Quantum Router for Qutrit Networks

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Abstract. Networks of quantum circuits or, more generally, networks transmitting quantum information will need, just like classical networks (e.g. internet), a mechanism for directing data to adequate nodes. Routing, understood as packet switching, is one of the most important processes in classical networks. The issue of routing is also present in quantum networks and an appropriate construction of a quantum router is required to transfer data to specific points in the network. We describe an implementation of a router for qutrits in this chapter. The router is four-qutrit quantum circuit (with one controlling unit). The efficiency and the accuracy of router's work is tested by the Fidelity measure. The circuit's dynamics is expressed by a Hamiltonian where the role of generalized Pauli operators is played by the Gell-Mann operators.

Keywords: Quantum networks · Quantum router · Qutrits

1 Introduction

Transferring information is not the only role of networks. They may be seen as something more, e.g. tensor networks [11], neural networks [19] or quantum circuits [22]. Undoubtedly, processing and transferring of information in quantum networks is a problem which should be solved to efficiently realize quantum computations [30,31], and utilize quantum communication protocols. It should be emphasized that, nowadays, quantum networks [27] are intensively studied, and many tools are constructed to investigate behavior of these structures, like quantum networks – or even quantum internet – simulators [8–10].

Presently, the transfer of quantum data is based on quantum spin-chains [5, 21], and entangled qubits [20,23,29]. Different physical elements are considered as components of future quantum networks. Many elements of classical networks, like switches, repeaters, and routers, have their quantum equivalents [1–4,14,28].

It should be emphasized that mentioned components of networks, especially routers, are not only discussed as theoretical devices. We can find their experimental physical implementations, for example with the use of coupled harmonic system [25], quantum tunneling effect [18], or superconducting circuits [7,26].

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In this work, we would like to show that a quantum router may be implemented for higher units of quantum information, i.e. qutrits. We describe the basis definition of a router, and its dynamics as a Hamiltonian where, because of qutrits, Pauli operators are substituted by the Gell-Mann matrices. The given router definition is consistent with the form of the qubit router, presented in professional literature, so it may be treated as a step in a direction of generalization to the qudit router. We present the values of Fidelity measure which proof that the proposed Hamiltonian correctly realizes tasks of the qutrit router.

The paper is organized as follows. In Sect. 2, we describe the basic information concerning qudits, and qutrits in particular. We adduce also the Gell-Mann operators which are required in the Hamiltonian construction (the Hamiltonian describes the dynamics of router's operating).

Section 3 contains the idea and definition of the qutrit router. The router is presented as a quantum circuit, and, what is more important, as the Hamiltonian. Experiments inspecting the router's operating are shown in Sect. 4. There, the values of Fidelity measure for the routing process are presented. The summary is positioned in Sect. 5. Acknowledgments and references section end the article.

2 Preliminaries

Both, in classical and quantum computing, the definition of a unit of information is required. The construction of presently used computers impose a bit as the basic unit. Naturally, the first algorithms for quantum computers were also proposed for quantum bits, so-called qubits. However, the technical development enables the utilization of quantum information units with a freedom level greater than two (higher freedom level causes higher informational content what allows obtaining the result of computation with fewer operations).

Let us define a qudit as a general unit of quantum information with the freedom level $d \geq 2$. A quantum state of a single qudit may be expressed as a normalized d-entity column vector. We denote this kind of vectors, in the Dirac notation, as $|\cdot\rangle$, e.g.

$$|\psi\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \dots \\ \alpha_{d-1} \end{pmatrix}, \tag{1}$$

where the normalization condition requires $\sum_{i=0}^{d-1} |\alpha_i|^2 = 1$, and α_i are the complex numbers. In next sections of the text, we denote a quantum state also as ψ what still means the correct state in the Dirac notation.

If the quantum state is created by more than one qudit, its states' vector is calculated as a tensor product of all one-qudit states vectors. For example, let us have two qudits: $|\psi\rangle, |\phi\rangle$, with different freedom levels: a and b, respectively. The state of these qudits, joined in one quantum register, is:

$$|\Psi\rangle = |\psi\rangle \otimes |\phi\rangle, \tag{2}$$

where the dimensionality of a vector $|\Psi\rangle$ is equal to $a \cdot b$ (the dot symbol represents the scalar product of two numbers). Of course, the joined qudits may have the same freedom levels. The symbol of tensor product is usually omitted, so the above state $|\Psi\rangle$ may be written as $|\psi\phi\rangle$.

Quantum states may be also described by the superposition equation. In this case, we need to define a concept of a computational basis. Just like in positional number system theory, we need to clearly point out a representative for each accepted value $i = 1 \dots d$. In quantum computing these values are substituted by vectors. The computational basis for a d-level single qudit contains d orthonormal vectors (the orthogonality ensures a possibility to distinguish the elements, and the normality guaranties obtaining a correct quantum state). The most popular computational basis is so-called standard basis. Vectors in this basis have one element equal to 1, and other $(d-1)$ elements equal 0. Of course, in each vector the non-zero element occupies different position:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ \cdots \\ 0 \end{pmatrix}, \ |1\rangle = \begin{pmatrix} 0 \\ 1 \\ \cdots \\ 0 \end{pmatrix}, \ \dots, \ |d-1\rangle = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 1 \end{pmatrix}. \tag{3}$$

The superposition is one of the characteristic features of quantum states. The superposition equation shows that a quantum state may be a mixture of basis states with the proportions described by probability amplitudes α_i:

$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle + \ldots + \alpha_{d-1}|d-1\rangle, \tag{4}$$

where $\sum_{i=0}^{d-1} |\alpha_i|^2 = 1$, and α_i are the complex numbers.

To realize the computation on quantum states, we need operators. These operators may be expressed as unitary matrices sized $d \times d$, if they act on single qudit with the freedom level d. If the state contains n qudits (all with the same freedom level), the size of an operator's matrix representation is $d^n \times d^n$, because matrices affecting sequent qudits are tensor multiplied just like in Eq. (2). Of course, if we do not want to change the state of one (or more) particular qudit in the register, we can use the identity matrix $I_{d \times d}$ in the tensor multiplication.

In this work, we describe a router acting on qudits with the freedom level $d = 3$ – called qutrits. Now, we would like to present basic quantum gates, but with the restriction to qutrit gates.

The fundamental rotations which may be realized on one qutrit are given by the Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{5}$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The above operators may be utilized to construct unitary counterparts of Pauli gates: X, Y, Z. The set of generalized operators contains more elements, e.g. there are three equivalents of the X gate for qutrits:

$$X_1 = e^{\frac{i}{3}\lambda_1},\ X_2 = e^{\frac{i}{3}\lambda_4},\ X_3 = e^{\frac{i}{3}\lambda_6}. \tag{6}$$

The counterparts of the Y gate are built with the use of λ_2, λ_5, λ_7 operators, and λ_3, λ_8 operators serve to define equivalents of the Z gate.

Another powerful feature of quantum systems, next to the superposition, is an entanglement [12]. This phenomenon is a kind of dependency between quantum states of qudits joined in one register. Colloquially speaking, modifying the state of one qudit (with the use of quantum gate) causes a change of other qudit/qudits which take a part in the entanglement. The entanglement takes place when the state of the register cannot be expressed as a tensor product of all single qudits involved in this system.

3 Quantum Router for Qutrits

In this work, the input qutrit is denoted as $|\psi_I\rangle$, and its state may expressed as:

$$|\psi_I\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle, \tag{7}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$. Naturally, the qutrit $|\psi_I\rangle$ is a data input for the router.

The output qutrits (and their states) are described as $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle$ or just $|\psi_1\psi_2\psi_3\rangle$. This three-qutrit register is an output of the router.

There is another qutrit in the router which is a controlling unit – its symbol is $|\psi_C\rangle$, and it accepts exclusively three quantum states: $|0\rangle$, $|1\rangle$, $|2\rangle$. The controlling qutrit's state decides about the position of $|\psi_I\rangle$ in the final state of the quantum register. Generally, the state of whole router may be denoted as the register:

$$|\Psi\rangle = |\psi_I\rangle|\psi_1\psi_2\psi_3\rangle|\psi_C\rangle. \tag{8}$$

The way the router operates, for the three fundamental states of controlling qutrit, may expressed as:

$$\begin{array}{l} |\psi_I\rangle|000\rangle|0\rangle \longrightarrow |0\rangle|\psi_I00\rangle|0\rangle, \\ |\psi_I\rangle|000\rangle|1\rangle \longrightarrow |0\rangle|0\psi_I0\rangle|1\rangle, \\ |\psi_I\rangle|000\rangle|2\rangle \longrightarrow |0\rangle|00\psi_I\rangle|2\rangle. \end{array} \tag{9}$$

If the controlling qutrit is in the superposition of standard basis states: $|\psi_C\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle$, then the router's construction affects the quantum state as follows:

$$|\psi_I\rangle|000\rangle|\psi_C\rangle \longrightarrow \alpha|0\rangle|\psi_I00\rangle|0\rangle + \beta|0\rangle|0\psi_I0\rangle|1\rangle + \gamma|00\psi_I\rangle|2\rangle, \tag{10}$$

and it means the entanglement of the controlling qutrit with the output qutrits.

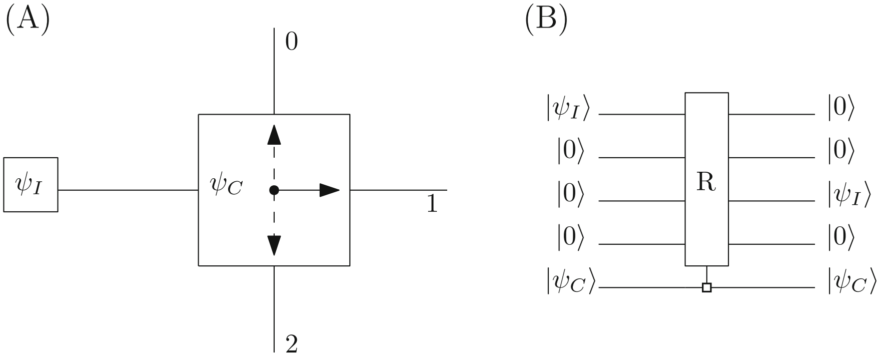

Fig. 1. The operation scheme (A) and the general form of the quantum circuit (B) for the router. Due to operating, one of the output qutrits accepts the state $|\psi_I\rangle$, and other output qutrits are equal $|0\rangle$. The circuit (B) realizes the transfer of information $|\psi_I\rangle$ to the output 1. The unitary operation R symbolizes the router which is controlled by the state of the fifth qutrit $|\psi_C\rangle$ (illustrated as the small empty square)

The description of the router, according to Eq. 9, is depict in Fig. 1. The information written in the state $|\psi_I\rangle$ is routed in a direction defined in $|\psi_C\rangle$ (it appears in one of the outputs: 0, 1 or 2).

It is also interesting to analyze a system built of a few routers. Such a bus of routers allows sending an information to particular nodes in the whole quantum network. Figure 2 depicts exemplary scheme of a five-router bus where the controlling qutrits states $|\psi_{C_0}\psi_{C_1}\psi_{C_2}\ldots\rangle$ point the router's output O_i for the information $|\psi_I\rangle$ to be transferred.

If the qutrit state ψ_I is expected to be routed to the output O_3, the controlling qutrits should be configured: $|111BB\rangle$ (letters B symbolize that qutrits ψ_{C_3} and ψ_{C_4} may accept any basis states – without any influence on the output O_3). The qutrits ψ_{C_3} and ψ_{C_4} are significant for the output O_{10}. To send ψ_I to O_{10}, the controlling qutrits should be $|12B02\rangle$ – as we can see now, the state of $|\psi_{C_2}\rangle$ may be one of three standard basis states, and the ψ_I will be still transferred to O_{10}.

Naturally, the state of controlling qutrits clearly defines the output. If the input information shows up in more than one output, we deal with a phenomenon of entanglement. It is not welcome if we discuss the basic function of the router. On the other hand, we can utilize the entangled states in different outputs as a background in solving other issues in the field of quantum computing.

It should be mentioned that the router transfers information from the input to one of the outputs, and just like for qubits, it is possible to induce entangled states during this process. Naturally, we can build a network of routers, but its structure is a chain or a two-dimensional grid (Fig. 2 depicts such a grid). An analysis of connections between qutrits in multidimensional grids seems very difficult because of the entanglement's presence – there are no methods of entanglement classification, especially for so-called multibody entanglement in real physical systems. There are works concerning routers for multidimensional quantum states, but they refer only to the router's operating on the entangled state [6,13,15,16].

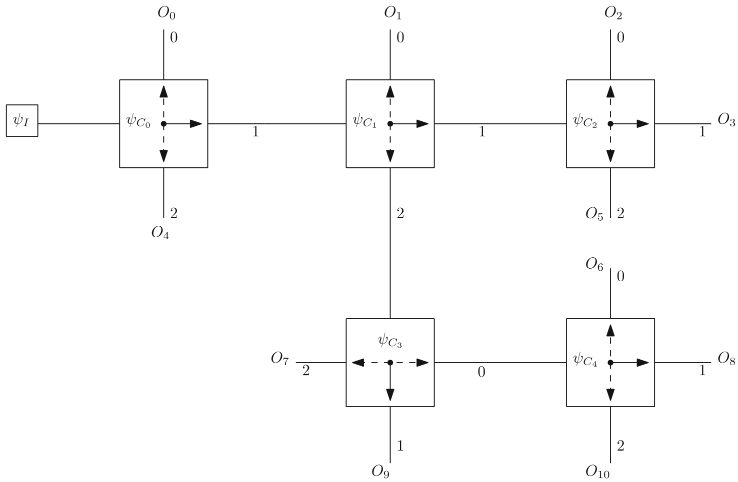

Fig. 2. The exemplary five-router bus. The input state ψ_I may be transferred to one of the outputs O_i, $i = 0, 1, \ldots$, by assigning proper values to controlling states ψ_{C_0}, ..., ψ_{C_4}. The state of first three qutrits $\psi_{C_0}\psi_{C_1}\psi_{C_2} = |111\rangle$ directs the state ψ_I to the output O_3 (the state of other units has no influence on the system)

In Fig. 1, we can see the controlled unitary operation R which describes the operating of the router. The matrix form of R may be expressed as a permutation operator:

$$U_0 = \begin{bmatrix} \cdot &&&&&&&&&&&&& \\ & \cdot &&&&&&& & 1 &&&& \\ && 1 &&&&&& 1 &&&&& \\ &&& 1 &&&& 1 &&&&&& \\ &&&& 1 &&&&&&& 1 && \\ &&&&& 0 &&&&& 1 &&& \\ &&&&&& 0 &&& 1 &&&& \\ &&& 1 &&&& \cdot &&&&&& \\ && 1 &&&&&& 0 &&&&& \\ & 1 &&&&& 1 &&& 1 &&&& \\ &&&&& 1 &&&&& 1 &&& \\ &&&& 1 &&&&&&& 1 && \\ &&&&&&&&&&&& \cdot & \\ &&&&&&&&&&&&& \cdot \end{bmatrix}, \quad (11)$$

because of the system's dimension, we present operator's abbreviated version, including omitted values 1 on the main diagonal and proportion of matrix is not preserved (the dimensions of the full matrix are 243×243).

Of course, the presented matrix is only a part of the permutation matrix, which realizes the task of the router, and it is directly defined, i.e. the digits "1" are placed in the crossings of rows and columns between which the transfer of information should occur.

However, the operator's description given in Eq. 11 does not show the inner actions between qutrits. This kind of insight offers a Hamiltonian (notation $\lambda_7^{(2)}$ means that the operator is used on the second controlling qutrit – the third router's output):

$$\begin{aligned} H = &-\frac{1}{2}(\Delta_1\lambda_3^{(1)}\lambda_8^{(1)} + \Delta_3\lambda_3^{(2)}\lambda_8^{(2)} + \Delta_3\lambda_3^{(3)}\lambda_8^{(3)}) \\ &+ J^Z(\lambda_3^{(1)}\lambda_8^{(1)} + \lambda_3^{(2)}\lambda_8^{(2)} + \lambda_3^{(3)}\lambda_8^{(3)})(\lambda_3^{(C)}\lambda_8^{(C)}) \\ &+ \frac{1}{2}J^X\left((\lambda_1^{(I)}\lambda_4^{(I)}\lambda_6^{(I)})\left(\lambda_1^{(1)}\lambda_4^{(1)}\lambda_6^{(1)} + \lambda_1^{(2)}\lambda_4^{(2)}\lambda_6^{(2)} + \lambda_1^{(3)}\lambda_4^{(3)}\lambda_6^{(3)}\right)\right. \\ &\left.+\lambda_2^{(I)}\lambda_5^{(I)}\lambda_7^{(I)})\left(\lambda_2^{(1)}\lambda_5^{(1)}\lambda_7^{(1)} + \lambda_2^{(2)}\lambda_5^{(2)}\lambda_7^{(2)} + \lambda_2^{(3)}\lambda_5^{(3)}\lambda_7^{(3)}\right)\right) \end{aligned} \tag{12}$$

The values Δ_1, Δ_2, Δ_3 symbolize frequencies of qutrit transitions between basis states. The frequency J^X denotes the coupling between input qutrit and output qutrits. While, J^Z is the frequency of coupling between output qutrits and controlling unit. Theoretically, these parameters may be selected independently one to another. However, if we want to send the input qutrit state to one of the outputs, the parameters have to meet:

$$\Delta_1 = -\Delta_2 = \frac{\Delta_3}{2} = 4J^Z. \tag{13}$$

Furthermore, we assume that $J^Z > J^X$, and J^Z have to be significantly greater than J^X.

Remark 1. The symbols Δ_i, J^X, J^Z keep their meaning just like for qubits [7]. However, the Pauli operators have to be replaced by the Gell-Mann operators. The given schema may be generalized for qudits, and then $SU(d)$ unitary group operators have to be used [17].

The unitary operator U, describing the router's operating, may be defined with the direct use of H:

$$U(t) = e^{-i\frac{\pi}{2}tH} \tag{14}$$

where $t \in \mathbb{R}$ is the time variable.

4 Numerical Experiments

One of the most important parameters of the router's operating is the accuracy. Of course, presenting quantum operation as the U_0 leads to the perfect results – during a simulation with the use of such a permutation operator, we obtain an output vector as a product of multiplication matrix by vector, and calculated value of Fidelity measure equals one. However, this procedure is purely theoretic. More realistic system's behaviour may be obtained by utilizing a Hamiltonian. Let U be a Hamiltonian-based operator, a value of the Fidelity measure (denoted by the capital letter F) in a moment t is calculated as:

$$F(t) = |\langle \psi_o | U(t) | \psi \rangle|^2, \tag{15}$$

where ψ_o represents the correct final quantum state (after the router's operating), $U(t)|\psi\rangle$ is the router's state for the moment t, if the initial state was ψ. The above definition of the Fidelity measure allow us to evaluate if the whole router works correctly.

Furthermore, in the case analyzed in this paper, it is important to employ the average Fidelity measure (denoted by the letter $\bar{F}$):

$$\bar{F} = \int \langle \psi | \hat{U}^{\dagger} \mathcal{E}(\psi) \hat{U} | \psi \rangle d\psi. \tag{16}$$

We integrate the area of all input states as a quantum map $\mathcal{E}$ for the router. The operator U denotes the final operation, correctly realizing the router's operating. In our work $\hat{U} = U_0$.

As in [24], the average Fidelity value may be calculated as:

$$\bar{F}(\psi, U_0, M) = \frac{1}{n(n+1)} \left(\mathrm{Tr}\left(MM^{\dagger}\right) + |\mathrm{Tr}\left(M\right)|^2 \right) \quad and \quad M = U_0^{\dagger} U(t). \tag{17}$$

This way of Fidelity computing does not require the state ψ value. It means that only the forms of U_0 and $U(t)$ influence the value of the average Fidelity measure. The dimension of the state ψ is n.

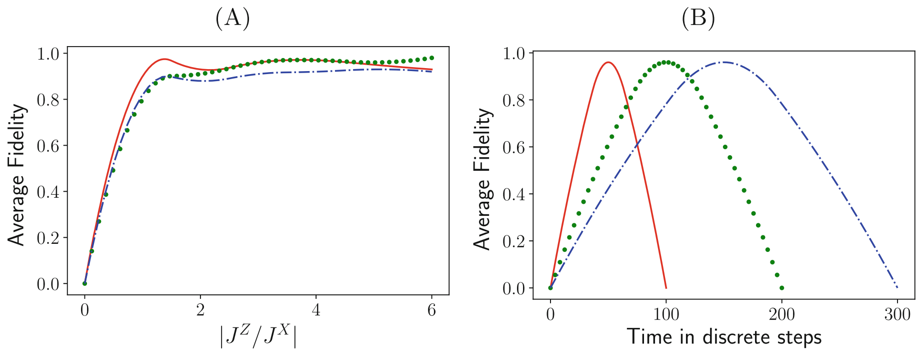

Fig. 3. The changes of average Fidelity value (A) during the router's operating for states $|0\rangle$ (red solid line), $|+\rangle$ (green dotted), $\frac{1}{\sqrt{2}}|0\rangle + |2\rangle$ (blue dash-dot line) for different values of $|J^Z/J^X|$ and the first 50 discrete time steps. The values of average Fidelity measure $\hat{F}$ for routing state $|0\rangle$ for there ration (red line $J^Z/J^X = 1$, green dotted $J^Z/J^X = 2$, blue dash-dot line $J^Z/J^X = 3$) are presented in plot (B) (Color figure online)

Figure 3 contains the values of Fidelity measure for three exemplary states. The time values are scaled to the time where the time variable is changed discretely each $\pi/2J^X$. It is possible to reach the Fidelity value ≈0.99 but it requires to select the parameters for each coupling.

5 Conclusions

The construction of quantum router was presented in this article. The router is a generalization of solutions working on qubits, and discussed in the literature in therms of spin interactions between quantum units of information. As the set of operators, we utilize the Gell-Mann matrices which are qutrit generalization of the Pauli operators. It is necessary to emphasize that the used Hamiltonian allows indicating the ways of possible physical implementation. The Hamiltonian describes interactions given by the generalized Pauli group, i.e. Gell-Mann operators for qutrits, to the physical realization of the router.

We have briefly shown that joining the routers allows building the structures able to transfer a quantum state to the defined node in a quantum network.

It is possible to achieve very high accuracy of information transfer in the router. However, it requires to carefully select the coupling parameters. The obtained values of the average Fidelity measure (≈0.99), show that the router operates correctly.

An interesting direction for further work is a hybrid system which could transfer the qubit state to one specific output from the available outputs with a qudit controlling state.

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