) \geq 1$; $\gamma_t(G)$ (min), SH; $\Gamma_t(G)$ (maximin);
(d) *paired domination*: $N[S] = V$ and $G[S]$ has a *perfect matching*, that is, an independent set of edges of cardinality $\frac{1}{2}|S|$; $\gamma_{pr}(G)$ (min), $\Gamma_{pr}(G)$ (maximin).
(e) *connected domination*: $N[S] = V$ and $G[S]$ is connected; $\gamma_c(G)$ (min), SH; $\Gamma_c(G)$ (maximin).

It is perhaps worth commenting why total domination and connected domination are both superhereditary properties (SH). Whereas a superset S^* of a connected set S might not be a connected set, if S is also a dominating set, then every vertex in

$\overline{S}$ is adjacent to a vertex in S implying that S^* is also a connected dominating set. Similarly, total domination is superhereditary.

2.2 Related Parameters

(a) *vertex covering*: every edge $e \in E$ is incident to a vertex in S; SH; $\beta(G)$ (min), $\beta^+(G)$ (maximin). Note that for any graph G of order $n = |V|$,

$$\alpha(G) + \beta(G) = n$$

and

$$i(G) + \beta^+(G) = n.$$

(b) *irredundance*: for every vertex $v \in S$, $N[v] \setminus N[S \setminus \{v\}] \neq \emptyset$; H; $ir(G)$ (minimax), $IR(G)$ (max).

(c) *enclaveless*: S does not contain an *enclave*, that is, a vertex $v \in S$, such that $N[v] \subseteq S$; H; $\psi(G)$ (minimax), $\Psi(G)$ (max). Note that for every graph G of order n,

$$\gamma(G) + \Psi(G) = n$$

and

$$\Gamma(G) + \psi(G) = n.$$

(d) *packing*: for every $u, v \in S$, $d(u, v) > 2$; H; $p_2(G)$ (minimax), $P_2(G)$ (max). The packing number $P_2(G)$ is also denoted $\rho(G)$ in the literature. Note that the packing number is a standard lower bound on the domination number for any graph G, that is, $P_2(G) \leq \gamma(G)$.

(e) *edge domination*: $F \subseteq E$ and every edge not in F is adjacent to some edge in F; SH; $\gamma'(G)$ (min), $\Gamma'(G)$ (maximin).

(f) *matching*: $F \subseteq E$ and F is an independent set of edges; H; $\alpha'^-(G)$ (minimax), $\alpha'(G)$ (max).

(g) *edge covering*: every vertex $v \in V$ is incident to an edge in $F \subseteq E$; SH; $\beta'(G)$ (min), $\beta'^+(G)$ (maximin). Note, it has been shown in [4] and [6], respectively, that for every graph G of order n,

$$\alpha'(G) + \beta'(G) = n,$$

and

$$\alpha'^-(G) + \beta'^+(G) = n.$$

3 Conditions on the Dominating Set

Many domination parameters are formed by combining domination with another graph theoretical property P. In this section, we consider the parameters defined by imposing an additional constraint on the dominating set. In the next section, we will see that a condition may also be placed on the dominated set or on the method of dominating.

We list samples of types of dominating sets S defined either by imposing a condition on the subgraph $G[S]$ induced by S or requiring that every vertex in S satisfy some added condition. Clearly, some of the basic types are defined within this framework. For example, if $G[S]$ has no edges, then the set S is an independent dominating set, if $G[S]$ has no isolated vertices, then S is a total dominating set, and if $G[S]$ is connected, then S is a connected dominating set. Since all of the pairs of parameters in this section consist of the smaller as a minimum and the larger as a maximum of minimal, the designations (min) and (maximin) are omitted.

(a) *acyclic domination*: $N[S] = V$ and $G[S]$ is acyclic (contains no cycles); $\gamma_a(G)$, $\Gamma_a(G)$.
(b) *bipartite domination*: $N[S] = V$ and $G[S]$ is bipartite; $\gamma_{\text{bip}}(G)$, $\Gamma_{\text{bip}}(G)$.
(c) *clique domination*: $N[S] = V$ and $G[S]$ is a complete graph: $\gamma_{cl}(G)$, $\Gamma_{cl}(G)$.
(d) *private domination*: $N[S] = V$ and for every $u \in S$ there exists a vertex $v \in \overline{S}$ such that $N(v) \cap S = \{u\}$; $\gamma_{\text{pvt}}(G)$, $\Gamma_{\text{pvt}}(G)$. Note that a well-known theorem of Bollobás and Cockayne [1] shows that for every graph G with no isolated vertices, $\gamma(G) = \gamma_{\text{pvt}}(G)$, that is, G has a γ-set S such that for each vertex $u \in S$, there is a vertex $v \in \overline{S}$ with $N(v) \cap S = \{u\}$.
(e) *semitotal domination*: $N[S] = V$ and for every vertex $u \in S$, there exists a vertex $v \in S$ with $d(u, v) \leq 2$; SH; $\gamma_{t2}(G)$, $\Gamma_{t2}(G)$.
(f) *weakly connected domination*: $N[S] = V$ and $G' = (V, E_S)$ is connected, where E_S is the set of edges of G incident to at least one vertex of S; SH; $\gamma_w(G)$, $\Gamma_w(G)$.
(g) *semipaired domination*: $N[S] = V$ and the vertices in S can be partitioned into $|S|/2$ pairs $\{u, v\}$ such that $d(u, v) \leq 2$; $\gamma_{\text{pr2}}(G)$, $\Gamma_{\text{pr2}}(G)$.
(h) *convex domination*: $N[S] = V$ and for any two vertices $u, v \in S$, the vertices contained in all shortest paths between u and v, called $u - v$ geodesics, belong to S; $\gamma_{\text{conv}}(G)$, $\Gamma_{\text{conv}}(G)$.
(i) *weakly convex domination*: $N[S] = V$ and for any two vertices $u, v \in S$, there exists at least one $u - v$ geodesic, all of whose vertices belong to S; $\gamma_{\text{wconv}}(G)$, $\Gamma_{\text{wconv}}(G)$.
(j) *cycle domination*: $N[S] = V$ and $G[S]$ has a Hamilton cycle; $\gamma_{\text{cy}}(G)$, $\Gamma_{\text{cy}}(G)$.
(k) *equivalence domination*: $N[S] = V$ and $G[S]$ is disjoint union of complete subgraphs; $\gamma_e(G)$, $\Gamma_e(G)$.
(l) *k-dependent domination*: $N[S] = V$ and $\Delta(G[S]) \leq k$; $\gamma_{[k]}(G)$, $\Gamma_{[k]}(G)$.

4 Conditions on $\overline{S} = V \setminus S$

The framework considered in this section encompasses dominating sets S for which some condition is imposed on the vertices in the set $\overline{S}$ or the subgraph $G[\overline{S}]$ induced by $\overline{S}$. As before, we list a sampling of types of dominating sets in this framework. In many of them, we do not mention the upper parameters, which indicates that in general they have not been studied. As before, the designations H hereditary and SH superhereditary are given whenever they apply.

(a) *distance k-domination*: for every $v \in \overline{S}$, there exists a vertex $u \in S$ with $d(u, v) \leq k$; SH; $\gamma_{\leq k}(G)$, $\Gamma_{\leq k}(G)$.
(b) *k-step domination*: for every $v \in \overline{S}$, there exists a vertex $u \in S$ and a (u, v)-path of length equal to k; SH; $\gamma_{=k}(G)$.
(c) *k-domination*: for every vertex $v \in \overline{S}$, $d_S(v) \geq k$; SH; $\gamma_k(G)$.
(d) *restrained domination*: $N[S] = V$ and for every $v \in \overline{S}$, $d_{\overline{S}}(v) \geq 1$; $\gamma_r(G)$.
(e) *geodetic domination*: every vertex in $\overline{S}$ lies on a shortest path between two vertices in S; SH; $\gamma_g(G)$.
(f) *locating domination*: $N[S] = V$ and for every $v, w \in \overline{S}$, $N(v) \cap S \neq N(w) \cap S$; SH; $\gamma_L(G)$.
(g) *secondary domination*: every vertex $w \in \overline{S}$ is adjacent to at least one vertex $u \in S$ and is distance at most k to a second vertex in S; SH; $\gamma_{(1,k)}(G)$. Note that for any nontrivial graph without isolated vertices, $\gamma(G) = \gamma_{(1,4)}(G)$ and $\gamma_2(G) = \gamma_{(1,1)}(G)$.
(h) *downhill domination*: for every vertex $v \in \overline{S}$, there exists a vertex $u \in S$ and a (downhill) path $u = v_1, v_2, \ldots v_k = v$ from u to v, such that $d(v_i) \geq d(v_{i+1})$ for all $i \in [k-1]$; SH; $\gamma_{\text{down}}(G)$.
(i) *uphill domination*: for every vertex $v \in \overline{S}$, there exists a vertex $u \in S$ and an (uphill) path $u = v_1, v_2, \ldots v_k = v$ from u to v, such that $d(v_i) \leq d(v_{i+1})$ for all $i \in [k-1]$; SH; $\gamma_{\text{up}}(G)$.
(j) *exponential domination*: for every vertex $v \in \overline{S}$, $\text{w}_s(v) \geq 1$, where

$$\text{w}_s(v) = \sum_{u \in S} \frac{1}{2^{\overline{d}(u,v)-1}},$$

and $\overline{d}(u, v)$ equals the length of a shortest (u, v)-path in $V \setminus (S \setminus \{u\})$ if such a path exists, and ∞ otherwise; SH; $\gamma_{\exp}(G)$.
(k) *fair domination*: $N[S] = V$ and every two vertices $u, v \in \overline{S}$ have the same number of neighbors in S; $\text{fdom}(G)$.
(l) *H-forming domination*: every vertex $v \in \overline{S}$ is contained in a copy of a graph H (not necessarily induced) with a subset of vertices in S; SH; $\gamma_H(G)$.
(m) *outer-connected domination*: $N[S] = V$ and $G[\overline{S}]$ is connected; $\gamma_{\tilde{c}}(G)$.
(n) *b-disjunctive domination*: for every $v \in \overline{S}$, either v is adjacent to a vertex $u \in S$ or there exist at least b vertices in S at distance 2 from v; SH; $\gamma_b^d(G)$.

(o) *secure domination*: $N[S] = V$ and for every vertex $u \in \overline{S}$, there is an adjacent vertex $v \in S$ such that the set $(S \setminus \{v\}) \cup \{u\}$ is a dominating set; SH; $\gamma_s(G)$.

5 Conditions on *V*

In this section, we consider a framework where the dominating set is defined by an added condition that is imposed on every vertex of G.

(a) *total domination*: $N(S) = V$, that is, for every vertex $v \in V$, $N(v) \cap S \neq \emptyset$; SH; $\gamma_t(G)$, $\Gamma_t(G)$.
(b) *odd domination*: $N[S] = V$, and for every $v \in V$, $|N[v] \cap S|$ is odd; $\gamma_{\text{odd}}(G)$. It is noteworthy that Sutner [7] was the first to observe that every graph G has an odd dominating set.
(c) *even domination*: $N[S] = V$, and for every $v \in V$, $|N[v] \cap S|$ is even; $\gamma_{\text{even}}(G)$.
(d) *identifying code number*: $N[S] = V$, and for every $v \in V$, $N[v] \cap S$ is unique; SH; $\gamma_{\text{id}}(G)$.
(e) *total distance k-dominating*: for every vertex $v \in V$, there exists a vertex $u \in S$, $u \neq v$, such that $d(u, v) \leq k$; SH; $\gamma_t^k(G)$.
(f) *k-tuple domination*: for every $v \in V$, $|N[v] \cap S| \geq k$; SH; $\gamma_{\times k}(G)$.

6 Conditions on Vertex Degrees

As we will see in this section, many types of dominating sets can be defined in terms of how many neighbors a vertex must have in either S or $\overline{S}$. These constraints are often perceived as requirements of access to the resources provided by members of a dominating set.

6.1 Degree Conditions on S and $\overline{S}$

Degree conditions as a framework of domination was first suggested by Telle [8]. We present a slightly different form of his framework here. There are four possible values under consideration, namely, $d_S(v)$ and $d_{\overline{S}}(v)$ for $v \in S$, and $d_S(v)$ and $d_{\overline{S}}(v)$ for $v \in \overline{S}$. Table 1 illustrates how with using combinations of these four values, different domination parameters are defined. We only include a few of the many parameters which can be defined by various combinations of the four degree values. A blank entry in Table 1 implies that this condition is not relevant to the definition. Let D-set, TD-set, ID-set, and RD-set denote dominating set, total dominating set, independent dominating set and restrained dominating set, respectively.

Table 1 Degree Conditions

S is	$v \in S, d_S(v)$	$v \in S, d_{\overline{S}}(v)$	$v \in \overline{S}, d_S(v)$	$v \in \overline{S}, d_{\overline{S}}(v)$
a D-set			≥ 1	
an ID-set	$= 0$		≥ 1	
a TD-set	≥ 1		≥ 1	
a perfect dominating set			$= 1$	
an RD-set			≥ 1	≥ 1
a k-dominating set			$\geq k$	
a D-set and $\overline{S}$ is a D-set		≥ 1	≥ 1	
a $[1, k]$-dominating set			≥ 1 and $\leq k$	
an odd D-set	even		odd	
an open odd D-set	odd		odd	
an efficient D-set	=0		$= 1$	
a 1-dependent D-set	≤ 1		≥ 1	

6.2 *Degree Conditions Per Vertex*

As in the previous section, the framework here is defined in terms of the minimum cardinality of a nonempty set S satisfying the stated conditions based on degree. The difference is that the constraints now depend on comparative comparative values of degrees. Recall that the boundary of a set S is $\partial(S) = N[S] \setminus S$.

(a) *alpha domination*: for every $v \in \overline{S}$, $d_S(v)/d(v) \geq \alpha$ where $0 < \alpha \leq 1$; SH; $\gamma_\alpha(G)$.
(b) *defensive alliance*: for every $v \in S$, $d_S[v] \geq d_{\overline{S}}(v)$; $a(G)$.
(c) *defensive k-alliance*: for every $v \in S$, $d_S(v) \geq d_{\overline{S}}(v) + k$; $a_k(G)$. Note that for $k = -1$, a defensive k-alliance is the standard defensive alliance, that is, $a_{-1}(G) = a(G)$.
(d) *global defensive alliance*: $N[S] = V$ and for every $v \in S$, $d_S[v] \geq d_{\overline{S}}(v)$; $\gamma_a(G)$.
(e) *offensive alliance*: for every $v \in \partial(S)$, $d_S(v) \geq d_{\overline{S}}[v]$; $a_o(G)$.
(f) *offensive k-alliance*: for every $v \in \partial(S)$, $d_S(v) \geq d_{\overline{S}}(v) + k$; $a_{ok}(G)$. Note that for $k = 1$, a k-offensive alliance is the normal offensive alliance.
(g) *global offensive alliance*: for every $v \in \overline{S}$, $d_S(v) \geq d_{\overline{S}}[v]$; $\gamma_{a_o}(G)$.
(h) *powerful alliance*: for every $u \in S$, $d_S[u] \geq d_{\overline{S}}(u)$ and for every $v \in \partial(S)$, $d_S(v) \geq d_{\overline{S}}[v]$; $a_p(G)$.
(i) *(static) monopoly*: for every vertex $v \in \overline{S}$, $d_S(v) \geq d_{\overline{S}}(v)$, that is, every vertex not in S has at least $\lceil d(v)/2 \rceil$ neighbors in S, or equivalently, every vertex in $\overline{S}$ has at least as many neighbors in S as it has in $\overline{S}$; SH; $m(G)$.
(j) *open, or total, monopoly*: for every vertex $v \in V$, $d_S(v) \geq d_{\overline{S}}(v)$, that is, every vertex in V has at least as many neighbors in S as it has in $\overline{S}$; SH; $m_t(G)$.
(k) *weak domination*: for every $v \in \overline{S}$, there exists a neighbor $u \in S$, $d(u) \leq d(v)$; SH; $\gamma_w(G)$.

(l) *strong domination*: for every $v \in \overline{S}$, there exists a neighbor $u \in S$, $d(u) \geq d(v)$; SH; $\gamma_s(G)$.
(m) *cost effective domination*: $N[S] = V$ and for every $v \in S$, $d_S(v) \leq d_{\overline{S}}(v)$; $\gamma_{\mathrm{ce}}(G)$.
(n) *very cost effective domination*: $N[S] = V$ and for every $v \in S$, $d_S(v) < d_{\overline{S}}(v)$; $\gamma_{\mathrm{vce}}(G)$.
(o) 1-*equitable domination*: $N[S] = V$ and for all $u, v \in S$, $|d_{\overline{S}}(u) - d_{\overline{S}}(v)| \leq 1$; $\gamma_{1eq}(G)$.
(p) 2-*equitable domination*: $N[S] = V$ and for all $u, v \in \overline{S}$, $|d_S(u) - d_S(v)| \leq 1$; $\gamma_{2eq}(G)$
(q) *equitable domination*: $N[S] = V$ and for all $u, v \in S$, $|d_{\overline{S}}(u) - d_{\overline{S}}(v)| \leq 1$, and for all $u, v \in \overline{S}$, $|d_S(u) - d_S(v)| \leq 1$; $\gamma_{eq}(G)$.
(r) *global distribution center*: $N[S] = V$ and for all $v \in \overline{S}$, there exists a vertex $u \in S$ such that $d_S[u] \geq d_{\overline{S}}[v]$; SH; $\mathrm{gdc}(G)$.

7 Functions $f: V \to \mathbb{N}$

For every set $S \subseteq V$, there is a corresponding *characteristic function* $f_S: V \to \{0, 1\}$, such that $f(v) = 1$ if $v \in S$, and $f(v) = 0$ if $v \in \overline{S}$. This suggests a variety of options for the range $\mathbb{N}$ of a function $f: V \to \mathbb{N}$, in terms of domination. In this section, we present a sample of the functions that have been considered under this framework. The value of each of the following parameters equals the minimum weight of a function of the given type, where the *weight* $\mathrm{w}(f)$ of such a function f is the sum of all assigned values,

$$\mathrm{w}(f) = \sum_{v \in V} f(v).$$

7.1 Dominating Functions

(a) *domination*: $f : V \to \{0, 1\}$, for every vertex $v \in V$, $f(N[v]) \geq 1$; $\gamma(G)$.
(b) *fractional domination*: $f : V \to [0, 1]$, for every vertex $v \in V$, $f(N[v]) \geq 1$; $\gamma_f(G)$.
(c) *signed domination*: $f : V \to \{-1, 1\}$, for every vertex $v \in V$, $f(N[v]) \geq 1$; $\gamma_s(G)$.
(d) *minus domination*: $f : V \to \{-1, 0, 1\}$, for every vertex $v \in V$, $f(N[v]) \geq 1$; $\gamma_m(G)$.
(e) $\{k\}$-*domination*: $f : V \to \{0, 1, \ldots, k\}$, for every vertex $v \in V$, $f(N[v]) \geq k$; $\gamma_{\{k\}}(G)$.

(f) *k-rainbow domination*: $f : V \to \mathcal{P}\{1, 2, \ldots, k\}$, every vertex $v \in V$ is assigned a subset of $\{1, 2, \ldots, k\}$ such that for every vertex $v \in V$ with $f(v) = \emptyset$, the union of the sets assigned to the closed neighborhood $N[v]$ equals $\{1, 2, \ldots, k\}$; $\gamma_{rk}(G)$.

7.2 Roman Dominating Functions

The types of domination in this section are models of a military defense strategy instituted by Emperor Constantine, between 306 and 337 AD, in which the regions in the Roman Empire were defended by armies stationed at key locations. A region was secured by armies stationed there, and a region without an army was protected by sending mobile armies from neighboring regions. But Constantine decreed that a mobile field army could not be sent to defend a region, if doing so left its original region unsecured. This defense strategy gave rise to what is called *Roman domination*, given below. As in the previous section, the value of each of the following domination parameters equals the minimum weight of a function of the given type.

Definition 1 Roman domination*: $f : V \to \{0, 1, 2\}$, for every vertex v with $f(v) = 0$, there is a vertex $u \in N(v)$ with $f(u) = 2$; $\gamma_R(G)$.*

It is easy to see, for example, that for every graph G, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$. From the initial definition of Roman domination as a framework, many varieties of domination can clearly be defined, and indeed, many have been defined. We only provide a sample here.

(a) *weak Roman domination*: $f : V \to \{0, 1, 2\}$, for every v with $f(v) = 0$, there is a vertex $u \in N(v)$ with $f(u) > 0$ such that the function f' with $f'(v) = 1$, $f'(u) = f(u) - 1$, and $f'(w) = f(w)$ otherwise, has no *undefended vertex*, meaning a vertex with $f'(N[w]) = 0$; $\gamma_r(G)$.
(b) *double Roman domination*: $f : V \to \{0, 1, 2, 3\}$, every vertex w with $f(w) = 0$ either has a neighbor u with $f(u) = 3$ or two neighbors u, v with $f(u) = f(v) = 2$, and if $f(w) = 1$, then w has at least one neighbor u with $2 \leq f(u) \leq 3$; $\gamma_{dR}(G)$.
(c) *Roman \{2\}-domination, also called Italian domination*: $f : V \to \{0, 1, 2\}$, every vertex v with $f(v) = 0$ has $f(N(v)) \geq 2$; $\gamma_{R2}(G)$ (also $\gamma_I(G)$).
(d) *Roman k-domination*: $f : V \to \{0, 1, 2\}$, every vertex v with $f(v) = 0$ is adjacent to at least k vertices u with $f(u) = 2$; $\gamma_{kR}(G)$.
(e) *independent Roman domination*: $f : V \to \{0, 1, 2\}$, every vertex v with $f(v) = 0$ has at least one neighbor u with $f(u) = 2$ and the set of vertices w with $f(w) > 0$ is an independent set; $i_R(G)$.
(f) *signed Roman domination*: $f : V \to \{-1, 1, 2\}$, for every vertex $v \in V$, $f(N[v]) \geq 1$, and every vertex v with $f(v) = -1$ has at least one neighbor u with $f(u) = 2$; $\gamma_{sR}(G)$.

(g) *total Roman domination*: $f : V \to \{0, 1, 2\}$, every vertex w with $f(w) = 0$ has at least one neighbor u with $f(u) = 2$ and every vertex u with $f(u) > 0$ has at least one neighbor v with $f(v) > 0$; $\gamma_{tR}(G)$.

8 Stratified Domination

A graph G together with a fixed partition of its vertex set V into nonempty subsets is called a *stratified graph*. If the partition is $V = \{V_1, V_2\}$, then G is a 2-stratified graph and the sets V_1 and V_2 are called the *strata* or sometimes the *color classes* of G. A framework for domination based on coloring the vertices of a graph was defined in [2] as follows. Let F be a 2-stratified graph with one fixed blue vertex v specified; F is said to be *rooted* at the blue vertex v. An *F-coloring* of a graph G is defined to be a red-blue coloring of the vertices of G such that every blue vertex v is a root of a copy of F (not necessarily induced) in G. The *F-domination number* $\gamma_F(G)$ of G is the minimum number of red vertices in an F-coloring of G.

We note that if F is a 2-stratified K_2 rooted at a blue vertex that is adjacent to a red vertex, then the set of red vertices in an F-coloring of G is a dominating set of G and $\gamma_F(G) = \gamma(G)$.

This extends to other 2-stratified graphs F and encapsulates many types of domination related parameters, including the domination, total domination, restrained, total restrained, and k-domination numbers. For example, let F be a 2-stratified P_3 rooted at a blue vertex v. The five possible choices for the graph F are shown in Figure 1.

Let G be a connected graph of order at least 3. It is shown in [2] that if $F = F_1$, then the set of red vertices of an F-coloring of G is a total dominating set and $\gamma_F(G) = \gamma_t(G)$, while if $F = F_2$, then the set of red vertices is a dominating set of G and $\gamma_F(G) = \gamma(G)$. Furthermore, if $F = F_4$, then the set of red vertices of an F-coloring of G is a restrained dominating set and $\gamma_F(G) = \gamma_r(G)$, and if $F = F_5$, then the set of red vertices of an F-coloring of G is a 2-dominating set and $\gamma_F(G) = \gamma_2(G)$.

On the other hand, the parameter $\gamma_{F_3}(G)$ defined a new domination parameter that had not been studied prior to considering domination from this framework.

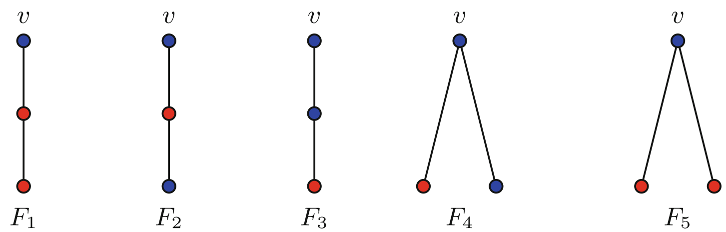

Fig. 1 The five 2-stratified graphs P_3

Stratified domination encompasses many known domination parameters and suggests new avenues for study.

9 Domination Chain

The domination chain expresses relationships that exist among dominating sets, independent sets, and irredundant sets in graphs. Irredundance is the concept that describes the minimality of a dominating set. If a dominating set S is *minimal*, then for every vertex $u \in S$ the set $S \setminus \{u\}$ is no longer a dominating set. This means that the vertex u dominates some vertex, which could be itself, that no other vertex in S dominates. Given a vertex set $S \subseteq V$ and a vertex $v \in S$, we make the following definitions.

(a) The vertex v is a *self-private neighbor* if v has no neighbors in S, that is, $N[v] \cap S = \{v\}$.
(b) The vertex v has an S-*external private neighbor* if there exists a vertex $w \in \overline{S}$ such that $N(w) \cap S = \{v\}$.
(c) The vertex v has an S-*internal private neighbor* if there exists a vertex $w \in S$ such that $N(w) \cap S = \{v\}$.

A nonempty set S is *irredundant* if and only if every vertex $v \in S$ either is a self-private neighbor or has an S-external private neighbor. The *irredundance numbers*, $ir(G)$ and $IR(G)$, are the minimum and maximum cardinalities, respectively, of a maximal irredundant set.

The following two properties of a minimal and maximal dominating set yield the domination chain:

Observation 1 *The following hold in a graph G.*

(a) *Every minimal dominating set in G is a maximal irredundant set of G.*
(b) *Every maximal independent set in G is a minimal dominating set of G.*

Theorem 2 (The Domination Chain) *For every graph G,*

$$ir(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G) \leq IR(G).$$

Since its introduction by Cockayne, Hedetniemi, and Miller [3] in 1978, the domination chain of Theorem 2 has become one of the major focal points in the study of domination in graphs, inspiring several hundred papers. As a framework, it is possible to obtain inequality chains similar to the domination chain starting from a suitable seed property. Thus, almost any property of subsets could be considered, for example, the seed property that S is a vertex cover.

10 Conditions Relating to Perfection

The concept of being dominated exactly once by the vertices in a set S is generally referred to as perfect or efficient domination. In this section, we list a few parameters related to this model of domination.

Given a set $S \subseteq V$, a vertex $v \in V$ is *perfect* (with respect to S) if $|N[v] \cap S| = 1$, and is *almost perfect* if it is either perfect or is adjacent to a perfect vertex. A vertex $v \in V$ is *open perfect* (with respect to S) if $|N(v) \cap S| = 1$, and is *almost open perfect* if it is either open perfect or adjacent to an open perfect vertex. A set S is a *perfect neighborhood set* if every vertex $v \in V$ is either perfect or almost perfect, with respect to S. A set S is called *internally perfect* if every vertex $v \in S$ is perfect, with respect to S, that is, if S is an independent set. And S is called *externally perfect* if every vertex $w \in \overline{S}$ is perfect, that is, every vertex w is adjacent to exactly one vertex in S. Externally perfect sets are also called *perfect dominating sets*. A set that is both internally and externally perfect is an *efficient dominating set*. A set S is called *nearly perfect* if for every vertex $v \in \overline{S}$, $|N(v) \cap S| \leq 1$, that is, every vertex in $\overline{S}$ is dominated at most once by the vertices in S, or every vertex in $\overline{S}$ has at most one neighbor in S.

(a) *perfect domination*: for every vertex $v \in \overline{S}$, $d_S(v) = 1$; $\gamma_p(G)$.
(b) *efficient domination*: for every $v \in V$, $d_S[v] = 1$; $\gamma(G)$. Note, it can be shown that all efficient dominating sets have the same cardinality, namely, $\gamma(G)$. Efficient dominating sets are also called *perfect codes*. This means that if $S = \{v_1, v_2, \ldots, v_k\}$ is an efficient dominating set, then $\pi = \{N[v_1], N[v_2], \ldots, N[v_k]\}$ is a partition of V.
(c) *efficiency*: $\varepsilon(S) = |\{v \in \overline{S} : d_S(v) = 1\}|$; $\varepsilon(G) = max_{S \subseteq V}\{\varepsilon(S)\}$.
(d) *total efficiency*: $\varepsilon_t(S) = |\{v \in V : d_S(v) = 1\}|$; $\varepsilon_t(G) = max_{S \subseteq V}\{\varepsilon_t(S)\}$.

11 Criticality Parameters

For any parameter, such as $\alpha(G)$ or $\gamma(G)$, it is natural to consider how the value changes when a small change is made in the graph G, for example, by the deletion of a vertex or edge, the addition of an edge, the subdivision of an edge, the identification of two non-adjacent vertices (an *elementary homomorphism*), or the identification of two adjacent vertices (an *elementary contraction*). Most of the study along these lines involves families of graphs whose domination number changes whenever the given modification is made arbitrarily in the graph. For example, *domination edge critical* graphs G have the property that the domination number decreases whenever any arbitrary edge is added, that is, $\gamma(G + e) < \gamma(G)$ for any $e \in E(\overline{G})$.

On the other hand, parameters that in some sense measure the degree of criticality have also been studied. Here we describe selected *criticality* parameters of this type that have been studied for domination. We note that this perspective deviates from our other frameworks in that it does not encompass dominating sets, but instead considers effects of a graph modification on the domination number.

(a) *reinforcement number*, $r(G)$: minimum number of edges that must be added to G in order to decrease the domination number.
(b) *bondage number*, $b(G)$: minimum number of edges that must be deleted from G in order to increase the domination number.
(c) *domination sensitivity*, $\gamma_{\pm}(G)$: minimum number of vertices that must be deleted to either increase or decrease the domination number.
(d) *domination subdivision number*, $sd_{\gamma}(G)$: minimum number of edges that must be subdivided in order to increase the domination number.
(e) *total domination subdivision number*, $sd_{\gamma_t}(G)$: minimum number of edges that must be subdivided in order to increase the total domination number.
(f) *paired domination subdivision number*, $sd_{\mathrm{pr}}(G)$: minimum number of edges that must be subdivided in order to increase the paired domination number.
(g) *forcing domination number*, $F_{\gamma}(G)$. A subset T of a minimum dominating set S is a *forcing subset* for S if S is the unique minimum dominating set containing T. The *forcing domination number* $F_{\gamma}(S)$ of a minimum dominating set S is the minimum cardinality among the forcing subsets of S, and the *forcing domination number* $F_{\gamma}(G)$ of G is the minimum forcing domination number among the minimum dominating sets S of G. It follows from the definition that $F_{\gamma}(G) \leq \gamma(G)$.

12 Partitions

For any property $\mathcal{P}$ of interest, it is natural to consider partitions of the vertex set $V = \{V_1, V_2, \ldots, V_k\}$ such that every set V_i, where $i \in [k]$, is a $\mathcal{P}$-set; these are generally referred to as $\mathcal{P}$-colorings. The most often studied partitions of this type are called *proper colorings*, in which each set V_i is an independent set.

In this section, we describe a variety of $\mathcal{P}$-colorings which have been studied, in which the property $\mathcal{P}$ is related to domination. As in the previous section, this perspective deviates from our other frameworks in that it does not encompass dominating sets, but instead considers parameters based on graph partitions involving dominating sets.

(a) *domatic number*, $d(G)$: maximum order of a vertex partition into dominating sets.
(b) *idomatic number*, $id(G)$: maximum order of a vertex partition into independent dominating sets, or the maximum number of vertex disjoint independent dominating sets.
(c) *capacitated domination*, $\gamma_{cap_k}(G)$: minimum order of a vertex partition into sets V_i such that $G[V_i]$ has a spanning star of order at most $k + 1$.

(d) *iterated independence numbers*, $i^*(G)$ and $\alpha^*(G)$: minimum and maximum orders of partitions resulting from repeated removals of maximal independent sets.
(e) *iterated domination numbers*, $\gamma^*(G)$, $\Gamma^*(G)$: minimum and maximum orders of partitions resulting from repeated removals of minimal dominating sets.
(f) *iterated irredundance numbers*, $ir^*(G)$, $IR^*(G)$: minimum and maximum orders of partitions resulting from repeated removals of maximal irredundant sets.
(g) *dominator coloring number*, $\chi_d(G)$: minimum order of a vertex partition, such that every vertex $v \in V$ dominates at least one set V_i.
(h) *gamma-gamma domination*, $\gamma\gamma(G)$, $\Gamma\Gamma(G)$: minimum and maximum of $|S_1| + |S_2|$ for two disjoint (minimal) dominating sets in G.
(i) *gamma-i domination*, $\gamma i(G)$: minimum of $|S_1| + |S_2|$ for two disjoint dominating sets in G, one of which is an independent dominating set.
(j) *defensive alliance partition number*, $\Psi_a(G)$: maximum order of a vertex partition into defensive alliances.

13 Summary

In the preceding sections we have seen a wide variety of contexts in which aspects of dominating sets in graphs can be expressed and studied.

If a condition can be imposed on the vertices only in S or only in $\overline{S}$, it can also be imposed to hold on all vertices in V. In this way we move from domination to total domination. If a condition can be imposed on the closed neighborhoods $N[v]$ of vertices, it can then be relaxed to hold only for open neighborhoods $N(v)$. All parameters involving sets $S \subseteq V$ can also be studied from the point of view of subsets of edges $F \subseteq E$.

One can consider the minimum cardinality of a set S having some property, and also consider the maximum cardinality of a minimal set having the same property. One can consider the maximum cardinality of a set S having some property, and also consider the minimum cardinality of a maximal set having the same property. Consider any hereditary property $\mathcal{P}$ of a set of vertices S, such as being an independent set. You can ask: what condition must exist for a set S to be a maximal $\mathcal{P}$-set? This condition is a property $\mathcal{P}'$ in its own right, and every maximal set having property $\mathcal{P}$ must then also have property $\mathcal{P}'$. In the same way you can consider any superhereditary property $\mathcal{Q}$ of a set of vertices, such as being a dominating set. You can then ask: what condition must exist for a set S to be a minimal $\mathcal{Q}$-set? This will then give rise to another property $\mathcal{Q}'$, which can be studied in its own right.

Among all subsets S having some property $\mathcal{P}$, one can impose an additional condition, often that the set S also be independent, but that the induced subgraph $G[S]$ have some common graph property, like having no isolated vertices, or being a connected subgraph. You can, of course impose an added condition on the set $\overline{S}$. In this way, for example, we get restrained domination and outer-connected domination. We have seen many examples where a condition is imposed on either $N_S(u)$ or $N_S[u]$, and likewise on $N_{\overline{S}}(v)$ or $N_{\overline{S}}[v]$, for vertices in either S or $\overline{S}$.

For every set $S \subseteq V$, there is a corresponding characteristic function $f_S : V \to \{0, 1\}$, such that $f(v) = 1$ if $v \in S$, and $f(v) = 0$ if $v \in \overline{S}$. This suggests a variety of options for the range of a function f, such as the closed unit interval, $f : V \to [0, 1]$, from which we get fractional domination, or $f : V \to \{0, 1, 2\}$, from which we get Roman domination, or $f : V \to \{-1, 1\}$ from which we get signed domination.

It is natural to consider partitions $\pi = \{V_1, V_2, \ldots, V_k\}$ such that every set V_i where $i \in [k]$ has some property $\mathcal{P}$, the most studied is that every set V_i is an independent set. Such partitions are sometimes called *$\mathcal{P}$-colorings* of graphs, and one seeks either the minimum order of such a partition or the maximum order, usually depending on whether the property $\mathcal{P}$ is hereditary (minimum order, e.g. chromatic number) or superhereditary (maximum order, e.g. domatic number).

Real-world applications of dominating sets often suggest new and interesting models of domination. This was the case with Roman domination, in which a vertex v with $f(v) = 2$ represents a location at which two armies are stationed, one of which can be used to defend a neighboring location by traveling along a single edge. This one application alone has suggested numerous other models for defending the vertices of a graph with different types of dominating sets.

In computer networks, a dominating set is viewed as a set of vertices, or nodes, each of which supplies, “in one hop” a needed resource to all neighboring vertices. But if one of these vertices becomes inoperative, or faulty, it might be helpful to have some sort of backup arrangement. One such arrangement could be to have a neighbor of the faulty node, also in the dominating set, so that a service could be provided in at most two hops while the fault can be fixed; this corresponds to a total dominating set, and is closely related to the model of a $(1, k)$-dominating set, in which every node either has one-hop service or secondary service at most k-hops away. Another arrangement might be to have a neighboring node to the faulty node serve temporarily as a backup in such a way that the resulting set of nodes is another dominating set. This leads to the model of secure dominating sets.

What models of domination have not we discussed? At the outset of this chapter, we said that space limitations would not permit us to be comprehensive in reviewing the many different models of domination that are being considered in the current literature. Some compensation for the limitations of this chapter, however, are provided by chapters in this volume and other books on domination. We list some of them here along with selected sources of information. Of course, there are many other application driven frameworks of domination, ranging from social networks to mathematical chemistry, that are beyond the scope of these sources.

(a) *Domination in hypergraphs*

Chapter 11 Domination in Hypergraphs, by M. A. Henning and A. Yeo, in *Structures of Domination in Graphs*, Springer, 2020.

(b) *Domination in linear and integer programming*

Chapter 1 LP-Duality, Complementarity, and Generality of Graphical Subset Parameters, by P. J. Slater, in *Domination in Graphs, Advanced Topics*, Marcel Dekker, 1998.

(c) *Domination in directed graphs and tournaments*

Chapter 15 Topics on Domination in Directed Graphs, by J. Ghoshal, R. C. Laskar, and D. Pillone, in *Domination in Graphs, Advanced Topics*, Marcel Dekker, 1998.

Chapter 13 Domination in Digraphs and Tournaments, by T. W. Haynes, S. T. Hedetniemi, and M. A. Henning, in *Structures of Domination in Graphs*, Springer, 2020.

(d) *Domination in chessboards*

Chapter 6 Combinatorial Problems on Chessboards: II, by S. M. Hedetniemi, S. T. Hedetniemi, and R. Reynolds, in *Domination in Graphs, Advanced Topics*, Marcel Dekker, 1998.

J.J. Watkins, *Across the Board: The Mathematics of Chessboard Problems*, Princeton University Press, 2004.

Chapter 12 Domination in Chessboards, by J. T. Hedetniemi and S. T. Hedetniemi, in *Structures of Domination in Graphs*, Springer, 2020.

(e) *Algorithms and complexity of domination in graphs*

Chapter 12 Domination Complexity and Algorithms, in *Fundamentals of Domination in Graphs*, Marcel Dekker, 1998.

Chapter 8 Algorithms, by D. Kratsch, in *Domination in Graphs, Advanced Topics*, Marcel Dekker, 1998.

Chapter 9 Complexity Results, by S. T. Hedetniemi, A. A. McRae, and D. A. Parks, in *Domination in Graphs, Advanced Topics*, Marcel Dekker, 1998.

Chapter 14 Algorithms and Complexity - Signed and Minus Domination, by S. T. Hedetniemi, A. A. McRae, and R. Mohan, in *Structures of Domination in Graphs*, Springer, 2020.

Chapter 15 Algorithms and Complexity - Power Domination, by S. T. Hedetniemi, A. A. McRae, and R. Mohan, in *Structures of Domination in Graphs*, Springer, 2020.

Chapter 16 Self-Stabilizing Domination Algorithms, by S. T. Hedetniemi, in *Structures of Domination in Graphs*, Springer, 2020.

(f) *Domination games on graphs*

Chapter 8 An Introduction to Game Domination in Graphs, by M. A. Henning, in *Structures of Domination in Graphs*, Springer, 2020.

(g) *Domination and eigenvalues in graph theory*

Chapter 9 Domination and Spectral Graph Theory, by C. Hoppen, D. Jacobs, and V. Trevisan, in *Structures of Domination in Graphs*, Springer, 2020.

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