Chapter 28
Challenging Mathematical Insights into Masonry Domes over the Centuries

Raffaella Pavani

Abstract We focus on masonry domes which are considered architectural landmarks either in different historical periods and in different cultural contexts. From a mathematical point of view, an approximation of a dome is provided by a rotation solid whose cross-section gives the generating curve. Obviously, a frequent generating curve is the semicircumference, but here we want to highlight the role of parabola and catenary used as generating curves to make the structural load lighter. At the present they are well-studied different curves, but until the 17th century, they were considered the same curve, even though they significantly differ from the point of view of structural properties. Actually, catenary is the curve of a hanging chain, which exhibits a tension strength only. When it is “frozen” and inverted it exhibits a compression strength only, which means that it supports itself. Parabola does not exhibit such structural property, but catenary may differ from a convenient parabola very slightly so that building approximation makes a catenary appear as a parabola and this parabola is so close to a catenary that it approximately retains its structural properties, point by point. Here, we investigate the mathematical connection between catenary and parabola in masonry dome structure, referring in particular to Brunelleshi’s dome in Florence, Saint Peter’s dome in Rome and San Gaudenzio’s dome in Novara.

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28.1 Introduction

Currently, we are accustomed to see buildings with curvilinear roofs of any material, of any curvature, of any regular or irregular shape. In the ancient centuries, instead,

R. Pavani
Department of Mathematics, Politecnico di Milano, Piazza L. da Vinci 32 - 20133 Milano, Italy
e-mail: raffaella.pavani@polimi.it

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curvilinear coverings were provided by masonry domes only. They were ideally generated by the rotation of an arch around its vertical symmetry axis. In this way, cross-sections become curves, mainly parabolas and catenaries (aside from, obviously, semicircumferences).

The most ancient still standing example of the curve (we now call) catenary, used as a cross-section of a masonry vault, goes back to the so-called Ctesiphon arch (3rd century A.D., Taq Kasra – Iraq). Previously, corbelled domes appeared in Minoan civilization about 1500 B.C., but true masonry domes were found at ancient Ur (in the present Iraq) and are dated back to 2500 B.C. (Chant and Goodman, 1999). This would place knowledge of true masonry dome long before the rise of Roman Empire. Even though the true dome was not a Roman invention, Romans were the first civilization to overcome the challenges associated with the true dome and perfect the form, enlarging the span of dome they could build (e.g. Caracalla's Baths and Diocletian's Baths). It is worth mentioning the Pantheon dome which has been a landmark in Rome panorama since the 1st century B.C.; remarkably, it is not a true dome, but a corbelled dome.

Here we consider parabolas and catenaries as cross-sections of ancient domes and from a mathematical point of view we will discuss features of the two curves when used in architecture. We add that when a surface is generated by the rotation of a catenary around its symmetry axis, it is also called *funicular surface*.

We remind that the relevance of catenary in architecture and structural engineering was firstly introduced by Hooke's studies on inverted chain and its stability properties, which will be treated in the next Section. We notice that our analytical-numerical approach has to be inserted in the more general geometrical approach which, more recently, was strengthened by Heyman's "*safe theorem*" (Heyman, 1995). According to this theorem, if by an elastic analysis a thrust line can be found which lies within the wall thickness of an arch in equilibrium, then this is sufficient to guarantee that the structure is stable under the given loading. Since mathematically the equilibrium of an arch is given by a convenient catenary, it is clear that such analytical curve is really significant to architecture. Within this context, we suggest to refer to (Cazzani et al, 2016a,b; Grillanda et al, 2019) for different efficient numerical methods, detailed discussions and significant examples.

28.2 Parabola vs. Catenary

The parabola curve is the graphic of any analytical function which is a polynomial of degree two. From the point of view of classical geometry, any parabola can be built as a conic section, as it is known since the 4th century B.C. by the Greek mathematician Menecmus. From the point of view of architecture, parabola has no stability property.

Instead, catenary has excellent stability properties, but its equation is more complicated, involving a hyperbolic cosine. This curve represents the shape of a hanging chain (or inextensible cable) of uniform mass, fixed at the ends and subject to its own weight only. This means that analytical expression of catenary is the solution of the differential equation providing the equilibrium of a hanging chain (or cable).

Suppose a cable with tension T. Let T_o be the tension in the cable at its lowest point. Let the origin be at this point. The horizontal force on the cable at that point is then T_o. Suppose we isolate a piece of the cable extending from the origin to the point (x, y) where the tension is T. Let ϑ be the tangent angle at (x, y).Then for horizontal equilibrium we have $T_o = T\cos(\vartheta)$.

Let s be the arc length from the origin up to the point (x, y). Let w be the weight of the cable per unit arc length. Then for the vertical equilibrium we have $ws = T\sin(\vartheta)$. Hence

$$\frac{dy}{dx} = \tan(\vartheta) = \frac{T\sin(\vartheta)}{T\cos(\vartheta)} = \frac{ws}{T_0} \tag{28.1}$$

Differentiating, we get

$$\frac{d^2y}{dx^2} = \frac{ws}{T_0}\frac{ds}{dx} = \frac{1}{a}\sqrt{1 + (dy/dx)^2} \tag{28.2}$$

Solution of this equation provides the equation of catenary

$$y(x) = a\cosh(x/a)\ , \tag{28.3}$$

where $a = T_o/ws$ is the *catenary constant.* For increasing values of a, catenary exhibits increasing span.

If the lowest point of the curve is in (x_0, y_0) then the catenary equation becomes

$$y(x) = a\cosh(\frac{x - x_0}{a}) + (y_0 - a) \tag{28.4}$$

If catenary is “*frozen*” and inverted, the chain (or cable) exhibits a compression strength only, which means that it supports itself. In 1671 Robert Hooke announced at the Royal Society in London he had found the shape of the optimal arch. In 1676 he published a book where he stated “*Ut continuum pendet flexible, sic stabit continuum rigidum inversum*,” i.e. as a flexible cable hangs, so, inverting it, a rigid body stands still. In Figure 28.1 we report a catenary and the corresponding inverted catenary. Then the inverted catenary, with the highest point in $(0, h)$, has the following equation

$$y(x) = -a\,(\cosh(x/a) - 1) + h \tag{28.5}$$

Given h and the intersection x with the axis of abscissas, we found the value of a by a numerical method which computes zeros of nonlinear functions.

Hooke did not provide any proof nor the analytical equation of the catenary. His assessment was based on experimental evidence only. However the topic was so interesting that the greatest contemporary mathematicians of that century (Leibniz, Huygens and Bernoulli brothers) studied the catenary curve in details, competing with each other, and succeeded in providing a complete mathematical description

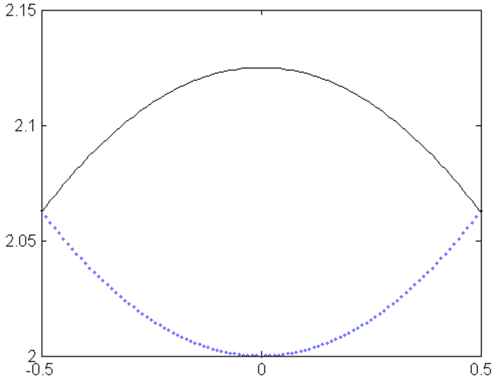

Fig. 28.1 Catenary and inverted catenary

of the properties of this curve related to its static equilibrium. It is worth noticing that it would impossible to get such results by the previous classical approach of mathematics. The new analytical approach allowed mathematics to enlarge its field: the analytical geometry was born. Later in (Gregory, 1697), it was investigated how the catenary form is the real shape of a stable arch, when it can be drawn within its section, since catenary can sustain itself.

Even though the parabola fails to exhibit the structural properties of catenary, the two curves are closely related from a mathematical point of view.

Firstly, from a geometrical point of view, the curve traced on the plane by the focus of a parabola rolling along a straight line, is exactly a catenary.

Then, from the analytical point of view, the equation of a catenary developed in power series provides a polynomial with even terms which provides a parabola, if the series is truncated at the second term. In more details, we have

$$a\cosh(x/a) = a + \frac{1}{2a}x^2 + \frac{1}{24a^3}x^4 + O\left(x^6\right) \tag{28.6}$$

Moreover, we notice that even the polynomial $p(x) = a + \frac{1}{2a}x^2 + \frac{1}{24a^3}x^4$ can be viewed as a parabola when the variable change $t = x^2$ is introduced. In order to enlighten this behavior, in Figure 28.2 the catenary through points $(0,34)$, $(26,0)$ is reported (in this case $a = 13.4496$) together with the approximating parabola $p(x)$ given above and plotted with respect to $\sqrt{t}$. In Figure 28.3 the same catenary is reported together with parabola interpolating at the maximum point and endpoints. In the first case the maximum relative error between the two curves is 3% and in the second case is 7% (in infinity norm). This means that, in some circumstances, a parabola can be a very good approximation of a catenary. So, in practice, given a catenary it is always possible to find a parabola which, point by point, resembles such catenary and its structural properties.

This result supports the hypothesis that a particular parabola was used in any masonry dome built before the 17th century in such a way that the parabolic cross-

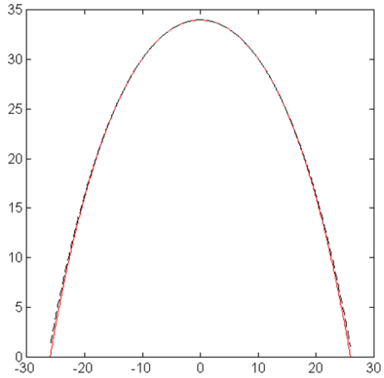

Fig. 28.2 Catenary and approximating parabola

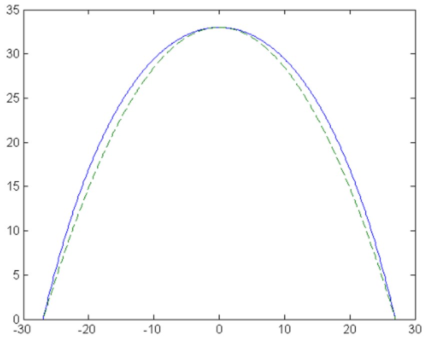

Fig. 28.3 Catenary and interpolating parabola

section of the rotation surface was actually resembling a catenary and its stability properties. Indeed, it is clear that the correct dimensioning of domes and arches was the result of empirical observations over a long period of time, when actually parabolas could be easily computed and built.

Moreover, there is experimental evidence that still in 19th century, catenaries as cross sections of rotation surfaces and vaults were not computed using the analytical approach (reported above) but using an analogical approach by oval curves, as pointed out in (Lluis-i Ginovart et al, 2017), referring to Spanish context.

Sometimes it may be hard to distinguish between a parabola and a catenary. In (Huylebrouck, 2007) a couple of significant examples are reported. In particular, here we present the example referring to Gaudi's Collegio Teresiano, reported in Figure 28.4.

In the cross-sections of those arches we can seen either catenaries or parabolas. As it was reported in Huylebrouck (2007), data are taken so that the top becomes the minimum in $(0, 1)$, for increasing x and y: a least squares fitting provides the (weighted) catenary $y = -0.7468 + 1.75 \cosh(2.8x)$ with $R^2 = 99.988\%$.

Alternatively, the parabola $y = 0.985 + 7.63x^2$ fits with $R^2 = 99.985\%$. The method used in (Huylebrouck, 2007) was a pseudo-inverse algorithm; he concludes that "No naked eye can catch the difference" between catenary and parabola. Actually, the relative error between fitted parabola and catenary is 0.6%.

Here we used a "trust-region-reflective" method by the function `lqscurvefit` provided by MATLAB. We found as a fitting (weighted) catenary $y = -0.9462 + 1.95\cosh(2.6684x)$ and as a fitting parabola $y = 1.0111 - 0.2771x + 8.2194x^2$. Figure 28.5 reports our fitting catenary in solid line, the experimental data by circles and our fitting parabola in dotted line. In both the cases the norm of the residual is equal to 0.4% , as well as the relative error between fitted catenary and parabola.

Again we can conclude that "No naked eye can catch the difference" between catenary and parabola. We remark that a fitting with a classical form of catenary does not provide good results.

Moreover, it is worth noticing that, in spite of appearance, our fitting parabola in practice overlaps the already published parabola in Huylebrouck (2007). This is due to the fact that both parabolas are close to the best approximation.

We remark that a parabola can be distinguished from a catenary by resorting also to geometric properties of parabola itself. In particular, one of these properties states:

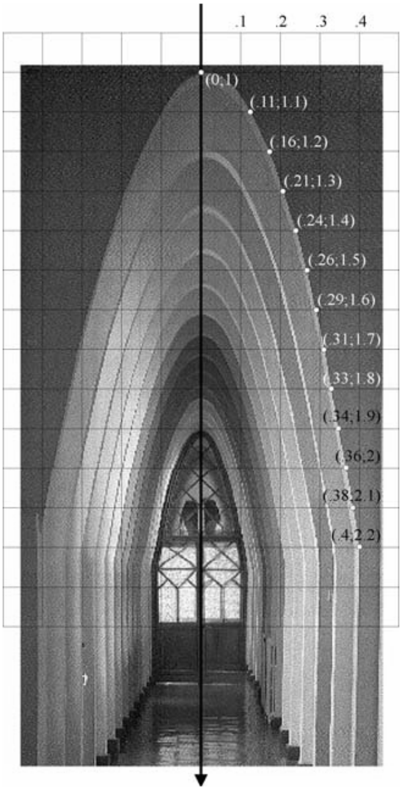

Fig. 28.4 Arches by Gaudi in Collegio Teresiano

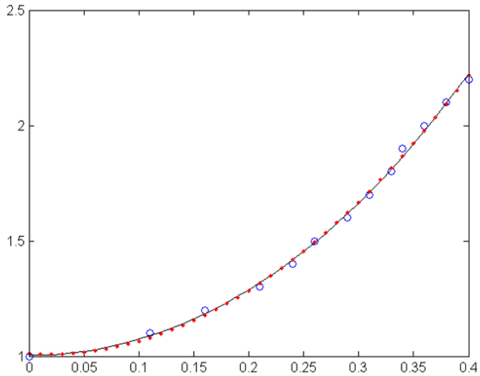

Fig. 28.5 Fitting experimental data by catenary and parabola

if a parabola has several parallel chords, their midpoints all lie on a line which is parallel to the axis of symmetry. Now, consider Gaudi's Paelle Guel as an example: the profile of the gate has to be considered as a catenary curve, since all the midpoints of parallel chords lie on a curve, which differs from a line parallel to the symmetry axis (e.g. see Ghione, 2009).

28.3 Case Study: Santa Maria del Fiore in Florence

The church of Santa Maria del Fiore was built according to a project by Arnolfo di Cambio, started in 1296 and grew up during a long period in the 14th century. The dome was built between 1420 and 1436 by Filippo Brunelleschi, who never described his method of building.

It is made up by eight membranes, based on an octagon at 55 m from the ground, so any horizontal section is octangular. However, within the thick octagonal dome, we can imagine a dome having circular section at every horizontal level; so safely the imaginary dome can be considered a possible thrust surface for the real structure (Heyman, 1995). The inner diameter of the dome is 45 m, the outer one is 54 m (e.g. see Como, 2017). Indeed the dome is built up by two shells with an inner space large about 1.2 m in between. The maximum height of intrados is 32.2 m and the maximum height of extrados is 35.75 m. The average height is 34 m over a 55 m high drum. These numbers remember the Fibonacci numbers: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ... which are characterized by property that the ratio of two consecutive number tends to the golden ratio (that is about 1.618034) as they increase. But Brunelleschi did not refer to Fibonacci numbers, even though he certainly knew them, because, obviously, he did not use a meter as a unit of measure. Actually, he used the Florentine arm = 0.5836 m. So the inner diameter is 77 arms. The diameter was then divided into 5 parts (each of them 15.4 arms = 8.98 m long); so that centers of the inner two curves of the intrados were found in order to have a ”pointed fifth” arch as a cross-section. For the extrados, instead, it was used a “pointed fourth” arch. Figure 28.6, provided by (Conti, 2014), enlightens the building method of the cross-section of the dome.

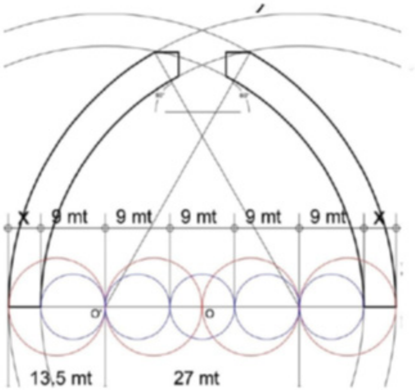

Fig. 28.6 Pointed arches

Here we propose that in the case of Brunelleschi’s dome, between the intrados, built by pointed fifth arch, and the extrados, built by pointed fourth arch, it is always possible to make a catenary run. Focusing on the masonry arch which represents the approximated cross-section of this dome, equilibrium can be visualized using a *thrust line*.

This theoretical line represents the path of the resultants of the compressive forces through the stone structure and has the shape of the inverted catenary discussed above. For a pure compression structure, equilibrium implies a thrust line that lies entirely within the masonry section. In Milankovitch (1907) it is provided an excellent mathematical treatment of this concept which was recently resumed and again studied in deep at MIT (Ma, USA), where new computational interactive equilibrium tools were produced (Block et al, 2006). In Figure 28.7 and Figure 28.8 we present our results about two possible catenaries running between intrados and extrados of Brunelleschi’s dome. Here we do not intend to study the optimal thrust line, but we aim just at presenting how thrust line (i.e. catenary) can be drawn by analytical formulas and approximated by convenient parabolas. Figure 28.7 reports catenary through $(0, 35)$, $(24, 0)$; Figure 28.8 reports catenary through $(0, 33.5)$, $(27, 0)$. In Figure 28.7 an Figure 28.8 catenary is given in solid line, the interpolating parabola through three points (the maximum and the ends points) is given in dashed line, whereas the approximating parabola from the series development is given in dotted line; its equation is

$$y(t) = a + \frac{1}{2a}t + \frac{1}{24a^3}t^2 \tag{28.7}$$

where $t = x^2$.

The maximum relative error between this second parabola and related catenary is 4% in Figure 28.7 and 2% in Figure 28.8. Again this means that we can find a parabola which completely overlaps related catenary. We remark that in Figure 28.7 the interpolating parabola through three points runs outside the wall thickness; instead the approximating parabola $y(t)$ always runs within the wall thickness and resembles the structural property of the catenary, point by point. From a numerical point of view, the approach by the approximating parabola is characterized mainly by two features: i) the computation of the catenary constant a solving numerically a nonlinear equation where a is the unknown; ii) the computation of $y(t)$ by a change of variable $t = x^2$ and then plot y vs $\sqrt{t}$.

Obviously, Brunelleschi did not follow this approach. However the Brunelleschi's choice was really effective even though he did not know catenary jet. He probably built and tested parabolas very close to a convenient unknown catenary and by intuition and experience he found very nice structural stability properties of his cross-section. Over the centuries Brunelleschi's dome substantially maintained a stable configuration (exhibiting a few minor structural problems only) and became a landmark of soundness and beauty.

28.4 Some More Ancient Masonry Domes

28.4.1 St. Peter's Dome in Rome

It was planned by Michelangelo, who worked on the construction of the renewed basilica beginning in 1547. The dome was concluded by Giacomo Della Porta, Michelangelo's disciple, in 1590. The dome has a double shell (following the example

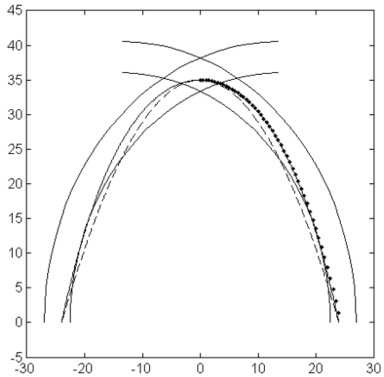

Fig. 28.7 Catenary through $(0,35)$, $(24,0)$

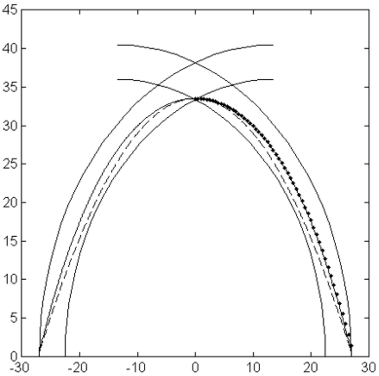

Fig. 28.8 Catenary through $(0, 33.5)$, $(27, 0)$

of more ancient Brunelleschi's dome) with an inner diameter of 42.56 m.; the height from the base to the top is 136.57 m.

As well as Brunelleschi, Michelangelo did not write anything about his project and this was a great disadvantage when by the end of the 17th century the dome started to show a serious chance of collapsing. In 1743 the Pope assigned to Giovanni Poleni the task of studying the structure and solving the problem. At that time Poleni was a famed engineer and mathematician and he knew very well the role of catenary in structure stability; he built small models in scale of the catenary running between the two shells of the dome.

His conclusion was that the shape of the Michelangelo's dome was satisfactory. So the structure was simply strengthened and even now we can admire the efficiency of that action based on the use of catenary as a mathematical model.

This was the very first example of conscious and documented use of catenary in Architecture. Indeed, many original drawings of catenaries referring to St. Peter's dome can be found in (Poleni, 1748).

28.4.2 St. Gaudenzio's Dome in Novara

The latest masonry dome in Italy was the St. Gaudenzio's Dome in Novara, built and designed by A. Antonelli between 1841 and 1878. It has a height from the floor level of the church to its top of 125 m, an internal diameter of 14 m and an external diameter of 22 m. Again we find two shells in the dome structure, but in this case the whole building exhibits a daring complex constructive system, astonishingly light. Unfortunately, the structure experienced many serious stability problems since the beginning of its life.

Nevertheless, investigations conducted on the structure of the dome had shown that the shape of thrust line is perfectly contained within the masonry section of the dome, with minimal variations. In fact, the shape of the arch of the internal cross-section becomes very similar to that of a catenary, with the difference of an average quadratic deviation well below 1% (Corradi et al, 2009).

28.5 Conclusions

Firstly, within the context of mathematics, we investigated numerical and analytical relations between parabola and catenary and we have shown that, given a catenary, it is always possible to find a parabola very close to the catenary which inherits the stability property of catenary, point by point.

Then, within the context of building structures, the Brunelleschi's dome was investigated in some details, using just the mathematical concepts previously provided. At last, some dedicated comments were presented relating St. Peter's dome and San Gaudenzio's dome.

Our results enlighten how good were the ancient builders in managing mathematical concepts, both consciously and intuitively.

References

Block P, DeJong M, Ochsendorf J (2006) As hangs the flexible line: Equilibrium of masonry arches. Nexus Network Journal 8:13–24

Cazzani A, Malagu M, Turco E (2016a) Isogeometric analysis: a powerful numerical tool for the elastic analysis of historical masonry arches. Continuum Mechanics and Termodynamics 28(1-2):139–156

Cazzani A, Malagu M, Turco E, Stochino F (2016b) Constitutive models for strongly curved beams in the frame of isogeometric analysis. Mathematics and Mechanics of Solids 21(2):182–209

Chant C, Goodman D (1999) Pre-Industrial Cities and Technology. Routledge, London

Como M (2017) Statics of Historic Masonry Constructions. Springer

Conti G (2014) La matematica nella cupola di santa maria del fiore a firenze. Itacha:Viaggio nella Scienza 4:5–11

Corradi M, Filemio V, Trenetti M (2009) Antonelli's dome for san gaudenzio: Geometry and statics. Nexus Network Journal 11(2):243–56

Ghione F (2009) Una non parabola: la catenaria. Quaderni di Laboratorio Universitá di Roma

Lluis-i Ginovart J, Costa-Jover A, Coll-Pla S, Lopez-Piquer M (2017) Layout of catenary arches in the spanish enlightenment and modernism. Nexus Network Journal 19:85–99

Gregory D (1697) Catenaria. Philosophical Transactions of the Royal Society 19:637–652

Grillanda N, Chiozzi A, Bondi F, Tralli A, Manconi F, Stochino F, Cazzani A (2019) Numerical insights on the structural assessment of historical masonry stellar vaults: the case of santa maria del monte in cagliari. Continuum Mechanics and Termodynamics pp 1–24

Heyman J (1995) The stone skeleton: structural engineering of masonry architecture. Cambridge University Press

Huylebrouck D (2007) Curve fitting in architecture. Nexus Network Journal 9:59–70

Milankovitch M (1907) Theorie der druckkurven. Zeitschrift fur Mathematik und Physik 55:1–27

Poleni G (1748) Memorie istoriche della gran cupola del tempio vaticano. Stamperia del Seminario di Padova