Chapter 5
Remarks on the Power Series in Quadratic Modules

This Chapter contains the translation of the paper:

M. Sce, *Osservazioni sulle serie di potenze nei moduli quadratici*, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat., (8) **23** (1957), 220–225.

Article by Michele Sce, presented during the meeting of 9 November 1957 by B. Segre, member of the Academy.

In this short paper we consider modules with units which are quadratic, that is, whose elements (with respect to the multiplicative structure induced in the module by their tensor algebra) satisfy a quadratic equation. We show that, in these modules, power series (positive o negative)—if the order of the module is even—are nullsolutions of a power of a generalized laplacian. This fact allows to generalize some results on quaternionic functions of Fueter and his school to Clifford algebras.

1. Let **M** be a module on a field **F** with characteristic not equal 2 and let $1 = i_0, i_1, \dots, i_n$ be a basis. After identifying the unit of **F** with the unit of **M**, we can write the elements in **M** in the form

$$x = x_0 + x_1 i_1 + \cdots + x_n i_n = x_0 + \mathbf{x} \qquad (x_i \in \mathbf{F}).$$

Let **T** be the tensor algebra over **M** and let us assume that for the elements x^2 in **T** one has

$$\mathbf{x}^2 = q(\mathbf{x}) = \sum_{j,k=1}^{n} a_{jk} x_j x_k \tag{5.1}$$

where $q(\mathbf{x})$ denotes a quadratic form on **F**; it follows that $x^2 (\in \mathbf{T})$ is in **M**. Thus **M** is closed with respect to the operation that to the pair x, y associates $\dfrac{xy + yx}{2}$, which gives a Jordan algebra $\mathbf{M}^+$. When one considers the module **M** in **T** equipped

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with the multiplicative structure of $\mathbf{M}^+$, one will say that $\mathbf{M}$ is a quadratic module and will denote it by $\mathbf{M}_q$.

Since, by reducing $q(\mathbf{x})$ to a canonical form, one notices that $\mathbf{M}^+$ is a Jordan algebra, central, simple, of degree 2, then $\mathbf{M}_q$ can be embedded only in algebras $\mathbf{A}$ such that $\mathbf{A}^+$ contains such a Jordan algebra. Among these algebras, those which may be obtained with the Cayley–Dickson process are particularly interesting; these algebras are themselves quadratic modules.[1] If, in addition, $\mathbf{A} \supset \mathbf{M}_q$ is associative, it contains the algebra quotient of $\mathbf{T}$ and of the ideal generated by (5.1); thus the smallest associative algebra containing a quadratic module is a Clifford algebra or an algebra whose semisimple part is a Clifford algebra and whose radical is an algebra with vanishing square—according to the fact that $q(\mathbf{x})$ is degenerate or not.[2]

2. We shall call conjugate of an element $x = x_0 + \mathbf{x}$ in $\mathbf{M}_q$ the element $\bar{x} = x_0 - \mathbf{x}$; it is immediate that

$$x + \bar{x} = 2x_0 = t(x) \qquad \text{(trace of } x\text{)}$$

$$x\bar{x} = x_0^2 - q(\mathbf{x}) = n(x) \qquad \text{(norm of } x\text{)}$$

are in $\mathbf{F}$ and that the elements x in $\mathbf{M}_q$ satisfy the equation in $\mathbf{F}$

$$z^2 - t(x)z + n(x) = 0. \tag{5.2}$$

If x is an element in $\mathbf{M}_q$ with nonzero norm, we can consider in $\mathbf{M}_q$

$$\frac{\bar{x}}{n(x)} \tag{5.3}$$

and verify that it is a solution to the equation $x \cdot y = 1$ in the variable y; moreover, since (5.3) possesses the formal properties of the inverse, we can call it inverse of x and denote it by x^{-1}.

3. Let us set

$$y^2 = \frac{1}{\varepsilon} q(\mathbf{x}) \qquad \text{and so} \qquad n(x) = x_0^2 - \varepsilon y^2$$

where y and ε belong to $\mathbf{F}$ or to one of its extensions $\mathbf{F}^o$; in the sequel, we shall consider $\mathbf{M}_q$ on $\mathbf{F}^o$ and we shall exclude the case y identically equal to zero.

[1] A. A. Albert, *Quadratic forms permitting composition*, Ann. of Math., **43** (1942), 161–177.

[2] C. C. Chevalley, *The algebraic theory of spinors*, New York 1954, Chapter 11, § 1.

We will say that a function $w(x)$ in $\mathbf{M}_q$ is *biholomorphic* if

$$w(x) = u(x_0, y) + \frac{1}{y} v(x_0, y)\mathbf{x} \tag{5.4}$$

where $u(x_0, y)$ and $v(x_0, y)$ are functions of x_0 and y [3] satisfying

$$\frac{\partial u}{\partial x_0} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = \varepsilon \frac{\partial v}{\partial x_0}. \text{ [4]} \tag{5.5}$$

Taking into account that

$$(m-2k)\binom{m}{2k} = (2k+1)\binom{m}{2k+1},$$

it is easy to verify that powers of a biholomorphic function

$$w^m = \left(\sum_{k=0}^{[m/2]} (\varepsilon)^k \binom{m}{2k} u^{m-2k} v^{2k}\right) + \frac{1}{y}\left(\sum_{k=0}^{[m/2]} (\varepsilon)^k \binom{m}{2k+1} u^{m-2k-1} v^{2k+1}\right)\mathbf{x}$$

($[m/2]$ is the integer part of $m/2$) are still biholomorphic functions. Since x and x^{-1} are evidently biholomorphic, it turns out that all linear combinations with constant coefficients of positive or negative powers of a variable are biholomorphic, and the property extends to series if $\mathbf{F}$ is finite or with evaluation.

4. Let us denote by ∂ the operator $i_1 \dfrac{\partial}{\partial x_1} + \cdots + i_n \dfrac{\partial}{\partial x_n}$ and let

$$q^{-1}(\mathbf{x}) = \sum_{j,k=1}^{n} \alpha_{jk} x_j x_k$$

be the quadratic form inverse of $q(\mathbf{x})$. Let us set

$$\Box w = \frac{\partial^2 w}{\partial x_0^2} - q^{-1}(\partial) w, \tag{5.6}$$

and let us show that, if $w_0 = u_0 + \dfrac{1}{y} v_0 \mathbf{x}$ is biholomorphic and n is odd, then:

$$\Box^{(n+1)/2} w_0 = 0. \tag{5.7}$$

[3]Note that $x_0 + y$ and $x_0 - y$ are solutions of (5.2). Thus we can presume that for an extension to cubic modules, etc. it will be more convenient to consider the expressions that appear when solving algebraic equations with the Lagrange method.

[4]Obviously, the derivations are meant as representations which have the usual formal properties.

To simplify the computations we set

$$u_s = \frac{\partial u_{s-1}}{\partial y}\frac{1}{y}, \qquad v_s = \frac{\partial v_{s-1}}{\partial y}\frac{1}{y} - \frac{v_{s-1}}{y^2} = \frac{\partial}{\partial y}\frac{v_{s-1}}{y}$$

$$w_s = u_s + \frac{1}{y}v_s\mathbf{x} \qquad (s = 1, 2, \ldots),$$

and we show that u_s, v_s satisfy the relations

$$\frac{\partial u_s}{\partial x_0} = \frac{\partial v_s}{\partial y} + 2s\frac{v_s}{y}, \qquad \frac{\partial u_s}{\partial y} = \varepsilon\frac{\partial v_s}{\partial x_0}. \tag{5.8}$$

For $s = 0$, (5.8) reduce to (5.5). So, let us suppose that (5.8) hold for $s - 1$; then

$$\frac{\partial u_s}{\partial x_0} = \frac{1}{y}\frac{\partial^2 u_{s-1}}{\partial x_0 \partial y} = \frac{1}{y}\frac{\partial}{\partial y}\left[\frac{\partial v_{s-1}}{\partial y} + 2(s-1)\frac{v_{s-1}}{y}\right] =$$
$$= \frac{1}{y}\frac{\partial}{\partial y}\left[y v_s + (2s-1)\frac{v_{s-1}}{y}\right] = \frac{\partial v_s}{\partial y} + 2s\frac{v_s}{y},$$
$$\frac{\partial u_s}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{y}\frac{\partial u_{s-1}}{\partial y}\right) = \varepsilon\frac{\partial}{\partial y}\left(\frac{1}{y}\frac{\partial v_{s-1}}{\partial x_0}\right) = \varepsilon\frac{\partial v_s}{\partial x_0},$$

which is what we had to prove.

With simple computations we then find

$$\frac{\partial^2 w_s}{\partial x_0^2} = \frac{\partial^2 u_s}{\partial x_0^2} + \frac{1}{y}\frac{\partial^2 v_s}{\partial x_0^2}\mathbf{x} = \frac{1}{\varepsilon}\left[y\frac{\partial u_{s+1}}{\partial y} + (2s+1)u_{s+1}\right] +$$
$$+ \frac{1}{\varepsilon}\left[\frac{\partial v_{s+1}}{\partial y} + (2s+2)\frac{v_{s+1}}{y}\right]\mathbf{x},$$
$$\frac{\partial w_s}{\partial x_j} = \frac{1}{\varepsilon}u_{s+1}\frac{\partial q}{\partial x_j} + \frac{1}{\varepsilon}\frac{v_{s+1}}{y}\frac{\partial q}{\partial x_j}\mathbf{x} + \frac{v_s}{y}i_j$$
$$\frac{\partial^2 w_s}{\partial x_j \partial x_k} = \frac{1}{\varepsilon^2 y}\frac{\partial u_{s+1}}{\partial y}\frac{\partial q}{\partial x_j}\frac{\partial q}{\partial x_k} + \frac{1}{\varepsilon}u_{s+1}a_{jk} - \frac{1}{\varepsilon^2}\frac{v_{s+1}}{y^3}\frac{\partial q}{\partial x_j}\frac{\partial q}{\partial x_k}\mathbf{x} +$$
$$+ \frac{1}{\varepsilon^2 y^2}\frac{\partial v_{s+1}}{\partial y}\frac{\partial q}{\partial x_j}\frac{\partial q}{\partial x_k}\mathbf{x} + \frac{1}{\varepsilon}\frac{v_{s+1}}{y}\left[a_{jk}\mathbf{x} + \frac{\partial q}{\partial x_j}i_k + \frac{\partial q}{\partial x_k}i_j\right].$$

[Editors' Note: the second line of the formula above was $+\frac{1}{\varepsilon}\left[\frac{\partial v_{s+1}}{\partial y} + 2s\frac{v_{s+1}}{y}\right]\mathbf{x}$ in the original text].

Finally

$$\sum_{j,k=1}^{n} \alpha_{jk} \frac{\partial^2 w_s}{\partial x_j \partial x_k} = \frac{1}{\varepsilon}\left[\frac{\partial u_{s+1}}{\partial y} y + n u_{s+1} + \frac{\partial v_{s+1}}{\partial y}\mathbf{x} + (n+1)\frac{v_{s+1}}{y}\mathbf{x}\right]$$

so that, taking into account (5.6),

$$\varepsilon \Box w_s = -(n-2s-1) w_{s+1}. \tag{5.9}$$

[Editors' Note: in the original text at the right hand side there was $(n-2s-1)w_{s+1}$.] Thus, if n is odd, for $s = (n-1)/2$ one has

$$\varepsilon \Box w_{(n-1)/2} = 0, \tag{5.10}$$

namely (5.7).

5. A function w in $\mathbf{M}_q$ will be called JB-*monogenic* (Jordan B-monogenic) if, for $B = \|b_{jk}\|$, $b_{jk} = b_{kj}$, $|B| \neq 0$ ($j, k = 1, \ldots, n$ Editors' note j was i in the original manuscript; recall also that $\|b_{jk}\|$ denotes the matrix with entries b_{ij}), one has

$$\frac{\partial w}{\partial x_0} - \frac{1}{2\varepsilon}\sum_{j,k=1}^{n} b_{jk}\left[\frac{\partial w}{\partial x_j} i_k + i_k \frac{\partial w}{\partial x_j}\right] = 0.$$

As w_0 is biholomorphic, we set

$$S = \frac{1}{2\varepsilon}\sum_{j,k=1}^{n} b_{jk} \frac{\partial q}{\partial x_j} i_k$$

[Editors' Note: in the original text it was $S = \frac{1}{\varepsilon}\sum_{j,k=1}^{n} b_{jk}\frac{\partial q}{\partial x_j} i_k$] and we take into account that

$$2 - \frac{1}{\varepsilon y^2}(S\mathbf{x} + \mathbf{x}S) = \frac{1}{\varepsilon y^2}[\mathbf{x}(\mathbf{x} - S) + (\mathbf{x} - S)\mathbf{x}].$$

Then the condition of JB-monogenicity for w_s can be written as

$$0 = \frac{1}{\varepsilon} u_{s+1}(\mathbf{x} - S) + \frac{\partial v_s}{\partial y}\left[1 - \frac{1}{2\varepsilon y^2}\mathbf{x}S + S\mathbf{x}\right] +$$

$$+ \frac{\partial v_s}{\partial y}\left[2s + \frac{1}{2\varepsilon y^2}(\mathbf{x}S + S\mathbf{x})\right] - \frac{v_s}{y}\frac{1}{\varepsilon}\sum_{j,k=1}^{n} b_{jk} i_j i_k =$$

[Editors' Note: it was $\frac{1}{2\varepsilon^2 y^2}$ in the original text]

$$= \frac{1}{2\varepsilon}[w_{s+1}(\mathbf{x} - S) + (\mathbf{x} - S)w_{s+1}] + \frac{v_s}{y}(2s+1) - \frac{v_s}{y}\frac{1}{\varepsilon}\sum_{jk} b_{jk} a_{jk};$$

and this is satisfied if $s = \dfrac{n-1}{2}$ and if

$$\mathbf{x} = S. \tag{5.11}$$

Thus: *if* (5.11) *holds in* $\mathbf{M}_q$*, the* $(n-1)/2$ *power of* $\Box$ *of all biholomorphic functions is JB-monogenic*.

6. If $\mathbf{M}_q$ is an algebra, or if B is a scalar and w is such that the jacobian matrix $\partial\mathbf{w}/\partial\mathbf{x}$ is symmetric, we can consider in $\mathbf{M}_q$ the equation

$$\frac{\partial w}{\partial x_0} - \frac{1}{\varepsilon}\sum_{j,k=1}^{n} b_{jk} i_j \frac{\partial w}{\partial x_k} = D_B w = 0. \text{ [5]} \tag{5.12}$$

Functions satisfying (5.12) will be called *B-monogenic on the left* (in a similar way one defines functions B-monogenic on the right).

With a computation similar to the one done in n. **5** one can see that, *if* (5.11) *holds, the* $(n-1)/2$ *power of* $\Box$ *of biholomorphic functions is* B*-monogenic on the left and on the right.*

Moreover if $\mathbf{M}_q$ is alternative, multiplying on the left (5.12) by $\bar{D}_B$, the conjugate operator of D_B, one finds that

$$\bar{D}_B D_B w = \left[\frac{\partial^2}{\partial x_0^2} - \frac{1}{\varepsilon^2} g(\partial)\right] w = 0, \tag{5.13}$$

where $g(\mathbf{x})$ is the quadratic form associated with the matrix BAB_{-1}. Thus, *if* B *satisfies the relation*

$$BAB_{-1} = \varepsilon^2 A^{-1}, \quad \text{[6]} \tag{5.14}$$

then (5.13) coincides with (5.6) and we can say that: *Functions* B*-monogenic are solutions of the equation* $\Box w = 0$.

[5]If $\mathbf{M}_q$ is not an algebra, in order that $D_B w$ is in $\mathbf{M}_q$ it is necessary and sufficient that $\partial\mathbf{w}/\partial\mathbf{x}B$ is symmetric; if x is such that the jacobian determinant is always nonzero, this implies that B must be scalar and $\partial\mathbf{w}/\partial\mathbf{x}$ is symmetric.

[6]It is easy to determine the matrices B, provided that one take sinto account that- since B is symmetric - (5.14) can be written as $(BA)^2 = \varepsilon^2 I$. All these relations become then particularly simple in the case of classical quadratic modules, namely for the modules such that

$$f(x) = -(x_1^2 + \cdots + x_n^2).$$

From this and from the preceding result one may reobtain the result of n. **4** (in the present particular case).

7. Let us assume now that **F** is with evaluation, and that the norm $n(x)$ is a definite quadratic form; then in $\mathbf{M}_q$ there are no zero divisors. In a future work, based on the results proved in the preceding sections, we shall show how the theory of quaternionic functions can be extended to functions in the (alternative) algebra of Cayley numbers [Editors' Note: here probably Sce refers to the paper that he eventually wrote with Dentoni, see Chapter 6]; here we will limit ourselves to some considerations on quadratic modules in associative algebras.

With reasonings nowadays classical, we prove first that for B-monogenic functions there is a bilateral integral theorem. Then in the representative space $\mathbf{M}_q$ one can construct an integral formula of Cauchy type, with kernel $\square^{(n-1)/2}$. From this fact one derives the possibility to develop in series,..., etc. Based on the penultimate paragraph of n. **6**, we also get in this way properties of of functions satisfying the (elliptic) equation $\square W = 0$.[7]

Let now $\mathbf{M}_q$ be a quadratic module on the real field, and **A** the smallest associative algebra containing it. The problem to extend to the elements of **A** an integral formula (of Cauchy type), once that it has been found for $\mathbf{M}_q$, is trivial if $\mathbf{M}_q$ is of order 2. On the other hand, the problem is not solvable as soon as the order of $\mathbf{M}_q$ is greater than 4, as it turns out from the classification of Clifford algebras[8] and from some simple considerations on the variety of zero divisors in algebras.[9] Thus it remains to be considered only the case in which $\mathbf{M}_q$ has order 4 (and it is not an algebra); but then **A** is a Clifford algebra of order 8 and one would go back to known results, at least in the classical case.[10]

[7]All the researches on these topics rely on R. Fueter, *Die Funktionentheorie der Differentialgleichungen $\Delta u = 0$ und $\Delta\Delta u = 0$ mit vier reellen Variablen*, Comment. Math. Helv., v. 7 pp. 307–330 (1934-35). Among the works of Fueter's school, those which treats topics near to ours are: W. Nef, *Funktionentheorie einer Klasse von hyperbolischen und ultrahyperbolischen Differentialgleichungen zweiter Ordnung*, ibid. vol. 17, pp. 83–107 (1944-45); H. G. Haefeli, *Hypercomplexe Differentiale*, ibid., vol. 20, pp. 382–420 (1947); A. Kriszten, *Elliptische systeme von partiellen Differentialgleichungen mit konstanten Koeffizienten*, ibid. vol. 23, pp. 243–271 (1949).

[8]See C. C. Chevalley cited in (2). The classification of classical Clifford algebras can already be found in the paper E. Study, E. Cartan, *Nombres complexes*, Encycl. Franc., I, 5, n. 36, pp. 463.

[9]See M. Sce, *Sulla varietàă dei divisori dello zero nelle algebre*, Rend. Lincei, August 1957

[10]G. B. Rizza, *Funzioni regolari nelle algebre di Clifford*, Rend., Roma, v. 15 pp. 53–79 (1956). The integral formulas established in this work hold in general for algebras which are direct sums of quaternions (but not for Clifford algebras of order greater than 8).

5.1 Comments and Historical Remarks

The starting point is a vector space $\mathbf{M}$ on a field $\mathbf{F}$ with characteristic different from 2 and with basis $1 = i_0, i_1, \ldots, i_n$. Following Chevalley [16], one can construct the tensor algebra

$$\mathbf{T} = \oplus_{j=0}^{\infty} \mathbf{M}^{\otimes j}$$

over $\mathbf{M}$ and then assume that $\mathbf{x} \otimes \mathbf{x} = Q(\mathbf{x})$ where

$$Q(\mathbf{x}) = \sum_{j,k=0}^{n} a_{jk} x_j x_k.$$

Note that below we use the symbol q to denote a quaternion, thus here we use Q to denote the quadratic form, although Sce uses q. Below we write $\mathbf{x}^2$ instead of $\mathbf{x} \otimes \mathbf{x}$ and $\mathbf{xy}$ instead of $\mathbf{x} \otimes \mathbf{y}$ and x denotes $x_0 + \mathbf{x}$, $x_0 \in \mathbb{R}$.

The fact that $x^2 \in \mathbf{M}$ implies that $(x + y)^2 \in \mathbf{M}$ and so $\dfrac{xy + yx}{2} \in \mathbf{M}$, thus $\mathbf{M}$ is closed with respect to the operation

$$(x, y) \mapsto x \cdot y = \frac{xy + yx}{2}. \tag{5.15}$$

This multiplicative structure on $\mathbf{M}$ gives in fact a (commutative) Jordan algebra $\mathbf{M}^+$.

Using Sce's terminology, although for us M_Q would be more appropriate, we give the following:

Definition 5.1 We call quadratic module, and we denote it by $\mathbf{M}_q$ the vector space $\mathbf{M}$ in which the multiplicative structure is given by (5.15).

For the sequel it can be useful to keep in mind the references [1, 16, 55, 61, 63, 71, 74, 77] also quoted in the original paper by Sce.

Remark 5.1 As a special case of the previous construction, we can take $\mathbf{F} = \mathbb{R}$ and we can consider, for example, $\mathbf{M} = \mathbb{R}^{n+1}$ identified with the set of paravectors, that is those $x \in \mathbb{R}_n$ that are of the form $x = x_0 + x_1 e_1 + \ldots + x_n e_n$, for $x_0, x_\ell \in \mathbb{R}$, where $e_0 = 1$, and e_ℓ, $\ell = 1, \ldots, n$ are the imaginary units generating a Clifford algebra over $\mathbb{R}$, i.e., $\mathbf{M}$ is the set of paravectors in $\mathbb{R}_n$. If the imaginary units satisfy a nondegenerate bilinear form $B(\cdot, \cdot)$, as in the case of a Clifford algebra, then we can set $Q(\mathbf{x}) = B(\mathbf{x}, \mathbf{x})$ and the construction above corresponds to the construction of a real universal Clifford algebra over n imaginary units. Note that $\mathbf{M}$ can be of signature (p, s), $p + s = n$ namely p units have positive square and s units have negative square. In this case, there exists a basis $e_1^*, \ldots, e_p^*, \ldots, e_n^*$ in which the bilinear form $B(\cdot, \cdot)$ satisfies

1. $B(e_i^*, e_i^*) = 1, i = 1, \dots, p$;
2. $B(e_i^*, e_i^*) = -1, i = p+1, \dots, n = p+s$;
3. $B(e_i^*, e_j^*) = 0, i \neq j$.

The product defined by (5.15) is the classical product of two paravectors in the Clifford algebra.

The Fueter mapping theorem was proved by Fueter in the Mid Thirthies, see [45], and provides an interesting way to generate Cauchy–Fueter regular functions starting from holomorphic functions. The idea is to start with a holomorphic function

$$f_0(u + iv) = \alpha(u, v) + i\beta(u, v)$$

defined in an open set of the upper half complex plane. Given a nonreal quaternion $q = x_0 + \underline{q}$, we define the function

$$f(q) = \alpha(x_0, |\underline{q}|) + \frac{\underline{q}}{|\underline{q}|}\beta(x_0, |\underline{q}|), \tag{5.16}$$

which is called the quaternionic valued function induced by f_0. Fueter's theorem can be stated as follows (the result was also surveyed in [43, 77]):

Theorem 5.1 (Fueter [45]) *Let $f_0(z) = \alpha(u, v) + i\beta(u, v)$ be a holomorphic function defined in a domain (open and connected) D in the upper-half complex plane and let*

$$\Omega_D = \{q = x_0 + ix_1 + jx_2 + kx_3 = x_0 + \underline{q} \ \mid \ (x_0, |\underline{q}|) \in D\}$$

be the open set induced by D in $\mathbb{H}$ and let $f(q)$ be the quaternionic valued function induced by f_0. Then Δf is both left and right Cauchy–Fueter regular in Ω_D, i.e.,

$$\frac{\partial}{\partial \bar{q}}\Delta f(q) = \Delta f(q)\frac{\partial}{\partial \bar{q}} = 0,$$

where Δ is the Laplacian in the four real variables x_ℓ, $\ell = 0, 1, 2, 3$ and $\frac{\partial}{\partial \bar{q}}$ is the Cauchy–Fueter operator.

Almost 20 years later, Sce extended this result in a very pioneering and general way. In the recent literature, Sce's result is known in the following form (see Theorem 5.2 below):

By applying $\Delta^{(n-1)/2}$ (Δ is the Laplacian in $n+1$ real variables) to a function induced on the set of paravectors by a holomorphic function, one obtains a monogenic one with values in the real Clifford algebra $\mathbb{R}_n$ over an odd number n of imaginary units.

In the sequel, we will discuss mainly the implications of the Fueter–Sce construction in the Clifford setting, so we fix here the notation. The imaginary units of the Clifford algebra $\mathbb{R}_n$ will be denoted by e_ℓ, $\ell = 1, \ldots, n$, and we set $e_0 = 1$. The paravectors are elements of the Clifford algebra that are of the form

$$x = x_0 + x_1 e_1 + \ldots + x_n e_n, \quad x_\ell \in \mathbb{R}, \quad \ell = 0, \ldots n,$$

x_0 is the real (or scalar) part of x also denoted by $\mathrm{Re}(x)$, the 1-vector part of x is defined by $\underline{x} = x_1 e_1 + \ldots + x_n e_n$, the conjugate of x is denoted by $\overline{x} = x_0 - \underline{x}$, and the Euclidean modulus of x is given by $|x|^2 = x_0^2 + \ldots + x_n^2$. The sphere of 1-vectors with modulus 1, is defined by

$$\mathbb{S} = \{\underline{x} = e_1 x_1 + \ldots + e_n x_n \mid x_1^2 + \ldots + x_n^2 = 1\}.$$

We can state now the Clifford algebra version of Sce's theorem in the version that is commonly known in the recent literature.

Theorem 5.2 (Sce [75]) *Consider the Euclidean space $\mathbb{R}^{n+1}$ whose elements are identified with paravectors $x = x_0 + \underline{x}$.*

Let $f_0(z) = f_0(u + iv) = \alpha(u, v) + i\beta(u, v)$ be a holomorphic function defined in a domain (open and connected) D in the upper-half complex plane and let

$$\Omega_D = \{x = x_0 + \underline{x} \mid (x_0, |\underline{x}|) \in D\}$$

be the open set induced by D in $\mathbb{H}$ and $f(x)$ be the Clifford-valued function induced by f_0. Then the function

$$\breve{f}(x) := \Delta^{\frac{n-1}{2}} \Big(\alpha(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|}\beta(x_0, |\underline{x}|)\Big)$$

is left and right monogenic.

For the sequel, it is convenient to define the following maps:

$$T_{FS1} : \alpha(u, v) + i\beta(u, v) \mapsto \alpha(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|}\beta(x_0, |\underline{x}|) \tag{5.17}$$

$$T_{FS2} : \alpha(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|}\beta(x_0, |\underline{x}|) \mapsto \Delta^{\frac{n-1}{2}} \Big(\alpha(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|}\beta(x_0, |\underline{x}|)\Big) \tag{5.18}$$

Sce's result requires some remarks, in fact it is broader than Theorem 5.2 from two different points of view: the algebra in which it is proven and the type of functions obtained.

Remark 5.2 As Sce observed, the quadratic module $\mathbf{M}_q$ can be embedded only in algebras of specific form: for example in the Cayley–Dickson algebras, in particular the octonions, and in the particular case of associative algebras, in all Clifford algebras or algebras whose semi-simple part is a Clifford algebra. However, Clifford algebras, of any signature, are only a special case of this construction. A natural question is then to ask what happens in the case one considers a module which is not quadratic, but instead cubic or else. It will be interesting to understand if a "Fueter–Sce mapping theorem" can be constructed in that case, and which operator has to be considered instead of the Laplacian.

Remark 5.3 As we said, Sce's extension of Fueter's result is broader than the one commonly quoted in the literature. In fact, with the above notations, it shows that given a function $f_0(z) = f_0(u + iv) = \alpha(u, v) + i\beta(u, v)$ which is holomorphic or anti-holomorphic, see (4.6), then the function

$$\Delta^{\frac{n-1}{2}} f(x) = \Delta^{\frac{n-1}{2}} \Big(u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v(x_0, |\underline{x}|) \Big) \tag{5.19}$$

is a JB-monogenic function namely it satisfies

$$\frac{\partial f}{\partial x_0} - \frac{1}{2\varepsilon} \sum_{j,k=1}^{n} b_{jk} \left[\frac{\partial f}{\partial x_j} i_k + i_k \frac{\partial f}{\partial x_j} \right] = 0,$$

where the matrix $B = [b_{jk}]$ is is symmetric and nondegenerate. The proof of this result, in the special case when B is scalar and the jacobian matrix of f is symmetric, gives that f is left and right B-monogenic.

Remark 5.4 The case in which one obtains a monogenic function or function in the kernel of the Dirac operator (in the sense of Clifford analysis) is very special and occurs when $B = I$, I being the identity matrix, and $\mathbf{M}_q$ is the set of paravectors in a Clifford algebra. However, the result is proved for an algebra generated by a module with unit and whose elements satisfy a quadratic equation. And again, according to this quadratic equation, the function f_0 satisfies the Cauchy–Riemann equation or a variation of it, see (5.5). Operators of Cauchy–Fueter type in which there are coefficients b_{jk} such that the matrix $B = [b_{jk}]$ is orthogonal have been considered by Shapiro and Vasilevski in [76].

Remark 5.5 About 40 years after Sce, Qian proved in [67] that the theorem of Sce holds in the case of a Clifford algebra over an even number n of imaginary units, using techniques of Fourier multipliers in the space of distributions in order to deal with the fractional powers of the Laplacian. He showed that also in this case, Sce's construction gives a monogenic function. After this paper there have been a number of generalizations and the interested reader may find more information in the survey [69] but see also the papers [38, 64–66, 73]. Qian also gave an interesting application of Fueter–Sce's theorem, see [68], to prove boundedness of singular integral operators.

Remark 5.6 It is also interesting to note that the language of stem functions and induced functions, later used by Sce in his paper on the octonionic case, see Chap. 6, was also previously used by Cullen and Rinehart, see [42, 70]. Also Sudbery in his paper [78] points out that Cullen used functions of the form (5.16) to define an alternative theory of functions of a quaternion variable. The concepts of stem functions, intrinsic and induced functions are relevant in the theory of functions nowadays called slice hyperholomorphic (also called slice regular when they are quaternionic functions and slice monogenic when they have values in a Clifford algebra). In the theory of slice hyperhomolomorphic functions, the two functions u and v (real-valued in the above discussion) have values in the hypercomplex algebra under consideration. In the language of slice hyperholomorphic functions, functions of the form (5.16) with u, v real are called intrinsic, according to the terminology introduced by Cullen and Rinehart [42, 70].

5.2 The Fueter–Sce Theorem: Function and Spectral Theories

The Fueter–Sce–Qian theorem is one of the most fundamental results in complex and hypercomplex analysis because it shows how to generalize complex analysis to the hypercomplex setting. The fact that the generalization procedure is done in two steps means that there are two function theories in such an extension. When we consider for example quaternionic valued functions, we obtain slice hyperholomorphicity for the quaternions at the first step, and Fueter regular functions at the second step. The other important example is the Clifford algebra valued functions where we obtain slice hyperholomorphicity for Clifford algebra, and monogenic functions, respectively. This fact has important consequences in operator theory, because in both steps of the Fueter–Sce–Qian contruction the two types of hyperholomorphic functions have a Cauchy formula. From the Cauchy formula of slice hyperholomorphic functions one deduces the notion of S-spectrum and, as a consequence, the spectral theory on the S-spectrum, while on the Cauchy formula of Fueter regular functions or monogenic functions one deduces the notion of monogenic spectrum and the related spectral theory. In this section we show how the two function theories are related, how they induce the associated spectral theories and the connections between them.

It is important to observe that quaternionic quantum mechanics was the main motivation to search for the S-spectrum but hypercomplex analysis has given the tools to identify this spectrum. In fact, in 1936 Birkhoff and von Neumann, see [13], showed that quantum mechanics can be formulated over the real, the complex and the quaternionic numbers. Since then, several papers and books treated this topic, however it is interesting, and somewhat surprising, that an appropriate notion of spectrum for quaternionic linear operators was not present in the literature. The way in which the so-called S-spectrum and the S-functional calculus were discovered in

2006 by Colombo and Sabadini is well explained in the introduction of the book [41], where it is shown how hypercomplex analysis methods allow to identify the notion of S-spectrum of a quaternionic linear operator which, from the physical point of view, seemed to be ineffable.

Before the works of the Italian mathematicians on slice hyperholomorphic functions, this function theory was simply seen an intermediate step in the Fueter–Sce–Qian's construction. These functions have various analogies with the theory of functions of one complex variable, but also crucial differences which make them very interesting. Moreover, they opened the way in the understanding of the spectral theories in the quaternionic and in the Clifford settings.

The literature on hyperholomorphic functions and related spectral theories is nowadays very large, so we mention only some monographs and the references therein. For the function theory of slice hyperholomorphic functions the main references are the books [33, 40, 46, 49], while for the spectral theory on the S-spectrum we mention the books [10, 20, 33, 41]. For the more classic quaternionic and monogenic function theory we refer to the books [14, 25, 44, 52, 54, 72], and for the monogenic spectral theory and applications we suggest the interested reader to consult [58].

It is also worthwhile to mention that also Schur analysis has been considered in the slice hyperholomorphic setting, see the book [8] and in the references therein. Schur analysis in the Fueter setting and related topics have been treated, for example, in the papers [2–4].

The Fueter–Sce Mapping Theorem and Function Theories

In the title of this section and below we will often refer to the Fueter–Sce–Qian mapping theorem as to Fueter–Sce mapping theorem because, for the sake of simplicity, the case of the fractional Laplacian considered by Qian will not be treated.

We start by discussing the recent research area of slice hyperholomorphic functions. The construction of Fueter is carried out for functions defined on open sets of the upper half complex plane but it can be generalized to the whole complex plane. Consider a stem function

$$f_0(z) = \alpha(u, v) + i\beta(u, v), \quad z = u + iv$$

defined in a set $D \subseteq \mathbb{C}$, symmetric with respect to the real axis, and set

$$f(x) = f(u + Iv) = \alpha(u, v) + I\beta(u, v), \tag{5.20}$$

where I is an element in the sphere $\mathbb{S}$ of purely imaginary quaternions or 1-vectors in the case of a Clifford algebra and x is either a quaternion or a paravector. This function is well defined if

$$\alpha(u,-v)=\alpha(u,v) \quad \text{and} \quad \beta(u,-v)=-\beta(u,v)$$

namely if α and β are, respectively, even and odd functions in the variable v. Additionally the pair (α,β) satisfies the Cauchy–Riemann system. This fact was already understood by Sce, see Chap. 6, no. 4, but was not taken into account until the work of Qian [68].

The theory of functions of the form (5.20) was somewhat abandoned until 2006 when Gentili and Struppa introduced in [48] the following definition:

Definition 5.2 Let U be an open set in $\mathbb{H}$ and let $f : U \to \mathbb{H}$ be real differentiable. The function f is said to be (left) slice regular or (left) slice hyperholomorphic in U if for every $I \in \mathbb{S}$, its restriction f_I to the complex plane $\mathbb{C}_I = \mathbb{R} + I\mathbb{R}$ passing through origin and containing I and 1 satisfies

$$\overline{\partial}_I f(u+Iv) := \frac{1}{2}\left(\frac{\partial}{\partial u} + I\frac{\partial}{\partial v}\right) f_I(u+Iv) = 0,$$

on $U \cap \mathbb{C}_I$.

Analogously, a function is said to be right slice regular (or right slice hyperholomorphic) in U if

$$(f_I\overline{\partial}_I)(u+Iv) := \frac{1}{2}\left(\frac{\partial}{\partial u} f_I(u+Iv) + \frac{\partial}{\partial v} f_I(u+Iv)I\right) = 0,$$

on $U \cap \mathbb{C}_I$, every $I \in \mathbb{S}$.

Further developments of the theory of slice regular functions were discussed also in [28] and the above definition was extended by Colombo, Sabadini and Struppa, in [27], (see also [21, 29, 30]) to the Clifford algebra setting for functions $f : U \to \mathbb{R}_n$, defined on an open set U contained in $\mathbb{R}^{n+1}$, where $\mathbb{R}_n$ is the Clifford Algebra over n imaginary units. Slice regular functions according to Definition 5.2 and their generalization to the Clifford algebra, called slice monogenic functions, possess good properties on specific open sets that are called axially symmetric slice domains. When it is not necessary to distinguish between the quaternionic case and the Clifford algebra case we call these functions slice hyperholomorphic.

On these domains, slice hyperholomorphic functions satisfy an important formula, called Representation Formula or Structure Formula, which allows to compute the values of the function once that we know its values on a complex plane $\mathbb{C}_I$.

Definition 5.3 Let $U \subseteq \mathbb{H}$ (or $U \subseteq \mathbb{R}^{n+1}$). We say that U is axially symmetric if, for every $u+Iv \in U$, all the elements $u+Jv$ for $J \in \mathbb{S}$ are contained in U. We say that U is a *slice domain* if $U \cap \mathbb{C}_I \neq \emptyset$ and $U \cap \mathbb{R}$ is a domain in $\mathbb{C}_I$ for every $I \in \mathbb{S}$.

The link with functions of the form (5.16) or (5.20) is provided by the Representation Formula or Structure Formula:

Theorem 5.3 *Let $f : U \to \mathbb{R}_n$ be a slice hyperholomorphic function defined on an axially symmetric slice domain $U \subseteq \mathbb{R}^{n+1}$. Let $J \in \mathbb{S}$ and let $x \pm Jy \in U \cap \mathbb{C}_J$. Then the following equality holds for all $x = u + Iv \in U$:*

$$\begin{aligned} f(u+Iv) &= \frac{1}{2}\Big[f(u+Iv) + f(u-Iv)\Big] + I\frac{1}{2}\Big[J[f(u-Iv) - f(u+Iv)]\Big] \\ &= \frac{1}{2}(1-IJ)f(u+Iv) + \frac{1}{2}(1+IJ)f(u-Iv). \end{aligned} \tag{5.21}$$

Moreover, for all $u + Kv \subseteq U$, $K \in \mathbb{S}$, there exist two functions α, β, independent of I, such that for any $K \in \mathbb{S}$ we have

$$\frac{1}{2}\Big[f(u+Kv)+f(u-Kv)\Big] = \alpha(u,v), \qquad \frac{1}{2}\Big[K[f(u-Kv)-f(u+Kv)]\Big] = \beta(u,v). \tag{5.22}$$

As a consequence we immediately have:

Corollary 5.1 *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric slice domain, let $D \subseteq \mathbb{R}^2$ be such that $u + Iv \in U$ whenever $(u, v) \in D$ and let $f : U \to \mathbb{R}_n$. The function f is slice hyperholomorphic if and only if there exist two differentiable functions $\alpha, \beta : D \subseteq \mathbb{R}^2 \to \mathbb{H}$, satisfying*

$$\alpha(u,v) = \alpha(u,-v), \quad \beta(u,v) = -\beta(u,-v)$$

and the Cauchy–Riemann system

$$\begin{cases} \partial_u \alpha - \partial_v \beta = 0 \\ \partial_u \beta + \partial_v \alpha = 0, \end{cases} \tag{5.23}$$

such that

$$f(u+Iv) = \alpha(u,v) + I\beta(u,v). \tag{5.24}$$

Thus, slice hyperholomorphic functions according to Definition 5.2 or the analogous definition for slice monogenic functions are in fact functions of the form (5.20) only on axially symmetric slice domains. However, if one defines a function to be slice hyperholomorphic if it is of the form (5.20) where α, β satisfy the above condition, one has that these functions are defined on axially symmetric open sets, not necessarily slice domains.

Thus, starting with functions of the form (5.20), called slice functions, has the advantage that they are defined on more general sets, moreover one can weaken the requests on the two functions α, β requiring, e.g. only continuity, or differentiability or to be of class $\mathscr{C}^k$, thus giving rise to the class of continuous or differentiable or $\mathscr{C}^k$ slice functions.

The class of slice functions can be considered over real alternative algebras, as done by Ghiloni and Perotti in [50, 51]. The idea of considering functions with values in an algebra more general than quaternions is the one followed by Sce in the paper translated in this Chapter. Although in his paper α, β are real valued, it is clear that his discussion involving the Laplacian, which is a real operator, extends to α, β with values in an algebra.

It is also possible to define slice hyperholomorphic functions, as functions in the kernel of the first order linear differential operator (introduced in [36])

$$Gf = \Big(|\underline{x}|^2 \frac{\partial}{\partial x_0} + \underline{x} \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j}\Big) f = 0,$$

where $\underline{x} = x_1 e_1 + \ldots + x_n e_n$. While, another way to introduce slice hyperholomorphicity, done by Laville and Ramadanoff in the paper [56], is inspired by the Fueter–Sce mapping theorem. They introduce the so called holomorphic Cliffordian functions defined by the differential equation $D\Delta^m f = 0$ over $\mathbb{R}^{2m+1}$, where D is the Dirac operator. Observe that the definition via the global operator G requires less regularity of the functions with respect to the definition in [56].

Here and in the following we will dedicate less attention to monogenic functions because they are very well known since long time. They are functions $f : U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$, with suitable regularity, that are in the kernel of the Dirac operator. Contrary to the monogenic case, slice hyperholomorphic functions can be defined in different ways, as shown above, not always equivalent, and also for this reason they require more comments.

Inversion of the Fueter–Sce–Qian Mapping Theorem

The inverse of the Fueter–Sce–Qian mapping can be obtained in at least two different ways. The first approach that has been introduced in the paper [32] is based on the Cauchy formula of monogenic functions and leads to an integral formula for the inverse Fueter–Sce–Qian mapping. In what follows we give some hints of the solution of the inversion problem because it is interesting to see how a partial differential equation is solved using methods of hypercomplex analysis. A second method to study the inverse of the Fueter–Sce–Qian mapping is based on the Radon and dual Radon transform. We will not presented this method here, but we refer the interested reader to the paper [39] for more details.

The Fueter–Sce–Qian mapping has range in the subset of monogenic functions given by the subclass of those functions which are axially monogenic. In simple words if U is an axially symmetric open set in $\mathbb{R}^{n+1}$ a left axially monogenic function on the open set U is a function of the form

$$F(x) = A(x_0, r) + IB(x_0, r)$$

where $x = x_0 + Ir$, $r = |\underline{x}| \neq 0$, $I = \underline{x}/|\underline{x}|$, and such that the functions $A = A(x_0, r)$ and $B = B(x_0, r)$ satisfy the Vekua's system, i.e.

$$\begin{cases} \partial_{x_0} A(x_0, r) - \partial_r B(x_0, r) = \frac{n-1}{r} B(x_0, r), \\ \partial_{x_0} B(x_0, r) + \partial_r A(x_0, r) = 0. \end{cases}$$

Thus, given an axially monogenic function F, we construct a Fueter–Sce primitive of F, namely a function f such that

$$\Delta^{\frac{n-1}{2}} f(x) = F(x).$$

This problem has been solved in [32] in the case n is odd and in [11] in the case of any $n \in \mathbb{N}$. It is interesting to observe that for the solution of this problem it is enough to construct a Fueter–Sce primitive of suitable functions constructed via the Cauchy kernel for monogenic functions. Precisely, we consider the Cauchy kernel of monogenic functions

$$\mathscr{G}(x) = \frac{1}{A_{n+1}} \frac{\overline{x}}{|x|^{n+1}}, \qquad x \in \mathbb{R}^{n+1} \setminus \{0\}, \tag{5.25}$$

where

$$A_{n+1} = \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})}.$$

and we define the kernels

$$\mathscr{N}_n^+(x) = \int_{\mathbb{S}} \mathscr{G}(x - J)\, dS(J), \qquad \mathscr{N}_n^-(x) = \int_{\mathbb{S}} \mathscr{G}(x - J)\, J\, dS(J),$$

where $\mathbb{S}$ is the unit $(n-1)$-dimensional sphere in $\mathbb{R}^{n+1}$, while $dS(J)$ is a scalar element of area of $\mathbb{S}$. The two functions $\mathscr{N}_n^{\pm}(x)$ are axially monogenic and their Fueter–Sce primitives, obviously not unique, can be obtained as the monogenic extension of the two functions:

$$\mathscr{W}_n^+(x_0) := \frac{\mathscr{C}_n}{\mathscr{K}_n}\, D^{-(n-1)} \frac{x_0}{(x_0^2 + 1)^{(n+1)/2}},$$

$$\mathscr{W}_n^-(x_0) := -\frac{\mathscr{C}_n}{\mathscr{K}_n}\, D^{-(n-1)} \frac{1}{(x_0^2 + 1)^{(n+1)/2}},$$

where the symbol $D^{-(n-1)}$ stands for the $(n-1)$ integrations with respect to x_0 and $\mathscr{C}_n$ and $\mathscr{K}_n$ are given constant that can be calculated explicitly. Then we used an extension lemma based on properties of the solutions of the Dirac equation, so the Fueter–Sce primitives $\mathscr{W}_n^{\pm}(x)$ are obtained by $\mathscr{W}_n^{\pm}(x_0)$ replacing x_0 by $x = x_0 + x_1e_1 + \ldots x_ne_n$. For example, in the case $n = 3$, we have

$$\mathscr{W}_3^{+}(x) = \frac{1}{2\pi} \arctan x, \qquad \mathscr{W}_3^{-}(x) = -\frac{1}{2\pi} x \arctan x.$$

So we can state the inverse Fueter–Sce mapping theorem:

Theorem 5.4 *Let us consider an axially monogenic function*

$$F(x) = A(x_0, r) + JB(x_0, r)$$

defined on an axially symmetric domain $U \subseteq \mathbb{R}^{n+1}$. Let Γ be the boundary of an open bounded subset $\mathscr{V}$ of the half plane $\mathbb{R} + J\mathbb{R}^+$ and let

$$V = \{x = u + Jv,\ (u, v) \in \mathscr{V},\ J \in \mathbb{S}\} \subset U.$$

Moreover suppose that Γ is a regular curve whose parametric equations $y_0 = y_0(s)$, $\rho = \rho(s)$ are expressed in terms of the arc-length $s \in [0, L]$, $L > 0$. Then, the function

$$f(x) = \int_\Gamma \mathscr{W}_n^{-}\Big(\frac{1}{\rho}(x - y_0)\Big)\rho^{n-2}(dy_0\, A(y_0, \rho) - d\rho\, B(y_0, \rho)) \\ - \int_\Gamma \mathscr{W}_n^{+}\Big(\frac{1}{\rho}(x - y_0)\Big)\rho^{n-2}(dy_0 B(y_0, \rho) - d\rho A(y_0, \rho)) \tag{5.26}$$

is a Fueter–Sce primitive of $F(x)$ on V, where $\mathscr{W}_n^{\pm}(x)$ are the Fueter–Sce primitive of $\mathscr{N}_n^{\pm}(x)$.

The proof of this result is rather involved and, in the general case, it requires Fourier multipliers in order to give meaning to fractional powers of the Laplacian. As the Fueter–Sce mapping theorem, also its inversion can be proved in various framework. It was proved for axially monogenic functions of degree k in [35] for n odd, and in the general case in [12]. The case of polyaxially monogenic functions seems to be more complicated and, at the moment, only the biaxial case has been considered in [37].

The Fueter–Sce Mapping Theorem and Spectral Theories

One of the most important motivations for the study hyperholomorphic functions theories is that they induce spectral theories through their Cauchy formulas.

In fact, in quaternionic operator theory a precise notion of spectrum for quaternionic linear operator was missing at least since the paper [13] of G. Birkhoff and J. von Neumann, where they proved that quantum mechanic can be formulated also on quaternionic numbers, but from the operator theory point of view the notion of spectrum of quaternionic linear operators was not made precise. In fact, in all the papers dealing with quaternionic quantum mechanic the notion of right eigenvalues is used, but as it is well known, a part from the finite dimensional case, the right eigenvalues alone are insufficient to construct a quaternionic spectral theory.

It was only in 2006 that, using techniques based solely on slice hyperholomorphic functions, the precise notion of spectrum of a quaternionic linear operator was identified. This spectrum was called the S-spectrum and since then the literature in quaternionic spectral theory has rapidly grown, see [41] for more information. Later in 2015 (and published in 2016) it was proved also the spectral theorem for quaternionic normal operators based on the S-spectrum, see [6, 7] and perturbation results of quaternionic normal operators can be found in [15]. Beyond the spectral theorem there are more recent developments in the direction of the characteristic operator functions, see [10] and the theory of spectral operators developed in [47].

The quaternionic Riesz–Dunford functional calculus based on the S-spectrum, called S-functional calculus (see for example [5, 22]), was extended also to the case of n-tuples of noncommuting operators using the notion of S-spectrum and the theory of slice monogenic functions, see [26] and the book [33].

An important extension of the S-functional calculus to unbounded sectorial operators is the H^∞-functional calculus which is one of the ways to define functions of unbounded operators. The H^∞-functional calculus has been used to define fractional powers of quaternionic linear operators that define fractional Fourier laws for nonhomogeneous material in the theory of heat propagation. For the original contributions see [9, 18, 19]. For a systematic and recent treatment of quaternionic spectral theory on the S-spectrum and the fractional diffusion problems based on these techniques, see the books [20, 41]. Moreover, in the monograph [33] one can find also the foundations of the spectral theory on the S-spectrum for n-tuples of noncommuting operators.

Below, we summarize in the following some of the applications and research directions of the hyperholomorphic function theories and relative spectral theories, induced by the two steps of the Fueter–Sce construction.

1. The *first step* generates *slice hyperholomorphic functions* and the *spectral theory of the S-spectrum*. Among the applications we mention:

- The mathematical tools for quaternionic quantum mechanics, related to the Spectral Theorem based on the S-spectrum.
- New classes of fractional diffusion problems that are based on the definition of the fractional powers of vector linear operators.
- The characteristic operator functions and applications to linear system theory.
- Quaternionic spectral operators, which allow to consider a class of nonself-adjoint problems.
- Spectral theory of Dirac operators on manifolds in the nonself-adjoint case.

2. The *second step* in the Fueter–Sce construction generates *Fueter regular or monogenic functions* and the *spectral theory on the monogenic spectrum*, and some of the applications are:

- Boundary value problems treated with quaternionic techniques, see the book of Gürlebeck and Sprössig [53] and the references therein.
- Quaternionic approach to div-rot systems of partial differential equations, see [34].
- Harmonic analysis in higher dimension, see the work of McIntosh, Qian, and many others [57, 59, 60, 62, 68].

For operator theory the most appropriate definition of slice hyperholomorphic functions is the one that comes from the Fueter–Sce mapping theorem because it allows to assume that the functions are defined only on axially symmetric open sets. The definition below generalizes Fueter's construction from open sets in the upper half complex plane to more general open sets.

Definition 5.4 Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric open set and let $\mathcal{U} \subseteq \mathbb{R}\times\mathbb{R}$ be such that $x = u+Jv \in U$ for all $(u, v) \in \mathcal{U}$. We say that a function $f : U \to \mathbb{R}_n$ of the form

$$f(x) = \alpha(u, v) + J\beta(u, v)$$

is left slice hyperholomorphic if α, β are $\mathbb{R}_n$-valued differentiable functions such that

$$\alpha(u, v) = \alpha(u, -v), \quad \beta(u, v) = -\beta(u, -v) \quad \text{for all} \ \ (u, v) \in \mathcal{U}$$

and if α and β satisfy the Cauchy–Riemann system

$$\partial_u\alpha - \partial_v\beta = 0, \qquad \partial_v\alpha + \partial_u\beta = 0.$$

It is called right slice hyperholomorphic when f is of the form

$$f(x) = \alpha(u, v) + \beta(u, v)J$$

and α, β satisfy the above conditions.

Since we will restrict just to left slice hyperholomorphic function on U we introduce the symbol $SH(U)$ to denote them.

Theorem 5.5 *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric open set such that $\partial(U \cap \mathbb{C}_I)$ is union of a finite number of continuously differentiable Jordan curves, for every $I \in \mathbb{S}$. Let f be an $\mathbb{R}_n$-valued slice hyperholomorphic function on an open set* *containing* $\overline{U}$ *and, for any* $I \in \mathbb{S}$*, we set* $ds_I = -Ids$*. Then, for every* $x \in U$*, we have:*

$$f(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} S_L^{-1}(s, x) ds_I f(s), \tag{5.27}$$

where

$$S_L^{-1}(s, x) = -(x^2 - 2\text{Re}(s)x + |s|^2)^{-1}(x - \overline{s}) \tag{5.28}$$

and the value of the integral (5.27) depends neither on U *nor on the imaginary unit* $I \in \mathbb{S}$.

It turns out that the kernel $S_L^{-1}(s, x)$ is slice hyperholomorphic in x and right slice hyperholomorphic in s for x, s such that $x^2 - 2\text{Re}(s)x + |s|^2 \neq 0$.

Denoting by $\mathscr{O}(D)$ the set of holomorphic functions on D, by $N(\Omega_D)$ the set of induced functions on Ω_D (which turn out to be intrinsic slice hyperholomorphic functions) and by $AM(\Omega_D)$ the set of axially monogenic functions on Ω_D the Fueter–Sce construction can be visualized by the diagram:

$$\mathscr{O}(D) \xrightarrow{T_{FS1}} N(\Omega_D) \xrightarrow{T_{FS2}=\Delta \ \ (or\ T_{FS2}=\Delta^{(n-1)/2})} AM(\Omega_D),$$

where T_{FS1} denotes the first operator of the Fueter–Sce construction and T_{FS2} the second one, see (5.17) and (5.18). The Fueter–Sce mapping theorem induces two spectral theories according to the classe of functions that we consider. The Cauchy formula of slice hyperholomorphic functions allows to define the notion of S-spectrum, while the Cauchy formula for monogenic functions induces the notion of monogenic spectrum, as illustrated by the diagram:

$$\begin{array}{ccc}
SH(U) & \xrightarrow{T_{FS2}} & AM(U) \\
{\scriptstyle Slice\ Cauchy\ formula}\downarrow & & {\scriptstyle Monogenic\ Cauchy\ formula}\downarrow \\
S-spectrum & & monogenic\ spectrum \\
\downarrow & & \downarrow \\
S-functional\ calculus & & monogenic\ functional\ calculus \\
\downarrow & & \downarrow \\
H^\infty - functional\ calculus & & H^\infty - monogenic\ functional\ calculus
\end{array}$$

In the above diagram we have replaced the set of intrinsic functions N by the larger set of slice hyperholomorphic functions SH. This is clearly possible because the map T_{FS2} is the Laplace operator or its powers.

Let us consider a Banach space V over $\mathbb{R}$ with norm $\|\cdot\|$. It is possible to endow V with an operation of multiplication by elements of $\mathbb{R}_n$ which gives a two-sided module over $\mathbb{R}_n$ and by V_n we indicate the two-sided Banach module over $\mathbb{R}_n$ given by $V \otimes \mathbb{R}_n$. Our aim is to construct a functional calculus for n-tuples of not necessarily commuting operators using slice hyperholomorphic functions. So we consider the so called paravector operator

$$T = T_0 + \sum_{j=1}^{n} e_j T_j,$$

where $T_\mu \in B(V)$ for $\mu = 0, 1, ..., n$, and where $B(V)$ is the space of all bounded $\mathbb{R}$-linear operators acting on V.

The notion of S-spectrum follows from the Cauchy formula of slice hyperholomorphic functions and from some not trivial considerations on the fact that we can replace in the Cauchy kernel $S_L^{-1}(s, x)$ the paravector x by the paravector operator T also in the case the components $(T_0, T_1, ..., T_n)$ of T do not commute among themselves. We have the following definition.

Definition 5.5 (S-Spectrum) Let $T \in B(V_n)$ be a paravector operator. We define the S-spectrum $\sigma_S(T)$ of T as:

$$\sigma_S(T) = \{s \in \mathbb{R}^{n+1} \ : \ T^2 - 2\,\mathrm{Re}\,(s)T + |s|^2 I \quad \text{is not invertible in } B(V_n)\}$$

where I denotes the identity operator. It's complement

$$\rho_S(T) = \mathbb{R}^{n+1} \setminus \sigma_S(T)$$

is called the S-resolvent set.

Definition 5.6 Let $T \in B(V_n)$ be a paravector operator and $s \in \rho_S(T)$. We define the left S-resolvent operator as

$$S_L^{-1}(s, T) := -(T^2 - 2\mathrm{Re}\,(s)T + |s|^2 I)^{-1}(T - \bar{s}I). \tag{5.29}$$

A similar definition can be given for the right resolvent operator.

Definition 5.7 We denote by $SH_{\sigma_S(T)}$ the set of slice hyperholomorphic functions defined on the axially symmetric set U that contains the S-spectrum of T.

A crucial result for the definition of the S-functional calculus is that integral

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s, T)\, ds_I\, f(s), \qquad \text{for} \qquad f \in SH_{\sigma_S(T)} \tag{5.30}$$

depends neither on U nor on the imaginary unit $I \in \mathbb{S}$, so the S-functional calculus turns out to be well defined.

Definition 5.8 (*S*-Functional Calculus) Let $T \in B(V_n)$ and let $U \subset \mathbb{H}$ be as above. We set $ds_I = -I ds$ and we define the S-functional calculus as

$$f(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} S_L^{-1}(s, T)\, ds_I\, f(s), \qquad \text{for} \qquad f \in SH_{\sigma_S(T)}. \tag{5.31}$$

The definition of the S-functional calculus is one of the most important results in noncommutative spectral theory.

Here we will not enter into the details of the monogenic functional calculus, we just point out that the starting point for its definition is the monogenic Cauchy formula and the fact that one has to give meaning to the monogenic Cauchy kernel (5.25)

$$\mathscr{G}(s-x) = \frac{1}{A_{n+1}} \frac{\overline{s-x}}{|s-x|^{n+1}}$$

when we replace the paravector x by the paravector operator T. In this case there are major differences with respect to the slice hyperholomorphic Cauchy kernel when the components $(T_0, T_1, ..., T_n)$ of T do not commute among themselves. Moreover, the operators $T_\mu : V \to V$, $\mu = 1, ...n$, must have real spectrum when considered as linear operators on the real Banach space V and we have to set $T_0 = 0$.

Since we are discussing the consequences of the Fueter–Sce theorem in the next subsection we will show how we can use an integral version of this theorem to define the F-functional calculus which is a version of the monogenic functional calculus for n-tuples of commuting operators but it is based on the S-spectrum.

The Fueter–Sce Theorem in Integral Form and the F-Functional Calculus

The Fueter–Sce mapping theorem in integral form and the F-functional calculus where introduced in [31] and further investigated in [17, 23, 24].

We now show how the Fueter–Sce mapping theorem provides an alternative way to define the functional calculus based on monogenic functions. The main idea is to apply the Fueter–Sce operator T_{FS2} to the slice hyperholomorphic Cauchy kernel as illustrated by the diagram:

$$\begin{array}{ccc} SH(U) & & AM(U) \\ \downarrow & & \\ Slice\ Cauchy\ formula & \xrightarrow{T_{FS2}} & Fueter\ Sce\ integral\ form \\ \downarrow & & \downarrow \\ S-Functional\ calculus & & F-functional\ calculus \end{array}$$

This procedure generates an integral transform, called the Fueter–Sce mapping theorem in integral form, that allows to define the so called F-functional calculus. This calculus uses slice hyperholomorphic functions and the commutative version of the S-spectrum, but defines a monogenic functional calculus. We just give an idea of how this works. We point out that the operator T_{FS2} has a kernel and one has to pay attention to this fact with the definition of the F-functional calculus, more details are given in [41]. Then, one has to observe that one can apply the powers of Laplacian to both sides of (5.27) obtaining:

$$\Delta^h f(x) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} \Delta^h S_L^{-1}(s,x) ds_I f(s)$$

which amounts to compute the powers of the Laplacian applied to the Cauchy kernel $S_L^{-1}(s,x)$. In general, it is not easy to compute $\Delta^h f$ and when we apply Δ^h to the Cauchy kernel written in the form (5.28), we do not get a simple formula. However, $S_L^{-1}(s,x)$ can be written in two equivalent ways as follows.

Proposition 5.1 *Let x, $s \in \mathbb{R}^{n+1}$ (or in $\mathbb{H}$ in the quaternionic case) be such that $x^2 - 2x\mathrm{Re}(s) + |s|^2 \neq 0$. Then the following identity holds:*

$$\begin{aligned} S_L^{-1}(s,x) &= -(x^2 - 2x\mathrm{Re}(s) + |s|^2)^{-1}(x - \overline{s}) \\ &= (s - \bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-1}. \end{aligned} \tag{5.32}$$

If we use the second expression for the Cauchy kernel we find a very simple expression for $\Delta^h S_L^{-1}(s,x)$.

Theorem 5.6 *Let x, $s \in \mathbb{R}^{n+1}$ be such that $x^2 - 2x\mathrm{Re}(s) + |s|^2 \neq 0$. Let*

$$S_L^{-1}(s,x) = (s - \bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-1}$$

be the slice monogenic Cauchy kernel and let $\Delta = \sum_{i=0}^{n} \frac{\partial^2}{\partial x_i^2}$ be the Laplace operator in the variable $x = x_0 + \sum_{i=1}^{n} x_i e_i$. Then, for $h \geq 1$, we have:

$$\Delta^h S_L^{-1}(s,x) = C_{n,h}\ (s - \bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-(h+1)}, \tag{5.33}$$

where

$$C_{n,h} := (-1)^h \prod_{\ell=1}^{h}(2\ell) \prod_{\ell=1}^{h}(n - (2\ell - 1)).$$

The function $\Delta^h S^{-1}(s,x)$ is slice hyperholomorphic in s for any $h \in \mathbb{N}$ but is monogenic in x only if and only if $h = (n+1)/2$, namely if and only if h equals the Sce exponent. We define the kernel

$$\mathscr{F}_L(s,x) := \Delta^{\frac{n-1}{2}} S_L^{-1}(s,x)$$
$$= \gamma_n (s-\bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-\frac{n+1}{2}},$$

where

$$\gamma_n := (-1)^{(n-1)/2} 2^{(n-1)/2} (n-1)! \Big(\frac{n-1}{2}\Big)!$$

which can be used to obtain the Fueter–Sce mapping theorem in integral form.

Theorem 5.7 *Let n be an odd number. Let f be a slice hyperholomorphic function defined in an open set that contains $\overline{U}$, where U is a bounded axially symmetric open set. Suppose that the boundary of $U \cap \mathbb{C}_I$ consists of a finite number of rectifiable Jordan curves for any $I \in \mathbb{S}$. Then, if $x \in U$, the function $\breve{f}(x)$, given by*

$$\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$$

is monogenic and it admits the integral representation

$$\breve{f}(x) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} \mathscr{F}_L(s,x) ds_I f(s), \quad ds_I = ds/I, \tag{5.34}$$

where the integral depends neither on U nor on the imaginary unit $I \in \mathbb{S}$.

Using the Fueter–Sce mapping theorem in integral form (5.34), one can define a functional calculus for monogenic functions $\breve{f} = \Delta^{\frac{n-1}{2}} f$ using slice hyperholomorphic functions and the S-spectrum. The F-functional calculus is based on (5.34) and it is a monogenic functional calculus in the spirit of the functional calculus based on the monogenic spectrum introduced by McIntosh (see the book of B. Jefferies [58]).

In the sequel, we will consider bounded paravector operators T, with commuting components $T_\ell \in B(V)$ for $\ell = 0, 1, \dots, n$. Such subset of $B(V_n)$ will be denoted by $BC^{0,1}(V_n)$. The F-functional calculus is based on the commutative version of the S-spectrum (often called F-spectrum in the literature). So we define the F-resolvent operators.

Definition 5.9 (F-Resolvent Operators) Let n be an odd number and let $T \in BC^{0,1}(V_n)$. For $s \in \rho_S(T)$ we define the left F-resolvent operator by

$$F_L(s,T) := \gamma_n (sI - \overline{T})(s^2 - (T+\overline{T})s + T\overline{T})^{-\frac{n+1}{2}}, \tag{5.35}$$

where the operator $\overline{T}$ is defined by

$$\overline{T} = -T_1 e_1 - \cdots - T_n e_n$$

the constants γ_n are given above.

Definition 5.10 (The F-Functional Calculus for Bounded Operators) Let n be an odd number, let $T \in BC^{0,1}(V_n)$ be such that $T = T_1 e_1 + \cdots + T_n e_n$, assume that the operators $T_\ell : V \to V$, $\ell = 1, .., n$ have real spectrum and set $ds_I = ds/I$, for $I \in \mathbb{S}$. Let $SH_{\sigma_S(T)}$ and U be as in Definition 5.7. We define

$$\breve{f}(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} F_L(s, T)\, ds_I\, f(s). \tag{5.36}$$

The definition of the F-functional calculus is well posed since the integrals in (5.36) depends neither on U and nor on the imaginary unit $I \in \mathbb{S}$.

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