Unreliable Single-Server Queueing System with Customers of Random Capacity

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Abstract. In the paper, we investigate one-server queueing system with stationary Poisson arrival process, non-homogeneous customers and unreliable server. As non-homogenity, we mean that each customer is characterized by some arbitrarily distributed random capacity that is called customer volume. Service time of a customer generally depends on his volume. The server can be broken when it is free or busy and the renewal period goes on for random time having an arbitrary distribution. During this period, customers present in the system and arriving to it are not served. Their service continues immediately after renewal period termination. For such systems, we determine the distribution of total volume of customers present in it. An analysis of some special cases and some numerical examples are attached as well.

Keywords: Queueing system with non-homogeneous customers · Unreliable queueing system · Total volume · Additional event method · Laplace-Stieltjes transform

1 Introduction

A single-server queueing system $M/G/1/\infty$ with unlimited queue is one of the basic models of classical queueing theory and its applications. Many real computer or telecommunication systems satisfy the assumptions of this model: they are composed of only one server, customers arriving to it form Poisson arrival process and service time of a customer is arbitrarily distributed. In some other cases, these assumptions are not strictly satisfied, but behavior of proper systems is very similar (e.g. the queue is limited but long enough or arrival process is close to Poisson one), and we can approximate their characteristics using this model. The results for the classical $M/G/1/\infty$ queueing model are widely known, especially Pollaczek-Khinchine formula for the generating function $P(z) = \sum_{k=0}^{\infty} \mathsf{P}\{\eta = k\} z^k$ of the stationary number of customers η present in the system [2].

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On the other hand, the headway in computer science leads to some new modifications of this model. Indeed, if we focus on analysis of real computer systems, we should take into account the following problems: 1) customers coming to queueing systems are not homogeneous: they usually transport some information (measured in bytes), i.e. different customers have as a rule different volumes (sizes); 2) service time of a customer depends on his volume; 3) servers are unreliable, they can be broken and then must be fixed for some random time. These additional assumptions lead to new queueing models called queueing systems with non-homogeneous customers (assumptions 1, 2) and unreliable servers (assumption 3). The main result for $M/G/1/\infty$ queueing system with non-homogeneous customers, unlimited total volume and customer's service time dependent on his volume is the expression for Laplace-Stieltjes transform $\delta(s) = \int_0^\infty e^{-sx}\mathrm{d}D(x)$ of steady-state customers' total volume σ, where $D(x)$ is distribution function of random variable σ [1,6], and its modifications [9–11,14] that can be treated as generalizations of Pollaczek-Khinchine formula obtained by tools of classical queueing theory [2,12,13].

The main purpose of this paper is an investigation of some modification of $M/G/1/\infty$ model in which we assume that: 1) customers that arrive to the system are characterized by random volume; 2) the server is unreliable, i.e. it can be paused both when it is free or when it is busy; 3) service time of a customer depends on his volume. The analyzed model is the generalization of classical single-server queueing system with unreliable server [3]. For this model, we obtain characteristics of the total volume of customers present in the system for its various versions and investigate some special cases.

The rest of the paper is organized as follows. In the next Sect. 2, we introduce some needed notation and present well known results from the theory of queueing systems with non-homogeneous customers that will help us to analyze the mentioned above model with unreliable server. Then, in Sect. 3, we present the method of additional event [3–5] that is rarely used in papers written in English and its modification for systems under consideration. We also show (as an example) how to use this modification to obtain well-known results for $M/G/1/\infty$ model with non-homogeneous customers and unlimited total volume (these results were obtained earlier with the use of other methods). Section 4 contains some preliminary results for the model with unreliable server. In Sect. 5, we present main statements for the system with unreliable in free state server. To obtain these results, we also use modified method of additional event. We also present here formulae for total volume characteristics (e.g. Laplace-Stieltjes transform of the steady-state total volume and its moments). Section 6 contains analysis of the analogous system, but this time we assume that the server can be also broken if it is busy. In Sect. 7, we present results for some special cases of the model together with numerical examples. The last Sect. 8 contains conclusions and final remarks.

2 Mathematical Description of Systems with Non-homogeneous Customers

We assume that each customer arriving to the system is characterized by some random volume ζ that is a non-negative random variable (RV). Let $\eta(t)$ be the number of customers present in the system at time instant t, $\sigma(t)$ be the sum of the volumes of all these customers (total volume). Our purpose is the determination of random process $\sigma(t)$ characteristics. We also assume that customer's service time ξ generally depends on his capacity ζ. This dependence is determined by the following joint distribution function (DF):

$$F(x,t) = \mathsf{P}\{\zeta < x, \xi < t\}.$$

Let $L(x) = F(x,\infty)$ be DF of customer's volume, $B(t) = F(\infty,t)$ be DF of service time. Let

$$\alpha(s,q) = \mathsf{E}e^{-s\zeta - q\xi} = \int_{x=0}^{\infty}\int_{u=0}^{\infty} e^{-sx-qu}\mathrm{d}F(x,u),$$

where $\mathrm{Re}\, s \geq 0$, $\mathrm{Re}\, q \geq 0$, be double Laplace-Stieltjes transform (LST) of DF $F(x,t)$. Denote by $\varphi(s) = \alpha(s,0)$ LST of DF $L(x)$ and by $\beta(q) = \alpha(0,q)$ – LST of DF $B(t)$. Let $i,j = 1,2,\ldots$. Denote by $\Delta(i,j)$ and $\Delta_q(i)$ the following differential operators:

$$\Delta(i,j) = (-1)^{i+j}\frac{\partial^{i+j}}{\partial s^i \partial q^j},\ \Delta_q(i) = (-1)^i\frac{\partial^i}{\partial q^i}.$$

Let $\alpha_{i,j} = \Delta(i,j)\alpha(s,q)|_{s=0,\,q=0}$, $\varphi_i = \Delta_s(i)\varphi(s)|_{s=0}$ and $\beta_i = \Delta_q(i)\beta(q)|_{q=0}$. Then, we have evidently that $\alpha_{i,j}$ is the mixed $(i+j)$th moment of DF $F(x,t)$ and φ_i, β_i are the ith moments of DF $L(x)$ and $B(t)$, respectively (if they exist).

We assume that the arrival process is a stationary Poisson one with parameter a. Assume also that service discipline does not depend on customer's volume ζ and system is empty at the initial time moment $t = 0$, i.e. $\sigma(0) = 0$. Introduce the notation $D(x,t) = \mathsf{P}\{\sigma(t) < x\}$.

Let

$$\overline{\delta}(s,t) = \mathsf{E}e^{-s\sigma(t)} = \int_{x=0}^{\infty} e^{-sx}\mathrm{d}_x D(x,t)$$

be LST of the function $D(x,t)$ with respect to x. It is clear that, for arbitrary $t > 0$, the ith moment of the random process $\sigma(t)$ (if it exists) takes the form:

$$\overline{\delta}_i(t) = \mathsf{E}\sigma^i(t) = \Delta_s(i)\overline{\delta}(s,t)\Big|_{s=0}.$$

Denote by

$$\delta(s,q) = \int_0^{\infty} e^{-qt}\overline{\delta}(s,t)\mathrm{d}t = \int_0^{\infty} e^{-qt}\mathsf{E}e^{-s\sigma(t)}\mathrm{d}t$$

the Laplace transform with respect to t of the function $\overline{\delta}(s,t)$.

Then, we obtain for Laplace transfom $\delta_i(q)$ of $\overline{\delta}_i(t)$ with respect to t that

$$\delta_i(q) = \int_0^\infty e^{-qt}\overline{\delta}_i(t)\mathrm{d}t = \Delta_s(i)\delta(s,q)\Big|_{s=0}.$$

If steady state exists for the system under consideration, i.e. $\sigma(t) \Rightarrow \sigma$ in the sense of a weak convergence, where σ is a steady-state total volume, we can introduce the following steady-state characteristics:

$$D(x) = \mathsf{P}\{\sigma < x\} = \lim_{t\to\infty} D(x,t),$$

$$\delta(s) = \int_0^\infty e^{-sx}\mathrm{d}D(x) = \lim_{t\to\infty}\overline{\delta}(s,t) = \lim_{q\to 0} q\delta(s,q).$$

For steady-state ith moments δ_i of the total volume σ, we obtain:

$$\delta_i = \mathsf{E}\sigma^i = \lim_{t\to\infty}\delta_i(t) = \Delta_s(i)\delta(s)\Big|_{s=0}.$$

Let $\chi(t)$ be the volume of a customer that is served at time instant t. Let $\xi^*(t)$ be the time from the service beginning to the moment t. The next statement was proved in [13].

Lemma 1. *Let $E_y(x) = \mathsf{P}\{\chi(t) < x \,|\, \xi^*(t) = y\}$ be conditional DF of the random variable $\chi(t)$ under condition $\xi^*(t) = y$. Then*

$$dE_y(x) = [1 - B(y)]^{-1}\int_{u=y}^\infty dF(x,u).$$

Hence, the function $E_y(x)$ takes the form:

$$E_y(x) = \mathsf{P}\{\zeta < x \,|\, \xi \geq y\} = \frac{\mathsf{P}\{\zeta < x, \xi \geq y\}}{\mathsf{P}\{\xi \geq y\}} = \frac{L(x) - F(x,y)}{1 - B(y)}.$$

Corollary. *LST of the function $E_y(x)$ has the form:*

$$e_y(s) = [1 - B(y)]^{-1}R(s,y),$$

where $R(s,y) = \int_{x=0}^\infty e^{-sx}\int_{u=y}^\infty dF(x,u)$.

3 Method of Additional Event and Its Modification

Firstly, we present short the classical method of additional event that is very rarely used in papers written in English. This method was introduced by G. P. Klimov (see [4]) and successfully used for analysis of priority queueing systems [3]. Its idea is to give a probability sense to the formal mathematical transforms: LST and generating function (GF). To clarify the idea of the method, we consider two simple examples.

Example 1. Let non-negative RV ξ with DF $A(t)$ is the duration of some random process under consideration. Let, independently of this process behavior, some events (so-called catastrophes) take place and form stationary Poisson arrival process with parameter $q > 0$. Then $a(q) = \int_0^\infty e^{-qt} \mathrm{d}A(t)$ is probability that catastrophes do not appear during duration of the process.

Example 2. Let ξ be a number of customers coming to a system during some fixed time interval and $p_k = \mathsf{P}\{\xi = k\}$. Assume that each customer is of red colour with probability z $(0 \leq z \leq 1)$ and of blue colour with probability $1 - z$, independently of other customers' colours. Then $P(z) = \sum_{k=0}^\infty p_k z^k$ is probability that only red customers come to the system during this interval (or blue customers do not come during it).

By this way, we give the probability sense to LST $a(q)$ and GF $P(z)$ when $q > 0$ and $0 \leq z \leq 1$, consequently. Now, we can calculate these functions as probabilities of proper events using, if it is necessary, the principle of analytic continuation.

Let e.g. $P(z,t) = \sum_{k=0}^\infty P_k(t) z^k$ be GF of number of customers present in a system at time instant t. Then, for $0 \leq z \leq 1$, function $P(z,t)$ is probability that there are no blue customers in the system at time instant t. Let $\pi(z,q) = \int_0^\infty e^{-qt} P(z,t) \mathrm{d}t$ be Laplace transform of $P(z,t)$ with respect to t. Then, $q\pi(z,q)$ is probability that the first catastrophe appears when there are no blue customers in the system.

Later on, for analysis of the system presented in Sect. 2, we shall use some modification of the classical method of additional event. We assume that: 1) some events (catastrophes) take place independently of the behavior of a system under consideration, they form Poisson arrival process with parameter q, $q > 0$ (this proposition does not distinguish from proper classical one); 2) an arbitrary customer of volume x has red colour with probability e^{-sx}, $s > 0$ or blue one with probability $1 - e^{-sx}$, independently of other customers' colours (this proposition is the generalization of proper classical one).

Then, the functions introduced in Sect. 2 have the following probability sense [13]: $\varphi(s)$ is probability that an arbitrary customer is red; $\beta(q)$ is probability that there are no catastrophes during arbitrary customer's service; $\alpha(s,q)$ is the joint probability that an arbitrary customer is red and there are no catastrophes during his service; $e_y(s) = [1 - B(y)]^{-1} R(s,y)$ is probability that a customer on service is red, under condition that time y has passed from the beginning of his service; $q\delta(s,q)$ is probability that the first catastrophe takes place in the system when there are no blue customers in it.

Presentation of the method of additional event and its modification is also the aim of this paper. Of course, it is possible to use other methods for analysis of the unreliable system with random volume customers, e.g. this system can be interpreted as a system with vacations (see e.g. [7]), but, in this paper, we demonstrate possibilities of our method that can be used also for analysis of other queueing models. Note that this method in its modification form was used for analysis of the system $M/G/1/\infty$ with random volume customers and preemptive service discipline (see [8]).

As an example, let us consider an application of this method to determine the function $\delta(s,q)$ for the system $M/G/1/\infty$ with reliable server. Note that this queue was investigated earlier by other method (see e.g. [12]). Below, we call 0-moments the moments of service beginning or termination (see [3]).

Assume that a busy period of the system begins at time instant 0 and continues at time t. Let $\mathit{\Pi}(x,y,t)\,\mathrm{d}y = \mathsf{P}\{\sigma(t) < x,\ \xi^*(t) \in [y; y+\mathrm{d}y)\}$ be probability that the total customers' volume $\sigma(t)$ is less than x at time instant t, and time y has passed from the last 0-moment to the instant t. Let

$$\pi(s,y,q) = \int_{x=0}^{\infty} \int_{t=0}^{\infty} e^{-sx-qt}\,\mathrm{d}_x \mathit{\Pi}(x,y,t)\,\mathrm{d}t.$$

Then, $q\pi(s,y,q)\mathrm{d}y$ is probability that the first catastrophe on the busy period occurs when there are no blue customers in the system and time y has passed from the last 0-moment. Denote by $\pi(s,q) = \int_0^\infty \pi(s,y,q)\mathrm{d}y$. Then, $q\pi(s,q)$ is probability that the first catastrophe on a separate busy period occurs when there are no blue customers in the system. Let $\mathit{\Pi}(t)$ be DF of busy period of the system.

Firstly, we determine probability $\pi(s,0,q)$ that there are no blue customers in the system at some epoch of service termination and catastrophes do not appear to this epoch inside of the busy period.

Lemma 2. *The function $\pi(s,0,q)$ is determined by the relation*

$$\pi(s,0,q) = \frac{\varphi(s)[\beta(q+a-a\varphi(s)) - \pi(q)]}{\varphi(s) - \beta(q+a-a\varphi(s))},$$

where $\pi(q) = \int_0^\infty e^{-qt} d\mathit{\Pi}(t)$ is LST of the busy period.

Proof of the lemma follows from the appropriate statement in [3] (p. 18), where z must be substituted by $\varphi(s)$. □

Lemma 3. *The functions $\pi(s,y,q)$ and $\pi(s,0,q)$ are connected by the following relation:*

$$\pi(s,y,q) = e^{-(q+a-a\varphi(s))y}\left[1 + \frac{\pi(s,0,q)}{\varphi(s)}\right] R(s,y).$$

Proof. The first catastrophe inside of the busy period occurs in the system at time instant when there are no blue customers in it and the time y passed from the last 0-moment, iff

1) either the first catastrophe occurs during the first red customer service when time y has passed from the beginning of his service (probability of this event is $R(s,y)qe^{-qy}\mathrm{d}y$), and only red customers arrived during time y (probability of this event is $e^{-a(1-\varphi(s))y}$); therefore the complete probability of this event is $qe^{-(q+a-a\varphi(s))y}R(s,y)\mathrm{d}y$;

2) or, at some 0-moment inside of the busy period, there were no blue customers in the system and there were no catastrophes before this moment (probability of this event is $\pi(s,0,q)$), the customer on service was red after time y has passed from his service beginning (probability of this event is $\{[1-B(y)]\varphi(s)\}^{-1}R(s,y)$), and a catastrophe appeared at this moment and during time y blue customers did not arrive; the complete probability of this event is

$$\frac{\pi(s,0,q)}{\varphi(s)} q e^{-(q+a-a\varphi(s))y} R(s,y)\mathrm{d}y.$$

By summing obtained probabilities, we obtain the statement of the lemma. □

If we substitute the function $\pi(s,0,q)$ from lemma 2 to the relation in the statement of lemma 3, we obtain the following theorem.

Theorem 1. *a) The function $\pi(s,y,q)$ is determined by the following relation:*

$$\pi(s,y,q) = \gamma(s,y,q)\frac{\varphi(s)-\pi(q)}{\varphi(s)-\beta(q+a-a\varphi(s))}, \tag{1}$$

where

$$\gamma(s,y,q) = e^{-(q+a-a\varphi(s))y}R(s,y). \tag{2}$$

b) The function $\pi(s,q)$ is determined by the relation

$$\pi(s,q) = \int_0^\infty \pi(s,y,q)\,dy$$
$$= \frac{\varphi(s)-\alpha(s,q+a-a\varphi(s))}{q+a-a\varphi(s)} \cdot \frac{\varphi(s)-\pi(q)}{\varphi(s)-\beta(q+a-a\varphi(s))}.$$

Denote by $P(x,y,t)\mathrm{d}y = \mathsf{P}\{\sigma(t) < x,\ \xi^*(t) \in [y; y+\mathrm{d}y)\}$ probability that the total customers volume is less than x at the time instant t and time y has passed from the last 0-moment to this instant. Then,

$$q\delta(s,y,q)\mathrm{d}y = q\int_{x=0}^\infty \int_{t=0}^\infty e^{-sx-qt} d_x P(x,y,t)\,\mathrm{d}y\,\mathrm{d}t$$

is the probability that the first catastrophe occurs when there are no blue customers in the system and time y has passed from the last 0-moment; $q\delta(s,q) = q\int_0^\infty \delta(s,y,q)\mathrm{d}y$ is the probability that the first catastrophe occurs when there are no blue customers in the system.

Theorem 2. *a) The function $\delta(s,y,q)$ is determined by the following relation:*

$$\delta(s,y,q) = \frac{e^{-(q+a)y}}{q+a-a\pi(q)}\left[q+a+\frac{ae^{a\varphi(s)y}(\varphi(s)-\pi(q))}{\varphi(s)-\beta(q+a-a\varphi(s))}R(s,y)\right].$$

b) The function $\delta(s,q)$ is determined by the following relation:

$$\delta(s,q)=\int_0^\infty \delta(s,y,q)\,dy=[q+a-a\pi(q)]^{-1}$$
$$\times\left\{1+\frac{\varphi(s)-\alpha(s,q+a-a\varphi(s))}{q+a-a\varphi(s)}\cdot\frac{a[\varphi(s)-\pi(q)]}{\varphi(s)-\beta(q+a-a\varphi(s))}\right\}.$$

Proof. Determine the probability $q\delta(s,y,q)\mathrm{d}y$. A proper event takes place iff:

1) either the first catastrophe occurs on the first interval when server is free (probability of this event is $e^{-(q+a)y}$);
2) or the first busy period begins earlier than a catastrophe appears (probability of this event is $a/(q+a)$) and the first catastrophe occurs on this period when there are no blue customers in the system, and time y has passed from the last 0-moment (probability of this event is $q\pi(s,y,q)\mathrm{d}y$);
3) or there were no catastrophes during the first interval when the server was free nor during the first busy period (probability of this event is $a\pi(q)/(q+a)$), and further the process behaves as from the start (it is clear that epochs of busy periods terminations are regeneration points of the process $\sigma(t)$).

As a result, we have:

$$q\delta(s,y,q)\mathrm{d}y=qe^{-(q+a)y}\mathrm{d}y+\frac{aq}{q+a}\pi(s,y,q)\mathrm{d}y+\frac{aq\pi(q)}{q+a}\delta(s,y,q)\mathrm{d}y,$$

whereas we obtain the first statement of the theorem. □

The last relation in the second statement coincides with results obtained earlier (see [12]).

4 The Model and Preliminary Results

Consider a system $M/G/1/\infty$ and assume that its server is reliable in busy state. If T is an epoch of service termination when there are no waiting customers and other customers do not arrive to the system during time t, the server can be broken on time interval $[T;T+t)$ with probability $E(t)$. After breakage, the server restores during some random time ψ. Denote by $H(t)=\mathsf{P}\{\psi<t\}$ DF of RV ψ. The volume of a customer ζ and his service time ξ are determined by the joint DF $F(x,t)=\mathsf{P}\{\zeta<x,\xi<t\}$. We assume that service time, the time of reliable state of the server and renewal time are independent RVs. For the considered system, we determine the function

$$\delta(s,q)=\int_0^\infty e^{-qt}\left[\int_{x=0}^\infty e^{-sx}\mathrm{d}_xD(x,t)\right]\mathrm{d}t,$$

where $D(x,t)=\mathsf{P}\{\sigma(t)<x\}$ is DF of total volume at time instant t.

Denote by $\pi(q)$ LST of busy period of the system under consideration when $E(t) \equiv 0$ (this is the busy period of the reliable system $M/G/1/\infty$). Let $\pi_n(q) = [\pi(q)]^n$ be LST of DF of so-called n-period (busy period that begins from the moment when there are n customers in the system, $n = 1, 2, \dots$). Let $\Pi^{(n)}(x, y, t)\mathrm{d}y$ be probability that the total customers volume $\sigma(t)$ is less than x at time instant t, and time y has passed from the last 0-moment, under assumption that this n-period does not terminate at time instant t.

Introduce the notations:

$$\pi_n(s, y, q) = \int_{x=0}^{\infty} \int_0^{\infty} e^{-sx-qt} \mathrm{d}_x \Pi^{(n)}(x, y, t)\mathrm{d}t,$$
$$\pi_n(s, q) = \int_0^{\infty} \pi_n(s, y, q)\mathrm{d}y.$$

Then, $q\pi_n(s, y, q)\mathrm{d}y$ is probability that the first catastrophe on n-period occurs when there are no blue customers in the system and time y has passed from the last 0-moment, $q\pi(s, y, q)\mathrm{d}y$ is probability of an analogous event for the reliable system.

Theorem 3. *The following relations take place:*

$$\begin{aligned} \pi_n(s, y, q) &= \frac{(\varphi(s))^n - (\pi(q))^n}{\varphi(s) - \beta(q + a - a\varphi(s))} e^{-(q+a-a\varphi(s))y} R(s, y), \\ \pi_n(s, q) &= \frac{(\varphi(s))^n - (\pi(q))^n}{\varphi(s) - \beta(q + a - a\varphi(s))} \cdot \frac{\varphi(s) - \alpha(s, q + a - a\varphi(s))}{q + a - a\varphi(s)}. \end{aligned} \tag{3}$$

Proof. As it follows from relations (1) and (2), the relation (3) can be presented as:

$$\pi_n(s, y, q) = \frac{(\varphi(s))^n - (\pi(q))^n}{\varphi(s) - \pi(q)} \pi(s, y, q),\ n \geq 1,$$

or, if we treat the fraction as a sum of n initial items of geometrical progression with the first item $(\varphi(s))^{n-1}$ and denominator $\pi(q)/\varphi(s)$,

$$\begin{aligned} q\pi_n(s, y, q)\mathrm{d}y &= (\varphi(s))^{n-1} q\pi(s, y, q)\mathrm{d}y + (\varphi(s))^{n-2}\pi(q) q\pi(s, y, q)\mathrm{d}y \\ &+ \dots + \varphi(s)(\pi(q))^{n-2} q\pi(s, y, q)\mathrm{d}y + (\pi(q))^{n-1} q\pi(s, y, q)\mathrm{d}y. \end{aligned} \tag{4}$$

Relation (4) can be proved by the modified method of additional event.

Assume that the first catastrophe inside of n-period occurs when all customers present in the system are red and time y has passed from the last 0-moment. This event takes place iff:

a) either the first catastrophe occurs inside of busy period connected with the first (from n) served customer, there are no blue customers in the system at this time instant, time y has passed from the last 0-moment (probability of this event is $q\pi(s, y, q)\mathrm{d}y$) and other $n-1$ customers present in the system at the moment of beginning of the n-period were red (probability of this event is $(\varphi(s))^{n-1}$);

b) or there were no catastrophes inside of busy period connected with the first served customer (probability of this event is $\pi(q)$), but the first catastrophe occurs inside of busy period connected with the second served customer (from presented ones at the beginning of the n-period) at time instant when all customers present in the system are red, time y has passed from the last 0-moment (probability of this event is $\pi(q)q\pi(s,y,q)\mathrm{d}y$) and other $n-2$ customers from those presented at the beginning of the n-period were red (probability of this event is $(\varphi(s))^{n-2}$);
c) or, finally, there were no catastrophes during initial $n-1$ busy periods and the first catastrophe occurs inside of the last one at time instant when all customers present in the system are red and time y has passed from the last 0-moment (probability of this event is $(\pi(q))^{n-1}q\pi(s,y,q)\mathrm{d}y$).

Summing obtained probabilities, we obtain the relation (4). The rest of the proof is evident. □

5 Main Statements

Denote by $\varepsilon(q)$ and $h(q)$ LSTs of DFs $E(t)$ and $H(t)$, respectively. As a regeneration period of the system we mean the time interval between neighbouring epochs when the system becomes empty after service termination. It is follows from [3, p. 46] that, for the system under consideration, LST $r(q)$ of DF of a regeneration period is determined by the following relation:

$$r(q) = \frac{a}{q+a}[1-\varepsilon(q+a)]\pi(q) + \varepsilon(q+a)h(q+a-a\pi(q)). \tag{5}$$

Denote by $\omega(t)$ the time that has passed from the beginning of a regeneration period to some time moment t inside it.

Let $P(x,y,t)\mathrm{d}y = \mathsf{P}\{\sigma(t)<x, \omega(t)\in[y;y+\mathrm{d}y)\}$. Then

$$\overline{\delta}(s,y,q) = \int_0^\infty e^{-qt}\left[\int_{x=0}^\infty e^{-sx}\mathrm{d}_x D(x,t\,|\,\omega(t)=y)\right]\mathrm{d}t$$

be the Laplace transform (with respect to t) of LST (with respect to x) of DF of total volume of customers present in the system t time units after beginning of a regeneration period, if time y has passed from the last 0-moment inside of the period.

Theorem 4. *The function $\overline{\delta}(s,y,q)$ is determined by the following relation:*

$$\begin{aligned}\overline{\delta}(s,y,q) &= [1-E(y)]e^{-(q+a)y} + \frac{\varepsilon(q+a)}{q+a}e^{-(q+a-a\varphi(s))y}\\ &\times\frac{\varphi(s)-\pi(q)}{\varphi(s)-\beta(q+a-a\varphi(s))}R(s,y)\\ &+\frac{\varepsilon(q+a)\,[h(q+a-a\varphi(s))-h(q+a-a\pi(q))]}{\varphi(s)-\beta(q+a-a\varphi(s))}e^{-(q+a-a\varphi(s))y}R(s,y)\\ &+\varepsilon(q+a)[1-H(y)]e^{-(q+a-a\varphi(s))y}.\end{aligned} \tag{6}$$

Proof. A time interval when the server is busy that begins either from beginning of service of an arriving customer, or from the epoch of server breakage, and terminates at the nearest time moment when the server is in good repair and there are no customers in the system, we call generalized busy period. Therefore, it is a time interval of two possible types [3]: 1) a generalized busy period begins from customer service; 2) it begins from server breakage. Probabilities that the regeneration period involves generalized busy periods of types 1 and 2 equal $a(q+a)^{-1}[1-\varepsilon(q+a)]$ and $\varepsilon(q+a)$, respectively.

Recall that $q\pi(s,y,q)\mathrm{d}y$ is probability that the first catastrophe on the busy period of the reliable system $M/G/1/\infty$ occurs when there are no blue customers in the system and time y has passed from the last 0-moment. It is clear that, for the system under consideration, probability of an analogous event on the generalized busy period of the first type equals $q\pi_n(s,y,q)\mathrm{d}y$, i.e. the distribution of the total customers volume on this period is determined by the function $\pi(s,y,q)$.

Consider a generalized busy period of the second type. It is clear that blue customers form Poisson arrival process with parameter $a(1-\varphi(s))$. Hence, $e^{-(q+a-a\varphi(s))y}$ is probability that, during time y, catastrophes do not occur and blue customers do not arrive; $q[1-H(y)]e^{-(q+a-a\varphi(s))y}\mathrm{d}y$ is probability that, during renewal period having duration greater than y, the first catastrophe occurs after time y from the beginning of the period, and only red customers arrive to the system before the catastrophe. Probability that catastrophes do not occur and n customers arrive to the system during the renewal period is equal to $\int_0^\infty \frac{(au)^n}{n!}e^{-(q+a)u}\mathrm{d}H(u)$. Then, probability $qG(s,y,q)dy$ that period of the second type involves service of customers and the first catastrophe inside it occurs when there no blue customers in the system, and time y has passed from the last 0-moment is equal to

$$qG(s,y,q)dy = \sum_{n=1}^{\infty}\int_{u=0}^{\infty}\frac{(au)^n}{n!}e^{-(q+a)u}q\pi_n(s,y,q)\mathrm{d}y\,\mathrm{d}H(u),$$

where function $\pi_n(s,y,q)$ is determined by relation (3), whereas we obtain after some transformations:

$$qG(s,y,q)dy = qe^{-(q+a-a\varphi(s))y}\mathrm{d}y\frac{h(q+a-a\varphi(s))-h(q+a-a\pi(q))}{\varphi(s)-\beta(q+a-a\varphi(s))}R(s,y).$$

Now, we can obtain the relation (6) using (5) and formula of total probability. □

Denote by $\delta(s,y,q)$ Laplace transform with respect to t of total customers volume $\sigma(t)$ under condition that time y has passed from the last 0-moment.

The next statement follows from Theorem 4 and can be proved analogously as Theorem 2.

Theorem 5. *a) Function $\delta(s,y,q)$ is determined by the following relation:*

$$\delta(s,y,q)=\left\{1-\frac{a}{q+a}[1-\varepsilon(q+a)]\pi(q)-\varepsilon(q+a)h(q+a-a\pi(q))\right\}^{-1}$$
$$\times\left\{[1-E(y)]e^{-(q+a)y}+\frac{ae^{-(q+a-a\varphi(s))y}}{q+a}[1-\varepsilon(q+a)]\frac{R(s,y)(\varphi(s)-\pi(q))}{\varphi(s)-\beta(q+a-a\varphi(s))}\right.$$
$$+\varepsilon(q+a)[1-H(y)]e^{-(q+a-a\varphi(s))y}$$
$$\left.+\varepsilon(q+a)e^{-(q+a-a\varphi(s))y}\frac{h(q+a-a\varphi(s))-h(q+a-a\pi(q))}{\varphi(s)-\beta(q+a-a\varphi(s))}R(s,y)\right\}.$$

b) Function $\delta(s,q)=\int_0^\infty e^{-qt}\mathsf{E}e^{-s\sigma(t)}dt$ is determined by the relation:

$$\delta(s,q)=\int_0^\infty\delta(s,y,q)dy$$
$$=\left\{1-\frac{a}{q+a}[1-\varepsilon(q+a)]\pi(q)-\varepsilon(q+a)h(q+a-a\pi(q))\right\}^{-1}\left\{\frac{1-\varepsilon(q+a)}{q+a}\right.$$
$$+\frac{a[1-\varepsilon(q+a)][\varphi(s)-\alpha(s,q+a-a\varphi(s))][\varphi(s)-\pi(q)]}{(q+a)(q+a-a\varphi(s))[\varphi(s)-\beta(q+a-a\varphi(s))]}\tag{7}$$
$$+\frac{\varepsilon(q+a)[\varphi(s)-\alpha(s,q+a-a\varphi(s))]}{q+a-a\varphi(s)}\cdot\frac{h(q+a-a\varphi(s))-h(q+a-a\pi(q))}{\varphi(s)-\beta(q+a-a\varphi(s))}$$
$$\left.+\frac{\varepsilon(q+a)[1-h(q+a-a\varphi(s))]}{q+a-a\varphi(s)}\right\}.$$

Corollary. *If $\rho=a\beta_1<1$, a steady state exists for the system under consideration, i.e. $\sigma(t)\Rightarrow\sigma$ in the sense of a weak convergence. LST $\delta(s)$ of DF $D(x)=\lim\limits_{t\to\infty}D(x,t)=\mathsf{P}\{\sigma<x\}$ of RV σ is determined by the following relation:*

$$\delta(s)=\lim_{q\to 0}q\delta(s,q)=\frac{1-\rho}{1-\varepsilon(a)(1-ah_1)}\left\{\left[1-\varepsilon(a)+\frac{\varepsilon(a)(1-h(a-a\varphi(s)))}{1-\varphi(s)}\right]\right.$$
$$\left.\times\left[1+\frac{\varphi(s)-\alpha(s,a-a\varphi(s))}{\beta(a-a\varphi(s))-\varphi(s)}\right]\right\},\tag{8}$$

where h_1 is the first moment of DF $H(t)$.

Using appropriate relations from Sect. 2, we can calculate moments of the total volume or their Laplace transforms. Note that, if $\varepsilon(q)\equiv 0$, we obtain the known relations for the reliable system $M/G/1/\infty$ (see [13]).

For example, Laplace transform $\delta_1(q) = \int_0^\infty e^{-qt}\delta_1(t)\mathrm{d}t$ of the mean total customers volume $\delta_1(t) = \mathsf{E}\sigma(t)$ has the following form:

$$\delta_1(q) = \left\{1 - \frac{a}{q+a}[1-\varepsilon(q+a)]\pi(q) - \varepsilon(q+a)h(q+a-a\pi(q))\right\}^{-1} \times\left[\varepsilon(q+a)(h(q)-h(q+a-a\pi(q))) + \frac{a(1-\varepsilon(q+a))(1-\pi(q))}{q+a}\right]\frac{\Delta_s(1)\alpha(s,q)|_{s=0}}{q(\beta(q)-1)} + \frac{a\varphi_1}{q^2}\left[\varepsilon(q+a)(1-h(q+a-a\pi(q))) + \frac{(1-\varepsilon(q+a))(q+a-a\pi(q))}{q+a}\right]\Bigg\},$$

whereas we obtain the first and second moments of RV σ in the following form:

$$\delta_1 = \mathsf{E}\sigma = a\alpha_{11} + \frac{a^2\beta_2\varphi_1}{2(1-\rho)} + \frac{a^2h_2\varepsilon(a)\varphi_1}{2(1-\varepsilon(a)+a\varepsilon(a)h_1)}, \tag{9}$$

$$\begin{aligned}\delta_2 = \mathsf{E}\sigma^2 &= a(\alpha_{21} + a\varphi_1\alpha_{12}) + \frac{a^3\beta_2\varphi_1\alpha_{11}}{1-\rho} + \frac{a^2\beta_2\varphi_2}{2(1-\rho)} + \frac{a^3\beta_3\varphi_1^2}{3(1-\rho)} \\ &+ \frac{a^4\beta_2^2\varphi_1^2}{2(1-\rho)^2} + \frac{a^3\varepsilon(a)h_2\varphi_1\alpha_{11}}{1-\varepsilon(a)+a\varepsilon(a)h_1} + \frac{a^2\varepsilon(a)h_2\varphi_2}{2(1-\varepsilon(a)+a\varepsilon(a)h_1)} \\ &+ \frac{a^3\varepsilon(a)h_3\varphi_1^2}{3(1-\varepsilon(a)+a\varepsilon(a)h_1)} + \frac{a^4\varepsilon(a)h_2\varphi_1^2}{2(1-\rho)(1-\varepsilon(a)+a\varepsilon(a)h_1)}.\end{aligned} \tag{10}$$

6 The Case of the System Unreliable Also When Server Is Busy

Our model can be generalized to the case of unreliable server also when it is busy. Let, in addition, the server can be broken on time interval $[T; T+t)$, where T is a moment of service beginning, with probability $G(t)$ (we assume that service of the customer does not terminate before time instant $T+t$). If this event takes place, the service of the customer is interrupted and will be continued after server's renewal. Denote by $X(t)$ DF of the renewal period. Let $g(q)$ and $\chi(q)$ be LSTs of the functions $G(t)$ and $X(t)$, respectively, and denote by χ_i, $i = 1, 2, \ldots$, the ith moment of the renewal time.

Obviously, in this case, the problem of determination of total customers volume distribution comes to previous one solved in Sect. 5, if service time of a customer is substituted by the time from beginning to termination of his service. This time is called the time of customer presence on server. Let $\kappa(q)$ be LST of DF of this time (taking into consideration possible breakages and renewals). Then, when $\varepsilon(q) \equiv 0$, instead of the equation $\pi(q) = \beta(q+a-a\pi(q))$, we have the following functional equation for busy period of the system under consideration: $\pi(q) = \kappa(q+a-a\pi(q))$.

Denote by $P(z,t)$ the generating function of number of server's breakages during customer service time t, under assumption that the total service time $y \geq t$. Let $B(t \mid \zeta = x) = \mathsf{P}\{\xi < t \mid \zeta = x\}$ is conditional DF of service time of a customer, under condition that his volume equals x, $\kappa(q \mid \zeta = x)$ is LST of DF of the time of customer presence on server, if his volume equals x. It is clear that

$$\kappa(q \mid \zeta = x) = \int_0^\infty e^{-qt} P(\chi(q), t)\, \mathrm{d}B(t \mid \zeta = x), \tag{11}$$

where the function $P(z,t)$ can be determined via its Laplace transform $p(z,q)$ (see e.g. [13]):

$$p(z,q) = \int_0^\infty e^{-qt} P(z,t)\, \mathrm{d}t = \frac{1-g(q)}{q[1-zg(q)]}.$$

Let ω be the time of customer presence on server, $\Gamma(x,t) = \mathsf{P}\{\zeta < x, \omega < t\}$ be joint DF of customer volume ζ and RV ω. We denote by $\gamma(s,q)$ double LST of the function $\Gamma(x,t)$:

$$\gamma(s,q) = \int_{x=0}^\infty \int_{t=0}^\infty e^{-sx-qt}\, \mathrm{d}\Gamma(x,t).$$

In particular, $\varphi(s) = \gamma(s,0)$, $\kappa(q) = \gamma(0,q)$. It follows from (11) that

$$\gamma(s,q) = \int_{x=0}^\infty e^{-sx} \kappa(q \mid \zeta = x)\, \mathrm{d}L(x) = \int_{x=0}^\infty \int_{t=0}^\infty e^{-sx-qt} P(\chi(q), t)\, \mathrm{d}F(x,t). \tag{12}$$

So, the problem of determination of process $\sigma(t)$ characteristics comes to analogous problem for the system $M/G/1/\infty$ with unreliable server in free state only, if we assume that the joint DF of customer volume and his service time is $\Gamma(x,t)$. Let κ_i be the ith moment of RV ω and γ_{ij} be the mixed $(i+j)$th moment of the random vector (ζ, ω). Then, for the system under consideration, the function $\delta(s,q)$ is determined by the relation (7), where we have to replace $\beta(q)$ by $\kappa(q)$ and $\alpha(s,q)$ by $\gamma(s,q)$.

It is clear that steady state exists for the system under consideration, if the inequality $\rho^* = a\kappa_1 < 1$ holds. The function $\delta(s)$ can be determined by relation (8) with the same previously made replacements.

7 Special Cases and Numerical Results

In this section we analyze some special cases of investigated in Sect. 6 model. Assume additionally that $G(t) = 1 - e^{-dt}$, $d > 0$. In this case, we obtain $P(z,t) = e^{-(1-z)dt}$, and it follows from (12) that

$$\gamma(s,q) = \int_{x=0}^\infty \int_{t=0}^\infty e^{-sx-(q+d-d\chi(q))t} \mathrm{d}F(x,t) = \alpha(s, q+d-d\chi(q)), \tag{13}$$

whereas we obtain:

$$\kappa(q) = \beta(q + d - d\chi(q)). \tag{14}$$

Then, for function $\delta(s)$ determination we can use relation (8), where $\alpha(s, q)$ is substituted by $\alpha(s, q + d - d\chi(q))$ and $\beta(q)$ by $\beta(q + d - d\chi(q))$.

The moments δ_1, δ_2 can be calculated by relations (9), (10), where α_{ij} is substituted by γ_{ij}, and β_i – by κ_i. We can easily obtain that

$$\gamma_{11} = (1 + d\chi_1)\alpha_{11},\ \gamma_{21} = (1 + d\chi_1)\alpha_{21},\ \gamma_{12} = (1 + d\chi_1)^2\alpha_{12} + d\chi_2\alpha_{11}, \tag{15}$$

$$\begin{aligned}\kappa_1 &= (1 + d\chi_1)\beta_1,\ \kappa_2 = (1 + d\chi_1)^2\beta_2 + d\chi_2\beta_1,\\ \kappa_3 &= (1 + d\chi_1)^3\beta_3 + 3d(1 + d\chi_1)\chi_2\beta_2 + d\chi_3\beta_1.\end{aligned} \tag{16}$$

Assume additionally that $E(t) = G(t) = 1 - e^{-dt}$ (probabilities of servers's breakage in free and busy state are determined exponentially with the same parameter d) and customer's service time ξ and volume ζ are connected by the relation $\xi = c\zeta$, $c > 0$. Then, we obtain $\alpha(s, q) = \varphi(s+cq)$ and $\beta(q) = \varphi(cq)$ (see e.g. [12,13]), and mixed moments α_{ij} are determined as $\alpha_{ij} = c^j\varphi_{i+j}$, moments β_i are determined as $\beta_i = c^i\varphi_i$, $i, j = 1, 2, \ldots$. In addition, we suppose that renewal periods distributions in server's free or busy state are also the same which means that $H(t) = X(t)$ (so $\chi(q) = h(q)$).

In this case, the relation (8) (taking into consideration replacements $\alpha(s, q)$ by $\gamma(s, q)$ and $\beta(q)$ by $\kappa(q)$) has the form

$$\begin{aligned}\delta(s) = &\left[\frac{1}{1 + dh_1} - ac\varphi_1\right]\left\{\left[1 + \frac{d(1 - h(a\psi(s)))}{a\psi(s)}\right]\right.\\ &\times\left.\left[1 + \frac{\varphi(s) - \varphi(s + ca\psi(s) + cd - cdh(a\psi(s)))}{\varphi(ca\psi(s) + cd - cdh(a\psi(s))) - \varphi(s)}\right]\right\},\end{aligned}$$

where $\psi(s) = 1 - \varphi(s)$. Initial moments of the total volume can be calculated by formulae (9), (10), where $\alpha_{ij} = \gamma_{ij}$, $\beta_i = \kappa_i$, and the values γ_{ij}, κ_i are calculated by formulae (15), (16).

If we additionally assume that $H(t) = 1 - e^{-rt}$, $L(x) = 1 - e^{-fx}$, then $h_i = i!/r^i$, $\varphi_i = i!/f^i$, $\gamma_{11} = 2c(d + r)/(rf^2)$, $\gamma_{21} = 6c(d + r)/(rf^3), \gamma_{12} = 2c[3c(d + r)^2 + 2df]/(r^2f^3)$, $\kappa_1 = c(d + r)/(rf)$, $\kappa_2 = 2c\left[c(d + r)^2 + df\right]/(rf)^2$, $\kappa_3 = 6c\left[c^2(d + r)^3 + 2cd(d + r)f + df^2\right]/(rf)^3$.

For example, in this case, the relations for the first moments of steady-state total volume have the form:

$$\delta_1 = \frac{a}{rf}\left\{\frac{d}{d+r} + \frac{c(d+r)[2rf - ca(d+r)] + acdf}{f[rf - ca(d+r)]}\right\}.$$

$$\begin{aligned}\delta_2 &= \frac{d(-2ad + 2ar - 2dr + a^2dr - 2r^2 + a^2r^2)}{c(d+r)^3(-acd - acr + fr)} + \frac{2(2a^2cd + acdr + acr^2)}{f^3r^2} \\ &+ \frac{2(a^2c^2d^2 + 2a^2c^2dr + a^2c^2r^2)}{f^4r^2} + \frac{2ad^2 - 2adr + 2d^2r - a^2d^2r + 2dr^2 - a^2dr^2}{cfr(d+r)^3} \\ &+ \frac{2(a^2d^2 - d^2r^2 - 2dr^3 - r^4)}{f^2r^2(d+r)^2} + \frac{2(a^2d^2 + 2ad^2r + 2adr^2 + d^2r^2 + 2dr^3 + r^4)}{(d+r)^2(-acd - acr + fr)^2}.\end{aligned} \tag{17}$$

Formulae (17) are important not only from the theoretical point of view. As it was discussed in [13] (p. 262–266), these characteristics are also used in approximation of loss characteristics for analogous (to the mentioned above) queueing system but with limited total volume. Assume now that we analyze single–server queueing model with non–homogeneous customers, unreliable server and limited (by value V) total volume (which means that $\sigma(t) \le V$). Then we introduce, for example, characteristic called loss probability, which is usually determined by the following relation:

$$P_{loss} = 1 - \int_0^V D_V(V - x)\, dL(x), \tag{18}$$

where $D_V(x)$ is the distribution function of the total customers' volume for this system and $L(x)$ – distribution function of the customer's volume. For the systems with limited memory, where service time of a customer and his volume are dependent, it is often impossible to determine function $D_V(x)$. Then we calculate estimators P^*_{loss} of P_{loss} substituting in (18) distribution $D(x)$ instead of $D_V(x)$ ($D(x)$ is analogous function for the system with unlimited total volume). Moreover, even in the case when total volume is unlimited, we rarely obtain relation for $D(x)$ in exact form. We usually obtain its LST and, on the base of its properties, we can calculate its first two moments δ_1, δ_2 that let us approximate convolution $\Phi(x) = \int_0^x D(V - u)\, dL(u)$ (that is present in (18)) by the function $\Phi^*(x) = \frac{\gamma(q,cx)}{\Gamma(q)}$, where $q = f_1^2/(f_2 - f_1^2), c = f_1/(f_2 - f_1^2)$, $f_1 = \delta_1 + \varphi_1$ and $f_2 = \delta_2 + \varphi_2 + 2\delta_1\varphi_1$. Finally, we use formula: $P^*_{loss} = 1 - \Phi^*(V)$. In Table 1, we present numerical computations for the model with limited total volume. Its characteristics are the same as characteristics of the model discussed at the beginning of this section, but this time total volume is limited by V. We present results for the following values: $a = 1$, $c = 1$, $d = 0,5$, $r = 1$, $f = 2$ (then $\rho = 0,75$); $a = 1$, $c = 1$, $d = 0,25$, $r = 1$, $f = 2$ ($\rho = 0,625$) and $a = 0,5$, $c = 1$, $d = 0,1$, $r = 1$, $f = 2$ ($\rho = 0,275$). As we can see, loss probability also depends strictly on the value of the parameter d which determines how often breakages in the system appear.

Note that our approach of P_{loss} estimation guarantees correct determination of buffer capacity, i.e. the determination of such V that this probability does not exceed a given value.

Table 1. Numerical values of P_{loss} for the system with limited memory and unreliable server

V	$\mathbf{P_{loss}}(\rho = \mathbf{0,75})$	$\mathbf{P_{loss}}(\rho = \mathbf{0,625})$	$\mathbf{P_{loss}}(\rho = \mathbf{0,275})$
1	0,7416	0,5980	0,3077
2	0,5330	0,3530	0,1042
3	0,3796	0,2077	0,0360
4	0,2690	0,1220	0,0125
5	0,1900	0,0716	0,0044
6	0,1339	0,0420	0,0015
7	0,0942	0,0246	0,0005
8	0,0662	0,0144	0,0002
9	0,0465	0,0084	0,0001
10	0,0326	0,0049	$2,4 \cdot 10^{-5}$

8 Conclusion and Final Remarks

In the paper, we present the modified method of additional event that is very rarely used in English scientific literature and use this method to investigate single-server queueing system with non-homogeneous customers, unreliable server and unlimited total volume. For the analyzed model, we obtain characteristics of the total volume both in stationary and nonstationary mode. We also calculate first two moments of the steady-state total volume and show loss probability estimators calculations for the analogous model with limited memory that gives us the possibility to determine buffer space capacity of the node of computer or communication network. Investigations show possibility of using method of additional event in the case of complicated models. In addition, we prove that the character of dependency between customer's volume and his service time has influence on characteristics of total volume and estimators of loss characteristics.

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