Chapter 3
On Systems of Partial Differential Equations Related to Real Algebras

This chapter contains the translation of the paper:

M. Sce, *Sui sistemi di equazioni differenziali a derivate parziali inerenti alle algebre reali*, (Italian) Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 32–38.

Article by Michele Sce, presented during the meeting of 11 December 1954 by B. Segre, member of the Academy.

In this Note, after some preliminaries in algebra and analysis, we classify systems of partial differential equations which give monogenicity conditions in algebras. These systems are elliptic for primitive algebras, parabolic with algebras with radical; the proof of this latter fact is based on a characterization, maybe unknown, of semisimple algebras via their determinant. Among hyperbolic systems, we highlight the one obtained from a regular algebra, and which has as characteristic hypersurface the cone of zero divisors of the algebra itself.

From these systems we deduce some partial differential equations of order equal to the order of the algebra which are of the same type of those satisfied by all monogenic functions. I hope that a further study of these equations would eventually lead me to solve, at least for monogenic functions in regular algebras, some problems analogous to the Cauchy problem.

1. Let $\mathscr{A}$ be an associative algebra over the real, with unit, and let $U = (u_1, \ldots, u_n)$ be a basis. The algebras of matrices $\mathscr{A}'$ ($\mathscr{A}''$) which gives the first (second) regular representation of $\mathscr{A}$ are such that for every x in $\mathscr{A}$ and X' (X'') in $\mathscr{A}'$ ($\mathscr{A}''$) one has

$$x u_{-1} = X'_{-1} u_{-1} \qquad (u x = u X''_{-1}), \tag{3.1}$$

the determinant of X', (X'') is also called *left (right) determinant* of x.[1]

The element $y = y_1 u_1 + \cdots + y_n u_n = \eta u_{-1}$ in $\mathscr{A}$ is said to be left (right) monogenic function of $x = x_1 u_1 + \cdots + x_n u_n$ if y_i are functions derivable with respect to x_k and such that

$$\sum_{i,k} \frac{\partial y_i}{\partial x_k} u_k u_i = 0 \qquad \left(\sum_{i,k} \frac{\partial y_i}{\partial x_k} u_i u_k = 0 \right) .[2]$$

By setting

$$u_k u_i = \sum_j c_{ki}^j u_j, \tag{3.2}$$

one gets a system of n linear differential equations of the first order:

$$\sum_{i,k} \frac{\partial y_i}{\partial x_k} c_{ki}^j = 0 \qquad (j = 1, 2, \ldots, n), \tag{3.3}$$

whose coefficients are the constants of multiplication in the algebra. By setting [Editors' Note: $\|a_{ij}\|$ denotes the matrix with entries a_{ij}]

$$\Omega = \left\| \sum_k c_{ki}^j \frac{\partial}{\partial x_k} \right\| = \|\gamma_{ij}\| \qquad (i, j = 1, 2, \ldots, n),$$

(3.3) can be written in a more compact form as:

$$\Omega \eta_{-1} = 0; \tag{3.4}$$

the determinant $|C|$ of the matrix

$$C = \left\| \sum_k c_{ki}^j z_k \right\| = \sum_k C_k z_k, \tag{3.5}$$

$C_k = \|c_{ki}^j\|$ (for $i, j = 1, \ldots, n$), obtained from Ω by substituting z_k instead of $\frac{\partial}{\partial x_k}$, is called *characteristic form* of the system. The system itself is called *elliptic* or *parabolic* if the *characteristic equation*

$$|C| = f(z_1, \ldots, z_n) = 0 \tag{3.6}$$

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[1] See G. Scorza, *Corpi numerici e algebre*, Messina (1921), Part II, n. 184 and 185.

[2] See B. Segre, *Forme differenziali e loro integrali*, (Roma, 1951), Ch. IV, n. 90. The indices of the infinite sums, unless otherwise stated, always run from 1 to n. Later we will always refer to left monogenicity, since analogous results can be obtained in a similar way for right monogenicity.

does not have real solutions different from the trivial solution $(0, \ldots, 0)$ or, by means of a change of variables, it can be reduced to an equation depending on less than n variables;[3] in the other cases the system is called *hyperbolic* and, in particular, *totally hyperbolic* if the matrix C is diagonalizable and its characteristic polynomial has real roots whatever are the z_k's.[4]

Finally, a hypersurface $\varphi(x_1, \ldots, x_n) = 0$ in the Euclidean space of n-tuples $(x_1, \ldots, x_n)$ is called *characteristic* if the directional cosines of its normal

$$\frac{\partial \varphi_1}{\partial x_1}, \ldots, \frac{\partial \varphi_n}{\partial x_n}$$

satisfy the characteristic equation (3.6).[5]

2. Let us multiply the matrix (3.5) on the right by the n-vector u_{-1}; in force of (3.2), one has:

$$Cu_{-1} = \begin{pmatrix} \sum_{j,k} c_{k1}^j z_k u_j \\ \vdots \\ \sum_{j,k} c_{kn}^j z_k u_j \end{pmatrix} = \begin{pmatrix} \sum_k z_k u_k u_1 \\ \vdots \\ \sum_k z_k u_k u_n \end{pmatrix}$$

and, setting $z = z_1 u_1 + \cdots + z_n u_n$, we deduce

$$z u_{-1} = C u_{-1}.$$

A comparison with (3.1) shows that C_{-1} is the matrix corresponding to z in the first regular representation of $\mathscr{A}$;[6] thus *the characteristic form of system* (3.3) *coincides with the left determinant of an element in* $\mathscr{A}$.

Since the determinants of elements in $\mathscr{A}$ are invariant with respect to change of basis in the algebra, one has also that *the characteristic form of the system expressing the monogenicity is invariant with respect to changes of basis in the algebra.*[7]

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[3] See R. Courant, D. Hilbert, *Methoden der Mathematischen Physik*, Band II (Berlin, 1937), Kap. III, § 4, n.2.

[4] A definition equivalent to ours can be found, for systems of quasi-linear equations, in R. Courant, K. O. Friedrichs, *Supersonic flow and shock waves*, Interscience Publishers, Inc., New York, N. Y., 1948, Chapt. II, n. 32; sometimes, as in R. Courant, P. Lax, *On nonlinear partial differential equations with two independent variables*, Comm. Pure Appl. Math., **2** (1949), 255–273, pp. 255–273, n.2, totally hyperbolic systems are called *hyperbolic*.

[5] See Courant-Friedrichs, cited in [(4)].

[6] In particular, Ω_{-1} corresponds to $\omega = \sum_k \frac{\partial}{\partial x_k} u_k$ considered as an element in $\mathscr{A}$ and the transpose of the C_k's in (3.5) correspond to the units in $\mathscr{A}$.

[7] On the contrary, monogenicity conditions depend on the basis of the algebra; see M. Sce, *Monogeneità e totale derivabilità nelle agebre reali e complesse*, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat., (8) **16** (1954), 30–35, Nota I, n. 1.

Primitive algebras have no zero divisors and so the determinants of their nonzero elements are always nonzero; and conversely. Based on the preceding arguments and on n. 1, this is equivalent to claim that *system* (3.3) *is elliptic for primitive algebras and only for those.*

3. We will prove that *the system* (3.3) *is parabolic for algebras not semi-simple and only for them*, by proving that *an algebra is semi-simple if and only if, with respect to any basis, the determinant of the elements depends on all their coordinates*.

Let us assume the $\mathscr{A}$ is semi-simple and that the determinant of the elements does not depend on all their coordinates. $\mathscr{A}$ is direct sum of simple algebras $\mathscr{A}_i$ and so the determinant of the elements in $\mathscr{A}$ is the product of the determinants of the elements in $\mathscr{A}_i$; thus some algebra $\mathscr{A}_i$ is such that the determinant of its elements $x = \sum_i x_i u_i$, $y = \sum_i y_i u_i$ does not depend, for example, on the coefficients of the units $u_m, \ldots, u_n$ but it depends on the coefficients of all the other units. Then also the determinant of

$$xy = \sum_{i,k=1}^{m-1} x_i y_k u_i u_k + \sum_{i=1,\ldots n, k=m,\ldots n}^{m-1} (x_i y_k u_i u_k + x_k y_i u_k u_i) + \sum_{i=m}^{n} x_i y_i u_i^2$$

does not depend on $x_m, \ldots, x_n$; $y_m, \ldots, y_n$; it turns out that all the products of units appearing in the second sum can be expressible as linear combinations of $u_m, \ldots, u_n$, namely

$$u_i u_k = \sum_{j=m}^{n} c_{ik}^{j} u_j \qquad (i = 1, \ldots, n;\ k = m, \ldots, n) \qquad (3.7)$$

must hold. The relations (3.7) allow to say that the set having basis $u_m, \ldots, u_n$ is a proper ideal of $\mathscr{A}_i$; this contradicts the assumption that $\mathscr{A}_i$ is simple and shows the necessity of the condition.

Let now $\mathscr{A}$ an algebra not semi-simple, that is, it has a nonzero radical $\mathscr{R}$. Let $U_1, \ldots, U_m, \ldots, U_n, (1 < m < n)$ be a basis for $\mathscr{A}'$ such that $U_m, \ldots, U_n$ is a basis for the image $\mathscr{R}'$ of $\mathscr{R}$ in the first regular representation of $\mathscr{A}$; then the trace of any element $X' = \sum_i x_i U_i$ in $\mathscr{A}'$ does not depend on $x_m, \ldots, x_n$ since the matrices $U_m, \ldots, U_n$—which are elements of $\mathscr{R}'$ are nilpotent. Since

$$X'^2 = \sum_{i,k=1}^{m-1} x_i x_k U_i U_k + \sum_{i=1,\ldots,n,\ k=m,\ldots,n} x_i x_k (U_i U_k + U_k U_i) - \sum_{i=m}^{n} x_i^2 u_i^2,$$

only the coefficients of the first sum—among which $x_m, \ldots, x_n$ do not appear—can give a nonzero contribution to the trace of X'^2 in fact $\mathscr{R}'$ is an ideal of $\mathscr{A}'$ and so a relation similar to (3.7) holds. Reasoning in this way we can prove that the traces of $X', X'^2, \ldots, X'^n$ do not depend on $x_m, \ldots, x_n$. Using the recurrence formulas—

which can be easily obtained[8]—expressing the coefficients of the characteristic equation of a matrix via the traces of a matrix and its powers, one gets that each coefficient of the characteristic equation of X' does not depend on $x_m, \ldots, x_n$. This is true, in particular, for the determinant of x and so the theorem is proved.

4. *The system* (3.3) *is completely hyperbolic for the algebras of n-real numbers and only for them.*

Let $u_1, \ldots, u_n$, $u_i^2 = u_i$, $u_i u_k = 0$, $(i \neq k)$, be the basis of n-real numbers; the matrix (3.5) turns out to be real and diagonal, so that the system (3.3) is totally hyperbolic.

Conversely, let us assume that the system (3.3) is totally hyperbolic, that is, C is diagonalizable and its characteristic roots are real. If C_i is diagonal and all the z_k except z_i, z_j are zero, the matrix $z_i C_i + z_j C_j$ is diagonal only if C_j is diagonal; it follows that, when i and j vary, a matrix which reduces a C_k to the diagonal form must reduce to the diagonal form all the other C_k. Thus the matrices C_k are pairwise commuting;[9] and since their transpose correspond to the units of $\mathscr{A}$ in the first regular representation, $\mathscr{A}$ is commutative. On the other hand, since the system (3.3) is not parabolic, $\mathscr{A}$ is semi-simple (n. **3**); a semi-simple algebra which is commutative is direct sum of the real field and of the algebra of complex numbers, and both can be repeated a certain number of times.[10] However, if $\mathscr{A}$ would have as a component the algebra of complex numbers, some C_k would have complex characteristic roots; thus $\mathscr{A}$ cannot be anything but the direct sum of the real field taken n times.

5. Let $e_{11}, e_{12}, \ldots, e_{nn}$ with $e_{ij} e_{jk} = e_{ik}$, $e_{ij} e_{lk} = 0$, $(i \neq l)$ the basis of a regular algebra $\mathscr{M}_n$ of order n^2; the matrices of order n^2 elements of $\mathscr{M}_n'$ are direct sums of n matrices equal to $\|x_{ik}\|$, of order n, in $\mathscr{M}_n$. Thus $\varphi(x_{ik}) = 0$ is a characteristic hypersurface of system (3.3) if:

$$\frac{\partial \varphi}{\partial x_{ik}} = 0. \tag{3.8}$$

Since $\dfrac{\partial |x_{ik}|}{\partial x_{ik}}$ is the adjoint of X_{ik} in $\|x_{ik}\|$, one has

$$\left| \frac{\partial |x_{ik}|}{\partial x_{ik}} \right| = |x_{ik}|^{n-1}$$

and the cone $|x_{ik}| = 0$ of zero divisors of $\mathscr{M}_n$ satisfies (3.8); thus *the cone of zero divisors of a regular algebra $\mathscr{M}_n$ counted $n - 1$ times, is a characteristic hypersurface of system* (3.3).

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[8]When a matrix is in canonic form, these formulas reduce to those of symmetric functions.

[9]See M. Sce, *Su alcune proprietà delle matrici permutabili e diagonalizzabili*, Rivista di Parma, vol. 1, (1950), pp. 363–374, n.5.

[10]See Scorza cited in $^{(1)}$, part II, n. 292.

Let us now consider the algebra $\mathscr{A}$ of order $2n^2$ direct product of the algebra of complex numbers and the regular algebra $\mathscr{M}_n$. The elements of $\mathscr{A}'$ with respect to $e_{ik}, i e_{ik}$ are matrices of order $2n^2$ direct sums of n matrices of order $2n$,

$$X = \begin{pmatrix} A & -B \\ B & A \end{pmatrix},$$

where A, B are arbitrary elements of $\mathscr{M}_n$. Denoting again by $X_{j\ell}$ the adjoint of $x_{j\ell}$ (where $j, \ell = 1, 2, \ldots, 2n$, a simple computation shows that—for $i, k = 1, \ldots, n$— one has:

$$\left| \frac{\partial |x_{j\ell}|}{\partial x_{j\ell}} \right| = \left| \begin{matrix} X_{ik} + X_{n+i,n+k} & X_{i,n+k} - X_{n+i,k} \\ X_{n+i,k} + X_{i,n+k} & X_{n+i,n+k} + X_{i,k} \end{matrix} \right| = |X_{j\ell}|;$$

thus the equation $|x_{j\ell}|^{2n^2-1} = 0$—in the complex case—represents a characteristic hypersurface of system (3.3). However, it should be noted that, in the real field, $|x_{j\ell}| = 0$ represents a cone (with vertex at the origin) of dimension $2(n^2 - 1)$ which certainly does not give a characteristic hypersurface.

Analogous remarks can be made in the case of an algebra direct product of the algebra of quaternions with a regular algebra.

6. Let $A = \|a_{ik}\|$ be a matrix of order n whose elements belong to an integral domain $\mathscr{D}$ with unit, but which is not a principal ideal ring. If A has maximal rank, its first column cannot be zero; thus, multiplying A on the left by a suitable matrix, we can assume that the element $(1, 1)$ is nonzero. Let us still denote by A the matrix obtained in this way, and let us multiply it by the nonsingular matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & \ldots \\ -a_{21} & a_{11} & 0 & \ldots \\ -a_{31} & 0 & a_{11} & \ldots \\ \vdots & \vdots & \vdots & \ldots \end{pmatrix};$$

one obtains

$$PA = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} & \ldots \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} & \ldots \\ \vdots & \vdots & \vdots & \ldots \end{pmatrix} = \begin{pmatrix} a_{11} & a \\ 0 & A_1 \end{pmatrix}. \tag{3.9}$$

Since also the first column of A_1 must be nonzero, we can act on A_1 as we did on A; iterating the procedure, we show that *every nonsingular matrix in $\mathscr{D}$ can be reduced in triangular form T, multiplying it on the left by a suitable nonsingular matrix.*

As it can be seen from (3.9), the first two elements on the principal diagonal of T are two minors of order 1 and 2 respectively, the first one contained in the second; reasoning by induction one finds that—by selecting a sequence of nonzero minors $\alpha_1, \alpha_2, \ldots, \alpha_n = |A|$ of all the orders from 1 to n, each of which contained in the following (complete chain)—*the elements on the principal diagonal of T are*

$$\alpha_1, \ \alpha_2, \ \alpha_1\alpha_3, \ \alpha_1^2\alpha_2\alpha_4, \ \ldots, \ \alpha_1^{n-2}\alpha_2^{n-3}\cdots\alpha_{n-2}\alpha_n.^{11} \tag{3.10}$$

The degree of the i-th element in the sequence (3.10) in the elements of A is clearly $\sum_{k=1}^{i-2} k2^{i-2-k} + i$; since

$$\sum_{k=i-1}^{\infty} k2^{i-2-k} = i,$$

we can conclude that *the i-th element of the sequence (3.10) is of degree 2^{i-1} in the elements of A.*

Let us consider the extension $\mathscr{F}$ of the field of the real numbers by means of the operators $\dfrac{\partial}{\partial x_1}, \ldots, \dfrac{\partial}{\partial x_n}$; defining formally, as usual, the operations of sums and product, $\mathscr{F}$ turns out to be a ring with unit. The set of operators $\mathscr{F}$ has no zero divisors; thus, if we assume that all the functions to which we apply elements of $\mathscr{F}$ have finite derivatives, continuous up to the order m, we can say that all the elements of order not greater than m behave like elements in an integral domain.

Thus, if we suppose that the y_i elements of η in (3.4) possess finite derivatives, continuous up to order 2^{n-1}, we can apply to the matrix Ω the considerations made above, this gives for the element y_n of η a partial differential equation of order at most 2^{n-1}. An analogous result can be obtained for the other y_i; but, in general, the equations obtained for the various y_i are different. To obtain a differential equation satisfied by all the y_i's, one needs to multiply between them the n complete chains, assuming that the operators which are their elements commute (namely, that the functions have derivatives finite and continuous up to the order of the equation);

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[11] Obviously for particular matrices one can obtain much more; for example, if

$$A = \begin{pmatrix} A' & A''' \\ -A''_{-1} & A'' \end{pmatrix}$$

is a symmetric matrix of the fourth order one has:

$$\begin{pmatrix} A'J & -A''J \\ -A''_{-1}J & A'J \end{pmatrix} \begin{pmatrix} A' & A'' \\ -A''_{-1} & A''' \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}, \text{ with } J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ D = \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}$$

where d is the pfaffian of A. Another example is given by the elemnts in the algebra which is the first regular representation of quaternions; these matrices multiplied by the transposed give the scalar matrix $|q|I_4$ (where $|q|$ is the norm of the quaternion q).

taking into account that the *n*-chains have all the same last element, which does not have to be necessarily repeated in the product, one has that *the single components of a monogenic function satisfy a same partial differential equation of order at most* $n(2^{n-1} - (n-1))$.

7. The procedure illustrated in the preceding section, is maybe of some interest; however, an equation satisfied by all the y_i's can be easily be obtained by multiplying (3.4) on the left by the matrix Ω^* adjoint of Ω. So we have that *the components of a monogenic function satisfy an equation of order at most n.*[12]

The characteristic form of the equation coincides with the characteristic form of the system; thus *the equations satisfied by the components of the monogenic functions are of the same kind of system* (3.3) *and the characteristic hypersurfaces of the system are the same of those of the equation.*

Since, as we have seen in n. **5**, the elements of the first regular representation of an algebra simple of order kn^2 are composed by *n* matrices and Ω, as we observed in the note 6, can be considered an element of the first regular representation, one has that *the single components of a monogenic function in a simple algebra of order* kn^2 *satisfy an equation of order at most kn*.

It is easy to extend the result to semisimple algebras.

3.1 Comments and Historical Remarks

In the paper translated in this chapter, Sce discusses the problem of characterising an algebra according to the properties of the system satisfied by the monogenic functions in that algebra see [8]. In his papers, he always has a special taste for algebraic questions, see also his paper [9], and in fact inspired by this problem, he also proves a new property of semi-simple algebras. Moreover, he also shows that each function, which is a component of a monogenic function, satisfies a suitable system of differential equations with order equal to the order of the algebra. Also this paper is an interesting combination of properties exquisitely algebraic in nature and analytical properties of functions.

Most of the properties of the algebras considered in this chapter are given in Chap. 2. The reader may refer to the books of Albert [1, 2] and of Scorza [10] which were also used by Sce.

Remark 3.1 In the paper we consider in this chapter, Sce makes use of the Italian term *"algebra primitiva"*, i.e., *primitive algebra*, that we keep in the translation.

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[12]Sometimes, by multiplying Ω on the left by matrices different from Ω^*, we obtain equations of lesser order; for example, in the case of quaternions multiplying Ω by its transpose one obtains the scalar matrix ΔI_n (Δ is the laplacian in four variables).

In more modern terms, one should translate the term as *division algebra*. In fact, as one can read in various old sources, see e.g. [11–13], a primitive algebra is an algebra which does not contain zero divisors. Such an algebra is semi-simple and also simple.

Another important definition is the following:

Definition 3.1 Let $\mathscr{A}_1, \ldots, \mathscr{A}_m$ be algebras over a field F. We say that $\mathscr{A}$ is the direct sum of $\mathscr{A}_1, \ldots, \mathscr{A}_m$ and we write

$$\mathscr{A} = \mathscr{A}_1 \oplus \ldots \oplus \mathscr{A}_m$$

if $v_i \, v_j = 0$ for $v_i \in \mathscr{A}_i$, $i \neq j$ and the order of $\mathscr{A}$ is the sum of the orders of $\mathscr{A}_i$, $i = 1, \ldots, m$.

Example 3.1 An instance of algebra which is a direct sum used in this chapter is the algebra of n-real numbers $\mathbb{R}^n = \mathbb{R} \oplus \cdots \oplus \mathbb{R}$ in which all the various copies of $\mathbb{R}$ are generated by u_i (the unit of $\mathbb{R}$) for $i = 1, \ldots, n$, and are such that $u_i u_j = 0$ when $i \neq j$.

An algebra is said to be decomposable if it can be written as sum of its (nontrivial) subalgebras, indecomposable if this is not possible.
These notions about the algebras are useful to characterise the systems of differential equations associated with the various notions of monogenicity. To provide concrete examples, we consider the monogenicity conditions in Chap. 1, for increasing dimension of the algebras considered.

Example 3.2 Let us start by considering second order algebras. We begin with the real algebra of complex numbers, which is clearly a division algebra. Then

$$C = \begin{pmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{pmatrix},$$

so that $|C| = x_1^2 + x_2^2$ and the system is, as it is well known, elliptic.
Again in dimension 2, we can consider the algebra of dual numbers and the left monogenicity condition, see (2.32), which are associated with the matrix

$$C = \begin{pmatrix} x_1 & 0 \\ x_2 & x_1 \end{pmatrix},$$

so that $|C| = x_1^2$ and the system is parabolic, which is consistent with the fact that the algebra of dual numbers is not semi-simple.

Example 3.3 In the case of hyperbolic numbers, the monogenicity is expressed by the matrix

$$C = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix}.$$

We have that $|C| = x_1^2 - x_2^2$. Moreover, the matrix is real symmetric with eigenvalues $x_1 \pm x_2$, thus the system is completely hyperbolic. According to the result proven in n. **4** the algebra of hyperbolic numbers is, up to a suitable isomorphism, the algebra $\mathbb{R}^2$. To see this fact, let us set

$$\mathsf{e} = \frac{u_1 + u_2}{2}, \quad \mathsf{e}^\dagger = \frac{u_1 - u_2}{2}.$$

Then $\mathsf{e}, \mathsf{e}^\dagger$ are two idempotents such that $\mathsf{e}\,\mathsf{e}^\dagger = \mathsf{e}^\dagger\mathsf{e} = 0$. The change of coordinates

$$x_1 = \frac{1}{2}(z_1 + z_2)$$

$$x_2 = \frac{1}{2}(z_1 - z_2)$$

allows to write $x = x_1 u_1 + x_2 u_2 = z_1 \mathsf{e} + z_2 \mathsf{e}^\dagger = z$. At this point we can identify the element $z = z_1 \mathsf{e} + z_2 \mathsf{e}^\dagger$ with the pair $(z_1, z_2) \in \mathbb{R}^2$. We note that given two elements $z = z_1 \mathsf{e} + z_2 \mathsf{e}^\dagger$, $z' = z_1' \mathsf{e} + z_2' \mathsf{e}^\dagger$ their sum and product are given by

$$z + z' = (z_1 + z_1')\mathsf{e} + (z_2 + z_2')\mathsf{e}^\dagger, \qquad zz' = (z_1 z_1')\mathsf{e} + (z_2 z_2')\mathsf{e}^\dagger$$

which, at level of pairs, corresponds to the sum and multiplication componentwise. In this new basis, the monogenicity condition (left or right) of a function $w = w(z)$ rewrites as

$$(\mathsf{e}\ \ \mathsf{e}^\dagger) \begin{pmatrix} \dfrac{\partial w_1}{\partial z_1} & \dfrac{\partial w_1}{\partial z_2} \\ \dfrac{\partial w_2}{\partial z_1} & \dfrac{\partial w_2}{\partial z_2} \end{pmatrix} \begin{pmatrix} \mathsf{e} \\ \mathsf{e}^\dagger \end{pmatrix} = 0,$$

which leads to

$$\frac{\partial w_1}{\partial z_1}\mathsf{e} + \frac{\partial w_2}{\partial z_2}\mathsf{e}^\dagger = 0,$$

that is

$$\frac{\partial w_1}{\partial z_1} = \frac{\partial w_2}{\partial z_2} = 0.$$

Example 3.4 As an example of third order algebra, we consider the ternions. The matrix C is in this case

$$C = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ x_3 & 0 & x_2 \end{pmatrix},$$

and $|C| = x_1 x_2^2$. The matrix is lower triangular, with eigenvalues equal to the diagonal elements. Easy arguments show that C can be diagonalized over the real and so the system is completely hyperbolic. We note that by multiplying Ω by its adjoint Ω^* we get that the components y_ℓ, $\ell = 1, 2, 3$ of a monogenic function $y_1 e_1 + y_2 e_2 + y_3 e_3$ satisfy the equation

$$\frac{\partial^3}{\partial x_1 \partial^2 x_2} y_\ell = 0, \qquad \ell = 1, 2, 3.$$

Example 3.5 Let us now consider two cases of four dimensional algebras. First we look at the case of the algebra of quaternions which is a division algebra. Thus we expect an elliptic system. And in fact the left monogenicity condition is associated with the matrix

$$C = \begin{pmatrix} x_1 & -x_2 & -x_3 & -x_4 \\ x_2 & x_1 & -x_4 & x_3 \\ x_3 & x_4 & x_1 & -x_2 \\ x_4 & -x_3 & x_2 & x_1 \end{pmatrix}.$$

We have that $|C| = (x_1^2 + x_2^2 + x_3^2 + x_4^2)^2$ which vanishes only at $(0, 0, 0, 0)$. In the case of the algebra of bicomplex numbers, we have:

$$C = \begin{pmatrix} x_1 & -x_2 & -x_3 & x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}$$

and

$$|C| = ((x_1 - x_4)^2 + (x_2 + x_3)^2)((x_1 + x_4)^2 + (x_2 - x_3)^2).$$

The determinant $|C|$ can vanish also for $(x_1, x_2, -x_2, x_1)$ and $(x_1, x_2, x_2, -x_1)$ with $x_1 x_2 \neq 0$. Thus the system is parabolic and it is possible to construct a change of basis for which the determinant of new matrix C associated with the monogenicity condition do not depend on all the four variables.

Remark 3.2 The work of Sce discussed in this chapter was, unfortunately, completely forgotten despite its relations with physical problems, see [3–5]. In the works of Krasnov, see e.g. [6, 7], the author discusses various properties of PDEs in algebras. In particular, in section 7.5 of [6] he considers ellipticity properties of the solutions of a generalized Cauchy–Riemann operator, i.e. monogenic functions, according to the type of algebras considered. In particular, it is shown that an

operator is elliptic if and only if the algebra is a division algebra. He also widely discusses the case of the Riccati equation. Among various results, he shows that

Proposition 3.1 (Proposition 2.1, [6]) *Any n-dimensional polynomial differential system $x' = P(x)$ with $\deg P = m$ can be embedded into a Riccati equation considered in an algebra $\mathscr{A}$ of dimension $\geq n$.*

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